

First Principles

1 (4 marks)

$$(a) \text{ Gradient of AB} = \frac{(3+h)^2 - 9}{h}$$

M1

$$= \frac{9 + 6h + h^2 - 9}{h} = \frac{6h + h^2}{h} = 6 + h$$

A1

Answer: 6 + h

2 marks

(b) As $h \rightarrow 0$, $6 + h \rightarrow 6$

B1

This is the gradient of the tangent (the curve) at A

E1

Answer: 6; gradient of the curve at A

2 marks

2 (3 marks)

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{5(x+h)^2 - 5x^2}{h}$$

M1

$$= \lim_{h \rightarrow 0} \frac{10xh + 5h^2}{h} = \lim_{h \rightarrow 0} (10x + 5h)$$

M1

As $h \rightarrow 0$, $f'(x) = 10x$

A1

Answer: 10x

3 marks

3 (3 marks)

$$\text{Chord gradient} = \frac{(3+h)^2 - 9}{h} = 6 + h$$

M1

$$h = \frac{1}{10}: \text{gradient} = 6.100$$

A1

Others: 6.01 and 6.001; the gradient approaches 6

A1

Answer: 6.1, 6.01, 6.001; approaches 6

3 marks

4 (4 marks)

$$f(x+h) - f(x) = 5(x+h) - 2 - (5x - 2) = 5h$$

M1

$$\frac{f(x+h) - f(x)}{h} = \frac{5h}{h} = 5$$

M1

As $h \rightarrow 0$, $f'(x) = 5$

A1

The graph is a straight line of gradient 5, so the gradient is the same at every point

B1

Answer: 5 (the gradient of the straight line)

4 marks

5 (3 marks)

$$f(1+h) = (1+h)^2 + 3(1+h) = 4 + 5h + h^2$$

M1

$$\text{Gradient of AB} = \frac{5h + h^2}{h} = 5 + h$$

A1

As $h \rightarrow 0$ the gradient tends to 5

B1

Answer: 5

3 marks

6 (4 marks)

$$f(x+h) = -h^2 - 2hx + 2h - x^2 + 2x$$

M1

$$f(x+h) - f(x) = -h^2 - 2hx + 2h$$

M1

$$\frac{f(x+h) - f(x)}{h} = -h - 2x + 2$$

A1

As $h \rightarrow 0$, $f'(x) = 2 - 2x$

A1

Answer: $f'(x) = 2 - 2x$

4 marks

7 (4 marks)

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^3 - x^3}{h}$$

M1

$$(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$$

M1

$$\text{So the gradient of the chord} = 3x^2 + 3xh + h^2$$

A1

$$\text{As } h \rightarrow 0, f'(x) = 3x^2$$

A1

$$\text{Answer: } 3x^2$$

4 marks

8 (4 marks)

$$f(x+h) = 2(x+h)^2 - 3(x+h) + 1 = 2x^2 + 4xh + 2h^2 - 3x - 3h + 1$$

M1

$$f(x+h) - f(x) = 4xh + 2h^2 - 3h$$

M1

$$\frac{f(x+h) - f(x)}{h} = 4x + 2h - 3$$

A1

$$\text{As } h \rightarrow 0, f'(x) = 4x - 3$$

A1

$$\text{Answer: } 4x - 3$$

4 marks

9 (4 marks)

$$\frac{f(-2+h) - f(-2)}{h}: f(-2) = 4 - 10 - 1 = -7$$

B1

$$f(-2+h) = (-2+h)^2 + 5(-2+h) - 1 = -7 + h + h^2$$

M1

$$\text{Gradient of chord} = \frac{h+h^2}{h} = 1+h$$

A1

$$\text{As } h \rightarrow 0 \text{ the gradient} \rightarrow 1$$

A1

$$\text{Answer: } 1$$

4 marks

10 (5 marks)

$$\text{(a) Gradient of PQ} = \frac{2^{2+h} - 4}{h}$$

M1

$$h = 0.1: 2.8709; h = 0.01: 2.7822; h = 0.001: 2.7735$$

A1 A1

$$\text{Answer: } 2.8709, 2.7822, 2.7735$$

3 marks

$$\text{(b) The gradients are decreasing towards a limit}$$

M1

$$\text{The gradient at P is approximately } 2.77$$

A1

$$\text{Answer: } 2.77$$

2 marks

11 (4 marks)

$$\frac{f(x+h) - f(x)}{h} = \frac{1}{h} \left(\frac{2}{x+h} - \frac{2}{x} \right) = \frac{1}{h} \times \frac{2x - 2(x+h)}{x(x+h)}$$

M1 A1

$$= \frac{-2h}{h x(x+h)} = -\frac{2}{x(x+h)}$$

M1

$$\text{As } h \rightarrow 0, f'(x) = -\frac{2}{x^2}$$

A1

$$\text{Answer: } -\frac{2}{x^2}$$

4 marks

12 (4 marks)

$$f(x+h) = h^2 + 2hx + 6h + x^2 + 6x + 9$$

M1

$$f(x+h) - f(x) = h^2 + 2hx + 6h$$

M1

$$\frac{f(x+h) - f(x)}{h} = h + 2x + 6$$

A1

$$\text{As } h \rightarrow 0, f'(x) = 2x + 6$$

A1

$$\text{Answer: } f'(x) = 2x + 6$$

4 marks

13 (5 marks)

$$(a) y + \delta y = (x + \delta x)^2 + (x + \delta x) = x^2 + 2x\delta x + (\delta x)^2 + x + \delta x$$

M1

$$\text{So } \delta y = 2x\delta x + (\delta x)^2 + \delta x$$

A1

$$(b) \frac{\delta y}{\delta x} = 2x + \delta x + 1$$

M1 A1

$$\text{As } \delta x \rightarrow 0, \frac{dy}{dx} = 2x + 1$$

A1

$$\text{Answer: } \delta y = 2x\delta x + (\delta x)^2 + \delta x; \frac{dy}{dx} = 2x + 1$$

5 marks

14 (5 marks)

$$\frac{f(x+h) - f(x)}{h} = \frac{1}{h} \left(\frac{1}{x+h} - \frac{1}{x} \right)$$

M1

$$= \frac{1}{h} \times \frac{x - (x+h)}{x(x+h)}$$

M1

$$= \frac{1}{h} \times \frac{-h}{x(x+h)} = -\frac{1}{x(x+h)}$$

A1 A1

$$\text{As } h \rightarrow 0, f'(x) = -\frac{1}{x^2}$$

A1

$$\text{Answer: } -\frac{1}{x^2}$$

5 marks

15 (5 marks)

$$\frac{f(x+h) - f(x)}{h} = \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

M1

Multiply top and bottom by $\sqrt{x+h} + \sqrt{x}$:

M1

$$= \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} = \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

M1 A1

$$\text{As } h \rightarrow 0, f'(x) = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

A1

$$\text{Answer: } \frac{1}{2\sqrt{x}}$$

5 marks

16 (6 marks)

$$(2+h)^3 = 8 + 12h + 6h^2 + h^3$$

M1 A1

$$\text{Gradient of chord} = \frac{12h + 6h^2 + h^3}{h} = 12 + 6h + h^2$$

M1

As $h \rightarrow 0$, the gradient tends to 12

A1

$$\text{Tangent: } y - 8 = 12(x - 2)$$

M1

$$y = 12x - 16$$

A1

$$\text{Answer: } y = 12x - 16$$

6 marks

17 (5 marks)

$$\frac{f(x+h) - f(x)}{h} = \frac{\sqrt{2x+2h+1} - \sqrt{2x+1}}{h}$$

M1

Multiply top and bottom by $\sqrt{2x+2h+1} + \sqrt{2x+1}$:

M1

$$= \frac{(2x+2h+1) - (2x+1)}{h(\sqrt{2x+2h+1} + \sqrt{2x+1})} = \frac{2}{\sqrt{2x+2h+1} + \sqrt{2x+1}}$$

M1 A1

$$\text{As } h \rightarrow 0: \frac{2}{2\sqrt{2x+1}} = \frac{1}{\sqrt{2x+1}}$$

A1

$$\text{Answer: } \frac{1}{\sqrt{2x+1}}$$

5 marks

18 (5 marks)

$$f(x+h) = a(x+h)^2 + b(x+h) + c$$

M1

$$f(x+h) - f(x) = 2axh + ah^2 + bh$$

M1 A1

$$\frac{f(x+h) - f(x)}{h} = 2ax + ah + b$$

A1

$$\text{As } h \rightarrow 0, f'(x) = 2ax + b$$

A1

Answer: $2ax + b$

5 marks

19 (4 marks)

$$\frac{f(x) - f(a)}{x - a} = \frac{x^4 - a^4}{x - a}$$

M1

$$x^4 - a^4 = (x - a)(x^3 + ax^2 + a^2x + a^3), \text{ so the quotient is } x^3 + ax^2 + a^2x + a^3$$

M1 A1

$$\text{As } x \rightarrow a \text{ this tends to } a^3 + a^3 + a^3 + a^3 = 4a^3$$

A1

Answer: $4a^3$

4 marks

20 (7 marks)

$$f(x+h) - f(x) = (3x^2h + 3xh^2 + h^3) - 3h$$

M1 A1

$$\frac{f(x+h) - f(x)}{h} = 3x^2 + 3xh + h^2 - 3$$

M1 A1

$$\text{As } h \rightarrow 0, f'(x) = 3x^2 - 3$$

A1

$$3x^2 - 3 = 0 \text{ so } x = \pm 1$$

M1

$$y = 1 - 3 = -2 \text{ and } y = -1 + 3 = 2, \text{ so the points are } (1, -2) \text{ and } (-1, 2)$$

A1

Answer: $(1, -2), (-1, 2)$

7 marks

21 (9 marks)

$$(a) y \text{ at } Q = (1+h)^3 - 4(1+h) = 1 + 3h + 3h^2 + h^3 - 4 - 4h$$

M1

$$\text{Difference in } y = -3 - h + 3h^2 + h^3 - (-3) = h^3 + 3h^2 - h$$

M1

$$\text{Gradient} = \frac{h^3 + 3h^2 - h}{h} = h^2 + 3h - 1$$

A1

Answer: $h^2 + 3h - 1$

3 marks

$$(b) \text{ As } h \rightarrow 0, \text{ gradient} \rightarrow -1$$

B1

Answer: -1

1 mark

$$(c) \frac{f(x+h) - f(x)}{h} = \frac{3x^2h + 3xh^2 + h^3 - 4h}{h} = 3x^2 + 3xh + h^2 - 4$$

M1 A1

$$\text{As } h \rightarrow 0, f'(x) = 3x^2 - 4$$

A1

$$3x^2 - 4 = 8 \text{ so } x = \pm 2, \text{ giving } y = 0$$

M1

$$\text{Points } (2, 0) \text{ and } (-2, 0)$$

A1

Answer: $f'(x) = 3x^2 - 4$; $(2, 0)$ and $(-2, 0)$

5 marks

22 (8 marks)

$$(a) \frac{f(x+h) - f(x)}{h} = \frac{1}{h} \left(\frac{1}{(x+h)^2} - \frac{1}{x^2} \right)$$

M1

$$= \frac{1}{h} \times \frac{x^2 - (x+h)^2}{x^2(x+h)^2}$$

M1

$$x^2 - (x+h)^2 = -2xh - h^2$$

A1

$$= \frac{-2x - h}{x^2(x+h)^2} \text{ (after cancelling } h)$$

M1 A1

$$\text{As } h \rightarrow 0: \frac{-2x}{x^4} = -\frac{2}{x^3}$$

A1

Answer: $-\frac{2}{x^3}$

6 marks

$$(b) f'\left(-\frac{1}{2}\right) = -\frac{2}{\left(-\frac{1}{2}\right)^3}$$

M1

$$= -\frac{2}{\left(-\frac{1}{8}\right)} = 16$$

A1

Answer: 16**2 marks****23 (6 marks)**

$$f(x+h) - f(x) = 2[(x+h)^3 - x^3] - 3\left(\frac{1}{x+h} - \frac{1}{x}\right)$$

M1

$$2[(x+h)^3 - x^3] = 6x^2h + 6xh^2 + 2h^3$$

M1 A1

$$-3\left(\frac{1}{x+h} - \frac{1}{x}\right) = -3 \times \frac{-h}{x(x+h)} = \frac{3h}{x(x+h)}$$

M1 A1

$$\text{Divide by } h \text{ and let } h \rightarrow 0: 6x^2 + 6xh + 2h^2 + \frac{3}{x(x+h)} \rightarrow 6x^2 + \frac{3}{x^2}$$

A1

$$\text{Answer: } 6x^2 + \frac{3}{x^2}$$

6 marks**24 (6 marks)**

$$\frac{f(x+h) - f(x)}{h} = \frac{1}{h}\left(\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}\right) = \frac{1}{h} \times \frac{\sqrt{x} - \sqrt{x+h}}{\sqrt{x}\sqrt{x+h}}$$

M1 A1

Multiply top and bottom of $\frac{\sqrt{x} - \sqrt{x+h}}{h}$ by $\sqrt{x} + \sqrt{x+h}$:

M1

$$\frac{\sqrt{x} - \sqrt{x+h}}{h} = \frac{x - (x+h)}{h(\sqrt{x} + \sqrt{x+h})} = -\frac{1}{\sqrt{x} + \sqrt{x+h}}$$

A1

$$\text{So the quotient} = -\frac{1}{\sqrt{x}\sqrt{x+h}(\sqrt{x} + \sqrt{x+h})}$$

M1

$$\text{As } h \rightarrow 0: -\frac{1}{\sqrt{x} \times \sqrt{x} \times 2\sqrt{x}} = -\frac{1}{2x\sqrt{x}}$$

A1

$$\text{Answer: } -\frac{1}{2x\sqrt{x}}$$

6 marks**25 (4 marks)**

$$(a) \frac{\sin 1.001 - \sin 1}{0.001} = 0.5399$$

M1 A1

(b) $f'(1) = \cos 1$, and the chord gradient approximates $f'(1)$, so $\cos 1 \approx 0.5399$

B1

True value $\cos 1 = 0.5403$, so the estimate is slightly too high (a chord with small h is only approximate)

E1

Answer: (a) 0.5399 (b) $\cos 1 \approx 0.5399$

4 marks**26 (6 marks)**

$$f(x+h) - f(x) = \frac{x+h}{x+h+1} - \frac{x}{x+1} = \frac{(x+h)(x+1) - x(x+h+1)}{(x+h+1)(x+1)}$$

M1 A1

$$\text{Numerator: } x^2 + x + hx + h - x^2 - xh - x = h$$

M1 A1

$$\frac{f(x+h) - f(x)}{h} = \frac{1}{(x+h+1)(x+1)}$$

A1

$$\text{As } h \rightarrow 0, f'(x) = \frac{1}{(x+1)^2}$$

A1

$$\text{Answer: } \frac{1}{(x+1)^2}$$

6 marks**27 (8 marks)**

$$(a) \frac{f(x+h) - f(x)}{h} = \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

M1

$$= \sin x \left(\frac{\cos h - 1}{h}\right) + \cos x \left(\frac{\sin h}{h}\right)$$

M1

$$\cos h - 1 \approx -\frac{h^2}{2} \text{ so } \frac{\cos h - 1}{h} \approx -\frac{h}{2}$$

M1

$$\text{As } h \rightarrow 0, -\frac{h}{2} \rightarrow 0$$

A1

$$\text{and } \frac{\sin h}{h} \approx 1$$

A1

$$\text{So } f'(x) = \sin x \times 0 + \cos x \times 1 = \cos x$$

A1

Answer: $\cos x$

6 marks

(b) The approximations $\sin h \approx h$ and $\cos h \approx 1 - \frac{h^2}{2}$ are only valid for h in radians

E1

In degrees, $f'(x) = \frac{\pi}{180} \cos x$

B1

Answer: radians needed; $\frac{\pi}{180} \cos x$

2 marks

28 (10 marks)

(a) Error: $(x + h)^2 \neq x^2 + h^2$, the term $2xh$ is missing

E1

Correct: $f'(x) = 2x$

B1

Answer: $2x$

2 marks

(b) $(x + h)^n = x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \dots + h^n$

M1

$$\frac{f(x+h) - f(x)}{h} = nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \dots + h^{n-1}$$

M1 A1

Every term after the first still contains a factor h

B1

As $h \rightarrow 0$ these terms tend to 0, so $f'(x) = nx^{n-1}$

A1

Answer: nx^{n-1}

5 marks

(c) $\frac{d}{dx} x^n = nx^{n-1}$

M1

At $x = 1$: $n - n = 0$, so stationary

A1

$y = 1 - n$

A1

Answer: $y = 1 - n$

3 marks

29 (8 marks)

(a) $f(x+h) - f(x) = e^{x+h} - e^x = e^x(e^h - 1)$

M1

$$\frac{f(x+h) - f(x)}{h} = e^x \times \frac{e^h - 1}{h}$$

A1

As $h \rightarrow 0$ this tends to $e^x \times 1 = e^x$

A1

Answer: e^x

3 marks

(b) $g(x+h) - g(x) = (x+h)e^{x+h} - xe^x = e^x[(x+h)e^h - x]$

M1

$$= e^x[x(e^h - 1) + he^h]$$

M1

$$\frac{g(x+h) - g(x)}{h} = e^x\left[x \times \frac{e^h - 1}{h} + e^h\right]$$

A1

As $h \rightarrow 0$: $\frac{e^h - 1}{h} \rightarrow 1$ and $e^h \rightarrow 1$

M1

So $g'(x) = e^x(x+1)$

A1

Answer: $(x+1)e^x$

5 marks

30 (8 marks)

(a) $\frac{(p+h)^3 - p^3}{h} = \frac{3p^2h + 3ph^2 + h^3}{h} = 3p^2 + 3ph + h^2$

M1 A1

As $h \rightarrow 0$ the gradient tends to $3p^2$

A1

Answer: $3p^2$

3 marks

(b) Tangent: $y - p^3 = 3p^2(x - p)$

M1

Meets C where $x^3 - p^3 = 3p^2(x - p)$

M1

$(x-p)(x^2 + px + p^2) - 3p^2(x-p) = 0$, so $(x-p)(x^2 + px - 2p^2) = 0$

M1 A1

$(x-p)^2(x+2p) = 0$, so Q has $x = -2p$ ($x = p$ is the point of tangency)

A1

Answer: $x = -2p$

5 marks

Basic Differentiation

1 (2 marks)

Differentiate term by term: $4 \times 3x^2 = 12x^2$, $-6 \times 2x = -12x$, constant gives 0

M1

$$\frac{d y}{d x} = 12x^2 - 12x$$

A1

Answer: $12x^2 - 12x$

2 marks

2 (3 marks)

$$f'(x) = 10x^4 - 2x + 7$$

M1 A1

$$f''(x) = 40x^3 - 2$$

A1

Answer: $f'(x) = 10x^4 - 2x + 7$, $f''(x) = 40x^3 - 2$

3 marks

3 (3 marks)

$$\frac{d S}{d r} = 8\pi r$$

M1 A1

$$\text{When } r = 5, \frac{d S}{d r} = 40\pi$$

A1

Answer: $40\pi \text{ cm}^2 \text{ per cm}$

3 marks

4 (3 marks)

$$f'(x) = 6x - 8$$

M1 A1

$$f'(3) = 10$$

A1

Answer: $f'(x) = 6x - 8$; $f'(3) = 10$

3 marks

5 (2 marks)

$$y = 6x^{1/2}$$

M1

$$\frac{d y}{d x} = 3x^{-1/2} = \frac{3}{\sqrt{x}}$$

A1

Answer: $\frac{3}{\sqrt{x}}$

2 marks

6 (2 marks)

$$y = 5x^{-3}$$

M1

$$\frac{d y}{d x} = -15x^{-4} = -\frac{15}{x^4}$$

A1

Answer: $-\frac{15}{x^4}$

2 marks

7 (3 marks)

Write as powers: $y = 3x^{1/2} - 4x^{-2}$

M1

$$\frac{d y}{d x} = \frac{3}{2}x^{-1/2} + 8x^{-3}$$

A1

$$= \frac{3}{2\sqrt{x}} + \frac{8}{x^3}$$

A1

Answer: $\frac{3}{2\sqrt{x}} + \frac{8}{x^3}$

3 marks

8 (4 marks)

Expand the numerator: $(2x + 3)(x - 1) = 2x^2 + x - 3$

M1

$$y = 2 + x^{-1} - 3x^{-2}$$

M1

$$\frac{d y}{d x} = -x^{-2} + 6x^{-3}$$

A1

$$= -\frac{1}{x^2} + \frac{6}{x^3}$$

A1

Answer: $-\frac{1}{x^2} + \frac{6}{x^3}$

4 marks

9 (4 marks)

(a) $\frac{ds}{dt} = 15t^2 - 4$

M1 A1

Answer: $15t^2 - 4$

2 marks

(b) $\frac{dv}{dt} = u + at$

M1 A1

Answer: $u + at$

2 marks

10 (3 marks)

$y = 3x - 2x^{-1}$

M1

$\frac{dy}{dx} = 3 + 2x^{-2} = 3 + \frac{2}{x^2}$

M1 A1

Answer: $3 + \frac{2}{x^2}$

3 marks

11 (3 marks)

$\frac{dy}{dx} = 2 \times \frac{3}{2}x^{1/2} - 4 \times \left(-\frac{1}{2}\right)x^{-3/2}$

M1 A1

$= 3x^{1/2} + 2x^{-3/2}$

A1

Answer: $3x^{1/2} + 2x^{-3/2}$

3 marks

12 (5 marks)

(a) $\frac{dy}{dx} = 3x^2 - 8x + 7$

M1

At $x = 2$: $12 - 16 + 7 = 3$

A1

Answer: 3

2 marks

(b) $3x^2 - 8x + 7 = 7$, so $3x^2 - 8x = 0$

M1

$x(3x - 8) = 0$

M1

$x = 0$ or $x = \frac{8}{3}$

A1

Answer: $x = 0$ or $x = \frac{8}{3}$

3 marks

13 (4 marks)

$\frac{dy}{dx} = 4x^3 - 12x$

M1 A1

$\frac{d^2y}{dx^2} = 12x^2 - 12$

A1

At $x = -1$: $12 - 12 = 0$

B1

Answer: $4x^3 - 12x$; $12x^2 - 12$; 0

4 marks

14 (5 marks)

$\frac{dy}{dx} = 3x^2 - 12x + 9$

M1 A1

Decreasing when $\frac{dy}{dx} < 0$: $3(x - 1)(x - 3) < 0$

M1

Critical values $x = 1$ and $x = 3$, and the quadratic is U-shaped

M1

$1 < x < 3$

A1

Answer: $1 < x < 3$

5 marks

15 (6 marks)

(a) $\frac{dh}{dt} = 0.06t^2 - 0.6t + 1.5$

M1 A1

Answer: $0.06t^2 - 0.6t + 1.5$

2 marks

(b) $\frac{dh}{dt} = 0.06(t^2 - 10t + 25)$

M1

$= 0.06(t - 5)^2$

A1

A square is never negative, so $\frac{dh}{dt} \geq 0$ and h never decreases

E1

Answer: $\frac{dh}{dt} = 0.06(t - 5)^2 \geq 0$

3 marks

(c) Smallest value of $0.06(t - 5)^2$ is 0 at $t = 5$: rate 0 m per hour at $t = 5$

B1

Answer: $t = 5$, rate 0 m per hour

1 mark

16 (5 marks)

Expand: $y = \frac{4x^2 - 4x + 1}{x^{1/2}}$

M1

$y = 4x^{3/2} - 4x^{1/2} + x^{-1/2}$

A1

$\frac{dy}{dx} = 6x^{1/2} - 2x^{-1/2} - \frac{1}{2}x^{-3/2}$

M1 A1 A1

Answer: $6x^{1/2} - 2x^{-1/2} - \frac{1}{2}x^{-3/2}$

5 marks

17 (4 marks)

$f'(x) = x^2 - 5x + 6$

M1 A1

f is increasing when $f'(x) > 0$: $(x - 2)(x - 3) > 0$

M1

$x < 2$ or $x > 3$

A1

Answer: $x < 2$ or $x > 3$

4 marks

18 (5 marks)

$\frac{dy}{dx} = 3x^2 - 12x + 5$

M1 A1

The gradient is a quadratic with its least value at $\frac{d}{dx}\left(\frac{dy}{dx}\right) = 6x - 12 = 0$, so $x = 2$

M1

Least gradient $= 12 - 24 + 5 = -7$

A1

$y = 8 - 24 + 10 = -6$, so the point is $(2, -6)$

A1

Answer: Minimum gradient -7 at $(2, -6)$

5 marks

19 (5 marks)

$\frac{dy}{dx} = 2kx - 6$

M1

At $x = 2$: $4k - 6 = 10$, so $k = 4$

M1 A1

Gradient zero: $8x - 6 = 0$, $x = \frac{3}{4}$; $y = 4\left(\frac{3}{4}\right)^2 - 6\left(\frac{3}{4}\right) + 1 = -\frac{5}{4}$

M1

Point $\left(\frac{3}{4}, -\frac{5}{4}\right)$

A1

Answer: $k = 4$; $\left(\frac{3}{4}, -\frac{5}{4}\right)$

5 marks

20 (5 marks)

$\frac{dP}{dt} = 40 - t$

M1 A1

When $t = 10$: $\frac{dP}{dt} = 30$ birds per year

A1

Population stops increasing when $\frac{dP}{dt} = 0$, so $t = 40$ years

M1 A1

Answer: 30 birds per year; $t = 40$

5 marks

21 (8 marks)

(a) Point $(1, 5)$: $a + b = 5$

M1 A1

$\frac{dy}{dx} = 2ax - \frac{b}{x^2}$, so $2a - b = 1$

M1 A1

Adding: $3a = 6$, so $a = 2$ and $b = 3$

A1

Answer: $a = 2$, $b = 3$

5 marks

(b) $\frac{d^2y}{dx^2} = 4 + \frac{6}{x^3}$

M1

At $x = 1$ the value is 10

A1

Answer: $\frac{d^2y}{dx^2} = 4 + \frac{6}{x^3}$; 10

2 marks

(c) $\frac{d^2y}{dx^2} = 10 > 0$, so the gradient is increasing

B1

Answer: increasing

1 mark

22 (7 marks)

(a) $(\sqrt{x})^2 + 2 \times \sqrt{x} \times \frac{2}{\sqrt{x}} + \left(\frac{2}{\sqrt{x}}\right)^2$

M1

$$= x + 4 + \frac{4}{x}$$

A1

Answer: $y = x + 4 + \frac{4}{x}$

2 marks

(b) $y = x + 4 + 4x^{-1}$

M1

$$\frac{dy}{dx} = 1 - \frac{4}{x^2}$$

A1

Answer: $1 - \frac{4}{x^2}$

2 marks

(c) $1 - \frac{4}{x^2} = \frac{3}{4}$ so $\frac{4}{x^2} = \frac{1}{4}$

M1

$$x^2 = 16, x = \pm 4$$

A1

As $x > 0$, $x = 4$

A1

Answer: $x = 4$

3 marks

23 (8 marks)

(a) $y = \frac{1}{2}x^{3/2} + 2x^{1/2} - 4x^{-1/2}$

M1 A1

Answer: $y = \frac{1}{2}x^{3/2} + 2x^{1/2} - 4x^{-1/2}$

2 marks

(b) $\frac{dy}{dx} = \frac{3}{4}x^{1/2} + x^{-1/2} + 2x^{-3/2}$

M1 A1

Answer: $\frac{3}{4}x^{1/2} + x^{-1/2} + 2x^{-3/2}$

2 marks

(c) At $x = 4$: $\frac{3}{4} \times 2 + \frac{1}{2} + 2 \times \frac{1}{8} = \frac{9}{4}$

M1 A1

For $x > 0$ every term of $\frac{dy}{dx}$ is positive, so $\frac{dy}{dx} > 0$

M1

Hence y is an increasing function for all $x > 0$

A1

Answer: $\frac{9}{4}$

4 marks

24 (7 marks)

(a) $\frac{dy}{dx} = 6x^{1/2} - 6x^{-1/2}$

M1 A1

$$\frac{d^2y}{dx^2} = 3x^{-1/2} + 3x^{-3/2}$$

M1 A1

Answer: $\frac{dy}{dx} = 6x^{1/2} - 6x^{-1/2}$; $\frac{d^2y}{dx^2} = 3x^{-1/2} + 3x^{-3/2}$

4 marks

(b) $6x^{1/2} = 6x^{-1/2}$ so $x = 1$

M1 A1

$\frac{d^2y}{dx^2}$ is a sum of positive terms for $x > 0$, so positive

E1

Answer: $x = 1$

3 marks

25 (6 marks)

$f'(x) = 3ax^2 + 2bx + 2$

M1 A1

$f''(x) = 6ax + 2b$

A1

$f'(1) = 2$: $3a + 2b + 2 = 2$, so $3a + 2b = 0$

M1

$f''(2) = 18$: $12a + 2b = 18$

M1

Subtract: $9a = 18$, so $a = 2$ and $b = -3$

A1

Answer: $a = 2$, $b = -3$

6 marks

26 (7 marks)

(a) $\frac{dV}{dt} = 12t - t^2$

M1 A1

Answer: $12t - t^2$ litres per minute

2 marks

(b) Let $r = 12t - t^2$; $\frac{dr}{dt} = 12 - 2t$

M1

$\frac{dr}{dt} = 0$ when $t = 6$

A1

$\frac{d^2r}{dt^2} = -2 < 0$, so it is a maximum

E1

$r = 72 - 36 = 36$

M1 A1

Answer: $t = 6$ min; 36 litres per minute

5 marks

27 (9 marks)

(a) $f'(x) = 3x^2 - 6x + 5$

M1

$= 3(x - 1)^2 + 2$

M1

$(x - 1)^2 \geq 0$ so $f'(x) \geq 2 > 0$ for all x

A1

Answer: $f'(x) = 3(x - 1)^2 + 2 > 0$

3 marks

(b) f is a cubic with positive leading coefficient so it takes both negative and positive values, hence crosses the x -axis at least once

E1

As f is always increasing it cannot cross more than once

E1

Answer: exactly one root

2 marks

(c) $g'(x) = 3x^2 - 6x + p$

M1

Need $g'(x) > 0$ for all x , so no real roots: $36 - 12p < 0$

M1 A1

$p > 3$

A1

Answer: $p > 3$

4 marks

28 (9 marks)(a) The expression has not been written as a power ($y = \frac{4}{3}x^{-2}$); the numerator and denominator cannot be differentiated separately

E1

The power has been lowered incorrectly and the constant multiplier changed wrongly

E1

Answer: Not written as a power, and numerator and denominator differentiated separately; the power is lowered incorrectly

(b) $y = \frac{4}{3}x^{-2}$

M1

$\frac{dy}{dx} = \frac{4}{3} \times (-2)x^{-3}$

M1

$= -\frac{8}{3x^3}$

A1

Answer: $-\frac{8}{3x^3}$

3 marks

(c) $\frac{8}{3x^3} = \frac{1}{3}$ so $x^3 = 8$

M1

$x = 2$

A1

Answer: $x = 2$

2 marks

(d) $\frac{d^2y}{dx^2} = 8x^{-4} = \frac{8}{x^4}$

M1

$\frac{8}{x^4} > 0$ for all $x > 0$, so the gradient is increasing

A1

Answer: $\frac{d^2y}{dx^2} = \frac{8}{x^4} > 0$

2 marks

29 (8 marks)

$$(a) \frac{dy}{dx} = 3x^2 + 2px + q$$

B1

Stationary points need $3x^2 + 2px + q = 0$

M1

$$\text{Discriminant} = 4p^2 - 12q$$

M1

If $p^2 < 3q$ then $4p^2 - 12q < 0$ so there are no real roots

M1

So $\frac{dy}{dx} \neq 0$ for every x : no stationary points

E1

Answer: Shown

5 marks

$$(b) \frac{dy}{dx} = 3x^2 + 4x + 5 = 3\left(x + \frac{2}{3}\right)^2 + \frac{11}{3}$$

M1 A1

Minimum $\frac{11}{3}$ when $x = -\frac{2}{3}$

A1

Answer: $\frac{11}{3}$ at $x = -\frac{2}{3}$

3 marks

30 (9 marks)

$$(a) \frac{dy}{dx} = nx^{n-1}$$

M1

$$\frac{d^2y}{dx^2} = n(n-1)x^{n-2}$$

A1

$$x^2 \times n(n-1)x^{n-2} = n(n-1)x^n = n(n-1)y$$

A1

Answer: Shown

3 marks

$$(b) n(n-1) = 12, \text{ so } n^2 - n - 12 = 0$$

M1

$$(n-4)(n+3) = 0$$

M1

$$n = 4 \text{ or } n = -3$$

A1

Answer: $n = 4$ or $n = -3$

3 marks

(c) y is a sum of the two power functions x^4 and x^{-3} , each satisfying the equation (differentiation is linear)

M1

$$\text{Directly: } \frac{d^2y}{dx^2} = 12x^2 + 12x^{-5}$$

M1

$$x^2 \frac{d^2y}{dx^2} = 12x^4 + 12x^{-3} = 12y$$

A1

Answer: Shown

3 marks

Tangents and Normals**1 (3 marks)**

$$\text{When } x = 2, y = 4 + 6 - 2 = 8$$

B1

$$\frac{dy}{dx} = 2x + 3 = 7 \text{ at } x = 2$$

M1

$$y - 8 = 7(x - 2) \text{ so } y = 7x - 6$$

A1

Answer: $y = 7x - 6$

3 marks

2 (3 marks)

$$\frac{dy}{dx} = 2x - 4 = 2 \text{ at } x = 3, \text{ so the normal has gradient } -\frac{1}{2}$$

M1

$$y + 2 = -\frac{1}{2}(x - 3)$$

M1

$$2y + 4 = -x + 3 \text{ so } x + 2y + 1 = 0$$

A1

Answer: $x + 2y + 1 = 0$

3 marks

3 (2 marks)

$$\frac{dy}{dx} = 3x^2 - 2$$

M1

$$\text{At } x = 2: 10$$

A1

Answer: 10

2 marks

4 (3 marks)When $x = 0$, $y = 1$, so the point is $(0, 1)$

B1

 $\frac{dy}{dx} = 3x^2 - 4$, so the gradient at $x = 0$ is -4

M1

 $y - 1 = -4x$, so $y = -4x + 1$

A1

Answer: $y = -4x + 1$

3 marks

5 (3 marks) $\frac{dy}{dx} = 3x^2$, so the gradient of the tangent at $x = 1$ is 3

M1 A1

Gradient of the normal $= -\frac{1}{3}$

A1

Answer: $-\frac{1}{3}$

3 marks

6 (3 marks) $\frac{dy}{dx} = -2x$, so the gradient at $x = -1$ is 2; $y = 4$

M1 A1

 $y - 4 = 2(x + 1)$, so $y = 2x + 6$

A1

Answer: $y = 2x + 6$

3 marks

7 (5 marks) $\frac{dy}{dx} = 2x + 2$

M1

The tangent has gradient 6, so $2x + 2 = 6$ and $x = 2$

M1 A1

 $y = 4 + 4 + 5 = 13$, so the point of contact is $(2, 13)$

A1

 $13 = 12 + c$, so $c = 1$

A1

Answer: $c = 1$; point of contact $(2, 13)$

5 marks

8 (5 marks)**(a)** $y = 4x^{-1}$ so $\frac{dy}{dx} = -4x^{-2} = -1$ at $x = 2$

M1 A1

 $y - 2 = -1(x - 2)$ so $y = 4 - x$

A1

Answer: $y = 4 - x$

3 marks

(b) Gradient of the normal $= 1$

M1

 $y - 2 = x - 2$ so $y = x$

A1

Answer: $y = x$

2 marks

9 (4 marks) $\frac{dy}{dx} = x^{-1/2} = \frac{1}{\sqrt{x}}$, so the gradient at $x = 4$ is $\frac{1}{2}$; $y = 4$

M1 A1

 $y - 4 = \frac{1}{2}(x - 4)$

M1

 $x - 2y + 4 = 0$

A1

Answer: $x - 2y + 4 = 0$

4 marks

10 (5 marks) $\frac{dy}{dx} = 1 - \frac{1}{x^2}$, so the tangent gradient at $x = 2$ is $\frac{3}{4}$; $y = \frac{5}{2}$

M1 A1

Normal gradient $= -\frac{4}{3}$

M1

 $y - \frac{5}{2} = -\frac{4}{3}(x - 2)$

M1

 $8x + 6y - 31 = 0$

A1

Answer: $8x + 6y - 31 = 0$

5 marks

11 (4 marks)

$$\frac{dy}{dx} = 2x - 3, \text{ so the tangent gradient at A is 1}$$

M1 A1

$$\text{Normal gradient} = -1: y - 3 = -(x - 2), \text{ so } y = 5 - x$$

M1

$$y = 0 \text{ when } x = 5, \text{ so B is } (5, 0)$$

A1

Answer: B(5, 0)

4 marks

12 (5 marks)

$$\frac{dy}{dx} = 3x^2 - 12x - 6$$

M1 A1

$$\text{Parallel to } y = 9x \text{ means gradient 9: } 3x^2 - 12x - 6 = 9$$

M1

$$x^2 - 4x - 5 = 0, \text{ so } (x - 5)(x + 1) = 0, x = 5 \text{ or } x = -1$$

M1

$$y = -55 \text{ when } x = 5; y = -1 \text{ when } x = -1; \text{ points } (5, -55) \text{ and } (-1, -1)$$

A1

Answer: (5, -55) and (-1, -1)

5 marks

13 (4 marks)

$$\frac{dy}{dx} = 3x^2, \text{ so the gradient at } (1, 1) \text{ is } 3$$

M1 A1

$$\text{Tangent: } y - 1 = 3(x - 1), \text{ so } y = 3x - 2$$

M1

$$\text{When } x = 0, y = -2, \text{ so the tangent passes through } (0, -2)$$

A1

Answer: Shown

4 marks

14 (6 marks)

$$\text{(a) } \frac{dy}{dx} = 3x^2 - 6x = 3 + 6 = 9 \text{ at } x = -1$$

M1 A1

$$y + 2 = 9(x + 1) \text{ so } y = 9x + 7$$

A1

Answer: y = 9x + 7

3 marks

$$\text{(b) x-axis: } 9x + 7 = 0 \text{ so } x = -\frac{7}{9}; \text{ y-axis: } y = 7$$

M1 A1

$$\text{Area} = \frac{1}{2} \times \frac{7}{9} \times 7 = \frac{49}{18}$$

A1

Answer: $\frac{49}{18}$

3 marks

15 (6 marks)

$$\text{(a) } \frac{dy}{dx} = 3x^2 - 6x - 9; \text{ the line has gradient } -9$$

M1 A1

$$3x^2 - 6x - 9 = -9 \text{ so } 3x(x - 2) = 0, x = 0 \text{ or } x = 2$$

M1

$$\text{Points } (0, 4) \text{ and } (2, -18)$$

A1

Answer: (0, 4) and (2, -18)

4 marks

$$\text{(b) } y - 4 = -9x \text{ gives } y = -9x + 4$$

$$y + 18 = -9(x - 2) \text{ gives } y = -9x$$

M1 A1

Answer: y = -9x + 4 and y = -9x

2 marks

16 (5 marks)

$$\frac{dy}{dx} = 2ax + b$$

M1

$$\text{Through } (2, 6): 4a + 2b = 6$$

M1

$$\text{Gradient 5 at } x = 2: 4a + b = 5$$

M1 A1

$$\text{Subtracting: } b = 1, \text{ so } a = 1$$

A1

Answer: a = 1, b = 1

5 marks

17 (6 marks)

$$3y + x = 7 \text{ has gradient } -\frac{1}{3}, \text{ so the tangent has gradient } 3$$

M1 A1

$$\frac{dy}{dx} = 4x - 5 = 3, \text{ so } x = 2$$

M1 A1

$$y = 8 - 10 = -2, \text{ so P is } (2, -2)$$

B1

$$\text{Tangent: } y + 2 = 3(x - 2), \text{ so } y = 3x - 8$$

A1

Answer: P(2, -2); y = 3x - 8

6 marks

18 (6 marks)

$$\frac{dy}{dx} = 6x - 2, \text{ so the tangent gradient at P is 4}$$

M1 A1

$$\text{Tangent: } y - 1 = 4(x - 1) \text{ meets } x = 0 \text{ at } y = -3, \text{ so A is } (0, -3)$$

M1

$$\text{Normal gradient } -\frac{1}{4}: y - 1 = -\frac{1}{4}(x - 1) \text{ meets } x = 0 \text{ at } y = \frac{5}{4}$$

M1 A1

$$AB = \frac{5}{4} + 3 = \frac{17}{4}$$

A1

$$\text{Answer: } \frac{17}{4}$$

6 marks

19 (4 marks)

$$\frac{dy}{dx} = 3e^x - 1$$

M1 A1

$$\text{At } x = 0: \text{ gradient} = 2 \text{ and } y = 3$$

B1

$$\text{Tangent: } y = 2x + 3$$

A1

$$\text{Answer: } y = 2x + 3$$

4 marks

20 (5 marks)

$$\frac{dy}{dx} = 2\cos x - \sin x$$

M1 A1

$$\text{At } x = \frac{\pi}{2}: \text{ gradient} = 0 - 1 = -1, \text{ and } y = 2 + 0 = 2$$

B1

$$y - 2 = -(x - \frac{\pi}{2})$$

M1

$$y = -x + 2 + \frac{\pi}{2}$$

A1

$$\text{Answer: } y = -x + 2 + \frac{\pi}{2}$$

5 marks

21 (8 marks)

$$\text{(a) } \frac{dy}{dx} = 2x - 5 = -3 \text{ at } x = 1$$

M1 A1

$$\text{Normal gradient } \frac{1}{3}: y - 2 = \frac{1}{3}(x - 1)$$

M1

$$3y = x + 5, \text{ so } x - 3y + 5 = 0$$

A1

$$\text{Answer: } x - 3y + 5 = 0$$

4 marks

$$\text{(b) } 3(x^2 - 5x + 6) = x + 5$$

M1

$$3x^2 - 16x + 13 = 0$$

A1

$$(3x - 13)(x - 1) = 0 \text{ so } x = \frac{13}{3} \text{ (} x = 1 \text{ is A)}$$

M1

$$y = (\frac{13}{3} + 5) \div 3 = \frac{28}{9}, \text{ B}(\frac{13}{3}, \frac{28}{9})$$

A1

$$\text{Answer: } \text{B}(\frac{13}{3}, \frac{28}{9})$$

4 marks

22 (8 marks)

$$\text{(a) } \frac{dy}{dx} = 2x - 2, \text{ so the gradient at P is } 2p - 2$$

M1

$$\text{Tangent: } y - (p^2 - 2p + 5) = (2p - 2)(x - p)$$

M1 A1

$$\text{Through } (0, 0): -p^2 + 2p - 5 = -2p^2 + 2p$$

M1

$$\text{So } p^2 = 5$$

A1

$$\text{Answer: } p^2 = 5$$

5 marks

$$\text{(b) } p = \pm\sqrt{5}; \text{ gradient} = 2p - 2$$

M1

$$p = \sqrt{5}: y = (2\sqrt{5} - 2)x$$

A1

$$p = -\sqrt{5}: y = (-2\sqrt{5} - 2)x$$

A1

$$\text{Answer: } y = (2\sqrt{5} - 2)x \text{ and } y = -(2\sqrt{5} + 2)x$$

3 marks

23 (7 marks)

$$\frac{dy}{dx} = 2x - 2, \text{ so the tangent gradient at P is 4}$$

M1 A1

$$\text{Tangent: } y - 4 = 4(x - 3); y = 0 \text{ gives } x = 2, \text{ so A is } (2, 0)$$

M1 A1

$$\text{Normal gradient } -\frac{1}{4}: y - 4 = -\frac{1}{4}(x - 3); y = 0 \text{ gives } x = 19, \text{ so B is } (19, 0)$$

M1 A1

$$\text{Area} = \frac{1}{2} \times (19 - 2) \times 4 = 34$$

A1

Answer: 34

7 marks

24 (9 marks)

$$\text{(a) } \frac{dy}{dx} = -2x, \text{ so the gradient at } x = p \text{ is } -2p$$

M1

$$y - (4 - p^2) = -2p(x - p)$$

M1

$$y = -2px + 2p^2 + 4 - p^2 = -2px + p^2 + 4$$

A1

Answer: Shown

3 marks

$$\text{(b) Through } (0, 5): p^2 + 4 = 5, \text{ so } p = \pm 1$$

M1 A1

$$\text{Tangents: } y = -2x + 5 \text{ and } y = 2x + 5$$

A1

$$\text{They meet the x-axis at } x = \frac{5}{2} \text{ and } x = -\frac{5}{2}$$

M1

$$\text{Area} = \frac{1}{2} \times 5 \times 5 = \frac{25}{2}$$

M1 A1

$$\text{Answer: } y = \pm 2x + 5; \frac{25}{2}$$

6 marks

25 (7 marks)

$$\text{(a) } \frac{dy}{dx} = 3x^2 - 6, \text{ so the gradient at P is } -3$$

M1 A1

$$y + 4 = -3(x - 1), \text{ so } y = -3x - 1$$

A1

$$\text{Answer: } y = -3x - 1$$

3 marks

$$\text{(b) } x^3 - 6x + 1 = -3x - 1, \text{ so } x^3 - 3x + 2 = 0$$

M1

$$(x - 1) \text{ is a repeated factor (tangent at } x = 1): (x - 1)^2(x + 2) = 0$$

M1 A1

$$x = -2, y = -3(-2) - 1 = 5, \text{ so Q is } (-2, 5)$$

A1

$$\text{Answer: Q}(-2, 5)$$

4 marks

26 (7 marks)

$$\text{(a) } x^2 = -x^2 + 4x - 2, \text{ so } 2x^2 - 4x + 2 = 0$$

M1

$$2(x - 1)^2 = 0, \text{ so } x = 1 \text{ only}$$

A1

$$y = 1, \text{ so T is } (1, 1)$$

A1

$$\text{Answer: T}(1, 1)$$

3 marks

$$\text{(b) C: } \frac{dy}{dx} = 2x = 2 \text{ at T}$$

B1

$$\text{D: } \frac{dy}{dx} = -2x + 4 = 2 \text{ at T}$$

B1

The curves have the same gradient at T so they touch there

E1

$$\text{Common tangent: } y - 1 = 2(x - 1), \text{ so } y = 2x - 1$$

A1

$$\text{Answer: } y = 2x - 1$$

4 marks

27 (10 marks)

$$\text{(a) For C: } \frac{dy}{dx} = -x^{-2} = -1 \text{ at } x = 1$$

M1 A1

$$\text{For D: } \frac{dy}{dx} = 2x + a = 2 + a, \text{ so } 2 + a = -1, a = -3$$

M1 A1

$$\text{P on D: } 1 - 3 + b = 1 \text{ so } b = 3$$

A1

$$\text{Answer: } a = -3, b = 3$$

5 marks

$$\text{(b) Tangent: } y - 1 = -(x - 1), y = 2 - x. \text{ Then } x^2 - 3x + 3 = 2 - x$$

M1

$$x^2 - 2x + 1 = (x - 1)^2 = 0: \text{ repeated root } x = 1 \text{ only}$$

A1

$$\text{Answer: } (x - 1)^2 = 0$$

2 marks

(c) Normal: gradient 1 through (1, 1) is $y = x$

B1

C: $\frac{1}{x} = x$ so $x = -1$, Q(-1, -1)

M1

D: $x^2 - 3x + 3 = x$ so $(x - 1)(x - 3) = 0$, R(3, 3)

A1

Answer: Q(-1, -1), R(3, 3)

3 marks

28 (9 marks)

(a) $\frac{dy}{dx} = 2x = 2p$ at P, so the normal has gradient $-\frac{1}{2p}$

M1 A1

Normal: $y - p^2 = -\frac{1}{2p}(x - p)$

M1

At $x = 0$: $y = p^2 + \frac{p}{2p} = p^2 + \frac{1}{2}$

A1

Answer: $(0, p^2 + \frac{1}{2})$

4 marks

(b) $p^2 + \frac{1}{2} = 5$ so $p^2 = \frac{9}{2}$

M1

$p = \pm \frac{3}{\sqrt{2}} = \pm \frac{3\sqrt{2}}{2}$

A1

$y = \frac{9}{2}$: points $(\pm \frac{3\sqrt{2}}{2}, \frac{9}{2})$

A1

Answer: $(\pm \frac{3\sqrt{2}}{2}, \frac{9}{2})$

3 marks

(c) The intercept is $p^2 + \frac{1}{2}$

E1

$p^2 \geq 0$ so the intercept is always at least $\frac{1}{2}$; the case $p = 0$ gives the y-axis

E1

Answer: intercept $\geq \frac{1}{2}$

2 marks

29 (10 marks)

(a) $\frac{dy}{dx} = -\frac{2}{x^2}$, gradient at P is $-\frac{2}{p^2}$

M1 A1

$y - \frac{2}{p} = -\frac{2}{p^2}(x - p)$

M1

$y = -\frac{2}{p^2}x + \frac{2}{p} + \frac{2}{p} = -\frac{2}{p^2}x + \frac{4}{p}$

A1

Answer: Shown

4 marks

(b) $y = 0$: $\frac{2}{p^2}x = \frac{4}{p}$, so $x = 2p$; A is $(2p, 0)$

M1 A1

$x = 0$: $y = \frac{4}{p}$; B is $(0, \frac{4}{p})$

A1

Answer: A(2p, 0), B(0, $\frac{4}{p}$)

3 marks

(c) Area = $\frac{1}{2} \times 2p \times \frac{4}{p}$

M1 A1

= 4, independent of p

A1

Answer: 4

3 marks

30 (10 marks)

(a) $\frac{dy}{dx} = 3x^2 - 3 = 3(x - 1)(x + 1)$

M1

This is zero when $x = \pm 1$

A1

Answer: Shown

2 marks

(b) The x-axis touches C at a point where the gradient is 0, so $y = 0$ at $x = 1$ or $x = -1$

M1

$x = 1$: $1 - 3 + k = 0$ gives $k = 2$

A1

$x = -1$: $-1 + 3 + k = 0$ gives $k = -2$, rejected since $k > 0$

M1 A1

Answer: $k = 2$

4 marks

(c) $y = x^3 - 3x + 2 = (x - 1)(x^2 + x - 2)$

M1

$$= (x - 1)^2(x + 2)$$

A1

Meets the x-axis at (1, 0) and (-2, 0)

A1

Meets the y-axis ($x = 0$) at (0, 2)

A1

Answer: (1, 0), (-2, 0), (0, 2)

4 marks

Stationary Points and Optimisation

1 (3 marks)

$$\frac{dy}{dx} = 2x - 6 = 0$$

M1

$$x = 3$$

A1

$$y = 9 - 18 + 5 = -4, \text{ so the point is } (3, -4)$$

A1

Answer: (3, -4)

3 marks

2 (3 marks)

$$\frac{dy}{dx} = 3x^2 - 12 = 0, \text{ so } x = \pm 2$$

M1 A1

$$x = 2: y = -16; x = -2: y = 16. \text{ Points } (2, -16) \text{ and } (-2, 16)$$

A1

Answer: (2, -16) and (-2, 16)

3 marks

3 (3 marks)

$$\frac{dy}{dx} = 3x^2 - 27$$

M1 A1

$$3x^2 = 27, \text{ so } x = \pm 3$$

A1

Answer: $x = \pm 3$

3 marks

4 (4 marks)

$$\frac{dy}{dx} = 4 - 2x = 0, \text{ so } x = 2$$

M1 A1

$$y = 5 + 8 - 4 = 9, \text{ so the point is } (2, 9)$$

A1

$$\frac{d^2y}{dx^2} = -2 < 0, \text{ so a maximum}$$

B1

Answer: (2, 9), maximum

4 marks

5 (3 marks)

$$\frac{dy}{dx} = 3x^2 - 12x, \frac{d^2y}{dx^2} = 6x - 12$$

M1 A1

$$\text{At } x = 4, \frac{d^2y}{dx^2} = 12 > 0, \text{ so it is a minimum}$$

A1

Answer: Minimum

3 marks

6 (3 marks)

$$\frac{dy}{dx} = 4x - 8 = 0, \text{ so } x = 2$$

M1 A1

$$y = 8 - 16 = -8$$

A1

Answer: -8

3 marks

7 (5 marks)

$$f'(x) = 6x^2 - 18x + 12 = 6(x - 1)(x - 2) = 0$$

M1 M1

$$x = 1, f(1) = 6; x = 2, f(2) = 5. \text{ Points } (1, 6) \text{ and } (2, 5)$$

A1

$$f''(x) = 12x - 18$$

M1

$$f''(1) = -6 < 0 \text{ so } (1, 6) \text{ is a maximum; } f''(2) = 6 > 0 \text{ so } (2, 5) \text{ is a minimum}$$

A1

Answer: (1, 6) maximum, (2, 5) minimum

5 marks

8 (4 marks)

$$\frac{dy}{dx} = 1 - \frac{4}{x^2} = 0 \text{ so } x^2 = 4, x = 2 \text{ (as } x > 0)$$

M1 A1

$$y = 2 + 2 = 4, \text{ point } (2, 4)$$

A1

$$\frac{d^2y}{dx^2} = \frac{8}{x^3} = 1 > 0 \text{ at } x = 2, \text{ so a minimum}$$

A1

Answer: (2, 4)

4 marks

9 (6 marks)

$$\frac{dy}{dx} = 3x^2 - 6x - 9 = 3(x - 3)(x + 1)$$

M1 A1

Stationary when $x = 3$ or $x = -1$; $y = -26$ and $y = 6$

M1 A1

$$\frac{d^2y}{dx^2} = 6x - 6: \text{ at } x = -1 \text{ it is } -12 < 0, \text{ a maximum; at } x = 3 \text{ it is } 12 > 0, \text{ a minimum}$$

M1 A1

Answer: (-1, 6) maximum; (3, -26) minimum

6 marks

10 (5 marks)

$$\frac{dP}{dx} = 40 - 0.4x = 0$$

M1 A1

$$x = 100$$

A1

$$\frac{d^2P}{dx^2} = -0.4 < 0, \text{ so a maximum}$$

B1

$$P = 4000 - 2000 - 300 = 1700, \text{ so the maximum profit is } \pounds 1700$$

A1

Answer: 100 items; maximum profit £1700

5 marks

11 (4 marks)

$$\frac{dy}{dx} = 4x^3 - 16x = 4x(x^2 - 4)$$

M1 A1

$$x = 0, 2 \text{ or } -2$$

A1

$$y = 0, -16 \text{ and } -16: \text{ points } (0, 0), (2, -16), (-2, -16)$$

A1

Answer: (0, 0), (2, -16), (-2, -16)

4 marks

12 (5 marks)Other side = $10 - x$ (half the perimeter minus x)

B1

$$A = x(10 - x) = 10x - x^2$$

M1

$$\frac{dA}{dx} = 10 - 2x = 0, \text{ so } x = 5$$

M1 A1

$$A = 50 - 25 = 25 \text{ cm}^2$$

A1

Answer: 25 cm²

5 marks

13 (4 marks)

$$\frac{dh}{dt} = 20 - 10t = 0, \text{ so } t = 2$$

M1 A1

$$h = 2 + 40 - 20 = 22$$

M1 A1

Answer: 22 m at $t = 2$ s

4 marks

14 (6 marks)**(a)** Side parallel to the wall = $80 - 2x$

M1

$$A = x(80 - 2x) = 80x - 2x^2$$

A1

Answer: $A = 80x - 2x^2$

2 marks

$$\textbf{(b)} \frac{dA}{dx} = 80 - 4x = 0 \text{ so } x = 20$$

M1 A1

$$\frac{d^2A}{dx^2} = -4 < 0 \text{ so a maximum}$$

B1

$$A = 80 \times 20 - 2 \times 400 = 800 \text{ m}^2$$

A1

Answer: $x = 20$, $A = 800 \text{ m}^2$

4 marks

15 (7 marks)**(a)** Base is a square of side $24 - 2x$ and the height is x M1

$$V = x(24 - 2x)^2 = x(576 - 96x + 4x^2) = 4x^3 - 96x^2 + 576x$$
A1

Answer: shown 2 marks

$$(b) \frac{dV}{dx} = 12x^2 - 192x + 576 = 12(x - 4)(x - 12)$$
M1 A1

$$0 < x < 12 \text{ so } x = 4$$
A1

$$\frac{d^2V}{dx^2} = 24x - 192 = -96 < 0 \text{ at } x = 4, \text{ a maximum}$$
B1

$$V = 4 \times 16^2 = 1024 \text{ cm}^3$$
A1

Answer: $x = 4$, $V = 1024 \text{ cm}^3$ 5 marks**16 (6 marks)**

$$y = 24 - x, \text{ so } P = 24x^2 - x^3$$
M1 A1

$$\frac{dP}{dx} = 48x - 3x^2 = 3x(16 - x) = 0, \text{ so } x = 16 \text{ (} x > 0 \text{)}$$
M1 A1

$$\frac{d^2P}{dx^2} = 48 - 6x = -48 < 0 \text{ when } x = 16, \text{ so a maximum}$$
M1

$$x = 16 \text{ and } y = 8 \text{ (} P = 2048 \text{)}$$
A1

Answer: $x = 16$, $y = 8$ 6 marks**17 (10 marks)**

$$(a) \text{ Area} = \frac{1}{2}r^2\theta = 50, \text{ so } \theta = \frac{100}{r^2}$$
M1

$$\text{Arc length} = r\theta = \frac{100}{r}$$
M1

$$P = 2r + \frac{100}{r}$$
A1

Answer: Shown 3 marks

$$(b) \frac{dP}{dr} = 2 - \frac{100}{r^2} = 0$$
M1 A1

$$r^2 = 50, \text{ so } r = 5\sqrt{2}$$
A1

$$\frac{d^2P}{dr^2} = \frac{200}{r^3} > 0 \text{ when } r > 0, \text{ so a minimum}$$
B1

$$P = 10\sqrt{2} + \frac{100}{5\sqrt{2}} = 10\sqrt{2} + 10\sqrt{2} = 20\sqrt{2}$$
M1 A1

Answer: $20\sqrt{2} \text{ cm}$ 6 marks

$$(c) \theta = \frac{100}{50} = 2$$
B1

Answer: $\theta = 2$ 1 mark**18 (10 marks)**

$$(a) \text{ Surface area: } 2\pi r^2 + 2\pi rh = 150\pi, \text{ so } h = \frac{75 - r^2}{r}$$
M1 A1

$$V = \pi r^2 h = \pi r(75 - r^2) = 75\pi r - \pi r^3$$
M1 A1

Answer: Shown 4 marks

$$(b) \frac{dV}{dr} = 75\pi - 3\pi r^2 = 0$$
M1 A1

$$r^2 = 25, \text{ so } r = 5$$
A1

$$\frac{d^2V}{dr^2} = -6\pi r = -30\pi < 0, \text{ so a maximum}$$
B1

$$V = 375\pi - 125\pi = 250\pi$$
M1 A1

Answer: $r = 5$; maximum volume $250\pi \text{ cm}^3$ 6 marks

19 (7 marks)

$$\frac{dy}{dx} = 2ax - \frac{b}{x^2} \quad \text{M1 A1}$$

$$\text{At } x = 2: 4a - \frac{b}{4} = 0, \text{ so } b = 16a \quad \text{M1}$$

$$y = 6 \text{ at } x = 2: 4a + \frac{b}{2} = 6, \text{ so } 4a + 8a = 6 \quad \text{M1}$$

$$a = \frac{1}{2}, b = 8 \quad \text{A1}$$

$$\frac{d^2y}{dx^2} = 2a + \frac{2b}{x^3} = 1 + \frac{16}{x^3} = 3 > 0 \text{ at } x = 2 \quad \text{M1}$$

A minimum A1

Answer: $a = \frac{1}{2}, b = 8$; minimum 7 marks

20 (5 marks)

$$\frac{dy}{dx} = 3x^2 - 6x + 3 = 3(x - 1)^2 \quad \text{M1 A1}$$

This is zero only when $x = 1$, so there is one stationary point (1, 1) A1

$$\text{Either side of } x = 1, \frac{dy}{dx} = 3(x - 1)^2 > 0 \quad \text{M1}$$

The gradient is positive on both sides, so it is neither a maximum nor a minimum: a point of inflection E1

Answer: Shown 5 marks

21 (9 marks)

$$\text{(a) } \pi r^2 h = 500 \text{ so } h = \frac{500}{\pi r^2} \quad \text{M1}$$

$$S = 2\pi r^2 + 2\pi r h \quad \text{M1}$$

$$= 2\pi r^2 + \frac{1000}{r} \quad \text{A1}$$

Answer: shown 3 marks

$$\text{(b) } \frac{dS}{dr} = 4\pi r - \frac{1000}{r^2} \quad \text{M1 A1}$$

$$= 0 \text{ so } r^3 = \frac{250}{\pi} \quad \text{M1}$$

$$r = 4.30 \text{ cm} \quad \text{A1}$$

Answer: $r = 4.30$ 4 marks

$$\text{(c) } \frac{d^2S}{dr^2} = 4\pi + \frac{2000}{r^3} > 0 \text{ for } r > 0, \text{ so a minimum} \quad \text{B1}$$

$$S = 2\pi(4.30)^2 + \frac{1000}{4.30} = 349 \text{ cm}^2 \quad \text{A1}$$

Answer: 349 cm^2 2 marks

22 (8 marks)

$$\text{(a) } \frac{dy}{dx} = 4x^3 - 12x^2 = 4x^2(x - 3) \quad \text{M1 A1}$$

$$x = 0 \text{ or } x = 3 \quad \text{A1}$$

Points (0, 0) and (3, -27) A1

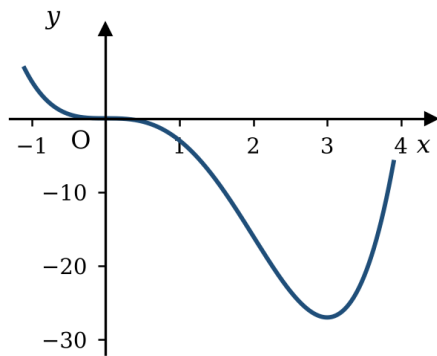
Answer: (0, 0) and (3, -27) 4 marks

$$\text{(b) } \frac{d^2y}{dx^2} = 12x^2 - 24x; \text{ at } x = 3 \text{ it is } 36 > 0, \text{ so } (3, -27) \text{ is a minimum} \quad \text{M1 A1}$$

$$\text{At } x = 0, \frac{d^2y}{dx^2} = 0 \text{ so the test is inconclusive} \quad \text{B1}$$

$$\frac{dy}{dx} = -16 \text{ at } x = -1 \text{ and } -8 \text{ at } x = 1: \text{ negative on both sides, so } (0, 0) \text{ is a point of inflection} \quad \text{A1}$$

Answer: (3, -27) minimum; (0, 0) point of inflection 4 marks

**23 (10 marks)**

(a) Perimeter: $2x + 2h + \pi x = 6$, so $h = 3 - x - \frac{\pi}{2}x$

M1 A1

$$A = 2xh + \frac{1}{2}\pi x^2$$

M1

$$= 6x - 2x^2 - \pi x^2 + \frac{1}{2}\pi x^2 = 6x - 2x^2 - \frac{\pi}{2}x^2$$

A1

Answer: Shown

4 marks

(b) $\frac{dA}{dx} = 6 - 4x - \pi x = 0$, so $x = \frac{6}{4 + \pi}$

M1 A1

$$\frac{d^2A}{dx^2} = -4 - \pi < 0, \text{ a maximum}$$

E1

$$A = 6x - (2 + \frac{\pi}{2})x^2 = \frac{36}{4 + \pi} - \frac{(4 + \pi)}{2} \times \frac{36}{(4 + \pi)^2}$$

M1 A1

$$= \frac{36}{4 + \pi} - \frac{18}{4 + \pi} = \frac{18}{4 + \pi}$$

A1

Answer: $x = \frac{6}{4 + \pi}$; maximum area $\frac{18}{4 + \pi} \text{ m}^2$

6 marks

24 (9 marks)

(a) $2x^2h = 108$, so $h = \frac{54}{x^2}$

M1

Top and base: $2 \times 2x^2 \times 30 = 120x^2$

M1

Sides: $2(2x \times h) + 2(x \times h) = 6xh = \frac{324}{x}$, so cost $20 \times \frac{324}{x} = \frac{6480}{x}$

M1

$$C = 120x^2 + \frac{6480}{x}$$

A1

Answer: Shown

4 marks

(b) $\frac{dC}{dx} = 240x - \frac{6480}{x^2} = 0$, so $x^3 = 27$, $x = 3$

M1 A1

$$\frac{d^2C}{dx^2} = 240 + \frac{12960}{x^3} > 0, \text{ a minimum}$$

E1

$$C = 120 \times 9 + \frac{6480}{3} = 1080 + 2160 = \text{£}3240$$

M1 A1

Answer: £3240

5 marks

25 (9 marks)

(a) Velocity $v = \frac{ds}{dt} = 6t^2 - 30t + 36 = 6(t - 2)(t - 3)$

M1 A1

$v = 0$ when $t = 2$ and $t = 3$

A1

$s(2) = 16 - 60 + 72 = 28 \text{ m}$; $s(3) = 54 - 135 + 108 = 27 \text{ m}$

M1 A1

Answer: $t = 2$ ($s = 28 \text{ m}$), $t = 3$ ($s = 27 \text{ m}$)

5 marks

(b) $\frac{dv}{dt} = 12t - 30 = 0$, so $t = \frac{5}{2}$

M1 A1

$$v = 6(\frac{5}{2} - 2)(\frac{5}{2} - 3) = 6 \times \frac{1}{2} \times (-\frac{1}{2})$$

M1

$$= -\frac{3}{2} \text{ m s}^{-1}$$

A1

Answer: $-\frac{3}{2} \text{ m s}^{-1}$

4 marks

26 (7 marks)

Let P be $(x, \sqrt{x})$. $AP^2 = (x - 4)^2 + x = x^2 - 7x + 16$

M1 A1

$\frac{d(AP^2)}{dx} = 2x - 7 = 0$, so $x = \frac{7}{2}$

M1 A1

Second derivative $2 > 0$, so a minimum

E1

P is $(\frac{7}{2}, \sqrt{\frac{7}{2}})$; $AP^2 = \frac{1}{4} + \frac{7}{2} = \frac{15}{4}$

M1

$AP = \frac{\sqrt{15}}{2}$

A1

Answer: $P(\frac{7}{2}, \sqrt{\frac{7}{2}})$; $AP = \frac{\sqrt{15}}{2}$

7 marks

27 (10 marks)

(a) Profit per item $= 30 - x - 12 = 18 - x$; items sold $= 300 + 20x$

B1 B1

$P = (18 - x)(300 + 20x) = 5400 + 60x - 20x^2$

M1

Answer: Shown

3 marks

(b) $\frac{dP}{dx} = 60 - 40x = 0$, so $x = 1.5$

M1 A1

$\frac{d^2P}{dx^2} = -40 < 0$, so a maximum

B1

Selling price $= 30 - 1.5 = \text{£}28.50$

A1

$P = 5400 + 90 - 45 = \text{£}5445$

A1

Answer: £28.50, maximum profit £5445

5 marks

(c) $60x - 20x^2 \geq 0$, so $20x(3 - x) \geq 0$

M1

$0 \leq x \leq 3$

A1

Answer: $0 \leq x \leq 3$

2 marks

28 (11 marks)

(a) Square: side $\frac{x}{4}$, area $\frac{x^2}{16}$

B1

Circle: circumference $100 - x$ so $r = \frac{100 - x}{2\pi}$

M1

Area $= \pi r^2 = \frac{(100 - x)^2}{4\pi}$

A1

Answer: shown

3 marks

(b) $\frac{dA}{dx} = \frac{x}{8} - \frac{100 - x}{2\pi}$

M1 A1

$= 0$ so $\pi x = 4(100 - x)$, $x = \frac{400}{\pi + 4} = 56.0$

A1

Answer: $x = 56.0$

3 marks

(c) $\frac{d^2A}{dx^2} = \frac{1}{8} + \frac{1}{2\pi} > 0$, so a minimum

B1

$A = 350 \text{ cm}^2$

A1

Answer: 350 cm^2

2 marks

(d) Greatest value is at an end of the interval since the only stationary point is a minimum

E1

$x = 0$: $A = \frac{10000}{4\pi} = 796$ (3 s.f.); $x = 100$: $A = 625$

M1

Greatest area is 796 cm^2 , when the whole wire forms a circle

A1

Answer: 796 cm^2

3 marks

29 (9 marks)

$$(a) \frac{dy}{dx} = 3x^2 - 6x - 24 = 3(x - 4)(x + 2)$$

M1 A1

$$x = -2: y = -8 - 12 + 48 + 20 = 48$$

M1 A1

$$x = 4: y = 64 - 48 - 96 + 20 = -60$$

A1

$$\frac{d^2y}{dx^2} = 6x - 6: -18 < 0 \text{ at } x = -2 \text{ (maximum) and } 18 > 0 \text{ at } x = 4 \text{ (minimum)}$$

A1

Answer: (-2, 48) maximum; (4, -60) minimum

6 marks

(b) Three roots when the line $y = k$ cuts C three times

M1

This needs k between the minimum and the maximum values

M1

$$-60 < k < 48$$

A1

Answer: $-60 < k < 48$

3 marks

30 (10 marks)**(a)** The centre of the sphere is at the midpoint of the axis; Pythagoras on a radius to the rim: $R^2 =$

M1 A1

$$r^2 + \left(\frac{h}{2}\right)^2$$

$$V = \pi r^2 h = \pi \left(R^2 - \frac{h^2}{4}\right)h$$

A1

Answer: Shown

3 marks

$$(b) V = \pi R^2 h - \frac{\pi}{4} h^3, \text{ so } \frac{dV}{dh} = \pi R^2 - \frac{3\pi}{4} h^2$$

M1 A1

$$\frac{dV}{dh} = 0 \text{ gives } h^2 = \frac{4R^2}{3}, h = \frac{2R}{\sqrt{3}}$$

M1 A1

$$\frac{d^2V}{dh^2} = -\frac{3\pi}{2} h < 0, \text{ so a maximum}$$

E1

$$V = \pi \left(R^2 - \frac{R^2}{3}\right) \frac{2R}{\sqrt{3}} = \pi \times \frac{2R^2}{3} \times \frac{2R}{\sqrt{3}}$$

M1

$$= \frac{4\pi R^3}{3\sqrt{3}}$$

A1

$$\text{Answer: } \frac{4\pi R^3}{3\sqrt{3}}$$

7 marks

Product Rule**1 (3 marks)**

$$\text{Let } u = x^3, v = 2x - 5, \text{ so } u' = 3x^2, v' = 2$$

M1 A1

$$\frac{dy}{dx} = 3x^2(2x - 5) + 2x^3 = 8x^3 - 15x^2$$

A1

$$\text{Answer: } 8x^3 - 15x^2$$

3 marks

2 (3 marks)

$$u = x, v = e^x, u' = 1, v' = e^x$$

M1 A1

$$\frac{dy}{dx} = e^x + xe^x = (1 + x)e^x$$

A1

$$\text{Answer: } (1 + x)e^x$$

3 marks

3 (4 marks)

$$u = 3x + 1, v = x^2 - 2, u' = 3, v' = 2x$$

B1

$$\frac{dy}{dx} = 3(x^2 - 2) + (3x + 1)(2x)$$

M1 A1

$$= 9x^2 + 2x - 6 \text{ (check: } y = 3x^3 + x^2 - 6x - 2, \text{ which gives the same derivative)}$$

A1

$$\text{Answer: } 9x^2 + 2x - 6$$

4 marks

4 (3 marks)

$$u = x, v = \sin x, \text{ so } u' = 1, v' = \cos x$$

B1

$$\frac{d y}{d x} = \sin x + x \cos x$$

M1 A1

$$\text{Answer: } \sin x + x \cos x$$

3 marks

5 (3 marks)

$$u = x^2, v = \cos x, \text{ so } u' = 2x, v' = -\sin x$$

B1

$$\frac{d y}{d x} = 2x \cos x - x^2 \sin x$$

M1 A1

$$\text{Answer: } 2x \cos x - x^2 \sin x$$

3 marks

6 (3 marks)

$$u = x^{1/2}, v = x + 4, \text{ so } u' = \frac{1}{2}x^{-1/2}, v' = 1$$

B1

$$\frac{d y}{d x} = \frac{1}{2}x^{-1/2}(x + 4) + x^{1/2}$$

M1

$$= \frac{3}{2}x^{1/2} + 2x^{-1/2} = \frac{3x + 4}{2\sqrt{x}}$$

A1

$$\text{Answer: } \frac{3}{2}x^{1/2} + 2x^{-1/2}$$

3 marks

7 (4 marks)

$$(a) u = x^2, v = \sin x: \frac{d y}{d x} = 2x \sin x + x^2 \cos x$$

M1 A1

$$\text{Answer: } 2x \sin x + x^2 \cos x$$

2 marks

$$(b) 2\pi \sin \pi + \pi^2 \cos \pi$$

M1

$$= 0 - \pi^2 = -\pi^2$$

A1

$$\text{Answer: } -\pi^2$$

2 marks

8 (5 marks)

$$\text{Write } y = (2x - 1)x^{1/2}; u = 2x - 1, v = x^{1/2}$$

M1

$$u' = 2, v' = \frac{1}{2}x^{-1/2}$$

A1

$$\frac{d y}{d x} = 2x^{1/2} + (2x - 1) \times \frac{1}{2\sqrt{x}}$$

A1

$$\text{Common denominator: } \frac{4x + 2x - 1}{2\sqrt{x}}$$

M1

$$= \frac{6x - 1}{2\sqrt{x}}$$

A1

$$\text{Answer: shown}$$

5 marks

9 (3 marks)

$$u = e^x, v = \ln x, \text{ so } u' = e^x, v' = \frac{1}{x}$$

B1

$$\frac{d y}{d x} = e^x \ln x + \frac{e^x}{x}$$

M1 A1

$$\text{Answer: } e^x \ln x + \frac{e^x}{x}$$

3 marks

10 (5 marks)

$$u = x^2 + 3, v = x^3 - x: u' = 2x, v' = 3x^2 - 1$$

B1

$$\frac{d y}{d x} = 2x(x^3 - x) + (x^2 + 3)(3x^2 - 1)$$

M1 A1

$$= 5x^4 + 6x^2 - 3$$

A1

$$\text{Check: } y = x^5 + 2x^3 - 3x, \text{ so } \frac{d y}{d x} = 5x^4 + 6x^2 - 3$$

B1

$$\text{Answer: } 5x^4 + 6x^2 - 3$$

5 marks

11 (4 marks)

$$u = x, v = \cos x: \frac{d y}{d x} = \cos x - x \sin x$$

M1 A1

$$\text{At } x = \pi: \cos \pi - \pi \sin \pi$$

M1

$$= -1 - 0 = -1$$

A1

Answer: -1

4 marks

12 (4 marks)

$$u = x^{1/2}, v = \ln x: u' = \frac{1}{2}x^{-1/2}, v' = \frac{1}{x}$$

B1

$$\frac{d y}{d x} = \frac{\ln x}{2\sqrt{x}} + \sqrt{x} \times \frac{1}{x}$$

M1 A1

$$= \frac{\ln x}{2\sqrt{x}} + \frac{1}{\sqrt{x}} = \frac{2 + \ln x}{2\sqrt{x}}$$

A1

$$\text{Answer: } \frac{2 + \ln x}{2\sqrt{x}}$$

4 marks

13 (4 marks)

$$\frac{d y}{d x} = \tan x + x \sec^2 x$$

M1 A1

$$\text{At } x = \frac{\pi}{4}: \tan \frac{\pi}{4} = 1 \text{ and } \sec^2 \frac{\pi}{4} = 2$$

B1

$$\frac{d y}{d x} = 1 + \frac{\pi}{4} \times 2 = 1 + \frac{\pi}{2}$$

A1

$$\text{Answer: } 1 + \frac{\pi}{2}$$

4 marks

14 (6 marks)

$$\text{(a) } u = x^3, v = \ln x, u' = 3x^2, v' = \frac{1}{x}$$

M1 A1

$$\frac{d y}{d x} = 3x^2 \ln x + x^2 = x^2(3 \ln x + 1)$$

A1

$$\text{Answer: } x^2(3 \ln x + 1)$$

3 marks

$$\text{(b) } x^2(3 \ln x + 1) = 0 \text{ with } x > 0, \text{ so } \ln x = -\frac{1}{3}$$

M1

$$x = e^{-1/3}$$

A1

$$y = e^{-1} \times \left(-\frac{1}{3}\right) = -\frac{1}{3e}; \text{ point } (e^{-1/3}, -\frac{1}{3e})$$

A1

$$\text{Answer: } (e^{-1/3}, -\frac{1}{3e})$$

3 marks

15 (7 marks)

$$\text{(a) } u = x^2, v = e^x$$

M1

$$\frac{d y}{d x} = 2xe^x + x^2e^x$$

A1

$$= xe^x(2 + x)$$

A1

$$\text{Answer: } xe^x(x + 2)$$

3 marks

$$\text{(b) At } x = 1: y = e, \text{ gradient} = 3e$$

B1

$$y - e = 3e(x - 1), \text{ so } y = 3ex - 2e$$

A1

$$\text{Answer: } y = 3ex - 2e$$

2 marks

$$\text{(c) } 0 = 3ex - 2e$$

M1

$$x = \frac{2}{3}$$

A1

$$\text{Answer: } x = \frac{2}{3}$$

2 marks

16 (7 marks)

$$(a) u = x^2, v = e^{-x}: u' = 2x, v' = -e^{-x}$$

B1

$$\frac{d y}{d x} = 2xe^{-x} - x^2e^{-x}$$

M1 A1

$$= x(2 - x)e^{-x}$$

A1

$$\text{Answer: } x(2 - x)e^{-x}$$

4 marks

$$(b) \frac{d y}{d x} = 0: x = 0 \text{ or } x = 2, \text{ and } e^{-x} \neq 0; P \text{ is at } x = 2 \text{ (the origin is the other stationary point)}$$

M1 A1

$$a = 2, y = 4e^{-2}$$

A1

$$\text{Answer: } a = 2; 4e^{-2}$$

3 marks

17 (6 marks)

$$(a) u = 3x + 2, v = e^{-2x}, u' = 3, v' = -2e^{-2x}$$

M1

$$\frac{d y}{d x} = 3e^{-2x} - 2(3x + 2)e^{-2x}$$

M1 A1

$$= (3 - 6x - 4)e^{-2x} = -(6x + 1)e^{-2x}$$

A1

Answer: Shown

4 marks

$$(b) y \text{ is decreasing when } \frac{d y}{d x} < 0; e^{-2x} > 0 \text{ for all } x, \text{ so need } 6x + 1 > 0$$

M1

$$x > -\frac{1}{6}$$

A1

$$\text{Answer: } x > -\frac{1}{6}$$

2 marks

18 (5 marks)

$$u = x, v = e^{2x}: u' = 1, v' = 2e^{2x}$$

B1

$$\frac{d y}{d x} = e^{2x} + 2xe^{2x} = (1 + 2x)e^{2x}$$

M1 A1

$$\text{Stationary: } 1 + 2x = 0, \text{ so } x = -\frac{1}{2}$$

M1

$$y = -\frac{1}{2}e^{-1}, \text{ so the point is } \left(-\frac{1}{2}, -\frac{1}{2}e^{-1}\right)$$

A1

$$\text{Answer: } \left(-\frac{1}{2}, -\frac{1}{2}e^{-1}\right)$$

5 marks

19 (5 marks)

$$\frac{d y}{d x} = \sin x + x \cos x$$

M1 A1

$$\frac{d^2 y}{d x^2} = \cos x + \cos x - x \sin x$$

M1 A1

$$= 2\cos x - x \sin x, \text{ so } \frac{d^2 y}{d x^2} + y = 2\cos x - x \sin x + x \sin x = 2\cos x$$

A1

Answer: Shown

5 marks

20 (5 marks)

$$\frac{d y}{d x} = e^{2x} + 2xe^{2x} = (1 + 2x)e^{2x}$$

M1 A1

$$\frac{d^2 y}{d x^2} = 2e^{2x} + 2(1 + 2x)e^{2x} = (4 + 4x)e^{2x}$$

M1 A1

$$\frac{d^2 y}{d x^2} - 4\frac{d y}{d x} + 4y = e^{2x}[(4 + 4x) - 4(1 + 2x) + 4x] = 0$$

A1

Answer: shown

5 marks

21 (8 marks)

$$(a) u = e^x, v = \sin x: u' = e^x, v' = \cos x$$

M1 A1

$$\frac{d y}{d x} = e^x \sin x + e^x \cos x = e^x(\sin x + \cos x)$$

A1

Answer: shown

3 marks

$$(b) \text{Product rule again with } u = e^x, v = \sin x + \cos x$$

M1

$$\frac{d^2y}{dx^2} = e^x(\sin x + \cos x) + e^x(\cos x - \sin x)$$

A1

$$= 2e^x \cos x$$

A1

Answer: shown

3 marks

(c) $e^x \neq 0$, so $\sin x + \cos x = 0$, $\tan x = -1$

M1

$$x = \frac{3\pi}{4}, \frac{7\pi}{4}$$

A1

Answer: $x = \frac{3\pi}{4}, \frac{7\pi}{4}$

2 marks

22 (8 marks)

(a) $\frac{dy}{dx} = e^{-x} - xe^{-x} = (1 - x)e^{-x}$

M1 A1

$$= 0 \text{ so } x = 1$$

A1

$$y = e^{-1}, P(1, \frac{1}{e})$$

A1

Answer: $P(1, \frac{1}{e})$

4 marks

(b) $\frac{d^2y}{dx^2} = -e^{-x} - (1 - x)e^{-x} = (x - 2)e^{-x}$

M1 A1

This is 0 at $x = 2$ and changes sign from negative to positive

A1

Answer: shown

3 marks

(c) $y = 2e^{-2}$

B1

Answer: $2e^{-2}$

1 mark

23 (5 marks)

Take $u = xe^x$, $v = \sin x$: $u' = e^x + xe^x$ (product rule again), $v' = \cos x$

M1 A1

$$\frac{dy}{dx} = (e^x + xe^x) \sin x + xe^x \cos x$$

M1 A1

$$= e^x(x \sin x + \sin x + x \cos x), \text{ so } a = 1, b = 1, c = 1$$

A1

Answer: $a = 1, b = 1, c = 1$

5 marks

24 (7 marks)

$$\frac{dy}{dx} = 2x \ln x + x^2 \times \frac{1}{x} = 2x \ln x + x = x(2 \ln x + 1)$$

M1 A1

$$\text{Stationary: } 2 \ln x + 1 = 0, \text{ so } \ln x = -\frac{1}{2}, x = e^{-1/2}$$

M1 A1

$$y = e^{-1} \times (-\frac{1}{2}) = -\frac{1}{2e}$$

A1

$$\frac{d^2y}{dx^2} = 2 \ln x + 3 \text{ (product rule on } 2x \ln x + x), \text{ which equals } 2 > 0 \text{ at } x = e^{-1/2}$$

M1

$$\text{So a minimum at } (e^{-1/2}, -\frac{1}{2e})$$

A1

Answer: $(e^{-1/2}, -\frac{1}{2e})$, minimum

7 marks

25 (8 marks)

(a) $u = 100t$, $v = e^{-t/2}$: $u' = 100$, $v' = -\frac{1}{2}e^{-t/2}$

B1

$$\frac{dB}{dt} = 100e^{-t/2} - 50te^{-t/2}$$

M1 A1

$$= 50(2 - t)e^{-t/2}$$

A1

Answer: Shown

4 marks

(b) $\frac{dB}{dt} = 0$ gives $t = 2$

M1 A1

Sign of $\frac{dB}{dt}$ changes from positive to negative, so a maximum

E1

$$B = 200e^{-1} \approx 73.6 \text{ thousand, i.e. about 73 600 bacteria}$$

A1

Answer: $t = 2$ h; $B = 200e^{-1} \approx 73\,600$

4 marks

26 (6 marks)

$$u = ax + b, v = e^{-x}: u' = a, v' = -e^{-x}$$

B1

$$\frac{dy}{dx} = ae^{-x} - (ax + b)e^{-x} = (a - ax - b)e^{-x}$$

M1 A1

$$\text{Stationary at } x = 2: a - 2a - b = 0, \text{ so } b = -a$$

M1

$$y = 3e^{-2} \text{ at } x = 2: (2a + b)e^{-2} = 3e^{-2}, \text{ so } 2a + b = 3$$

M1

$$a = 3, b = -3$$

A1

Answer: $a = 3, b = -3$

6 marks

27 (9 marks)

$$(a) \frac{\delta y}{\delta x} = \frac{u(x+h)v(x+h) - u(x)v(x)}{h}$$

M1

$$= u(x+h) \frac{v(x+h) - v(x)}{h} + v(x) \frac{u(x+h) - u(x)}{h}$$

M1 A1

$$\text{As } h \rightarrow 0, \frac{v(x+h) - v(x)}{h} \rightarrow \frac{dv}{dx} \text{ and } \frac{u(x+h) - u(x)}{h} \rightarrow \frac{du}{dx}$$

B1

$$\text{and } u(x+h) \rightarrow u(x) \text{ (} u \text{ is continuous), so } \frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$$

A1

Answer: shown

5 marks

$$(b) u = x^3, v = \sin 2x, \text{ so } u' = 3x^2 \text{ and } v' = 2\cos 2x$$

M1 A1

$$\frac{dy}{dx} = x^3 \times 2\cos 2x + \sin 2x \times 3x^2$$

M1

$$= x^2(2x \cos 2x + 3 \sin 2x)$$

A1

Answer: $x^2(2x \cos 2x + 3 \sin 2x)$

4 marks

28 (10 marks)

$$(a) u = 20t, v = e^{-0.5t}, v' = -0.5e^{-0.5t}$$

M1 A1

$$\frac{dC}{dt} = 20e^{-0.5t} - 10te^{-0.5t} = 20e^{-0.5t}(1 - 0.5t)$$

A1

Answer: $20e^{-0.5t}(1 - 0.5t)$

3 marks

$$(b) \frac{dC}{dt} = 0 \text{ so } t = 2$$

M1 A1

$$C = 40e^{-1} = 14.7 \text{ mg per litre}$$

A1

Answer: $t = 2$ hours, 14.7 mg per litre

3 marks

$$(c) \frac{d^2C}{dt^2} = -10e^{-0.5t} - 10e^{-0.5t}(1 - 0.5t) = 5(t - 4)e^{-0.5t}$$

M1

$$\text{At } t = 2 \text{ this is } -10e^{-1} < 0, \text{ so a maximum}$$

A1

Answer: shown

2 marks

$$(d) \frac{dC}{dt} = 20e^{-3}(1 - 3) = -40e^{-3} = -1.99$$

M1 A1

Answer: -1.99 mg per litre per hour (concentration falling)

2 marks

29 (11 marks)

$$(a) \text{Product rule: } u = x^n, v = e^{-x}, u' = nx^{n-1}, v' = -e^{-x}$$

B1

$$f'(x) = nx^{n-1}e^{-x} - x^n e^{-x}$$

M1

$$= x^{n-1}(n - x)e^{-x}$$

A1

Answer: Shown

3 marks

$$(b) x^{n-1}e^{-x} > 0 \text{ for } x > 0, \text{ so } f'(x) \text{ has the sign of } (n - x)$$

M1

$$f'(x) > 0 \text{ for } x < n \text{ and } f'(x) < 0 \text{ for } x > n$$

A1

$$\text{So } f \text{ has a maximum at } x = n$$

A1

$$\text{Maximum value } f(n) = n^n e^{-n}$$

A1

Answer: $x = n; n^n e^{-n}$

4 marks

$$(c) \text{From (b) with } n = 3: x^3 e^{-x} \leq 27e^{-3}$$

M1 A1

$$\text{Multiply both sides by } e^x > 0: x^3 \leq 27e^{-3}e^x$$

M1

$$\text{So } e^x \geq \frac{e^3 x^3}{27} = \left(\frac{e}{3}\right)^3 x^3$$

A1

Answer: Shown**4 marks****30 (10 marks)**

$$(a) \frac{dy}{dx} = 2xe^x + x^2e^x = (x^2 + 2x)e^x$$

M1 A1

$$\frac{d^2y}{dx^2} = (2x + 2)e^x + (x^2 + 2x)e^x = (x^2 + 4x + 2)e^x$$

M1 A1

$$\text{Answer: } (x^2 + 2x)e^x; (x^2 + 4x + 2)e^x$$

4 marks

$$(b) \text{ Pattern: } \frac{d^ny}{dx^n} = (x^2 + 2nx + n(n-1))e^x$$

B1

$$\text{True for } n = 1: x^2 + 2x + 0$$

B1

$$\text{Assume true for } n = k. \text{ Differentiating: } (2x + 2k)e^x + (x^2 + 2kx + k(k-1))e^x$$

M1 A1

$$= (x^2 + 2(k+1)x + k(k-1) + 2k)e^x, \text{ and } k(k-1) + 2k = (k+1)k$$

M1

So true for $n = k + 1$; true for $n = 1$, hence true for all positive integers n by induction

A1

$$\text{Answer: } (x^2 + 2nx + n(n-1))e^x$$

6 marks**Quotient Rule****1 (3 marks)**

$$u = x, v = x + 1, u' = 1, v' = 1$$

M1 A1

$$\frac{dy}{dx} = \frac{(x+1) - x}{(x+1)^2} = \frac{1}{(x+1)^2}$$

A1

$$\text{Answer: } \frac{1}{(x+1)^2}$$

3 marks**2 (3 marks)**

$$u = e^x, v = x$$

M1 A1

$$\frac{dy}{dx} = \frac{xe^x - e^x}{x^2} = \frac{e^x(x-1)}{x^2}$$

A1

$$\text{Answer: } \frac{e^x(x-1)}{x^2}$$

3 marks**3 (3 marks)**

$$u = 2x + 1, v = x - 3, \text{ so } u' = 2, v' = 1$$

B1

$$\frac{dy}{dx} = \frac{u'v - uv'}{v^2} = \frac{2 \times (x-3) - (2x+1) \times 1}{(x-3)^2} = -\frac{7}{(x-3)^2}$$

M1 A1

$$\text{Answer: } -\frac{7}{(x-3)^2}$$

3 marks**4 (3 marks)**

$$u = \cos x, v = x, \text{ so } u' = -\sin x, v' = 1$$

B1

$$\frac{dy}{dx} = \frac{u'v - uv'}{v^2} = \frac{-\sin x \times x - \cos x \times 1}{(x)^2} = -\frac{x \sin x + \cos x}{x^2}$$

M1 A1

$$\text{Answer: } -\frac{x \sin x + \cos x}{x^2}$$

3 marks**5 (3 marks)**

$$u = x, v = e^x, \text{ so } u' = 1, v' = e^x$$

B1

$$\frac{dy}{dx} = \frac{u'v - uv'}{v^2} = \frac{1 \times e^x - x \times e^x}{(e^x)^2} = -(x-1)e^{-x}$$

M1 A1

$$\text{Answer: } -(x-1)e^{-x}$$

3 marks

6 (3 marks)

$$u = 1, v = x^2 + 1, \text{ so } u' = 0, v' = 2x$$

B1

$$\frac{dy}{dx} = \frac{0 \times (x^2 + 1) - 1 \times 2x}{(x^2 + 1)^2} = -\frac{2x}{(x^2 + 1)^2}$$

M1 A1

$$\text{Answer: } -\frac{2x}{(x^2 + 1)^2}$$

3 marks

7 (4 marks)

$$(a) u = \sin x, v = x: \frac{dy}{dx} = \frac{x \cos x - \sin x}{x^2}$$

M1 A1

$$\text{Answer: } \frac{x \cos x - \sin x}{x^2}$$

2 marks

$$(b) \frac{\pi \cos \pi - \sin \pi}{\pi^2} = \frac{-\pi}{\pi^2}$$

M1

$$= -\frac{1}{\pi}$$

A1

$$\text{Answer: } -\frac{1}{\pi}$$

2 marks

8 (4 marks)

$$u = 2x + 3, v = x - 1, u' = 2, v' = 1$$

M1

$$f'(x) = \frac{2(x - 1) - (2x + 3)}{(x - 1)^2}$$

A1

$$= -\frac{5}{(x - 1)^2}$$

A1

$$(x - 1)^2 > 0 \text{ for } x \neq 1, \text{ so } f'(x) < 0 \text{ and } f \text{ is decreasing}$$

E1

$$\text{Answer: } f'(x) = -\frac{5}{(x - 1)^2} < 0$$

4 marks

9 (4 marks)

$$u = \ln x, v = x + 1, \text{ so } u' = \frac{1}{x}, v' = 1$$

B1

$$\frac{dy}{dx} = \frac{\frac{1}{x}(x + 1) - \ln x}{(x + 1)^2}$$

M1 A1

$$= \frac{x + 1 - x \ln x}{x(x + 1)^2}, \text{ multiplying top and bottom by } x$$

A1

$$\text{Answer: } \frac{x + 1 - x \ln x}{x(x + 1)^2}$$

4 marks

10 (6 marks)

$$u = x^2 - 3x + 6, v = x - 1, \text{ so } u' = 2x - 3 \text{ and } v' = 1$$

M1

$$f'(x) = \frac{(2x - 3)(x - 1) - (x^2 - 3x + 6)}{(x - 1)^2}$$

A1

$$= \frac{x^2 - 2x - 3}{(x - 1)^2}$$

A1

$$\frac{x^2 - 2x - 3}{(x - 1)^2} = \frac{3}{4}, \text{ so } 4(x^2 - 2x - 3) = 3(x - 1)^2$$

M1

$$x^2 - 2x - 15 = 0, \text{ so } (x - 5)(x + 3) = 0$$

A1

$$x = 5 \text{ or } x = -3$$

A1

$$\text{Answer: } x = 5 \text{ or } x = -3$$

6 marks

11 (4 marks)

$$u = e^x, v = x^2 + 1, \text{ so } u' = e^x, v' = 2x$$

B1

$$\frac{dy}{dx} = \frac{e^x(x^2 + 1) - 2xe^x}{(x^2 + 1)^2}$$

M1 A1

$$= \frac{e^x(x^2 - 2x + 1)}{(x^2 + 1)^2} = \frac{e^x(x - 1)^2}{(x^2 + 1)^2}$$

A1

Answer: Shown

4 marks

12 (5 marks)

$$u = \sin x, v = 1 + \cos x, \text{ so } u' = \cos x, v' = -\sin x$$

B1

$$\frac{dy}{dx} = \frac{\cos x(1 + \cos x) + \sin^2 x}{(1 + \cos x)^2}$$

M1 A1

$$\text{Numerator} = \cos x + \cos^2 x + \sin^2 x$$

M1

$$= \cos x + 1, \text{ so } \frac{dy}{dx} = \frac{1 + \cos x}{(1 + \cos x)^2} = \frac{1}{1 + \cos x}$$

A1

Answer: Shown

5 marks

13 (5 marks)

$$u = x^{1/2}, v = x + 2, \text{ so } u' = \frac{1}{2}x^{-1/2}, v' = 1$$

B1

$$\frac{dy}{dx} = \frac{\frac{1}{2\sqrt{x}}(x + 2) - \sqrt{x}}{(x + 2)^2}$$

M1 A1

$$\text{Multiply top and bottom by } 2\sqrt{x}: \text{ numerator } (x + 2) - 2x = 2 - x$$

M1

$$\frac{dy}{dx} = \frac{2 - x}{2\sqrt{x}(x + 2)^2}$$

A1

$$\text{Answer: } \frac{2 - x}{2\sqrt{x}(x + 2)^2}$$

5 marks

14 (6 marks)

$$(a) u = \ln x, v = x^2, u' = \frac{1}{x}, v' = 2x$$

M1 A1

$$\frac{dy}{dx} = \frac{x - 2x \ln x}{x^4} = \frac{1 - 2 \ln x}{x^3}$$

A1

Answer: shown

3 marks

$$(b) 1 - 2 \ln x = 0 \text{ so } \ln x = \frac{1}{2}$$

M1

$$x = e^{1/2} = \sqrt{e}$$

A1

$$y = \left(\frac{1}{2}\right) \div e = \frac{1}{2e}; \text{ point } \left(\sqrt{e}, \frac{1}{2e}\right)$$

A1

$$\text{Answer: } \left(\sqrt{e}, \frac{1}{2e}\right)$$

3 marks

15 (6 marks)

$$(a) u = \sin x, v = \cos x, u' = \cos x, v' = -\sin x$$

M1 A1

$$\frac{dy}{dx} = \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x}$$

A1

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

A1

Answer: $\sec^2 x$

4 marks

$$(b) \sec^2\left(\frac{\pi}{4}\right) = \frac{1}{\left(\frac{1}{\sqrt{2}}\right)^2}$$

M1

$$= 2$$

A1

Answer: 2

2 marks

16 (5 marks)

$$u = x^2 + 1, v = x - 1, \text{ so } u' = 2x, v' = 1$$

B1

$$\frac{d y}{d x} = \frac{2x(x-1) - (x^2+1)}{(x-1)^2} = \frac{x^2 - 2x - 1}{(x-1)^2}$$

M1 A1

$$\text{Stationary: } x^2 - 2x - 1 = 0$$

M1

$$x = \frac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}$$

A1

$$\text{Answer: } x = 1 \pm \sqrt{2}$$

5 marks

17 (5 marks)

$$\frac{d y}{d x} = \frac{e^x \times x - e^x \times 1}{x^2} = \frac{e^x(x-1)}{x^2}$$

M1 A1

$$\text{At } x = 2: \text{ gradient} = \frac{e^2}{4}, y = \frac{e^2}{2}$$

B1

$$\text{Tangent: } y - \frac{e^2}{2} = \frac{e^2}{4}(x - 2)$$

M1

$$y = \frac{e^2}{4}x - \frac{e^2}{2} + \frac{e^2}{2} = \frac{e^2}{4}x, \text{ which passes through } (0, 0)$$

A1

$$\text{Answer: Shown}$$

5 marks

18 (6 marks)

$$\frac{d y}{d x} = \frac{2x(x-a)^2 - x^2 \times 2(x-a)}{(x-a)^4}$$

M1 A1

$$= \frac{2x(x-a) - 2x^2}{(x-a)^3} = -\frac{2ax}{(x-a)^3}$$

M1 A1

$$\text{At } x = 2a: \frac{d y}{d x} = -\frac{4a^2}{a^3} = -\frac{4}{a}$$

M1

$$-\frac{4}{a} = -2, \text{ so } a = 2$$

A1

$$\text{Answer: } a = 2$$

6 marks

19 (5 marks)

$$\operatorname{cosec} x = \frac{1}{\sin x}: u = 1, v = \sin x, u' = 0, v' = \cos x$$

B1

$$\frac{d}{d x}(\operatorname{cosec} x) = \frac{0 \times \sin x - 1 \times \cos x}{\sin^2 x}$$

M1 A1

$$= -\frac{\cos x}{\sin^2 x}$$

A1

$$= -\frac{1}{\sin x} \times \frac{\cos x}{\sin x} = -\operatorname{cosec} x \cot x$$

A1

$$\text{Answer: Proved}$$

5 marks

20 (6 marks)

$$\text{Through } (0, 3): \frac{b}{1} = 3, \text{ so } b = 3$$

B1

$$u = ax + b, v = x + 1: \frac{d y}{d x} = \frac{a(x+1) - (ax+b)}{(x+1)^2}$$

M1 A1

$$= \frac{a-b}{(x+1)^2}$$

A1

$$\text{At } x = 0: a - b = -2, \text{ so } a - 3 = -2$$

M1

$$a = 1$$

A1

$$\text{Answer: } a = 1, b = 3$$

6 marks

21 (8 marks)

$$(a) u = 4x, v = x^2 + 1, u' = 4, v' = 2x$$

M1 A1

$$\frac{d y}{d x} = \frac{4(x^2+1) - 4x \cdot 2x}{(x^2+1)^2} = \frac{4-4x^2}{(x^2+1)^2}$$

A1

$$\text{Answer: shown}$$

3 marks

(b) $1 - x^2 = 0$ so $x = \pm 1$

M1

(1, 2) and (-1, -2)

A1

Answer: (1, 2) and (-1, -2)

2 marks

(c) Maximum value 2 at $x = 1$ and minimum value -2 at $x = -1$

B1

As $x \rightarrow \pm\infty$, $y \rightarrow 0$ and the curve is continuous, with no other turning points

E1

$-2 \leq y \leq 2$

A1

Answer: $-2 \leq y \leq 2$

3 marks

22 (8 marks)

(a) $u = x^2$, $v = x - 2$, $u' = 2x$, $v' = 1$

M1 A1

$$\frac{dy}{dx} = \frac{2x(x-2) - x^2}{(x-2)^2} = \frac{x^2 - 4x}{(x-2)^2}$$

A1

Answer: shown

3 marks

(b) $x(x-4) = 0$ so $x = 0$ or $x = 4$

M1

(0, 0) and (4, 8)

A1

Answer: (0, 0) and (4, 8)

2 marks

(c) The denominator $(x-2)^2$ is positive, so the sign of $\frac{dy}{dx}$ is the sign of $x(x-4)$

M1

$x < 0$: positive; $0 < x < 2$ and $2 < x < 4$: negative, so (0, 0) is a maximum

A1

$x > 4$: positive, so (4, 8) is a minimum

A1

Answer: (0, 0) maximum, (4, 8) minimum

3 marks

23 (8 marks)

(a) $u = \ln x$, $v = x$: $\frac{dy}{dx} = \frac{\frac{1}{x} \times x - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$

M1 A1

Stationary: $\ln x = 1$, so $x = e$, $y = \frac{1}{e}$

M1 A1

For $x < e$, $\ln x < 1$ so $\frac{dy}{dx} > 0$; for $x > e$, $\frac{dy}{dx} < 0$

M1

Gradient changes from positive to negative: a maximum at $(e, \frac{1}{e})$

A1

Answer: $(e, \frac{1}{e})$ maximum

6 marks

(b) The greatest value of $\frac{\ln x}{x}$ is $\frac{1}{e}$, so $\frac{\ln x}{x} \leq \frac{1}{e}$

M1

Multiply by $x > 0$: $\ln x \leq \frac{x}{e}$

A1

Answer: Shown

2 marks

24 (9 marks)

(a) $u = \sin x$, $v = 2 - \cos x$, $u' = \cos x$, $v' = \sin x$

M1

$$\frac{dy}{dx} = \frac{\cos x(2 - \cos x) - \sin x \times \sin x}{(2 - \cos x)^2}$$

M1 A1

$$= \frac{2\cos x - (\cos^2 x + \sin^2 x)}{(2 - \cos x)^2} = \frac{2\cos x - 1}{(2 - \cos x)^2}$$

A1

Answer: Shown

4 marks

(b) $\frac{dy}{dx} = 0$ when $\cos x = \frac{1}{2}$, so $x = \frac{\pi}{3}$ or $x = \frac{5\pi}{3}$

M1 A1

At $x = \frac{\pi}{3}$: $y = \frac{\sqrt{3}}{2} \div \frac{3}{2} = \frac{\sqrt{3}}{3}$; at $x = \frac{5\pi}{3}$: $y = -\frac{\sqrt{3}}{3}$

M1 A1

$\frac{dy}{dx} > 0$ for $x < \frac{\pi}{3}$, $\frac{dy}{dx} < 0$ between, so $(\frac{\pi}{3}, \frac{\sqrt{3}}{3})$ is the maximum and $(\frac{5\pi}{3}, -\frac{\sqrt{3}}{3})$ the minimum

A1

Answer: maximum $(\frac{\pi}{3}, \frac{\sqrt{3}}{3})$; minimum $(\frac{5\pi}{3}, -\frac{\sqrt{3}}{3})$

5 marks

25 (8 marks)

(a) $u = xe^x$, so $u' = e^x + xe^x = (x + 1)e^x$ (product rule); $v = x + 1$, $v' = 1$

M1 A1

$$\frac{dy}{dx} = \frac{(x+1)e^x(x+1) - xe^x}{(x+1)^2}$$

M1 A1

$$= \frac{e^x(x^2 + 2x + 1 - x)}{(x+1)^2} = \frac{e^x(x^2 + x + 1)}{(x+1)^2}$$

A1

Answer: Shown

5 marks

(b) $\frac{dy}{dx} = 0$ needs $x^2 + x + 1 = 0$ since $e^x > 0$

M1

Discriminant $1 - 4 = -3 < 0$, so no real roots

M1

So there are no stationary points

A1

Answer: Shown

3 marks

26 (10 marks)

(a) $u = x$, $v = x^2 + 1$: $\frac{dy}{dx} = \frac{1 \times (x^2 + 1) - x \times 2x}{(x^2 + 1)^2}$

M1 A1

$$= \frac{1 - x^2}{(x^2 + 1)^2}$$

A1

Answer: Shown

3 marks

(b) $u = 1 - x^2$, $v = (x^2 + 1)^2$: $u' = -2x$, $v' = 4x(x^2 + 1)$

B1

$$\frac{d^2y}{dx^2} = \frac{-2x(x^2 + 1)^2 - (1 - x^2) \times 4x(x^2 + 1)}{(x^2 + 1)^4}$$

M1 A1

$$\text{Cancel } (x^2 + 1): \frac{-2x(x^2 + 1) - 4x(1 - x^2)}{(x^2 + 1)^3}$$

M1

$$= \frac{2x^3 - 6x}{(x^2 + 1)^3} = \frac{2x(x^2 - 3)}{(x^2 + 1)^3}$$

A1

Answer: Shown

5 marks

(c) $2x(x^2 - 3) = 0$

M1

$$x = 0, x = \sqrt{3} \text{ or } x = -\sqrt{3}$$

A1

Answer: $x = 0, \pm\sqrt{3}$

2 marks

27 (10 marks)

(a) $u = 1$, $v = \cos x$, $u' = 0$, $v' = -\sin x$

M1 A1

$$\frac{dy}{dx} = \frac{0 - 1 \cdot (-\sin x)}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \sec x \tan x$$

A1

Answer: shown

3 marks

(b) $\cot x = \frac{\cos x}{\sin x}$: $u = \cos x$, $v = \sin x$

M1

$$\frac{dy}{dx} = \frac{-\sin x \cdot \sin x - \cos x \cdot \cos x}{\sin^2 x}$$

A1

$$= -\frac{1}{\sin^2 x} = -\operatorname{cosec}^2 x$$

A1

Answer: shown

3 marks

(c) $u = 1 + \sin x$, $v = \cos x$: $\frac{dy}{dx} = \frac{\cos x \cdot \cos x - (1 + \sin x)(-\sin x)}{\cos^2 x}$

M1 A1

$$= \frac{\cos^2 x + \sin^2 x + \sin x}{\cos^2 x} = \frac{1 + \sin x}{\cos^2 x}$$

A1

$$\text{At } x = \frac{\pi}{3}: \frac{(1 + \frac{\sqrt{3}}{2})}{(\frac{1}{2})^2} = 4 + 2\sqrt{3}$$

A1

Answer: $4 + 2\sqrt{3}$

4 marks

28 (10 marks)

(a) The quotient rule is $\frac{v u' - u v'}{v^2}$; the student has used $u v' - v u'$ (numerator reversed) E1

so the sign of the whole numerator is wrong E1

Answer: numerator reversed 2 marks

(b) $u = x^2$, $v = x + 1$ M1

$$\frac{d y}{d x} = \frac{(x+1) \cdot 2x - x^2 \cdot 1}{(x+1)^2}$$
A1

$$= \frac{x^2 + 2x}{(x+1)^2} = \frac{x(x+2)}{(x+1)^2}$$
A1

Answer: $\frac{x(x+2)}{(x+1)^2}$ 3 marks

(c) $x = 0$ or $x = -2$ M1

$(0, 0)$ and $(-2, -4)$ A1

Answer: $(0, 0)$ and $(-2, -4)$ 2 marks

$$(d) \frac{x^2}{x+1} = \frac{(x+1)(x-1)+1}{x+1} = x - 1 + \frac{1}{x+1}$$
B1

$$\frac{d y}{d x} = 1 - \frac{1}{(x+1)^2}$$
M1

$$= \frac{(x+1)^2 - 1}{(x+1)^2} = \frac{x^2 + 2x}{(x+1)^2}, \text{ as in (b)}$$
A1

Answer: agrees 3 marks

29 (12 marks)

(a) $u = x^2 + k$, $v = x + 1$: $u' = 2x$, $v' = 1$ B1

$$\frac{d y}{d x} = \frac{2x(x+1) - (x^2 + k)}{(x+1)^2}$$
M1

$$= \frac{x^2 + 2x - k}{(x+1)^2}$$
A1

Answer: Shown 3 marks

(b) Need $x^2 + 2x - k = 0$ to have two distinct real roots M1

Discriminant: $4 + 4k > 0$ M1

$k > -1$ A1

Answer: $k > -1$ 3 marks

(c) $x^2 + 2x - 3 = 0$ gives $x = 1$ or $x = -3$ M1 A1

A(1, 2) and B(-3, -6) A1

Gradient $\frac{x^2 + 2x - 3}{(x+1)^2}$ is negative for $-3 < x < 1$ ($x \neq -1$) and positive for $x > 1$ and for $x < -3$ M1 A1

So A is a minimum and B is a maximum A1

Answer: A(1, 2) minimum, B(-3, -6) maximum 6 marks

30 (10 marks)

(a) $u = e^x$, $v = 1 + e^x$: $u' = v' = e^x$ B1

$$f'(x) = \frac{e^x(1 + e^x) - e^x \times e^x}{(1 + e^x)^2}$$
M1

$$= \frac{e^x}{(1 + e^x)^2}$$
A1

Answer: Shown 3 marks

$$(b) 1 - f(x) = \frac{1 + e^x - e^x}{1 + e^x} = \frac{1}{1 + e^x}$$
M1 A1

$$f(x)(1 - f(x)) = \frac{e^x}{1 + e^x} \times \frac{1}{1 + e^x} = f'(x)$$
A1

Answer: Shown 3 marks

$$(c) f' = f - f^2 = \frac{1}{4} - \left(f - \frac{1}{2}\right)^2$$
M1 A1

Maximum $\frac{1}{4}$, when $f(x) = \frac{1}{2}$

A1

$\frac{e^x}{1+e^x} = \frac{1}{2}$ gives $e^x = 1$, so $x = 0$

A1

Answer: Maximum $\frac{1}{4}$ at $x = 0$

4 marks

Chain Rule

1 (2 marks)

Let $u = 3x + 2$: $\frac{dy}{dx} = 5(3x + 2)^4 \times 3$

M1

$= 15(3x + 2)^4$

A1

Answer: $15(3x + 2)^4$

2 marks

2 (3 marks)

Let $u = x^2 + 1$: $\frac{dy}{dx} = 4(x^2 + 1)^3 \times 2x$

M1 A1

At $x = 1$: $4 \times 8 \times 2 = 64$

A1

Answer: 64

3 marks

3 (2 marks)

$\frac{dy}{dx} = 7(4 - x)^6 \times -1$

M1

$= -7(4 - x)^6$

A1

Answer: $-7(4 - x)^6$

2 marks

4 (2 marks)

$\frac{dy}{dx} = -\sin(2x + 1) \times 2$

M1

$= -2 \sin(2x + 1)$

A1

Answer: $-2 \sin(2x + 1)$

2 marks

5 (2 marks)

$\frac{dy}{dx} = e^{5x} \times 5$

M1

$= 5e^{5x}$

A1

Answer: $5e^{5x}$

2 marks

6 (2 marks)

$\frac{dy}{dx} = \frac{1}{(3x + 4)} \times 3$

M1

$= \frac{3}{3x + 4}$

A1

Answer: $\frac{3}{3x + 4}$

2 marks

7 (4 marks)

(a) $\frac{dy}{dx} = (6x - 1)e^{3x^2 - x}$

M1 A1

Answer: $(6x - 1)e^{3x^2 - x}$

2 marks

(b) $y = (\cos x)^3$: $\frac{dy}{dx} = 3(\cos x)^2 \times (-\sin x)$

M1

$= -3 \cos^2 x \sin x$

A1

Answer: $-3 \cos^2 x \sin x$

2 marks

8 (5 marks)

$$(a) y = (2x + 1)^{-3}, \text{ so } \frac{dy}{dx} = -3(2x + 1)^{-4} \times 2$$

M1

$$= -\frac{6}{(2x + 1)^4}$$

A1

$$\text{Answer: } -\frac{6}{(2x + 1)^4}$$

2 marks

$$(b) \text{ When } x = 0, y = 1$$

B1

$$\text{gradient} = -6$$

M1

$$y - 1 = -6x \text{ so } y = 1 - 6x$$

A1

$$\text{Answer: } y = 1 - 6x$$

3 marks

9 (3 marks)

$$\text{Let } u = 4x + 1, \text{ so } y = u^{1/2}$$

B1

$$\frac{dy}{du} = \frac{1}{2}u^{-1/2}, \frac{du}{dx} = 4$$

M1

$$\frac{dy}{dx} = 2u^{-1/2} = \frac{2}{\sqrt{4x + 1}}$$

A1

$$\text{Answer: } \frac{2}{\sqrt{4x + 1}}$$

3 marks

10 (3 marks)

$$\text{Let } u = \sin 3x, \text{ so } y = e^u \text{ and } \frac{du}{dx} = 3\cos 3x$$

M1 A1

$$\frac{dy}{dx} = 3\cos 3x e^{\sin 3x}$$

A1

$$\text{Answer: } 3\cos 3x e^{\sin 3x}$$

3 marks

11 (3 marks)

$$\text{Let } u = x^2 + 2x, \text{ so } y = e^u$$

B1

$$\frac{dy}{du} = e^u, \frac{du}{dx} = 2x + 2$$

M1

$$\frac{dy}{dx} = (2x + 2)e^{x^2 + 2x}$$

A1

$$\text{Answer: } (2x + 2)e^{x^2 + 2x}$$

3 marks

12 (4 marks)

$$\text{Let } u = \sin x, \text{ so } y = u^2$$

B1

$$\frac{dy}{du} = 2u, \frac{du}{dx} = \cos x$$

M1

$$\frac{dy}{dx} = 2 \sin x \cos x$$

A1

$$= \sin 2x \text{ (double angle formula)}$$

A1

$$\text{Answer: } \sin 2x$$

4 marks

13 (4 marks)

$$\text{Let } u = x^2 - 2x + 3; \frac{dy}{dx} = 3u^2 \times (2x - 2)$$

M1 A1

$$\text{At } x = 2: u = 4 - 4 + 3 = 3 \text{ and } 2x - 2 = 2$$

B1

$$\frac{dy}{dx} = 3 \times 9 \times 2 = 54$$

A1

$$\text{Answer: } 54$$

4 marks

14 (6 marks)

(a) $y = (x^2 + 9)^{1/2}$

B1

$$\frac{dy}{dx} = \frac{1}{2}(x^2 + 9)^{-1/2} \times 2x$$

M1

$$= \frac{x}{\sqrt{x^2 + 9}}$$

A1

Answer: shown

3 marks

(b) $\frac{4}{\sqrt{25}} = \frac{4}{5}$

B1

Answer: $\frac{4}{5}$

1 mark

(c) At $x = 4$: $y = 5$; normal gradient $-\frac{5}{4}$, so $y - 5 = -\frac{5}{4}(x - 4)$

M1

$4y - 20 = -5x + 20$ so $5x + 4y = 40$

A1

Answer: $5x + 4y = 40$

2 marks

15 (6 marks)

(a) $\frac{dy}{dx} = e^{-x^2/2} \times (-x)$

M1

$= -xe^{-x^2/2}$

A1

Answer: $-xe^{-x^2/2}$

2 marks

(b) Product rule: $\frac{d^2y}{dx^2} = -e^{-x^2/2} + (-x)(-x)e^{-x^2/2}$

M1

$= (x^2 - 1)e^{-x^2/2}$

A1

Answer: shown

2 marks

(c) $\frac{d^2y}{dx^2} = 0$ so $x^2 - 1 = 0$

M1

$x = \pm 1$ (the sign of $x^2 - 1$ changes at both)

A1

Answer: $x = \pm 1$

2 marks

16 (5 marks)

(a) $gf(x) = (2 \ln 3x)^2 + 4 = 4(\ln 3x)^2 + 4$

B1

Answer: Shown

1 mark

(b) $\frac{d}{dx}(\ln 3x) = \frac{1}{x}$, so $\frac{dy}{dx} = 8 \ln 3x \times \frac{1}{x}$

M1 A1

At $x = 2$: $\frac{dy}{dx} = \frac{8 \ln 6}{2}$

M1

$= 4 \ln 6$

A1

Answer: $4 \ln 6$

4 marks

17 (5 marks)

Left side: $\frac{d}{dx}(\sin 3x) = 3 \cos 3x$

B1

Right side: $\frac{d}{dx}(3 \sin x - 4 \sin^3 x) = 3 \cos x - 12 \sin^2 x \cos x$

M1 A1

So $\cos 3x = \cos x - 4 \sin^2 x \cos x = \cos x - 4(1 - \cos^2 x) \cos x$

M1

$= 4 \cos^3 x - 3 \cos x$

A1

Answer: shown

5 marks

18 (7 marks)

$$\text{Let } u = x^2 - 2x: \frac{dy}{dx} = 3(x^2 - 2x)^2(2x - 2)$$

M1 A1

$$\text{Stationary: } x^2 - 2x = 0 \text{ (} x = 0 \text{ or } 2) \text{ or } 2x - 2 = 0 \text{ (} x = 1)$$

M1 A1

$$\text{Points } (0, 0), (2, 0), (1, -1)$$

A1

$$\frac{dy}{dx} = 6x^2(x - 2)^2(x - 1): \text{ the factor } x^2(x - 2)^2 \geq 0, \text{ so the sign changes only at } x = 1$$

M1

Only $(1, -1)$ is a turning point (a minimum); $(0, 0)$ and $(2, 0)$ are points of inflection

A1

Answer: $(0, 0)$, $(2, 0)$ inflections; $(1, -1)$ minimum

7 marks

19 (4 marks)

$$\frac{dy}{dx} = -\frac{1}{2}(1 + x^2)^{-3/2} \times 2x = -x(1 + x^2)^{-3/2}$$

M1 A1

$$(1 + x^2) \frac{dy}{dx} = -x(1 + x^2)^{-1/2} = -xy$$

M1

$$\text{So } (1 + x^2) \frac{dy}{dx} + xy = 0$$

A1

Answer: Shown

4 marks

20 (6 marks)

$$\frac{dy}{dx} = \frac{2x - 4}{x^2 - 4x + 5}$$

M1 A1

$$\text{Zero only when } 2x - 4 = 0, \text{ so } x = 2, y = \ln 1 = 0$$

M1 A1

$$x^2 - 4x + 5 = (x - 2)^2 + 1 \geq 1$$

M1

$$\text{So } y = \ln((x - 2)^2 + 1) \geq 0 \text{ with equality at } x = 2: \text{ a minimum at } (2, 0)$$

A1

Answer: $(2, 0)$, minimum

6 marks

21 (8 marks)

$$\text{(a) Outer power: } 3(1 + \sin 2x)^2$$

M1

$$\text{inner function: derivative of } \sin 2x \text{ is } 2 \cos 2x$$

A1

$$\frac{dy}{dx} = 6 \cos 2x(1 + \sin 2x)^2$$

A1

Answer: $6 \cos 2x(1 + \sin 2x)^2$

3 marks

$$\text{(b) } 6 \cos\left(\frac{\pi}{3}\right)(1 + \sin\left(\frac{\pi}{3}\right))^2 = 3\left(1 + \frac{\sqrt{3}}{2}\right)^2$$

M1 A1

$$= 3\left(\frac{7}{4} + \sqrt{3}\right) = \frac{21}{4} + 3\sqrt{3}$$

A1

Answer: $\frac{21}{4} + 3\sqrt{3}$

3 marks

$$\text{(c) } \cos 2x = 0 \text{ or } 1 + \sin 2x = 0: 2x = \frac{\pi}{2}, \frac{3\pi}{2}$$

M1

$$x = \frac{\pi}{4}, \frac{3\pi}{4}$$

A1

Answer: $x = \frac{\pi}{4}, \frac{3\pi}{4}$

2 marks

22 (8 marks)

$$\text{(a) Product rule: } u = x^2, v = (1 - x^2)^{1/2}$$

M1

$$v' = \frac{1}{2}(1 - x^2)^{-1/2} \times (-2x) = -\frac{x}{\sqrt{1 - x^2}}$$

M1 A1

$$\frac{dy}{dx} = 2x\sqrt{1 - x^2} - \frac{x^3}{\sqrt{1 - x^2}} = \frac{2x(1 - x^2) - x^3}{\sqrt{1 - x^2}} = \frac{x(2 - 3x^2)}{\sqrt{1 - x^2}}$$

A1

Answer: shown

4 marks

$$\text{(b) } 2 - 3x^2 = 0 \text{ so } x = \sqrt{\frac{2}{3}}$$

M1 A1

$$y = \frac{2}{3} \times \sqrt{\frac{1}{3}} = \frac{2\sqrt{3}}{9}$$

A1

Answer: $(\sqrt{\frac{2}{3}}, \frac{2\sqrt{3}}{9})$

3 marks

(c) $\frac{dy}{dx} > 0$ for x slightly less than $\sqrt{\frac{2}{3}}$ and < 0 slightly above, so a maximum

B1

Answer: maximum

1 mark

23 (5 marks)

Let $u = x^2$: $\frac{dy}{dx} = \cos(x^2) \times 2x = 2x \cos(x^2)$

M1 A1

Product rule on $2x \cos(x^2)$: $u = 2x$, $v = \cos(x^2)$, $v' = -2x \sin(x^2)$

M1

$\frac{d^2y}{dx^2} = 2\cos(x^2) + 2x \times (-2x \sin(x^2))$

A1

$= 2\cos(x^2) - 4x^2 \sin(x^2)$

A1

Answer: $2x \cos(x^2)$; $2\cos(x^2) - 4x^2 \sin(x^2)$

5 marks

24 (6 marks)

Product rule: $u = x^3$, $v = (1 - x^2)^{3/2}$, $u' = 3x^2$

B1

Chain rule: $v' = \frac{3}{2}(1 - x^2)^{1/2} \times (-2x) = -3x(1 - x^2)^{1/2}$

M1 A1

$\frac{dy}{dx} = 3x^2(1 - x^2)^{3/2} - 3x^4(1 - x^2)^{1/2}$

M1

Take out the common factor $3x^2(1 - x^2)^{1/2}$: $3x^2(1 - x^2)^{1/2}[(1 - x^2) - x^2]$

M1

$= 3x^2(1 - 2x^2)\sqrt{1 - x^2}$

A1

Answer: Shown

6 marks

25 (6 marks)

(a) $\frac{dN}{dt} = 800 \times (-0.04)e^{-0.04t} = -32e^{-0.04t}$

M1 A1

At $t = 10$: $-32e^{-0.4} = -21.5$ million per day

A1

Answer: -21.5 million per day

3 marks

(b) $32e^{-0.04t} = 8$, so $e^{-0.04t} = \frac{1}{4}$

M1

$-0.04t = -\ln 4$

M1

$t = \frac{\ln 4}{0.04} = 34.7$ days

A1

Answer: 34.7 days

3 marks

26 (7 marks)

(a) $\frac{dx}{dy} = 3y^2 + 1$

M1 A1

$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{3y^2 + 1}$

A1

Answer: Shown

3 marks

(b) At $y = 1$: $\frac{dy}{dx} = \frac{1}{4}$

M1 A1

$y - 1 = \frac{1}{4}(x - 2)$

M1

$x - 4y + 2 = 0$

A1

Answer: $x - 4y + 2 = 0$

4 marks

27 (9 marks)

(a) $a^x = e^{x \ln a}$

B1

$\frac{d}{dx}(e^{x \ln a}) = \ln a \times e^{x \ln a}$

M1

$= a^x \ln a$

A1

Answer: shown

3 marks

(b) Chain rule: $\frac{dy}{dx} = 2^{x^2-1} \ln 2 \times 2x$

M1

$$= 2x \ln 2 \times 2^{x^2-1}$$

A1

Answer: $2x \ln 2 \times 2^{x^2-1}$

2 marks

(c) At $x = 1$: $y = 1$, gradient $= 2 \ln 2$

B1

$$y - 1 = 2 \ln 2 (x - 1)$$

A1

Answer: $y - 1 = 2 \ln 2 (x - 1)$

2 marks

(d) $-1 = 2 \ln 2 (x - 1)$

M1

$$x = 1 - \frac{1}{2 \ln 2}$$

A1

Answer: $x = 1 - \frac{1}{2 \ln 2}$

2 marks

28 (10 marks)

(a) Let $u = 3x^2 + 1$ and $w = \sin u$, so $y = w^2$; $\frac{du}{dx} = 6x$, $\frac{dw}{du} = \cos u$, $\frac{dy}{dw} = 2w$

M1

$$\frac{dy}{dx} = 2 \sin u \cos u \times 6x$$

M1 A1

$$= 6x \sin 2u = 6x \sin(6x^2 + 2)$$

A1

Answer: Shown

4 marks

(b) $\frac{dy}{dx} = 0$ with $x > 0$ requires $\sin(6x^2 + 2) = 0$

M1

For $0 < x < 1$, $2 < 6x^2 + 2 < 8$, so $6x^2 + 2 = \pi$ or 2π

M1

$$x = \sqrt{\frac{\pi - 2}{6}} \text{ or } x = \sqrt{\frac{\pi - 1}{3}}$$

A1

Answer: $x = \sqrt{\frac{\pi - 2}{6}}$, $x = \sqrt{\frac{\pi - 1}{3}}$

3 marks

(c) $\frac{d^2y}{dx^2} = 6 \sin(6x^2 + 2) + 72x^2 \cos(6x^2 + 2)$

M1 A1

At $x = \sqrt{\frac{\pi - 2}{6}}$: $\frac{d^2y}{dx^2} = 0 + 72x^2 \cos \pi = -12(\pi - 2) < 0$, so a maximum

A1

Answer: maximum

3 marks

29 (10 marks)

(a) Let $u = -\frac{(x - a)^2}{2}$: $\frac{du}{dx} = -(x - a)$

M1 A1

$$\frac{dy}{dx} = e^u \times (-(x - a)) = -(x - a)y$$

A1

Answer: Shown

3 marks

(b) Product rule: $\frac{d^2y}{dx^2} = -y - (x - a)\frac{dy}{dx}$

M1 A1

$$= -y + (x - a)^2 y = ((x - a)^2 - 1)y$$

A1

Answer: Shown

3 marks

(c) Inflection needs $\frac{d^2y}{dx^2} = 0$ with a sign change; $y > 0$ so $(x - a)^2 = 1$

M1 A1

$$x = a \pm 1$$

A1

$y = e^{-1/2}$ at both points: $(a - 1, e^{-1/2})$ and $(a + 1, e^{-1/2})$ (the sign of $((x - a)^2 - 1)$ changes at each)

A1

Answer: $(a \pm 1, e^{-1/2})$

4 marks

30 (8 marks)

(a) Let $u = \sin x$, so $y = u^n$: $\frac{dy}{dx} = nu^{n-1} \times \cos x$

M1

$$= n \sin^{n-1} x \cos x$$

A1

Answer: Shown

2 marks

(b) Product rule on $n \sin^{n-1} x \times \cos x$

M1

$$\frac{d}{dx}(\sin^{n-1} x) = (n - 1) \sin^{n-2} x \cos x$$

M1 A1

$$\frac{d^2y}{dx^2} = n(n-1) \sin^{n-2}x \cos^2x - n \sin^n x$$

A1

$$\text{Use } \cos^2x = 1 - \sin^2x: n(n-1) \sin^{n-2}x - n(n-1) \sin^n x - n \sin^n x$$

M1

$$= n(n-1) \sin^{n-2}x - n^2 \sin^n x$$

A1

Answer: Shown

6 marks

Implicit Differentiation

1 (3 marks)

$$\text{Differentiate both sides with respect to } x: 2x + 2y \frac{dy}{dx} = 0$$

M1 A1

$$\frac{dy}{dx} = -\frac{x}{y}$$

A1

$$\text{Answer: } -\frac{x}{y}$$

3 marks

2 (3 marks)

$$2y \frac{dy}{dx} = 4$$

M1

$$\frac{dy}{dx} = \frac{2}{y}$$

A1

$$\text{At } (1, 2): \text{gradient} = 1$$

A1

$$\text{Answer: } \frac{2}{y}; 1$$

3 marks

3 (3 marks)

$$\text{Differentiate both sides: } 2x + 6y \frac{dy}{dx} = 0$$

M1 A1

$$\frac{dy}{dx} = -\frac{x}{3y}$$

A1

$$\text{Answer: } -\frac{x}{3y}$$

3 marks

4 (3 marks)

$$\text{Product rule on } xy: y + x \frac{dy}{dx} = 0$$

M1 A1

$$\frac{dy}{dx} = -\frac{y}{x}$$

A1

$$\text{Answer: } -\frac{y}{x}$$

3 marks

5 (3 marks)

$$6y^2 \frac{dy}{dx} + 1 = 0$$

M1 A1

$$\frac{dy}{dx} = -\frac{1}{6y^2}$$

A1

$$\text{Answer: } -\frac{1}{6y^2}$$

3 marks

6 (4 marks)

$$2x + 8y \frac{dy}{dx} = 0$$

M1 A1

$$\frac{dy}{dx} = -\frac{x}{4y}$$

A1

$$\text{At } (2, 1): -\frac{2}{4} = -\frac{1}{2}$$

A1

$$\text{Answer: } -\frac{1}{2}$$

4 marks

7 (4 marks)

$$(a) 2x + 2y \frac{dy}{dx} - 6 + 4 \frac{dy}{dx} = 0$$

M1 A1

$$\frac{dy}{dx}(2y + 4) = 6 - 2x \text{ so } \frac{dy}{dx} = \frac{3 - x}{y + 2}$$

A1

$$\text{Answer: } \frac{3 - x}{y + 2}$$

3 marks

$$(b) \frac{3 - 6}{2 + 2} = -\frac{3}{4}$$

B1

$$\text{Answer: } -\frac{3}{4}$$

1 mark

8 (5 marks)

$$(a) 3y^2 \frac{dy}{dx} + y + x \frac{dy}{dx} = 0$$

M1 A1

$$\frac{dy}{dx} = -\frac{y}{3y^2 + x}$$

A1

$$\text{Answer: } -\frac{y}{3y^2 + x}$$

3 marks

$$(b) \text{Gradient at P} = -\frac{2}{13}$$

M1

$$y - 2 = -\frac{2}{13}(x - 1) \text{ so } 2x + 13y = 28$$

A1

$$\text{Answer: } 2x + 13y = 28$$

2 marks

9 (4 marks)

$$\text{Differentiate: } 3x^2 + 3y^2 \frac{dy}{dx} = 3y + 3x \frac{dy}{dx}$$

M1 A1 A1

$$\text{Collect: } (\frac{dy}{dx})(3y^2 - 3x) = 3y - 3x^2, \text{ so } \frac{dy}{dx} = \frac{y - x^2}{y^2 - x}$$

A1

$$\text{Answer: Shown}$$

4 marks

10 (5 marks)

$$\text{At } (2, 1): 4 + 4 - 1 = 7, \text{ so P lies on C}$$

B1

$$2x + 2y + 2x \frac{dy}{dx} - 2y \frac{dy}{dx} = 0$$

M1 A1

$$\frac{dy}{dx} = -\frac{x + y}{x - y}; \text{ at P: } -\frac{3}{1} = -3$$

A1

$$y - 1 = -3(x - 2), \text{ so } y = -3x + 7$$

A1

$$\text{Answer: } y = -3x + 7$$

5 marks

11 (4 marks)

$$\text{Differentiate: } 2e^{2y} \frac{dy}{dx} = 1 + \frac{dy}{dx}$$

M1 A1 A1

$$\frac{dy}{dx}(2e^{2y} - 1) = 1, \text{ so } \frac{dy}{dx} = \frac{1}{2e^{2y} - 1}$$

A1

$$\text{Answer: } \frac{1}{2e^{2y} - 1}$$

4 marks

12 (3 marks)

$$-\sin y \frac{dy}{dx} + 2x = 0$$

M1 A1

$$\frac{dy}{dx} = \frac{2x}{\sin y}$$

A1

$$\text{Answer: } \frac{2x}{\sin y}$$

3 marks

13 (5 marks)

(a) Differentiate both sides: $\ln y + \frac{x}{y} \frac{dy}{dx} + \frac{dy}{dx} = 0$

M1 A1

$$\frac{dy}{dx} \left(\frac{x}{y} + 1 \right) = -\ln y, \text{ so } \frac{dy}{dx} = -\frac{y \ln y}{x + y}$$

A1

Answer: $-\frac{y \ln y}{x + y}$

3 marks

(b) At (0, 3): $\frac{dy}{dx} = -\frac{3 \ln 3}{3} = -\ln 3$

M1

$$y - 3 = -(\ln 3)x, \text{ so } y = 3 - x \ln 3$$

A1

Answer: $y = 3 - x \ln 3$

2 marks

14 (6 marks)

(a) $2x + e^y \frac{dy}{dx} = 0$

M1 A1

$$\frac{dy}{dx} = -\frac{2x}{e^y} = -2xe^{-y}$$

A1

Answer: shown

3 marks

(b) Gradient at (3, 0) = -6

B1

Normal gradient $\frac{1}{6}$: $y = \frac{1}{6}(x - 3)$

M1

$$x - 6y - 3 = 0$$

A1

Answer: $x - 6y - 3 = 0$

3 marks

15 (6 marks)

(a) $2(x - 2) + 2(y - 1) \frac{dy}{dx} = 0$

M1 A1

$$\frac{dy}{dx} = -\frac{x - 2}{y - 1} = \frac{2 - x}{y - 1}$$

A1

Answer: shown

3 marks

(b) $\frac{dy}{dx} = 0$ so $x = 2$

M1

$$(y - 1)^2 = 9 \text{ so } (2, 4) \text{ and } (2, -2)$$

A1

Answer: (2, 4) and (2, -2)

2 marks

(c) Denominator zero: $y = 1$, so $(x - 2)^2 = 9$: (5, 1) and (-1, 1)

B1

Answer: (5, 1) and (-1, 1)

1 mark

16 (6 marks)

Differentiate: $2x + 4y \frac{dy}{dx} - 4 + 4 \frac{dy}{dx} = 0$

M1 A1

$$\frac{dy}{dx} = \frac{4 - 2x}{4y + 4} = \frac{2 - x}{2y + 2}$$

A1

Parallel to x-axis: $\frac{dy}{dx} = 0$, so $x = 2$

M1

$$4 + 2y^2 - 8 + 4y = 4, \text{ so } y^2 + 2y - 4 = 0$$

M1

$$y = -1 \pm \sqrt{5}: \text{ points } (2, -1 + \sqrt{5}) \text{ and } (2, -1 - \sqrt{5})$$

A1

Answer: (2, -1 ± √5)

6 marks

17 (5 marks)

Product rule both sides: $\frac{dy}{dx} e^x + ye^x = e^y + xe^y \frac{dy}{dx}$

M1 A1 A1

$$\frac{dy}{dx} (e^x - xe^y) = e^y - ye^x, \text{ so } \frac{dy}{dx} = \frac{e^y - ye^x}{e^x - xe^y}$$

A1

At (0, 0): $\frac{1 - 0}{1 - 0} = 1$

A1

Answer: Gradient 1

5 marks

18 (4 marks)

$$\text{Differentiate: } \sec^2 y \frac{dy}{dx} = 2xy + x^2 \frac{dy}{dx}$$

M1 A1 A1

$$\frac{dy}{dx}(\sec^2 y - x^2) = 2xy, \text{ so } \frac{dy}{dx} = \frac{2xy}{\sec^2 y - x^2}$$

A1

$$\text{Answer: } \frac{2xy}{\sec^2 y - x^2}$$

4 marks

19 (6 marks)

$$\text{Substitute } (1, -1): a + 3 + 2 = 7, \text{ so } a = 2$$

M1 A1

$$4x + 6y \frac{dy}{dx} - 2y - 2x \frac{dy}{dx} = 0$$

M1 A1

$$\frac{dy}{dx} = \frac{2y - 4x}{6y - 2x}; \text{ at P: } \frac{-2 - 4}{-6 - 2} = \frac{3}{4}$$

A1

$$y + 1 = \frac{3}{4}(x - 1), \text{ so } 3x - 4y - 7 = 0$$

A1

$$\text{Answer: } a = 2; 3x - 4y - 7 = 0$$

6 marks

20 (6 marks)

$$\text{Differentiate: } 2ax + 3y + 3x \frac{dy}{dx} + 2y \frac{dy}{dx} = 0, \text{ so } \frac{dy}{dx} = -\frac{2ax + 3y}{3x + 2y}$$

M1 A1

$$\text{The normal has gradient } \frac{7}{10}, \text{ so the tangent at P has gradient } -\frac{10}{7}$$

M1

$$\text{At P: } -\frac{2a + 6}{7} = -\frac{10}{7}, \text{ so } 2a + 6 = 10 \text{ and } a = 2$$

M1 A1

$$b = 2(1) + 3(2) + 4 = 12$$

A1

$$\text{Answer: } a = 2, b = 12$$

6 marks

21 (8 marks)

$$(a) 2x + y + x \frac{dy}{dx} + 2y \frac{dy}{dx} = 0$$

M1 A1

$$\frac{dy}{dx}(x + 2y) = -(2x + y), \text{ giving the result}$$

A1

$$\text{Answer: shown}$$

3 marks

$$(b) 1 + y + y^2 = 7 \text{ so } y^2 + y - 6 = 0, y = 2 \text{ or } y = -3$$

M1 A1

$$(1, 2): -\frac{4}{5}; (1, -3): -\frac{-1}{-5} = -\frac{1}{5}$$

A1

$$\text{Answer: } -\frac{4}{5} \text{ at } (1, 2) \text{ and } -\frac{1}{5} \text{ at } (1, -3)$$

3 marks

$$(c) 2x + y = 0 \text{ so } y = -2x: x^2 - 2x^2 + 4x^2 = 7, x = \pm\sqrt{\frac{7}{3}}$$

M1

$$(\sqrt{\frac{7}{3}}, -2\sqrt{\frac{7}{3}}) \text{ and } (-\sqrt{\frac{7}{3}}, 2\sqrt{\frac{7}{3}})$$

A1

$$\text{Answer: } (\pm\sqrt{\frac{7}{3}}, \mp 2\sqrt{\frac{7}{3}})$$

2 marks

22 (8 marks)

$$(a) 2x + 2y \frac{dy}{dx} = 0$$

M1

$$\frac{dy}{dx} = -\frac{x}{y}$$

A1

$$\text{Answer: } -\frac{x}{y}$$

2 marks

$$(b) \frac{d^2y}{dx^2} = -\frac{y - x \frac{dy}{dx}}{y^2}$$

M1 A1

$$\text{Substitute } \frac{dy}{dx} = -\frac{x}{y}: -\frac{y + \frac{x^2}{y}}{y^2} = -\frac{y^2 + x^2}{y^3}$$

M1

$$x^2 + y^2 = 25, \text{ so } \frac{d^2y}{dx^2} = -\frac{25}{y^3}$$

A1

Answer: shown**4 marks**

$$(c) \frac{dy}{dx} = 0 \text{ at } (0, 5) \text{ and } \frac{d^2y}{dx^2} = -\frac{25}{125} = -\frac{1}{5}$$

M1

Negative, so a maximum

A1**Answer: maximum****2 marks****23 (12 marks)**

$$(a) \text{ Product rule on each term: } y^2 + 2xy \frac{dy}{dx} - 2xy - x^2 \frac{dy}{dx} = 0$$

M1 A1 A1

$$\frac{dy}{dx}(2xy - x^2) = 2xy - y^2, \text{ so } \frac{dy}{dx} = \frac{y(2x - y)}{x(2y - x)}$$

A1**Answer: Shown****4 marks**(b) At (1, 3): $9 - 3 = 6$, so P lies on C**B1**

$$\frac{dy}{dx} = \frac{3(2 - 3)}{1(6 - 1)} = -\frac{3}{5}$$

M1 A1

$$y - 3 = -\frac{3}{5}(x - 1), \text{ so } 3x + 5y = 18$$

A1**Answer: $3x + 5y = 18$** **4 marks**(c) Vertical tangent: the denominator $x(2y - x) = 0$, so $x = 2y$ ($x \neq 0$)**M1 A1**

$$\text{Substitute: } 2y \times y^2 - 4y^2 \times y = 6, \text{ so } -2y^3 = 6$$

M1

$$y = -\sqrt[3]{3}, x = -2\sqrt[3]{3}: \text{ the point is } (-2\sqrt[3]{3}, -\sqrt[3]{3})$$

A1**Answer: $(-2\sqrt[3]{3}, -\sqrt[3]{3})$** **4 marks****24 (8 marks)**

$$(a) \text{ Horizontal tangent: } \frac{dy}{dx} = 0, \text{ so } y = x^2$$

M1 A1

$$\text{Substitute: } x^3 + x^6 = 3x^3, \text{ so } x^3(x^3 - 2) = 0$$

M1

$$x \neq 0, \text{ so } x^3 = 2, x = \sqrt[3]{2}$$

A1

$$y = x^2 = \sqrt[3]{4}; \text{ P is } (\sqrt[3]{2}, \sqrt[3]{4})$$

A1**Answer: $P(\sqrt[3]{2}, \sqrt[3]{4})$** **5 marks**(b) Vertical tangent: $y^2 - x = 0$, so $x = y^2$ **M1**By the symmetry of the equation in x and y, $y^3 = 2$ **M1**

$$\text{Q is } (\sqrt[3]{4}, \sqrt[3]{2})$$

A1**Answer: $Q(\sqrt[3]{4}, \sqrt[3]{2})$** **3 marks****25 (6 marks)**

$$C_1: 2x - 2y \frac{dy}{dx} = 0, \text{ so } \frac{dy}{dx} = \frac{x}{y} = 3 \text{ at P}$$

M1 A1

$$C_2: y + x \frac{dy}{dx} = 0, \text{ so } \frac{dy}{dx} = -\frac{y}{x} = -\frac{1}{3} \text{ at P}$$

M1 A1

$$\text{Product of gradients} = 3 \times \left(-\frac{1}{3}\right) = -1$$

M1

So the tangents are perpendicular

A1**Answer: Shown****6 marks****26 (6 marks)**

$$\text{Product rule: } \cos x \cos y - \sin x \sin y \frac{dy}{dx} = 0$$

M1 A1 A1

$$\frac{dy}{dx} = \frac{\cos x \cos y}{\sin x \sin y}$$

A1

$$\text{At } \left(\frac{\pi}{6}, \frac{\pi}{3}\right): \frac{\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)} = 1$$

M1

$$y - \frac{\pi}{3} = x - \frac{\pi}{6}, \text{ so } y = x + \frac{\pi}{6}$$

A1**Answer: $y = x + \frac{\pi}{6}$** **6 marks**

27 (9 marks)

(a) $2x - y - x \frac{dy}{dx} + 2y \frac{dy}{dx} = 0$

M1 A1

$\frac{dy}{dx}(2y - x) = y - 2x$, giving the result

A1

Answer: shown

3 marks

(b) $x = 0: y^2 = 3$ so $y = \pm\sqrt{3}$

M1

gradient $= \frac{y}{2y} = \frac{1}{2}$ at both points

A1

Equal gradients, so the tangents are parallel

E1

Answer: both gradients $\frac{1}{2}$

3 marks

(c) $2y - x = 0$ so $x = 2y$

M1

$4y^2 - 2y^2 + y^2 = 3, y^2 = 1$

A1

$(2, 1)$ and $(-2, -1)$

A1

Answer: $(2, 1)$ and $(-2, -1)$

3 marks

28 (10 marks)

(a) Differentiate $x = e^y$ with respect to x : $1 = e^y \frac{dy}{dx}$

M1 A1

$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$

A1

Answer: shown

3 marks

(b) $\ln y = x \ln x$

B1

$\frac{1}{y} \frac{dy}{dx} = \ln x + x \times \frac{1}{x} = \ln x + 1$

M1 A1

$\frac{dy}{dx} = x^x(1 + \ln x)$

A1

Answer: shown

4 marks

(c) $1 + \ln x = 0$ so $x = e^{-1}$

M1

$y = (e^{-1})^{e^{-1}} = e^{-1/e}$, so the point is $(e^{-1}, e^{-1/e})$

A1

$1 + \ln x < 0$ for $x < e^{-1}$ and > 0 for $x > e^{-1}$, so a minimum

B1

Answer: $(e^{-1}, e^{-1/e})$, minimum

3 marks

29 (9 marks)

(a) $2x - 2y \frac{dy}{dx} = 0$, so $\frac{dy}{dx} = \frac{x}{y}$

M1 A1

Answer: Shown

2 marks

(b) Let the point of contact be (p, q) : tangent $y - q = \frac{p}{q}(x - p)$

M1

Through $(0, 4)$: $4 - q = -\frac{p^2}{q}$, so $p^2 - q^2 + 4q = 0$

M1 A1

(p, q) lies on C so $p^2 - q^2 = 8$; hence $4q = -8$ and $q = -2$

M1

$p^2 = 8 + 4 = 12$, so the points are $(2\sqrt{3}, -2)$ and $(-2\sqrt{3}, -2)$

A1

Answer: $(2\sqrt{3}, -2)$ and $(-2\sqrt{3}, -2)$

5 marks

(c) The gradients $\frac{p}{q}$ are $-\sqrt{3}$ and $\sqrt{3}$; tangents: $y = -\sqrt{3}x + 4$ and $y = \sqrt{3}x + 4$

M1 A1

Answer: $y = 4 - \sqrt{3}x$ and $y = 4 + \sqrt{3}x$

2 marks

30 (10 marks)

(a) Differentiate: $\frac{2x}{a^2} + \frac{2y}{b^2} \frac{dy}{dx} = 0$, so $\frac{dy}{dx} = -\frac{b^2x}{a^2y}$

M1 A1

At P: gradient = $-\frac{b^2p}{a^2q}$

A1

Tangent: $y - q = -\frac{b^2p}{a^2q}(x - p)$

M1

Multiply by a^2q : $a^2qy - a^2q^2 = -b^2px + b^2p^2$

A1

$b^2px + a^2qy = b^2p^2 + a^2q^2 = a^2b^2$ since P lies on E

M1

Divide by a^2b^2 : $\frac{px}{a^2} + \frac{qy}{b^2} = 1$

A1

Answer: Proved

7 marks

(b) Check: $\frac{12}{16} + \frac{1}{4} = 1$

B1

$\frac{2\sqrt{3}x}{16} + \frac{y}{4} = 1$

M1

Multiply by 16: $2\sqrt{3}x + 4y = 16$, i.e. $\sqrt{3}x + 2y = 8$

A1

Answer: $\sqrt{3}x + 2y = 8$

3 marks

Parametric Differentiation**1 (3 marks)**

$\frac{dx}{dt} = 2t$, $\frac{dy}{dt} = 2$

M1 A1

$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{2}{2t} = \frac{1}{t}$

A1

Answer: $\frac{1}{t}$

3 marks

2 (3 marks)

$\frac{dx}{dt} = 3$, $\frac{dy}{dt} = 2t$

M1

$\frac{dy}{dx} = \frac{2t}{3}$

A1

At $t = 3$: gradient = 2

A1

Answer: $\frac{2t}{3}$; 2

3 marks

3 (3 marks)

$\frac{dx}{dt} = 2e^{2t}$ and $\frac{dy}{dt} = 4e^t$

M1 A1

$\frac{dy}{dx} = \frac{4e^t}{2e^{2t}} = 2e^{-t}$

A1

Answer: $2e^{-t}$

3 marks

4 (3 marks)

$\frac{dx}{dt} = 3t^2$, $\frac{dy}{dt} = 6t$

B1

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2}{t}$

M1 A1

Answer: $\frac{2}{t}$

3 marks

5 (3 marks)

$$\frac{dx}{dt} = 2, \frac{dy}{dt} = 3t^2 - 1$$

B1

$$\frac{dy}{dx} = \frac{3t^2 - 1}{2}$$

M1

$$\text{At } t = 2: \frac{11}{2}$$

A1

$$\text{Answer: } \frac{11}{2}$$

3 marks

6 (3 marks)

$$\frac{dx}{dt} = -\sin t, \frac{dy}{dt} = \cos t$$

B1

$$\frac{dy}{dx} = \frac{\cos t}{-\sin t}$$

M1

$$= -\cot t$$

A1

Answer: Shown

3 marks

7 (4 marks)

$$\frac{dx}{dt} = -2 \sin t, \frac{dy}{dt} = 3 \cos t$$

M1

$$\frac{dy}{dx} = -\frac{3 \cos t}{2 \sin t}$$

A1

$$\text{At } t = \frac{\pi}{6}: -\frac{3}{2} \times \frac{\cos(\frac{\pi}{6})}{\sin(\frac{\pi}{6})}$$

M1

$$= -\frac{3}{2} \times \sqrt{3} = -\frac{3\sqrt{3}}{2}$$

A1

$$\text{Answer: } -\frac{3\sqrt{3}}{2}$$

4 marks

8 (5 marks)

$$\text{At } t = 2: x = 4, y = 2$$

B1

$$\frac{dy}{dx} = \frac{3t^2 - 3}{2t}$$

M1 A1

$$\text{At } t = 2: \frac{9}{4}$$

A1

$$y - 2 = \frac{9}{4}(x - 4), \text{ so } 9x - 4y - 28 = 0$$

A1

$$\text{Answer: } 9x - 4y - 28 = 0$$

5 marks

9 (6 marks)

$$\text{(a) } 3t^2 - 1 = 2, \text{ so } t^2 = 1 \text{ and } t = 1$$

M1 A1

$$2 + k = 7, \text{ so } k = 5$$

A1

$$\text{Answer: } t = 1, k = 5$$

3 marks

$$\text{(b) } \frac{dx}{dt} = 6t, \frac{dy}{dt} = 6t^2, \text{ so } \frac{dy}{dx} = t$$

M1

$$\text{At P the gradient is 1, so } y - 7 = x - 2, \text{ which gives } y = x + 5$$

A1 A1

$$\text{Answer: gradient 1; } y = x + 5$$

3 marks

10 (4 marks)

$$\frac{dx}{dt} = 1 - \frac{1}{t^2}, \frac{dy}{dt} = 1 + \frac{1}{t^2}$$

B1 B1

$$\frac{dy}{dx} = \frac{1 + \frac{1}{t^2}}{1 - \frac{1}{t^2}}$$

M1

$$\text{Multiply top and bottom by } t^2: \frac{t^2 + 1}{t^2 - 1}$$

A1

Answer: Shown

4 marks

11 (4 marks)

$$\frac{dx}{dt} = e^t, \frac{dy}{dt} = 2e^{2t}$$

B1

$$\frac{dy}{dx} = \frac{2e^{2t}}{e^t} = 2e^t$$

M1 A1

$$\text{Check: } y = x^2, \text{ so } \frac{dy}{dx} = 2x = 2e^t$$

B1

Answer: $2e^t$

4 marks

12 (5 marks)

$$\frac{dx}{dt} = \cos t, \frac{dy}{dt} = -2 \sin 2t$$

B1

$$\frac{dy}{dx} = -\frac{2 \sin 2t}{\cos t} = -4 \sin t$$

M1 A1

$$\text{At } t = \frac{\pi}{6}: \text{gradient} = -2; \text{point } \left(\frac{1}{2}, \frac{1}{2}\right)$$

B1

$$y - \frac{1}{2} = -2\left(x - \frac{1}{2}\right), \text{ so } y = -2x + \frac{3}{2}$$

A1

$$\text{Answer: } y = -2x + \frac{3}{2}$$

5 marks

13 (5 marks)

$$\frac{dx}{dt} = 4t, \frac{dy}{dt} = 4, \text{ so } \frac{dy}{dx} = \frac{1}{t}$$

B1 M1

$$\text{At } t = 1: \text{gradient } 1, \text{ point } (2, 4); \text{ normal gradient } -1$$

A1

$$y - 4 = -(x - 2)$$

M1

$$x + y = 6$$

A1

$$\text{Answer: } x + y = 6$$

5 marks

14 (6 marks)

$$\text{(a) } y = 4t^{-1}, \text{ so } \frac{dy}{dt} = -\frac{4}{t^2}, \frac{dx}{dt} = 2t$$

M1 A1

$$\frac{dy}{dx} = -\frac{4}{t^2} \times \frac{1}{2t} = -\frac{2}{t^3}$$

A1

$$\text{Answer: } -\frac{2}{t^3}$$

3 marks

$$\text{(b) At } t = 1: (1, 4), \text{ gradient } -2$$

B1

$$\text{Normal gradient} = \frac{1}{2}: y - 4 = \frac{1}{2}(x - 1)$$

M1

$$x - 2y + 7 = 0$$

A1

$$\text{Answer: } x - 2y + 7 = 0$$

3 marks

15 (7 marks)

$$\text{(a) } \frac{dy}{dt} = 3t^2 - 3, \frac{dx}{dt} = 2$$

M1

$$\frac{dy}{dx} = \frac{3t^2 - 3}{2}$$

A1

$$\text{Answer: } \frac{3t^2 - 3}{2}$$

2 marks

$$\text{(b) } 3t^2 - 3 = 0 \text{ so } t = \pm 1$$

M1 A1

$$t = 1: (2, -2); t = -1: (-2, 2)$$

A1

$$\text{Answer: } (2, -2) \text{ and } (-2, 2)$$

3 marks

$$\text{(c) } \frac{d^2y}{dx^2} = \left(\frac{d}{dt} \frac{dy}{dx}\right) \div \frac{dx}{dt} = 3t \div 2 = \frac{3t}{2}$$

M1

$$t = 1: \text{positive, minimum at } (2, -2); t = -1: \text{negative, maximum at } (-2, 2)$$

A1

$$\text{Answer: } (2, -2) \text{ minimum, } (-2, 2) \text{ maximum}$$

2 marks

16 (7 marks)

$$(a) \cos t = \frac{x-1}{2}, \sin t = \frac{y+2}{3}$$

M1 A1

$$\text{Use } \cos^2 t + \sin^2 t = 1$$

A1

Answer: Shown

3 marks

$$(b) \frac{dx}{dt} = -2 \sin t, \frac{dy}{dt} = 3 \cos t$$

B1

$$\frac{dy}{dx} = -\frac{3 \cos t}{2 \sin t} = -\frac{3}{2} \cot t$$

M1 A1

$$\text{At } t = \frac{\pi}{3}: -\frac{3}{2} \times \frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{2}$$

A1

$$\text{Answer: } -\frac{3}{2} \cot t; -\frac{\sqrt{3}}{2}$$

4 marks

17 (6 marks)

$$\frac{dx}{dt} = 2t + 1, \frac{dy}{dt} = 3t^2 - 3$$

B1

$$(a) \text{ Horizontal: } 3t^2 - 3 = 0, t = \pm 1$$

M1 A1

$$t = 1: (2, -2); t = -1: (0, 2)$$

A1

$$(b) \text{ Vertical: } 2t + 1 = 0, t = -\frac{1}{2}$$

M1

$$\text{Point } \left(-\frac{1}{4}, \frac{11}{8}\right)$$

A1

$$\text{Answer: (a) } (2, -2), (0, 2) \text{ (b) } \left(-\frac{1}{4}, \frac{11}{8}\right)$$

6 marks

18 (5 marks)

$$\frac{dy}{dx} = \frac{3t^2}{2}$$

M1 A1

$$\frac{d^2y}{dx^2} = \frac{d}{dt} \left(\frac{dy}{dx} \right) \div \frac{dx}{dt}$$

M1

$$= \frac{3t}{2}$$

A1

$$\text{At } t = 2: 3$$

A1

$$\text{Answer: } \frac{3t}{2}; 3$$

5 marks

19 (5 marks)

$$\frac{dx}{dt} = \sec t \tan t, \frac{dy}{dt} = \sec^2 t$$

B1 B1

$$\frac{dy}{dx} = \frac{\sec^2 t}{\sec t \tan t} = \frac{\sec t}{\tan t}$$

M1

$$= \frac{1}{\cos t} \times \frac{\cos t}{\sin t} = \frac{1}{\sin t} = \operatorname{cosec} t$$

A1

$$\text{At } t = \frac{\pi}{6}: \operatorname{cosec} \frac{\pi}{6} = 2$$

A1

$$\text{Answer: } \operatorname{cosec} t; 2$$

5 marks

20 (5 marks)

$$\frac{dy}{dx} = \frac{2}{2t} = \frac{1}{t}, \text{ so the gradient at P is } \frac{1}{p}$$

M1 A1

$$\text{P is } (p^2, 2p): y - 2p = \frac{1}{p}(x - p^2)$$

M1

$$\text{Multiply by p: } py - 2p^2 = x - p^2$$

A1

$$py = x + p^2$$

A1

$$\text{Answer: Shown}$$

5 marks

21 (8 marks)

$$(a) \frac{dx}{dt} = 2 \cos t, \frac{dy}{dt} = -6 \sin 2t$$

M1 A1

$$\frac{dy}{dx} = -\frac{6(2 \sin t \cos t)}{2 \cos t} = -6 \sin t$$

A1

Answer: shown

3 marks

$$(b) \cos 2t = 1 - 2 \sin^2 t, \text{ so } y = 3 - 6 \sin^2 t$$

M1

$$\sin t = \frac{x}{2}, \text{ so } y = 3 - \frac{6x^2}{4} = 3 - \frac{3}{2}x^2$$

A1

Answer: shown

2 marks

$$(c) \frac{dy}{dx} = -3x = -6 \sin t$$

B1

Answer: confirmed

1 mark

$$(d) \text{ At } t = \frac{\pi}{6}: (1, \frac{3}{2}), \text{ gradient } -3$$

B1

$$y - \frac{3}{2} = -3(x - 1), \text{ so } y = -3x + \frac{9}{2}$$

A1

$$\text{Answer: } y = -3x + \frac{9}{2}$$

2 marks

22 (8 marks)

$$(a) \frac{dy}{dx} = \frac{3t^2}{2t} = \frac{3t}{2} = 3 \text{ at } t = 2$$

M1 A1

$$y - 8 = 3(x - 4), \text{ so } y = 3x - 4$$

A1

Answer: y = 3x - 4

3 marks

$$(b) t^3 = 3t^2 - 4, \text{ so } t^3 - 3t^2 + 4 = 0$$

M1

$$(t - 2)^2(t + 1) = 0 \text{ (} t = 2 \text{ is a repeated root)}$$

M1 A1

$$t = -1: (1, -1)$$

A1

Answer: (1, -1)

4 marks

$$(c) \frac{3t}{2} = -\frac{3}{2}$$

B1

$$\text{Answer: } -\frac{3}{2}$$

1 mark

23 (8 marks)

$$(a) \frac{dy}{dx} = \frac{2t}{2} = t, \text{ so the gradient at P is 2; P is (4, 4)}$$

M1 A1

$$\text{Normal gradient } -\frac{1}{2}: y - 4 = -\frac{1}{2}(x - 4)$$

M1

$$x + 2y = 12$$

A1

Answer: x + 2y = 12

4 marks

$$(b) \text{ Substitute } x = 2t, y = t^2: 2t + 2t^2 = 12$$

M1

$$t^2 + t - 6 = 0, \text{ so } (t - 2)(t + 3) = 0$$

M1 A1

$$t = -3: Q \text{ is } (-6, 9)$$

A1

Answer: Q(-6, 9)

4 marks

24 (9 marks)

$$(a) \frac{dx}{dt} = \frac{3(1+t^3) - 3t \times 3t^2}{(1+t^3)^2} = \frac{3(1-2t^3)}{(1+t^3)^2}$$

M1 A1 A1

$$\frac{dy}{dt} = \frac{6t(1+t^3) - 3t^2 \times 3t^2}{(1+t^3)^2} = \frac{3t(2-t^3)}{(1+t^3)^2}$$

M1 A1

$$\frac{dy}{dx} = \frac{3t(2-t^3)}{3(1-2t^3)} = \frac{t(2-t^3)}{1-2t^3}$$

A1

Answer: Shown

6 marks

$$(b) \text{ Horizontal: } t(2-t^3) = 0, \text{ so } t^3 = 2 \text{ (} t > 0 \text{)}$$

M1 A1

$$x = \frac{3\sqrt[3]{2}}{3} = \sqrt[3]{2}, y = \frac{3\sqrt[3]{4}}{3} = \sqrt[3]{4}: \text{ the point } (\sqrt[3]{2}, \sqrt[3]{4})$$

A1

$$\text{Answer: } (\sqrt[3]{2}, \sqrt[3]{4})$$

3 marks

25 (9 marks)

(a) $\frac{dx}{dt} = 3$ and $\frac{dy}{dt} = -\frac{12}{t^2}$, so $\frac{dy}{dx} = -\frac{4}{t^2}$

M1 A1

Tangent: $y - \frac{12}{t} = -\frac{4}{t^2}(x - 3t)$

M1

Multiply by t^2 : $t^2y - 12t = -4x + 12t$, so $4x + t^2y = 24t$

A1

Answer: Shown

4 marks

(b) Substitute (4, 8): $16 + 8t^2 = 24t$

M1

$t^2 - 3t + 2 = 0$, so $(t - 1)(t - 2) = 0$

A1

$t = 1$: $4x + y = 24$; $t = 2$: $4x + 4y = 48$, so $x + y = 12$

A1 A1 A1

Answer: $4x + y = 24$ and $x + y = 12$

5 marks

26 (8 marks)

(a) $\cos \theta = \frac{x-3}{5}$, $\sin \theta = \frac{y+1}{5}$

M1

$(x-3)^2 + (y+1)^2 = 25(\cos^2 \theta + \sin^2 \theta) = 25$

M1

Circle with centre (3, -1) and radius 5

A1

Answer: Centre (3, -1), radius 5

3 marks

(b) At (6, 3): $5 \cos \theta = 3$ and $5 \sin \theta = 4$, so $\cos \theta = \frac{3}{5}$, $\sin \theta = \frac{4}{5}$

M1 A1

$\frac{dy}{dx} = \frac{5 \cos \theta}{-5 \sin \theta} = -\cot \theta = -\frac{3}{4}$

M1 A1

$y - 3 = -\frac{3}{4}(x - 6)$, so $3x + 4y = 30$

A1

Answer: $3x + 4y = 30$

5 marks

27 (10 marks)

(a) $\frac{dx}{dt} = 1 - \cos t$, $\frac{dy}{dt} = \sin t$

M1 A1

$\frac{dy}{dx} = \frac{\sin t}{1 - \cos t} = \frac{2 \sin(\frac{t}{2}) \cos(\frac{t}{2})}{2 \sin^2(\frac{t}{2})}$

M1

$= \cot(\frac{t}{2})$

A1

Answer: shown

4 marks

(b) $\cot(\frac{\pi}{4}) = 1$

B1

Answer: 1

1 mark

(c) $\frac{dy}{dx} = 0$ so $t = \pi$

M1

$(\pi, 2)$

A1

Answer: (π , 2)

2 marks

(d) At $t = \frac{\pi}{2}$: $x = \frac{\pi}{2} - 1$, $y = 1$

B1

$y - 1 = 1 \times (x - \frac{\pi}{2} + 1)$

M1

$y = x + 2 - \frac{\pi}{2}$

A1

Answer: $y = x + 2 - \frac{\pi}{2}$

3 marks

28 (9 marks)

(a) $t = \frac{x}{20}$

M1

$y = \frac{15x}{20} - \frac{5x^2}{400} = \frac{3}{4}x - \frac{x^2}{80}$

A1

Answer: shown

2 marks

(b) $\frac{dy}{dt} = 15 - 10t$, $\frac{dx}{dt} = 20$

M1

$$\frac{dy}{dx} = \frac{15 - 10t}{20} = \frac{3 - 2t}{4}$$

A1

Answer: $\frac{3 - 2t}{4}$

2 marks

(c) At $t = 0$, gradient $= \frac{3}{4}$

M1

$\tan \theta = \frac{3}{4}$ so $\theta = 36.9^\circ$

A1

Answer: 36.9°

2 marks

(d) $\frac{dy}{dt} = 0$ so $t = 1.5$

M1

$y = 22.5 - 11.25 = 11.25$ m

A1

Answer: 11.25 m

2 marks

(e) Lands at $t = 3$: gradient $= -\frac{3}{4}$, so 36.9° below the horizontal

B1

Answer: 36.9° below horizontal

1 mark

29 (12 marks)

(a) $\frac{dx}{dt} = -3a \cos^2 t \sin t$, $\frac{dy}{dt} = 3a \sin^2 t \cos t$

M1 A1 A1

$\frac{dy}{dx} = \frac{3a \sin^2 t \cos t}{-3a \cos^2 t \sin t} = -\frac{\sin t}{\cos t} = -\tan t$

A1

Answer: Shown

4 marks

(b) $y - a \sin^3 t = -\tan t(x - a \cos^3 t)$

M1

Multiply by $\cos t$: $y \cos t - a \sin^3 t \cos t = -x \sin t + a \sin t \cos^3 t$

M1

$x \sin t + y \cos t = a \sin t \cos t(\sin^2 t + \cos^2 t)$

A1

$= a \sin t \cos t$

A1

Answer: Shown

4 marks

(c) $y = 0$: $x = a \cos t$, so A is $(a \cos t, 0)$

M1 A1

$x = 0$: $y = a \sin t$, so B is $(0, a \sin t)$

A1

$AB^2 = a^2 \cos^2 t + a^2 \sin^2 t = a^2$, so $AB = a$, independent of t

A1

Answer: $AB = a$

4 marks

30 (11 marks)

(a) $\frac{dx}{dt} = \frac{1}{t}$, $\frac{dy}{dt} = 1 - \frac{1}{t^2}$

B1 B1

$\frac{dy}{dx} = (1 - \frac{1}{t^2}) \times t = t - \frac{1}{t}$

A1

Answer: Shown

3 marks

(b) $\frac{d^2y}{dx^2} = \frac{d}{dt}(\frac{dy}{dx}) \div \frac{dx}{dt}$

M1

$\frac{d}{dt}(t - \frac{1}{t}) = 1 + \frac{1}{t^2}$

A1

$(1 + \frac{1}{t^2}) \times t = t + \frac{1}{t}$

M1

$= y$

A1

Answer: Shown

4 marks

(c) $t - \frac{1}{t} = 0$, so $t = 1$ ($t > 0$)

M1 A1

$x = \ln 1 = 0$ and $y = 2$: $(0, 2)$

A1

$\frac{d^2y}{dx^2} = y = 2 > 0$, so a minimum

A1

Answer: $(0, 2)$, minimum

4 marks

Trigonometric, Exponential and Logarithmic Differentiation

1 (2 marks)

$$\frac{d y}{d x} = 3 \cos x - 2(-\sin x)$$

M1

$$= 3 \cos x + 2 \sin x$$

A1

$$\text{Answer: } 3 \cos x + 2 \sin x$$

2 marks

2 (3 marks)

$$\frac{d}{d x}(3e^{4x}) = 12e^{4x}$$

M1 A1

$$\frac{d}{d x}(\ln 2x) = \frac{1}{x}, \text{ so } f'(x) = 12e^{4x} - \frac{1}{x}$$

A1

$$\text{Answer: } 12e^{4x} - \frac{1}{x}$$

3 marks

3 (2 marks)

$$\frac{d y}{d x} = 2 \sec^2 x - 3$$

M1 A1

$$\text{Answer: } 2 \sec^2 x - 3$$

2 marks

4 (2 marks)

$$\frac{d y}{d x} = 6 \times \frac{1}{2} e^{x/2}$$

M1

$$= 3e^{x/2}$$

A1

$$\text{Answer: } 3e^{x/2}$$

2 marks

5 (2 marks)

$$f'(x) = \frac{5}{x} - 2$$

M1 A1

$$\text{Answer: } \frac{5}{x} - 2$$

2 marks

6 (3 marks)

$$\frac{d y}{d x} = 4 \cos 4x - (-2 \sin 2x)$$

M1 A1

$$= 4 \cos 4x + 2 \sin 2x$$

A1

$$\text{Answer: } 4 \cos 4x + 2 \sin 2x$$

3 marks

7 (4 marks)

$$(a) \frac{d y}{d x} = 2 \sec^2 2x$$

M1 A1

$$\text{Answer: } 2 \sec^2 2x$$

2 marks

$$(b) \text{ Using } \frac{d}{d x} \ln f(x) = \frac{f'(x)}{f(x)}: \frac{d y}{d x} = \frac{6x}{3x^2 - 1}$$

M1 A1

$$\text{Answer: } \frac{6x}{3x^2 - 1}$$

2 marks

8 (5 marks)

$$(a) \frac{d y}{d x} = 2e^{2x} - 3 = -1 \text{ at } x = 0$$

M1 A1

$$y = 1 \text{ at } x = 0, \text{ so } y = 1 - x$$

A1

$$\text{Answer: } y = 1 - x$$

3 marks

$$(b) 2e^{2x} = 3, e^{2x} = \frac{3}{2}$$

M1

$$x = \frac{1}{2} \ln\left(\frac{3}{2}\right)$$

A1

$$\text{Answer: } x = \frac{1}{2} \ln\left(\frac{3}{2}\right)$$

2 marks

9 (4 marks)

$$\frac{dy}{dx} = \frac{12}{x} - 2x$$

M1 A1

y is decreasing when $\frac{dy}{dx} < 0$, so $\frac{12}{x} < 2x$; since $x > 0$ this gives $x^2 > 6$

M1

$$x > \sqrt{6}$$

A1

Answer: $x > \sqrt{6}$

4 marks

10 (4 marks)

$$y = 5^x = e^{x \ln 5}$$

M1

$$\frac{dy}{dx} = \ln 5 \times e^{x \ln 5} = 5^x \ln 5$$

M1 A1

$$\text{At } x = 0: \ln 5$$

A1

Answer: $\ln 5$

4 marks

11 (7 marks)

$$(a) \frac{dy}{dx} = 2e^{2x} - 6e^x - 8 = 0, \text{ so } e^{2x} - 3e^x - 4 = 0$$

M1 A1

Answer: Shown

2 marks

$$(b) (e^x - 4)(e^x + 1) = 0$$

M1

$e^x = -1$ is impossible, so $e^x = 4$ and $x = \ln 4$

A1

$$\frac{d^2y}{dx^2} = 4e^{2x} - 6e^x$$

M1

$$\text{At } x = \ln 4: \frac{d^2y}{dx^2} = 64 - 24 = 40 > 0$$

A1

So the stationary point is a minimum

B1

Answer: $x = \ln 4$

5 marks

12 (3 marks)

$$\frac{dy}{dx} = 6e^{-2x}$$

M1 A1

$$3e^{-2x} = 4 - y, \text{ so } \frac{dy}{dx} = 2(4 - y) = 8 - 2y$$

A1

Answer: shown

3 marks

13 (3 marks)

$$\text{Let } u = \sin x: \frac{dy}{dx} = \frac{1}{\sin x} \times \cos x$$

M1 A1

$$= \cot x$$

A1

Answer: Shown

3 marks

14 (6 marks)

$$(a) \text{ Chain rule: } \frac{dy}{dx} = \frac{1}{\cos 2x} \times (-2 \sin 2x)$$

M1 A1

$$= -2 \tan 2x$$

A1

Answer: shown

3 marks

$$(b) -2 \tan\left(\frac{\pi}{4}\right) = -2$$

B1

Answer: -2

1 mark

$$(c) \tan 2x \geq 0 \text{ on the interval so } \frac{dy}{dx} \leq 0: y \text{ is decreasing}$$

E1

$y(0) = \ln 1 = 0$ is therefore the greatest value

E1

Answer: 0 at $x = 0$

2 marks

15 (6 marks)**(a)** Product rule: $u = e^{-x}$, $v = \cos 2x$

M1

$$\frac{d y}{d x} = -e^{-x} \cos 2x + e^{-x}(-2 \sin 2x)$$

A1

$$= -e^{-x}(\cos 2x + 2 \sin 2x)$$

A1

Answer: shown

3 marks

(b) $\cos 2x + 2 \sin 2x = 0$ so $\tan 2x = -\frac{1}{2}$

M1 A1

$$2x = \pi - 0.4636 = 2.678, \text{ so } x = 1.34$$

A1

Answer: $x = 1.34$

3 marks

16 (5 marks)Product rule: $\frac{d y}{d x} = e^x \cos x - e^x \sin x = e^x(\cos x - \sin x)$

M1 A1

Stationary: $\cos x = \sin x$ since $e^x \neq 0$, so $\tan x = 1$

M1 A1

$$x = \frac{\pi}{4} \text{ or } x = \frac{5\pi}{4}$$

A1

Answer: $x = \frac{\pi}{4}, \frac{5\pi}{4}$

5 marks

17 (4 marks)

$$\frac{d y}{d x} = \frac{e^x}{1 + e^x}$$

M1 A1

Divide top and bottom by e^x : $\frac{1}{e^{-x} + 1}$

M1 A1

Answer: Shown

4 marks

18 (5 marks) $\sec x = (\cos x)^{-1}$: $\frac{d}{d x}(\sec x) = -(\cos x)^{-2} \times (-\sin x)$

M1 A1

$$= \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x} \times \frac{\sin x}{\cos x} = \sec x \tan x$$

A1

$$\text{At } x = \frac{\pi}{3}: \sec \frac{\pi}{3} = 2 \text{ and } \tan \frac{\pi}{3} = \sqrt{3}$$

M1

$$\text{Gradient} = 2\sqrt{3}$$

A1

Answer: $\sec x \tan x$; $2\sqrt{3}$

5 marks

19 (3 marks) $a = e^{\ln a}$, so $y = e^{x \ln a}$

M1

$$\frac{d y}{d x} = \ln a \times e^{x \ln a}$$

M1

$$= a^x \ln a$$

A1

Answer: Shown

3 marks

20 (4 marks) $y = \ln(x+1) - \ln(x-1)$ (laws of logarithms)

M1

$$\frac{d y}{d x} = \frac{1}{x+1} - \frac{1}{x-1}$$

M1 A1

$$= \frac{(x-1) - (x+1)}{x^2 - 1} = -\frac{2}{x^2 - 1}$$

A1

Answer: Shown

4 marks

21 (8 marks)

$$\text{(a)} \quad \frac{d y}{d x} = \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}$$

M1 A1 A1

$$= \frac{\sec x(\tan x + \sec x)}{\sec x + \tan x} = \sec x$$

A1

Answer: shown

4 marks

(b) $\sec\left(\frac{\pi}{3}\right) = 2$

B1

Answer: 2

1 mark

(c) $y = \ln(2 + \sqrt{3})$ when $x = \frac{\pi}{3}$

B1

$y - \ln(2 + \sqrt{3}) = 2\left(x - \frac{\pi}{3}\right)$

M1

$y = 2x - \frac{2\pi}{3} + \ln(2 + \sqrt{3})$

A1

Answer: $y = 2x - \frac{2\pi}{3} + \ln(2 + \sqrt{3})$

3 marks

22 (7 marks)

(a) $\ln y = 3 \ln(x - 1) + 2 \ln(2x + 1) - 4 \ln(x + 2)$

M1 A1

$\frac{1}{y} \frac{dy}{dx} = \frac{3}{x-1} + \frac{4}{2x+1} - \frac{4}{x+2}$

M1 A1

Multiply by y to obtain the result

A1

Answer: Shown

5 marks

(b) At $x = 2$: $y = \frac{1 \times 25}{256} = \frac{25}{256}$

B1

Bracket $= 3 + \frac{4}{5} - 1 = \frac{14}{5}$, so $\frac{dy}{dx} = \frac{25}{256} \times \frac{14}{5} = \frac{35}{128}$

A1

Answer: $\frac{35}{128}$

2 marks

23 (9 marks)

(a) $\ln y = \frac{1}{x} \ln x = \frac{\ln x}{x}$

M1

Differentiate: $\frac{1}{y} \frac{dy}{dx} = \frac{\frac{1}{x} \times x - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$

M1 A1

$\frac{dy}{dx} = x^{1/x} \times \frac{1 - \ln x}{x^2}$

A1

Answer: Shown

4 marks

(b) $x^{1/x} > 0$ and $x^2 > 0$, so stationary when $\ln x = 1$, $x = e$

M1 A1

$y = e^{1/e}$

A1

For $x < e$, $\ln x < 1$ so $\frac{dy}{dx} > 0$; for $x > e$, $\frac{dy}{dx} < 0$

M1

The gradient changes from positive to negative: a maximum at $(e, e^{1/e})$

A1

Answer: $(e, e^{1/e})$, maximum

5 marks

24 (10 marks)

(a) Product rule: $u = e^{-x}$, $v = \sin x$

M1

$\frac{dy}{dx} = -e^{-x} \sin x + e^{-x} \cos x$

A1

$= e^{-x}(\cos x - \sin x)$

A1

Answer: Shown

3 marks

(b) $\cos x = \sin x$, so $\tan x = 1$ and $x = \frac{\pi}{4}$

M1 A1

$y = e^{-\pi/4} \times \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2} e^{-\pi/4}$

A1

Answer: $P\left(\frac{\pi}{4}, \frac{\sqrt{2}}{2} e^{-\pi/4}\right)$

3 marks

(c) $\frac{d^2y}{dx^2} = -e^{-x}(\cos x - \sin x) + e^{-x}(-\sin x - \cos x)$

M1 A1

$= e^{-x}(-\cos x + \sin x - \sin x - \cos x) = -2e^{-x} \cos x$

A1

At $x = \frac{\pi}{4}$ this is $-2e^{-\pi/4} \times \frac{\sqrt{2}}{2} < 0$, so a maximum

A1

Answer: Shown

4 marks

25 (9 marks)

(a) $t = 0: T = 20 + 70 = 90\text{ }^{\circ}\text{C}$

B1

As $t \rightarrow \infty, e^{-0.1t} \rightarrow 0$, so $T \rightarrow 20\text{ }^{\circ}\text{C}$

B1

Answer: 90 °C; 20 °C

2 marks

(b) $\frac{dT}{dt} = 70 \times (-0.1)e^{-0.1t} = -7e^{-0.1t}$

M1 A1

$T - 20 = 70e^{-0.1t}$, so $\frac{dT}{dt} = -0.1(T - 20)$

A1

Answer: Shown

3 marks

(c) $-7e^{-0.1t} = -2$, so $e^{-0.1t} = \frac{2}{7}$

M1

$t = 10 \ln 3.5 = 12.5$ minutes

A1

$T = 20 + 70 \times \frac{2}{7}$

M1

$= 40\text{ }^{\circ}\text{C}$

A1

Answer: t = 12.5 min; T = 40 °C

4 marks

26 (9 marks)

(a) $\frac{dy}{dx} = -e^{-x} \sin 3x + 3e^{-x} \cos 3x = e^{-x}(3\cos 3x - \sin 3x)$

M1 A1

$R \cos \alpha = 3$ and $R \sin \alpha = 1$, since $R \cos(3x + \alpha) = R \cos \alpha \cos 3x - R \sin \alpha \sin 3x$

M1

$R = \sqrt{10}$

A1

$\tan \alpha = \frac{1}{3}$

A1

Answer: $R = \sqrt{10}$, $\tan \alpha = \frac{1}{3}$

5 marks

(b) Stationary when $\cos(3x + \alpha) = 0$, so $3x + \alpha = \frac{\pi}{2}$ for the smallest positive x

M1 A1

$3x = \frac{\pi}{2} - \arctan \frac{1}{3} = \arctan 3$, so $x = \frac{1}{3} \arctan 3$

M1 A1

Answer: $x = \frac{1}{3} \arctan 3$

4 marks

27 (9 marks)

(a) $\frac{d}{dx}(x + (x^2 + 1)^{1/2}) = 1 + \frac{x}{\sqrt{x^2 + 1}}$

M1 A1

$\frac{dy}{dx} = \frac{1 + \frac{x}{\sqrt{x^2 + 1}}}{x + \sqrt{x^2 + 1}}$

M1

Numerator = $\frac{\sqrt{x^2 + 1} + x}{\sqrt{x^2 + 1}}$

A1

so $\frac{dy}{dx} = \frac{1}{\sqrt{x^2 + 1}}$

A1

Answer: shown

5 marks

(b) $\frac{1}{\sqrt{4}} = \frac{1}{2}$

B1

Answer: $\frac{1}{2}$

1 mark

(c) $\frac{dy}{dx} = (x^2 + 1)^{-1/2}$

B1

$\frac{d^2y}{dx^2} = -\frac{1}{2}(x^2 + 1)^{-3/2} \times 2x = -x(x^2 + 1)^{-3/2}$

M1

This is negative for $x > 0$, so the gradient is decreasing

A1

Answer: shown

3 marks

28 (10 marks)

(a) The rule $\frac{d}{dx}(x^n) = nx^{n-1}$ applies when the base is the variable and the power is a constant **E1**

Here the variable is in the power **E1**

Answer: wrong rule used **2 marks**

(b) $2^x = e^{x \ln 2}$ **M1**

so $\frac{d}{dx}(2^x) = \ln 2 \times e^{x \ln 2} = 2^x \ln 2$ **A1**

Answer: shown **2 marks**

(c) $\frac{dy}{dx} = 2^x \ln 2 - 2x = f(x)$, which is 0 at a stationary point **B1**

$f(0) = \ln 2 > 0$ and $f(1) = 2 \ln 2 - 2 \approx -0.61 < 0$ **M1**

f is continuous and changes sign, so there is a root in $(0, 1)$ **A1**

Answer: shown **3 marks**

(d) $2^x \ln 2 = 2x$ so $x = \frac{2^x \ln 2}{2} = 2^{x-1} \ln 2$ **B1**

$x_1 = 0.490, x_2 = 0.487$ **M1**

$x_3 = 0.486$ **A1**

Answer: $x_1 = 0.490, x_2 = 0.487, x_3 = 0.486$ **3 marks**

29 (12 marks)

(a) $\frac{dy}{dx} = -e^{-x} \sin 2x + 2e^{-x} \cos 2x = e^{-x}(2\cos 2x - \sin 2x)$ **M1 A1**

$\frac{d^2y}{dx^2} = -e^{-x}(2\cos 2x - \sin 2x) + e^{-x}(-4 \sin 2x - 2\cos 2x)$ **M1 A1**

$= e^{-x}(-3 \sin 2x - 4\cos 2x)$ **A1**

Answer: Shown **5 marks**

(b) $2\frac{dy}{dx} = e^{-x}(4\cos 2x - 2 \sin 2x)$ and $5y = 5e^{-x} \sin 2x$ **M1**

Sum: $e^{-x}(-3 \sin 2x - 4\cos 2x + 4\cos 2x - 2 \sin 2x + 5 \sin 2x)$ **M1**

$= 0$ **A1**

Answer: Shown **3 marks**

(c) Stationary: $2\cos 2x = \sin 2x$, so $\tan 2x = 2$ **M1 A1**

$2x = 1.107\dots, x = 0.554$ **A1**

$y = e^{-0.554} \sin 1.107 = 0.514$; point $(0.554, 0.514)$ **A1**

Answer: (0.554, 0.514) **4 marks**

30 (9 marks)

(a) $\ln y = \sin x \ln x$ **M1**

$\frac{1}{y} \frac{dy}{dx} = \cos x \ln x + \sin x \times \frac{1}{x}$ **M1 A1 A1**

$\frac{dy}{dx} = x^{\sin x}(\cos x \ln x + \frac{\sin x}{x})$ **A1**

Answer: Shown **5 marks**

(b) At $x = \frac{\pi}{2}$: $y = (\frac{\pi}{2})^1 = \frac{\pi}{2}$ **B1**

$\cos \frac{\pi}{2} = 0$ and $\sin \frac{\pi}{2} = 1$, so $\frac{dy}{dx} = \frac{\pi}{2}(0 + \frac{1}{\frac{\pi}{2}}) = 1$ **M1 A1**

Tangent: $y - \frac{\pi}{2} = 1 \times (x - \frac{\pi}{2})$, so $y = x$, which passes through $(0, 0)$ **A1**

Answer: Shown **4 marks**

Connected Rates of Change

1 (3 marks)

$$A = \pi r^2 \text{ so } \frac{dA}{dr} = 2\pi r = 10\pi \quad \text{M1}$$

$$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt} = 10\pi \times 0.2 \quad \text{M1}$$

$$= 2\pi \text{ cm}^2 \text{ s}^{-1} \quad \text{A1}$$

Answer: $2\pi \text{ cm}^2 \text{ s}^{-1}$ 3 marks

2 (3 marks)

$$V = x^3, \frac{dV}{dx} = 3x^2 = 48 \text{ at } x = 4 \quad \text{M1 A1}$$

$$\frac{dV}{dt} = 48 \times 0.5 = 24 \text{ cm}^3 \text{ s}^{-1} \quad \text{A1}$$

Answer: $24 \text{ cm}^3 \text{ s}^{-1}$ 3 marks

3 (3 marks)

$$C = 2\pi r, \text{ so } \frac{dC}{dr} = 2\pi \quad \text{B1}$$

$$\frac{dC}{dt} = \frac{dC}{dr} \times \frac{dr}{dt} = 2\pi \times 0.4 \quad \text{M1}$$

$$= 0.8\pi \text{ cm s}^{-1} \quad \text{A1}$$

Answer: $0.8\pi \text{ cm s}^{-1}$ 3 marks

4 (3 marks)

$$\frac{dP}{dx} = -\frac{60}{x^2}, \text{ and } \frac{dx}{dt} = -0.5 \quad \text{M1 A1}$$

$$\frac{dP}{dt} = -\frac{60}{16} \times (-0.5) = 1.875 \text{ kPa s}^{-1} \text{ (the pressure is increasing)} \quad \text{A1}$$

Answer: 1.875 kPa s^{-1} (increasing) 3 marks

5 (3 marks)

$$\frac{dy}{dx} = 3x^2 + 2 = 5 \text{ at } x = 1 \quad \text{B1}$$

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt} = 5 \times 2 \quad \text{M1}$$

$$= 10 \text{ units per second} \quad \text{A1}$$

Answer: 10 units per second 3 marks

6 (3 marks)

$$\frac{dV}{dr} = 4\pi r^2 \quad \text{B1}$$

$$\frac{dV}{dt} = 4\pi \times 25 \times 0.1 \quad \text{M1}$$

$$= 10\pi \text{ mm}^3 \text{ s}^{-1} \quad \text{A1}$$

Answer: $10\pi \text{ mm}^3 \text{ s}^{-1}$ 3 marks

7 (5 marks)

(a) After 5 s the area is 10 cm^2 ; $\pi r^2 = 10 \quad \text{M1}$

$$r = \sqrt{\frac{10}{\pi}} = 1.78 \text{ cm} \quad \text{A1}$$

Answer: 1.78 cm 2 marks

(b) $A = \pi r^2 \text{ so } \frac{dA}{dt} = 2\pi r \frac{dr}{dt} \quad \text{M1}$

$$2 = 2\pi(1.784) \frac{dr}{dt} \quad \text{M1}$$

$$\frac{dr}{dt} = 0.178 \text{ cm s}^{-1} \quad \text{A1}$$

Answer: 0.178 cm s^{-1} **3 marks****8 (5 marks)**

$$(a) A = \pi r^2, \text{ so } \frac{dA}{dr} = 2\pi r$$

B1

$$\frac{dr}{dt} = \frac{dA}{dt} \div \frac{dA}{dr}$$

M1

$$= \frac{3}{2\pi r}$$

A1**Answer: shown****3 marks**

$$(b) r = \sqrt{\frac{50}{\pi}} = 3.989$$

M1

$$\frac{dr}{dt} = \frac{3}{2\pi \times 3.989} = 0.120 \text{ m s}^{-1}$$

A1**Answer: 0.120 m s^{-1}** **2 marks****9 (4 marks)**

$$\text{With } h = 12: V = 4\pi r^2, \text{ so } \frac{dV}{dr} = 8\pi r$$

M1 A1

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt} = 8\pi \times 3 \times 0.5$$

M1

$$= 12\pi \text{ cm}^3 \text{ s}^{-1}$$

A1**Answer: $12\pi \text{ cm}^3 \text{ s}^{-1}$** **4 marks****10 (5 marks)**

$$V = x^3: \frac{dV}{dt} = 3x^2 \frac{dx}{dt}$$

M1

$$24 = 12 \frac{dx}{dt}, \text{ so } \frac{dx}{dt} = 2 \text{ cm s}^{-1}$$

A1

$$S = 6x^2: \frac{dS}{dx} = 12x$$

B1

$$\frac{dS}{dt} = 12x \times \frac{dx}{dt} = 24 \times 2$$

M1

$$= 48 \text{ cm}^2 \text{ s}^{-1}$$

A1**Answer: (a) 2 cm s^{-1} (b) $48 \text{ cm}^2 \text{ s}^{-1}$** **5 marks****11 (4 marks)**

$$\frac{dy}{dx} = 2x - \frac{1}{x^2}$$

M1 A1

$$\text{At } x = 2: 4 - \frac{1}{4} = \frac{15}{4}$$

B1

$$\frac{dy}{dt} = \frac{15}{4} \times 3 = \frac{45}{4} \text{ units per second}$$

A1**Answer: $\frac{45}{4}$ units per second****4 marks****12 (3 marks)**

$$V = \pi \times 10^2 h = 100\pi h, \text{ so } \frac{dV}{dh} = 100\pi$$

B1

$$200 = 100\pi \times \frac{dh}{dt}$$

M1

$$\frac{dh}{dt} = \frac{2}{\pi} \text{ cm s}^{-1}$$

A1**Answer: $\frac{2}{\pi} \text{ cm s}^{-1} \approx 0.637 \text{ cm s}^{-1}$** **3 marks**

13 (5 marks)

$$\frac{dS}{dt} = 8\pi r \frac{dr}{dt}, \text{ so } 8\pi = 32\pi \frac{dr}{dt}$$

M1 A1

$$\frac{dr}{dt} = \frac{1}{4} \text{ cm s}^{-1}$$

A1

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} = 4\pi \times 16 \times \frac{1}{4}$$

M1

$$= 16\pi \text{ cm}^3 \text{ s}^{-1}$$

A1

Answer: (a) $\frac{1}{4} \text{ cm s}^{-1}$ (b) $16\pi \text{ cm}^3 \text{ s}^{-1}$

5 marks

14 (6 marks)

(a) Radius of the surface $r = h \tan 30^\circ = \frac{h}{\sqrt{3}}$

B1

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \times \frac{h^2}{3} \times h$$

M1

$$= \frac{\pi h^3}{9}$$

A1

Answer: shown

3 marks

(b) $\frac{dV}{dh} = \frac{\pi h^2}{3} = 12\pi$ at $h = 6$

M1

$$\frac{dh}{dt} = \frac{8}{12\pi}$$

M1

$$= \frac{2}{3\pi} = 0.212 \text{ cm s}^{-1}$$

A1

Answer: 0.212 cm s^{-1}

3 marks

15 (7 marks)

(a) $x^2 + y^2 = 25$

B1

Differentiate with respect to t : $2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$

M1 A1

When $x = 3$: $y = 4$

B1

$$\frac{dx}{dt} = 0.4 \text{ so } 2(3)(0.4) + 2(4)\frac{dy}{dt} = 0$$

M1 A1

$$\frac{dy}{dt} = -0.3, \text{ so the top moves down at } 0.3 \text{ m s}^{-1}$$

A1

Answer: 0.3 m s^{-1} downwards

7 marks

16 (6 marks)

$$V = x^3, \text{ so } \frac{dV}{dt} = 3x^2 \frac{dx}{dt}$$

M1

$$-6 = 27 \frac{dx}{dt}, \text{ so } \frac{dx}{dt} = -\frac{2}{9} \text{ cm per minute}$$

A1

$$S = 6x^2, \text{ so } \frac{dS}{dt} = 12x \frac{dx}{dt}$$

M1 A1

$$= 36 \times \left(-\frac{2}{9}\right) = -8$$

M1

The area is decreasing at 8 cm^2 per minute

A1

Answer: 8 cm^2 per minute

6 marks

17 (5 marks)

(a) Cross-sectional area $= \frac{1}{2} \times 2h \times h = h^2$

M1

$$V = 3 \times h^2 = 3h^2$$

A1

Answer: Shown

2 marks

(b) $\frac{dV}{dh} = 6h = 3$ when $h = 0.5$

M1

$$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}, \text{ so } 0.06 = 3 \frac{dh}{dt}$$

M1

$$\frac{dh}{dt} = 0.02 \text{ m per minute}$$

A1

Answer: 0.02 m per minute

3 marks

18 (4 marks)

$$A = \pi r^2, \text{ so } \frac{dA}{dr} = 2\pi r$$

B1

$$\frac{dr}{dt} = 2; \text{ when } t = 3, r = 7$$

B1

$$\frac{dA}{dt} = 2\pi \times 7 \times 2 = 28\pi$$

M1

$$= 88.0 \text{ m}^2 \text{ s}^{-1}$$

A1

Answer: 88.0 m² s⁻¹

4 marks

19 (5 marks)

$$\text{(a) In triangle LOP, } \tan \theta = \frac{x}{100}$$

B1

Answer: Shown

1 mark

$$\text{(b) } \frac{dx}{d\theta} = 100 \sec^2 \theta$$

M1 A1

$$\frac{dx}{dt} = \frac{dx}{d\theta} \times \frac{d\theta}{dt} = 100 \sec^2 \theta \times 0.05$$

M1

$$\text{At } \theta = \frac{\pi}{4}: \sec^2 \theta = 2, \text{ so } \frac{dx}{dt} = 10 \text{ m s}^{-1}$$

A1

Answer: 10 m s⁻¹

4 marks

20 (5 marks)

$$V = \frac{4}{3}\pi r^3, \text{ so } \frac{dV}{dr} = 4\pi r^2$$

B1

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}, \text{ so } 40\pi = 4\pi r^2 \frac{dr}{dt}$$

M1

$$\frac{dr}{dt} = \frac{10}{r^2}$$

A1

$$0.1 = \frac{10}{r^2}, \text{ so } r^2 = 100$$

M1

$$r = 10 \text{ cm}$$

A1

Answer: r = 10 cm

5 marks

21 (10 marks)

$$\text{(a) } \frac{dr}{dt} = 2e^{-t/2}$$

M1 A1

$$A = \pi r^2 \text{ so } \frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

M1

$$= 2\pi \times 4(1 - e^{-t/2}) \times 2e^{-t/2} = 16\pi(e^{-t/2} - e^{-t})$$

M1 A1

Answer: Shown

5 marks

$$\text{(b) } \frac{d^2A}{dt^2} = 16\pi(-\frac{1}{2}e^{-t/2} + e^{-t}) = 0$$

M1 A1

$$e^{-t/2}(e^{-t/2} - \frac{1}{2}) = 0, \text{ so } e^{-t/2} = \frac{1}{2} \text{ and } t = 2 \ln 2$$

A1

Second derivative changes from positive to negative, so a maximum rate

B1

$$\frac{dA}{dt} = 16\pi(\frac{1}{2} - \frac{1}{4}) = 4\pi \text{ cm}^2 \text{ s}^{-1}$$

A1

Answer: t = 2 ln 2 seconds; 4π cm² s⁻¹

5 marks

22 (8 marks)

$$\text{(a) } A = xy, \text{ so } \frac{dA}{dt} = y \frac{dx}{dt} + x \frac{dy}{dt}$$

M1 A1

$$= 10 \times 2 + 6 \times (-1) = 14 \text{ cm}^2 \text{ s}^{-1}$$

A1

Answer: 14 cm² s⁻¹

3 marks

$$\text{(b) } P = 2x + 2y: 2(2) + 2(-1) = 2 \text{ cm s}^{-1}$$

B1

Answer: 2 cm s^{-1} **1 mark**

$$(c) d^2 = x^2 + y^2, \text{ so } 2d \frac{d d}{d t} = 2x \frac{d x}{d t} + 2y \frac{d y}{d t}$$

M1

$$d = \sqrt{136}; \frac{d d}{d t} = \frac{6(2) + 10(-1)}{\sqrt{136}}$$

A1

$$= \frac{2}{\sqrt{136}} = 0.171 \text{ cm s}^{-1}$$

A1**Answer: 0.171 cm s^{-1}** **3 marks**

$$(d) x = 6 + 2t, y = 10 - t: \frac{d A}{d t} = 14 - 4t = 0, t = 3.5 \text{ s}$$

B1**Answer: 3.5 s** **1 mark****23 (8 marks)**

$$(a) \text{ Triangles with the lamp and the man are similar: } \frac{s}{s+x} = \frac{1.8}{4}$$

M1 A1

$$4s = 1.8s + 1.8x, \text{ so } 2.2s = 1.8x \text{ and } s = \frac{9}{11}x$$

A1**Answer: Shown****3 marks**

$$(b) \frac{d s}{d t} = \frac{9}{11} \frac{d x}{d t}$$

M1

$$= \frac{9}{11} \times 1.5 = \frac{27}{22} \text{ m s}^{-1}$$

A1

$$\text{Answer: } \frac{27}{22} \text{ m s}^{-1} \approx 1.23 \text{ m s}^{-1}$$

2 marks

$$(c) \text{ Distance of tip from the lamp} = x + s = \frac{20}{11}x$$

M1 A1

$$\text{Speed} = \frac{20}{11} \times 1.5 = \frac{30}{11} \text{ m s}^{-1}$$

A1

$$\text{Answer: } \frac{30}{11} \text{ m s}^{-1} \approx 2.73 \text{ m s}^{-1}$$

3 marks**24 (8 marks)**

$$(a) V = \pi r^2 h: \frac{d V}{d t} = \pi(2rh \frac{d r}{d t} + r^2 \frac{d h}{d t})$$

M1 A1

$$= \pi(2 \times 5 \times 12 \times 0.02 + 25 \times (-0.05))$$

M1

$$= \pi(2.4 - 1.25) = 1.15\pi \approx 3.61 \text{ cm}^3 \text{ s}^{-1}$$

A1

Positive, so the volume is increasing

A1**Answer: $1.15\pi \approx 3.61 \text{ cm}^3 \text{ s}^{-1}$, increasing****5 marks**

$$(b) \frac{d A}{d t} = 2\pi(h \frac{d r}{d t} + r \frac{d h}{d t})$$

M1 A1

$$= 2\pi(12 \times 0.02 - 5 \times 0.05) = 2\pi(-0.01) = -0.02\pi \text{ cm}^2 \text{ s}^{-1}$$

A1**Answer: $-0.02\pi \text{ cm}^2 \text{ s}^{-1}$** **3 marks****25 (8 marks)**

$$(a) 2x + y + x \frac{d y}{d x} + 2y \frac{d y}{d x} = 0, \text{ so } \frac{d y}{d x} = -\frac{2x+y}{x+2y}$$

M1 A1

$$\text{At } (1, 2): \frac{d y}{d x} = -\frac{4}{5}$$

A1

$$\frac{d y}{d t} = \frac{d y}{d x} \times \frac{d x}{d t} = -\frac{4}{5} \times 0.5 = -0.4 \text{ units per second}$$

M1 A1**Answer: $-0.4 \text{ units per second}$** **5 marks**

$$(b) \frac{d y}{d t} = 0 \text{ requires } 2x + y = 0, \text{ so } y = -2x$$

M1

$$x^2 - 2x^2 + 4x^2 = 7, \text{ so } 3x^2 = 7$$

M1

$$x = \pm \sqrt{\frac{7}{3}}$$

A1

$$\text{Answer: } x = \pm \sqrt{\frac{7}{3}}$$

3 marks

26 (7 marks)

$$(a) \tan \theta = \frac{h}{50}, \text{ so } \sec^2 \theta \frac{d\theta}{dt} = \frac{1}{50} \frac{dh}{dt}$$

M1 A1

$$\frac{d\theta}{dt} = \frac{2}{50} \cos^2 \theta = \frac{1}{25} \cos^2 \theta$$

A1

Answer: Shown

3 marks

$$(b) h = 50: \tan \theta = 1, \theta = \frac{\pi}{4}, \cos^2 \theta = \frac{1}{2}$$

M1

$$\frac{d\theta}{dt} = \frac{1}{50} \text{ rad s}^{-1}$$

A1

$$\frac{d\theta}{dt} \text{ is greatest when } \cos^2 \theta \text{ is greatest, i.e. } \theta = 0$$

M1

$$(\text{at the start, when the balloon is on the ground}), \text{ where the rate is } \frac{1}{25} \text{ rad s}^{-1}$$

A1

$$\text{Answer: } \frac{1}{50} \text{ rad s}^{-1}; \text{ greatest at } \theta = 0$$

4 marks

27 (10 marks)

$$(a) \frac{dV}{dh} = \pi(20h - h^2)$$

M1

$$= \pi h(20 - h)$$

A1

Answer: shown

2 marks

$$(b) \frac{dV}{dh} = \pi \times 5 \times 15 = 75\pi$$

M1

$$\frac{dh}{dt} = \frac{30}{75\pi}$$

M1

$$= \frac{2}{5\pi} = 0.127 \text{ cm s}^{-1}$$

A1

$$\text{Answer: } 0.127 \text{ cm s}^{-1}$$

3 marks

$$(c) \frac{dh}{dt} = \frac{30}{\pi h(20 - h)}, \text{ smallest when } h(20 - h) \text{ is greatest}$$

M1

$$h(20 - h) \text{ is greatest at } h = 10$$

A1

$$\frac{30}{100\pi} = 0.0955 \text{ cm s}^{-1}$$

A1

$$\text{Answer: } h = 10 \text{ cm, } 0.0955 \text{ cm s}^{-1}$$

3 marks

$$(d) \text{ Net inflow} = 10 \text{ cm}^3 \text{ s}^{-1}$$

M1

$$\frac{dh}{dt} = \frac{10}{75\pi} = 0.0424 \text{ cm s}^{-1}$$

A1

$$\text{Answer: } 0.0424 \text{ cm s}^{-1}$$

2 marks

28 (10 marks)

$$(a) OP^2 = x^2 + x^4$$

B1

$$2(OP) \frac{d(OP)}{dt} = (2x + 4x^3) \frac{dx}{dt}$$

M1 A1

$$\text{At } x = 2: OP = 2\sqrt{5}, \text{ so } 4\sqrt{5} \frac{d(OP)}{dt} = 36 \times 2, \text{ giving } \frac{d(OP)}{dt} = \frac{18}{\sqrt{5}} = 8.05 \text{ units s}^{-1}$$

A1

$$\text{Answer: } \frac{18}{\sqrt{5}} = 8.05 \text{ units s}^{-1}$$

4 marks

$$(b) \frac{dy}{dt} = 2x \frac{dx}{dt}$$

M1

$$= 2(2)(2) = 8 \text{ units s}^{-1}$$

A1

$$\text{Answer: } 8 \text{ units s}^{-1}$$

2 marks

$$(c) \tan \theta = \frac{y}{x} = \frac{x^2}{x} = x$$

B1

$$\sec^2 \theta \frac{d\theta}{dt} = \frac{dx}{dt} = 2$$

M1

$$\sec^2 \theta = 1 + \tan^2 \theta = 5 \text{ when } x = 2$$

A1

$$\frac{d\theta}{dt} = \frac{2}{5} = 0.4 \text{ rad s}^{-1}$$

A1

Answer: 0.4 rad s^{-1} **4 marks****29 (11 marks)**

(a) $\frac{4}{3}\pi r^3 = 36\pi t^2$, so $r^3 = 27t^2$

M1 A1

$r = 3t^{2/3}$

A1**Answer: Shown****3 marks**

(b) $\frac{dr}{dt} = 3 \times \frac{2}{3}t^{-1/3} = 2t^{-1/3}$

M1 A1

At $t = 8$: $2 \times \frac{1}{2} = 1 \text{ cm s}^{-1}$

A1**Answer: 1 cm s^{-1}** **3 marks**

(c) $S = 4\pi \times 9t^{4/3} = 36\pi t^{4/3}$, so $\frac{dS}{dt} = 48\pi t^{1/3}$

M1 A1

At $t = 8$: $96\pi \text{ cm}^2 \text{ s}^{-1}$

A1

$\frac{dV}{dt} = 72\pi t$, so the ratio = $\frac{48\pi t^{1/3}}{72\pi t} = \frac{2}{3}t^{-2/3}$

M1

$r = 3t^{2/3}$ gives $t^{-2/3} = \frac{3}{r}$, so the ratio = $\frac{2}{3} \times \frac{3}{r} = \frac{2}{r}$

A1**Answer: $96\pi \text{ cm}^2 \text{ s}^{-1}$; shown****5 marks****30 (13 marks)**

(a) Width = $20 - x$

B1

$A = x(20 - x) = 20x - x^2$, so $\frac{dA}{dx} = 20 - 2x$

M1 A1

$\frac{dA}{dt} = (20 - 2x) \times 0.5 = 10 - x$

A1**Answer: Shown****4 marks**

(b) $\frac{dA}{dt} = 0$ when $x = 10$

M1For $x < 10$ the area is increasing and for $x > 10$ it is decreasing**M1**So the area is a maximum (a square) at $x = 10$ **A1****Answer: $x = 10$ (a square)****3 marks**

(c) $D^2 = x^2 + (20 - x)^2$, so $2D \frac{dD}{dt} = (2x - 2(20 - x)) \frac{dx}{dt}$

M1 A1

$= (4x - 40) \times 0.5 = 2x - 20$

A1

$x = 6$: $D = \sqrt{36 + 196} = \sqrt{232}$, so $\frac{dD}{dt} = \frac{-8}{2\sqrt{232}} = -\frac{4}{\sqrt{232}} = -\frac{2}{\sqrt{58}} \approx -0.263 \text{ cm s}^{-1}$

M1 A1 $\frac{dD}{dt} = 0$ only when $x = 10$, negative for $x < 10$ and positive for $x > 10$: D is least at $x = 10$, the same value at which A is greatest**A1****Answer: $-\frac{2}{\sqrt{58}} \approx -0.263 \text{ cm s}^{-1}$; shown****6 marks**