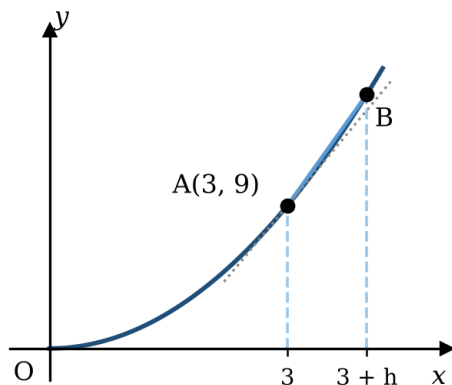
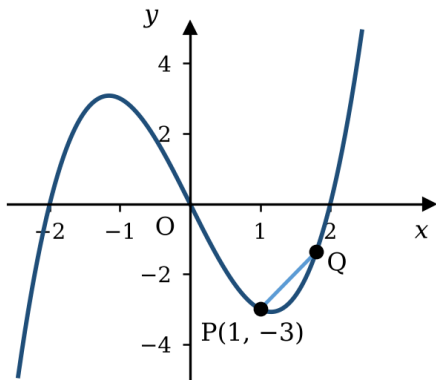


First Principles

- 1 The point A(3, 9) and the point B with x-coordinate $3 + h$ lie on the curve $y = x^2$, as shown.



- (a) Show that the gradient of the chord AB is $6 + h$. [2]
 - (b) State what the gradient of AB tends to as $h \rightarrow 0$, and explain what this represents. [2]
- 2 Given that $f(x) = 5x^2$, find $f'(x)$ from first principles. [3]
- 3 Let $f(x) = x^2$. Copy and complete the table of gradients of the chord joining the points on the curve with $x = 3$ and $x = 3 + h$, and say what value the gradient approaches as h gets smaller.
 $h = 0.1, h = 0.01, h = 0.001$ [3]
- 4 Differentiate $f(x) = 5x - 2$ from first principles, and explain why your answer is what you would expect. [4]
- 5 The points A(1, 4) and B(1 + h, $f(1 + h)$) lie on the curve $y = f(x)$, where $f(x) = x^2 + 3x$. Show that the gradient of the chord AB is $5 + h$, and state the gradient of the curve at A. [3]
- 6 Differentiate $f(x) = -x^2 + 2x$ from first principles. [4]
- 7 Differentiate $f(x) = x^3$ from first principles. [4]
- 8 Use differentiation from first principles to show that the derivative of $f(x) = 2x^2 - 3x + 1$ is $4x - 3$. [4]
- 9 Use the definition of the derivative to find the gradient of the curve $y = x^2 + 5x - 1$ at the point where $x = -2$. [4]
- 10 The points P(2, 4) and Q(2 + h, 2^{2+h}) lie on the curve $y = 2^x$.
 - (a) Calculate the gradient of the chord PQ for $h = 0.1, h = 0.01$ and $h = 0.001$, giving each answer to 4 decimal places. [3]
 - (b) Hence estimate the gradient of the curve at P, giving your answer to 2 decimal places. [2]
 - 11 Prove from first principles that the derivative of $f(x) = \frac{2}{x}$ is $-\frac{2}{x^2}$. [4]
 - 12 Use differentiation from first principles to find $f'(x)$ when $f(x) = (x + 3)^2$. [4]
 - 13 The points P(x, y) and Q(x + δx , y + δy) lie on the curve $y = x^2 + x$.
 - (a) Find an expression for δy in terms of x and δx .
 - (b) Hence find $\frac{dy}{dx}$ by considering the limit of $\frac{\delta y}{\delta x}$ as $\delta x \rightarrow 0$. [5]

- 14** Prove from first principles that the derivative of $f(x) = \frac{1}{x}$ is $-\frac{1}{x^2}$. [5]
- 15** Show, from first principles, that the derivative of $f(x) = \sqrt{x}$ is $\frac{1}{2\sqrt{x}}$ for $x > 0$. [5]
- 16** Use first principles to find the gradient of the curve $y = x^3$ at the point $(2, 8)$, and hence find an equation of the tangent to the curve at this point. [6]
- 17** Show, from first principles, that the derivative of $f(x) = \sqrt{2x + 1}$ is $\frac{1}{\sqrt{2x + 1}}$ for $x > -\frac{1}{2}$. [5]
- 18** Let $f(x) = ax^2 + bx + c$, where a , b and c are constants. Prove from first principles that $f'(x) = 2ax + b$. [5]
- 19** The derivative of f at $x = a$ can be defined as $f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$. Use this definition to prove that if $f(x) = x^4$ then $f'(a) = 4a^3$. [4]
- 20** Use first principles to show that the gradient function of $y = x^3 - 3x$ is $3x^2 - 3$, and hence find the coordinates of the points on the curve where the tangent is parallel to the x -axis. [7]
- 21** The curve C has equation $y = x^3 - 4x$. The points $P(1, -3)$ and Q , with x -coordinate $1 + h$, lie on C .
- 
- (a) Show that the gradient of the chord PQ is $h^2 + 3h - 1$. [3]
- (b) Hence state the gradient of C at P . [1]
- (c) Use first principles to find the gradient function of C , and hence the coordinates of the points on C where the gradient is 8. [5]
- 22** A curve has equation $f(x) = \frac{1}{x^2}$, $x \neq 0$.
- (a) Prove from first principles that $f'(x) = -\frac{2}{x^3}$. [6]
- (b) Find the gradient of the curve where $x = -\frac{1}{2}$. [2]
- 23** Prove from first principles that the derivative of $f(x) = 2x^3 - \frac{3}{x}$ is $6x^2 + \frac{3}{x^2}$. [6]
- 24** Prove from first principles that the derivative of $f(x) = \frac{1}{\sqrt{x}}$ is $-\frac{1}{2x\sqrt{x}}$, for $x > 0$. [6]

- 25** Let $f(x) = \sin x$, where x is in radians.
 (a) Calculate the gradient of the chord joining the points with $x = 1$ and $x = 1.001$. Give your answer to 4 decimal places.
 (b) Use your answer to (a) to estimate the value of $\cos 1$, and comment on the accuracy of the estimate. [4]
- 26** Prove from first principles that the derivative of $f(x) = \frac{x}{x+1}$ is $\frac{1}{(x+1)^2}$, for $x \neq -1$. [6]
- 27** Starting from the definition of the derivative, prove that if $f(x) = \sin x$, where x is in radians, then $f'(x) = \cos x$.
 (a) Use $\sin(x+h) = \sin x \cos h + \cos x \sin h$ and the small angle approximations for h in radians ($\cos h \approx 1 - \frac{h^2}{2}$, $\sin h \approx h$) to prove that $f'(x) = \cos x$. [6]
 (b) Explain why this proof only works if x is measured in radians, and state what the derivative of $\sin x$ would be if x were in degrees. [2]
- 28** A student attempts to differentiate $f(x) = x^2$ from first principles and writes:
 (a) $f'(x) = \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \rightarrow 0} \frac{x^2 + h^2 - x^2}{h} = \lim_{h \rightarrow 0} h = 0$
 Identify the error in the student's working and write down the correct value of $f'(x)$. [2]
 (b) Use the binomial theorem to prove from first principles that, for any positive integer n , if $f(x) = x^n$ then $f'(x) = nx^{n-1}$. [5]
 (c) Hence show that, for every integer $n \geq 2$, the curve $y = x^n - nx$ has a stationary point at $x = 1$, and find its y -coordinate. [3]
- 29** You may use the results $\frac{e^h - 1}{h} \rightarrow 1$ and $e^h \rightarrow 1$ as $h \rightarrow 0$.
 (a) Prove from first principles that the derivative of $f(x) = e^x$ is e^x . [3]
 (b) Prove from first principles that the derivative of $g(x) = xe^x$ is $(x+1)e^x$. [5]
- 30** The curve C has equation $y = x^3$. The tangent to C at the point $P(p, p^3)$, $p \neq 0$, meets C again at Q .
 (a) Use first principles to show that the gradient of C at P is $3p^2$. [3]
 (b) Show that the x -coordinate of Q is $-2p$. [5]

Basic Differentiation

- 1** Find $\frac{dy}{dx}$ when $y = 4x^3 - 6x^2 + 5$. [2]
- 2** Given that $f(x) = 2x^5 - x^2 + 7x$, find $f'(x)$ and $f''(x)$. [3]
- 3** The surface area $S \text{ cm}^2$ of a sphere of radius $r \text{ cm}$ is given by $S = 4\pi r^2$. Find the rate of change of S with respect to r when $r = 5$, giving your answer in terms of π . [3]
- 4** Given that $f(x) = 3x^2 - 8x + 1$, find $f'(x)$ and the value of $f'(3)$. [3]
- 5** Differentiate $y = 6\sqrt{x}$ with respect to x . [2]
- 6** Find $\frac{dy}{dx}$ when $y = \frac{5}{x^3}$. [2]
- 7** Find $\frac{dy}{dx}$ when $y = 3\sqrt{x} - \frac{4}{x^2}$. [3]

- 8** Find $\frac{dy}{dx}$ when $y = \frac{(2x+3)(x-1)}{x^2}$. [4]
-
- 9** Differentiate with respect to t :
 (a) $s = 5t^3 - 4t + 7$ [2]
 (b) $v = ut + \frac{1}{2}at^2$, where u and a are constants. [2]
-
- 10** Given that $y = \frac{3x^3 - 2x}{x^2}$, $x \neq 0$, find $\frac{dy}{dx}$. [3]
-
- 11** Find $\frac{dy}{dx}$ when $y = 2x^{3/2} - 4x^{-1/2}$, $x > 0$. Give each term in your answer in the form kx^n . [3]
-
- 12** The curve C has equation $y = x^3 - 4x^2 + 7x - 2$.
 (a) Find the gradient of C at the point where $x = 2$. [2]
 (b) Find the values of x at which the gradient of C is 7. [3]
-
- 13** Given that $y = x^4 - 6x^2$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. Find the value of $\frac{d^2y}{dx^2}$ when $x = -1$. [4]
-
- 14** The curve C has equation $y = x^3 - 6x^2 + 9x$. Find the set of values of x for which y is decreasing. [5]
-
- 15** The depth, h metres, of water in a reservoir t hours after it starts to fill is modelled by $h = 0.02t^3 - 0.3t^2 + 1.5t$, for $0 \leq t \leq 10$.
 (a) Find $\frac{dh}{dt}$. [2]
 (b) Show that the depth never decreases during the first 10 hours. [3]
 (c) State when the depth is increasing most slowly and the rate at that time. [1]
-
- 16** Given that $y = \frac{(2x-1)^2}{\sqrt{x}}$, $x > 0$, find $\frac{dy}{dx}$, giving your answer in the form $ax^{1/2} + bx^{-1/2} + cx^{-3/2}$, where a , b and c are constants. [5]
-
- 17** Find the set of values of x for which $f(x) = \frac{x^3}{3} - \frac{5x^2}{2} + 6x$ is an increasing function. [4]
-
- 18** The curve C has equation $y = x^3 - 6x^2 + 5x$. Find the minimum value of the gradient of C , and the coordinates of the point on C where this occurs. [5]
-
- 19** The gradient of the curve $y = kx^2 - 6x + 1$ at the point where $x = 2$ is 10, where k is a constant. Find the value of k , and the coordinates of the point on the curve where the gradient is zero. [5]
-
- 20** The population P of a colony of birds on an island, t years after observations began, is modelled by $P = 500 + 40t - \frac{1}{2}t^2$, for $0 \leq t \leq 50$. Find the rate of change of the population when $t = 10$, and the time at which the population stops increasing. [5]
-
- 21** The curve C has equation $y = ax^2 + \frac{b}{x}$, where a and b are constants. The point $(1, 5)$ lies on C and the gradient of C at this point is 1.
 (a) Find the value of a and the value of b . [5]
 (b) Find $\frac{d^2y}{dx^2}$ and its value at $x = 1$. [2]
 (c) State, with a reason, whether the gradient of C is increasing or decreasing at $x = 1$. [1]

- 22** For $x > 0$, $y = (\sqrt{x} + \frac{2}{\sqrt{x}})^2$.
- (a) Show that $y = x + 4 + \frac{4}{x}$. [2]
- (b) Find $\frac{dy}{dx}$. [2]
- (c) Find the value of x at which the gradient of the curve is $\frac{3}{4}$. [3]
-
- 23** For $x > 0$, $y = \frac{x^2 + 4x - 8}{2\sqrt{x}}$.
- (a) Express y in the form $ax^p + bx^q + cx^r$, where a , b , c , p , q and r are constants. [2]
- (b) Find $\frac{dy}{dx}$. [2]
- (c) Find the gradient of the curve at the point where $x = 4$, and show that y is an increasing function of x for all $x > 0$. [4]
-
- 24** For $x > 0$, $y = 4x^{3/2} - 12x^{1/2} + 5$.
- (a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. [4]
- (b) Find the value of x for which $\frac{dy}{dx} = 0$, and show that $\frac{d^2y}{dx^2} > 0$ for all $x > 0$. [3]
-
- 25** The function f is given by $f(x) = ax^3 + bx^2 + 2x$, where a and b are constants. Given that $f'(1) = 2$ and $f'(2) = 18$, find the values of a and b . [6]
-
- 26** Water flows into a tank. The volume of water, V litres, in the tank t minutes after the flow starts is modelled by $V = 6t^2 - \frac{1}{3}t^3$, for $0 \leq t \leq 12$.
- (a) Find an expression for the rate at which water is entering the tank. [2]
- (b) Find the time at which the rate of flow is greatest, and the greatest rate. Justify that it is a maximum. [5]
-
- 27** The function f is defined by $f(x) = x^3 - 3x^2 + 5x + 1$, for $x \in \mathbb{R}$.
- (a) Show that f is an increasing function for all real x . [3]
- (b) Hence explain why the equation $f(x) = 0$ has exactly one real root. [2]
- (c) The function g is given by $g(x) = x^3 - 3x^2 + px + 1$, where p is a constant. Find the set of values of p for which $g'(x) > 0$ for all x . [4]
-
- 28** A student is asked to differentiate $y = \frac{4}{3x^2}$ and writes: " $y = \frac{4}{3x^2}$, so $\frac{dy}{dx} = \frac{4}{6x}$ ".
- (a) Describe two errors in the student's working. [2]
- (b) Find the correct value of $\frac{dy}{dx}$. [3]
- (c) Find the value of x for which the gradient is $-\frac{1}{3}$. [2]
- (d) Show that the gradient of the curve is increasing for all $x > 0$. [2]
-
- 29** The curve C has equation $y = x^3 + px^2 + qx + r$, where p , q and r are constants.
- (a) Show that C has no stationary points if $p^2 < 3q$. [5]
- (b) Given that $p = 2$ and $q = 5$, find the minimum value of $\frac{dy}{dx}$, and state the corresponding value of x . [3]

30 Throughout this question $x > 0$.

(a) Given that $y = x^n$, where n is a constant, show that $x^2 \frac{d^2y}{dx^2} = n(n-1)y$. [3]

(b) Find the two values of n for which $x^2 \frac{d^2y}{dx^2} = 12y$. [3]

(c) Hence show that $y = x^4 + x^{-3}$ satisfies $x^2 \frac{d^2y}{dx^2} = 12y$. [3]

Tangents and Normals

1 Find an equation of the tangent to the curve $y = x^2 + 3x - 2$ at the point where $x = 2$. Give your answer in the form $y = mx + c$. [3]

2 Find an equation of the normal to the curve $y = x^2 - 4x + 1$ at the point $(3, -2)$, giving your answer in the form $ax + by + c = 0$, where a , b and c are integers. [3]

3 Find the gradient of the curve $y = x^3 - 2x$ at the point where $x = 2$. [2]

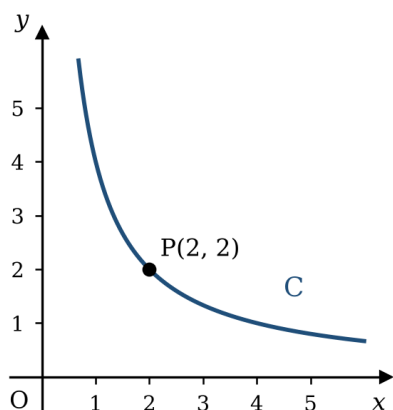
4 Find an equation of the tangent to the curve $y = x^3 - 4x + 1$ at the point where the curve crosses the y -axis. [3]

5 Find the gradient of the normal to the curve $y = x^3 + 1$ at the point where $x = 1$. [3]

6 Find an equation of the tangent to the curve $y = 5 - x^2$ at the point where $x = -1$. [3]

7 The line $y = 6x + c$, where c is a constant, is a tangent to the curve $y = x^2 + 2x + 5$. Find the value of c and the coordinates of the point where the line touches the curve. [5]

8 The diagram shows part of the curve C with equation $y = \frac{4}{x}$ and the point $P(2, 2)$ on C .



(a) Find an equation of the tangent to C at P . [3]

(b) Find an equation of the normal to C at P . [2]

9 Find an equation of the tangent to the curve $y = 2\sqrt{x}$ at the point where $x = 4$. Give your answer in the form $ax + by + c = 0$, where a , b and c are integers. [4]

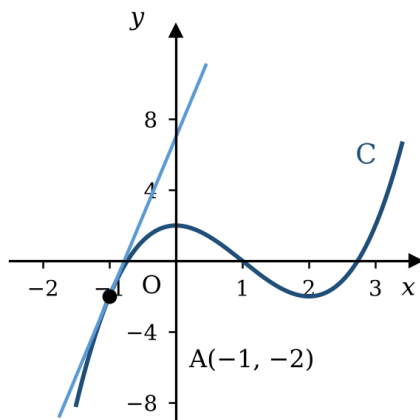
10 Find an equation of the normal to the curve $y = x + \frac{1}{x}$ at the point where $x = 2$. Give your answer in the form $ax + by + c = 0$, where a , b and c are integers. [5]

11 The normal to the curve $y = x^2 - 3x + 5$ at the point $A(2, 3)$ meets the x -axis at B . Find the coordinates of B . [4]

12 Find the coordinates of the points on the curve $y = x^3 - 6x^2 - 6x$ at which the tangent is parallel to the line $y = 9x$. [5]

13 Show that the tangent to the curve $y = x^3$ at the point $(1, 1)$ passes through the point $(0, -2)$. [4]

14 The curve C has equation $y = x^3 - 3x^2 + 2$ and passes through the point $A(-1, -2)$, as shown.



(a) Find an equation of the tangent to C at A . [3]

(b) The tangent at A meets the coordinate axes at the points B and D . Find the area of the triangle OBD , where O is the origin. [3]

15 The curve C has equation $y = x^3 - 3x^2 - 9x + 4$.
(a) Find the coordinates of the points on C at which the tangent is parallel to the line $9x + y = 5$. [4]

(b) Find the equations of these two tangents. [2]

16 The curve with equation $y = ax^2 + bx$ passes through the point $(2, 6)$ and the gradient of the curve at this point is 5. Find the values of the constants a and b . [5]

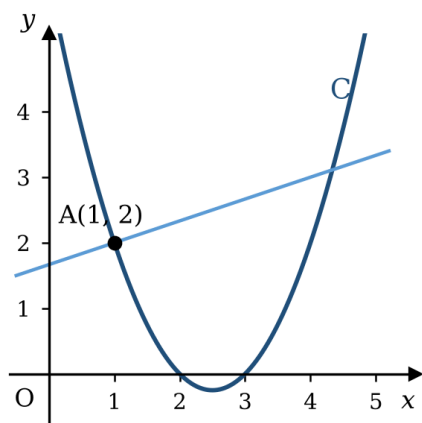
17 The tangent to the curve $y = 2x^2 - 5x$ at the point P is perpendicular to the line $3y + x = 7$. Find the coordinates of P and an equation of the tangent at P . [6]

18 The curve C has equation $y = 3x^2 - 2x$. The tangent and the normal to C at the point $P(1, 1)$ meet the y -axis at A and B respectively. Find the length AB . [6]

19 Find an equation of the tangent to the curve $y = 3e^x - x$ at the point where $x = 0$. [4]

20 Find an equation of the tangent to the curve $y = 2 \sin x + \cos x$ at the point where $x = \frac{\pi}{2}$.
Give your answer in the form $y = mx + c$, where m and c are exact. [5]

- 21** The curve C has equation $y = x^2 - 5x + 6$ and passes through the point A(1, 2), as shown.



- (a) Find an equation of the normal to C at A, giving your answer in the form $ax + by + c = 0$ where a, b and c are integers. [4]
 (b) The normal meets C again at the point B. Find the coordinates of B. [4]

- 22** The curve C has equation $y = x^2 - 2x + 5$. The tangent to C at the point P, with x-coordinate p, passes through the origin O.

- (a) Show that $p^2 = 5$. [5]
 (b) Hence find, in exact form, the equations of the two tangents to C that pass through O. [3]

- 23** The point P(3, 4) lies on the curve $y = x^2 - 2x + 1$. The tangent and the normal to the curve at P meet the x-axis at A and B respectively. Find the area of triangle APB. [7]

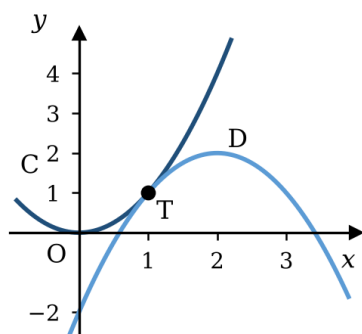
- 24** The curve C has equation $y = 4 - x^2$. Two tangents to C pass through the point A(0, 5).

- (a) Show that the tangent to C at the point with x-coordinate p has equation $y = -2px + p^2 + 4$. [3]
 (b) Find equations of the two tangents through A, and the area of the triangle formed by these tangents and the x-axis. [6]

- 25** The curve C has equation $y = x^3 - 6x + 1$ and the point P(1, -4) lies on C.

- (a) Find an equation of the tangent to C at P. [3]
 (b) The tangent to C at P meets C again at the point Q. Find the coordinates of Q. [4]

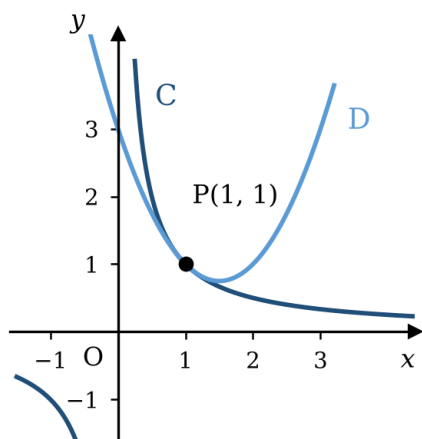
- 26** The curve C has equation $y = x^2$ and the curve D has equation $y = -x^2 + 4x - 2$, as shown.



- (a) Show that C and D meet at exactly one point T, and find its coordinates. [3]
 (b) Show that C and D have the same gradient at T, and find an equation of their common tangent. [4]

27

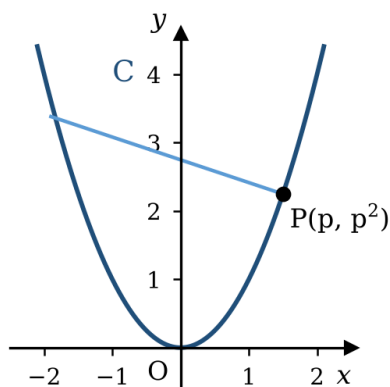
The curves C and D are shown. C has equation $y = \frac{1}{x}$ ($x \neq 0$) and D has equation $y = x^2 + ax + b$, where a and b are constants. The curves meet at the point $P(1, 1)$ and have the same tangent at P .



- Find the value of a and the value of b . [5]
- Show that the common tangent meets D at no other point. [2]
- The normal to both curves at P meets C again at Q and D again at R . Find the coordinates of Q and of R . [3]

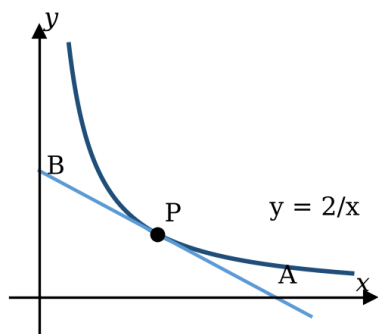
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The curve C has equation $y = x^2$. The point $P(p, p^2)$ lies on C , where $p \neq 0$, as shown.



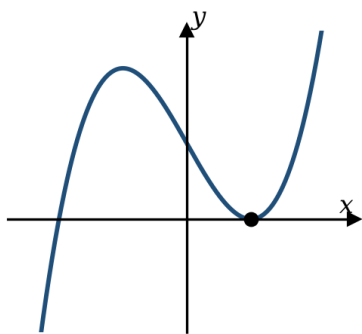
- Show that the normal to C at P meets the y -axis at the point $(0, p^2 + \frac{1}{2})$. [4]
- Hence find the exact coordinates of the points on C at which the normal passes through $(0, 5)$. [3]
- Explain why, other than the y -axis itself, no normal to C meets the y -axis below the point $(0, \frac{1}{2})$. [2]

- 29** The curve C has equation $y = \frac{2}{x}$, $x > 0$. The tangent to C at the point $P(p, \frac{2}{p})$ meets the x -axis at A and the y -axis at B , as shown.



- (a) Show that the tangent at P has equation $y = -\frac{2}{p^2}x + \frac{4}{p}$. [4]
 (b) Find the coordinates of A and B in terms of p . [3]
 (c) Prove that the area of triangle OAB is the same for every value of p , and state this area. [4]

- 30** The curve C has equation $y = x^3 - 3x + k$, where k is a positive constant. The x -axis is a tangent to C .



- (a) Show that C has gradient zero when $x = 1$ or $x = -1$. [2]
 (b) Hence find the value of k . [4]
 (c) For this value of k , find the coordinates of all the points where C meets the axes. [4]

Stationary Points and Optimisation

- 1** Find the coordinates of the stationary point of the curve $y = x^2 - 6x + 5$. [3]
2 Find the coordinates of the stationary points of the curve $y = x^3 - 12x$. [3]
3 Find the values of x for which $\frac{dy}{dx} = 0$, given that $y = x^3 - 27x$. [3]
4 Find the coordinates of the stationary point on the curve $y = -x^2 + 4x + 5$ and determine whether it is a maximum or a minimum. [4]
5 The curve $y = x^3 - 6x^2$ has a stationary point where $x = 4$. Use the second derivative to determine its nature. [3]
6 Find the minimum value of $y = 2x^2 - 8x$. [3]
7 The function f is given by $f(x) = 2x^3 - 9x^2 + 12x + 1$. Find the coordinates of the stationary points of f and determine their nature. [5]

8 Given that $y = x + \frac{4}{x}$ for $x > 0$, find the coordinates of the minimum point of the curve, justifying that it is a minimum. [4]

9 Find the coordinates of the stationary points of the curve $y = x^3 - 3x^2 - 9x + 1$ and use the second derivative to determine their nature. [6]

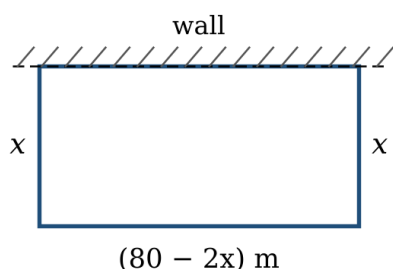
10 The weekly profit £P of a small business that sells x items is modelled by $P = 40x - 0.2x^2 - 300$. Find the number of items sold that gives the maximum weekly profit, justify that it is a maximum, and find the maximum profit. [5]

11 Find the coordinates of the three stationary points of the curve $y = x^4 - 8x^2$. [4]

12 A rectangle has a perimeter of 20 cm and one side of length x cm. Show that its area is $A = 10x - x^2$, and find the maximum area. [5]

13 A ball is thrown upwards from a point 2 m above the ground. Its height h metres above the ground after t seconds is modelled by $h = 2 + 20t - 5t^2$. Find the greatest height reached by the ball and the time at which this occurs. [4]

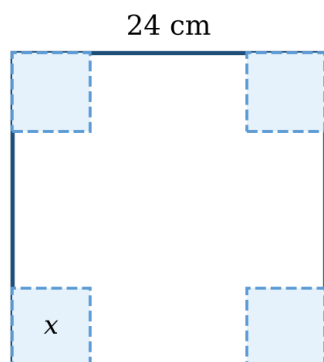
14 A farmer has 80 m of fencing to make three sides of a rectangular pen. The fourth side is a straight wall, as shown. The sides perpendicular to the wall have length x metres.



(a) Show that the area $A \text{ m}^2$ of the pen is $A = 80x - 2x^2$. [2]

(b) Find the value of x for which A is a maximum, justify that it is a maximum, and find the maximum area. [4]

15 A square sheet of card has sides of length 24 cm. A square of side x cm is cut from each corner, as shown, and the remaining card is folded to make an open box of volume $V \text{ cm}^3$.



(a) Show that $V = 4x^3 - 96x^2 + 576x$. [2]

(b) Find the value of x for which V is a maximum, justify that it is a maximum, and find the maximum volume. [5]

16 Two positive numbers x and y satisfy $x + y = 24$. Find the values of x and y for which $P = x^2y$ is a maximum, and show that it is a maximum. [6]

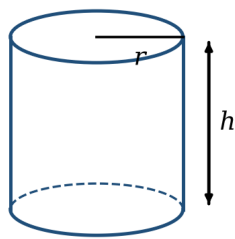
- 17** A sector of a circle has radius r cm and angle θ radians. The area of the sector is 50 cm^2 and its perimeter is P cm.
- (a) Show that $P = 2r + \frac{100}{r}$. [3]
- (b) Find the minimum value of P , in exact form, justifying that it is a minimum. [6]
- (c) Find the value of θ when P is a minimum. [1]

- 18** A closed cylinder has radius r cm and height h cm, and its total surface area is $150\pi \text{ cm}^2$.
- (a) Show that the volume of the cylinder is $V = 75\pi r - \pi r^3$. [4]
- (b) Find the value of r for which V is a maximum, justify that it is a maximum, and find the maximum volume. [6]

- 19** The curve with equation $y = ax^2 + \frac{b}{x}$, $x > 0$, has a stationary point at $(2, 6)$. Find the values of the constants a and b , and determine the nature of the stationary point. [7]

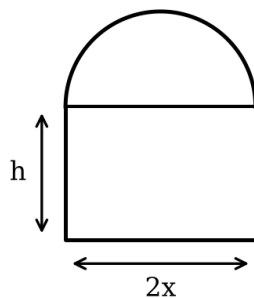
- 20** Show that the curve $y = x^3 - 3x^2 + 3x$ has exactly one stationary point, and that it is a point of inflection. [5]

- 21** A closed cylindrical can has radius r cm, height h cm and volume 500 cm^3 , as shown. The total surface area of the can is $S \text{ cm}^2$.



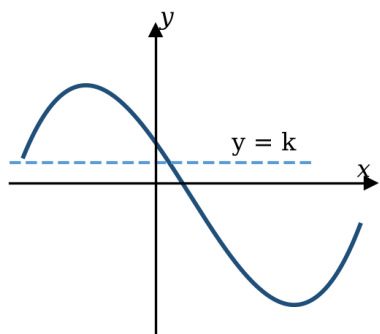
- (a) Show that $S = 2\pi r^2 + \frac{1000}{r}$. [3]
- (b) Find the value of r for which S is a minimum, giving your answer to 3 significant figures. [4]
- (c) Justify that this value gives a minimum and find the minimum value of S to 3 significant figures. [2]
- 22** The curve C has equation $y = x^4 - 4x^3$.
- (a) Find the coordinates of the stationary points of C . [4]
- (b) Determine the nature of each stationary point. (Note that $\frac{d^2y}{dx^2}$ alone is not enough for one of them.) [4]

- 23** A window is in the shape of a rectangle of width $2x$ metres and height h metres, with a semicircle of radius x metres on top, as shown. The perimeter of the whole window is 6 m.



- (a) Show that the area of the window is $A = 6x - 2x^2 - \frac{\pi}{2}x^2$. [4]
- (b) Find the exact value of x for which A is a maximum, justifying that it is a maximum, and find the maximum area in the form $\frac{a}{b + \pi}$. [6]
-
- 24** A closed rectangular tank has a base measuring $2x$ metres by x metres and a height of h metres. Its volume is 108 m^3 . The top and base cost £30 per m^2 and the four sides cost £20 per m^2 .
- (a) Show that the total cost, £ C , of the tank is $C = 120x^2 + \frac{6480}{x}$. [4]
- (b) Find the minimum cost, justifying that it is a minimum. [5]
-
- 25** A particle moves along a straight line so that its displacement s metres from a fixed point O after t seconds is $s = 2t^3 - 15t^2 + 36t$, for $t \geq 0$.
- (a) Find the times at which the particle is instantaneously at rest, and its displacement at each of these times. [5]
- (b) Find the minimum value of the velocity. [4]
-
- 26** The point P lies on the curve $y = \sqrt{x}$, and A is the point $(4, 0)$. Find the coordinates of P for which the distance AP is least, and find this least distance, giving it in exact form. [7]
-
- 27** A wholesaler buys a product for £12 per item and sells it for £30 per item. In a typical week 300 items are sold. For every £1 reduction in the selling price, 20 extra items are sold each week. The price is reduced by £ x .
- (a) Show that the weekly profit, £ P , is given by $P = 5400 + 60x - 20x^2$. [3]
- (b) Find the selling price that gives the maximum weekly profit, and find this maximum profit. [5]
- (c) The wholesaler wants the weekly profit to be at least £5400. Find the range of values of x . [2]
-
- 28** A wire of length 100 cm is cut into two pieces. One piece, of length x cm, is bent to form a square. The other piece is bent to form a circle. The total area enclosed is $A \text{ cm}^2$.
- (a) Show that $A = \frac{x^2}{16} + \frac{(100 - x)^2}{4\pi}$. [3]
- (b) Find the value of x for which A is stationary, to 3 significant figures. [3]
- (c) Show that this gives the minimum area and find it, to 3 significant figures. [2]
- (d) Given that $0 \leq x \leq 100$, find the greatest possible total area and explain how it arises. [3]

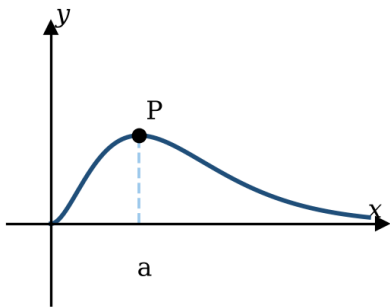
- 29** The curve C has equation $y = x^3 - 3x^2 - 24x + 20$, as shown, together with the line $y = k$, where k is a constant.



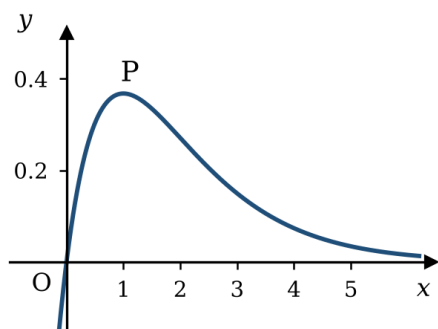
- (a) Find the coordinates of the two stationary points of C and determine their nature. [6]
- (b) Find the set of values of k for which the equation $x^3 - 3x^2 - 24x + 20 = k$ has three distinct real roots. [3]
- 30** A solid right circular cylinder of radius r and height h is inscribed in a sphere of radius R (a constant), so that the circumference of each end of the cylinder lies on the sphere.
- (a) Show that the volume of the cylinder is $V = \pi(R^2 - \frac{h^2}{4})h$. [3]
- (b) Find the exact maximum value of V , in terms of R , justifying that it is a maximum. [7]

Product Rule

- 1** Use the product rule to differentiate $y = x^3(2x - 5)$. [3]
- 2** Differentiate $y = xe^x$. [3]
- 3** Use the product rule to differentiate $y = (3x + 1)(x^2 - 2)$. Check your answer by expanding first. [4]
- 4** Differentiate $y = x \sin x$. [3]
- 5** Differentiate $y = x^2 \cos x$. [3]
- 6** Find the gradient function of $y = \sqrt{x}(x + 4)$ for $x > 0$. [3]
- 7** The curve C has equation $y = x^2 \sin x$.
- (a) Find $\frac{dy}{dx}$. [2]
- (b) Find the gradient of C at the point where $x = \pi$. [2]
- 8** Given that $y = (2x - 1)\sqrt{x}$, $x > 0$, show that $\frac{dy}{dx} = \frac{6x - 1}{2\sqrt{x}}$. [5]
- 9** Differentiate $y = e^x \ln x$, for $x > 0$. [3]
- 10** Given that $y = (x^2 + 3)(x^3 - x)$, use the product rule to find $\frac{dy}{dx}$, and verify your answer by expanding the brackets first. [5]
- 11** Find the exact gradient of the curve $y = x \cos x$ at the point where $x = \pi$. [4]
- 12** Differentiate $y = \sqrt{x} \ln x$, $x > 0$, giving your answer in the form $\frac{a + b \ln x}{c\sqrt{x}}$. [4]

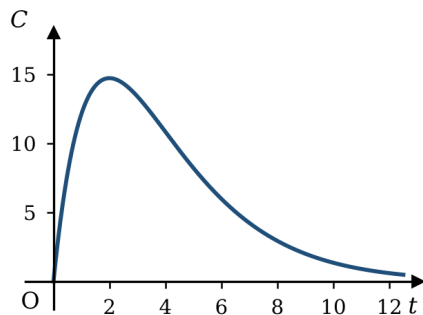
- 13** The curve C has equation $y = x \tan x$. Find the exact gradient of C at the point where $x = \frac{\pi}{4}$. [4]
-
- 14** The curve C has equation $y = x^3 \ln x$, $x > 0$.
(a) Find $\frac{dy}{dx}$. [3]
(b) Find the exact coordinates of the stationary point of C. [3]
-
- 15** The curve C has equation $y = x^2 e^x$.
(a) Find $\frac{dy}{dx}$. [3]
(b) Find an equation of the tangent to C at the point where $x = 1$. [2]
(c) Find the exact x-coordinate of the point where this tangent crosses the x-axis. [2]
-
- 16** The diagram shows the curve with equation $y = x^2 e^{-x}$, for $x \geq 0$. The curve has a maximum point P with x-coordinate a.
- 
- (a) Find $\frac{dy}{dx}$, and show that it can be written as $x(2 - x)e^{-x}$. [4]
(b) Find the value of a and the exact y-coordinate of P. [3]
-
- 17** The curve C has equation $y = (3x + 2)e^{-2x}$.
(a) Show that $\frac{dy}{dx} = -(6x + 1)e^{-2x}$. [4]
(b) Find the set of values of x for which y is a decreasing function of x. [2]
-
- 18** Given that $y = xe^{2x}$, show that $\frac{dy}{dx} = (1 + 2x)e^{2x}$, and hence find the exact coordinates of the stationary point of the curve. [5]
-
- 19** Given that $y = x \sin x$, show that $\frac{d^2y}{dx^2} + y = 2\cos x$. [5]
-
- 20** Given that $y = xe^{2x}$, show that $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$. [5]
-
- 21** The curve C has equation $y = e^x \sin x$.
(a) Show that $\frac{dy}{dx} = e^x(\sin x + \cos x)$. [3]
(b) Hence show that $\frac{d^2y}{dx^2} = 2e^x \cos x$. [3]
(c) Find the x-coordinates of the stationary points of C in the interval $0 \leq x \leq 2\pi$. [2]
-

- 22** The diagram shows the curve with equation $y = xe^{-x}$, for $x \geq -0.6$. The point P is the maximum point.



- (a) Find the exact coordinates of P. [4]
 (b) Show that the curve has a point of inflection at $x = 2$. [3]
 (c) Find the y-coordinate of the point of inflection. [1]
-
- 23** Differentiate $y = xe^x \sin x$, giving your answer in the form $e^x(ax \sin x + b \sin x + cx \cos x)$ where a , b and c are constants to be found. [5]
-
- 24** The curve C has equation $y = x^2 \ln x$, for $x > 0$. Find the exact coordinates of the stationary point of C and determine its nature. [7]
-
- 25** The number of bacteria, B thousand, in a culture t hours after treatment is modelled by $B = 100te^{-t/2}$, for $t \geq 0$.
 (a) Show that $\frac{dB}{dt} = 50(2 - t)e^{-t/2}$. [4]
 (b) Find the maximum number of bacteria predicted by the model, and the time at which it occurs. [4]
-
- 26** The curve with equation $y = (ax + b)e^{-x}$ has a stationary point at $(2, 3e^{-2})$. Find the values of the constants a and b . [6]
-
- 27** In this question u and v are differentiable functions of x .
 (a) Prove from first principles that if $y = uv$, where u and v are differentiable functions of x , then $\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$. [5]
 (b) Hence differentiate $y = x^3 \sin 2x$, simplifying your answer. [4]

- 28** The concentration, C mg per litre, of a drug in the blood t hours after it is taken is modelled by $C = 20te^{-0.5t}$, for $t \geq 0$.



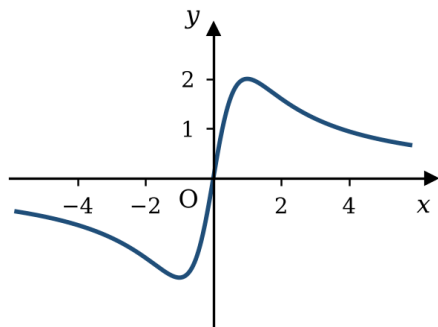
- (a) Find $\frac{dC}{dt}$. [3]
 - (b) Find the time at which the concentration is greatest, and the greatest concentration, to 3 significant figures. [3]
 - (c) Show that $\frac{d^2C}{dt^2} = 5(t - 4)e^{-0.5t}$ and use it to verify that your answer to (b) is a maximum. [2]
 - (d) Find the rate of change of concentration at $t = 6$, to 3 significant figures, and interpret your answer. [2]
-
- 29** Let n be a positive integer and let $f(x) = x^n e^{-x}$, for $x > 0$.
- (a) Show that $f'(x) = x^{n-1}(n - x)e^{-x}$. [3]
 - (b) Show that f has a maximum at $x = n$, and state the maximum value. [4]
 - (c) Hence prove that $e^x \geq (\frac{e}{3})^3 x^3$ for all $x > 0$. [4]
-
- 30** Let $y = x^2 e^x$.
- (a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$, expressing each in the form $(x^2 + px + q)e^x$. [4]
 - (b) Conjecture a formula for the n th derivative and prove it by induction. [6]

Quotient Rule

- 1** Use the quotient rule to differentiate $y = \frac{x}{x+1}$. [3]
- 2** Differentiate $y = \frac{e^x}{x}$, $x \neq 0$. [3]
- 3** Use the quotient rule to differentiate $y = \frac{2x+1}{x-3}$. [3]
- 4** Differentiate $y = \frac{\cos x}{x}$. [3]
- 5** Differentiate $y = \frac{x}{e^x}$ using the quotient rule. [3]
- 6** Differentiate $y = \frac{1}{x^2+1}$ using the quotient rule. [3]

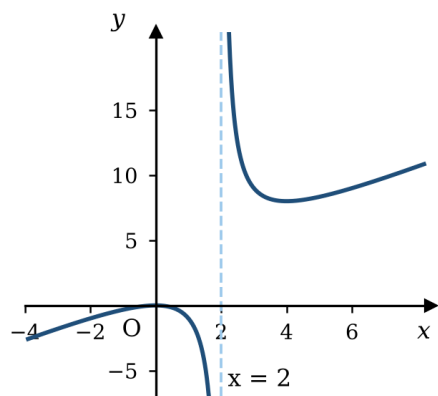
- 7 The curve C has equation $y = \frac{\sin x}{x}$, $x > 0$.
 (a) Find $\frac{dy}{dx}$. [2]
 (b) Find the gradient of C at $x = \pi$. [2]
-
- 8 Show that the function $f(x) = \frac{2x+3}{x-1}$, $x \neq 1$, is decreasing for all x in its domain. [4]
-
- 9 Differentiate $y = \frac{\ln x}{x+1}$, for $x > 0$. [4]
-
- 10 Given that $f(x) = \frac{x^2 - 3x + 6}{x - 1}$, $x \neq 1$, find the values of x for which $f'(x) = \frac{3}{4}$. [6]
-
- 11 Given that $y = \frac{e^x}{x^2 + 1}$, show that $\frac{dy}{dx} = \frac{e^x(x-1)^2}{(x^2+1)^2}$. [4]
-
- 12 Show that the derivative of $\frac{\sin x}{1 + \cos x}$ is $\frac{1}{1 + \cos x}$. [5]
-
- 13 Find the derivative of $y = \frac{\sqrt{x}}{x+2}$ for $x > 0$, giving your answer in the form $\frac{a-x}{b\sqrt{x}(x+2)^2}$. [5]
-
- 14 The curve C has equation $y = \frac{\ln x}{x^2}$, $x > 0$.
 (a) Show that $\frac{dy}{dx} = \frac{1 - 2 \ln x}{x^3}$. [3]
 (b) Find the exact coordinates of the stationary point of C. [3]
-
- 15 Using $\sin x$ and $\cos x$, prove that if $y = \tan x$ then $\frac{dy}{dx} = \sec^2 x$.
 (a) Write $\tan x = \frac{\sin x}{\cos x}$ and use the quotient rule to prove that $\frac{d}{dx}(\tan x) = \sec^2 x$. [4]
 (b) Find the exact gradient of $y = \tan x$ at $x = \frac{\pi}{4}$. [2]
-
- 16 The curve C has equation $y = \frac{x^2 + 1}{x - 1}$, $x \neq 1$. Find the exact x -coordinates of the stationary points of C. [5]
-
- 17 The curve C has equation $y = \frac{e^x}{x}$, $x > 0$. Show that the tangent to C at the point where $x = 2$ passes through the origin. [5]
-
- 18 The curve C has equation $y = \frac{x^2}{(x-a)^2}$, $x \neq a$, where a is a non-zero constant. Given that the gradient of C at the point where $x = 2a$ is -2 , find the value of a . [6]
-
- 19 Use the quotient rule to prove that $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$. [5]
-
- 20 The curve with equation $y = \frac{ax+b}{x+1}$ passes through the point $(0, 3)$, and the gradient of the curve at this point is -2 . Find the values of the constants a and b . [6]
-

- 21 The diagram shows the curve C with equation $y = \frac{4x}{x^2 + 1}$, $x \in \mathbb{R}$.



- (a) Show that $\frac{dy}{dx} = \frac{4(1 - x^2)}{(x^2 + 1)^2}$. [3]
 (b) Find the coordinates of the stationary points of C. [2]
 (c) Hence state the range of $y = \frac{4x}{x^2 + 1}$, giving a reason. [3]

- 22 The diagram shows the curve C with equation $y = \frac{x^2}{x - 2}$, $x \neq 2$. The line $x = 2$ is an asymptote.



- (a) Show that $\frac{dy}{dx} = \frac{x(x - 4)}{(x - 2)^2}$. [3]
 (b) Find the coordinates of the stationary points of C. [2]
 (c) Determine the nature of each stationary point. [3]

- 23 The curve C has equation $y = \frac{\ln x}{x}$, $x > 0$.

- (a) Find the exact coordinates of the stationary point of C and show that it is a maximum. [6]
 (b) Hence deduce that $\ln x \leq \frac{x}{e}$ for all $x > 0$. [2]

- 24 The curve C has equation $y = \frac{\sin x}{2 - \cos x}$, for $0 \leq x < 2\pi$.

- (a) Show that $\frac{dy}{dx} = \frac{2\cos x - 1}{(2 - \cos x)^2}$. [4]
 (b) Hence find the exact coordinates of the maximum point and the minimum point of C. [5]

- 25 The curve C has equation $y = \frac{xe^x}{x + 1}$, $x \neq -1$.

- (a) Show that $\frac{dy}{dx} = \frac{e^x(x^2 + x + 1)}{(x + 1)^2}$. [5]
 (b) Hence show that C has no stationary points. [3]

26

The curve C has equation $y = \frac{x}{x^2 + 1}$.

(a) Show that $\frac{dy}{dx} = \frac{1 - x^2}{(x^2 + 1)^2}$. [3]

(b) Show that $\frac{d^2y}{dx^2} = \frac{2x(x^2 - 3)}{(x^2 + 1)^3}$. [5]

(c) Hence find the values of x at which $\frac{d^2y}{dx^2} = 0$. [2]

27

Use the quotient rule in this question. The derivatives of $\sin x$ and $\cos x$ may be quoted.

(a) By writing $\sec x = \frac{1}{\cos x}$, show that $\frac{d}{dx}(\sec x) = \sec x \tan x$. [3]

(b) Show that $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$. [3]

(c) Given that $y = \frac{1 + \sin x}{\cos x}$, show that $\frac{dy}{dx} = \frac{1 + \sin x}{\cos^2 x}$ and find the exact gradient at $x = \frac{\pi}{3}$. [4]

28

A student differentiates $y = \frac{x^2}{x + 1}$ using the quotient rule and writes $\frac{dy}{dx} = \frac{x^2 \cdot 1 - (x + 1) \cdot 2x}{(x + 1)^2}$.

(a) Explain the error in the student's working. [2]

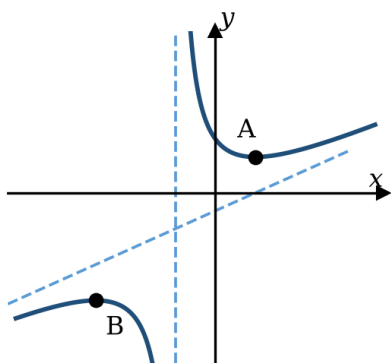
(b) Find the correct $\frac{dy}{dx}$, in the form $\frac{x(x + a)}{(x + 1)^2}$. [3]

(c) Find the coordinates of the stationary points of the curve. [2]

(d) Show that $y = x - 1 + \frac{1}{x + 1}$ and differentiate this form to check your answer to (b). [3]

29

The curve C has equation $y = \frac{x^2 + k}{x + 1}$, $x \neq -1$, where k is a constant.



(a) Show that $\frac{dy}{dx} = \frac{x^2 + 2x - k}{(x + 1)^2}$. [3]

(b) Find the set of values of k for which C has two stationary points. [3]

(c) For $k = 3$, as shown, find the coordinates of the stationary points A and B and determine their nature. [6]

30

A population model uses the function $f(x) = \frac{e^x}{1 + e^x}$, $x \in \mathbb{R}$.

(a) Show that $f'(x) = \frac{e^x}{(1 + e^x)^2}$. [3]

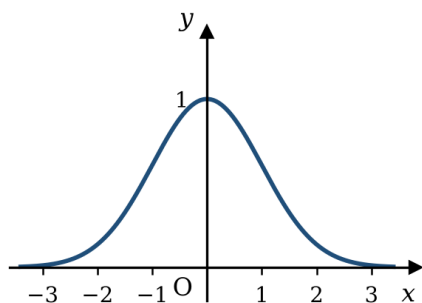
(b) Hence show that $f'(x) = f(x)(1 - f(x))$. [3]

(c) By writing $f'(x)$ in terms of $f(x)$ and completing the square, find the maximum value of $f'(x)$, and the value of x at which it occurs. [4]

Chain Rule

- 1 Differentiate $y = (3x + 2)^5$. [2]
- 2 The curve C has equation $y = (x^2 + 1)^4$. Find the gradient of C at the point where $x = 1$. [3]
- 3 Differentiate $y = (4 - x)^7$. [2]
- 4 Find $\frac{dy}{dx}$ when $y = \cos(2x + 1)$. [2]
- 5 Differentiate $y = e^{5x}$. [2]
- 6 Differentiate $y = \ln(3x + 4)$, for $x > -\frac{4}{3}$. [2]
- 7 Differentiate with respect to x :
 - (a) $y = e^{3x^2 - x}$ [2]
 - (b) $y = \cos^3 x$ [2]
- 8 The curve C has equation $y = \frac{1}{(2x + 1)^3}$, $x > -\frac{1}{2}$.
 - (a) Find $\frac{dy}{dx}$. [2]
 - (b) Find an equation of the tangent to C where $x = 0$. [3]
- 9 Find $\frac{dy}{dx}$ when $y = \sqrt{4x + 1}$, $x > -\frac{1}{4}$. [3]
- 10 Differentiate $y = e^{\sin 3x}$ with respect to x . [3]
- 11 Find the derivative of $y = e^{x^2 + 2x}$. [3]
- 12 Given that $y = \sin^2 x$, use the chain rule to show that $\frac{dy}{dx} = \sin 2x$. [4]
- 13 Find the gradient of the curve $y = (x^2 - 2x + 3)^3$ at the point where $x = 2$. [4]
- 14 The curve C has equation $y = \sqrt{x^2 + 9}$.
 - (a) Show that $\frac{dy}{dx} = \frac{x}{\sqrt{x^2 + 9}}$. [3]
 - (b) Find the gradient of C at the point where $x = 4$. [1]
 - (c) Find an equation of the normal to C at this point, in the form $ax + by = c$ where a , b and c are integers. [2]

- 15 The diagram shows the curve with equation $y = e^{-x^2/2}$.

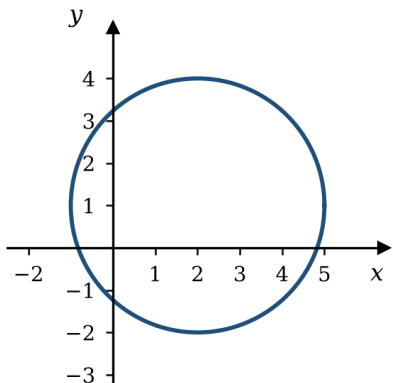


- (a) Find $\frac{dy}{dx}$. [2]
- (b) Show that $\frac{d^2y}{dx^2} = (x^2 - 1)e^{-x^2/2}$. [2]
- (c) Hence find the x-coordinates of the points of inflection of the curve. [2]
-
- 16 The functions f and g are defined by $f(x) = 2 \ln(3x)$, $x > 0$, and $g(x) = x^2 + 4$, $x \in \mathbb{R}$.
- (a) Write down an expression for $gf(x)$. [1]
- (b) Find the exact gradient of the curve $y = gf(x)$ at the point where $x = 2$. [4]
-
- 17 By differentiating both sides of the identity $\sin 3x \equiv 3 \sin x - 4 \sin^3 x$, show that $\cos 3x \equiv 4 \cos^3 x - 3 \cos x$. [5]
-
- 18 The curve C has equation $y = (x^2 - 2x)^3$. Find the coordinates of the stationary points of C and determine which of them is a turning point. [7]
-
- 19 Given that $y = (1 + x^2)^{-1/2}$, show that $(1 + x^2) \frac{dy}{dx} + xy = 0$. [4]
-
- 20 The curve C has equation $y = \ln(x^2 - 4x + 5)$. Show that C has exactly one stationary point, and that it is a minimum. [6]
-
- 21 The curve C has equation $y = (1 + \sin 2x)^3$.
- (a) Find $\frac{dy}{dx}$. [3]
- (b) Find the exact gradient of C at $x = \frac{\pi}{6}$. [3]
- (c) Find the x-coordinates of the stationary points of C in the interval $0 \leq x \leq \pi$. [2]
-
- 22 The curve C has equation $y = x^2 \sqrt{1 - x^2}$, for $0 \leq x \leq 1$.
- (a) Show that $\frac{dy}{dx} = \frac{x(2 - 3x^2)}{\sqrt{1 - x^2}}$. [4]
- (b) Find the exact coordinates of the stationary point of C for $0 < x < 1$. [3]
- (c) Justify that this point is a maximum. [1]
-
- 23 Given that $y = \sin(x^2)$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. [5]
-
- 24 Given that $y = x^3(1 - x^2)^{3/2}$, for $0 < x < 1$, show that $\frac{dy}{dx} = 3x^2(1 - 2x^2)\sqrt{1 - x^2}$. [6]
-
- 25 A radioactive sample has N atoms (in millions) at time t days, modelled by $N = 800e^{-0.04t}$.
- (a) Find the rate at which the number of atoms is changing when $t = 10$, giving your answer to 3 significant figures. [3]
- (b) Find the value of t at which the rate of decrease is 8 million per day. Give your answer to 3 significant figures. [3]

- 26** A curve is defined by $x = y^3 + y$.
- (a) Find $\frac{dx}{dy}$ and hence show that $\frac{dy}{dx} = \frac{1}{3y^2 + 1}$. [3]
- (b) The point P(2, 1) lies on the curve. Find an equation of the tangent at P. [4]
-
- 27** Throughout, use the chain rule and the derivative of e^x as e^x .
- (a) By writing $a^x = e^{x \ln a}$, show that $\frac{d}{dx}(a^x) = a^x \ln a$ for $a > 0$. [3]
- (b) Hence differentiate $y = 2^{x^2 - 1}$. [2]
- (c) Find an equation of the tangent to the curve $y = 2^{x^2 - 1}$ at the point where $x = 1$. [2]
- (d) Find the exact x-coordinate of the point where this tangent crosses the x-axis. [2]
-
- 28** The curve C has equation $y = \sin^2(3x^2 + 1)$.
- (a) Show that $\frac{dy}{dx} = 6x \sin(6x^2 + 2)$. [4]
- (b) Find the exact x-coordinates of the stationary points of C for which $0 < x < 1$. [3]
- (c) Determine the nature of the stationary point with the smaller x-coordinate. [3]
-
- 29** The curve C has equation $y = e^{-(x-a)^2/2}$, where a is a constant.
- (a) Show that $\frac{dy}{dx} = -(x-a)y$. [3]
- (b) Hence show that $\frac{d^2y}{dx^2} = ((x-a)^2 - 1)y$. [3]
- (c) Find the exact coordinates of the points of inflection of C, in terms of a. [4]
-
- 30** Let $y = \sin^n x$, where n is a constant and $0 < x < \pi$.
- (a) Show that $\frac{dy}{dx} = n \sin^{n-1} x \cos x$. [2]
- (b) Hence show that $\frac{d^2y}{dx^2} = n(n-1) \sin^{n-2} x - n^2 \sin^n x$. [6]

Implicit Differentiation

- 1** Find $\frac{dy}{dx}$ in terms of x and y for the curve $x^2 + y^2 = 25$. [3]
-
- 2** A curve has equation $y^2 = 4x$. Find $\frac{dy}{dx}$ in terms of y, and hence the gradient at the point (1, 2). [3]
-
- 3** Find $\frac{dy}{dx}$ in terms of x and y, given that $x^2 + 3y^2 = 12$. [3]
-
- 4** A curve has equation $xy = 6$. Use implicit differentiation to find $\frac{dy}{dx}$. [3]
-
- 5** Find $\frac{dy}{dx}$ given that $2y^3 + x = 7$. [3]
-
- 6** The curve C has equation $x^2 + 4y^2 = 8$. Find the gradient of C at the point (2, 1). [4]
-
- 7** The curve C has equation $x^2 + y^2 - 6x + 4y = 12$.
- (a) Find $\frac{dy}{dx}$ in terms of x and y. [3]
- (b) Find the gradient of C at the point (6, 2). [1]

- 8** The curve C has equation $y^3 + xy = 10$ and passes through the point P(1, 2).
 (a) Find $\frac{dy}{dx}$ in terms of x and y. [3]
 (b) Find an equation of the tangent to C at P in the form $ax + by = c$. [2]
-
- 9** A curve has equation $x^3 + y^3 = 3xy$. Show that $\frac{dy}{dx} = \frac{y - x^2}{y^2 - x}$. [4]
-
- 10** The curve C has equation $x^2 + 2xy - y^2 = 7$. Show that the point P(2, 1) lies on C and find an equation of the tangent to C at P. [5]
-
- 11** Find $\frac{dy}{dx}$ in terms of y when $e^{2y} = x + y$. [4]
-
- 12** Find $\frac{dy}{dx}$ in terms of x and y when $\cos y + x^2 = 3$. [3]
-
- 13** The curve C has equation $x \ln y + y = 3$.
 (a) Find $\frac{dy}{dx}$ in terms of x and y. [3]
 (b) Find an equation of the tangent to C at the point (0, 3). [2]
-
- 14** The curve C has equation $x^2 + e^y = 10$ and passes through the point (3, 0).
 (a) Show that $\frac{dy}{dx} = -2xe^{-y}$. [3]
 (b) Find an equation of the normal to C at (3, 0), in the form $ax + by + c = 0$ where a, b and c are integers. [3]
-
- 15** The circle C has equation $(x - 2)^2 + (y - 1)^2 = 9$, as shown.
- 
- (a) Use implicit differentiation to show that $\frac{dy}{dx} = \frac{2 - x}{y - 1}$. [3]
 (b) Find the coordinates of the points on C where the tangent is parallel to the x-axis. [2]
 (c) Find the coordinates of the points where the tangent is parallel to the y-axis. [1]
-
- 16** The curve C has equation $x^2 + 2y^2 - 4x + 4y = 4$. Find the coordinates of the points on C where the tangent is parallel to the x-axis. [6]
-
- 17** A curve has equation $ye^x = xe^y$. Show that $\frac{dy}{dx} = \frac{e^y - ye^x}{e^x - xe^y}$, and find the gradient of the curve at the origin. [5]
-
- 18** A curve has equation $\tan y = x^2y$. Find $\frac{dy}{dx}$ in terms of x and y. [4]
-
- 19** The point P(1, -1) lies on the curve with equation $ax^2 + 3y^2 - 2xy = 7$, where a is a constant. Find the value of a and an equation of the tangent to the curve at P. [6]

- 20** The curve C has equation $ax^2 + 3xy + y^2 = b$, where a and b are constants. The point P(1, 2) lies on C and the normal to C at P has equation $7x - 10y + 13 = 0$. Find the value of a and the value of b. [6]

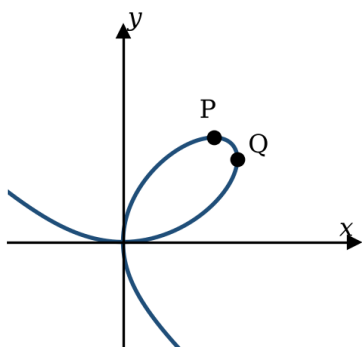
- 21** The curve C has equation $x^2 + xy + y^2 = 7$.
 (a) Show that $\frac{dy}{dx} = -\frac{2x+y}{x+2y}$. [3]
 (b) Find the gradient of C at each point where $x = 1$. [3]
 (c) Find the exact coordinates of the points on C where the tangent is parallel to the x-axis. [2]

- 22** The curve C has equation $x^2 + y^2 = 25$.
 (a) Find $\frac{dy}{dx}$ in terms of x and y. [2]
 (b) Show that $\frac{d^2y}{dx^2} = -\frac{25}{y^3}$. [4]
 (c) Hence determine the nature of the stationary point of C at (0, 5). [2]

- 23** The curve C has equation $xy^2 - x^2y = 6$.
 (a) Show that $\frac{dy}{dx} = \frac{y(2x-y)}{x(2y-x)}$. [4]
 (b) Show that the point P(1, 3) lies on C and find an equation of the tangent to C at P. [4]
 (c) Find the exact coordinates of the point on C where the tangent is vertical. [4]

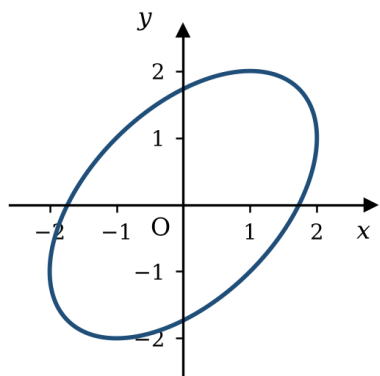
- 24** The curve C has equation $x^3 + y^3 = 3xy$, as shown. The points P and Q are the points of C, other than the origin, at which the tangent is horizontal and vertical respectively.

You may use the result $\frac{dy}{dx} = \frac{y-x^2}{y^2-x}$.



- (a) Find the exact coordinates of P. [5]
 (b) Find the exact coordinates of Q. [3]
- 25** The curves C_1 and C_2 have equations $x^2 - y^2 = 8$ and $xy = 3$ respectively. They meet at the point P(3, 1). Show that the tangents to the two curves at P are perpendicular. [6]
- 26** The curve C has equation $\sin x \cos y = \frac{1}{4}$, for $0 < x < \frac{\pi}{2}$ and $0 < y < \frac{\pi}{2}$. Find an equation of the tangent to C at the point $(\frac{\pi}{6}, \frac{\pi}{3})$, giving your answer in the form $y = x + c$. [6]

- 27 The curve C has equation $x^2 - xy + y^2 = 3$, as shown.



- (a) Show that $\frac{dy}{dx} = \frac{y - 2x}{2y - x}$. [3]
- (b) Show that the tangents to C at the two points where C meets the y-axis are parallel. [3]
- (c) Find the coordinates of the points on C where the tangent is parallel to the y-axis. [3]
-
- 28 In this question $x > 0$ where required.
- (a) Let $y = \ln x$. By writing $x = e^y$, use implicit differentiation to prove that $\frac{dy}{dx} = \frac{1}{x}$. [3]
- (b) Given $y = x^x$ for $x > 0$, take natural logarithms and differentiate implicitly to show that $\frac{dy}{dx} = x^x(1 + \ln x)$. [4]
- (c) Find the exact coordinates of the stationary point of $y = x^x$ and show that it is a minimum. [3]
-
- 29 The curve C has equation $x^2 - y^2 = 8$. Two tangents to C pass through the point A(0, 4).
- (a) Show that $\frac{dy}{dx} = \frac{x}{y}$. [2]
- (b) Find the coordinates of the two points at which these tangents touch C. [5]
- (c) Find equations of the two tangents. [2]
-
- 30 The ellipse E has equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where a and b are positive constants.
- (a) Prove that the tangent to E at the point P(p, q), where $q \neq 0$, has equation $\frac{px}{a^2} + \frac{qy}{b^2} = 1$. [7]
- (b) Hence find an equation of the tangent to $\frac{x^2}{16} + \frac{y^2}{4} = 1$ at the point $(2\sqrt{3}, 1)$. [3]

Parametric Differentiation

- 1 A curve has parametric equations $x = t^2$, $y = 2t$. Find $\frac{dy}{dx}$ in terms of t. [3]
-
- 2 A curve has parametric equations $x = 3t + 1$, $y = t^2 - 4$. Find $\frac{dy}{dx}$ in terms of t, and its value when $t = 3$. [3]
-
- 3 A curve has parametric equations $x = e^{2t}$, $y = 4e^t$. Find $\frac{dy}{dx}$ in terms of t. [3]
-
- 4 A curve has parametric equations $x = t^3$, $y = 3t^2 + 1$. Find $\frac{dy}{dx}$ in terms of t. [3]
-
- 5 A curve has parametric equations $x = 2t + 3$, $y = t^3 - t$. Find the gradient of the curve at the point where $t = 2$. [3]

6 A curve has parametric equations $x = \cos t$, $y = \sin t$. Show that $\frac{dy}{dx} = -\cot t$. [3]

7 A curve has parametric equations $x = 2 \cos t$, $y = 3 \sin t$. Find the exact gradient of the curve at the point where $t = \frac{\pi}{6}$. [4]

8 A curve has parametric equations $x = t^2$, $y = t^3 - 3t$. Find an equation of the tangent to the curve at the point where $t = 2$, giving your answer in the form $ax + by + c = 0$. [5]

9 A curve has parametric equations $x = 3t^2 - 1$, $y = 2t^3 + k$, where k is a constant. The point $P(2, 7)$ lies on the curve, and $t > 0$ at P .

(a) Find the value of t at P and the value of k . [3]

(b) Find the gradient of the curve at P , and hence an equation of the tangent at P . [3]

10 A curve has parametric equations $x = t + \frac{1}{t}$, $y = t - \frac{1}{t}$, $t \neq 0$. Show that $\frac{dy}{dx} = \frac{t^2 + 1}{t^2 - 1}$. [4]

11 A curve has parametric equations $x = e^t$, $y = e^{2t}$. Find $\frac{dy}{dx}$ in terms of t , and verify your answer by finding the Cartesian equation of the curve. [4]

12 A curve has parametric equations $x = \sin t$, $y = \cos 2t$. Find an equation of the tangent to the curve at the point where $t = \frac{\pi}{6}$. [5]

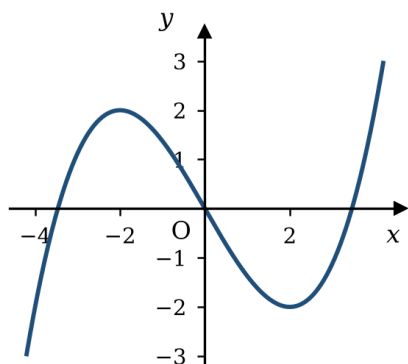
13 A curve has parametric equations $x = 2t^2$, $y = 4t$. Find an equation of the normal to the curve at the point where $t = 1$. [5]

14 A curve has parametric equations $x = t^2$, $y = \frac{4}{t}$, $t > 0$.

(a) Find $\frac{dy}{dx}$ in terms of t . [3]

(b) Find an equation of the normal to the curve at the point where $t = 1$. [3]

15 The curve C has parametric equations $x = 2t$, $y = t^3 - 3t$, for $-2 \leq t \leq 2$, and is shown.



(a) Find $\frac{dy}{dx}$ in terms of t . [2]

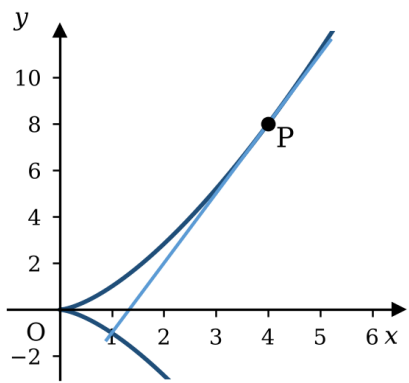
(b) Find the coordinates of the stationary points of C . [3]

(c) Show that $\frac{d^2y}{dx^2} = \frac{3t}{2}$ and determine the nature of the stationary points. [2]

16 A curve has parametric equations $x = 2 \cos t + 1$, $y = 3 \sin t - 2$, $0 \leq t < 2\pi$.

(a) Show that the Cartesian equation of the curve is $\frac{(x-1)^2}{4} + \frac{(y+2)^2}{9} = 1$. [3]

(b) Find $\frac{dy}{dx}$ in terms of t , and the exact gradient of the curve where $t = \frac{\pi}{3}$. [4]

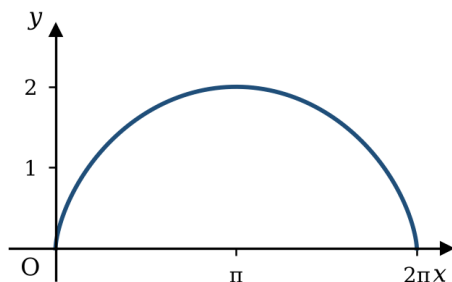
- 17** A curve has parametric equations $x = t^2 + t$, $y = t^3 - 3t$. Find the coordinates of the points on the curve where the tangent is (a) horizontal, (b) vertical. [6]
- 18** A curve has parametric equations $x = 2t + 1$, $y = t^3$. Find $\frac{d^2y}{dx^2}$ in terms of t , and its value when $t = 2$. [5]
- 19** A curve has parametric equations $x = \sec t$, $y = \tan t$, $0 < t < \frac{\pi}{2}$. Show that $\frac{dy}{dx} = \operatorname{cosec} t$, and find the exact gradient at the point where $t = \frac{\pi}{6}$. [5]
- 20** A parabola has parametric equations $x = t^2$, $y = 2t$. Show that the tangent at the point P with parameter p , $p \neq 0$, has equation $py = x + p^2$. [5]
- 21** A curve has parametric equations $x = 2 \sin t$, $y = 3 \cos 2t$, for $0 \leq t < 2\pi$.
- Show that $\frac{dy}{dx} = -6 \sin t$. [3]
 - Show that the Cartesian equation of the curve is $y = 3 - \frac{3}{2}x^2$. [2]
 - Differentiate the Cartesian equation to confirm the result in (a). [1]
 - Find an equation of the tangent at $t = \frac{\pi}{6}$. [2]
- 22** The curve C has parametric equations $x = t^2$, $y = t^3$. The point P(4, 8) lies on C and the tangent at P is shown.
- 
- Find an equation of the tangent to C at P. [3]
 - Show that the tangent meets C again, and find the coordinates of this point. [4]
 - Find the gradient of C at this point. [1]
- 23** The parabola C has parametric equations $x = 2t$, $y = t^2$. The normal to C at the point P where $t = 2$ meets C again at Q.
- Find an equation of the normal at P. [4]
 - Find the coordinates of Q. [4]
- 24** A curve has parametric equations $x = \frac{3t}{1+t^3}$, $y = \frac{3t^2}{1+t^3}$, for $t > 0$.
- Show that $\frac{dy}{dx} = \frac{t(2-t^3)}{1-2t^3}$. [6]
 - Find the exact coordinates of the point on the curve where the tangent is horizontal. [3]
- 25** The curve C has parametric equations $x = 3t$, $y = \frac{12}{t}$, $t \neq 0$.
- Show that an equation of the tangent to C at the point with parameter t is $4x + t^2y = 24t$. [4]
 - Two tangents to C pass through the point A(4, 8). Find equations of these two tangents. [5]

26 A curve has parametric equations $x = 3 + 5 \cos \theta$, $y = -1 + 5 \sin \theta$.

(a) Show that the curve is a circle, and state its centre and radius. [3]

(b) Find an equation of the tangent to the curve at the point (6, 3). [5]

27 The curve C is defined by $x = t - \sin t$, $y = 1 - \cos t$, for $0 \leq t \leq 2\pi$, as shown.



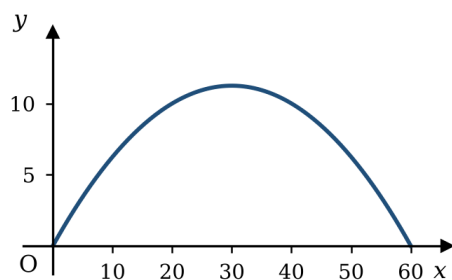
(a) Show that $\frac{dy}{dx} = \cot\left(\frac{t}{2}\right)$. [4]

(b) Find the gradient of C at $t = \frac{\pi}{2}$. [1]

(c) Find the coordinates of the highest point of C. [2]

(d) Find an equation of the tangent at $t = \frac{\pi}{2}$. [3]

28 A ball is hit so that its position t seconds later is $x = 20t$, $y = 15t - 5t^2$, where x and y are in metres, as shown. The ground is the x -axis.



(a) Show that the Cartesian equation of the path is $y = \frac{3}{4}x - \frac{x^2}{80}$. [2]

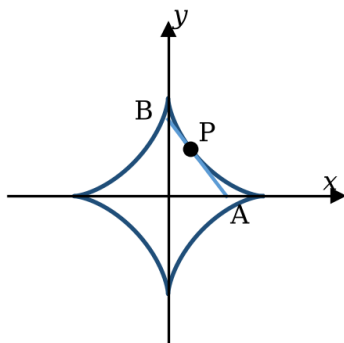
(b) Find $\frac{dy}{dx}$ in terms of t . [2]

(c) Find the angle the path makes with the horizontal at the start, to 1 decimal place. [2]

(d) Find the greatest height reached. [2]

(e) Find the angle at which the ball reaches the ground. [1]

29 The curve C has parametric equations $x = a \cos^3 t$, $y = a \sin^3 t$, where $a > 0$ and $0 < t < \frac{\pi}{2}$. The tangent at the point P on C meets the x -axis at A and the y -axis at B, as shown.



(a) Show that $\frac{dy}{dx} = -\tan t$. [4]

(b) Show that the tangent at P has equation $x \sin t + y \cos t = a \sin t \cos t$. [4]

(c) Hence prove that the length AB is the same for every position of P, and find it. [4]

30 A curve has parametric equations $x = \ln t$, $y = t + \frac{1}{t}$, for $t > 0$.

(a) Show that $\frac{dy}{dx} = t - \frac{1}{t}$. [3]

(b) Show that $\frac{d^2y}{dx^2} = y$. [4]

(c) Find the coordinates of the stationary point and use (b) to justify that it is a minimum. [4]

Trigonometric, Exponential and Logarithmic Differentiation

1 Differentiate $y = 3 \sin x - 2 \cos x + 5$ with respect to x . [2]

2 Given that $f(x) = 3e^{4x} - \ln 2x$, $x > 0$, find $f'(x)$. [3]

3 Differentiate $y = 2 \tan x - 3x$ with respect to x . [2]

4 Differentiate $y = 6e^{x/2}$. [2]

5 Find $f'(x)$ when $f(x) = 5 \ln x - 2x$, $x > 0$. [2]

6 Differentiate $y = \sin 4x - \cos 2x$. [3]

7 Differentiate with respect to x :

(a) $y = \tan 2x$ [2]

(b) $y = \ln(3x^2 - 1)$ [2]

8 The curve C has equation $y = e^{2x} - 3x$.

(a) Find an equation of the tangent to C at the point where $x = 0$. [3]

(b) Find the exact x -coordinate of the stationary point of C . [2]

9 The curve C has equation $y = 12 \ln x - x^2$, for $x > 0$. Find the set of values of x for which y is a decreasing function of x . [4]

10 Given that $y = 5^x$, find the exact gradient of the curve at the point where $x = 0$. [4]

11 The curve C has equation $y = e^{2x} - 6e^x - 8x$.

(a) Show that the x -coordinates of the stationary points of C satisfy $e^{2x} - 3e^x - 4 = 0$. [2]

(b) Hence find the exact x -coordinate of the stationary point, and show that it is a minimum. [5]

12 Given that $y = 4 - 3e^{-2x}$, show that $\frac{dy}{dx} = 8 - 2y$. [3]

13 Show that the derivative of $y = \ln(\sin x)$, $0 < x < \pi$, is $\cot x$. [3]

14 The curve C has equation $y = \ln(\cos 2x)$, $0 \leq x < \frac{\pi}{4}$.

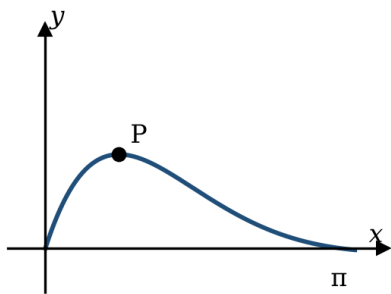
(a) Show that $\frac{dy}{dx} = -2 \tan 2x$. [3]

(b) Find the gradient of C at $x = \frac{\pi}{8}$. [1]

(c) Explain why the greatest value of y on this interval is 0. [2]

- 15** The curve C has equation $y = e^{-x} \cos 2x$.
- (a) Show that $\frac{dy}{dx} = -e^{-x}(\cos 2x + 2 \sin 2x)$. [3]
- (b) Find the smallest positive x-coordinate of a stationary point of C, to 3 significant figures. [3]
-
- 16** Find the values of x in the interval $0 < x < 2\pi$ for which the curve $y = e^x \cos x$ has a stationary point. Give your answers as exact multiples of π . [5]
-
- 17** Given that $y = \ln(1 + e^x)$, show that $\frac{dy}{dx} = \frac{1}{1 + e^{-x}}$. [4]
-
- 18** Prove that $\frac{d}{dx}(\sec x) = \sec x \tan x$, and hence find the gradient of the curve $y = \sec x$ at $x = \frac{\pi}{3}$. [5]
-
- 19** Given that a is a positive constant and $y = a^x$, show that $\frac{dy}{dx} = a^x \ln a$. [3]
-
- 20** Given that $y = \ln\left(\frac{x+1}{x-1}\right)$, $x > 1$, show that $\frac{dy}{dx} = -\frac{2}{x^2 - 1}$. [4]
-
- 21** The curve C has equation $y = \ln(\sec x + \tan x)$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$.
- (a) Show that $\frac{dy}{dx} = \sec x$. [4]
- (b) Find the gradient of C at $x = \frac{\pi}{3}$. [1]
- (c) Find an equation of the tangent to C at $x = \frac{\pi}{3}$. [3]
-
- 22** A curve C has equation $y = \frac{(x-1)^3(2x+1)^2}{(x+2)^4}$, for $x > 1$.
- (a) By taking natural logarithms of both sides, show that $\frac{dy}{dx} = y\left[\frac{3}{x-1} + \frac{4}{2x+1} - \frac{4}{x+2}\right]$. [5]
- (b) Find the exact gradient of C at the point where $x = 2$. [2]
-
- 23** The curve C has equation $y = x^{1/x}$, for $x > 0$.
- (a) By taking natural logarithms of both sides, show that $\frac{dy}{dx} = x^{1/x} \times \frac{1 - \ln x}{x^2}$. [4]
- (b) Find the exact coordinates of the stationary point of C and show that it is a maximum. [5]
-

- 24** The diagram shows the curve with equation $y = e^{-x} \sin x$, for $0 \leq x \leq \pi$. The curve has a maximum at the point P.



- (a) Show that $\frac{dy}{dx} = e^{-x}(\cos x - \sin x)$. [3]
 (b) Find the exact coordinates of P. [3]
 (c) Show that $\frac{d^2y}{dx^2} = -2e^{-x} \cos x$, and use it to confirm that P is a maximum. [4]

- 25** A mug of coffee cools so that its temperature, T °C, after t minutes is modelled by $T = 20 + 70e^{-0.1t}$, for $t \geq 0$.

- (a) Find the initial temperature, and the temperature the coffee approaches in the long term. [2]
 (b) Show that $\frac{dT}{dt} = -0.1(T - 20)$. [3]
 (c) Find the time when the coffee is cooling at 2 °C per minute, and its temperature then. Give your answers to 3 significant figures. [4]

- 26** The curve C has equation $y = e^{-x} \sin 3x$.

- (a) Show that $\frac{dy}{dx}$ can be written in the form $Re^{-x} \cos(3x + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$, giving the exact value of R and the value of $\tan \alpha$. [5]
 (b) Hence find the exact x -coordinate of the stationary point of C with the smallest positive x -coordinate. [4]

- 27** Use the chain rule and standard derivatives in this question.

- (a) Show that if $y = \ln(x + \sqrt{x^2 + 1})$ then $\frac{dy}{dx} = \frac{1}{\sqrt{x^2 + 1}}$. [5]
 (b) Find the gradient of the curve at $x = \sqrt{3}$. [1]
 (c) Show that $\frac{d^2y}{dx^2} = -x(x^2 + 1)^{-3/2}$ and deduce that the gradient is decreasing for $x > 0$. [3]

- 28** A student claims that $\frac{d}{dx}(2^x) = x \times 2^{x-1}$.

- (a) Explain the student's error. [2]
 (b) Prove that $\frac{d}{dx}(2^x) = 2^x \ln 2$. [2]
 (c) The curve $y = 2^x - x^2$ has a stationary point with x -coordinate α , where $0 < \alpha < 1$. Show that α is a root of $f(x) = 2^x \ln 2 - 2x = 0$ and that this root lies between 0 and 1. [3]
 (d) Show that $f(x) = 0$ can be rearranged as $x = 2^{x-1} \ln 2$. Use the iteration $x_{n+1} = 2^{x_n-1} \ln 2$ with $x_0 = 0.5$ to find x_1 , x_2 and x_3 to 3 decimal places. [3]

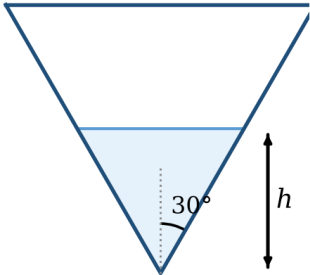
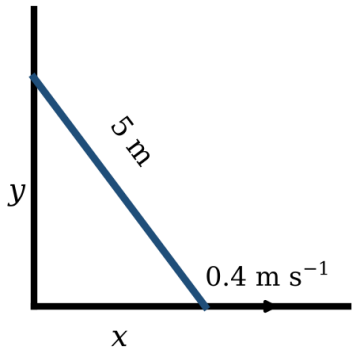
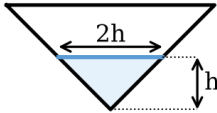
- 29** The curve C has equation $y = e^{-x} \sin 2x$.

- (a) Show that $\frac{dy}{dx} = e^{-x}(2\cos 2x - \sin 2x)$ and that $\frac{d^2y}{dx^2} = e^{-x}(-3\sin 2x - 4\cos 2x)$. [5]
 (b) Hence show that $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 0$. [3]
 (c) Find, to 3 significant figures, the coordinates of the first stationary point with $x > 0$. [4]

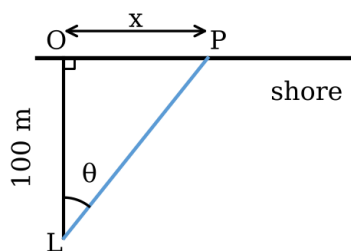
- 30** The curve C has equation $y = x^{\sin x}$, for $x > 0$.
- (a) By taking natural logarithms, show that $\frac{dy}{dx} = x^{\sin x}(\cos x \ln x + \frac{\sin x}{x})$. [5]
- (b) Show that the tangent to C at the point where $x = \frac{\pi}{2}$ passes through the origin. [4]

Connected Rates of Change

- 1** The radius of a circular ripple increases at a constant rate of 0.2 cm s^{-1} . Find the rate at which the area of the ripple is increasing when the radius is 5 cm. [3]
- 2** The side length $x \text{ cm}$ of a cube is increasing at 0.5 cm s^{-1} . Find the rate of increase of the volume of the cube when $x = 4$. [3]
- 3** The radius $r \text{ cm}$ of a circle is increasing at a rate of 0.4 cm s^{-1} . Find the rate at which the circumference is increasing. [3]
- 4** The pressure $P \text{ kPa}$ of a gas trapped in a cylinder is given by $P = \frac{60}{x}$, where $x \text{ cm}$ is the length of the cylinder. The piston is pushed in so that x decreases at a rate of 0.5 cm s^{-1} . Find the rate at which the pressure is changing when $x = 4$. [3]
- 5** Variables x and y are related by $y = x^3 + 2x$. Given that x is increasing at a rate of 2 units per second, find the rate of change of y when $x = 1$. [3]
- 6** The radius of a sphere is increasing at 0.1 mm s^{-1} . Given that $V = \frac{4}{3}\pi r^3$, find the rate at which the volume is increasing when $r = 5 \text{ mm}$. [3]
- 7** Liquid dye is poured onto a flat cloth and forms a circular stain. The area of the stain increases at a steady $2 \text{ cm}^2 \text{ s}^{-1}$, and the stain has zero area at time $t = 0$.
- (a) Find the radius of the stain after 5 seconds, to 3 significant figures. [2]
- (b) Find the rate at which the radius is increasing at that instant, to 3 significant figures. [3]
- 8** The area of a circular oil slick increases at a constant rate of $3 \text{ m}^2 \text{ s}^{-1}$.
- (a) Show that the radius $r \text{ metres}$ of the slick satisfies $\frac{dr}{dt} = \frac{3}{2\pi r}$. [3]
- (b) Find the rate at which the radius is increasing when the area of the slick is 50 m^2 , to 3 significant figures. [2]
- 9** A cone has a fixed height of 12 cm. Its base radius $r \text{ cm}$ is increasing at a constant rate of 0.5 cm s^{-1} . Find the rate at which the volume $V = \frac{1}{3}\pi r^2 h$ is increasing when $r = 3$. Give your answer in terms of π . [4]
- 10** The volume of a cube is increasing at $24 \text{ cm}^3 \text{ s}^{-1}$. Find, when the side length x is 2 cm, (a) the rate of increase of x , (b) the rate of increase of the total surface area. [5]
- 11** Variables x and y are related by $y = x^2 + \frac{1}{x}$. Given that x is increasing at a rate of 3 units per second, find the rate of change of y when $x = 2$. [4]
- 12** Water is poured into a cylindrical tank of radius 10 cm at a constant rate of $200 \text{ cm}^3 \text{ s}^{-1}$. Find the rate at which the depth of water is rising. [3]

- 13** The surface area S of a spherical balloon increases at $8\pi \text{ cm}^2 \text{ s}^{-1}$. Given that $S = 4\pi r^2$ and $V = \frac{4}{3}\pi r^3$, find, when $r = 4 \text{ cm}$, (a) the rate of increase of r , (b) the rate of increase of V . [5]
- 14** Water is poured at a constant rate of $8 \text{ cm}^3 \text{ s}^{-1}$ into a container in the shape of an inverted cone. The cone has semi-vertical angle 30° and axis vertical, and the depth of water is $h \text{ cm}$, as shown.
- 
- (a) Show that the volume of water $V \text{ cm}^3$ is given by $V = \frac{\pi h^3}{9}$. [3]
 (b) Find the rate at which the depth is increasing when $h = 6$, to 3 significant figures. [3]
- 15** A ladder of length 5 m rests with its top against a smooth vertical wall and its foot on horizontal ground. The foot of the ladder slides away from the wall at 0.4 m s^{-1} . At time t seconds, the foot is $x \text{ m}$ from the wall and the top is $y \text{ m}$ above the ground, as shown.
- 
- (a) Find the rate at which the top of the ladder is moving down the wall at the instant when $x = 3$. [7]
- 16** A cube of ice melts so that its volume decreases at a constant rate of 6 cm^3 per minute. Find the rate at which the total surface area is decreasing when the side length is 3 cm . Assume that the ice stays a cube. [6]
- 17** A water trough has length 3 m . Its cross-section is a right-angled isosceles triangle with the right angle at the bottom, so when the depth is $h \text{ m}$ the surface width is $2h \text{ m}$, as shown. Water is poured in at a constant rate of 0.06 m^3 per minute.
- 
- (a) Show that the volume of water is $V = 3h^2$. [2]
 (b) Find the rate at which the depth is increasing when $h = 0.5 \text{ m}$. [3]
- 18** A circular ripple spreads so that its radius r metres at time t seconds is $r = 2t + 1$. Find the rate at which the area of the ripple is increasing when $t = 3$. Give your answer to 3 significant figures. [4]

- 19** A lighthouse L is 100 m from a straight shore, with O the nearest point on the shore. The beam from L rotates at a constant rate of 0.05 radians per second, and the spot P of light on the shore is a distance x metres from O, as shown.

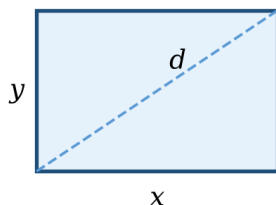


- (a) Show that $x = 100 \tan \theta$. [1]
 (b) Find the speed of the spot of light along the shore when $\theta = \frac{\pi}{4}$. [4]

- 20** The volume of a spherical balloon is increasing at a constant rate of $40\pi \text{ cm}^3 \text{ s}^{-1}$. Show that $\frac{dr}{dt} = \frac{10}{r^2}$, and find the radius of the balloon at the instant when its radius is increasing at 0.1 cm s^{-1} . [5]

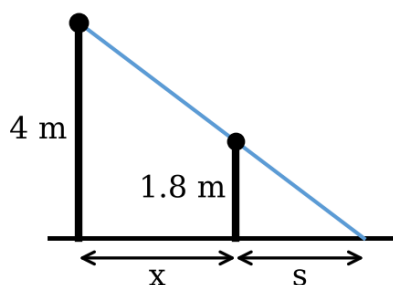
- 21** The radius $r \text{ cm}$ of a circular patch of oil on water, t seconds after it forms, is modelled by $r = 4(1 - e^{-t/2})$.
 (a) Show that the area $A \text{ cm}^2$ of the patch satisfies $\frac{dA}{dt} = 16\pi(e^{-t/2} - e^{-t})$. [5]
 (b) Find the exact time at which the area is increasing most rapidly, and the exact rate of increase of the area at that time. [5]

- 22** The sides of a rectangle are $x \text{ cm}$ and $y \text{ cm}$. At a certain instant $x = 6$ and $y = 10$, with x increasing at 2 cm s^{-1} and y decreasing at 1 cm s^{-1} , as shown. The area of the rectangle is $A \text{ cm}^2$ and the length of its diagonal is $d \text{ cm}$.

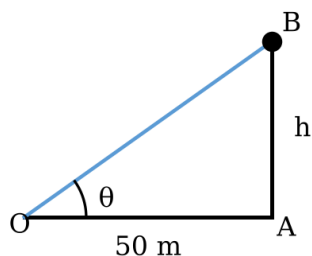


- (a) Find the rate of change of A at this instant. [3]
 (b) Find the rate of change of the perimeter. [1]
 (c) Find the rate of change of d , to 3 significant figures. [3]
 (d) Assuming the rates stay constant, find the time at which the area stops increasing. [1]

- 23** A street lamp is 4 m above horizontal ground. A man of height 1.8 m walks directly away from the base of the lamp at a constant speed of 1.5 m s^{-1} . At time t the man is x m from the base of the lamp and his shadow has length s m, as shown.

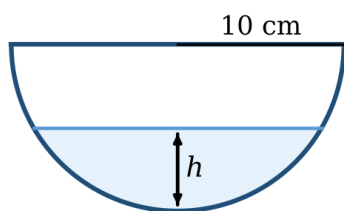


- (a) Use similar triangles to show that $s = \frac{9}{11}x$. [3]
- (b) Find the rate at which the length of the shadow is increasing. [2]
- (c) Find the speed at which the tip of the shadow moves away from the lamp. [3]
-
- 24** A solid cylinder has radius r cm and height h cm. At a certain instant $r = 5$ and $h = 12$, the radius is increasing at 0.02 cm s^{-1} and the height is decreasing at 0.05 cm s^{-1} .
- (a) Find the rate of change of the volume at this instant, and state whether the volume is increasing or decreasing. [5]
- (b) Find the rate of change of the curved surface area, $2\pi rh$, at this instant. [3]
-
- 25** The variables x and y satisfy $x^2 + xy + y^2 = 7$, and x is increasing at a constant rate of 0.5 units per second.
- (a) Find the rate of change of y at the instant when $x = 1$ and $y = 2$. [5]
- (b) Find the exact values of x at the instants when y is neither increasing nor decreasing. [3]
-
- 26** A balloon B rises vertically from a point A on horizontal ground at a constant speed of 2 m s^{-1} . An observer at O is on the ground, 50 m from A. The angle of elevation of the balloon from O is θ , and the height of the balloon is h m, as shown.



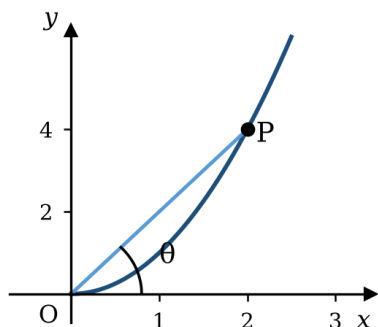
- (a) Show that $\frac{d\theta}{dt} = \frac{1}{25}\cos^2\theta$. [3]
- (b) Find the rate of change of θ when the balloon is 50 m above the ground, and the value of θ at which the angle of elevation is increasing most rapidly. [4]

- 27** A bowl is a hemisphere of radius 10 cm, with its rim horizontal. When the depth of water in the bowl is h cm, the volume of water is $V = \pi(10h^2 - \frac{h^3}{3})$ cm³, as shown.



- Show that $\frac{dV}{dh} = \pi h(20 - h)$. [2]
- Water flows in at a constant $30 \text{ cm}^3 \text{ s}^{-1}$. Find the rate at which the depth is increasing when $h = 5$, to 3 significant figures. [3]
- Find the depth at which the water level is rising most slowly, and the rate at that depth. [3]
- A small hole now lets water leave at $20 \text{ cm}^3 \text{ s}^{-1}$ while the inflow stays at $30 \text{ cm}^3 \text{ s}^{-1}$. Find $\frac{dh}{dt}$ when $h = 5$. [2]

- 28** The point P moves along the curve $y = x^2$, for $x > 0$, so that its x-coordinate increases at a constant rate of 2 units per second. O is the origin and θ is the angle between OP and the x-axis, as shown.



- Find the rate of change of the distance OP when $x = 2$, in exact form and to 3 significant figures. [4]
- Find the rate at which the y-coordinate is increasing when $x = 2$. [2]
- Show that $\tan \theta = x$ and hence find the rate of change of θ when $x = 2$. [4]

- 29** The volume $V \text{ cm}^3$ of a spherical balloon at time t seconds after inflation begins is $V = 36\pi t^2$, $t > 0$.

- Show that the radius is $r = 3t^{2/3}$. [3]
- Find the rate at which the radius is increasing when $t = 8$. [3]
- Find the rate at which the surface area $S = 4\pi r^2$ is increasing when $t = 8$, and show that the ratio $\frac{dS}{dt} \div \frac{dV}{dt}$ is $\frac{2}{r}$ for all t . [5]

- 30** A rectangle has a fixed perimeter of 40 cm. Its length x cm is increasing at a constant rate of 0.5 cm s^{-1} , where $0 < x < 20$.

- Show that the area $A \text{ cm}^2$ satisfies $\frac{dA}{dt} = 10 - x$. [4]
- Find the value of x at which the area stops increasing, and explain why the area is then a maximum. [3]
- Let the diagonal of the rectangle have length D cm. Find $\frac{dD}{dt}$ when $x = 6$, and show that D is least when the area is greatest. [6]