

Introduction to Complex Numbers

1 (2 marks)

$$\sqrt{-25} = 5i$$

B1

$$\sqrt{-18} = \sqrt{18}i = 3\sqrt{2}i$$

B1

Answer: (a) 5i (b) $3\sqrt{2}i$

2 marks

2 (2 marks)

(a) Real part 4, imaginary part -7

B1

(b) Real part 0, imaginary part 3

B1

Answer: (a) 4 and -7 (b) 0 and 3

2 marks

3 (3 marks)

$$i^3 = i^2 \times i = -i$$

B1

$$i^4 = (i^2)^2 = 1$$

B1

$$i^5 = i^4 \times i = i$$

B1

Answer: $-i$, 1, i

3 marks

4 (2 marks)

$$x^2 = -16$$

M1

$$x = \pm 4i$$

A1

Answer: $x = \pm 4i$

2 marks

5 (2 marks)

$$\sqrt{-36} = 6i \text{ and } \sqrt{-9} = 3i$$

B1

$$\text{Sum} = 9i$$

B1

Answer: $9i$

2 marks

6 (2 marks)

Equating real parts: $x = 5$

B1

Equating imaginary parts: $y = -3$

B1

Answer: $x = 5$, $y = -3$

2 marks

7 (3 marks)

$$i^{27} = (i^4)^6 \times i^3$$

M1

$$= -i$$

A1

$$i^{-1} = \frac{1}{i} = \frac{i}{i^2} = -i$$

B1

Answer: $i^{27} = -i$, $i^{-1} = -i$

3 marks

8 (3 marks)

$$x = \frac{-2 \pm \sqrt{4 - 20}}{2}$$

M1

$$= \frac{-2 \pm 4i}{2}$$

A1

$$x = -1 \pm 2i$$

A1

Answer: $x = -1 + 2i$ or $x = -1 - 2i$

3 marks

9 (2 marks)

Real parts: $x + 3 = 7$, so $x = 4$

B1

Imaginary parts: $y - 2 = 5$, so $y = 7$

B1

Answer: $x = 4$, $y = 7$

2 marks

10 (2 marks)

(a) $(2i)^3 = 8i^3 = -8i$

B1

(b) $(-3i)^2 = 9i^2 = -9$

B1

Answer: (a) $-8i$ (b) -9

2 marks

11 (3 marks)The rule $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ is only valid when at least one of a, b is positive.

E1

$\sqrt{-4} = 2i$ and $\sqrt{-9} = 3i$

M1

Product = $6i^2 = -6$

A1

Answer: -6

3 marks

12 (3 marks)

$2i(3 - i) = 6i - 2i^2$

M1

$= 2 + 6i$

A1

$p = 2, q = 6$

A1

Answer: $p = 2, q = 6$

3 marks

13 (3 marks)

$$x = \frac{-1 \pm \sqrt{1 - 24}}{4}$$

M1

$\sqrt{-23} = i\sqrt{23}$

A1

$$x = \frac{-1 \pm i\sqrt{23}}{4}$$

A1

Answer: $x = \frac{-1 \pm i\sqrt{23}}{4}$

3 marks

14 (3 marks)

$i^{101} = i, i^{102} = -1, i^{103} = -i, i^{104} = 1$ (powers repeat every 4)

M1 A1

Sum = $i - 1 - i + 1 = 0$

A1

Answer: 0

3 marks

15 (4 marks)

Real parts: $2x + y = 8$ (1)

B1

Imaginary parts: $x - 3y = -3$ (2)

B1

(1) $\times 3 +$ (2): $7x = 21$, so $x = 3$

M1

$y = 8 - 6 = 2$

A1

Answer: $x = 3, y = 2$

4 marks

16 (4 marks)

$(1 + i)^2 = 1 + 2i + i^2$

M1

$= 2i$ (shown)

A1

$(1 + i)^8 = ((1 + i)^2)^4 = (2i)^4$

M1

$= 16i^4 = 16$

A1

Answer: 16

4 marks

17 (4 marks)

$$x = \frac{6 \pm \sqrt{36 - 52}}{2}$$

M1

$x = 3 \pm 2i$

A1

Sum = $(3 + 2i) + (3 - 2i) = 6$

B1

Product = $9 - 4i^2 = 13$

B1

Answer: $x = 3 \pm 2i$

4 marks

18 (4 marks)

$$i^{15} = i^3 = -i$$

B1

$$i^{10} = i^2 = -1$$

B1

$$i^{-3} = \frac{1}{i^3} = \frac{1}{-i} = i$$

M1

$$\text{Total} = -i + 1 + 3i = 1 + 2i$$

A1

Answer: $1 + 2i$

4 marks

19 (5 marks)

$$\text{No real roots: } k^2 - 16 < 0$$

M1

$$-4 < k < 4$$

A1

$$k = 2: x^2 + 2x + 4 = 0, x = \frac{-2 \pm \sqrt{4 - 16}}{2}$$

M1

$$= \frac{-2 \pm 2i\sqrt{3}}{2}$$

A1

$$x = -1 \pm i\sqrt{3}$$

A1

Answer: $-4 < k < 4$; $x = -1 \pm i\sqrt{3}$

5 marks

20 (4 marks)

$$(x + 2i)^2 = x^2 - 4 + 4xi$$

M1 A1

$$(a) \text{ Real when } 4x = 0, \text{ so } x = 0$$

A1

$$(b) \text{ Purely imaginary when } x^2 - 4 = 0, \text{ so } x = \pm 2$$

A1

Answer: (a) $x = 0$ (b) $x = \pm 2$

4 marks

21 (6 marks)

$$(x + yi)^2 = x^2 - y^2 + 2xyi$$

M1

$$x^2 - y^2 = 5 \text{ and } 2xy = 12$$

A1

$$y = \frac{6}{x}: x^2 - \frac{36}{x^2} = 5, \text{ so } x^4 - 5x^2 - 36 = 0$$

M1

$$(x^2 - 9)(x^2 + 4) = 0$$

A1

$$x^2 = 9 \text{ (x real), so } x = \pm 3$$

A1

$$x = 3, y = 2 \text{ or } x = -3, y = -2$$

A1

Answer: (3, 2) or (-3, -2)

6 marks

22 (5 marks)

$$\text{Any four consecutive powers sum to } i - 1 - i + 1 = 0$$

M1 A1

$$(a) \text{ The 40 terms form 10 complete cycles, so the sum is 0}$$

A1

$$(b) \text{ Extra terms } i^{41} + i^{42} = i + (-1)$$

M1

$$\text{Sum} = -1 + i$$

A1

Answer: (a) 0 (b) $-1 + i$

5 marks

23 (5 marks)

$$z = \frac{4 \pm \sqrt{16 - 4k}}{2} = 2 \pm i\sqrt{k - 4}$$

M1 A1

$$\text{Imaginary parts are } \pm\sqrt{k - 4}, \text{ which differ by } 2\sqrt{k - 4}$$

M1

$$2\sqrt{k - 4} = 6, \text{ so } k - 4 = 9 \text{ and } k = 13$$

A1

$$\alpha, \beta = 2 \pm 3i$$

A1

Answer: $k = 13$, roots $2 \pm 3i$

5 marks

24 (6 marks)

$$(a) (1 + i)^2 = 2i, \text{ so } (1 + i)^4 = (2i)^2 = -4$$

M1

$$(1 + i)^4 + 4 = 0$$

A1

Answer: Shown

2 marks

$$(b) \text{ Check: } (-1 + i)^2 = -2i, \text{ so } (-1 + i)^4 = -4; (1 - i)^2 = -2i; (-1 - i)^2 = 2i$$

M1

$$\text{Each of } 1 + i, 1 - i, -1 + i, -1 - i \text{ has fourth power } -4$$

A1

A quartic has at most four roots, so these are all of them

E1

Roots: $\pm 1 \pm i$

A1

Answer: $z = \pm 1 \pm i$

4 marks

25 (5 marks)

$$(a + i)^3 = a^3 + 3a^2i + 3ai^2 + i^3$$

M1

$$= (a^3 - 3a) + (3a^2 - 1)i$$

A1

Purely imaginary: real part zero, $a^3 - 3a = 0$

M1

$$a(a^2 - 3) = 0$$

A1

$$a = 0 \text{ or } a = \pm\sqrt{3}$$

A1

Answer: $a = 0, \pm\sqrt{3}$

5 marks

26 (5 marks)

i^n takes the values 1, i , -1 , $-i$ as $n = 0, 1, 2, 3 \pmod{4}$

B1

$$n \equiv 0: 1 + 1 = 2$$

M1 A1

$n \equiv 1 \text{ or } 3: i^{-1} = -i$, so $i + (-i) = 0$; likewise for $n \equiv 3$

A1

$$n \equiv 2: -1 + (-1) = -2$$

A1

Answer: 2 (n multiple of 4), 0 (n odd), -2 ($n \equiv 2 \pmod{4}$)

5 marks

27 (9 marks)

$$(a + i)^2 = a^2 - 1 + 2ai$$

B1

$$(a + i)^4 = (a^2 - 1)^2 - 4a^2 + 4a(a^2 - 1)i$$

M1 A1

$$(a) \text{ Real: } 4a(a^2 - 1) = 0$$

M1

$$a = 0, 1 \text{ or } -1$$

A1

$$(b) \text{ Purely imaginary: } (a^2 - 1)^2 = 4a^2$$

M1

$$a^2 - 1 = 2a \text{ or } a^2 - 1 = -2a$$

M1

$$a^2 - 2a - 1 = 0 \text{ gives } a = 1 \pm \sqrt{2}; a^2 + 2a - 1 = 0 \text{ gives } a = -1 \pm \sqrt{2}$$

M1

$$a = \pm 1 \pm \sqrt{2} \text{ (four values)}$$

A1

Answer: (a) $a = 0, \pm 1$ (b) $a = \pm 1 \pm \sqrt{2}$

9 marks

28 (11 marks)

$$(a) (x + yi)^2 = x^2 - y^2 + 2xyi$$

M1

Equate real and imaginary parts

A1

Answer: Shown

2 marks

$$(b) y = \frac{q}{2x}, \text{ so } x^2 - \frac{q^2}{4x^2} = p$$

M1

$$\text{Multiply by } 4x^2: 4x^4 - q^2 = 4px^2$$

A1

$$x^2 = \frac{4p \pm \sqrt{16p^2 + 16q^2}}{8} = \frac{p \pm \sqrt{p^2 + q^2}}{2}$$

M1 A1

x^2 must be positive, so the + sign is taken

E1

Answer: Shown

5 marks

$$(c) p = 21, q = 20: \sqrt{441 + 400} = 29$$

M1

$$x^2 = \frac{21 + 29}{2} = 25, \text{ so } x = \pm 5$$

A1

$$y = \frac{q}{2x} = \frac{10}{x} = \pm 2$$

M1

Square roots: $\pm(5 + 2i)$. Check: $(5 + 2i)^2 = 25 - 4 + 20i$

A1

Answer: $\pm(5 + 2i)$

4 marks

29 (9 marks)

(a) $(a + bi)^2 = a^2 - b^2 + 2abi$

M1

$(a - bi)^2 = a^2 - b^2 - 2abi$

M1

Equal iff the imaginary parts are equal: $2ab = -2ab$

A1

$4ab = 0$, so $a = 0$ or $b = 0$

A1

 $a = 0$: purely imaginary; $b = 0$: real

E1

Conversely if $a = 0$ or $b = 0$ then $2ab = 0$ and the two squares are equal.

E1

Answer: Shown

6 marks

(b) From (a), $a = 0$ or $b = 0$

M1

$a = 0$: $b^2 = 25$, so $b = \pm 5$, giving $\pm 5i$

A1

$b = 0$: $a = \pm 5$

A1

Answer: ± 5 and $\pm 5i$

3 marks

30 (9 marks)

(a) $u_1 = i - i = 0$, $u_2 = -1 + (-1) = -2$

B1

$u_3 = -i + i = 0$, $u_4 = 1 + 1 = 2$

B1

Answer: 0, -2, 0, 2

2 marks

(b) $(-i)^n = (-1)^n i^n$, so $u_n = i^n(1 + (-1)^n)$

M1 A1

n odd: $1 + (-1)^n = 0$, so $u_n = 0$

A1

n even: $u_n = 2i^n = 2(i^2)^{\frac{n}{2}} = 2(-1)^{\frac{n}{2}}$

A1

Answer: Shown

4 marks

(c) Only even $n = 2m$ contribute, giving $2(-1)^m$ for $m = 1, \dots, 50$

M1

These terms alternate $-2, +2, \dots$ with 50 terms, in 25 cancelling pairs

A1

Sum = 0

A1

Answer: 0

3 marks

Complex Arithmetic and Conjugates**1 (2 marks)**

$z + w = 4 - 2i$

B1

$z - w = 2 + 6i$

B1

Answer: $4 - 2i$; $2 + 6i$

2 marks

2 (3 marks)

$2z = 4 - 2i$

B1

$2z - w = -5i$

B1

$z + 3w = 14 + 8i$

B1

Answer: $-5i$; $14 + 8i$

3 marks

3 (3 marks)

(a) $5 + 2i$

B1

(b) $-7i$

B1

(c) -3 (a real number is its own conjugate)

B1

Answer: (a) $5 + 2i$ (b) $-7i$ (c) -3

3 marks

4 (2 marks)

$(2 + i)(3 - i) = 6 - 2i + 3i - i^2$

M1

$= 7 + i$

A1

Answer: $7 + i$

2 marks

5 (2 marks)

$= 9 - 12i + 12i - 16i^2$

M1

$= 9 + 16 = 25$

A1

Answer: 25

2 marks

6 (2 marks)

$$\frac{1}{i} = \frac{i}{i \times i} = \frac{i}{i^2}$$

M1

$$= -i$$

A1

Answer: $-i$

2 marks

7 (2 marks)

$$(1 + 3i)^2 = 1 + 6i + 9i^2$$

M1

$$= -8 + 6i$$

A1

Answer: $-8 + 6i$

2 marks

8 (3 marks)

Multiply numerator and denominator by $1 + i$: $\frac{(3 + i)(1 + i)}{(1 - i)(1 + i)}$

M1

Numerator = $3 + 3i + i + i^2 = 2 + 4i$; denominator = $1 - i^2 = 2$

A1

$$= 1 + 2i$$

A1

Answer: $1 + 2i$

3 marks

9 (3 marks)

$$z^* = 4 + 3i$$

B1

$$z + z^* = 8$$

B1

$$zz^* = 16 + 9 = 25$$

B1

Answer: $z + z^* = 8$, $zz^* = 25$

3 marks

10 (3 marks)

$$z = \frac{4 + 3i}{1 + 2i} = \frac{(4 + 3i)(1 - 2i)}{(1 + 2i)(1 - 2i)}$$

M1

$$= \frac{4 - 8i + 3i + 6}{5} = \frac{10 - 5i}{5}$$

A1

$$z = 2 - i$$

A1

Answer: $z = 2 - i$

3 marks

11 (3 marks)

$$(2 + 3i)^2 = 4 + 12i - 9 = -5 + 12i$$

B1

$$(2 - 3i)^2 = -5 - 12i$$

B1

$$\text{Difference} = 24i$$

B1

Answer: $24i$

3 marks

12 (4 marks)

(a) $z^2 = 9 - 6i + i^2$

M1

$$= 8 - 6i$$

A1

(b) $(z^*)^2 = (3 + i)^2 = 8 + 6i$

A1

$(z^*)^2$ is the conjugate of z^2

B1

Answer: $8 - 6i$; $8 + 6i$

4 marks

13 (4 marks)

$$(x + iy)(2 + i) = 2x - y + (x + 2y)i$$

M1 A1

$$2x - y = 3 \text{ and } x + 2y = 4$$

A1

$$x = 2, y = 1$$

A1

Answer: $x = 2$, $y = 1$

4 marks

14 (4 marks)

$$\frac{(2+i)(3+i)}{(3-i)(3+i)} = \frac{6+2i+3i-1}{10}$$

M1

$$= \frac{5+5i}{10} = \frac{1}{2} + \frac{1}{2}i$$

A1

$$\text{Square} = \frac{1}{4}(1+i)^2 = \frac{1}{4} \times 2i$$

M1

$$= \frac{1}{2}i$$

A1

$$\text{Answer: } \frac{1}{2} + \frac{1}{2}i; \frac{1}{2}i$$

4 marks

15 (4 marks)

$$\text{Let } z = x + iy: 3x + 3iy + 2x - 2iy = 5x + iy$$

M1 A1

$$5x = 10 \text{ and } y = -1$$

M1

$$z = 2 - i$$

A1

$$\text{Answer: } z = 2 - i$$

4 marks

16 (5 marks)

$$\frac{a+2i}{1-i} = \frac{(a+2i)(1+i)}{2}$$

M1

$$= \frac{a+ai+2i-2}{2} = \frac{a-2}{2} + \frac{a+2}{2}i$$

A1

$$(a) \text{ Purely imaginary: real part } 0, a = 2$$

A1

$$(b) \text{ Real: imaginary part } 0, \text{ so } a+2=0$$

M1

$$a = -2$$

A1

$$\text{Answer: (a) } a = 2 \text{ (b) } a = -2$$

5 marks

17 (4 marks)

$$\text{Subtract: } 3w = (4+i) - (1+4i)$$

M1

$$3w = 3 - 3i, \text{ so } w = 1 - i$$

A1

$$z = 4 + i - (1 - i)$$

M1

$$z = 3 + 2i$$

A1

$$\text{Answer: } z = 3 + 2i, w = 1 - i$$

4 marks

18 (4 marks)

$$z^2 = 4 + 4i - 1 = 3 + 4i$$

M1

$$z^2 - 4z + 5 = 3 + 4i - 8 - 4i + 5$$

M1

$$= 0$$

A1

$$2 - i \text{ is also a root: the coefficients are real, so roots come in conjugate pairs.}$$

E1

$$\text{Answer: Shown; } 2 - i$$

4 marks

19 (4 marks)

$$\frac{2+i}{1-i} = \frac{(2+i)(1+i)}{2} = \frac{1+3i}{2}$$

M1 A1

$$\frac{2-i}{1+i} \text{ is the conjugate of the first, } = \frac{1-3i}{2}$$

A1

$$\text{Sum} = 1 \text{ (real)}$$

A1

$$\text{Answer: } 1$$

4 marks

20 (3 marks)

$$z + z^* = (a + bi) + (a - bi) = 2a$$

B1

$$zz^* = (a + bi)(a - bi) = a^2 - b^2i^2 = a^2 + b^2$$

M1

$$a \text{ and } b \text{ are real, so } 2a \text{ and } a^2 + b^2 \text{ are real.}$$

E1

$$\text{Answer: Shown}$$

3 marks

21 (6 marks)

$$\text{Let } (x + iy)^2 = -8 + 6i, \text{ so } x^2 - y^2 = -8 \text{ and } 2xy = 6$$

M1 A1

$$y = \frac{3}{x}: x^2 - \frac{9}{x^2} = -8, \text{ so } x^4 + 8x^2 - 9 = 0$$

M1

$$(x^2 + 9)(x^2 - 1) = 0, \text{ so } x^2 = 1 \text{ (x real)}$$

A1

$$x = 1, y = 3 \text{ or } x = -1, y = -3$$

A1

$$\text{Square roots: } \pm(1 + 3i)$$

A1

$$\text{Answer: } \pm(1 + 3i)$$

6 marks

22 (5 marks)

$$\text{Let } z = x + iy: zz^* = x^2 + y^2 \text{ and } z - z^* = 2iy$$

M1 A1

$$x^2 + y^2 + 6iy = 13 + 12i$$

A1

$$\text{Imaginary parts: } 6y = 12, y = 2$$

M1

$$x^2 + 4 = 13, \text{ so } x = \pm 3: z = 3 + 2i \text{ or } z = -3 + 2i$$

A1

$$\text{Answer: } z = \pm 3 + 2i$$

5 marks

23 (5 marks)

$$zw = (ac - bd) + (ad + bc)i$$

M1 A1

$$(zw)^* = (ac - bd) - (ad + bc)i$$

A1

$$z^*w^* = (a - bi)(c - di) = ac - adi - bci + bdi^2$$

M1

$$= (ac - bd) - (ad + bc)i = (zw)^*$$

A1

$$\text{Answer: Proved}$$

5 marks

24 (5 marks)

$$\frac{1}{Z_1} = \frac{2 - i}{5} \text{ and } \frac{1}{Z_2} = \frac{1 + i}{2}$$

M1 A1

$$\frac{1}{Z} = \frac{2(2 - i) + 5(1 + i)}{10} = \frac{9 + 3i}{10}$$

M1

$$Z = \frac{10}{9 + 3i} = \frac{10(9 - 3i)}{90}$$

M1

$$Z = 1 - \frac{1}{3}i$$

A1

$$\text{Answer: } Z = 1 - \frac{1}{3}i$$

5 marks

25 (6 marks)

$$(a) w = \frac{(z + 1)(z^* - 1)}{(z - 1)(z^* - 1)}$$

M1

$$\text{Denominator} = (x - 1)^2 + y^2$$

A1

$$\text{Numerator} = zz^* - z + z^* - 1 = x^2 + y^2 - 1 - 2iy$$

M1

$$\text{Re } w = \frac{x^2 + y^2 - 1}{(x - 1)^2 + y^2}, \text{ Im } w = \frac{-2y}{(x - 1)^2 + y^2}$$

A1

$$\text{Answer: Re } w = \frac{x^2 + y^2 - 1}{(x - 1)^2 + y^2}, \text{ Im } w = \frac{-2y}{(x - 1)^2 + y^2}$$

4 marks

$$(b) \text{ If } x^2 + y^2 = 1 \text{ the real part is } 0$$

M1

$$\text{and the imaginary part is } \frac{-y}{1 - x} \text{ which is non-zero when } y \neq 0, \text{ so } w \text{ is purely imaginary}$$

A1

$$\text{Answer: Shown}$$

2 marks

26 (5 marks)

$$\text{From the second equation, } w = i - iz$$

M1

$$\text{Substitute: } z + i(i - iz) = 3 + 2i$$

M1

$$z - 1 + z = 3 + 2i$$

A1

$$z = 2 + i$$

A1

$$w = i - i(2 + i) = i - 2i + 1 = 1 - i$$

A1

$$\text{Answer: } z = 2 + i, w = 1 - i$$

5 marks

27 (12 marks)

$$(a) \frac{1}{z} = \frac{a - bi}{a^2 + b^2}$$

M1 A1

$$\text{Imaginary part of } z + \frac{1}{z} = b - \frac{b}{a^2 + b^2}$$

A1

$$= b(1 - \frac{1}{a^2 + b^2}) = 0$$

M1

$$b = 0 \text{ or } a^2 + b^2 = 1$$

A1

Answer: Shown

5 marks

$$(b) z^2 - z + 1 = 0$$

M1

$$z = \frac{1 \pm \sqrt{1 - 4}}{2}$$

M1

$$z = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

A1

(check: $a^2 + b^2 = 1$, consistent with part (a))

A1

$$\text{Answer: } z = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

4 marks

(c) Non-real z needs $a^2 + b^2 = 1$, then $z + 1/z = 2a = \frac{5}{2}$, so $a = \frac{5}{4}$, but $a^2 \leq 1$: impossible

M1 A1

$$\text{Real: } 2z^2 - 5z + 2 = 0, z = 2 \text{ or } z = \frac{1}{2}$$

A1

$$\text{Answer: } z = 2 \text{ or } \frac{1}{2}$$

3 marks

28 (10 marks)

$$(a) \text{ Let } z_1 = a + bi, z_2 = c + di$$

B1

$$(z_1 + z_2)^* = (a + c) - (b + d)i = z_1^* + z_2^*$$

M1 A1

$$(z_1 z_2)^* = z_1^* z_2^* \text{ (expand both sides: } (ac - bd) - (ad + bc)i)$$

A1

Answer: Proved

4 marks

$$(b) \text{ Repeated use of } (z_1 z_2)^* = z_1^* z_2^*, \text{ with } z_1 = z^{n-1} \text{ and } z_2 = z \text{ (induction)}$$

M1

$$\text{gives } (z^n)^* = (z^*)^n$$

A1

Answer: Shown

2 marks

$$(c) p(z)^* = (p z^2 + q z + r)^* = p(z^*)^2 + q z^* + r \text{ (p, q, r real, parts (a) and (b))}$$

M1 A1

$$= p(z^*)$$

A1

$$\text{If } p(\alpha) = 0 \text{ then } p(\alpha^*) = p(\alpha)^* = 0^* = 0$$

A1

Answer: Shown

4 marks

29 (9 marks)

$$\text{Let } z = x + iy: x^2 - y^2 + 2xyi = x - iy$$

M1

$$\text{Real parts: } x^2 - y^2 = x \text{ (1)}$$

A1

$$\text{Imaginary parts: } 2xy = -y \text{ (2)}$$

A1

$$\text{From (2): } y(2x + 1) = 0, \text{ so } y = 0 \text{ or } x = -\frac{1}{2}$$

M1

$$y = 0: \text{ (1) gives } x^2 = x, \text{ so } x = 0 \text{ or } 1$$

M1 A1

$$x = -\frac{1}{2}: \text{ (1) gives } \frac{1}{4} - y^2 = -\frac{1}{2}, \text{ so } y^2 = \frac{3}{4}$$

M1

$$y = \pm \frac{\sqrt{3}}{2}$$

A1

$$z = 0, 1, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

A1

$$\text{Answer: } z = 0, 1, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

9 marks

30 (9 marks)

$$(a) z = \frac{1 - it}{(1 + it)(1 - it)}$$

M1

$$= \frac{1 - it}{1 + t^2}$$

A1

Equate real and imaginary parts

A1

Answer: Shown

3 marks

$$(b) x^2 + y^2 = \frac{1 + t^2}{(1 + t^2)^2}$$

M1 A1

$$= \frac{1}{1 + t^2} = x$$

A1

Answer: Shown

3 marks

$$(c) x^2 - x + y^2 = 0$$

M1

$$(x - \frac{1}{2})^2 + y^2 = \frac{1}{4}$$

A1

Circle with centre $(\frac{1}{2}, 0)$ and radius $\frac{1}{2}$

A1

Answer: Centre $(\frac{1}{2}, 0)$, radius $\frac{1}{2}$

3 marks

Argand Diagrams**1 (3 marks)**A is (2, 3), so $2 + 3i$

B1

B is $(-3, 1)$, so $-3 + i$

B1

C is (1, -2), so $1 - 2i$

B1

Answer: A: $2 + 3i$, B: $-3 + i$, C: $1 - 2i$

3 marks

2 (2 marks)Reflection in the real axis changes the sign of the imaginary part: $2 - 3i$

B1

This is the conjugate z^*

B1

Answer: $2 - 3i = z^*$

2 marks

3 (3 marks) $4 - i$ is (4, -1)

B1

 $-2i$ is (0, -2)

B1

5 is (5, 0)

B1

Answer: (4, -1), (0, -2), (5, 0)

3 marks

4 (2 marks)

$$OP = \sqrt{3^2 + 4^2}$$

M1

$$= 5$$

A1

Answer: 5

2 marks

5 (2 marks)

$$-z = -2 + 3i$$

B1

A rotation of 180° about the origin (reflection in O)

B1

Answer: $-2 + 3i$; rotation through 180° about O

2 marks

6 (2 marks)

$$\text{Midpoint} = \frac{1}{2}((2 + i) + (6 + 5i))$$

M1

$$= 4 + 3i$$

A1

Answer: $4 + 3i$

2 marks

7 (3 marks)

$$z_1 + z_2 = 4 + i$$

B1

OC is the diagonal of the parallelogram formed by OA and OB: the vector from B to C is z_1 , equal to OA

M1

so BC is equal and parallel to OA, and OACB is a parallelogram.

E1

Answer: $4 + i$

3 marks

8 (3 marks)

$$iz = i(3 + i) = 3i + i^2$$

M1

$$= -1 + 3i$$

A1

OB is OA rotated through 90° anticlockwise about O (same length).

B1

Answer: $iz = -1 + 3i$; rotation of 90° anticlockwise

3 marks

9 (3 marks)

$$\text{Difference} = (5 + 7i) - (2 + 3i) = 3 + 4i$$

M1

$$\text{Distance} = |3 + 4i|$$

M1

$$= \sqrt{9 + 16} = 5$$

A1

Answer: 5

3 marks

10 (3 marks)

$|z - (1 + 2i)|$ is the distance from z to the point $1 + 2i$

M1

so the locus is a circle with centre $1 + 2i$

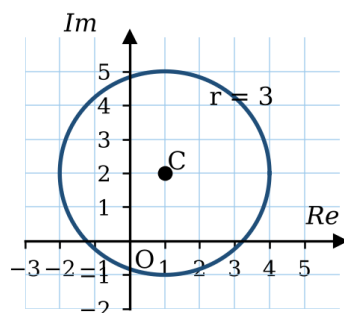
A1

and radius 3

A1

Answer: Circle, centre $1 + 2i$, radius 3

3 marks



11 (3 marks)

A half-line from the origin (not including O)

B1

making an angle of $\frac{\pi}{4}$ with the positive real axis

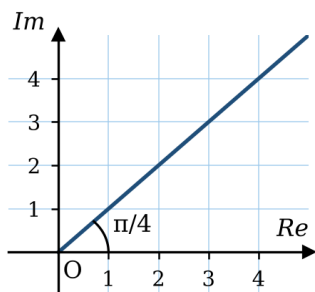
B1

Cartesian equation $y = x$ with $x > 0$

B1

Answer: $y = x, x > 0$

3 marks



12 (3 marks)

$$\text{Re } z = x = t \text{ and } \text{Im } z = y = 2t + 1$$

B1

$$\text{Eliminate } t: y = 2x + 1$$

M1

P lies on the straight line $y = 2x + 1$ (gradient 2, passing through the point i)

A1

Answer: $y = 2x + 1$

3 marks

13 (3 marks)

z is equidistant from 3 and -1

M1

so the locus is the perpendicular bisector of the line joining (3, 0) and $(-1, 0)$

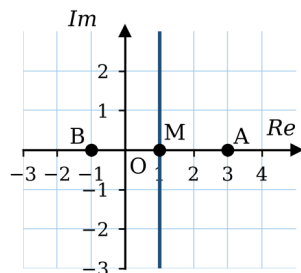
A1

Cartesian equation $x = 1$ (the line $\operatorname{Re} z = 1$)

A1

Answer: $\operatorname{Re} z = 1$ (the line $x = 1$)

3 marks



14 (4 marks)

Since ABCD is a parallelogram, $d = a + c - b$

M1

$$d = (1 + i) + (4 + 4i) - (4 + i) = 1 + 4i$$

A1

$$AC = |(4 + 4i) - (1 + i)| = |3 + 3i|$$

M1

$$= \sqrt{18} = 3\sqrt{2}$$

A1

Answer: $D = 1 + 4i$, $AC = 3\sqrt{2}$

4 marks

15 (4 marks)

$|z - 1| \leq 2$ is the disc with centre 1 and radius 2

B1

$\operatorname{Im} z \geq 0$ is the half-plane on and above the real axis

B1

The region is the upper half of the disc (the centre lies on the real axis)

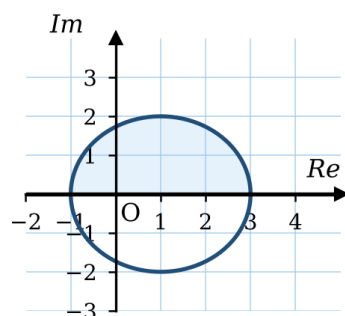
B1

$$\text{Area} = \frac{1}{2} \times \pi \times 2^2 = 2\pi$$

B1

Answer: 2π

4 marks



16 (4 marks)

The locus is a circle with centre C(3, 4) and radius 2

B1

$$OC = |3 + 4i| = 5$$

B1

$$\text{Greatest } |z| = 5 + 2 = 7$$

B1

$$\text{Least } |z| = 5 - 2 = 3$$

B1

Answer: Greatest 7, least 3

4 marks

17 (4 marks)

$|z - 2| \leq 3$ is the disc with centre 2 and radius 3

B1

$|z - 2| \geq 1$ excludes the interior of the disc with centre 2 and radius 1

B1

The region is the ring (annulus) between the two circles

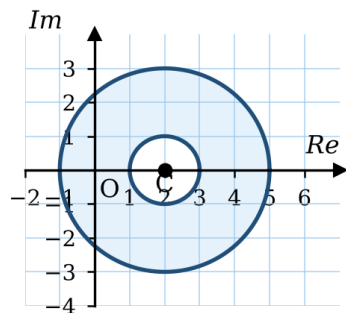
B1

$$\text{Area} = \pi \times 3^2 - \pi \times 1^2 = 8\pi$$

B1

Answer: 8π

4 marks



18 (4 marks)

$$\text{Vector PA} = (4 + i) - (1 + i) = 3$$

M1

Rotating by 90° anticlockwise multiplies by i : $\text{PB} = 3i$

M1 A1

$$b = (1 + i) + 3i = 1 + 4i$$

A1

Answer: $1 + 4i$

4 marks

19 (4 marks)

$$(a) (x - 2)^2 + (y - 1)^2 = 9$$

B1 B1

$$(b) z - 2 = (x - 2) + iy \text{ has argument } \frac{\pi}{3}, \text{ so } y = (x - 2)\tan \frac{\pi}{3}$$

M1

$$y = \sqrt{3}(x - 2) \text{ with } x > 2$$

A1

$$\text{Answer: (a) } (x - 2)^2 + (y - 1)^2 = 9 \text{ (b) } y = \sqrt{3}(x - 2), x > 2$$

4 marks

20 (4 marks)

$$\text{Let } z = x + 3i: x^2 + 9 = 25$$

M1 A1

$$x^2 = 16$$

A1

$$z = 4 + 3i \text{ or } z = -4 + 3i$$

A1

Answer: $z = \pm 4 + 3i$

4 marks

21 (4 marks)

The locus is a circle, centre $C(4, 0)$, radius 2. The greatest argument is at the tangent from O

B1

In the right-angled triangle OCT (T the point of tangency): $\sin \theta = \frac{2}{4}$

M1 A1

$$\theta = \frac{\pi}{6}$$

A1

Answer: $\frac{\pi}{6}$

4 marks

22 (6 marks)

$$\text{AB: } (4 + 2i) - (1 + i) = 3 + i$$

M1

$$\text{AC: } 4i - (1 + i) = -1 + 3i$$

M1

$$i(3 + i) = 3i - 1 = -1 + 3i, \text{ so } AC = i \times AB$$

A1

$$|AC| = |AB| = \sqrt{10}, \text{ so isosceles}$$

A1

Multiplying by i rotates through 90° , so angle $BAC = 90^\circ$

E1

$$\text{Area} = \frac{1}{2} \times \sqrt{10} \times \sqrt{10} = 5$$

A1

Answer: Shown; area 5

6 marks

23 (5 marks)

$$|z - 4| = |z - 4i|: \text{perpendicular bisector of } 4 \text{ and } 4i, \text{ the line } y = x$$

B1

$$|z - 2| = 2: \text{circle centre } 2, \text{ radius } 2$$

B1

$$\text{Substitute } y = x \text{ into } (x - 2)^2 + y^2 = 4$$

M1

$$2x^2 - 4x = 0, \text{ so } x = 0 \text{ or } x = 2$$

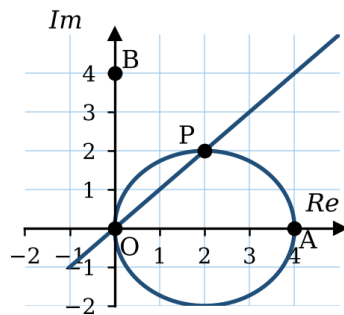
A1

$$z = 0 \text{ and } z = 2 + 2i$$

A1

Answer: $z = 0, z = 2 + 2i$

5 marks



24 (5 marks)

L_1 is a half-line from 2 at $\frac{\pi}{3}$ to the real axis; L_2 is a circle, centre 2, radius 4

B1 B1

Intersection: $z - 2 = 4(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3})$

M1

$$= 4(\frac{1}{2} + \frac{\sqrt{3}}{2}i) = 2 + 2\sqrt{3}i$$

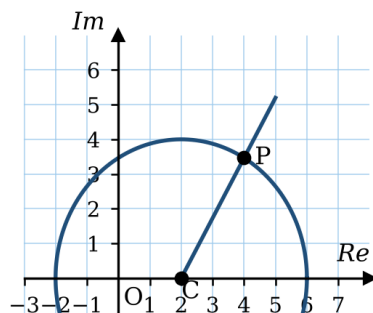
A1

$$z = 4 + 2\sqrt{3}i$$

A1

Answer: $z = 4 + 2\sqrt{3}i$

5 marks



25 (5 marks)

$$w = (1 + i)(3 + i) = 3 + i + 3i - 1 = 2 + 4i$$

M1 A1

$$w - z = (2 + 4i) - (3 + i) = -1 + 3i = i(3 + i) = iz$$

A1

ZW is OZ rotated through 90° , so ZW is perpendicular to OZ

E1

and $|ZW| = |i||z| = |OZ|$, so triangle OZW is isosceles with a right angle at Z.

E1

Answer: $w = 2 + 4i$; shown

5 marks

26 (6 marks)

$$\text{Centre } M = \frac{1}{2}((1 + 2i) + (5 + 4i)) = 3 + 3i$$

M1 A1

$$MA = (1 + 2i) - (3 + 3i) = -2 - i$$

A1

The diagonals are equal and perpendicular, so $MB = i \times MA$ (or $-i \times MA$): $MB = i(-2 - i) = 1 - 2i$

M1

$$B = 3 + 3i + (1 - 2i) = 4 + i \text{ and } D = 3 + 3i - (1 - 2i) = 2 + 5i \text{ (labels may be swapped)}$$

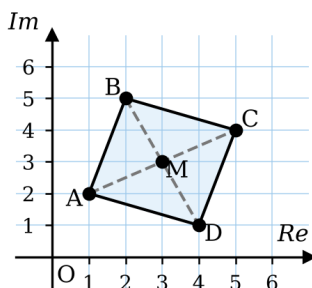
A1

$$\text{Side}^2 = |B - A|^2 = |3 - i|^2 = 10, \text{ area} = 10$$

A1

Answer: B, D = 2 + 5i and 4 + i; area 10

6 marks



27 (10 marks)

(a) Let $z = x + iy$: $(x - 4)^2 + y^2 = 4((x - 1)^2 + y^2)$ M1

$$x^2 - 8x + 16 + y^2 = 4x^2 - 8x + 4 + 4y^2$$
A1

$$3x^2 + 3y^2 = 12, \text{ so } x^2 + y^2 = 4$$
A1

Circle, centre O, radius 2 A1

Answer: Circle, centre O, radius 2 4 marks

(b) $z = 2(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3})$ M1

$$= 1 + \sqrt{3}i$$
A1

$$|z - 4| = |-3 + \sqrt{3}i| = \sqrt{9 + 3} = 2\sqrt{3}$$
M1

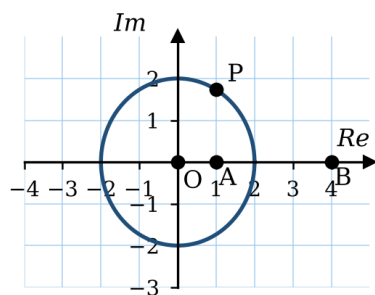
$$|z - 1| = |\sqrt{3}i| = \sqrt{3}, \text{ and } 2\sqrt{3} = 2 \times \sqrt{3}, \text{ so verified}$$
A1

Answer: $z = 1 + \sqrt{3}i$ 4 marks

(c) Greatest $\text{Re } z = 2$ at $z = 2$ B1

$$|z - 4| = 2$$
B1

Answer: 2; 2 2 marks

**28 (11 marks)**

(a) Let $z = x + iy$: $x^2 + y^2 \geq (x - 2)^2 + y^2$ M1

$$0 \geq -4x + 4, \text{ so } x \geq 1$$
A1

$|z - 2| \leq 2$ is the disc, centre 2, radius 2 B1

R is the part of the disc on or to the right of the line $x = 1$ B1

Answer: $\text{Re } z \geq 1$ 4 marks

(b) $(1 - 2)^2 + y^2 = 4$, so $y^2 = 3$ M1

$$z = 1 \pm \sqrt{3}i$$
A1

Answer: $z = 1 \pm i\sqrt{3}$ 2 marks

(c) The chord PQ is at distance 1 from the centre C(2, 0), so $\cos \alpha = \frac{1}{2}$ where α is the half angle M1 A1

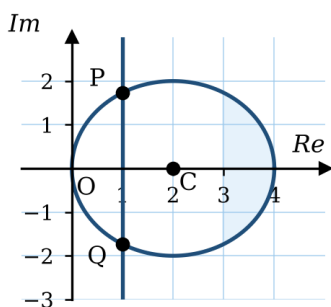
at C, $\alpha = \frac{\pi}{3}$

Segment on the left of the chord: $\frac{1}{2}r^2(2\alpha - \sin 2\alpha) = 2(\frac{2\pi}{3} - \frac{\sqrt{3}}{2})$ M1

$$= \frac{4\pi}{3} - \sqrt{3}$$
A1

Area of R = $4\pi - (\frac{4\pi}{3} - \sqrt{3}) = \frac{8\pi}{3} + \sqrt{3}$ A1

Answer: $\frac{8\pi}{3} + \sqrt{3}$ 5 marks



29 (8 marks)

$$(a) |z - 1|^2 = (z - 1)(z^* - 1) = 1$$

M1

$$zz^* - z - z^* + 1 = 1$$

A1

$$zz^* = z + z^*$$

A1

Answer: Shown

3 marks

$$(b) \frac{1}{z} = \frac{z^*}{zz^*}$$

M1

$$\operatorname{Re} \frac{1}{z} = \frac{\operatorname{Re} z}{zz^*} = \frac{\frac{1}{2}(z + z^*)}{zz^*}$$

$$= \frac{1}{2}$$

M1

A1

Answer: Shown

3 marks

$$(c) \operatorname{Re} w = \frac{1}{2} \text{ for every such point}$$

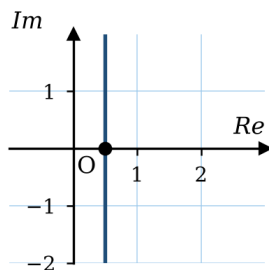
M1

so w lies on the line $\operatorname{Re} w = \frac{1}{2}$, a line parallel to the imaginary axis (whole line, as z ranges over the circle).

A1

$$\text{Answer: The line } \operatorname{Re} w = \frac{1}{2}$$

2 marks

**30 (8 marks)**

$$(a) |z_1 + z_2|^2 = (a + c)^2 + (b + d)^2$$

M1

$$= a^2 + 2ac + c^2 + b^2 + 2bd + d^2$$

A1

$$= |z_1|^2 + |z_2|^2 + 2(ac + bd)$$

A1

Answer: Shown

3 marks

$$(b) ac + bd \leq |ac + bd| \leq |z_1||z_2|$$

M1

$$|z_1 + z_2|^2 \leq |z_1|^2 + |z_2|^2 + 2|z_1||z_2| = (|z_1| + |z_2|)^2$$

M1

Take positive square roots

A1

Answer: Proved

3 marks

(c) In triangle OAC ($C = A + B$) one side is not longer than the sum of the other two.

E1

Equality when O, A, B are collinear with A and B on the same side of O (arguments equal).

E1

Answer: Triangle inequality; equal when z_1, z_2 have the same argument

2 marks

Modulus and Argument**1 (2 marks)**

$$|3 + 4i| = \sqrt{9 + 16} = 5$$

B1

$$|5 - 12i| = \sqrt{25 + 144} = 13$$

B1

Answer: (a) 5 (b) 13

2 marks

2 (2 marks)

$$|-7 + 24i| = \sqrt{49 + 576}$$

M1

$$= 25$$

A1

Answer: 25

2 marks

3 (4 marks)

(a) 0

B1

(b) π

B1

(c) $\frac{\pi}{2}$

B1

(d) $-\frac{\pi}{2}$

B1

Answer: (a) 0 (b) π (c) $\frac{\pi}{2}$ (d) $-\frac{\pi}{2}$

4 marks

4 (2 marks)

$\tan \theta = 1$ and the number is in the first quadrant: $\arg(1 + i) = \frac{\pi}{4}$

B1

$\arg(1 - i) = -\frac{\pi}{4}$

B1

Answer: $\frac{\pi}{4}$ and $-\frac{\pi}{4}$

2 marks

5 (2 marks)

$\cos \frac{\pi}{3} = \frac{1}{2}$ and $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$

B1

$z = 1 + \sqrt{3}i$

B1

Answer: $1 + \sqrt{3}i$

2 marks

6 (2 marks)

$|zw| = |z||w| = 10$

B1

$|\frac{z}{w}| = \frac{|z|}{|w|} = \frac{5}{2}$

B1

Answer: 10 and $\frac{5}{2}$

2 marks

7 (3 marks)

$|z| = \sqrt{1+3} = 2$

B1

Acute angle: $\tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$

M1

z is in the second quadrant, so $\arg z = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

A1

Answer: $|z| = 2$, $\arg z = \frac{2\pi}{3}$

3 marks

8 (3 marks)

Acute angle = $\tan^{-1}(1) = \frac{\pi}{4}$

M1

Third quadrant: $\arg = -(\pi - \frac{\pi}{4})$

M1

$= -\frac{3\pi}{4}$

A1

Answer: $-\frac{3\pi}{4}$

3 marks

9 (3 marks)

$r = \sqrt{1+3} = 2$

B1

$\theta = -\tan^{-1}(\sqrt{3}) = -\frac{\pi}{3}$ (fourth quadrant)

M1

$1 - \sqrt{3}i = 2(\cos(-\frac{\pi}{3}) + i \sin(-\frac{\pi}{3}))$

A1

Answer: $2(\cos(-\frac{\pi}{3}) + i \sin(-\frac{\pi}{3}))$

3 marks

10 (4 marks)

$$|zw| = 2 \times 3 = 6$$

B1

$$\arg(zw) = \frac{\pi}{6} + \frac{\pi}{3} = \frac{\pi}{2}$$

M1 A1

$$zw = 6(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}) = 6i$$

A1

$$\text{Answer: } 6(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}) = 6i$$

4 marks

11 (4 marks)

$$|\frac{z}{w}| = \frac{8}{2} = 4$$

B1

$$\arg(\frac{z}{w}) = \frac{5\pi}{6} - \frac{\pi}{3} = \frac{\pi}{2}$$

M1 A1

$$\frac{z}{w} = 4i$$

A1

$$\text{Answer: } 4i$$

4 marks

12 (3 marks)

$$|-4 + 3i| = \sqrt{16 + 9} = 5$$

B1

$$\text{Acute angle} = \tan^{-1}(\frac{3}{4}) = 0.6435$$

M1

$$\arg = \pi - 0.6435 = 2.50 \text{ (second quadrant)}$$

A1

$$\text{Answer: } 5, 2.50 \text{ rad}$$

3 marks

13 (3 marks)

$$\frac{5\pi}{4} \text{ is outside the range } -\pi < \theta \leq \pi; \text{ subtract } 2\pi: \frac{5\pi}{4} - 2\pi = -\frac{3\pi}{4}$$

M1

$$z = 4(\cos(-\frac{3\pi}{4}) + i \sin(-\frac{3\pi}{4}))$$

A1

$$z^* \text{ has the opposite argument: } \arg z^* = \frac{3\pi}{4}$$

B1

$$\text{Answer: } z = 4(\cos(-\frac{3\pi}{4}) + i \sin(-\frac{3\pi}{4})); \arg z^* = \frac{3\pi}{4}$$

3 marks

14 (6 marks)

$$\text{(a) } |1 + i| = \sqrt{2}, |\sqrt{3} - i| = 2, \text{ so the modulus is } 2\sqrt{2}$$

B1

$$\arg(1 + i) = \frac{\pi}{4}, \arg(\sqrt{3} - i) = -\frac{\pi}{6}$$

B1

$$\arg \text{ of product} = \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}$$

B1

$$\text{Answer: } 2\sqrt{2}, \frac{\pi}{12}$$

3 marks

$$\text{(b) Expand: } (1 + i)(\sqrt{3} - i) = (\sqrt{3} + 1) + (\sqrt{3} - 1)i$$

M1

$$\tan \frac{\pi}{12} = \frac{\sqrt{3} - 1}{\sqrt{3} + 1}$$

A1

$$= \frac{(\sqrt{3} - 1)^2}{2} = \frac{4 - 2\sqrt{3}}{2} = 2 - \sqrt{3}$$

A1

$$\text{Answer: Shown}$$

3 marks

15 (4 marks)

$$\arg z = \frac{3\pi}{4}$$

B1

$$z^2 = 1 - 2i - 1 = -2i, \text{ so } \arg(z^2) = -\frac{\pi}{2}$$

M1 A1

$$2 \arg z = \frac{3\pi}{2}, \text{ which differs from } -\frac{\pi}{2} \text{ by } 2\pi; \text{ the argument is only defined to within multiples of}$$

E1

$$2\pi, \text{ and } -\frac{\pi}{2} \text{ is the value in the principal range.}$$

$$\text{Answer: } \arg z = \frac{3\pi}{4}; \arg(z^2) = -\frac{\pi}{2}$$

4 marks

16 (4 marks)

$$|1 + \sqrt{3}i| = 2 \text{ and } \arg = \frac{\pi}{3}$$

B1

$$(1 + \sqrt{3}i)^6 \text{ has modulus } 2^6 = 64$$

M1

$$\text{and argument } 6 \times \frac{\pi}{3} = 2\pi$$

M1

$$\text{so } (1 + \sqrt{3}i)^6 = 64$$

A1

Answer: 64

4 marks

17 (4 marks)

$$\frac{1}{1+i} = \frac{1-i}{2}$$

M1 A1

$$\text{Modulus} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

A1

$$\text{Argument} = -\frac{\pi}{4}, \text{ so } \frac{1}{\sqrt{2}}(\cos(-\frac{\pi}{4}) + i \sin(-\frac{\pi}{4}))$$

A1

$$\text{Answer: } \frac{1}{\sqrt{2}}(\cos(-\frac{\pi}{4}) + i \sin(-\frac{\pi}{4}))$$

4 marks

18 (4 marks)

$$z = 2\sqrt{2}(\cos(-\frac{\pi}{4}) + i \sin(-\frac{\pi}{4}))$$

M1

$$= 2\sqrt{2}(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i) = 2 - 2i$$

A1

$$z^2 \text{ has modulus } 8 \text{ and argument } -\frac{\pi}{2}$$

M1

$$z^2 = -8i$$

A1

$$\text{Answer: } z = 2 - 2i, z^2 = -8i$$

4 marks

19 (5 marks)

$$|z_1| = |z_2| = 2, \arg z_1 = \frac{\pi}{3}, \arg z_2 = \frac{\pi}{6}$$

B1

$$\frac{z_1}{z_2} = \cos \frac{\pi}{6} + i \sin \frac{\pi}{6}$$

M1

$$= \frac{\sqrt{3}}{2} + \frac{1}{2}i$$

A1

$$(b) \frac{(1 + \sqrt{3}i)(\sqrt{3} - i)}{(\sqrt{3} + i)(\sqrt{3} - i)} = \frac{\sqrt{3} - i + 3i + \sqrt{3}}{4}$$

M1

$$= \frac{2\sqrt{3} + 2i}{4} = \frac{\sqrt{3}}{2} + \frac{1}{2}i, \text{ which agrees}$$

A1

$$\text{Answer: } \frac{\sqrt{3}}{2} + \frac{1}{2}i$$

5 marks

20 (4 marks)

$$\tan \frac{2\pi}{3} = -\sqrt{3} = \frac{3}{a}$$

M1

$$a = -\sqrt{3}$$

A1

$$\tan(-\frac{\pi}{6}) = -\frac{1}{\sqrt{3}} = \frac{b}{4}$$

M1

$$b = -\frac{4}{\sqrt{3}} = -\frac{4\sqrt{3}}{3}$$

A1

$$\text{Answer: } a = -\sqrt{3}, b = -\frac{4\sqrt{3}}{3}$$

4 marks

21 (5 marks)

$\sqrt{3} + i$ has modulus 2 and argument $\frac{\pi}{6}$, so $(\sqrt{3} + i)^n$ has argument $\frac{n\pi}{6}$

B1 B1

(a) Real when $\frac{n\pi}{6}$ is a multiple of π : $n = 6$, value $2^6(\cos \pi + i \sin \pi) = -64$

M1 A1

(b) Purely imaginary when $\frac{n\pi}{6} = \frac{\pi}{2} + k\pi$: $n = 3$, value $2^3(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}) = 8i$

A1

Answer: (a) $n = 6$, -64 (b) $n = 3$, $8i$

5 marks

22 (5 marks)

$|1 - i| = \sqrt{2}$, $\arg(1 - i) = -\frac{\pi}{4}$

B1

$|(1 - i)^{10}| = (\sqrt{2})^{10} = 32$

M1 A1

$\arg = -\frac{10\pi}{4} = -\frac{5\pi}{2}$, equivalent to $-\frac{\pi}{2}$ (adding 2π)

M1

So $(1 - i)^{10} = 32(\cos(-\frac{\pi}{2}) + i \sin(-\frac{\pi}{2})) = -32i$

A1

Answer: Shown

5 marks

23 (5 marks)

$\arg(\frac{z}{w}) = \frac{3\pi}{4} - (-\frac{\pi}{3}) = \frac{13\pi}{12}$

M1 A1

This exceeds π , so subtract 2π : $-\frac{11\pi}{12}$

A1

$\arg(zw) = \frac{3\pi}{4} - \frac{\pi}{3} = \frac{5\pi}{12}$

M1

which is in the principal range

A1

Answer: $\arg(z/w) = -\frac{11\pi}{12}$, $\arg(zw) = \frac{5\pi}{12}$

5 marks

24 (5 marks)

$zz^* = |z|^2 = 1$, so $z^* = \frac{1}{z}$

M1 A1

If $z = \cos \theta + i \sin \theta$ then $z^* = \cos \theta - i \sin \theta = \cos(-\theta) + i \sin(-\theta)$, so $\arg z^* = -\theta$

A1

$z + \frac{1}{z} = z + z^* = (\cos \theta + i \sin \theta) + (\cos \theta - i \sin \theta)$

M1

$= 2 \cos \theta$

A1

Answer: Shown

5 marks

25 (4 marks)

$(1 + i)^8$: modulus $(\sqrt{2})^8 = 16$, argument $8 \times \frac{\pi}{4} = 2\pi$, so $(1 + i)^8 = 16$

M1 A1

$(\sqrt{3} - i)^6$: modulus $2^6 = 64$, argument $6 \times (-\frac{\pi}{6}) = -\pi$, so $= -64$

M1

Value $= \frac{16}{-64} = -\frac{1}{4}$

A1

Answer: $-\frac{1}{4}$

4 marks

26 (6 marks)

$z^2 = 1 + 2i - 1 = 2i$

B1

$z^3 = 2i(1 + i) = -2 + 2i$

M1 A1

$z^4 = (2i)^2 = -4$

A1

$|z| = \sqrt{2}$, so $|z^n| = (\sqrt{2})^n$: the modulus is multiplied by $\sqrt{2}$ each time

B1

$\arg z = \frac{\pi}{4}$, so $\arg z^n = \frac{n\pi}{4}$: the argument increases by $\frac{\pi}{4}$ each time (anticlockwise spiral)

B1

Answer: $z^2 = 2i$, $z^3 = -2 + 2i$, $z^4 = -4$

6 marks

27 (9 marks)

$$(a) 1 + \cos \theta = 2\cos^2 \frac{\theta}{2} \quad \text{B1}$$

$$\sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} \quad \text{B1}$$

$$\text{So the number} = 2\cos \frac{\theta}{2} (\cos \frac{\theta}{2} + i \sin \frac{\theta}{2}) \quad \text{A1}$$

Answer: Shown 3 marks

$$(b) 2\cos \frac{\theta}{2} > 0 \text{ because } 0 < \frac{\theta}{2} < \frac{\pi}{2}, \text{ so it is the modulus} \quad \text{B1}$$

$$\text{Argument} = \frac{\theta}{2} \quad \text{B1}$$

$$\text{Answer: Modulus } 2\cos \frac{\theta}{2}, \text{ argument } \frac{\theta}{2} \quad \text{2 marks}$$

$$(c) \text{ Modulus } 8\cos^3 \frac{\theta}{2}, \text{ argument } \frac{3\theta}{2} \quad \text{M1 A1}$$

$$\theta = \frac{\pi}{3}: \text{ modulus } 8 \times \left(\frac{\sqrt{3}}{2}\right)^3 = 3\sqrt{3} \quad \text{M1}$$

$$\text{argument } \frac{\pi}{2}, \text{ so the value is } 3\sqrt{3}i \quad \text{A1}$$

$$\text{Answer: } 3\sqrt{3}i \quad \text{4 marks}$$

28 (9 marks)

$$(a) z_1 z_2 = r_1 r_2 (\cos \alpha + i \sin \alpha)(\cos \beta + i \sin \beta) \quad \text{M1}$$

$$= r_1 r_2 [(\cos \alpha \cos \beta - \sin \alpha \sin \beta) + i(\sin \alpha \cos \beta + \cos \alpha \sin \beta)] \quad \text{A1}$$

$$\cos \alpha \cos \beta - \sin \alpha \sin \beta = \cos(\alpha + \beta) \quad \text{B1}$$

$$\sin \alpha \cos \beta + \cos \alpha \sin \beta = \sin(\alpha + \beta) \quad \text{B1}$$

Answer: Proved 4 marks

$$(b) \text{ Let } \frac{z_1}{z_2} = \rho(\cos \varphi + i \sin \varphi), \text{ so } z_1 = \rho(\cos \varphi + i \sin \varphi) z_2 \quad \text{M1}$$

$$\text{By (a), } r_1 = \rho r_2 \text{ and } \alpha = \varphi + \beta \pmod{2\pi} \quad \text{A1}$$

$$\rho = \frac{r_1}{r_2}, \varphi = \alpha - \beta \quad \text{A1}$$

Answer: Proved 3 marks

$$(c) z = r(\cos \alpha + i \sin \alpha) \text{ gives } z^* = r(\cos \alpha - i \sin \alpha) \quad \text{M1}$$

$$= r(\cos(-\alpha) + i \sin(-\alpha)), \text{ so } \arg z^* = -\alpha \quad \text{A1}$$

Answer: Shown 2 marks

29 (9 marks)

$$(a) 1 + \sqrt{3}i = 2(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}) \quad \text{M1 A1}$$

$$1 - i = \sqrt{2}(\cos(-\frac{\pi}{4}) + i \sin(-\frac{\pi}{4})) \quad \text{A1}$$

$$\text{Answer: } 2(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}); \sqrt{2}(\cos(-\frac{\pi}{4}) + i \sin(-\frac{\pi}{4})) \quad \text{3 marks}$$

$$(b) \text{ Product: modulus } 2\sqrt{2}, \text{ argument } \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12} \quad \text{M1 A1}$$

$$\text{Product} = 2\sqrt{2}(\cos \frac{\pi}{12} + i \sin \frac{\pi}{12}) \quad \text{A1}$$

$$\text{Expanding directly: } (1 + \sqrt{3}i)(1 - i) = (1 + \sqrt{3}) + (\sqrt{3} - 1)i \quad \text{M1}$$

$$\text{Equate: } 2\sqrt{2}\cos \frac{\pi}{12} = 1 + \sqrt{3}, \text{ so } \cos \frac{\pi}{12} = \frac{1 + \sqrt{3}}{2\sqrt{2}} = \frac{\sqrt{6} + \sqrt{2}}{4} \quad \text{A1}$$

$$\sin \frac{\pi}{12} = \frac{\sqrt{3} - 1}{2\sqrt{2}} = \frac{\sqrt{6} - \sqrt{2}}{4} \quad \text{A1}$$

$$\text{Answer: } \cos \frac{\pi}{12} = \frac{\sqrt{6} + \sqrt{2}}{4}, \sin \frac{\pi}{12} = \frac{\sqrt{6} - \sqrt{2}}{4} \quad \text{6 marks}$$

30 (9 marks)

$$(a) |z_1 \pm z_2|^2 = (z_1 \pm z_2)(z_1^* \pm z_2^*)$$

M1

$$= |z_1|^2 \pm z_1 z_2^* \pm z_2 z_1^* + |z_2|^2$$

A1

Adding the + and – versions, the cross terms cancel

M1

$$= 2|z_1|^2 + 2|z_2|^2$$

A1

Answer: Proved

4 marks

(b) The diagonals have lengths $|z_1 + z_2|$ and $|z_1 - z_2|$, the sides $|z_1|$ and $|z_2|$

B1

The sum of the squares of the diagonals equals the sum of the squares of the four sides.

E1

Answer: Parallelogram law

2 marks

$$(c) 49 + |z_1 - z_2|^2 = 2 \times 9 + 2 \times 25$$

M1

$$|z_1 - z_2|^2 = 68 - 49 = 19$$

A1

$$|z_1 - z_2| = \sqrt{19}$$

A1

Answer: $\sqrt{19}$

3 marks

Complex Roots of Equations

1 (2 marks)

$$z^2 = -25$$

M1

$$z = \pm 5i$$

A1

Answer: $z = \pm 5i$

2 marks

2 (1 mark)

Complex roots of a real quadratic come in conjugate pairs: $3 - 2i$

B1

Answer: $3 - 2i$

1 mark

3 (3 marks)

Sum of roots = 2, product = $(1 + i)(1 - i) = 2$

B1 B1

$$z^2 - 2z + 2 = 0$$

B1

Answer: $z^2 - 2z + 2 = 0$

3 marks

4 (2 marks)

$$z = \frac{2 \pm \sqrt{4 - 20}}{2}$$

M1

$$z = 1 \pm 2i$$

A1

Answer: $z = 1 \pm 2i$

2 marks

5 (3 marks)

$$\alpha + \beta = 6 \text{ and } \alpha\beta = 25$$

B1

$$z = \frac{6 \pm \sqrt{36 - 100}}{2}$$

M1

$$z = 3 \pm 4i$$

A1

Answer: 6, 25; roots $3 \pm 4i$

3 marks

6 (2 marks)

$$i^3 + i^2 + i + 1 = -i - 1 + i + 1$$

M1

$$= 0$$

A1

Answer: Shown

2 marks

7 (3 marks)

$$z = \frac{-4 \pm \sqrt{16 - 52}}{2}$$

M1

$$= \frac{-4 \pm 6i}{2}$$

A1

$$z = -2 \pm 3i$$

A1

Answer: $z = -2 \pm 3i$

3 marks

8 (3 marks)The other root is $2 + 3i$

B1

Sum = $4 = -p$, so $p = -4$

M1

Product = $4 + 9 = 13 = q$

A1

Answer: $p = -4$, $q = 13$

3 marks

9 (4 marks)

$$(1 + 2i)^2 - 2(1 + 2i) + 5 = 1 + 4i - 4 - 2 - 4i + 5$$

M1

$$= 0$$

A1

The other root is $1 - 2i$

B1

$$z^2 - 2z + 5 = (z - 1 - 2i)(z - 1 + 2i)$$

A1

Answer: $(z - 1 - 2i)(z - 1 + 2i)$

4 marks

10 (3 marks)

$$z = \frac{2 \pm \sqrt{4 - 40}}{4}$$

M1

$$= \frac{2 \pm 6i}{4}$$

A1

$$z = \frac{1}{2} \pm \frac{3}{2}i$$

A1

Answer: $z = \frac{1}{2} \pm \frac{3}{2}i$

3 marks

11 (4 marks)

$$z^3 - 1 = (z - 1)(z^2 + z + 1)$$

M1

 $z = 1$ is one root

A1

$$z^2 + z + 1 = 0: z = \frac{-1 \pm \sqrt{-3}}{2}$$

M1

$$z = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

A1

Answer: $z = 1, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

4 marks

12 (4 marks)

$$\text{Discriminant} = (-2a)^2 - 4(a^2 + 1)$$

M1

 $= -4$, which is negative for every a , so the roots are non-real

A1

$$z = \frac{2a \pm \sqrt{-4}}{2} = \frac{2a \pm 2i}{2}$$

M1

$$z = a \pm i$$

A1

Answer: $z = a \pm i$

4 marks

13 (3 marks)

$$z^4 - 16 = (z^2 - 4)(z^2 + 4) = 0$$

M1

$$z^2 = 4 \text{ gives } z = \pm 2$$

A1

$$z^2 = -4 \text{ gives } z = \pm 2i$$

A1

Answer: $z = \pm 2, \pm 2i$

3 marks

14 (5 marks)

$$1 - 3 + 4 - 2 = 0, \text{ so } z = 1 \text{ is a root}$$

B1

$$z^3 - 3z^2 + 4z - 2 = (z - 1)(z^2 - 2z + 2)$$

M1 A1

$$z^2 - 2z + 2 = 0: z = \frac{2 \pm \sqrt{4 - 8}}{2}$$

M1

$$z = 1 \pm i$$

A1

$$\text{Answer: } z = 1, 1 \pm i$$

5 marks

15 (5 marks)

$$(1 + i)^2 = 2i \text{ and } (1 + i)^3 = -2 + 2i$$

B1

$$-2 + 2i - 4(2i) + 6(1 + i) - 4 = (-2 + 6 - 4) + (2 - 8 + 6)i = 0$$

M1 A1

$$1 - i \text{ is also a root (real coefficients)}$$

B1

$$\text{Sum of roots} = 4, \text{ so the third root is } 4 - 2 = 2$$

A1

$$\text{Answer: } 1 - i \text{ and } 2$$

5 marks

16 (5 marks)

$$k^2 - 36 < 0$$

M1

$$-6 < k < 6$$

A1

$$k = 2: z = \frac{-2 \pm \sqrt{4 - 36}}{2}$$

M1

$$= \frac{-2 \pm 4\sqrt{2}i}{2}$$

A1

$$z = -1 \pm 2\sqrt{2}i$$

A1

$$\text{Answer: } -6 < k < 6; z = -1 \pm 2\sqrt{2}i$$

5 marks

17 (4 marks)

$$(-1 + 2i)^2 = 1 - 4i - 4 = -3 - 4i$$

B1

$$-3 - 4i + p(-1 + 2i) + q = 0: \text{real part } -3 - p + q = 0, \text{imaginary part } -4 + 2p = 0$$

M1 A1

$$p = 2, q = 5$$

A1

$$\text{Answer: } p = 2, q = 5$$

4 marks

18 (5 marks)

$$\text{Let } z = x + iy: x^2 - y^2 = -5 \text{ and } 2xy = 12$$

M1 A1

$$y = \frac{6}{x}: x^4 + 5x^2 - 36 = 0$$

M1

$$(x^2 - 4)(x^2 + 9) = 0, \text{ so } x = \pm 2$$

A1

$$z = 2 + 3i \text{ or } z = -2 - 3i$$

A1

$$\text{Answer: } z = \pm(2 + 3i)$$

5 marks

19 (4 marks)

$$z = \frac{2i \pm \sqrt{(-2i)^2 + 20}}{2}$$

M1

$$(-2i)^2 + 20 = -4 + 20 = 16$$

A1

$$z = \frac{2i \pm 4}{2}$$

M1

$$z = 2 + i \text{ or } z = -2 + i$$

A1

$$\text{Answer: } z = \pm 2 + i$$

4 marks

20 (5 marks)

$$\alpha + \beta = 4, \alpha\beta = 13$$

B1

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = 16 - 26 = -10$$

M1 A1

$$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta} = \frac{4}{13}$$

A1

$$\text{Check: roots } 2 \pm 3i, \text{ squares } -5 \pm 12i, \text{ sum } -10$$

B1

$$\text{Answer: } -10 \text{ and } \frac{4}{13}$$

5 marks

21 (5 marks)

The third root is $1 + 2i$ (conjugate)

B1

Quadratic factor from the complex pair: sum 2, product 5, so $z^2 - 2z + 5$

M1 A1

$$(z - 3)(z^2 - 2z + 5) = z^3 - 2z^2 + 5z - 3z^2 + 6z - 15$$

M1

$$z^3 - 5z^2 + 11z - 15 = 0$$

A1

Answer: $z^3 - 5z^2 + 11z - 15 = 0$; third root $1 + 2i$

5 marks

22 (7 marks)

$1 - 2i$ is also a root (real coefficients)

B1

$(z - 1 - 2i)(z - 1 + 2i) = z^2 - 2z + 5$ is a factor

M1 A1

Divide: $z^4 - 4z^3 + 11z^2 - 14z + 10 = (z^2 - 2z + 5)(z^2 + az + 2)$

M1

Compare z^3 terms: $a - 2 = -4$, so $a = -2$

A1

$$z^2 - 2z + 2 = 0 \text{ gives } z = \frac{2 \pm \sqrt{4 - 8}}{2}$$

M1

Roots: $1 \pm 2i$ and $1 \pm i$

A1

Answer: $z = 1 \pm 2i, 1 \pm i$

7 marks

23 (6 marks)

$$(2 + i)^2 = 3 + 4i \text{ and } (2 + i)^3 = (3 + 4i)(2 + i) = 2 + 11i$$

B1

$$2(2 + 11i) - 9(3 + 4i) + 14(2 + i) - 5 = (4 - 27 + 28 - 5) + (22 - 36 + 14)i = 0$$

M1 A1

$2 - i$ is a root, giving the factor $z^2 - 4z + 5$

B1

$$2z^3 - 9z^2 + 14z - 5 = (z^2 - 4z + 5)(2z - 1)$$

M1

Other roots: $2 - i$ and $\frac{1}{2}$

A1

Answer: $2 - i$ and $\frac{1}{2}$

6 marks

24 (6 marks)

$$(1 + i)^3 = -2 + 2i$$

B1

$-2 + 2i + a(1 + i) + b = 0$: real part $-2 + a + b = 0$, imaginary part $2 + a = 0$

M1 A1

$$a = -2, b = 4$$

A1

$$z^3 - 2z + 4 = (z^2 - 2z + 2)(z + 2), \text{ using the conjugate root } 1 - i$$

M1

Other roots: $1 - i$ and -2

A1

Answer: $a = -2, b = 4$; roots $1 - i, -2$

6 marks

25 (6 marks)

$$z^3 - 8 = (z - 2)(z^2 + 2z + 4)$$

M1

$$z^2 + 2z + 4 = 0: z = \frac{-2 \pm \sqrt{4 - 16}}{2} = -1 \pm \sqrt{3}i$$

M1 A1

$$|-1 \pm \sqrt{3}i| = \sqrt{1 + 3} = 2 \text{ and } |2| = 2$$

B1

$$AB^2 = (-1 - 2)^2 + (\sqrt{3})^2 = 12, \text{ and } AC^2 = 12$$

M1

$BC = 2\sqrt{3}$ so $BC^2 = 12$: all sides equal, so equilateral

A1

Answer: $z = 2, -1 \pm \sqrt{3}i$; shown

6 marks

26 (6 marks)

$$z = \frac{3 \pm \sqrt{9 - 4(3 + i)}}{2} = \frac{3 \pm \sqrt{-3 - 4i}}{2}$$

M1 A1

Find $\sqrt{-3 - 4i}$: $(x + iy)^2 = -3 - 4i$ gives $x^2 - y^2 = -3, xy = -2$

M1

$$x^4 + 3x^2 - 4 = 0, \text{ so } x^2 = 1; x = 1, y = -2 \text{ (or } x = -1, y = 2)$$

M1

$$\sqrt{-3 - 4i} = \pm(1 - 2i)$$

A1

$$z = \frac{3 + 1 - 2i}{2} = 2 - i \text{ or } z = \frac{3 - 1 + 2i}{2} = 1 + i$$

A1

Answer: $z = 2 - i$ or $z = 1 + i$

6 marks

27 (8 marks)

(a) $p(\alpha) = 0$, so $p(\alpha)^* = 0^* = 0$

B1

By the properties of conjugates, $p(\alpha)^* = \alpha^{*3} + a\alpha^{*2} + b\alpha^* + c = p(\alpha^*)$ since a, b, c are real

M1

So $p(\alpha^*) = 0$

A1

Answer: Shown

3 marks

(b) There are three roots (counting multiplicity)

B1

Non-real roots occur in conjugate pairs, so the number of non-real roots is even (0 or 2)

M1

So at least one of the three roots is real.

A1

Answer: Proved

3 marks

(c) For example $(z - 1)(z^2 + 1) = z^3 - z^2 + z - 1$

M1

Roots 1, i and $-i$

A1

Answer: $z^3 - z^2 + z - 1 = 0$

2 marks

28 (10 marks)

(a) Roots: $1 + 2i$, $1 - 2i$ and a real root r

B1

Product of the roots = -10 : $(1 + 4)r = -10$, so $r = -2$

M1 A1

Sum of roots = $-a$: $2 - 2 = 0$, so $a = 0$

M1 A1

Sum of products of pairs = b : $5 + r(2) = 5 - 4 = 1$, so $b = 1$

A1

Answer: $a = 0$, $b = 1$; roots $1 - 2i$ and -2

6 marks

(b) $\Sigma \alpha^2 = (\Sigma \alpha)^2 - 2\Sigma \alpha\beta$

M1

$= 0^2 - 2(1) = -2$

A1

Check: $(1 + 2i)^2 + (1 - 2i)^2 + (-2)^2 = -3 + 4i - 3 - 4i + 4$

M1

$= -2$

A1

Answer: -2

4 marks

29 (11 marks)

(a) $(z^2 + 2)^2 - 4z^2 = z^4 + 4z^2 + 4 - 4z^2 = z^4 + 4$

M1 A1

Difference of two squares: $(z^2 + 2 - 2z)(z^2 + 2 + 2z)$

M1

$z^4 + 4 = (z^2 - 2z + 2)(z^2 + 2z + 2)$

A1

Answer: $(z^2 - 2z + 2)(z^2 + 2z + 2)$

4 marks

(b) $z^2 - 2z + 2 = 0$ gives $z = 1 \pm i$; $z^2 + 2z + 2 = 0$ gives $z = -1 \pm i$

M1 A1

The roots are $1 + i$, $-1 + i$, $-1 - i$, $1 - i$

A1

They are $(\pm 1, \pm 1)$: a square with sides parallel to the axes and side length 2

M1

Area = 4

A1

Answer: $z = \pm 1 \pm i$; area 4

5 marks

(c) The squares are $(1 + i)^2 = 2i$ and $(1 - i)^2 = -2i$

M1

Equation: $w^2 + 4 = 0$

A1

Answer: $w^2 + 4 = 0$

2 marks

30 (10 marks)

(a) $z = \frac{-4 \pm \sqrt{16 - 4k}}{2} = -2 \pm i\sqrt{k - 4}$

M1 A1

$|OA|^2 = 4 + (k - 4) = k$, so $|OA| = \sqrt{k}$

M1

A and B are $(-2, \pm\sqrt{k - 4})$, so $|AB| = 2\sqrt{k - 4}$

A1

Answer: $|OA| = \sqrt{k}$, $|AB| = 2\sqrt{k - 4}$

4 marks

(b) $|OA| = |AB|$, so $k = 4(k - 4)$

M1 A1

$3k = 16$

M1

$k = \frac{16}{3}$

A1

Answer: $k = \frac{16}{3}$

4 marks

(c) $\text{Side}^2 = \frac{16}{3}$, area = $\frac{\sqrt{3}}{4} \times \frac{16}{3}$

M1

$= \frac{4\sqrt{3}}{3}$

A1

Answer: $\frac{4\sqrt{3}}{3}$

2 marks