

A LEVEL MATHS • EXTREME PAPERS

MATHS



MODEL ANSWERS

EXTREME PAPER: MECHANICS

SCORE

/90

PERCENTAGE

%

STEMUP

Paper: Mechanics (90 marks)

1 (10 marks) Edexcel | AQA | OCR A | OCR MEI | CIE

(a) Key idea: study the cubic $g(t) = t(t - 6)^2$ as a curve and slide the line $g = 3c$ across it.

$$x_A = 4t^2 - \frac{t^3}{3} \quad (\text{since } x_A = 0 \text{ at } t = 0). \quad \text{M1}$$

$$x_B = 12t - c. \text{ Meeting: } 4t^2 - \frac{t^3}{3} = 12t - c, \text{ so } t^3 - 12t^2 + 36t = 3c, \text{ that is } t(t - 6)^2 = 3c. \quad \text{A1}$$

$$g'(t) = 3t^2 - 24t + 36 = 3(t - 2)(t - 6). \quad \text{M1}$$

$$g(0) = 0, \text{ local maximum } g(2) = 32, \text{ local minimum } g(6) = 0, \text{ and } g(t) \rightarrow \infty \text{ as } t \rightarrow \infty. \quad \text{A1}$$

$$\text{Three distinct positive roots (one in each of } (0, 2), (2, 6), (6, \infty)) \text{ exactly when } 0 < 3c < 32, \text{ so } 0 < c < \frac{32}{3}. \quad \text{A1}$$

Answer: $0 < c < 32/3$ 5 marks

$$(b) t^3 - 12t^2 + 36t - 27 = 0. t = 3 \text{ works: } 27 - 108 + 108 - 27 = 0. \quad \text{M1}$$

$$(t - 3)(t^2 - 9t + 9) = 0, \text{ so } t = 3 \text{ or } t = \frac{9 \pm 3\sqrt{5}}{2}. \quad \text{A1}$$

Answer: $t = 3, (9 - 3\sqrt{5})/2, (9 + 3\sqrt{5})/2$ 2 marks

(c) Key idea: A may turn round, so check its velocity; then use B, whose position at the meetings equals A's.

$$v_A = t(8 - t) > 0 \text{ for } 0 < t < 8, \text{ and the third meeting is at } t_3 = \frac{9 + 3\sqrt{5}}{2} < 8 \text{ because } 3\sqrt{5} < 7 \text{ (} 45 < 49 \text{). So A does not reverse and distance = displacement.}$$

$$\text{Distance} = x_A(t_3) - x_A(t_1) = x_B(t_3) - x_B(t_1) = 12(t_3 - t_1). \quad \text{M1}$$

$$t_3 - t_1 = 3\sqrt{5}, \text{ so distance} = 36\sqrt{5} \text{ m (about 80.5 m).} \quad \text{A1}$$

Answer: $36\sqrt{5} \text{ m}$ 3 marks

2 (12 marks) Edexcel | AQA | OCR A | OCR MEI | CIE

(a) Key idea: resolve parallel and perpendicular to the plane; 'at right angles' means the velocity component along the plane is zero on arrival.

$$\text{Perpendicular to the plane (deceleration } g \cos \beta): 0 = U \sin \alpha T - \frac{1}{2} g \cos \beta T^2, \text{ so the flight} \quad \text{M1 A1}$$

$$\text{time is } T = \frac{2U \sin \alpha}{g \cos \beta}.$$

$$\text{Along the plane (deceleration } g \sin \beta): \text{ the component of velocity on arrival is } U \cos \alpha - g \sin \beta T, \text{ which must be 0.} \quad \text{M1}$$

$$\text{So } U \cos \alpha = g \sin \beta \times \frac{2U \sin \alpha}{g \cos \beta} = 2U \sin \alpha \tan \beta. \quad \text{M1}$$

$$\text{Hence } \cos \alpha = 2 \sin \alpha \tan \beta, \text{ that is } \tan \alpha \tan \beta = \frac{1}{2}. \quad \text{A1}$$

Answer: shown 5 marks

(b) Along the plane the particle decelerates uniformly from $U \cos \alpha$ to 0 in time T , so $OP =$

$$\frac{(U \cos \alpha)^2}{2g \sin \beta}. \quad \text{M1}$$

$$\tan \alpha = \frac{1}{2 \tan \beta}, \text{ so } \cos^2 \alpha = \frac{1}{1 + \tan^2 \alpha} = \frac{4 \tan^2 \beta}{4 \tan^2 \beta + 1} = \frac{4 \sin^2 \beta}{4 \sin^2 \beta + \cos^2 \beta} = \frac{4 \sin^2 \beta}{1 + 3 \sin^2 \beta}. \quad \text{M1}$$

$$OP = \frac{U^2}{2g \sin \beta} \times \frac{4 \sin^2 \beta}{1 + 3 \sin^2 \beta} = \frac{2U^2 \sin \beta}{g(1 + 3 \sin^2 \beta)}. \quad \text{A1}$$

Answer: shown 3 marks

$$(c) \text{ Let } s = \sin \beta \text{ and } f(s) = \frac{s}{1 + 3s^2}. \text{ Then } f'(s) = \frac{(1 + 3s^2) - 6s^2}{(1 + 3s^2)^2} = \frac{1 - 3s^2}{(1 + 3s^2)^2}. \quad \text{M1}$$

$$f'(s) = 0 \text{ gives } s = \frac{1}{\sqrt{3}}; f' > 0 \text{ for smaller } s \text{ and } f' < 0 \text{ for larger } s, \text{ so this is a maximum.} \quad \text{A1 B1}$$

$$\sin \beta = \frac{1}{\sqrt{3}}, \text{ so } \beta = 35.3^\circ \text{ (1 d.p.) and } OP_{\max} = \frac{2U^2}{g} \times \frac{1}{\sqrt{3}} / (1 + 1) = \frac{U^2}{\sqrt{3} g}. \quad \text{A1}$$

Answer: $\sin \beta = 1/\sqrt{3}$ ($\beta = 35.3^\circ$); $OP_{\max} = U^2/(\sqrt{3} g)$ 4 marks

3 (11 marks) Edexcel | AQA | OCR A | OCR MEI | CIE

(a) $R = mg \cos \alpha = \frac{4mg}{5}$, so friction is $\frac{1}{4}R = \frac{mg}{5}$.

B1

Moving down: $ma = mg \sin \alpha - \frac{mg}{5} = \frac{2mg}{5}$, so $a_d = \frac{2g}{5}$. From $d = \frac{1}{2}a_d t^2$: $t = \sqrt{2d/a_d} = \sqrt{5d/g}$.

M1 A1

Speed at O: $v^2 = 2a_d d = \frac{4gd}{5}$. Moving up: friction acts down the plane, so $a_u = g \sin \alpha + \frac{g}{5} = \frac{4g}{5}$.

A1

Distance up = $\frac{v^2}{2a_u} = \frac{d}{2}$.

Answer: $t = \sqrt{(5d/g)}$; rises $d/2$

4 marks

(b) Key idea: the same two accelerations apply every time, so each rise is half the distance of the previous fall.

At the top of a rise, $mg \sin \alpha = \frac{3mg}{5}$ exceeds the maximum friction $\frac{mg}{5}$, so P slides back down.

B1

Falling from a distance s gives $v^2 = 2a_d s$ and a rise of $\frac{a_d}{a_u}s = \frac{s}{2}$. So the successive rises are $\frac{d}{2}, \frac{d}{4}, \frac{d}{8}, \dots$

M1

Total distance = $d + 2(\frac{d}{2} + \frac{d}{4} + \dots) = d + 2d = 3d$.

A1

Answer: $3d$

3 marks

(c) Key idea: time scales with the square root of the distance, so the times form a geometric series with ratio $\frac{1}{\sqrt{2}}$.

Rise n has length $d/2^n$. Time up = $\sqrt{5d/2g} \times (1/\sqrt{2})^n$ and time down = $\sqrt{5d/g} \times (1/\sqrt{2})^n$, using $t = \sqrt{2s/a}$.

M1 A1

Sum over $n \geq 1$ (the series $1/\sqrt{2} + 1/2 + \dots$ has sum $\sqrt{2} + 1$): total = $(\sqrt{5d/2g} + \sqrt{5d/g})(\sqrt{2} + 1) = \sqrt{5d/g}(1 + \frac{1}{\sqrt{2}})(\sqrt{2} + 1) = \sqrt{5d/g}(2 + \frac{3\sqrt{2}}{2})$.

M1

Total time = $\sqrt{5d/g} + \sqrt{5d/g}(2 + \frac{3\sqrt{2}}{2}) = \frac{3(2 + \sqrt{2})}{2}\sqrt{5d/g}$.

A1

Answer: $(3(2 + \sqrt{2})/2) \sqrt{(5d/g)}$

4 marks**4 (13 marks) Edexcel | AQA | OCR A | OCR MEI | CIE**

(a) Key idea: sliding at both ends means friction is limiting at both ends ($F = \mu R$ at A and at B).

Let R and $F = \mu_1 R$ act at A (F towards the wall), S and $\mu_2 S$ at B (friction upwards).

Horizontal: $S = \mu_1 R$.

M1

Vertical: $R + \mu_2 S = W$, so $R(1 + \mu_1 \mu_2) = W$.

M1

Moments about A: $W a \cos \alpha = S(2a \sin \alpha) + \mu_2 S(2a \cos \alpha)$.

M1

Substitute $S = \mu_1 R$ and $R = \frac{W}{1 + \mu_1 \mu_2}$: $\frac{1}{2} = \frac{\mu_1}{1 + \mu_1 \mu_2}(\tan \alpha + \mu_2)$.

A1

$\tan \alpha = \frac{1 + \mu_1 \mu_2}{2\mu_1} - \mu_2 = \frac{1 - \mu_1 \mu_2}{2\mu_1}$.

A1

Answer: $\tan \alpha = (1 - \mu_1 \mu_2)/(2\mu_1)$

5 marks

(b) Key idea: the largest x corresponds to limiting friction at both ends, with total weight $2W$.

Horizontal: $S = \frac{1}{2}R$. Vertical: $R + \frac{1}{4}S = 2W$, so $R(1 + \frac{1}{8}) = 2W$ and $R = \frac{16W}{9}$.

M1

Moments about A ($\sin \alpha = \cos \alpha$): $W a \cos \alpha + W x \cos \alpha = S(2a \sin \alpha) + \frac{1}{4}S(2a \cos \alpha)$.

M1

$W(a + x) = 2aS(1 + \frac{1}{4}) = 2a \times \frac{8W}{9} \times \frac{5}{4} = \frac{20Wa}{9}$.

A1

$x = \frac{11a}{9}$ (about $1.22a$, which is less than $2a$).

A1

Answer: $x = 11a/9$

4 marks

(c) Key idea: friction only takes the value needed, so it is enough to exhibit one admissible set of forces. Take wall friction $G = \mu_2 S$ (limiting), ground normal reaction R and ground friction F

towards the wall.

Moments about A: $W a \cos \alpha = S(2a \sin \alpha) + \mu_2 S(2a \cos \alpha)$, so $S = \frac{W \cos \alpha}{2(\sin \alpha + \mu_2 \cos \alpha)}$. M1

Vertical: $R = W - \mu_2 S$. Horizontal: $F = S$. Equilibrium is possible if $F \leq \mu_1 R$, that is $S(1 + \mu_1 \mu_2) \leq \mu_1 W$. M1

Substituting S and simplifying: $\cos \alpha(1 + \mu_1 \mu_2) \leq 2\mu_1(\sin \alpha + \mu_2 \cos \alpha)$, i.e. $(1 - \mu_1 \mu_2) \cos \alpha \leq 2\mu_1 \sin \alpha$. A1

Since $\mu_1 \mu_2 \geq 1$ the left side is not positive while the right side is positive for $0^\circ < \alpha < 90^\circ$, so the condition holds. (Also $S < \frac{W}{2\mu_2}$, so $R > \frac{W}{2} > 0$.) A1

Answer: shown

4 marks

5 (12 marks) CIE

(a) Upwards, both forces act downwards: $m v \frac{dv}{dx} = -mg - mkv^2$. M1

$\int_u^0 \frac{v}{g + kv^2} dv = -\int_0^H dx$, using x measured upwards from O . M1

$\frac{1}{2k} [\ln(g + kv^2)]_u^0 = -H$, so $\frac{1}{2k} (\ln g - \ln(g + ku^2)) = -H$. A1

$H = \frac{1}{2k} \ln\left(\frac{g + ku^2}{g}\right) = \frac{1}{2k} \ln\left(1 + \frac{ku^2}{g}\right)$. A1

Answer: $H = (1/2k) \ln(1 + ku^2/g)$

4 marks

(b) Key idea: use the height from (a) to eliminate H .

Downwards: $m v \frac{dv}{dx} = mg - mkv^2$, so $\int_0^w \frac{v}{g - kv^2} dv = H$, giving $-\frac{1}{2k} \ln\left(1 - \frac{kw^2}{g}\right) = H$. M1

$1 - \frac{kw^2}{g} = e^{-2kH} = \frac{g}{g + ku^2}$ (using part (a)). M1

$\frac{kw^2}{g} = \frac{ku^2}{g + ku^2}$, so $w^2 = \frac{gu^2}{g + ku^2}$ and $\frac{1}{w^2} = \frac{1}{u^2} + \frac{k}{g}$. A1

Answer: shown

3 marks

(c) $w^2 = \frac{g \times \frac{3g}{k}}{4g} = \frac{3g}{4k}$. Time down: $t_d = \int_0^w \frac{dv}{g - kv^2}$. M1

$\frac{1}{g - kv^2} = \frac{1}{2\sqrt{gk}} \left(\frac{1}{\sqrt{g} - \sqrt{k}v} + \frac{1}{\sqrt{g} + \sqrt{k}v} \right)$, so $t_d = \frac{1}{2\sqrt{gk}} \ln\left(\frac{\sqrt{g} + \sqrt{k}w}{\sqrt{g} - \sqrt{k}w}\right)$. M1

$\sqrt{k}w/\sqrt{g} = \frac{\sqrt{3}}{2}$, and $\frac{2 + \sqrt{3}}{2 - \sqrt{3}} = (2 + \sqrt{3})^2$, so $t_d = \frac{\ln(2 + \sqrt{3})}{\sqrt{gk}}$. A1

Answer: $t = \ln(2 + \sqrt{3})/\sqrt{gk}$

3 marks

(d) $t_u = \int_0^u \frac{dv}{g + kv^2} = \frac{1}{\sqrt{gk}} \arctan(u\sqrt{\frac{k}{g}}) = \frac{1}{\sqrt{gk}} \arctan \sqrt{3} = \frac{\pi}{3\sqrt{gk}}$. M1

$t_d - t_u = \frac{\ln(2 + \sqrt{3}) - \frac{\pi}{3}}{\sqrt{gk}} \approx \frac{0.270}{\sqrt{gk}}$. A1

Answer: $0.270/\sqrt{gk}$

2 marks

6 (11 marks) Edexcel (IAL) | CIE

(a) Key idea: record velocities (towards the wall positive) collision by collision; every number is exact.

A and B collide with velocities a and b ($a > b$): $3a + b = 3a' + b'$ and $b' - a' = a - b$. M1

Solving: $a' = \frac{a+b}{2}$ and $b' = \frac{3a-b}{2}$. A1

Collision 1 ($a = 6, b = 0$): A has velocity 3, B has velocity 9. A1

B reaches the wall and returns at velocity -9 , then meets A (still moving at 3): collision 2 ($a = 3, b = -9$) gives $a' = -3, b' = 9$. A1

B again reaches the wall and returns at -9 . A is moving away at 3 and B follows at 9, so collision 3 ($a = -3, b = -9$) gives $a' = -6, b' = 0$. A1

After collision 3, A moves away from the wall at 6 m s^{-1} and B is at rest, so B never reaches the wall again and A and B never meet again. M1

Collisions in order: A-B, B-wall, A-B, B-wall, A-B: five in total. Final velocities: A 6 m s^{-1} away from the wall, B at rest. A1

Answer: A: 6 m/s away from the wall; B: at rest 7 marks

(b) Collision 1 is at 9 m from the wall (B has not moved), at $t = 3 \text{ s}$. B reaches the wall in 1 s (A is then 6 m away); B and A close at 12 m s^{-1} , so collision 2 is 0.5 s later at 4.5 m. B reaches the wall in 0.5 s, A is then 6 m from it; B follows A at relative speed 6 m s^{-1} : 1 s later B has travelled 9 m, so collision 3 is 9 m from the wall. M1

Time between first and last collisions = $1 + 0.5 + 0.5 + 1 = 3 \text{ s}$, at the original position of B. A1

Answer: 3 s 2 marks

(c) At each wall collision B's velocity changes from $+9$ to -9 : impulse magnitude $m(9 + 9) = 18m$. Total = $36m \text{ N s}$. M1

Check: momentum of A and B towards the wall changes from $3m(6) = 18m$ to $3m(-6) = -18m$, a change of $36m$. A1

Answer: $36m \text{ N s}$ 2 marks

7 (10 marks) Edexcel | AQA | OCR A | OCR MEI | CIE

(a) Along a smooth wire the acceleration is $g \sin \alpha$. B1

Starting from rest, $s = \frac{1}{2}(g \sin \alpha)t^2$. M1

Answer: shown 2 marks

(b) Key idea: take O as the origin and write the position in terms of α ; the factor $\sin \alpha$ appears twice.

Let $D = \frac{1}{2} g t^2$. The bead is at $(x, y) = (D \sin \alpha \cos \alpha, -D \sin^2 \alpha)$. M1

Then $x^2 + y^2 = D^2 \sin^2 \alpha (\cos^2 \alpha + \sin^2 \alpha) = D^2 \sin^2 \alpha = -Dy$, so $x^2 + (y + \frac{D}{2})^2 = \frac{D^2}{4}$. M1

A circle with centre $(0, -\frac{D}{2})$ on the vertical through O, and radius $\frac{D}{2} = \frac{g t^2}{4}$. A1

Answer: radius $g t^2/4$ 3 marks

(c) Key idea: at time t the beads are on the circle from (b); L is first reached when that circle just touches L.

The least time is when the circle of radius $r = \frac{g t^2}{4}$, with centre r below O on the vertical, touches L. M1

The vertical distance from the centre to L is $9.8 - r$, so its perpendicular distance to L is $(9.8 - r) \cos \beta$. Tangency: $(9.8 - r) \frac{3}{5} = r$. M1 A1

So $r = \frac{3 \times 9.8}{8} = 3.675$ and $t^2 = \frac{4r}{g} = \frac{3}{2}$, $t = \sqrt{3/2} = 1.22 \text{ s}$ (3 s.f.). A1

The wire is the chord from O to the point of contact. The radius to the contact point is perpendicular to L, so it makes angle β with the downward vertical, and the chord from O (the top of the circle's vertical diameter) makes half this angle: $\tan(\beta/2) = \frac{\sin \beta}{1 + \cos \beta} = \frac{4}{5/8/5} = \frac{1}{2}$. The wire B1

is at $\arctan \frac{1}{2} = 26.6^\circ$ to the downward vertical.

Answer: $t = \sqrt{(3/2)} = 1.22$ s; 26.6° to the vertical ($\tan = 1/2$)

5 marks

8 (11 marks) Edexcel (IAL) | CIE

(a) System: $3mg - mg = 4m a$, so $a = \frac{g}{2} = 4.9 \text{ m s}^{-2}$.

M1

$9.8 = \frac{1}{2}(4.9)t^2$ gives $t = 2$, and $v = 4.9 \times 2 = 9.8 \text{ m s}^{-1}$.

A1

Answer: $t = 2$ s, $v = 9.8$ m/s

2 marks

(b) Key idea: while the string is slack B is a projectile; use symmetry, then impulse and momentum along the string.

At $t = 2$ s, B is 9.8 m above the ground moving upwards at 9.8 m s^{-1} and is in free flight. It

M1

returns to this height (where the string is taut again) after $\frac{2 \times 9.8}{9.8} = 2$ s, moving downwards at 9.8 m s^{-1} .

Let J be the impulsive tension and w the common speed afterwards. A (upwards): $3m w = J$.

M1

B (downwards): $m(9.8) - J = m w$.

M1

So $3w = 9.8 - w$, giving $w = 2.45 \text{ m s}^{-1}$.

A1

Answer: 2.45 m/s

4 marks

(c) After the jerk the system moves with A rising and B falling and decelerates at $\frac{(3m - m)g}{4m} = 4.9 \text{ m s}^{-2}$.

M1

It stops after $\frac{2.45}{4.9} = 0.5$ s and returns, so A reaches the ground after 1 s with speed 2.45 m s^{-1} ,

A1

and B is then moving upwards at 2.45 m s^{-1} at height 9.8 m with the string slack.

This is the situation of part (b) with 9.8 replaced by 2.45, so all speeds and times are multiplied by

M1

$\frac{1}{4}$. The intervals between successive landings of A are 3 s ($= 2 + 1$), then $\frac{3}{4}$ s, $\frac{3}{16}$ s, ... a

geometric progression with ratio $\frac{1}{4}$.

A first lands at $t = 2$ s; the motion ceases at $2 + \frac{3}{1 - 1/4} = 6$ s.

A1

Answer: 6 s

4 marks

(d) For example: A does not rebound from the ground (real impacts have some rebound), or the string is inextensible (real strings stretch and absorb the jerks), or air resistance and pulley friction are neglected.

B1

Answer: e.g. no rebound of A / inextensible string

1 mark