

Exponential Functions

- 1 The function f is defined by $f(x) = 3^x$.
Find the exact value of (a) $f(4)$, (b) $f(-2)$. [2]

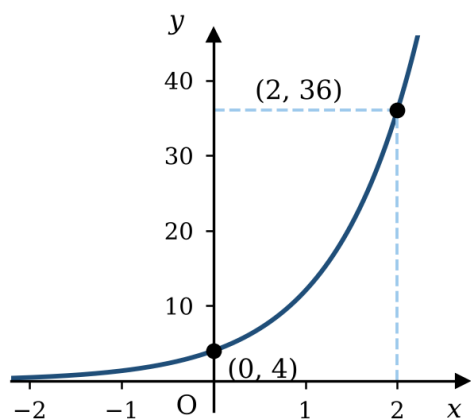
- 2 The function f is defined by $f(x) = 5 \times 2^x$.
(a) Find $f(0)$ and $f(3)$. [2]
(b) Find the value of x for which $f(x) = 160$. [2]

- 3 Evaluate (a) 2^5 , (b) 10^{-3} , (c) $9^{3/2}$. [3]

- 4 Simplify (a) $e^{3x} \times e^{2x}$, (b) $(e^x)^4$, (c) $e^{6x} \div e^{2x}$. [3]

- 5 State, with a reason, whether each function is increasing or decreasing for all real x .
(a) $y = 3^x$ (b) $y = (0.5)^x$ (c) $y = e^{-x}$ (d) $y = 1.01^x$ [4]

- 6 The diagram shows the curve with equation $y = k a^x$, where k and a are positive constants.
The curve passes through the points $(0, 4)$ and $(2, 36)$.



- (a) Find the value of k and the value of a . [3]
(b) Find the value of y when $x = -1$. [2]

- 7 Answer the following.
(a) Simplify $\frac{e^{5x} \times e^{-2x}}{e^x}$. [2]
(b) The function f is given by $f(x) = e^{2x} + 3$. Find the exact value of $f(\ln 4)$. [2]

- 8 Expand and simplify (a) $(e^x + 2)^2$, (b) $(e^x - e^{-x})^2$. [4]

- 9 £2500 is invested at 3% per year compound interest. Find the value of the investment after 8 years, to the nearest pound. [2]

- 10 A car costs £18 000 new and loses 12% of its value each year. Find a formula for its value £ V after t years and its value after 5 years, to the nearest £100. [3]

- 11 The function f is defined by $f(x) = 2^x + 3$, $x \in \mathbb{R}$.
(a) Write down the range of f . (b) Solve $f(x) = 11$. [3]

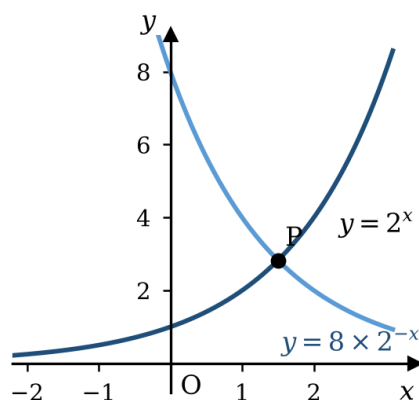
- 12** A colony of bacteria is modelled by $P = 800 \times 1.05^t$, where P is the number of bacteria t hours after the start of an experiment.
- State the initial number of bacteria. [1]
 - Find the number of bacteria after 10 hours, giving your answer to the nearest whole number. [2]
 - Use trial and improvement to find the smallest whole number of hours after which the colony first exceeds 1500 bacteria. Show the values you test. [3]
-
- 13** Answer the following.
- Show that $4^{x+1} - 4^x = 3 \times 4^x$. [2]
 - Hence solve $4^{x+1} - 4^x = 192$. [3]
-
- 14** The table gives values of y for three values of x . The variables are related by $y = a b^x$, where a and b are constants.
Find a and b , and hence the value of y when $x = 6$, to 3 significant figures.
- | $x = 0$ | $x = 1$ | $x = 2$ |
|---------|---------|---------|
| 5 | 7.5 | 11.25 |
- [6]
-
- 15** The curve C has equation $y = 4e^{2x}$. Find $\frac{dy}{dx}$, and the gradient of C at the point where $x = \ln 3$. [4]
-
- 16** Simplify $\frac{9^x \times 3^{x+1}}{27^x}$, showing that it does not depend on x . [3]
-
- 17** The number of fish in a lake is modelled by $P = 1200 \times 0.94^t$, where t is the number of years after the start of 2020.
- Find the model's prediction for P at the start of 2023, to the nearest whole number. [2]
 - Find the percentage decrease in the population over 10 years, to 1 decimal place. [3]
 - Find, by trial, the first year in which the model gives fewer than 500 fish. [3]
-
- 18** $f(x) = 3^x$ and $g(x) = x^2 - 1$, for $x \in \mathbb{R}$.
- Find $fg(2)$ and $gf(2)$. [3]
 - Show that $gf(x) = 9^x - 1$ and solve $gf(x) = 80$. [3]
-
- 19** The functions f and g are defined for all real x by $f(x) = 6e^{-x}$ and $g(x) = x + 3$.
- Find an expression for $gf(x)$. [2]
 - State the range of gf . [2]
 - Solve the equation $gf(x) = 4.5$, giving the answer in exact form. [3]
-
- 20** The function f is defined by $f(x) = 7 - 2^x$, $x \in \mathbb{R}$.
- State the range of f . [2]
 - Solve $f(x) = -9$. [2]
 - Find the coordinates of the points where the graph of $y = f(x)$ crosses the axes, giving any decimal value to 3 significant figures. [3]
-
- 21** The functions f and g are defined for all real x by
- $$f(x) = \frac{e^x + e^{-x}}{2}, \quad g(x) = \frac{e^x - e^{-x}}{2}.$$
- Show that $[f(x)]^2 - [g(x)]^2 = 1$ for all x . [3]
 - Show that $f(2x) = 2[f(x)]^2 - 1$. [3]
 - Given that $f(a) = \frac{5}{4}$, find the possible values of $g(a)$. [2]

- 22** The value, £V, of an investment t years after it was made is modelled by $V = A k^t$, where A and k are positive constants.
The value is £1650 after 1 year and £1996.50 after 3 years.
- Find the value of k and the value of A . [4]
 - Find the value of the investment after 10 years, to the nearest penny. [2]
 - Give one reason why the model may not be suitable for large values of t . [1]

- 23** Show that $(2^x + 2^{-x})^2 - (2^x - 2^{-x})^2 = 4$ for all real x . [4]

- 24** The number V of visitors to a website in month t is modelled by $V = V_0 k^t$, where V_0 and k are constants.
There were 4500 visitors in month 2 and 7776 in month 5.
- Find the values of k and V_0 . [4]
 - Interpret k in context and find the month when the model first predicts more than 20 000 visitors. [3]

- 25** The diagram shows the curves $y = 2^x$ and $y = 8 \times 2^{-x}$, which meet at the point P .
Find the exact coordinates of P .



[5]

- 26** The curve C has equation $y = p + q e^{2x}$, where p and q are constants. C passes through the point $(0, 7)$ and the gradient of C at this point is 10.
- Find the value of p and the value of q . [4]
 - Find the exact x -coordinate of the point on C where the gradient is 40. [3]

- 27** A student claims that $3^x + 3^x = 9^x$ for all values of x .
- Show, by substituting $x = 1$, that the claim is not true for all x . [2]
 - Describe the error the student has made and write down the correct simplification of $3^x + 3^x$. [2]
 - Find the value of x for which the two sides are equal, giving your answer to 3 significant figures. [5]

- 28** Let $f(n) = 2^n$.
- Show that $f(n + 1) - f(n) = f(n)$. [2]
 - Hence show that $\sum_{n=1}^k 2^n = 2^{k+1} - 2$. [3]
 - Find the least integer k for which $\sum_{n=1}^k 2^n$ exceeds 1 000 000. [3]

- 29** Let $f(x) = a^x$, where $a > 0$ and $a \neq 1$.
- Show that $f(x + y) = f(x) f(y)$ and $f(x) f(-x) = 1$. [4]
 - Hence, by considering $(a^{x/2} - a^{-x/2})^2$, prove that $f(x) + f(-x) \geq 2$ for all real x , and state when equality holds. [5]

- 30** Investment A is £5000 at 4% compound interest per year. Investment B is £4000 at 6% compound interest per year.
- (a) Write down expressions for the values of A and B after n years. [2]
- (b) Show, by trial, that B first exceeds A after 12 years. [4]
- (c) Show that $B > A$ when $n > \frac{\ln 1.25}{\ln(1.06/1.04)}$, and explain how this confirms your answer to part (b). [4]

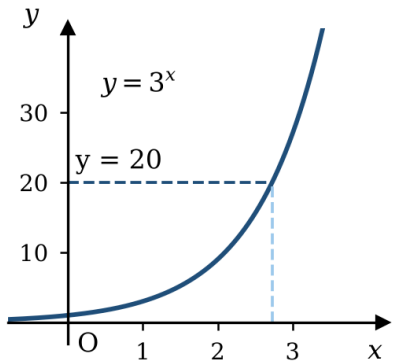
Laws of Logarithms

- 1** Evaluate, without a calculator,
- (a) $\log_2 32$ [1]
- (b) $\log_{10} 0.001$ [1]
- (c) $\log_5 \frac{1}{25}$ [1]
- 2** Write $3 \log_a 2 + \log_a 5 - \log_a 10$ as a single logarithm, simplifying your answer. [3]
- 3** Write each as a single logarithm.
- (a) $\log_3 4 + \log_3 5$ (b) $\log_7 40 - \log_7 8$ [2]
- 4** Without using a calculator, evaluate (a) $\log_3 54 - \log_3 2$, (b) $\log_6 4 + \log_6 9$, (c) $3 \log_2 6 - \log_2 27$. [3]
- 5** Write each of the following in the form $k \ln 2$, where k is a constant.
- (a) $\ln 8$ (b) $\ln \frac{1}{4}$ (c) $\ln \sqrt{2}$ [3]
- 6** Given that $\log_a 2 = p$ and $\log_a 3 = q$, express in terms of p and q
- (a) $\log_a 12$ [2]
- (b) $\log_a \frac{8}{9}$ [2]
- 7** Answer the following.
- (a) Use the change of base formula to show that $\log_4 x = \frac{1}{2} \log_2 x$. [2]
- (b) Hence, or otherwise, find the exact value of $\log_8 32$. [2]
- 8** By writing both in base 3, show that $\log_9 27 = \frac{3}{2}$. [3]
- 9** Simplify (a) $\log_a a^5$, (b) $10^{\log_{10} 7}$, (c) $\ln e^{3x}$, (d) $e^{\ln 5}$. [4]
- 10** Write (a) $2 + \log_{10} 3$ and (b) $1 - \log_{10} 4$ each as a single logarithm to base 10. [4]
- 11** Use the change of base formula to evaluate $\log_5 20$, giving your answer to 3 significant figures. [3]
- 12** Given that $y = 5 \times 3^{2x}$, express x in terms of y , giving your answer in terms of logarithms to base 3. [3]
- 13** Find the value of the constant k such that $\frac{5}{6} \ln 64 - \frac{3}{2} \ln 16 + \ln 8 = k \ln 2$. [4]
- 14** Show that $2 \log_2 6 - \log_2 9 + 3 = 5$. [4]

- 15** Given that $\ln x = a$ and $\ln y = b$, where x and y are positive.
- (a) Express $\ln\left(\frac{x^2\sqrt{y}}{e}\right)$ in terms of a and b . [3]
- (b) Given also that $\ln\left(\frac{x^2}{y}\right) = 3$ and $\ln(xy) = 6$, find x and y in terms of e . [4]
-
- 16** In this question $x > 3$.
- (a) Show that $\log_{10}(x^2 - 9) - \log_{10}(x + 3) = \log_{10}(x - 3)$. [3]
- (b) Hence solve $\log_{10}(x^2 - 9) - \log_{10}(x + 3) = 1$. [2]
-
- 17** Prove that $\log_a b \times \log_b a = 1$ for all positive a , b not equal to 1. [4]
-
- 18** Given that $\log_2 x = p$, find $\log_4(8x)$ in terms of p . [3]
-
- 19** Given that $\log_a x = 3$ and $\log_a y = -2$, find the value of $\log_a\left(\frac{x^2\sqrt{y}}{a}\right)$. [4]
-
- 20** The magnitude M of an earthquake is defined by $M = \log_{10}\left(\frac{I}{I_0}\right)$, where I is the intensity of the earthquake and I_0 is a constant reference intensity.
- (a) Find, in terms of I_0 , the intensity of an earthquake of magnitude 5.8. [2]
- (b) Show that an earthquake of magnitude 7.2 is about 25 times as intense as one of magnitude 5.8. [3]
-
- 21** Simplify $\ln(16e^3) - \ln\left(\frac{8}{e}\right)$, giving your answer in the form $a + b \ln 2$, where a and b are integers. [4]
-
- 22** Answer the following.
- (a) Prove that $\log_a b = \frac{1}{\log_b a}$, where a and b are positive and not equal to 1. [3]
- (b) Hence simplify $\log_2 3 \times \log_3 4 \times \log_4 5 \times \log_5 8$. [2]
- (c) Solve $\log_x 2 + \log_2 x = \frac{5}{2}$. [4]
-
- 23** Prove that $\frac{1}{\log_2 x} + \frac{1}{\log_3 x} + \frac{1}{\log_5 x} = \frac{1}{\log_{30} x}$, for $x > 0$, $x \neq 1$. [5]
-
- 24** The positive numbers x and y , with $x > y$, satisfy $2 \log_{10}(x - y) = \log_{10} x + \log_{10} y$.
- (a) Show that $x^2 - 3xy + y^2 = 0$. [3]
- (b) Hence find the exact value of $\frac{x}{y}$. [4]
-
- 25** Use the laws of logarithms for the following.
- (a) Prove that $\log_{a^n} b = \frac{1}{n} \log_a b$, where $a, b > 0$, $a \neq 1$ and n is a positive integer. [3]
- (b) Hence evaluate $\log_{27} 9 + \log_{125} 25$. [3]
-
- 26** The positive numbers x and y satisfy $\log_2 x + \log_4 y = 4$, $\log_2 y - \log_4 x = \frac{1}{2}$.
- (a) Show that $\log_4 y = \frac{1}{2} \log_2 y$. [2]
- (b) Find the values of x and y . [5]
- (c) Hence find the exact value of $\log_x y$. [2]

- 27** The positive numbers a and b satisfy $a^2 + b^2 = 7ab$.
- (a) Show that $\ln\left(\frac{a+b}{3}\right) = \frac{1}{2}(\ln a + \ln b)$. [5]
- (b) Given also that $\ln a + \ln b = 6$, find $\ln(a+b)$ in the form $p + \ln q$ where p and q are integers. [3]
-
- 28** Let x , y and z be positive numbers.
- (a) Given that x , y , z are consecutive terms of a geometric sequence, prove that $\ln x$, $\ln y$, $\ln z$ are consecutive terms of an arithmetic sequence. [4]
- (b) The numbers 2, 6, 18 are in geometric sequence. Find the common difference of $\ln 2$, $\ln 6$, $\ln 18$. [2]
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- 29** Prove that $\log_2 3 + \log_3 2 > 2$. [5]
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- 30** Prove by contradiction that $\log_2 5$ is irrational. [5]
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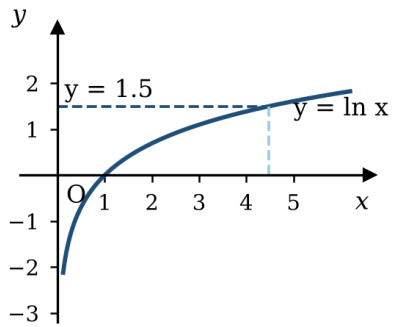
Exponential Equations

- 1** Solve (a) $2^x = 32$, (b) $3^x = \frac{1}{27}$. [2]
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- 2** Solve $5^x = 12$, giving your answer to 3 significant figures. [2]
-
- 3** Solve (a) $2^{x+1} = 16$, (b) $5^x = \frac{1}{5}$, (c) $4^x = 2$. [3]
-
- 4** Solve, giving exact answers: (a) $e^x = 7$, (b) $e^{2x} = 9$. [2]
-
- 5** The diagram shows the curve $y = 3^x$ and the line $y = 20$. Find the x -coordinate of their intersection, to 3 significant figures.
- 
- [2]
-
- 6** Solve $1.5^x = 10$, giving your answer to 3 significant figures. [2]
-
- 7** Solve $4^x = 8^{x-1}$. [3]
-
- 8** Solve each equation.
- (a) $e^{2x} = 7$, giving your answer in an exact form. [2]
- (b) $2^{3x-1} = 10$, giving your answer to 3 significant figures. [3]
-
- 9** Solve $27^x = 9^{x+2}$. [3]
-
- 10** Solve $2^{x+1} + 2^x = 48$. [3]
-

- 11** Solve $e^{3x-1} = 5$, giving the answer as an exact value and then to 3 significant figures. [3]
- 12** Solve $5^{2x+1} = 40$, giving your answer to 3 significant figures. [3]
- 13** Solve $9^x - 4 \times 3^x + 3 = 0$. [4]
- 14** A cup of coffee cools so that its temperature, T °C, at time t minutes after it was poured is modelled by $T = 20 + 70e^{-0.05t}$.
 (a) Find the initial temperature of the coffee. [1]
 (b) Find the time taken for the coffee to cool to 50 °C, giving your answer to 3 significant figures. [4]
 (c) Explain what the model predicts about the temperature in the long term. [1]
- 15** Solve $e^{2x} - 5e^x + 6 = 0$, giving exact answers. [4]
- 16** Solve the simultaneous equations $2^{x+y} = 64$ and $3^{x-y} = 81$. [5]
- 17** Solve $e^x + 12e^{-x} = 7$, giving exact answers. [4]
- 18** Solve $3^x e^{2x-1} = 12$, giving your answer as an exact fraction in terms of natural logarithms, and then to 3 significant figures. [5]
- 19** Without using a calculator, solve $x \ln 12 + \ln 45 = \ln 5 + x \ln 4$, giving x as an integer. [4]
- 20** Consider the equation $3^{2x+1} = 5^{x+2}$.
 (a) Show that the equation can be written as $(\frac{9}{5})^x = \frac{25}{3}$. [3]
 (b) Hence find the value of x , giving your answer to 3 significant figures. [3]
- 21** Show that the solution of $3^{x+1} = 2^{x+3}$ can be written as $x = \frac{\ln(8)}{3/\ln(3/2)}$, and evaluate x to 3 significant figures. [5]
- 22** Solve the inequalities, giving the smallest integer n in each case.
 (a) $1.05^n > 2$ [3]
 (b) $0.8^n < 0.01$ [4]
- 23** Solve $2^{2x+1} - 5 \times 2^x + 2 = 0$. [6]
- 24** The size of a bacterial population is modelled by $P = P_0 e^{kt}$, where t is in hours. The population doubles every 12 hours.
 (a) Show that $k = \frac{\ln 2}{12}$. [2]
 (b) Find the time for the population to treble, to 3 significant figures. [3]
 (c) Find the percentage increase in population every hour, to 3 significant figures. [2]
- 25** Solve $e^{3x} - 3e^{2x} - 6e^x + 8 = 0$, giving exact answers. [5]
- 26** Solve $e^{2x+2} - 7e^{x+1} + 10 = 0$, giving exact answers. [4]
- 27** Answer the following.
 (a) Solve the inequality $3^{2x} - 10 \times 3^x + 9 < 0$. [5]
 (b) A student solves $3^x + 3^{x+1} = 36$ by writing $\ln(3^x) + \ln(3^{x+1}) = \ln 36$. Explain the error, then solve the equation correctly. [5]

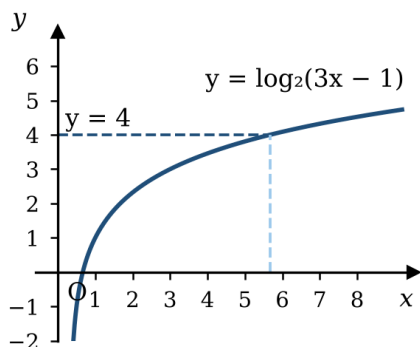
- 28** The first term of a geometric series is 5 and its common ratio is 3.
- (a) Show that the sum of the first n terms is $\frac{5}{2}(3^n - 1)$. [2]
- (b) Find the least value of n for which the sum exceeds 10 000 000. [4]
- (c) Solve $4^x - 6 \times 2^x - 16 = 0$, explaining why one of the roots of your quadratic is rejected. [4]
-
- 29** Consider the equation $4^x - 2^{x+1} + k = 0$, where k is a constant.
- (a) Show that when $k = 1$ the equation has exactly one solution and find it. [3]
- (b) Find the set of values of k for which the equation has two distinct real solutions. Justify your answer. [5]
- (c) Solve the equation when $k = -3$. [3]
-
- 30** Consider the equation $e^x - e^{-x} = 2k$, where $k > 0$ is a constant.
- (a) Show that when $k = 1$ the equation has exactly one real solution and find it in exact form. [5]
- (b) Hence, or otherwise, solve $e^x - e^{-x} = 2k$ for $k > 0$ in terms of k . [3]
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Logarithmic Equations

- 1** Solve (a) $\log_3 x = 4$, (b) $\log_x 64 = 2$, where $x > 0$. [2]
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- 2** Solve (a) $\ln x = 3$, giving your answer in terms of e , (b) $\log_{10}(x + 1) = 2$. [2]
-
- 3** The line $y = 1.5$ meets the curve $y = \ln x$ at the point P , as shown. Find the x -coordinate of P , as an exact value and to 3 significant figures.
- 
- [3]
-
- 4** Solve $\log_5(2x + 1) = 2$. [2]
-
- 5** Solve $\log_3 x + \log_3 4 = \log_3 20$. [2]
-
- 6** Answer the following.
- (a) Given that $\log_a 64 = 3$, find the value of a . [2]
- (b) For this value of a , solve $\log_a x = -2$. [2]
-
- 7** Solve each equation.
- (a) $\log_2(3x - 1) = 4$ [2]
- (b) $\log_5(x + 4) - \log_5 x = 1$ [3]
-
- 8** Solve $\ln(3x - 2) = \ln x + \ln 2$. [2]
-
- 9** Solve $\log_{10}(x + 6) - \log_{10}(x - 3) = 1$. [3]
-

- 10** Solve $2 \log_3 x = \log_3(x + 6)$. [3]
- 11** Solve $\ln(3x + 4) = 2 \ln x$, where $x > 0$. [5]
- 12** The sound level, L decibels, of a sound of intensity I watts per square metre is given by $L = 10 \log_{10}\left(\frac{I}{10^{-12}}\right)$.
 (a) Find the sound level when $I = 10^{-5}$. [2]
 (b) Find the intensity of a sound of level 95 decibels, giving your answer in standard form to 3 significant figures. [3]
 (c) Show that doubling the intensity of any sound increases its level by about 3 decibels. [2]
- 13** Solve $(\ln x)^2 - 4 \ln x + 3 = 0$, giving exact answers. [4]
- 14** The positive numbers x and y satisfy $\log_{10} x + \log_{10} y = 3$ and $y = 10x$. Find x and y . [5]
- 15** The pH of a solution is given by $\text{pH} = -\log_{10} h$, where h is the concentration of hydrogen ions in mol per litre.
 (a) Find the pH when $h = 3.2 \times 10^{-5}$, to 2 decimal places. [2]
 (b) Find h when the pH is 6.3, giving your answer in standard form to 3 significant figures. [3]
- 16** Solve $\log_x(2x + 3) = 2$, where $x > 0$ and $x \neq 1$. [4]
- 17** The time t days for a culture of bacteria to grow from 250 to N bacteria is modelled by $t = 8 \ln\left(\frac{N}{250}\right)$.
 (a) Find t when $N = 4000$, to 3 significant figures. [2]
 (b) Find N when $t = 20$, to the nearest whole number. [3]
- 18** Find the exact solution of the simultaneous equations
 $x + e^y = 7$, $\ln(x + 1) = y$. [4]
- 19** Solve $\ln(2x - 1) = 1 + \ln(x - 2)$, giving the exact answer. [5]
- 20** Solve each equation, showing clearly how you decide which values of x are valid.
 (a) $\log_3(x^2) - \log_3(x - 2) = 2$ [4]
 (b) $\log_3(x - 2) = \log_9 x$ [4]
- 21** The positive numbers x and y satisfy $\log_{10} x + \log_{10} y = 2$ and $x - y = 15$.
 (a) Find the values of x and y . [5]
 (b) The pair $x = -5$, $y = -20$ also satisfies $xy = 100$ and $x - y = 15$. Explain why it is not a solution of the original equations. [2]
- 22** Solve $\log_2(x + 4) = 3 - \log_2(x - 1)$, giving your answer in exact form. [6]
- 23** Solve $2 \log_{10}(x + 2) = 1 + \log_{10} x$. [6]

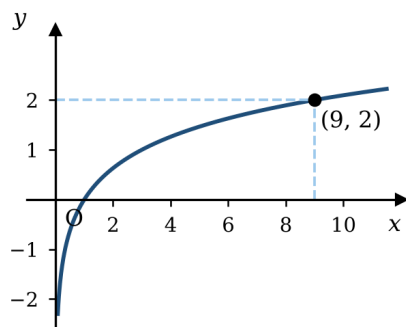
- 24** The diagram shows the curve $y = \log_2(3x - 1)$ and the line $y = 4$.



- (a) Find the exact x-coordinate of their intersection. [2]
 (b) Hence solve the inequality $\log_2(3x - 1) > 4$. [2]
 (c) Solve $\ln(5 - x) \leq \ln(x + 1)$, stating the full range of x . [3]
-
- 25** A student solves $\log_2(x^2) = 4$ as follows: ' $2 \log_2 x = 4$, so $\log_2 x = 2$, so $x = 4$ '.
 (a) Show that $x = -4$ is also a solution. [2]
 (b) Explain the error in the student's working. [2]
 (c) Solve $\log_2(x^2) = 4$ correctly. [2]
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- 26** Solve $3 \ln x + \frac{2}{\ln x} = 7$, giving exact answers. [5]
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- 27** A student is asked to solve $\log_2(x - 1) + \log_2(x - 3) = 3$ and writes: ' $\log_2[(x - 1) + (x - 3)] = 3$, so $2x - 4 = 8$, so $x = 6$ '.
 (a) Identify the error in the student's first step. [2]
 (b) Solve the equation correctly. [5]
 (c) Show that for every real value of k the equation $\log_2(x - 1) + \log_2(x - 3) = k$ has exactly one solution. [4]
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- 28** Answer the following.
 (a) Solve $(\log_2 x)^2 - 4 \log_2 x - 5 = 0$. [4]
 (b) Solve $x^{\log_2 x} = 16x^3$, where $x > 0$. [6]
-
- 29** Solve $2 \log_x 3 - \log_9 x = 1$, where $x > 0$ and $x \neq 1$, giving your answers in the form 3^k . [8]
-
- 30** The positive numbers x and y , both different from 1, satisfy $\log_x y + \log_y x = \frac{10}{3}$ and $xy = 64$.
 Find all pairs (x, y) in exact form. [8]

Exponential and Logarithmic Graphs

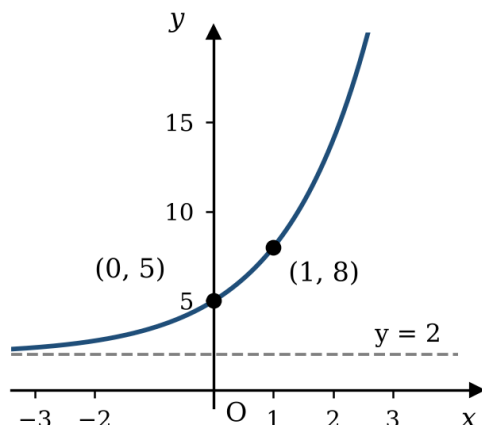
- 1** The diagram shows the curve $y = \log_a x$, where $a > 1$. The curve passes through the point (9, 2). Find the value of a , and the value of y when $x = 27$.



[3]

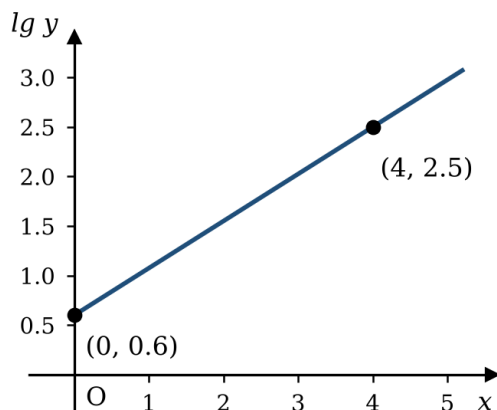
- 2** For each curve write down the coordinates of the point where it crosses the y-axis and the equation of its horizontal asymptote.
(a) $y = 2^x$ (b) $y = e^x + 3$ (c) $y = 5e^{-x}$ [3]
- 3** Sketch the curve $y = 2^{-x}$, showing the coordinates of the point where it crosses the y-axis. [2]
- 4** The curve C has equation $y = e^x$. Write down the equation of the curve obtained when C is (a) translated 2 units to the right, (b) reflected in the x-axis, (c) translated 3 units up. [3]
- 5** Sketch the curve $y = e^x - 4$. Show the coordinates of the points where the curve crosses the axes and state the equation of the asymptote. [4]
- 6** Sketch the curve $y = \ln(x + 3)$. Show the coordinates of the points where the curve crosses the axes and state the equation of the asymptote. [4]
- 7** Sketch the curve $y = \ln(2x)$, showing where it crosses the x-axis, and state the equation of its asymptote. Describe the transformation that maps $y = \ln x$ onto $y = \ln(2x)$. [4]
- 8** The function f is defined by $f(x) = e^x + 1$, $x \in \mathbb{R}$. Find $f^{-1}(x)$ and state its domain. [3]
- 9** Describe the single transformation that maps the graph of $y = e^x$ onto each of (a) $y = e^{2x}$, (b) $y = 3e^x$, (c) $y = e^{-x}$. [3]
- 10** The variables x and y are related by $y = a b^x$. A graph of $\log_{10} y$ against x is a straight line through (0, 0.70) and (3, 1.60). Find a and b , each to 2 significant figures. [5]
- 11** Make x the subject of each equation, where x and y are real.
(a) $y = \ln(x - 2) + 3$ [2]
(b) $y = \frac{1}{2} e^{x-4} + 3$ [3]

- 12** The diagram shows the curve with equation $y = a b^x + c$, where a , b and c are constants and $b > 0$. The curve has the horizontal asymptote $y = 2$ and passes through $(0, 5)$ and $(1, 8)$.



- (a) Find the values of a , b and c . [4]
 (b) Find the value of x for which $y = 50$. [3]

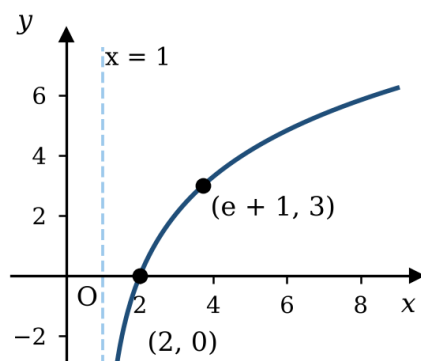
- 13** Two variables x and y are related by $y = a b^x$, where a and b are constants. The graph of $\lg y$ against x (logarithms to base 10) is a straight line passing through $(0, 0.6)$ and $(4, 2.5)$, as shown.



- (a) Show that $\lg y = \lg a + x \lg b$. [2]
 (b) Find the values of a and b , giving each to 2 significant figures. [4]

- 14** The function f is defined by $f(x) = 3e^{2x} - 1$, $x \in \mathbb{R}$. Find the range of f , and find $f^{-1}(x)$ with its domain. [5]

- 15** The diagram shows the curve $y = a \ln(x - 1)$, where a is a constant. The curve passes through the point $(e + 1, 3)$.



- (a) Find the value of a . [2]
 (b) Find the exact x -coordinate of the point on the curve where $y = 6$. [2]

- 16** By sketching the graphs of $y = \ln x$ and $y = 2 - x$ on the same axes, show that the equation $\ln x + x - 2 = 0$ has exactly one real root, and show that it lies between 1 and 2. [4]

- 17** The curve C has equation $y = \ln(Ax + B)$, where A and B are positive constants. C passes through the points $(1, \ln 5)$ and $(3, \ln 11)$.

- (a) Find the value of A and the value of B . [4]
 (b) Write down the equation of the vertical asymptote of C , and find the exact coordinates of the points where C crosses the axes. [4]

- 18** The curve C has equation $y = \ln x$. The curve D has equation $y = 4 - \ln(x + 2)$.

- (a) Describe a sequence of three transformations that maps C onto D . [3]
 (b) Write down the equation of the vertical asymptote of D . [1]
 (c) Find the exact coordinates of the points where D crosses the axes. [3]

- 19** The variables x and y are related by $y = ax^n$, where a and n are constants. Some values are shown in the table (y is given to 2 decimal places).

x	2	4	6	8
y	8.49	24.00	44.09	67.88

- (a) Show that the graph of $\ln y$ against $\ln x$ is a straight line, and state its gradient and intercept in terms of a and n . [2]
 (b) Use the values at $x = 4$ and $x = 8$ to find n and a . [5]
 (c) Predict the value of y when $x = 10$, giving your answer to 3 significant figures. [2]

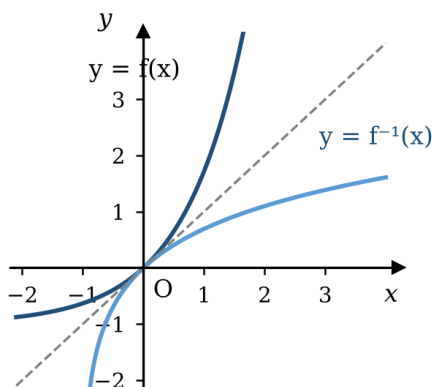
- 20** The curve C_1 has equation $y = 2^x$. The curve C_2 is obtained by reflecting C_1 in the y -axis and then translating the result by 3 units in the positive y direction.

- (a) Write down the equation of C_2 . [2]
 (b) Sketch C_1 and C_2 on the same axes, showing the coordinates of their intersections with the y -axis and the equation of any asymptote. [3]
 (c) Find the exact x -coordinate of the point where C_1 and C_2 intersect. [4]

- 21** The curve C has equation $y = e^x$.

- (a) C is translated by 3 units in the positive x -direction and then stretched by scale factor 2 parallel to the y -axis. Find the equation of the resulting curve. [2]
 (b) Show that the curve $y = 2e^{x-3}$ can also be obtained from C by a single stretch parallel to the y -axis, and give the scale factor. [2]

- 22** The diagram shows the graphs of $y = f(x)$, where $f(x) = e^x - 1$ for $x \in \mathbb{R}$, and $y = f^{-1}(x)$, together with the line $y = x$.



- (a) Find $f^{-1}(x)$ and state its domain and range. [5]
 (b) Explain why the graphs of f and f^{-1} are reflections of each other in $y = x$, and state the number of intersections of the two curves. [4]

- 23** Let $f(x) = |2^x - 4|$.
- (a) Sketch the graph of $y = f(x)$, showing the coordinates of any points where it meets the axes and the equation of its asymptote. [5]
- (b) Solve $f(x) = 4$. [3]

- 24** The table shows experimental values of x and y (first row x , second row y). It is suggested that $y = a x^n$.
By plotting $\ln y$ against $\ln x$, or otherwise, show that this model fits the data, and find a and n .

x	1	2	3	4	5
y	3.00	8.49	15.59	24.00	33.54

[6]

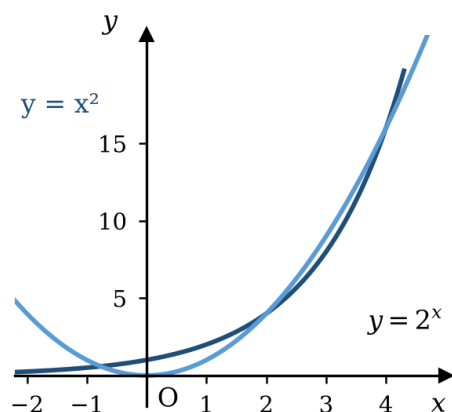
- 25** The curve C has equation $y = \ln x$.
- (a) Show that the tangent to C at the point $P(e, 1)$ passes through the origin. [4]
- (b) Deduce that $\ln x \leq \frac{x}{e}$ for all $x > 0$. [2]

- 26** The curve C has equation $y = e^{2x} - e^x - 6x$.
- (a) Find $\frac{dy}{dx}$. [2]
- (b) Show that C has exactly one stationary point and find its exact coordinates. [7]
- (c) Determine the nature of the stationary point. [2]

- 27** The function f is defined by $f(x) = 3e^x - 2$, for $x \in \mathbb{R}$.
- (a) Write down the range of f . [1]
- (b) Find $f^{-1}(x)$ and state its domain. [3]
- (c) Sketch $y = f(x)$ and $y = f^{-1}(x)$ on the same axes, showing the coordinates of any points where the curves cross the axes and the equations of any asymptotes. [4]
- (d) Given that the solutions of $f(x) = f^{-1}(x)$ are the solutions of $f(x) = x$, show that the curves $y = f(x)$ and $y = f^{-1}(x)$ do not intersect. [4]

- 28** Answer the following about transformations and graphs of logarithmic functions.
- (a) Show that the graph of $y = \ln(3x)$ can be obtained from the graph of $y = \ln x$ by a translation, and state the translation vector. [3]
- (b) A student claims that the graphs of $y = \ln(x^2)$ and $y = 2 \ln x$ are identical. Sketch both graphs on the same axes and explain why the claim is false. [4]
- (c) Find, in exact form, the x -coordinates of the points where the line $y = 4$ meets $y = \ln(x^2)$. [2]

- 29 The diagram shows the graphs of $y = 2^x$ and $y = x^2$.



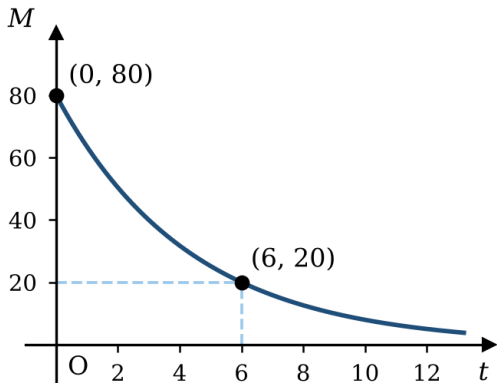
- (a) Verify that $x = 2$ and $x = 4$ are solutions of $2^x = x^2$. [2]
 (b) Use the diagram to explain why the equation has exactly one other solution, and show that it lies between -0.8 and -0.7 . [4]
 (c) Solve $2^x < x^2$ for $x > 0$. [2]

- 30 Let $f(x) = x - 1 - \ln x$, for $x > 0$.

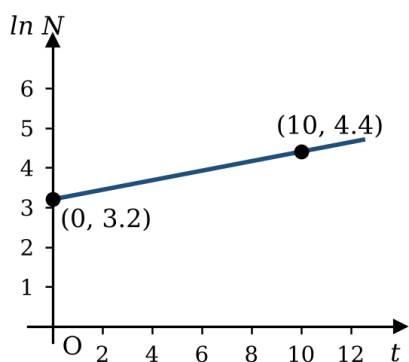
- (a) Show that f has a single stationary point and that it is a minimum. Find its value. [5]
 (b) Hence prove that $\ln x \leq x - 1$ for all $x > 0$. [2]
 (c) Deduce that $e^t \geq 1 + t$ for all real t . [3]

Exponential Growth and Decay

- 1 The number of insects in a population is modelled by $N = 200e^{0.03t}$, where t is the time in days after the start of observation. Find the value of N when (a) $t = 0$, (b) $t = 10$, giving your answer to the nearest whole number. [2]
- 2 A radioactive sample has mass m grams at time t years, where $m = 50e^{-0.1t}$. Find the mass of the sample after 5 years, giving your answer to 3 significant figures. [2]
- 3 £1000 is invested at 4% compound interest per year. Find its value after 5 years, to the nearest penny. [2]
- 4 The mass of a radioactive sample is modelled by $m = m_0e^{-0.02t}$, where t is in years. Find the percentage of the original mass remaining after 10 years, to 3 significant figures. [2]
- 5 A radioactive isotope has a half-life of 5 hours. A sample has mass 80 g. Find its mass after 15 hours. [2]
- 6 The mass M of a radioactive isotope at time t days is modelled by $M = M_0e^{-kt}$, where M_0 and k are positive constants. The half-life of the isotope is 8 days.
 (a) Show that $k = \frac{\ln 2}{8}$. [3]
 (b) Find the percentage of the original mass that remains after 20 days, giving your answer to 3 significant figures. [2]
- 7 The population of a town is modelled by $P = P_0e^{kt}$, where t is the number of years after 1 January 2020. The population was 12 000 on 1 January 2020 and 13 500 on 1 January 2023.
 (a) Find the value of k , giving your answer to 3 significant figures. [2]
 (b) Predict the population on 1 January 2030, to the nearest hundred. [2]

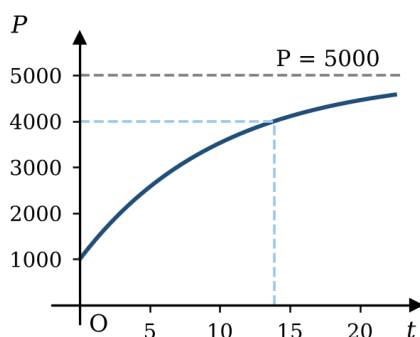
- 8** A population of bacteria doubles every 5 days. It is modelled by $N = N_0 e^{kt}$, with t in days. Find the exact value of k and its value to 3 significant figures. [3]
- 9** A population is modelled by $P = 300e^{0.04t}$. Find the time for P to reach 1000, to 3 significant figures. [3]
- 10** The value £ V of a machine t years after it was bought is modelled by $V = 25\,000e^{-0.15t}$.
 (a) Find the value after 3 years, to the nearest £10. [2]
 (b) Find when the machine's value is half its original value. [2]
- 11** A population is modelled by $N = 200e^{0.03t}$. Find $\frac{dN}{dt}$ and the rate of increase at $t = 10$, to 2 significant figures. [3]
- 12** The concentration of a drug in the blood is $C = C_0 e^{-0.2t}$, t in hours. Find the time for the concentration to fall to 10% of its initial value, to 3 significant figures. [3]
- 13** The graph shows the mass, M grams, of a decaying substance at time t hours, where $M = M_0 e^{-kt}$. The curve passes through $(0, 80)$ and $(6, 20)$.

 (a) Find the values of M_0 and k , giving k to 3 significant figures. [4]
 (b) Find the half-life of the substance. [2]
- 14** The number of bacteria in a culture is modelled by $N = 500e^{0.2t}$, where t is the time in hours since the culture was set up.
 (a) Show that $\frac{dN}{dt} = 0.2N$. [2]
 (b) Find the rate of increase in the number of bacteria when $N = 2000$. [1]
 (c) Find the time taken for the number of bacteria to reach 10 000, giving your answer to 2 decimal places. [3]
- 15** The number N of bacteria satisfies $N = Ae^{kt}$. When $t = 2$, $N = 12$ and when $t = 5$, $N = 75$. Find A and k to 3 significant figures. [5]

- 16** A graph of $\ln N$ against t is a straight line through $(0, 3.2)$ and $(10, 4.4)$, where N is the number of animals in a colony and t is in years. The colony is modelled by $N = N_0 e^{kt}$. Find N_0 and k , and the predicted value of N when $t = 25$, to the nearest whole number.



[5]

- 17** The population P of a species on an island t years after it was introduced is modelled by $P = 5000 - 4000e^{-0.1t}$.



- (a) Find the initial population and the long-term limit of the population. [2]
 (b) Find the time for the population to reach 4000, to 3 significant figures. [2]

- 18** Carbon-14 has a half-life of 5730 years. A bone is found to contain 63% of the carbon-14 of a living sample. Assuming exponential decay, estimate the age of the bone to the nearest 100 years. [5]

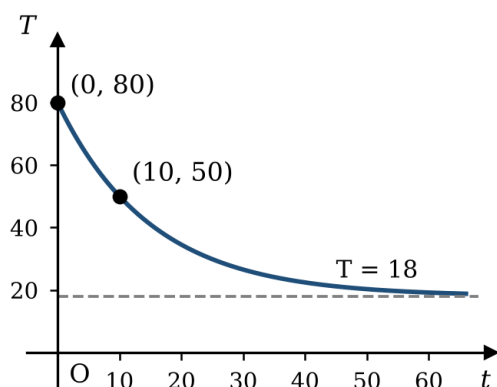
- 19** A colony of insects is modelled by $P = Ae^{kt}$, where t is the time in days. When $t = 10$ the population is 300 and when $t = 30$ it is 1200.

- (a) Show that $e^{20k} = 4$ and find the exact value of k . [2]
 (b) Find the value of A . [2]
 (c) Find the population predicted when $t = 50$. [2]

- 20** The concentration, C mg per litre, of a drug in a patient's blood t hours after an injection is modelled by $C = C_0 e^{-kt}$. The concentration is 64 mg per litre after 2 hours and 8 mg per litre after 5 hours.

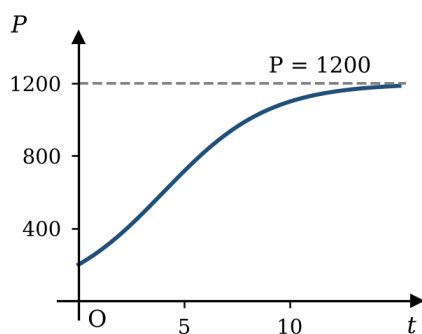
- (a) Find the exact value of k and the value of C_0 . [4]
 (b) A second injection is given when the concentration first falls below 5 mg per litre. Find, to the nearest minute, the time after the first injection at which this happens. [3]
 (c) Show that the time for the concentration to halve is the same whatever the starting concentration, and state its value for this drug. [2]

- 21** A cup of tea is left in a room of constant temperature 18°C . Its temperature, $T^\circ\text{C}$, at time t minutes after it was made is modelled by $T = 18 + 62e^{-kt}$, where k is a positive constant. After 10 minutes the temperature is 50°C .



- (a) Show that $k = \frac{1}{10} \ln\left(\frac{31}{16}\right)$. [3]
 (b) Find the time at which the tea reaches 25°C , to 3 significant figures. [3]
 (c) Find the rate at which the tea is cooling when $t = 10$, giving your answer to 3 significant figures. [2]

- 22** The number P of members of a club t months after it opened is modelled by $P = \frac{1200}{1 + 5e^{-0.4t}}$.



- (a) Find the initial membership and the limit of P as $t \rightarrow \infty$. [2]
 (b) Find when the membership first exceeds 900. [3]
 (c) Explain why this model is more realistic than $P = P_0e^{kt}$ for a club. [1]

- 23** The population of a town is modelled by $P = P_0e^{kt}$ with $k > 0$.

- (a) Prove that the time T for the population to double is the same whatever the starting time, and find T in terms of k . [4]
 (b) The population doubled between 1990 and 2010. Estimate the year in which it will have trebled from its 1990 value. [4]

- 24** A capacitor discharges so that the voltage across it is $V = V_0e^{-kt}$ volts, t seconds after it starts to discharge. The voltage falls from 12 V to 4.5 V in 3 seconds.

- (a) Find the exact value of k . [3]
 (b) Find the voltage after 5 seconds, to 3 significant figures, and the time at which it falls to 0.5 V. [4]

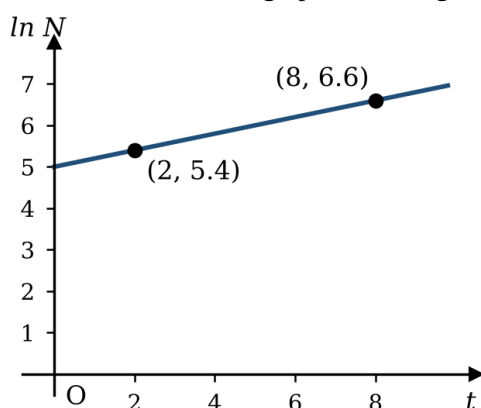
- 25** Two accounts each start with £5000. Account A pays 3.5% per year compound interest, added once a year. Account B grows continuously according to $B = 5000e^{0.035t}$.

- (a) Find the value of each account after 20 years, to the nearest pound. [4]
 (b) Find the time, in years to 3 significant figures, for each account to double. [4]

- 26** The value, £V, of a painting t years after it was bought is modelled by $V = 600 + 200 \ln(2t + 1)$, $t \geq 0$.
- Find the value of the painting when it was bought. [1]
 - Find the value of the painting after 10 years, to the nearest pound. [2]
 - Find the exact time at which the model predicts that the value is £1000, and give it to 3 significant figures. [3]
 - Find $\frac{dV}{dt}$ and hence explain why the model predicts that the value keeps increasing, but at a decreasing rate. [3]

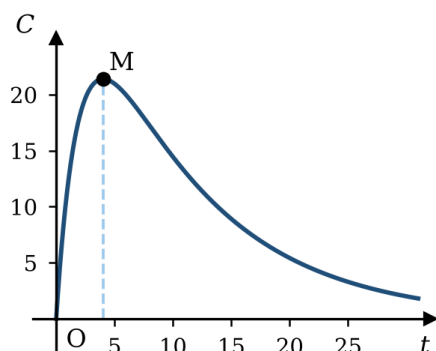
- 27** The populations of two towns t years after 1 January 2020 are modelled by $A = 80\,000e^{-0.02t}$ and $B = 50\,000e^{0.03t}$.
- Find the time when the two populations are predicted to be equal, giving your answer to 3 significant figures. [4]
 - Show that the ratio $\frac{A}{B} = 1.6e^{-0.05t}$. [2]
 - Find the time at which the total population $A + B$ is a minimum, and show that it is a minimum. Give the minimum total population to the nearest hundred. [5]

- 28** A biologist records the number N of cells in a sample at time t hours and suspects that $N = Ae^{kt}$, where A and k are constants. A graph of $\ln N$ against t is a straight line passing through (2, 5.4) and (8, 6.6).



- Show that the biologist's suspicion is consistent with a straight line graph of $\ln N$ against t . [2]
- Find the values of k and A , giving A to 3 significant figures. [4]
- Find the time for the number of cells to double, giving your answer to 3 significant figures. [2]
- Find when the number of cells is predicted to reach 1 000 000, to 3 significant figures, and comment on whether the model is likely to be valid then. [3]

- 29** The concentration C mg per litre of a drug in a patient's blood t hours after it was taken by mouth is modelled by $C = 40(e^{-0.1t} - e^{-0.5t})$, $t \geq 0$. The curve has a maximum at the point M .



- Show that $C = 0$ at $t = 0$ and explain what happens to C as $t \rightarrow \infty$. [2]
- Show that the maximum occurs when $e^{0.4t} = 5$, and find the exact value of t at M . [5]
- Find the maximum concentration, to 3 significant figures. [3]

- 30** The table shows the population P (in thousands) of a town at the start of each of six years, $t = 0$ to 5.

$t = 0$	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
50	61	75	91	105	116

- (a) A student models the population by $P = Ae^{kt}$, using the values at $t = 0$ and $t = 2$. Find A and k , and the predictions at $t = 4$ and $t = 5$. [5]
- (b) Find the percentage error of the model at $t = 5$, and comment on how well the model fits the data. [3]
- (c) Give one reason why an exponential model is unlikely to be suitable for the long-term population of a town. [1]