

Exponential Functions

1 (2 marks)

$$f(4) = 3^4 = 81$$

B1

$$f(-2) = 3^{-2} = \frac{1}{3^2} = \frac{1}{9}$$

B1

Answer: (a) 81, (b) $\frac{1}{9}$

2 marks

2 (4 marks)

(a) $f(0) = 5 \times 2^0 = 5$

B1

$$f(3) = 5 \times 8 = 40$$

B1

Answer: $f(0) = 5$, $f(3) = 40$

2 marks

(b) $5 \times 2^x = 160$, so $2^x = 32$

M1

$$2^x = 2^5, \text{ so } x = 5$$

A1

Answer: $x = 5$

2 marks

3 (3 marks)

$$2^5 = 32$$

B1

$$10^{-3} = \frac{1}{10^3} = \frac{1}{1000}$$

B1

$$9^{3/2} = (\sqrt{9})^3 = 27$$

B1

Answer: (a) 32, (b) $\frac{1}{1000}$, (c) 27

3 marks

4 (3 marks)

Add indices: $e^{3x} \times e^{2x} = e^{5x}$

B1

Multiply indices: $(e^x)^4 = e^{4x}$

B1

Subtract indices: $e^{6x} \div e^{2x} = e^{4x}$

B1

Answer: (a) e^{5x} , (b) e^{4x} , (c) e^{4x}

3 marks

5 (4 marks)

$3 > 1$, so $y = 3^x$ is increasing

B1

$0 < 0.5 < 1$, so $y = (0.5)^x$ is decreasing

B1

$e^{-x} = (e^{-1})^x = (0.368\dots)^x$ with base below 1, so decreasing

B1

$1.01 > 1$, so increasing

B1

Answer: (a) increasing, (b) decreasing, (c) decreasing, (d) increasing

4 marks

6 (5 marks)

(a) At (0, 4): $k a^0 = 4$, so $k = 4$

B1

At (2, 36): $4a^2 = 36$, so $a^2 = 9$

M1

$a > 0$, so $a = 3$

A1

Answer: $k = 4$, $a = 3$

3 marks

(b) $y = 4 \times 3^{-1}$

M1

$$y = \frac{4}{3}$$

A1

Answer: $y = \frac{4}{3}$

2 marks

7 (4 marks)

(a) Add the indices on the top: $e^{5x-2x} = e^{3x}$

M1

Subtract the index below: $\frac{e^{3x}}{e^x} = e^{2x}$

A1

Answer: e^{2x}

2 marks

(b) $f(\ln 4) = e^{2 \ln 4} + 3 = (e^{\ln 4})^2 + 3$

M1

$$= 4^2 + 3 = 19$$

Answer: 19

A1

2 marks

8 (4 marks)

$$(e^x + 2)^2 = e^{2x} + 4e^x + 4$$

M1 A1

$$(e^x - e^{-x})^2 = e^{2x} - 2e^x e^{-x} + e^{-2x} = e^{2x} - 2 + e^{-2x}$$

M1 A1

Answer: (a) $e^{2x} + 4e^x + 4$, (b) $e^{2x} - 2 + e^{-2x}$

4 marks

9 (2 marks)

Multiplier 1.03 each year: 2500×1.03^8

M1

= £3166.93..., so £3167

A1

Answer: £3167

2 marks

10 (3 marks)

Multiplier 0.88: $V = 18000 \times 0.88^t$

B1

$t = 5$: $V = 18000 \times 0.88^5 = 9499.17$

M1

= £9500 (nearest £100)

A1

Answer: $V = 18000 \times 0.88^t$; £9500

3 marks

11 (3 marks)

$2^x > 0$ for all x , so $f(x) > 3$

B1

$2^x + 3 = 11$, so $2^x = 8 = 2^3$

M1

$x = 3$

A1

Answer: (a) $f(x) > 3$, (b) $x = 3$

3 marks

12 (6 marks)

(a) $t = 0$: $P = 800 \times 1 = 800$

B1

Answer: 800

1 mark

(b) $P = 800 \times 1.05^{10}$

M1

= 1303.1..., so 1303 bacteria

A1

Answer: 1303

2 marks

(c) $t = 12$: $P = 800 \times 1.05^{12} = 1437 < 1500$

M1

$t = 13$: $P = 800 \times 1.05^{13} = 1509 > 1500$

M1

So the smallest whole number of hours is 13

A1

Answer: 13 hours

3 marks

13 (5 marks)

(a) $4^{x+1} = 4^x \times 4^1 = 4 \times 4^x$

M1

$4 \times 4^x - 4^x = 3 \times 4^x$

A1

Answer: shown

2 marks

(b) $3 \times 4^x = 192$

M1

$4^x = 64 = 4^3$

A1

$x = 3$

A1

Answer: $x = 3$

3 marks

14 (6 marks)

$x = 0$: $a = 5$

B1

$x = 1$: $5b = 7.5$, so $b = 1.5$

M1 A1

Check $x = 2$: $5 \times 1.5^2 = 11.25$

B1

$x = 6$: $y = 5 \times 1.5^6 = 56.953$

M1

$y = 57.0$ (3 s.f.)

A1

Answer: $a = 5$, $b = 1.5$, $y = 57.0$

6 marks

15 (4 marks)

$$y = 4e^{2x} \text{ so } \frac{dy}{dx} = 8e^{2x}$$

M1 A1

$$\text{At } x = \ln 3: 8e^{2 \ln 3} = 8 \times e^{\ln 9} = 8 \times 9$$

M1

$$\text{Gradient} = 72$$

A1

$$\text{Answer: } \frac{dy}{dx} = 8e^{2x}; \text{ gradient } 72$$

4 marks

16 (3 marks)

$$9^x = 3^{2x}, 27^x = 3^{3x}$$

M1

$$\text{Numerator: } 3^{2x} \times 3^{x+1} = 3^{3x+1}$$

M1

$$\frac{3^{3x+1}}{3^{3x}} = 3^1 = 3$$

A1

$$\text{Answer: } 3$$

3 marks

17 (8 marks)

$$(a) P = 1200 \times 0.94^3 = 996.70$$

M1

$$= 997$$

A1

$$\text{Answer: } 997$$

2 marks

$$(b) P(10) = 1200 \times 0.94^{10} = 646.34$$

M1

$$\text{Decrease} = (1 - 0.94^{10}) \times 100 = 46.1\%$$

M1 A1

$$\text{Answer: } 46.1\%$$

3 marks

$$(c) P(14) = 504.6 > 500, P(15) = 474.4 < 500$$

M1 A1

$$15 \text{ years after the start of 2020, so 2035}$$

A1

$$\text{Answer: } 2035$$

3 marks

18 (6 marks)

$$(a) g(2) = 3, \text{ so } fg(2) = 3^3 = 27$$

M1 A1

$$f(2) = 9, \text{ so } gf(2) = 81 - 1 = 80$$

A1

$$\text{Answer: } fg(2) = 27, gf(2) = 80$$

3 marks

$$(b) gf(x) = (3^x)^2 - 1 = 9^x - 1$$

B1

$$9^x = 81 = 9^2$$

M1

$$x = 2$$

A1

$$\text{Answer: } x = 2$$

3 marks

19 (7 marks)

$$(a) gf(x) = g(6e^{-x})$$

M1

$$= 6e^{-x} + 3$$

A1

$$\text{Answer: } gf(x) = 6e^{-x} + 3$$

2 marks

$$(b) e^{-x} > 0 \text{ for all } x, \text{ so } 6e^{-x} > 0$$

B1

$$\text{Range: } gf(x) > 3$$

B1

$$\text{Answer: } gf(x) > 3$$

2 marks

$$(c) 6e^{-x} + 3 = 4.5, \text{ so } e^{-x} = \frac{1}{4}$$

M1

$$-x = \ln\left(\frac{1}{4}\right) = -\ln 4$$

M1

$$x = \ln 4$$

A1

$$\text{Answer: } x = \ln 4$$

3 marks

20 (7 marks)

$$(a) 2^x > 0 \text{ for all } x, \text{ so } 7 - 2^x < 7$$

B1

$$\text{Range: } f(x) < 7$$

B1

$$\text{Answer: } f(x) < 7$$

2 marks

$$(b) 7 - 2^x = -9, \text{ so } 2^x = 16$$

M1

$$x = 4$$

A1

$$\text{Answer: } x = 4$$

2 marks

(c) $x = 0: y = 7 - 1 = 6$, so the point (0, 6)

B1

$y = 0: 2^x = 7$, so $x = \frac{\ln 7}{\ln 2}$

M1

$x = 2.81$, so the point (2.81, 0)

A1

Answer: (0, 6) and (2.81, 0)

3 marks

21 (8 marks)

(a) $[f(x)]^2 = \frac{e^{2x} + 2 + e^{-2x}}{4}$

M1

$[g(x)]^2 = \frac{e^{2x} - 2 + e^{-2x}}{4}$

M1

Difference = $\frac{4}{4} = 1$

A1

Answer: shown

3 marks

(b) $f(2x) = \frac{e^{2x} + e^{-2x}}{2}$

B1

$2[f(x)]^2 - 1 = \frac{e^{2x} + 2 + e^{-2x}}{2} - 1$

M1

$= \frac{e^{2x} + e^{-2x}}{2} = f(2x)$

A1

Answer: shown

3 marks

(c) $[g(a)]^2 = [f(a)]^2 - 1 = \frac{25}{16} - 1 = \frac{9}{16}$

M1

$g(a) = \pm \frac{3}{4}$

A1

Answer: $g(a) = \pm \frac{3}{4}$

2 marks

22 (7 marks)

(a) $A k = 1650$ and $A k^3 = 1996.5$

M1

Divide: $k^2 = \frac{1996.5}{1650} = 1.21$

A1

$k > 0$, so $k = 1.1$

A1

$A = \frac{1650}{1.1} = 1500$

A1

Answer: $k = 1.1$, $A = 1500$

4 marks

(b) $V = 1500 \times 1.1^{10}$

M1

$= £3890.61$

A1

Answer: £3890.61

2 marks

(c) The model gives unlimited growth at a fixed rate of 10% per year, but interest rates change over long periods

E1

Answer: fixed growth rate unrealistic over long periods

1 mark

23 (4 marks)

Expand: $(2^x + 2^{-x})^2 = 2^{2x} + 2 \times 2^x \times 2^{-x} + 2^{-2x} = 2^{2x} + 2 + 2^{-2x}$

M1 A1

$(2^x - 2^{-x})^2 = 2^{2x} - 2 \times 2^x \times 2^{-x} + 2^{-2x} = 2^{2x} - 2 + 2^{-2x}$

M1

Subtract: $(2^{2x} + 2 + 2^{-2x}) - (2^{2x} - 2 + 2^{-2x}) = 4$

A1

Answer: shown

4 marks

24 (7 marks)

(a) $V_0 k^2 = 4500$ and $V_0 k^5 = 7776$

M1

Divide: $k^3 = \frac{7776}{4500} = 1.728$

M1

$k = 1.2$, $V_0 = \frac{4500}{1.44} = 3125$

A1 A1

Answer: $k = 1.2$, $V_0 = 3125$

4 marks

(b) $k = 1.2$ means visitors increase by 20% each month

E1

$t = 10: 3125 \times 1.2^{10} = 19349$; $t = 11: 23219$

M1

Month 11

A1

Answer: 20% growth per month; month 11

3 marks

25 (5 marks)

At P: $2^x = 8 \times 2^{-x}$

M1

Multiply by 2^x : $2^{2x} = 8 = 2^3$

M1

$2x = 3$, so $x = \frac{3}{2}$

A1

$y = 2^{3/2} = 2\sqrt{2}$

M1

$P = (\frac{3}{2}, 2\sqrt{2})$

A1

Answer: P = ($\frac{3}{2}$, $2\sqrt{2}$)

5 marks

26 (7 marks)

(a) $x = 0$: $p + q = 7$

B1

$\frac{dy}{dx} = 2q e^{2x}$, so at $x = 0$: $2q = 10$

M1 A1

$q = 5$, $p = 2$

A1

Answer: p = 2, q = 5

4 marks

(b) $10e^{2x} = 40$, so $e^{2x} = 4$

M1

$2x = \ln 4 = 2 \ln 2$

M1

$x = \ln 2$

A1

Answer: x = ln 2

3 marks

27 (9 marks)

(a) $x = 1$: LHS = $3 + 3 = 6$

B1

RHS = 9, and $6 \neq 9$

B1

Answer: $6 \neq 9$

2 marks

(b) The student has treated adding the terms as if it were multiplying them (adding indices); $3^x \times$

E1

$3^x = 3^{2x} = 9^x$

$3^x + 3^x = 2 \times 3^x$

E1

Answer: 2×3^x

2 marks

(c) $2 \times 3^x = 9^x = (3^2)^x$

M1

Let $y = 3^x$: $y^2 - 2y = 0$, so $y(y - 2) = 0$

M1

$y > 0$, so $y = 2$, that is $3^x = 2$

A1

$x \ln 3 = \ln 2$

M1

$x = \frac{\ln 2}{\ln 3} = 0.631$

A1

Answer: x = 0.631

5 marks

28 (8 marks)

(a) $f(n+1) - f(n) = 2^{n+1} - 2^n = 2 \times 2^n - 2^n$

M1

$= 2^n = f(n)$

A1

Answer: shown

2 marks

(b) From (a), $2^n = 2^{n+1} - 2^n$, so the sum is $\sum (2^{n+1} - 2^n)$

M1

Writing out the terms, $(2^2 - 2^1) + (2^3 - 2^2) + \dots + (2^{k+1} - 2^k)$: all middle terms cancel

M1

Sum = $2^{k+1} - 2^1 = 2^{k+1} - 2$

A1

Answer: shown

3 marks

(c) $2^{k+1} - 2 > 1\,000\,000$, so $2^{k+1} > 1\,000\,002$

M1

$2^{19} = 524\,288$ and $2^{20} = 1\,048\,576$

M1

$k+1 = 20$, so $k = 19$

A1

Answer: k = 19

3 marks

29 (9 marks)

(a) $f(x+y) = a^{x+y} = a^x \times a^y = f(x)f(y)$

M1 A1

$f(x)f(-x) = a^x \times a^{-x} = a^0 = 1$

M1 A1

Answer: shown

4 marks

(b) $(a^{x/2} - a^{-x/2})^2 \geq 0$ as it is a square

M1

Expand: $a^x - 2a^0 + a^{-x} \geq 0$

M1 A1

So $f(x) + f(-x) \geq 2$

E1

Equality when $a^{x/2} = a^{-x/2}$, i.e. $x = 0$

A1

Answer: proved; equality at $x = 0$

5 marks

30 (10 marks)

(a) $A = 5000 \times 1.04^n$

B1

$B = 4000 \times 1.06^n$

B1

Answer: $A = 5000 \times 1.04^n$, $B = 4000 \times 1.06^n$

2 marks

(b) $n = 11$: $A = 7697.27$, $B = 7593.19$, so $B < A$

M1 A1

$n = 12$: $A = 8005.16$, $B = 8048.79$, so $B > A$

M1 A1

Answer: shown

4 marks

(c) $4000 \times 1.06^n > 5000 \times 1.04^n$, so $(\frac{1.06}{1.04})^n > 1.25$

M1 A1

Take logs: $n \ln(\frac{1.06}{1.04}) > \ln 1.25$, and $\ln(\frac{1.06}{1.04}) > 0$

M1

$n > 11.71$, so the first whole number is $n = 12$

E1

Answer: $n > 11.7$, so 12 years

4 marks

Laws of Logarithms**1 (3 marks)**

(a) $2^5 = 32$, so $\log_2 32 = 5$

B1

Answer: 5

1 mark

(b) $0.001 = 10^{-3}$, so $\log_{10} 0.001 = -3$

B1

Answer: -3

1 mark

(c) $\frac{1}{25} = 5^{-2}$, so $\log_5 \frac{1}{25} = -2$

B1

Answer: -2

1 mark

2 (3 marks)

$3 \log_a 2 = \log_a 2^3 = \log_a 8$

M1

$\log_a 8 + \log_a 5 - \log_a 10 = \log_a \frac{8 \times 5}{10}$

M1

$= \log_a 4$

A1

Answer: $\log_a 4$

3 marks

3 (2 marks)

Add logs: $\log_3(4 \times 5) = \log_3 20$

B1

Subtract logs: $\log_7(\frac{40}{8}) = \log_7 5$

B1

Answer: (a) $\log_3 20$, (b) $\log_7 5$

2 marks

4 (3 marks)

$\log_3(\frac{54}{2}) = \log_3 27 = 3$

B1

$\log_6(4 \times 9) = \log_6 36 = 2$

B1

$\log_2 6^3 - \log_2 27 = \log_2(\frac{216}{27}) = \log_2 8 = 3$

B1

Answer: (a) 3, (b) 2, (c) 3

3 marks

5 (3 marks)

$$\ln 8 = \ln 2^3 = 3 \ln 2$$

B1

$$\ln 2^{-2} = -2 \ln 2$$

B1

$$\ln 2^{1/2} = \frac{1}{2} \ln 2$$

B1

Answer: (a) $3 \ln 2$, (b) $-2 \ln 2$, (c) $\frac{1}{2} \ln 2$

3 marks

6 (4 marks)

(a) $12 = 2^2 \times 3$, so $\log_a 12 = 2 \log_a 2 + \log_a 3$

M1

$$= 2p + q$$

A1

Answer: $2p + q$

2 marks

(b) $\log_a \frac{8}{9} = 3 \log_a 2 - 2 \log_a 3$

M1

$$= 3p - 2q$$

A1

Answer: $3p - 2q$

2 marks

7 (4 marks)

(a) $\log_4 x = \frac{\log_2 x}{\log_2 4}$

M1

$$\log_2 4 = 2, \text{ so } \log_4 x = \frac{1}{2} \log_2 x$$

A1

Answer: shown

2 marks

(b) $\log_8 32 = \frac{\log_2 32}{\log_2 8}$

M1

$$= \frac{5}{3}$$

A1

Answer: $\frac{5}{3}$

2 marks

8 (3 marks)

Let $\log_9 27 = x$, so $9^x = 27$

M1

$$3^{2x} = 3^3, \text{ so } 2x = 3$$

M1

$$x = \frac{3}{2}$$

A1

Answer: $\frac{3}{2}$

3 marks

9 (4 marks)

$$\log_a a^5 = 5$$

B1

$$10^{\log_{10} 7} = 7$$

B1

$$\ln e^{3x} = 3x$$

B1

$$e^{\ln 5} = 5$$

B1

Answer: (a) 5, (b) 7, (c) $3x$, (d) 5

4 marks

10 (4 marks)

$$2 = \log_{10} 100, \text{ so the sum is } \log_{10} 300$$

M1 A1

$$1 = \log_{10} 10, \text{ so } \log_{10} \left(\frac{10}{4}\right) = \log_{10} 2.5$$

M1 A1

Answer: (a) $\log_{10} 300$, (b) $\log_{10} 2.5$

4 marks

11 (3 marks)

$$\log_5 20 = \frac{\ln 20}{\ln 5}$$

M1

$$= \frac{2.9957}{1.6094} = 1.861$$

M1 A1

Answer: 1.86

3 marks

12 (3 marks)

$$\frac{y}{5} = 3^{2x} \quad \text{M1}$$

$$2x = \log_3\left(\frac{y}{5}\right) \quad \text{M1}$$

$$x = \frac{1}{2} \log_3\left(\frac{y}{5}\right) \quad \text{A1}$$

$$\text{Answer: } x = \frac{1}{2} \log_3\left(\frac{y}{5}\right) \quad \text{3 marks}$$

13 (4 marks)

$$\ln 64 = 6 \ln 2, \ln 16 = 4 \ln 2, \ln 8 = 3 \ln 2 \quad \text{M1}$$

$$\frac{5}{6} \times 6 \ln 2 - \frac{3}{2} \times 4 \ln 2 + 3 \ln 2 \quad \text{M1}$$

$$= 5 \ln 2 - 6 \ln 2 + 3 \ln 2 = 2 \ln 2 \quad \text{A1}$$

$$k = 2 \quad \text{A1}$$

$$\text{Answer: } k = 2 \quad \text{4 marks}$$

14 (4 marks)

$$2 \log_2 6 = \log_2 36 \quad \text{M1}$$

$$\log_2 36 - \log_2 9 = \log_2 \frac{36}{9} = \log_2 4 \quad \text{M1}$$

$$\log_2 4 = 2 \quad \text{A1}$$

$$2 + 3 = 5 \quad \text{A1}$$

$$\text{Answer: shown} \quad \text{4 marks}$$

15 (7 marks)

$$\text{(a)} \ln\left(\frac{x^2 \sqrt{y}}{e}\right) = \ln x^2 + \ln y^{1/2} - \ln e \quad \text{M1}$$

$$= 2 \ln x + \frac{1}{2} \ln y - 1 \quad \text{M1}$$

$$= 2a + \frac{1}{2} b - 1 \quad \text{A1}$$

$$\text{Answer: } 2a + \frac{1}{2} b - 1 \quad \text{3 marks}$$

$$\text{(b)} 2a - b = 3 \text{ and } a + b = 6 \quad \text{M1 M1}$$

$$\text{Adding: } 3a = 9, \text{ so } a = 3 \text{ and } b = 3 \quad \text{A1}$$

$$x = e^3, y = e^3 \quad \text{A1}$$

$$\text{Answer: } x = e^3, y = e^3 \quad \text{4 marks}$$

16 (5 marks)

$$\text{(a)} x^2 - 9 = (x - 3)(x + 3) \quad \text{M1}$$

$$\log_{10}\left(\frac{(x-3)(x+3)}{x+3}\right) = \log_{10}(x-3) \quad \text{M1 A1}$$

$$\text{Answer: shown} \quad \text{3 marks}$$

$$\text{(b)} \log_{10}(x-3) = 1 \quad \text{M1}$$

$$x - 3 = 10, \text{ so } x = 13 \quad \text{A1}$$

$$\text{Answer: } x = 13 \quad \text{2 marks}$$

17 (4 marks)

$$\text{Let } \log_a b = m, \text{ so } a^m = b \quad \text{M1}$$

$$\text{Take } \log_b: m \log_b a = \log_b b = 1 \quad \text{M1 A1}$$

$$\text{So } \log_a b \times \log_b a = m \times \frac{1}{m} = 1 \quad \text{E1}$$

$$\text{Answer: proved} \quad \text{4 marks}$$

18 (3 marks)

$$\log_4(8x) = \frac{\log_2(8x)}{\log_2 4}$$

M1

$$= \frac{\log_2 8 + \log_2 x}{2} = \frac{3 + p}{2}$$

M1 A1

Answer: $\frac{3 + p}{2}$

3 marks

19 (4 marks)

$$\log_a x^2 + \log_a y^{1/2} - \log_a a$$

M1

$$= 2(3) + \frac{1}{2}(-2) - 1$$

M1 A1

$$= 4$$

A1

Answer: 4

4 marks

20 (5 marks)

(a) $\frac{I}{I_0} = 10^{5.8}$

M1

$$I = 10^{5.8} I_0$$

A1

Answer: $I = 10^{5.8} I_0$

2 marks

(b) Ratio = $\frac{10^{7.2} I_0}{10^{5.8} I_0} = 10^{1.4}$

M1 A1

$$= 25.1 \approx 25$$

A1

Answer: ratio $10^{1.4} \approx 25.1$

3 marks

21 (4 marks)

$$\ln(16e^3) = \ln 16 + \ln e^3 = 4 \ln 2 + 3$$

M1 A1

$$\ln\left(\frac{8}{e}\right) = \ln 8 - \ln e = 3 \ln 2 - 1$$

M1

$$\text{Difference} = 4 \ln 2 + 3 - 3 \ln 2 + 1 = 4 + \ln 2$$

A1

Answer: $4 + \ln 2$

4 marks

22 (9 marks)

(a) Let $\log_a b = p$, so $a^p = b$

M1

Take $\log_b$ of both sides: $p \log_b a = \log_b b = 1$

A1

So $p = \frac{1}{\log_b a}$, that is $\log_a b = \frac{1}{\log_b a}$

A1

Answer: proved

3 marks

(b) Using natural logs: $\frac{\ln 3}{\ln 2} \times \frac{\ln 4}{\ln 3} \times \frac{\ln 5}{\ln 4} \times \frac{\ln 8}{\ln 5}$

M1

Everything cancels except $\frac{\ln 8}{\ln 2} = \log_2 8 = 3$

A1

Answer: 3

2 marks

(c) Let $t = \log_2 x$; then $\log_x 2 = \frac{1}{t}$

M1

$$\frac{1}{t} + t = \frac{5}{2}, \text{ so } 2t^2 - 5t + 2 = 0$$

M1

$$(2t - 1)(t - 2) = 0, \text{ so } t = \frac{1}{2} \text{ or } t = 2$$

A1

$$x = 2^{1/2} = \sqrt{2} \text{ or } x = 2^2 = 4$$

A1

Answer: $x = \sqrt{2}$ or $x = 4$

4 marks

23 (5 marks)

$$\frac{1}{\log_a x} = \log_x a \text{ (change of base to base } x) \quad \text{M1}$$

$$\text{LHS} = \log_x 2 + \log_x 3 + \log_x 5 \quad \text{A1}$$

$$= \log_x(2 \times 3 \times 5) = \log_x 30 \quad \text{M1 A1}$$

$$= \frac{1}{\log_{30} x} = \text{RHS} \quad \text{A1}$$

Answer: proved 5 marks

24 (7 marks)

$$\text{(a)} \log_{10}(x - y)^2 = \log_{10}(xy) \quad \text{M1}$$

$$(x - y)^2 = xy \quad \text{M1}$$

$$x^2 - 2xy + y^2 = xy, \text{ so } x^2 - 3xy + y^2 = 0 \quad \text{A1}$$

Answer: shown 3 marks

$$\text{(b)} \text{ Divide by } y^2: r^2 - 3r + 1 = 0 \text{ where } r = \frac{x}{y} \quad \text{M1}$$

$$r = \frac{3 \pm \sqrt{5}}{2} \quad \text{A1}$$

$$x > y \text{ so } r > 1: r = \frac{3 + \sqrt{5}}{2} \quad \text{E1 A1}$$

Answer: $\frac{3 + \sqrt{5}}{2}$ 4 marks

25 (6 marks)

$$\text{(a)} \text{ Change base to } a: \log_{a^n} b = \frac{\log_a b}{\log_a a^n} \quad \text{M1}$$

$$= \frac{\log_a b}{n} \quad \text{M1 A1}$$

Answer: proved 3 marks

$$\text{(b)} \log_{27} 9 = \frac{\log_3 9}{\log_3 27} = \frac{2}{3} \quad \text{M1}$$

$$\log_{125} 25 = \frac{2}{3} \quad \text{A1}$$

$$\text{Sum} = \frac{4}{3} \quad \text{A1}$$

Answer: $\frac{4}{3}$ 3 marks

26 (9 marks)

$$\text{(a)} \log_4 y = \frac{\log_2 y}{\log_2 4} \quad \text{M1}$$

$$= \frac{\log_2 y}{2} \quad \text{A1}$$

Answer: shown 2 marks

$$\text{(b)} \text{ Let } X = \log_2 x \text{ and } Y = \log_2 y. \text{ Then } X + \frac{1}{2} Y = 4 \text{ and } Y - \frac{1}{2} X = \frac{1}{2} \quad \text{M1 M1}$$

$$\text{Multiply: } 2X + Y = 8 \text{ and } 2Y - X = 1 \quad \text{A1}$$

$$\text{Eliminate: } 2(2Y - 1) + Y = 8, \text{ so } Y = 2 \text{ and } X = 3 \quad \text{A1}$$

$$x = 2^3 = 8 \text{ and } y = 2^2 = 4 \quad \text{A1}$$

Answer: $x = 8, y = 4$ 5 marks

$$\text{(c)} \log_8 4 = \frac{\log_2 4}{\log_2 8} \quad \text{M1}$$

$$= \frac{2}{3} \quad \text{A1}$$

Answer: $\frac{2}{3}$ 2 marks

27 (8 marks)

$$(a) (a + b)^2 = a^2 + b^2 + 2ab$$

M1

$$= 7ab + 2ab = 9ab$$

M1

$$\text{So } \left(\frac{a+b}{3}\right)^2 = ab$$

A1

$$\text{Take natural logs: } 2 \ln\left(\frac{a+b}{3}\right) = \ln(ab) = \ln a + \ln b$$

M1

$$\ln\left(\frac{a+b}{3}\right) = \frac{1}{2}(\ln a + \ln b)$$

A1

Answer: shown

5 marks

$$(b) \ln\left(\frac{a+b}{3}\right) = \frac{1}{2} \times 6 = 3$$

M1

$$\ln(a + b) - \ln 3 = 3$$

M1

$$\ln(a + b) = 3 + \ln 3$$

A1

Answer: 3 + ln 3

3 marks

28 (6 marks)

$$(a) y^2 = xz \text{ (geometric)}$$

M1

$$2 \ln y = \ln x + \ln z$$

M1 A1

$$\text{So } \ln y - \ln x = \ln z - \ln y: \text{ arithmetic}$$

E1

Answer: proved

4 marks

$$(b) \ln 6 - \ln 2 = \ln 3$$

M1 A1

Answer: ln 3

2 marks

29 (5 marks)

$$\text{Let } t = \log_2 3. \text{ Then } \log_3 2 = \frac{1}{t} \text{ (reciprocal of change of base)}$$

M1

$$t + \frac{1}{t} - 2 = \frac{(t-1)^2}{t}$$

M1 A1

$$t > 0, \text{ and } t \neq 1 \text{ since } 2 \neq 3, \text{ so } (t-1)^2 > 0$$

M1

$$\text{Hence } t + \frac{1}{t} > 2$$

E1

Answer: proved

5 marks

30 (5 marks)

$$\text{Suppose } \log_2 5 = \frac{p}{q} \text{ with } p, q \text{ positive integers}$$

M1

$$\text{Then } 2^{p/q} = 5, \text{ so } 2^p = 5^q$$

M1 A1

$$2^p \text{ is even } (p \geq 1) \text{ but } 5^q \text{ is odd}$$

M1

$$\text{This is a contradiction, so } \log_2 5 \text{ is irrational}$$

E1

Answer: proved

5 marks

Exponential Equations**1 (2 marks)**

$$32 = 2^5, \text{ so } x = 5$$

B1

$$\frac{1}{27} = 3^{-3}, \text{ so } x = -3$$

B1

Answer: (a) x = 5, (b) x = -3

2 marks

2 (2 marks)

$$x \ln 5 = \ln 12$$

M1

$$x = \frac{\ln 12}{\ln 5} = 1.54$$

A1

Answer: x = 1.54

2 marks

3 (3 marks)

$$2^{x+1} = 2^4, \text{ so } x + 1 = 4 \text{ and } x = 3$$

B1

$$5^x = 5^{-1}, \text{ so } x = -1$$

B1

$$2^{2x} = 2^1, \text{ so } x = \frac{1}{2}$$

B1

Answer: (a) 3, (b) -1, (c) $\frac{1}{2}$

3 marks

4 (2 marks)

$$x = \ln 7$$

B1

$$2x = \ln 9, \text{ so } x = \frac{1}{2} \ln 9 = \ln 3$$

B1

Answer: (a) $\ln 7$, (b) $\ln 3$

2 marks

5 (2 marks)

$$3^x = 20, \text{ so } x = \frac{\ln 20}{\ln 3}$$

M1

$$x = 2.727$$

A1

Answer: 2.73

2 marks

6 (2 marks)

$$\text{Take logs: } x \ln 1.5 = \ln 10$$

M1

$$x = \frac{\ln 10}{\ln 1.5} = 5.679$$

A1

Answer: 5.68

2 marks

7 (3 marks)

$$4 = 2^2 \text{ and } 8 = 2^3, \text{ so } 2^{2x} = 2^{3(x-1)}$$

M1

$$2x = 3x - 3$$

M1

$$x = 3$$

A1

Answer: $x = 3$

3 marks

8 (5 marks)

(a) $2x = \ln 7$

M1

$$x = \frac{1}{2} \ln 7$$

A1

Answer: $x = \frac{1}{2} \ln 7$

2 marks

(b) $(3x - 1) \ln 2 = \ln 10$

M1

$$3x - 1 = \frac{\ln 10}{\ln 2} = 3.3219...$$

M1

$$x = 1.44$$

A1

Answer: $x = 1.44$

3 marks

9 (3 marks)

$$3^{3x} = 3^{2(x+2)}$$

M1

$$3x = 2x + 4$$

M1

$$x = 4$$

A1

Answer: $x = 4$

3 marks

10 (3 marks)

$$2^{x+1} = 2 \times 2^x, \text{ so } 3 \times 2^x = 48$$

M1

$$2^x = 16$$

M1

$$x = 4$$

A1

Answer: $x = 4$

3 marks

11 (3 marks)

$$3x - 1 = \ln 5$$

M1

$$x = \frac{1 + \ln 5}{3}$$

A1

$$x = 0.870$$

A1

$$\text{Answer: } x = \frac{1 + \ln 5}{3} = 0.870$$

3 marks

12 (3 marks)

$$(2x + 1) \ln 5 = \ln 40$$

M1

$$2x + 1 = 2.2920$$

M1

$$x = 0.646$$

A1

$$\text{Answer: } 0.646$$

3 marks

13 (4 marks)

$$\text{Let } y = 3^x, \text{ so } 9^x = y^2: y^2 - 4y + 3 = 0$$

M1

$$(y - 1)(y - 3) = 0, \text{ so } y = 1 \text{ or } y = 3$$

A1

$$3^x = 1 \text{ gives } x = 0$$

A1

$$3^x = 3 \text{ gives } x = 1$$

A1

$$\text{Answer: } x = 0 \text{ or } x = 1$$

4 marks

14 (6 marks)

$$(a) t = 0: T = 20 + 70 = 90^\circ\text{C}$$

B1

$$\text{Answer: } 90^\circ\text{C}$$

1 mark

$$(b) 20 + 70e^{-0.05t} = 50, \text{ so } 70e^{-0.05t} = 30$$

M1

$$e^{-0.05t} = \frac{3}{7}$$

A1

$$-0.05t = \ln\left(\frac{3}{7}\right)$$

M1

$$t = 20 \ln\left(\frac{7}{3}\right) = 16.9 \text{ minutes}$$

A1

$$\text{Answer: } 16.9 \text{ minutes}$$

4 marks

$$(c) \text{ As } t \rightarrow \infty, e^{-0.05t} \rightarrow 0, \text{ so } T \rightarrow 20^\circ\text{C}, \text{ the room temperature}$$

E1

$$\text{Answer: } T \text{ tends to } 20^\circ\text{C}$$

1 mark

15 (4 marks)

$$\text{Let } u = e^x: u^2 - 5u + 6 = 0$$

M1

$$(u - 2)(u - 3) = 0$$

A1

$$e^x = 2 \text{ or } 3$$

A1

$$x = \ln 2 \text{ or } x = \ln 3$$

A1

$$\text{Answer: } x = \ln 2, x = \ln 3$$

4 marks

16 (5 marks)

$$2^{x+y} = 2^6, \text{ so } x + y = 6$$

M1 A1

$$3^{x-y} = 3^4, \text{ so } x - y = 4$$

M1 A1

$$x = 5, y = 1$$

A1

$$\text{Answer: } x = 5, y = 1$$

5 marks

17 (4 marks)

$$\text{Multiply by } e^x: e^{2x} + 12 = 7e^x$$

M1

$$e^{2x} - 7e^x + 12 = 0, (e^x - 3)(e^x - 4) = 0$$

M1 A1

$$x = \ln 3 \text{ or } x = \ln 4$$

A1

$$\text{Answer: } x = \ln 3, x = \ln 4$$

4 marks

18 (5 marks)

$$\text{Take natural logs: } \ln 3^x + \ln e^{2x-1} = \ln 12$$

M1

$$x \ln 3 + 2x - 1 = \ln 12$$

M1

$$x(2 + \ln 3) = 1 + \ln 12$$

M1

$$x = \frac{1 + \ln 12}{2 + \ln 3}$$

A1

$$x = 1.12$$

A1

$$\text{Answer: } x = \frac{1 + \ln 12}{2 + \ln 3} = 1.12$$

5 marks

19 (4 marks)

$$\text{Collect the terms in } x: x \ln 12 - x \ln 4 = \ln 5 - \ln 45$$

M1

$$x \ln\left(\frac{12}{4}\right) = \ln\left(\frac{5}{45}\right)$$

M1

$$x \ln 3 = \ln\left(\frac{1}{9}\right) = -2 \ln 3$$

M1

$$x = -2$$

A1

$$\text{Answer: } x = -2$$

4 marks

20 (6 marks)

$$\text{(a) } 3^{2x+1} = 3 \times 9^x \text{ and } 5^{x+2} = 25 \times 5^x$$

M1

$$3 \times 9^x = 25 \times 5^x$$

M1

$$\text{Divide by } 3 \times 5^x: \left(\frac{9}{5}\right)^x = \frac{25}{3}$$

A1

$$\text{Answer: shown}$$

3 marks

$$\text{(b) } x \ln\left(\frac{9}{5}\right) = \ln\left(\frac{25}{3}\right)$$

M1

$$x = \frac{\ln\left(\frac{25}{3}\right)}{\ln\left(\frac{9}{5}\right)}$$

A1

$$x = 3.61$$

A1

$$\text{Answer: } x = 3.61$$

3 marks

21 (5 marks)

$$(x + 1) \ln 3 = (x + 3) \ln 2$$

M1

$$x \ln 3 - x \ln 2 = 3 \ln 2 - \ln 3$$

M1

$$x \ln\left(\frac{3}{2}\right) = \ln 8 - \ln 3 = \ln\left(\frac{8}{3}\right)$$

M1 A1

$$x = \frac{\ln(8)}{3/\ln(3/2)} = 2.419$$

A1

$$\text{Answer: } x = 2.42$$

5 marks

22 (7 marks)

$$\text{(a) } n \ln 1.05 > \ln 2$$

M1

$$n > \frac{\ln 2}{\ln 1.05} = 14.21$$

A1

$$n = 15$$

A1

$$\text{Answer: } n = 15$$

3 marks

$$\text{(b) } n \ln 0.8 < \ln 0.01$$

M1

$$\ln 0.8 < 0, \text{ so dividing reverses the inequality: } n > \frac{\ln 0.01}{\ln 0.8} = 20.64$$

M1 E1

$$n = 21$$

A1

$$\text{Answer: } n = 21$$

4 marks

23 (6 marks)

$$2^{2x+1} = 2(2^x)^2; \text{ let } u = 2^x: 2u^2 - 5u + 2 = 0$$

M1 A1

$$(2u - 1)(u - 2) = 0$$

M1

$$u = \frac{1}{2} \text{ or } 2$$

A1

$$x = -1 \text{ or } x = 1$$

A1 A1

Answer: $x = -1$ or 1

6 marks

24 (7 marks)

$$(a) 2P_0 = P_0 e^{12k}, \text{ so } e^{12k} = 2$$

M1

$$12k = \ln 2, \text{ so } k = \frac{\ln 2}{12}$$

A1

Answer: shown

2 marks

$$(b) 3 = e^{kt}, \text{ so } t = \frac{\ln 3}{k} = \frac{12 \ln 3}{\ln 2}$$

M1 A1

$$t = 19.0 \text{ hours}$$

A1

Answer: 19.0 hours

3 marks

$$(c) \text{ Hourly multiplier } e^k = 2^{1/12} = 1.0595$$

M1

$$\text{Increase} = 5.95\%$$

A1

Answer: 5.95%

2 marks

25 (5 marks)

$$\text{Let } u = e^x: u^3 - 3u^2 - 6u + 8 = 0$$

M1

$$u = 1 \text{ gives } 1 - 3 - 6 + 8 = 0, \text{ so } (u - 1) \text{ is a factor}$$

M1

$$(u - 1)(u^2 - 2u - 8) = (u - 1)(u - 4)(u + 2)$$

A1

$$e^x > 0, \text{ so } u = -2 \text{ is rejected}$$

E1

$$e^x = 1 \text{ or } e^x = 4, \text{ so } x = 0 \text{ or } x = \ln 4$$

A1

Answer: $x = 0, x = \ln 4$

5 marks

26 (4 marks)

$$\text{Let } u = e^{x+1}, \text{ so } e^{2x+2} = u^2: u^2 - 7u + 10 = 0$$

M1

$$(u - 2)(u - 5) = 0$$

A1

$$e^{x+1} = 2 \text{ or } e^{x+1} = 5$$

A1

$$x + 1 = \ln 2 \text{ or } \ln 5, \text{ so } x = \ln 2 - 1 \text{ or } x = \ln 5 - 1$$

A1

Answer: $x = \ln 2 - 1, x = \ln 5 - 1$

4 marks

27 (10 marks)

$$(a) \text{ Let } y = 3^x: y^2 - 10y + 9 < 0$$

M1

$$(y - 1)(y - 9) < 0$$

A1

$$\text{So } 1 < y < 9, \text{ that is } 1 < 3^x < 9$$

M1

$$3^x > 1 \text{ gives } x > 0 \text{ and } 3^x < 9 \text{ gives } x < 2$$

M1

$$0 < x < 2$$

A1

Answer: $0 < x < 2$

5 marks

(b) $\ln(A + B)$ is not $\ln A + \ln B$; the log of a sum cannot be split. The correct rule is $\ln(AB) = \ln A + \ln B$

E1 E1

$$3^{x+1} = 3 \times 3^x, \text{ so } 3^x + 3 \times 3^x = 36$$

M1

$$4 \times 3^x = 36, \text{ so } 3^x = 9$$

A1

$$x = 2$$

A1

Answer: $x = 2$

5 marks

28 (10 marks)

$$(a) S_n = \frac{a(r^n - 1)}{r - 1} \text{ with } a = 5, r = 3$$

M1

$$S_n = \frac{5(3^n - 1)}{2}$$

A1

Answer: shown

2 marks

$$(b) \frac{5}{2}(3^n - 1) > 10^7, \text{ so } 3^n > 4\,000\,001$$

M1

$$n \ln 3 > \ln 4\,000\,001$$

M1

$$n > 13.8\dots$$

A1

Least integer $n = 14$

A1

Answer: $n = 14$

4 marks

$$(c) \text{ Let } y = 2^x: y^2 - 6y - 16 = 0$$

M1

$$(y - 8)(y + 2) = 0, \text{ so } y = 8 \text{ or } y = -2$$

A1

$$2^x > 0 \text{ for all } x, \text{ so } y = -2 \text{ is rejected}$$

E1

$$2^x = 8, \text{ so } x = 3$$

A1

Answer: $x = 3$

4 marks

29 (11 marks)

$$(a) u = 2^x: u^2 - 2u + k = 0$$

M1

$$k = 1: (u - 1)^2 = 0, \text{ so } u = 1$$

A1

$$2^x = 1, \text{ so } x = 0 \text{ (only one)}$$

A1

Answer: $x = 0$

3 marks

$$(b) \text{ Two distinct real } u \text{ needs discriminant } 4 - 4k > 0, \text{ so } k < 1$$

M1 A1

$$\text{Each solution must be positive } (2^x > 0): \text{ product of roots } = k > 0 \text{ and sum } = 2 > 0$$

M1 A1

$$\text{So } 0 < k < 1$$

A1

Answer: $0 < k < 1$

5 marks

$$(c) u^2 - 2u - 3 = 0, (u - 3)(u + 1) = 0$$

M1

$$u = -1 \text{ is rejected since } 2^x > 0$$

E1

$$2^x = 3, x = \frac{\ln 3}{\ln 2}$$

A1

$$\text{Answer: } x = \frac{\ln 3}{\ln 2}$$

3 marks

30 (8 marks)

$$(a) u = e^x: u - \frac{1}{u} = 2, \text{ so } u^2 - 2u - 1 = 0$$

M1 A1

$$u = 1 \pm \sqrt{2}$$

M1

$$u > 0 \text{ so } u = 1 + \sqrt{2} \text{ only } (1 - \sqrt{2} < 0)$$

E1

$$x = \ln(1 + \sqrt{2})$$

A1

$$\text{Answer: } x = \ln(1 + \sqrt{2})$$

5 marks

$$(b) u^2 - 2ku - 1 = 0, \text{ so } u = k \pm \sqrt{k^2 + 1}$$

M1 A1

$$\text{Positive root only: } x = \ln(k + \sqrt{k^2 + 1})$$

A1

$$\text{Answer: } x = \ln(k + \sqrt{k^2 + 1})$$

3 marks

Logarithmic Equations**1 (2 marks)**

$$x = 3^4 = 81$$

B1

$$x^2 = 64, \text{ and } x > 0 \text{ so } x = 8$$

B1

Answer: (a) $x = 81$, (b) $x = 8$

2 marks

2 (2 marks)

$$x = e^3$$

B1

$$x + 1 = 10^2 = 100, \text{ so } x = 99$$

B1

Answer: (a) $x = e^3$, (b) $x = 99$

2 marks

3 (3 marks)

$$\ln x = 1.5$$

M1

$$x = e^{1.5}$$

A1

$$x = 4.48$$

A1

$$\text{Answer: } x = e^{1.5} = 4.48$$

3 marks

4 (2 marks)

$$2x + 1 = 5^2 = 25$$

M1

$$x = 12$$

A1

$$\text{Answer: } x = 12$$

2 marks

5 (2 marks)

$$\log_3 4x = \log_3 20$$

M1

$$4x = 20, x = 5$$

A1

$$\text{Answer: } x = 5$$

2 marks

6 (4 marks)

$$\text{(a) } a^3 = 64$$

M1

$$a = 4$$

A1

$$\text{Answer: } a = 4$$

2 marks

$$\text{(b) } x = 4^{-2}$$

M1

$$x = \frac{1}{16}$$

A1

$$\text{Answer: } x = \frac{1}{16}$$

2 marks

7 (5 marks)

$$\text{(a) } 3x - 1 = 2^4 = 16$$

M1

$$x = \frac{17}{3}$$

A1

$$\text{Answer: } x = \frac{17}{3}$$

2 marks

$$\text{(b) } \log_5\left(\frac{x+4}{x}\right) = 1$$

M1

$$\frac{x+4}{x} = 5, \text{ so } x+4 = 5x$$

M1

$$x = 1$$

A1

$$\text{Answer: } x = 1$$

3 marks

8 (2 marks)

$$\ln(3x - 2) = \ln 2x$$

M1

$$3x - 2 = 2x, \text{ so } x = 2$$

A1

$$\text{Answer: } x = 2$$

2 marks

9 (3 marks)

$$\log_{10}\left(\frac{x+6}{x-3}\right) = 1$$

M1

$$x + 6 = 10(x - 3)$$

M1

$$9x = 36, x = 4$$

A1

$$\text{Answer: } x = 4$$

3 marks

10 (3 marks)

$$\log_3 x^2 = \log_3(x + 6)$$

M1

$$x^2 = x + 6, (x - 3)(x + 2) = 0$$

M1

$$x > 0, \text{ so } x = 3 \text{ (} x = -2 \text{ rejected)}$$

A1

$$\text{Answer: } x = 3$$

3 marks

11 (5 marks)

$$2 \ln x = \ln x^2$$

M1

$$3x + 4 = x^2$$

M1

$$x^2 - 3x - 4 = 0$$

A1

$$(x - 4)(x + 1) = 0$$

M1

$$x > 0, \text{ so } x = 4 \text{ (} x = -1 \text{ is rejected)}$$

A1

$$\text{Answer: } x = 4$$

5 marks

12 (7 marks)

$$\text{(a) } L = 10 \log_{10}\left(\frac{10^{-5}}{10^{-12}}\right) = 10 \log_{10} 10^7$$

M1

$$= 70 \text{ decibels}$$

A1

$$\text{Answer: } 70 \text{ decibels}$$

2 marks

$$\text{(b) } 95 = 10 \log_{10}\left(\frac{I}{10^{-12}}\right), \text{ so } \log_{10}\left(\frac{I}{10^{-12}}\right) = 9.5$$

M1

$$I = 10^{-12} \times 10^{9.5}$$

M1

$$I = 3.16 \times 10^{-3} \text{ W m}^{-2}$$

A1

$$\text{Answer: } 3.16 \times 10^{-3} \text{ W m}^{-2}$$

3 marks

$$\text{(c) New level} = 10 \log_{10}\left(\frac{2I}{10^{-12}}\right) = 10 \log_{10} 2 + 10 \log_{10}\left(\frac{I}{10^{-12}}\right)$$

M1

$$= L + 3.01, \text{ an increase of about 3 decibels}$$

A1

$$\text{Answer: increase of } 10 \log_{10} 2 \approx 3.01$$

2 marks

13 (4 marks)

$$\text{Let } u = \ln x: u^2 - 4u + 3 = 0$$

M1

$$(u - 1)(u - 3) = 0$$

A1

$$\ln x = 1 \text{ or } 3$$

M1

$$x = e \text{ or } x = e^3$$

A1

$$\text{Answer: } x = e, x = e^3$$

4 marks

14 (5 marks)

$$\log_{10}(xy) = 3, \text{ so } xy = 1000$$

M1 A1

$$x(10x) = 1000, \text{ so } x^2 = 100$$

M1

$$x = 10 \text{ (} x > 0 \text{), } y = 100$$

A1 A1

$$\text{Answer: } x = 10, y = 100$$

5 marks

15 (5 marks)

$$\text{(a) pH} = -\log_{10}(3.2 \times 10^{-5})$$

M1

$$= 4.49$$

A1

$$\text{Answer: } 4.49$$

2 marks

$$\text{(b) } \log_{10} h = -6.3$$

M1

$$h = 10^{-6.3}$$

M1

$$h = 5.01 \times 10^{-7}$$

A1

$$\text{Answer: } 5.01 \times 10^{-7}$$

3 marks

16 (4 marks)

$$x^2 = 2x + 3$$

M1

$$(x - 3)(x + 1) = 0$$

A1

$$\text{A base must be positive, so } x = 3 \text{ (} x = -1 \text{ rejected)}$$

E1 A1

$$\text{Answer: } x = 3$$

4 marks

17 (5 marks)

$$\text{(a) } t = 8 \ln 16$$

M1

$$t = 22.2 \text{ days}$$

A1

$$\text{Answer: } 22.2 \text{ days}$$

2 marks

$$(b) \ln\left(\frac{N}{250}\right) = 2.5$$

M1

$$N = 250e^{2.5}$$

M1

$$N = 3046$$

A1

Answer: 3046

3 marks

18 (4 marks)

$$\ln(x + 1) = y, \text{ so } x + 1 = e^y$$

M1

$$x + (x + 1) = 7$$

M1

$$x = 3$$

A1

$$e^y = 4, \text{ so } y = \ln 4$$

A1

Answer: $x = 3, y = \ln 4$

4 marks

19 (5 marks)

$$\ln(2x - 1) - \ln(x - 2) = 1$$

M1

$$\frac{2x - 1}{x - 2} = e$$

M1

$$2x - 1 = ex - 2e$$

M1

$$x(2 - e) = 1 - 2e, \text{ so } x = \frac{2e - 1}{e - 2}$$

A1

$$x = 6.18 > 2, \text{ so both logarithms are defined}$$

E1

Answer: $x = \frac{2e - 1}{e - 2}$

5 marks

20 (8 marks)

$$(a) \log_3\left(\frac{x^2}{x - 2}\right) = 2, \text{ so } \frac{x^2}{x - 2} = 9$$

M1 A1

$$x^2 - 9x + 18 = 0, \text{ so } (x - 3)(x - 6) = 0$$

M1

$$\text{Both values are greater than 2, so both are valid: } x = 3 \text{ or } x = 6$$

A1

Answer: $x = 3$ or $x = 6$

4 marks

$$(b) \log_9 x = \frac{\log_3 x}{\log_3 9} = \frac{1}{2} \log_3 x, \text{ so } \log_3(x - 2) = \log_3 \sqrt{x}$$

M1

$$(x - 2) = \sqrt{x}, \text{ so } (x - 2)^2 = x$$

A1

$$x^2 - 5x + 4 = 0, \text{ so } x = 1 \text{ or } x = 4$$

M1

$$x = 1 \text{ gives } x - 2 < 0, \text{ so reject it; } x = 4$$

A1

Answer: $x = 4$

4 marks

21 (7 marks)

$$(a) \log_{10}(xy) = 2, \text{ so } xy = 100$$

M1 A1

$$\text{Substitute } x = y + 15: y(y + 15) = 100, \text{ so } y^2 + 15y - 100 = 0$$

M1

$$(y - 5)(y + 20) = 0, \text{ and } y > 0 \text{ so } y = 5$$

A1

$$x = 20$$

A1

Answer: $x = 20, y = 5$

5 marks

$$(b) \log_{10} x \text{ and } \log_{10} y \text{ are not defined for negative numbers}$$

E1

The step $\log_{10} x + \log_{10} y = \log_{10}(xy)$ only holds when $x > 0$ and $y > 0$, so $xy = 100$ has introduced extra solutions

E1

Answer: logs of negative numbers undefined

2 marks

22 (6 marks)

$$\log_2(x + 4) + \log_2(x - 1) = 3$$

M1

$$(x + 4)(x - 1) = 8$$

M1 A1

$$x^2 + 3x - 12 = 0, \text{ so } x = \frac{-3 \pm \sqrt{57}}{2}$$

M1 A1

$$x > 1 \text{ is needed, so } x = \frac{-3 + \sqrt{57}}{2}$$

E1

Answer: $x = \frac{-3 + \sqrt{57}}{2}$

6 marks

23 (6 marks)

$$\log_{10}(x+2)^2 = \log_{10} 10 + \log_{10} x = \log_{10} 10x$$

M1 A1

$$(x+2)^2 = 10x, \text{ so } x^2 - 6x + 4 = 0$$

M1 A1

$$x = 3 \pm \sqrt{5}$$

A1

Both are positive, so both satisfy $x > 0$ and the logarithms exist: $x = 3 + \sqrt{5}$ or $3 - \sqrt{5}$

E1

Answer: $x = 3 \pm \sqrt{5}$

6 marks

24 (7 marks)

(a) $\log_2(3x - 1) = 4$, so $3x - 1 = 16$

M1

$$x = \frac{17}{3}$$

A1

Answer: $\frac{17}{3}$

2 marks

(b) The curve is above the line to the right of the intersection

M1

$$x > \frac{17}{3}$$

A1

Answer: $x > \frac{17}{3}$

2 marks

(c) Logs defined: $5 - x > 0$ and $x + 1 > 0$, so $-1 < x < 5$

B1

ln increasing: $5 - x \leq x + 1$, so $x \geq 2$

M1

$$2 \leq x < 5$$

A1

Answer: $2 \leq x < 5$

3 marks

25 (6 marks)

(a) $\log_2((-4)^2) = \log_2 16 = 4$

M1 A1

Answer: shown

2 marks

(b) The rule $\log x^2 = 2 \log x$ needs $x > 0$

E1

$\log_2(x^2)$ is defined for all $x \neq 0$, so this step loses the negative solution; the correct rule is $\log x^2 = 2 \log|x|$

E1

Answer: the step $2 \log x$ requires $x > 0$, losing $x = -4$

2 marks

(c) $x^2 = 2^4 = 16$

M1

$$x = \pm 4$$

A1

Answer: $x = \pm 4$

2 marks

26 (5 marks)

Let $t = \ln x$, with $t \neq 0$: $3t + \frac{2}{t} = 7$

M1

$$3t^2 - 7t + 2 = 0$$

M1

$$(3t - 1)(t - 2) = 0$$

A1

$$\ln x = \frac{1}{3} \text{ or } \ln x = 2$$

A1

$$x = e^{1/3} \text{ or } x = e^2$$

A1

Answer: $x = e^{1/3}$, $x = e^2$

5 marks

27 (11 marks)

(a) The sum of two logarithms is the log of the product, not of the sum

E1

The correct step is $\log_2[(x-1)(x-3)] = 3$

E1

Answer: should multiply, not add

2 marks

(b) $\log_2[(x-1)(x-3)] = 3$, so $(x-1)(x-3) = 8$

M1 A1

$$x^2 - 4x - 5 = 0$$

A1

$$(x-5)(x+1) = 0$$

M1

Need $x > 3$ for both logarithms to exist, so $x = 5$

A1

Answer: $x = 5$

5 marks

(c) $(x-1)(x-3) = 2^k$, so $x^2 - 4x + 3 - 2^k = 0$

M1

$$x = 2 \pm \sqrt{1 + 2^k}$$

A1

$2^k > 0$, so $\sqrt{1+2^k} > 1$ and $2 - \sqrt{1+2^k} < 1$, which is rejected as $x > 3$ is needed

M1

$2 + \sqrt{1+2^k} > 3$ is always valid: exactly one solution

A1

Answer: $x = 2 + \sqrt{1+2^k}$

4 marks

28 (10 marks)

(a) Let $t = \log_2 x$: $t^2 - 4t - 5 = 0$

M1

$(t - 5)(t + 1) = 0$, so $t = 5$ or $t = -1$

A1

$x = 2^5 = 32$

A1

$x = 2^{-1} = \frac{1}{2}$

A1

Answer: $x = 32$ or $x = \frac{1}{2}$

4 marks

(b) Take $\log_2$ of both sides: $\log_2(x^{\log_2 x}) = \log_2(16x^3)$

M1

$(\log_2 x)^2 = 4 + 3 \log_2 x$

A1 A1

With $t = \log_2 x$: $t^2 - 3t - 4 = 0$

M1

$(t - 4)(t + 1) = 0$, so $t = 4$ or $t = -1$

A1

$x = 16$ or $x = \frac{1}{2}$

A1

Answer: $x = 16$ or $x = \frac{1}{2}$

6 marks

29 (8 marks)

Let $t = \log_3 x$. $\log_x 3 = \frac{1}{t}$ and $\log_9 x = \frac{t}{2}$

M1 A1

$\frac{2}{t} - \frac{t}{2} = 1$

M1

$4 - t^2 = 2t$, so $t^2 + 2t - 4 = 0$

M1 A1

$t = -1 \pm \sqrt{5}$

A1

Both give $x > 0$, $x \neq 1$: $x = 3^{-1+\sqrt{5}}$ or $x = 3^{-1-\sqrt{5}}$

E1 A1

Answer: $x = 3^{-1+\sqrt{5}}$, $3^{-1-\sqrt{5}}$

8 marks

30 (8 marks)

Let $t = \log_x y$: $\log_y x = \frac{1}{t}$

M1

$t + \frac{1}{t} = \frac{10}{3}$, so $3t^2 - 10t + 3 = 0$

M1 A1

$(3t - 1)(t - 3) = 0$, so $t = 3$ or $t = \frac{1}{3}$

A1

$t = 3$: $y = x^3$, so $x^4 = 64$ and $x = 2\sqrt{2}$, $y = 16\sqrt{2}$

M1 A1

$t = \frac{1}{3}$: $x = y^3$, so $(x, y) = (16\sqrt{2}, 2\sqrt{2})$

M1 A1

Answer: $(2\sqrt{2}, 16\sqrt{2})$ and $(16\sqrt{2}, 2\sqrt{2})$

8 marks

Exponential and Logarithmic Graphs

1 (3 marks)

$\log_a 9 = 2$, so $a^2 = 9$

M1

$a > 1$ so $a = 3$

A1

$x = 27$: $y = \log_3 27 = 3$

A1

Answer: $a = 3$; $y = 3$

3 marks

2 (3 marks)

$x = 0$: $y = 1$, so $(0, 1)$; asymptote $y = 0$

B1

$x = 0$: $y = 1 + 3 = 4$, so $(0, 4)$; asymptote $y = 3$

B1

$x = 0$: $y = 5$, so $(0, 5)$; asymptote $y = 0$

B1

Answer: (a) $(0, 1)$, $y = 0$; (b) $(0, 4)$, $y = 3$; (c) $(0, 5)$, $y = 0$

3 marks

3 (2 marks)

Reflection of $y = 2^x$ in the y-axis: decreasing curve, asymptote $y = 0$ as $x \rightarrow \infty$

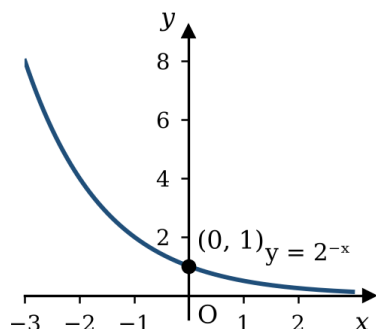
B1

Crosses the y-axis at $(0, 1)$

B1

Answer: decreasing curve through $(0, 1)$, asymptote $y = 0$

2 marks



4 (3 marks)

$y = e^{x-2}$

B1

$y = -e^x$

B1

$y = e^x + 3$

B1

Answer: (a) $y = e^{x-2}$, (b) $y = -e^x$, (c) $y = e^x + 3$

3 marks

5 (4 marks)

Translation of $y = e^x$ by 4 units down: correct increasing shape

B1

$x = 0$: $y = 1 - 4 = -3$, so the curve crosses the y-axis at $(0, -3)$

B1

$y = 0$: $e^x = 4$, so $x = \ln 4$, giving the point $(\ln 4, 0)$

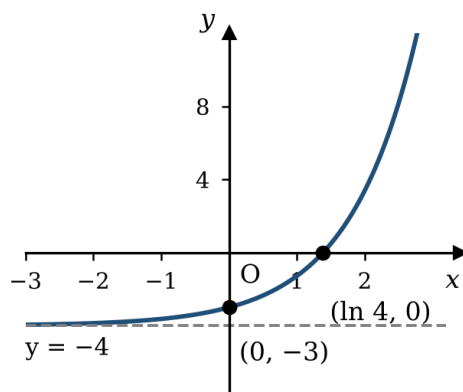
B1

Asymptote $y = -4$

B1

Answer: see sketch: $(0, -3)$, $(\ln 4, 0)$, asymptote $y = -4$

4 marks



6 (4 marks)

Translation of $y = \ln x$ by 3 units to the left: correct shape

B1

$x = 0$: $y = \ln 3$, giving $(0, \ln 3)$

B1

$y = 0$: $x + 3 = 1$, so $x = -2$, giving $(-2, 0)$

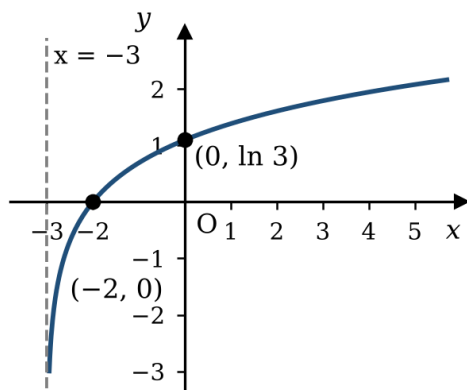
B1

Asymptote $x = -3$

B1

Answer: see sketch: $(0, \ln 3)$, $(-2, 0)$, asymptote $x = -3$

4 marks

**7 (4 marks)**

Crosses the x-axis where $2x = 1$, so at $(\frac{1}{2}, 0)$

B1

Asymptote $x = 0$ (the y-axis)

B1

Stretch parallel to the x-axis with scale factor $\frac{1}{2}$

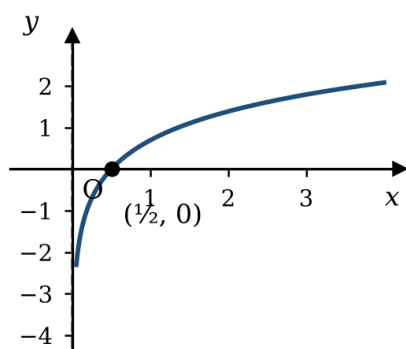
B1

Increasing curve of the correct shape

B1

Answer: $(\frac{1}{2}, 0)$; $x = 0$; stretch in x, scale factor $\frac{1}{2}$

4 marks

**8 (3 marks)**

$y = e^x + 1$, so $e^x = y - 1$

M1

$x = \ln(y - 1)$, so $f^{-1}(x) = \ln(x - 1)$

A1

Domain of f^{-1} = range of f : $x > 1$

B1

Answer: $f^{-1}(x) = \ln(x - 1)$, $x > 1$

3 marks

9 (3 marks)

Stretch parallel to the x-axis, scale factor $\frac{1}{2}$

B1

Stretch parallel to the y-axis, scale factor 3

B1

Reflection in the y-axis

B1

Answer: (a) x-stretch sf $\frac{1}{2}$, (b) y-stretch sf 3, (c) reflection in y-axis

3 marks

10 (5 marks)

$\log_{10} y = \log_{10} a + x \log_{10} b$

M1

Intercept: $\log_{10} a = 0.70$, so $a = 10^{0.70} = 5.0$

M1 A1

Gradient = $\frac{1.60 - 0.70}{3} = 0.30 = \log_{10} b$, so $b = 10^{0.3} = 2.0$

M1 A1

Answer: $a = 5.0$, $b = 2.0$

5 marks

11 (5 marks)

(a) $\ln(x - 2) = y - 3$

M1

$x = e^{y-3} + 2$

A1

Answer: $x = e^{y-3} + 2$

2 marks

$$(b) \frac{1}{2} e^{x-4} = y - 3, \text{ so } e^{x-4} = 2(y - 3)$$

M1

$$x - 4 = \ln(2y - 6)$$

M1

$$x = 4 + \ln(2y - 6)$$

A1

$$\text{Answer: } x = 4 + \ln(2y - 6)$$

3 marks

12 (7 marks)

$$(a) \text{ Asymptote } y = 2 \text{ gives } c = 2$$

B1

$$x = 0: a + 2 = 5, \text{ so } a = 3$$

M1

$$x = 1: 3b + 2 = 8, \text{ so } b = 2$$

A1 A1

$$\text{Answer: } a = 3, b = 2, c = 2$$

4 marks

$$(b) 3 \times 2^x + 2 = 50, \text{ so } 2^x = 16$$

M1 A1

$$x = 4$$

A1

$$\text{Answer: } x = 4$$

3 marks

13 (6 marks)

$$(a) \text{ Take lg of both sides: } \lg y = \lg(a b^x) = \lg a + \lg b^x$$

M1

$$= \lg a + x \lg b$$

A1

$$\text{Answer: shown}$$

2 marks

$$(b) \text{ Gradient} = \lg b = \frac{2.5 - 0.6}{4} = 0.475$$

M1

$$b = 10^{0.475} = 3.0$$

A1

$$\text{Intercept} = \lg a = 0.6$$

M1

$$a = 10^{0.6} = 4.0$$

A1

$$\text{Answer: } a = 4.0, b = 3.0$$

4 marks

14 (5 marks)

$$e^{2x} > 0, \text{ so } 3e^{2x} - 1 > -1; \text{ range } f(x) > -1$$

B1

$$y = 3e^{2x} - 1, \text{ so } e^{2x} = \frac{y+1}{3}$$

M1

$$2x = \ln\left(\frac{y+1}{3}\right), \text{ so } f^{-1}(x) = \frac{1}{2} \ln\left(\frac{x+1}{3}\right)$$

M1 A1

$$\text{Domain } x > -1$$

B1

$$\text{Answer: } f(x) > -1; f^{-1}(x) = \frac{1}{2} \ln\left(\frac{x+1}{3}\right), x > -1$$

5 marks

15 (4 marks)

$$(a) x = e + 1: a \ln(e) = 3$$

M1

$$a = 3$$

A1

$$\text{Answer: } a = 3$$

2 marks

$$(b) 3 \ln(x - 1) = 6, \text{ so } \ln(x - 1) = 2$$

M1

$$x - 1 = e^2, \text{ so } x = 1 + e^2$$

A1

$$\text{Answer: } x = 1 + e^2$$

2 marks

16 (4 marks)

Sketch: $y = \ln x$ is increasing and $y = 2 - x$ is decreasing, so they meet exactly once

M1 A1

$$g(x) = \ln x + x - 2: g(1) = -1 < 0$$

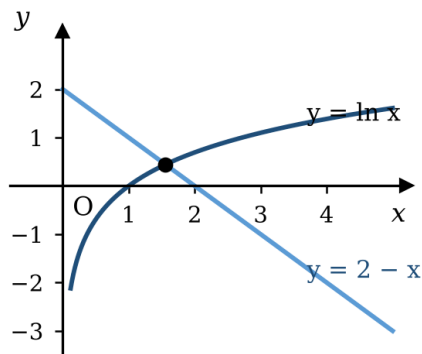
M1

$$g(2) = \ln 2 = 0.693 > 0, \text{ so by sign change } 1 < \text{root} < 2$$

A1

$$\text{Answer: one root, } 1 < x < 2$$

4 marks



17 (8 marks)

(a) $A + B = 5$ and $3A + B = 11$

Subtract: $2A = 6$, so $A = 3$

$B = 2$

Answer: $A = 3, B = 2$

M1 A1

M1

A1

4 marks

(b) Asymptote: $3x + 2 = 0$, so $x = -\frac{2}{3}$

$x = 0$: $y = \ln 2$, so $(0, \ln 2)$

$y = 0$: $3x + 2 = 1$, so $x = -\frac{1}{3}$, giving $(-\frac{1}{3}, 0)$

Answer: $x = -\frac{2}{3}$; $(0, \ln 2)$ and $(-\frac{1}{3}, 0)$

B1

B1

M1 A1

4 marks

18 (7 marks)

(a) Translation of $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$ (2 units in the negative x-direction): $y = \ln(x + 2)$

Reflection in the x-axis: $y = -\ln(x + 2)$

Translation of $\begin{pmatrix} 0 \\ 4 \end{pmatrix}$ (4 units in the positive y-direction): $y = 4 - \ln(x + 2)$

Answer: translate $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$, reflect in the x-axis, translate $\begin{pmatrix} 0 \\ 4 \end{pmatrix}$

B1

B1

B1

3 marks

(b) $x = -2$

Answer: $x = -2$

B1

1 mark

(c) $x = 0$: $y = 4 - \ln 2$, so $(0, 4 - \ln 2)$

$y = 0$: $\ln(x + 2) = 4$, so $x + 2 = e^4$

$x = e^4 - 2$, so $(e^4 - 2, 0)$

Answer: $(0, 4 - \ln 2)$ and $(e^4 - 2, 0)$

B1

M1

A1

3 marks

19 (9 marks)

(a) $\ln y = \ln a + n \ln x$

A straight line of gradient n with intercept $\ln a$ on the vertical axis

Answer: gradient n , intercept $\ln a$

M1

A1

2 marks

(b) $n = \frac{\ln 67.88 - \ln 24.00}{\ln 8 - \ln 4}$

$= \frac{1.0397...}{0.6931...} = 1.50$

$24.00 = a \times 4^{1.5} = 8a$

$a = 3$

Answer: $n = 1.5, a = 3$

M1

A1 A1

M1

A1

5 marks

(c) $y = 3 \times 10^{1.5}$

$= 94.9$

Answer: 94.9

M1

A1

2 marks

20 (9 marks)

(a) Reflection in the y-axis: $y = 2^{-x}$

M1

Translation: $y = 2^{-x} + 3$

A1

Answer: $y = 2^{-x} + 3$

2 marks

(b) C_1 : increasing curve through (0, 1) with asymptote $y = 0$

B1

C_2 : decreasing curve through (0, 4)

B1

C_2 has asymptote $y = 3$, and the curves cross once

B1

Answer: see sketch

3 marks

(c) $2^x = 2^{-x} + 3$. Let $u = 2^x$: $u = \frac{1}{u} + 3$, so $u^2 - 3u - 1 = 0$

M1 A1

$$u = \frac{3 \pm \sqrt{13}}{2}, \text{ and } u > 0, \text{ so } u = \frac{3 + \sqrt{13}}{2}$$

M1

$$x = \log_2\left(\frac{3 + \sqrt{13}}{2}\right)$$

A1

Answer: $x = \log_2\left(\frac{3 + \sqrt{13}}{2}\right) (\approx 1.72)$

4 marks

21 (4 marks)

(a) Translation: $y = e^{x-3}$

M1

Stretch in y: $y = 2e^{x-3}$

A1

Answer: $y = 2e^{x-3}$

2 marks

(b) $2e^{x-3} = 2e^{-3} \times e^x$

M1

So it is a stretch parallel to the y-axis, scale factor $2e^{-3}$ ($= 0.0996$)

A1

Answer: stretch parallel to y-axis, scale factor $2e^{-3}$

2 marks

22 (9 marks)

(a) $y = e^x - 1$, so $e^x = y + 1$, $x = \ln(y + 1)$

M1 A1

$$f^{-1}(x) = \ln(x + 1)$$

A1

Domain $x > -1$, range $f^{-1}(x) \in \mathbb{R}$

B1 B1

Answer: $f^{-1}(x) = \ln(x + 1)$, $x > -1$

5 marks

(b) f^{-1} undoes f , so (a, b) on f means (b, a) on f^{-1}

E1

Both curves pass through the origin ($f(0) = 0$); f has gradient 1 there

M1

The curves meet only at (0, 0) as $e^x - 1 \geq x$ with equality only at $x = 0$: 1 intersection

E1 A1

Answer: reflection since f^{-1} swaps x and y; one intersection (the origin)

4 marks

23 (8 marks)

(a) Sketch $y = 2^x - 4$ then reflect the part below the x-axis

M1 A1

y-intercept: $|1 - 4| = 3$, so (0, 3)

B1

x-intercept: $2^x = 4$, so (2, 0); asymptote $y = 4$

B1 B1

Answer: (0, 3), (2, 0), asymptote $y = 4$

5 marks

(b) $2^x - 4 = 4$ or $2^x - 4 = -4$

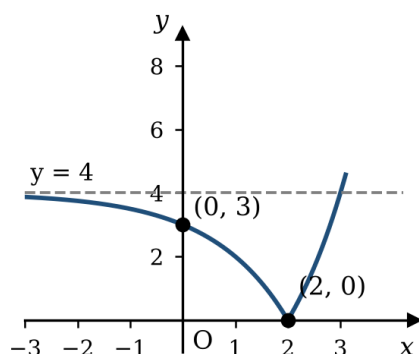
M1

$2^x = 8$ gives $x = 3$; $2^x = 0$ has no solution

A1 E1

Answer: $x = 3$

3 marks



24 (6 marks)

$\ln y = n \ln x + \ln a$, so $\ln y$ against $\ln x$ is linear if the model holds M1

$\ln x$: 0, 0.693, 1.10, 1.39, 1.61; $\ln y$: 1.10, 2.14, 2.75, 3.18, 3.51 M1

Gradient from first and last points = $\frac{3.513 - 1.099}{1.609 - 0} = 1.50$ M1 A1

$x = 1$: $\ln a = \ln 3.00$, so $a = 3.0$; points lie on a line, so the model fits A1 E1

Answer: $n = 1.5$, $a = 3.0$ 6 marks

25 (6 marks)

(a) $\frac{dy}{dx} = \frac{1}{x}$, so gradient at $P = \frac{1}{e}$ M1 A1

Tangent: $y - 1 = \frac{1}{e}(x - e)$, so $y = \frac{x}{e}$ M1

At $x = 0$, $y = 0$: passes through the origin A1

Answer: shown 4 marks

(b) C is concave (second derivative $-\frac{1}{x^2} < 0$), so lies below its tangents M1

Hence $\ln x \leq \frac{x}{e}$, with equality only at $x = e$ A1

Answer: shown 2 marks

26 (11 marks)

(a) $\frac{dy}{dx} = 2e^{2x} - e^x - 6$ M1 A1

Answer: $\frac{dy}{dx} = 2e^{2x} - e^x - 6$ 2 marks

(b) Let $u = e^x$: $2u^2 - u - 6 = 0$ M1

$(2u + 3)(u - 2) = 0$ M1 A1

$u = -\frac{3}{2}$ is rejected as $e^x > 0$, so $e^x = 2$ and $x = \ln 2$ E1 A1

$y = e^{2 \ln 2} - e^{\ln 2} - 6 \ln 2 = 4 - 2 - 6 \ln 2$ M1

$y = 2 - 6 \ln 2$ A1

Answer: $(\ln 2, 2 - 6 \ln 2)$ 7 marks

(c) $\frac{d^2y}{dx^2} = 4e^{2x} - e^x$ M1

At $x = \ln 2$: $16 - 2 = 14 > 0$, so it is a minimum A1

Answer: minimum 2 marks

27 (12 marks)

(a) $e^x > 0$, so $f(x) > -2$ B1

Answer: $f(x) > -2$ 1 mark

(b) $y = 3e^x - 2$, so $e^x = \frac{y+2}{3}$ M1

$x = \ln\left(\frac{y+2}{3}\right)$, so $f^{-1}(x) = \ln\left(\frac{x+2}{3}\right)$ A1

Domain: $x > -2$ B1

Answer: $f^{-1}(x) = \ln\left(\frac{x+2}{3}\right)$, $x > -2$ 3 marks

(c) f : increasing curve through $(0, 1)$ with asymptote $y = -2$ B1

f^{-1} : reflection of f in $y = x$, through $(1, 0)$ with asymptote $x = -2$ B1

f crosses the x -axis at $(\ln \frac{2}{3}, 0)$ and f^{-1} crosses the y -axis at $(0, \ln \frac{2}{3})$ B1

Curves symmetrical about $y = x$ B1

Answer: see sketch 4 marks

(d) Let $g(x) = 3e^x - 2 - x$; need to show $g(x) = 0$ has no solution M1

$g'(x) = 3e^x - 1 = 0$ at $x = -\ln 3$ M1

$g(-\ln 3) = 1 - 2 + \ln 3 = \ln 3 - 1 > 0$ (as $\ln 3 > 1$) A1

This is the minimum value of g , so $g(x) > 0$ for all x and there are no intersections E1

Answer: minimum of g is $\ln 3 - 1 > 0$ **4 marks****28 (9 marks)**

(a) $\ln(3x) = \ln 3 + \ln x$

M1So $y = \ln(3x)$ is $y = \ln x$ shifted up by $\ln 3$ **A1**

Translation $\begin{pmatrix} 0 \\ \ln 3 \end{pmatrix}$

A1**Answer: translation $\begin{pmatrix} 0 \\ \ln 3 \end{pmatrix}$** **3 marks**

(b) $y = 2 \ln x$ exists only for $x > 0$, giving one branch through $(1, 0)$

B1 $y = \ln(x^2)$ exists for all $x \neq 0$ and has two branches, symmetrical about the y-axis**B1**The two graphs coincide for $x > 0$ only**B1**So the claim is false: the identity $\ln(x^2) = 2 \ln x$ holds only for $x > 0$ **E1****Answer: identical only for $x > 0$** **4 marks**

(c) $\ln(x^2) = 4$, so $x^2 = e^4$

M1

$x = \pm e^2$

A1**Answer: $x = \pm e^2$** **2 marks****29 (8 marks)**

(a) $2^2 = 4 = 2^2$

B1

$2^4 = 16 = 4^2$

B1**Answer: verified****2 marks****(b)** For $x < 0$ the curve $y = x^2$ falls to the left while $2^x \rightarrow 0$, and they cross exactly once there; for $x > 4$, 2^x grows faster**E1 E1**

Let $h(x) = 2^x - x^2$: $h(-0.8) = -0.0657 < 0$

M1

$h(-0.7) = 0.126 > 0$, so a root lies in $(-0.8, -0.7)$

A1**Answer: third solution in $(-0.8, -0.7)$** **4 marks****(c)** Between the intersections the curve $y = 2^x$ is below $y = x^2$ **M1**

$2 < x < 4$

A1**Answer: $2 < x < 4$** **2 marks****30 (10 marks)**

(a) $f'(x) = 1 - \frac{1}{x}$

M1 A1

$f'(x) = 0$ at $x = 1$; $f''(x) = \frac{1}{x^2} > 0$, so a minimum

M1 A1

$f(1) = 1 - 1 - 0 = 0$

A1**Answer: minimum value 0 at $x = 1$** **5 marks**

(b) $f(x) \geq f(1) = 0$ for all $x > 0$

M1

So $x - 1 - \ln x \geq 0$

A1**Answer: proved****2 marks**

(c) Put $x = e^t$ (> 0): $t \leq e^t - 1$

M1 A1

So $e^t \geq 1 + t$

A1**Answer: proved****3 marks****Exponential Growth and Decay****1 (2 marks)**

$t = 0$: $N = 200e^0 = 200$

B1

$t = 10$: $N = 200e^{0.3} = 270$

B1**Answer: (a) 200, (b) 270****2 marks**

2 (2 marks)

$$m = 50e^{-0.1 \times 5} = 50e^{-0.5}$$

M1

$$= 30.3 \text{ g}$$

A1

Answer: 30.3 g

2 marks

3 (2 marks)

$$1000 \times 1.04^5$$

M1

$$= £1216.65$$

A1

Answer: £1216.65

2 marks

4 (2 marks)

$$m/m_0 = e^{-0.2}$$

M1

$$= 0.8187, \text{ so } 81.9\%$$

A1

Answer: 81.9%

2 marks

5 (2 marks)

15 hours is 3 half-lives

M1

$$80 \times \left(\frac{1}{2}\right)^3 = 10 \text{ g}$$

A1

Answer: 10 g

2 marks

6 (5 marks)

(a) After 8 days $M = \frac{1}{2}M_0$, so $\frac{1}{2}M_0 = M_0e^{-8k}$

M1

$$e^{-8k} = \frac{1}{2}, \text{ so } -8k = \ln\left(\frac{1}{2}\right) = -\ln 2$$

M1

$$k = \frac{\ln 2}{8}$$

A1

Answer: shown

3 marks

(b) $\frac{M}{M_0} = e^{-20k} = e^{-2.5 \ln 2} = 2^{-2.5}$

M1

$$= 0.177, \text{ so } 17.7\%$$

A1

Answer: 17.7%

2 marks

7 (4 marks)

(a) $13\,500 = 12\,000e^{3k}$, so $e^{3k} = 1.125$

M1

$$k = \frac{\ln 1.125}{3} = 0.0393$$

A1

Answer: k = 0.0393

2 marks

(b) $P = 12\,000e^{10k} = 12\,000 \times 1.125^{10/3}$

M1

$$= 17770, \text{ so } 17\,800 \text{ to the nearest hundred}$$

A1

Answer: 17 800

2 marks

8 (3 marks)

$$2N_0 = N_0e^{5k}, \text{ so } e^{5k} = 2$$

M1

$$k = \frac{\ln 2}{5}$$

A1

$$k = 0.139$$

A1

Answer: k = $\frac{\ln 2}{5} = 0.139$

3 marks

9 (3 marks)

$$300e^{0.04t} = 1000, \text{ so } e^{0.04t} = \frac{10}{3}$$

M1

$$t = \frac{\ln(10/3)}{0.04}$$

M1

$$t = 30.1$$

A1

Answer: 30.1

3 marks

10 (4 marks)

$$(a) V = 25\,000e^{-0.45}$$

$$= £15940$$

M1

A1

Answer: £15940

2 marks

$$(b) e^{-0.15t} = \frac{1}{2}$$

M1

$$t = \frac{\ln 2}{0.15} = 4.62 \text{ years}$$

A1

Answer: 4.62 years

2 marks

11 (3 marks)

$$\frac{dN}{dt} = 6e^{0.03t} = 0.03N$$

M1 A1

$$t = 10: 6e^{0.3} = 8.10$$

A1

Answer: 8.1 per unit time

3 marks

12 (3 marks)

$$e^{-0.2t} = 0.1$$

M1

$$t = \frac{\ln 10}{0.2}$$

M1

$$t = 11.5 \text{ hours}$$

A1

Answer: 11.5 hours

3 marks

13 (6 marks)

$$(a) t = 0: M_0 = 80$$

B1

$$20 = 80e^{-6k}, \text{ so } e^{-6k} = \frac{1}{4}$$

M1

$$6k = \ln 4$$

M1

$$k = \frac{\ln 4}{6} = 0.231$$

A1

Answer: $M_0 = 80$, $k = 0.231$

4 marks

$$(b) \frac{1}{2} = e^{-kt}, \text{ so } t = \frac{\ln 2}{k}$$

M1

$$= \frac{\ln 2}{(\ln 4/6)} = 3 \text{ hours}$$

A1

Answer: 3 hours

2 marks

14 (6 marks)

$$(a) \frac{dN}{dt} = 500 \times 0.2e^{0.2t}$$

M1

$$= 0.2 \times 500e^{0.2t} = 0.2N$$

A1

Answer: shown

2 marks

$$(b) \frac{dN}{dt} = 0.2 \times 2000 = 400 \text{ bacteria per hour}$$

B1

Answer: 400 bacteria per hour

1 mark

$$(c) 500e^{0.2t} = 10\,000, \text{ so } e^{0.2t} = 20$$

M1

$$0.2t = \ln 20$$

M1

$$t = 5 \ln 20 = 14.98 \text{ hours}$$

A1

Answer: 14.98 hours

3 marks

15 (5 marks)

$$Ae^{2k} = 12 \text{ and } Ae^{5k} = 75$$

M1

$$\text{Divide: } e^{3k} = \frac{75}{12}$$

M1

$$k = \frac{1}{3} \ln 6.25 = 0.611$$

A1

$$A = 12e^{-2k} = 3.54$$

M1 A1

Answer: $k = 0.611$, $A = 3.54$

5 marks

16 (5 marks)

$$\ln N = \ln N_0 + kt$$

M1

$$\ln N_0 = 3.2, \text{ so } N_0 = e^{3.2} = 24.53$$

A1

$$k = \text{gradient} = \frac{4.4 - 3.2}{10} = 0.12$$

M1 A1

$$t = 25: \ln N = 3.2 + 3 = 6.2, N = e^{6.2} = 493$$

A1

$$\text{Answer: } N_0 = 24.5, k = 0.12, N = 493$$

5 marks

17 (4 marks)

$$(a) t = 0: P = 5000 - 4000 = 1000$$

B1

$$\text{As } t \rightarrow \infty, e^{-0.1t} \rightarrow 0, \text{ so } P \rightarrow 5000$$

B1

$$\text{Answer: } 1000; 5000$$

2 marks

$$(b) 4000e^{-0.1t} = 1000, \text{ so } e^{-0.1t} = \frac{1}{4}$$

M1

$$t = 10 \ln 4 = 13.9 \text{ years}$$

A1

$$\text{Answer: } 13.9 \text{ years}$$

2 marks

18 (5 marks)

$$N = N_0 e^{-kt} \text{ with } k = \frac{\ln 2}{5730}$$

M1 A1

$$0.63 = e^{-kt}$$

M1

$$t = \frac{-\ln 0.63}{k} = 3819$$

M1

$$\approx 3800 \text{ years}$$

A1

$$\text{Answer: } 3800 \text{ years}$$

5 marks

19 (6 marks)

$$(a) \frac{P(30)}{P(10)} = e^{20k} = \frac{1200}{300} = 4$$

M1

$$20k = \ln 4, \text{ so } k = \frac{\ln 4}{20} = \frac{\ln 2}{10}$$

A1

$$\text{Answer: } k = \frac{\ln 2}{10}$$

2 marks

$$(b) e^{10k} = 2, \text{ so } 300 = A \times 2$$

M1

$$A = 150$$

A1

$$\text{Answer: } A = 150$$

2 marks

$$(c) e^{50k} = 2^5 = 32$$

M1

$$P = 150 \times 32 = 4800$$

A1

$$\text{Answer: } 4800$$

2 marks

20 (9 marks)

$$(a) \text{Dividing: } \frac{64}{8} = e^{3k}, \text{ so } e^{3k} = 8$$

M1

$$3k = \ln 8 = 3 \ln 2, \text{ so } k = \ln 2$$

A1

$$64 = C_0 e^{-2 \ln 2} = \frac{C_0}{4}$$

M1

$$C_0 = 256$$

A1

$$\text{Answer: } k = \ln 2, C_0 = 256$$

4 marks

$$(b) 256e^{-t \ln 2} = 256 \times 2^{-t} = 5$$

M1

$$2^t = 51.2, \text{ so } t = \frac{\ln 51.2}{\ln 2} = 5.678...$$

M1

$$0.678 \times 60 = 40.7, \text{ so } 5 \text{ hours } 41 \text{ minutes}$$

A1

$$\text{Answer: } 5 \text{ hours } 41 \text{ minutes}$$

3 marks

$$(c) C_0 e^{-k(t+h)} = \frac{1}{2} C_0 e^{-kt} \text{ gives } e^{-kh} = \frac{1}{2}, \text{ independent of } C_0 \text{ and } t$$

E1

$$h = \frac{\ln 2}{k} = 1 \text{ hour}$$

E1

$$\text{Answer: } 1 \text{ hour}$$

2 marks

21 (8 marks)

(a) $50 = 18 + 62e^{-10k}$, so $62e^{-10k} = 32$

M1

$$e^{-10k} = \frac{16}{31}$$

A1

$$-10k = \ln\left(\frac{16}{31}\right) = -\ln\left(\frac{31}{16}\right), \text{ so } k = \frac{1}{10} \ln\left(\frac{31}{16}\right)$$

A1

Answer: shown

3 marks

(b) $62e^{-kt} = 7$

M1

$$t = \frac{1}{k} \ln\left(\frac{62}{7}\right)$$

M1

$$= 33.0 \text{ minutes}$$

A1

Answer: 33.0 minutes

3 marks

(c) $\frac{dT}{dt} = -62ke^{-kt}$

M1

$$t = 10: -k \times 32 = -2.12 \text{ }^\circ\text{C per minute}$$

A1

Answer: $-2.12 \text{ }^\circ\text{C per minute}$ (cooling at $2.12 \text{ }^\circ\text{C per minute}$)

2 marks

22 (6 marks)

(a) $t = 0: P = \frac{1200}{6} = 200$

B1

$$e^{-0.4t} \rightarrow 0, \text{ so } P \rightarrow 1200$$

B1

Answer: 200; 1200

2 marks

(b) $\frac{1200}{1 + 5e^{-0.4t}} = 900$, so $1 + 5e^{-0.4t} = \frac{4}{3}$

M1

$$e^{-0.4t} = \frac{1}{15}$$

M1

$$t = \frac{\ln 15}{0.4} = 6.77, \text{ so after 6.77 months}$$

A1

Answer: 6.77 months

3 marks

(c) Exponential growth is unbounded, but a club has a maximum size; this model levels off at 1200

E1

Answer: it has an upper limit

1 mark

23 (8 marks)

(a) Doubling from time t_1 : $P_0 e^{k(t_1+T)} = 2P_0 e^{kt_1}$

M1

$$e^{kT} = 2 \text{ (independent of } t_1)$$

M1 A1

$$T = \frac{\ln 2}{k}$$

A1

Answer: $T = \frac{\ln 2}{k}$

4 marks

(b) $T = 20$, so $k = \frac{\ln 2}{20}$

M1

$$e^{kt} = 3, t = \frac{\ln 3}{k} = \frac{20 \ln 3}{\ln 2} = 31.70$$

M1 A1

$$1990 + 31.7 = 2021.7, \text{ so during 2021}$$

A1

Answer: 2021

4 marks

24 (7 marks)

(a) $4.5 = 12e^{-3k}$

M1

$$e^{3k} = \frac{12}{4.5} = \frac{8}{3}$$

M1

$$k = \frac{1}{3} \ln\left(\frac{8}{3}\right)$$

A1

Answer: $k = \frac{1}{3} \ln\left(\frac{8}{3}\right)$

3 marks

(b) $V = 12e^{-5k} = 12\left(\frac{3}{8}\right)^{5/3} = 2.34 \text{ V}$

M1 A1

$$0.5 = 12e^{-kt}, t = \frac{\ln 24}{k}$$

M1

$$t = 9.72 \text{ seconds}$$

A1

Answer: 2.34 V; 9.72 s

4 marks

25 (8 marks)

(a) $A = 5000 \times 1.035^{20} = 9948.94 \approx \text{£}9949$

M1 A1

$$B = 5000e^{0.7} = 10068.8 \approx \text{£}10069$$

M1 A1

Answer: A: £9949, B: £10069

4 marks

(b) $B: e^{0.035t} = 2, t = \frac{\ln 2}{0.035} = 19.8$

M1 A1

$$A: 1.035^t = 2, t = \frac{\ln 2}{\ln 1.035} = 20.1$$

M1 A1

Answer: B: 19.8 years; A: 20.1 years

4 marks

26 (9 marks)

(a) $t = 0: V = 600 + 200 \ln 1 = 600$

B1

Answer: £600

1 mark

(b) $V = 600 + 200 \ln 21$

M1

$$= \text{£}1209$$

A1

Answer: £1209

2 marks

(c) $600 + 200 \ln(2t + 1) = 1000$, so $\ln(2t + 1) = 2$

M1

$$2t + 1 = e^2$$

M1

$$t = \frac{e^2 - 1}{2} = 3.19 \text{ years}$$

A1

Answer: $t = \frac{e^2 - 1}{2} = 3.19$

3 marks

(d) $\frac{dV}{dt} = \frac{400}{2t + 1}$

M1 A1

This is positive, so V keeps increasing; it decreases as t increases, so the rate of increase falls

E1

Answer: $\frac{dV}{dt} = \frac{400}{2t + 1}$

3 marks

27 (11 marks)

(a) $80\,000e^{-0.02t} = 50\,000e^{0.03t}$

M1

$$e^{0.05t} = \frac{80\,000}{50\,000} = 1.6$$

M1 A1

$$t = 20 \ln 1.6 = 9.40 \text{ years}$$

A1

Answer: t = 9.40 years

4 marks

(b) $\frac{A}{B} = \frac{80\,000e^{-0.02t}}{50\,000e^{0.03t}}$

M1

$$= 1.6e^{-0.02t - 0.03t} = 1.6e^{-0.05t}$$

A1

Answer: shown

2 marks

(c) $\frac{d}{dt}(A + B) = -1600e^{-0.02t} + 1500e^{0.03t}$

M1

$$= 0 \text{ gives } e^{0.05t} = \frac{16}{15}, \text{ so } t = 20 \ln\left(\frac{16}{15}\right) = 1.29$$

M1 A1

$$\text{Second derivative } 32e^{-0.02t} + 45e^{0.03t} > 0, \text{ so a minimum}$$

A1

$$A + B = 129935, \text{ so } 129\,900 \text{ to the nearest hundred}$$

B1

Answer: t = 1.29 years; minimum total population about 129 900

5 marks

28 (11 marks)

(a) $\ln N = \ln(A e^{kt}) = \ln A + \ln e^{kt}$

M1

$$= \ln A + kt, \text{ a straight line with gradient } k \text{ and intercept } \ln A$$

A1

Answer: $\ln N = \ln A + kt$

2 marks

(b) $k = \frac{6.6 - 5.4}{8 - 2} = 0.2$

M1 A1

$$\ln A = 5.4 - 0.2 \times 2 = 5.0$$

M1

$$A = e^5 = 148$$

A1

Answer: $k = 0.2$, $A = 148$

4 marks

(c) $Ae^{k(t+h)} = 2Ae^{kt}$ gives $e^{kh} = 2$

M1

$$h = 5 \ln 2 = 3.47 \text{ hours}$$

A1

Answer: 3.47 hours

2 marks

(d) $\ln 10^6 = 5 + 0.2t$

M1

$$t = \frac{13.8155... - 5}{0.2} = 44.1 \text{ hours}$$

A1

Growth cannot continue unchecked indefinitely as space and nutrients become limited, so the model is unlikely to be valid

E1

Answer: $t = 44.1$ hours; unlikely to be valid

3 marks

29 (10 marks)

(a) $C(0) = 40(1 - 1) = 0$

B1

Both exponentials tend to 0, so $C \rightarrow 0$

B1

Answer: $C(0) = 0$; $C \rightarrow 0$

2 marks

(b) $\frac{dC}{dt} = 40(-0.1e^{-0.1t} + 0.5e^{-0.5t})$

M1 A1

$$= 0: 0.5e^{-0.5t} = 0.1e^{-0.1t}, \text{ so } e^{0.4t} = 5$$

M1 A1

$$t = 2.5 \ln 5$$

A1

Answer: $t = 2.5 \ln 5 = 4.02$

5 marks

(c) $e^{-0.1t} = 5^{-1/4}$, $e^{-0.5t} = 5^{-5/4}$

M1

$$C = 40(5^{-1/4} - 5^{-5/4}) = 40 \times 5^{-1/4}(1 - \frac{1}{5})$$

M1

$$= 21.4 \text{ mg per litre}$$

A1

Answer: 21.4 mg per litre

3 marks

30 (9 marks)

(a) $t = 0$: $A = 50$

B1

$$50e^{2k} = 75, \text{ so } k = \frac{1}{2} \ln 1.5 = 0.2027$$

M1 A1

$$t = 4: 50 \times 1.5^2 = 112.5; t = 5: 50e^{5k} = 137.8$$

M1 A1

Answer: $A = 50$, $k = 0.203$; $P(4) = 112.5$, $P(5) = 137.8$

5 marks

(b) $\text{Error} = \frac{137.8 - 116}{116} \times 100 = 18.8\%$

M1 A1

The model fits well for small t but overestimates later: the data shows growth slowing

E1

Answer: 18.8% too high; model fits poorly for larger t

3 marks

(c) Exponential growth is unbounded, but resources and space limit the population

E1

Answer: growth cannot continue without limit

1 mark