

Indices and Surds

1 (2 marks)

$$81^{3/4} = (\sqrt[4]{81})^3$$

$$= 3^3 = 27$$

M1

A1

Answer: 27

2 marks

2 (2 marks)

$$\sqrt{75} = 5\sqrt{3}, \sqrt{12} = 2\sqrt{3}, \sqrt{27} = 3\sqrt{3}$$

M1

$$\text{So the expression is } (5 + 2 - 3)\sqrt{3} = 4\sqrt{3}$$

A1

Answer: $4\sqrt{3}$

2 marks

3 (2 marks)

$$8^{-2/3} = \frac{1}{(\sqrt[3]{8})^2} = \frac{1}{2^2}$$

M1

$$= \frac{1}{4}$$

A1

Answer: $\frac{1}{4}$

2 marks

4 (3 marks)

$$\frac{4}{\sqrt{x}} = 4x^{-1/2}$$

M1

$$5x\sqrt{x} = 5x \times x^{1/2} = 5x^{3/2}$$

M1

$$\text{So the expression is } 4x^{-1/2} + 5x^{3/2}$$

A1

Answer: $4x^{-1/2} + 5x^{3/2}$

3 marks

5 (2 marks)

$$\text{Difference of two squares: } 3^2 - (\sqrt{5})^2$$

M1

$$= 9 - 5 = 4$$

A1

Answer: 4

2 marks

6 (2 marks)

$$\text{Square each factor: } 4^2 x^{3 \times 2} y^{-2 \times 2}$$

M1

$$= 16x^6y^{-4}$$

A1

Answer: $16x^6y^{-4}$

2 marks

7 (3 marks)

$$(2x^3)^2 = 4x^6, \text{ so the numerator is } 4x^6 \times 3x^{-1} = 12x^5$$

M1

$$\frac{12x^5}{6x^4} = 2x^{5-4}$$

M1

$$= 2x$$

A1

Answer: 2x

3 marks

8 (3 marks)

$$\text{Multiply top and bottom by } (3 + \sqrt{2}): \frac{(6 + \sqrt{2})(3 + \sqrt{2})}{(3 - \sqrt{2})(3 + \sqrt{2})}$$

M1

$$\text{Numerator: } 18 + 6\sqrt{2} + 3\sqrt{2} + 2 = 20 + 9\sqrt{2}$$

A1

$$\text{Denominator: } 9 - 2 = 7, \text{ so the result is } \frac{20 + 9\sqrt{2}}{7}$$

A1

Answer: $\frac{20 + 9\sqrt{2}}{7}$

3 marks

9 (3 marks)

Write both sides as powers of 2: $(2^2)^x = 2^5$

M1

$$2^{2x} = 2^5, \text{ so } 2x = 5$$

M1

$$x = \frac{5}{2}$$

A1

Answer: $x = \frac{5}{2}$

3 marks

10 (3 marks)

$$(2 - 3\sqrt{5})(2 - 3\sqrt{5}) = 4 - 6\sqrt{5} - 6\sqrt{5} + 9 \times 5$$

M1 A1

$$= 49 - 12\sqrt{5}$$

A1

Answer: $49 - 12\sqrt{5}$

3 marks

11 (3 marks)

$$\sqrt{x} = x^{1/2}, \text{ so the numerator is } 2x^{7/2}$$

M1

Divide the numbers: $\frac{2}{4} = \frac{1}{2}$; subtract the indices: $\frac{7}{2} - (-2) = \frac{11}{2}$

M1

$$= \frac{1}{2}x^{11/2}$$

A1

Answer: $\frac{1}{2}x^{11/2}$

3 marks

12 (3 marks)

$$x\sqrt{12} = 2\sqrt{3}x \text{ and } \frac{3x}{\sqrt{3}} = \sqrt{3}x$$

M1

$$2\sqrt{3}x - 5 = \sqrt{3}x, \text{ so } \sqrt{3}x = 5$$

M1

$$x = \frac{5}{\sqrt{3}} = \frac{5\sqrt{3}}{3}$$

A1

Answer: $x = \frac{5}{3}\sqrt{3}$

3 marks

13 (3 marks)

$$8^{x+1} = (2^3)^{x+1} = 2^{3x+3}$$

M1

$$= (2^x)^3 \times 2^3$$

M1

$$= 8a^3$$

A1

Answer: $8a^3$

3 marks

14 (5 marks)

Let $y = 2^x$. Then $2^{2x+1} = 2 \times (2^x)^2 = 2y^2$

M1

$$2y^2 - 5y + 2 = 0$$

A1

$$(2y - 1)(y - 2) = 0$$

M1

$$y = \frac{1}{2} \text{ or } y = 2$$

A1

$$2^x = 2^{-1} \text{ or } 2^x = 2^1, \text{ so } x = -1 \text{ or } x = 1$$

A1

Answer: $x = -1 \text{ or } x = 1$

5 marks

15 (5 marks)

$$32 = 2^5, \text{ so } x + y = 5$$

M1 A1

$$9^x = 3^{2x}, \text{ so } 3^{2x} \div 3^y = 3^{2x-y} = 3^1$$

M1

$$2x - y = 1$$

A1

Adding the equations: $3x = 6$, so $x = 2$ and $y = 3$

A1

Answer: $x = 2, y = 3$

5 marks

16 (5 marks)

$$12 = a \times 4^n \text{ and } 18 = a \times 9^n$$

M1

$$\text{Dividing the equations: } \frac{3}{2} = \left(\frac{9}{4}\right)^n = \left(\frac{3}{2}\right)^{2n}$$

M1 A1

$$\text{So } 2n = 1 \text{ and } n = \frac{1}{2}$$

A1

$$12 = a \times 4^{1/2} = 2a, \text{ so } a = 6$$

A1

$$\text{Answer: } n = \frac{1}{2}, a = 6$$

5 marks

17 (4 marks)

$$\frac{1}{2 - \sqrt{3}} = \frac{2 + \sqrt{3}}{(2 - \sqrt{3})(2 + \sqrt{3})} = \frac{2 + \sqrt{3}}{1}$$

M1 A1

$$\frac{1}{2 + \sqrt{3}} = \frac{2 - \sqrt{3}}{(2 + \sqrt{3})(2 - \sqrt{3})} = 2 - \sqrt{3}$$

M1

$$(2 + \sqrt{3}) - (2 - \sqrt{3}) = 2\sqrt{3}$$

A1

Answer: Shown

4 marks

18 (5 marks)

$$\text{Let } y = \sqrt{x}, \text{ so that } y > 0. \text{ Then } y - \frac{12}{y} = 4$$

M1

$$y^2 - 4y - 12 = 0$$

A1

$$(y - 6)(y + 2) = 0$$

M1

$$y = -2 \text{ is rejected because } \sqrt{x} \text{ cannot be negative, so } y = 6$$

A1

$$x = 36$$

A1

Answer: x = 36

5 marks

19 (5 marks)

$$\text{(a) } 3^{n+2} = 3^n \times 3^2 = 9 \times 3^n$$

M1

$$9 \times 3^n - 3^n = 8 \times 3^n$$

A1

Answer: Shown

2 marks

$$\text{(b) Using (a) with } n = 99: 3^{101} - 3^{99} = 8 \times 3^{99}$$

M1

$$\frac{8 \times 3^{99}}{3^{97}} = 8 \times 3^2$$

M1

$$= 72$$

A1

Answer: 72

3 marks

20 (5 marks)

$$\text{Let } y = x^{1/3}, \text{ so } x^{2/3} = y^2$$

M1

$$y^2 - 5y + 4 = 0$$

A1

$$(y - 1)(y - 4) = 0, \text{ so } y = 1 \text{ or } y = 4$$

M1 A1

$$x = y^3: x = 1 \text{ or } x = 64$$

A1

Answer: x = 1 or x = 64

5 marks

21 (8 marks)

$$\text{(a) Width} = \frac{21 + 11\sqrt{3}}{5 + 2\sqrt{3}}; \text{ multiply top and bottom by } (5 - 2\sqrt{3})$$

M1

$$\text{Numerator: } 105 - 42\sqrt{3} + 55\sqrt{3} - 66 = 39 + 13\sqrt{3}$$

A1

$$\text{Denominator: } 25 - 12 = 13, \text{ so width} = 3 + \sqrt{3}$$

A1

Answer: 3 + \sqrt{3}

3 marks

$$\text{(b) Perimeter} = 2[(5 + 2\sqrt{3}) + (3 + \sqrt{3})]$$

M1

$$= 16 + 6\sqrt{3} \text{ cm}$$

A1

Answer: 16 + 6\sqrt{3} cm

2 marks

$$\text{(c) } d^2 = (5 + 2\sqrt{3})^2 + (3 + \sqrt{3})^2$$

M1

$$(5 + 2\sqrt{3})^2 = 25 + 20\sqrt{3} + 12 = 37 + 20\sqrt{3}$$

M1

$$(3 + \sqrt{3})^2 = 12 + 6\sqrt{3}, \text{ so } d^2 = 49 + 26\sqrt{3}$$

Answer: $49 + 26\sqrt{3}$

A1

3 marks

22 (8 marks)

(a) The middle term has been missed: $(a + b)^2$ is not $a^2 + b^2$; there is also a $2ab$ term

E1

Answer: The cross term $2ab$ is missing

1 mark

$$(b) (2 + \sqrt{3})(2 + \sqrt{3}) = 4 + 2\sqrt{3} + 2\sqrt{3} + 3 \\ = 7 + 4\sqrt{3}$$

M1

A1

Answer: $7 + 4\sqrt{3}$

2 marks

$$(c) (2 - \sqrt{3})^2 = 7 - 4\sqrt{3}$$

M1

$$\text{Sum} = (7 + 4\sqrt{3}) + (7 - 4\sqrt{3}) = 14$$

A1

Answer: Shown

2 marks

$$(d) (2 + \sqrt{3})^3 = (2 + \sqrt{3})(7 + 4\sqrt{3})$$

M1

$$= 14 + 8\sqrt{3} + 7\sqrt{3} + 12$$

A1

$$= 26 + 15\sqrt{3}$$

A1

Answer: $26 + 15\sqrt{3}$

3 marks

23 (6 marks)

$$(a) \text{ Expanding: } a^2 + 2ab\sqrt{3} + 3b^2 = 19 + 8\sqrt{3}$$

M1

Comparing rational and irrational parts: $a^2 + 3b^2 = 19$ and $2ab = 8$, so $ab = 4$

A1

Positive integer pairs with $ab = 4$: (1, 4), (2, 2), (4, 1); only (4, 1) gives $16 + 3 = 19$

M1

$a = 4$, $b = 1$

A1

Answer: $a = 4$, $b = 1$

4 marks

$$(b) \sqrt{19 + 8\sqrt{3}} = 4 + \sqrt{3} \text{ (positive root)}$$

M1 A1

Answer: $4 + \sqrt{3}$

2 marks

24 (6 marks)

$$(a) M = 96 \times 2^{-10/4} = 96 \times 2^{-5/2}$$

M1

$$2^{-5/2} = \frac{1}{2^2 \times \sqrt{2}} = \frac{1}{4\sqrt{2}}, \text{ so } M = \frac{24}{\sqrt{2}}$$

A1

$$= 12\sqrt{2} \text{ g}$$

A1

Answer: $12\sqrt{2} \text{ g}$

3 marks

$$(b) 96 \times 2^{-t/4} = 12, \text{ so } 2^{-t/4} = \frac{1}{8} = 2^{-3}$$

M1 A1

$$\frac{t}{4} = 3, \text{ so } t = 12 \text{ hours}$$

A1

Answer: 12 hours

3 marks

25 (5 marks)

$$\frac{6\sqrt{2}}{\sqrt{3} - 1} \times \frac{\sqrt{3} + 1}{\sqrt{3} + 1} = \frac{6\sqrt{2}(\sqrt{3} + 1)}{2} = 3\sqrt{6} + 3\sqrt{2}$$

M1 A1

$$\frac{\sqrt{6}}{\sqrt{2} + 1} \times \frac{\sqrt{2} - 1}{\sqrt{2} - 1} = \sqrt{6}(\sqrt{2} - 1) = 2\sqrt{3} - \sqrt{6}$$

M1 A1

$$\text{Subtracting: } 3\sqrt{6} + 3\sqrt{2} - 2\sqrt{3} + \sqrt{6} = 3\sqrt{2} - 2\sqrt{3} + 4\sqrt{6}$$

A1

Answer: $a = 3$, $b = -2$, $c = 4$

5 marks

26 (6 marks)

$$(a) \text{ By Pythagoras, } h^2 = (1 + \sqrt{3})^2 - \left(\frac{1 + \sqrt{3}}{2}\right)^2$$

M1

$$= \frac{3}{4}(1 + \sqrt{3})^2, \text{ so } h = \frac{\sqrt{3}}{2}(1 + \sqrt{3})$$

M1

$$= \frac{\sqrt{3} + 3}{2}$$

A1

Answer: Shown

3 marks

$$\begin{aligned} \text{(b) Area} &= \frac{1}{2}(1 + \sqrt{3}) \times \frac{3 + \sqrt{3}}{2} \\ &= \frac{3 + \sqrt{3} + 3\sqrt{3} + 3}{4} = \frac{6 + 4\sqrt{3}}{4} \\ &= \frac{3}{2} + \sqrt{3} \text{ cm}^2 \end{aligned}$$

M1

A1

A1

Answer: $\frac{3}{2} + \sqrt{3} \text{ cm}^2$

3 marks

27 (9 marks)

(a) Multiply top and bottom by $(\sqrt{n+1} - \sqrt{n})$: $u_n = \frac{\sqrt{n+1} - \sqrt{n}}{(\sqrt{n+1} + \sqrt{n})(\sqrt{n+1} - \sqrt{n})}$

M1

Denominator $= (n+1) - n = 1$

A1

So $u_n = \sqrt{n+1} - \sqrt{n}$

A1

Answer: Shown

3 marks

(b) Using (a): sum $= (\sqrt{2} - \sqrt{1}) + (\sqrt{3} - \sqrt{2}) + \dots + (\sqrt{N+1} - \sqrt{N})$

M1

Every term except the first and last cancels

M1

Sum $= \sqrt{N+1} - \sqrt{1} = \sqrt{N+1} - 1$

A1

Answer: Shown

3 marks

(c) $\sqrt{N+1} - 1 > 9$, so $\sqrt{N+1} > 10$

M1

$N+1 > 100$, so $N > 99$

A1

The least integer value is $N = 100$

A1

Answer: $N = 100$

3 marks

28 (9 marks)

(a) Assume $\sqrt{2} = \frac{a}{b}$ where a and b are integers with no common factor

B1

Then $a^2 = 2b^2$, so a^2 is even and hence a is even

M1

Let $a = 2c$: $4c^2 = 2b^2$, so $b^2 = 2c^2$ and b is also even

M1

a and b both even contradicts 'no common factor', so $\sqrt{2}$ is irrational

E1

Answer: Proved

4 marks

(b) Assume $3 + 5\sqrt{2} = \frac{p}{q}$ with p, q integers, $q \neq 0$

M1

Then $\sqrt{2} = \frac{p - 3q}{5q}$, a ratio of integers

M1

So $\sqrt{2}$ would be rational, contradicting (a); hence $3 + 5\sqrt{2}$ is irrational

E1

Answer: Proved

3 marks

(c) $\sqrt{2} \times \sqrt{2} = 2$

B1

$\sqrt{2}$ is irrational by (a) but 2 is rational

E1

Answer: $\sqrt{2} \times \sqrt{2} = 2$

2 marks

29 (7 marks)

From the second equation $y = \frac{3x}{2}$, so $x + y\sqrt{2} = x + \frac{3\sqrt{2}}{2}x = \frac{x}{2}(2 + 3\sqrt{2})$

M1 A1

Squaring: $\frac{x^2}{4}(2 + 3\sqrt{2})^2 = \frac{x^2}{4}(4 + 12\sqrt{2} + 18) = \frac{x^2}{4}(22 + 12\sqrt{2})$

M1 A1

So $\frac{x^2}{4}(22 + 12\sqrt{2}) = 22 + 12\sqrt{2}$, and since $22 + 12\sqrt{2} \neq 0$ we get $x^2 = 4$

M1 A1

$x = 2, y = 3$ or $x = -2, y = -3$

A1

Answer: $x = 2, y = 3$ or $x = -2, y = -3$

7 marks

30 (11 marks)

(a) $u_0 = 1 + 1 = 2$

B1

$u_1 = (3 + \sqrt{5}) + (3 - \sqrt{5}) = 6$

B1

$u_2 = (14 + 6\sqrt{5}) + (14 - 6\sqrt{5}) = 28$

B1

Answer: 2, 6, 28

3 marks

(b) Sum of the numbers = 6 and product = $9 - 5 = 4$

M1

so they are the roots of $t^2 - 6t + 4 = 0$

A1

Answer: Shown

2 marks

(c) For each root t : $t^2 = 6t - 4$, so $t^{n+1} = 6t^n - 4t^{n-1}$

M1 A1

Adding the two results gives $u_{n+1} = 6u_n - 4u_{n-1}$

A1

Answer: Shown

3 marks

(d) $u_3 = 6 \times 28 - 4 \times 6 = 144$

B1

$u_1 = 6$ and $u_2 = 28$ are even integers

M1

If u_n and u_{n-1} are even integers then $6u_n - 4u_{n-1}$ is an even integer, so all terms are even integers (induction)

E1

Answer: $u_3 = 144$; each term is $6u_n - 4u_{n-1}$

3 marks

Expanding and Factorising

1 (2 marks)

$$x^2 - 5x + 3x - 15$$

M1

$$= x^2 - 2x - 15$$

A1

Answer: $x^2 - 2x - 15$

2 marks

2 (2 marks)

The highest common factor is $3xy$

M1

$$6x^2y - 9xy^2 = 3xy(2x - 3y)$$

A1

Answer: $3xy(2x - 3y)$

2 marks

3 (3 marks)

$$4(x^2 - 3x) = 4x^2 - 12x$$

B1

$$(x + 1)(x + 4) = x^2 + 5x + 4$$

M1

$$4x^2 - 12x - x^2 - 5x - 4 = 3x^2 - 17x - 4$$

A1

Answer: $3x^2 - 17x - 4$

3 marks

4 (2 marks)

The highest common factor is $5x^2$

M1

$$15x^3 - 10x^2 = 5x^2(3x - 2)$$

A1

Answer: $5x^2(3x - 2)$

2 marks

5 (1 mark)

Difference of two squares: $(x - 7)(x + 7)$

A1

Answer: $(x - 7)(x + 7)$

1 mark

6 (2 marks)

Two numbers with product -20 and sum 1 are 5 and -4

M1

$$(x + 5)(x - 4)$$

A1

Answer: $(x + 5)(x - 4)$

2 marks

7 (3 marks)

$$(x + 4)(x - 3) = x^2 + x - 12$$

M1

$$(2x - 1)(x^2 + x - 12) = 2x^3 + 2x^2 - 24x - x^2 - x + 12$$

M1

$$= 2x^3 + x^2 - 25x + 12$$

A1

Answer: $2x^3 + x^2 - 25x + 12$

3 marks

8 (3 marks)

Take out the common factor x : $x(4x^2 - 25)$

B1

$$4x^2 - 25 = (2x)^2 - 5^2$$

M1

$$= x(2x - 5)(2x + 5)$$

A1

Answer: $x(2x - 5)(2x + 5)$

3 marks

9 (3 marks)

Find two numbers with product $6 \times (-12) = -72$ and sum 1: 9 and -8

M1

$$6x^2 + 9x - 8x - 12 = 3x(2x + 3) - 4(2x + 3)$$

M1

$$= (2x + 3)(3x - 4)$$

A1

Answer: $(2x + 3)(3x - 4)$

3 marks

10 (3 marks)

Write $(2x + ay)(x + by)$: $ab = -1$ and $2b + a = -1$

M1

$$a = 1 \text{ and } b = -1$$

A1

$$(2x + y)(x - y)$$

A1

Answer: $(2x + y)(x - y)$

3 marks

11 (3 marks)

Take out the common factor x : $x(x^2 - x - 6)$

M1

$$x^2 - x - 6 = (x - 3)(x + 2)$$

M1

$$x(x - 3)(x + 2)$$

A1

Answer: $x(x - 3)(x + 2)$

3 marks

12 (3 marks)

$$(x + 3)^2 = x^2 + 6x + 9 \text{ and } (x - 3)^2 = x^2 - 6x + 9$$

M1 A1

Subtracting: $12x$

A1

Answer: Shown

3 marks

13 (4 marks)

Group in pairs: $x^2(2x + 3) - 4(2x + 3)$

M1 A1

$$= (2x + 3)(x^2 - 4)$$

A1

$$= (2x + 3)(x - 2)(x + 2)$$

A1

Answer: $(2x + 3)(x - 2)(x + 2)$

4 marks

14 (4 marks)

$$(x + 2)^3 = x^3 + 6x^2 + 12x + 8$$

M1

$$(x - 2)^3 = x^3 - 6x^2 + 12x - 8$$

A1

$$\text{Subtracting: } (x^3 - x^3) + (6x^2 + 6x^2) + (12x - 12x) + (8 + 8)$$

M1

$$= 12x^2 + 16$$

A1

Answer: Shown

4 marks

15 (4 marks)

(a) Treat as a quadratic in x^2 : $(x^2 - 4)(x^2 - 9)$

M1 A1

$$= (x - 2)(x + 2)(x - 3)(x + 3)$$

A1

Answer: $(x - 2)(x + 2)(x - 3)(x + 3)$

3 marks

(b) $x = \pm 2$ or $x = \pm 3$

A1

Answer: $x = \pm 2, \pm 3$

1 mark

16 (4 marks)

Use $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ with $a = 3x$, $b = -2y$

M1

$$(3x)^3 = 27x^3 \text{ and } 3(3x)^2(-2y) = -54x^2y$$

A1

$$3(3x)(-2y)^2 = 36xy^2 \text{ and } (-2y)^3 = -8y^3$$

A1

$$27x^3 - 54x^2y + 36xy^2 - 8y^3$$

A1

Answer: $27x^3 - 54x^2y + 36xy^2 - 8y^3$

4 marks

17 (4 marks)

$$81x^4 - 16 = (9x^2)^2 - 4^2 = (9x^2 - 4)(9x^2 + 4)$$

M1 A1

$$9x^2 - 4 = (3x - 2)(3x + 2)$$

M1

$$(3x - 2)(3x + 2)(9x^2 + 4)$$

A1

$$\text{Answer: } (3x - 2)(3x + 2)(9x^2 + 4)$$

4 marks

18 (4 marks)

Let the odd integers be $2k - 1$ and $2k + 1$ for an integer k

B1

$$(2k + 1)^2 - (2k - 1)^2 = (4k^2 + 4k + 1) - (4k^2 - 4k + 1)$$

M1

$$= 8k$$

A1

$8k$ is a multiple of 8 since k is an integer

E1

$$\text{Answer: } 8k, \text{ a multiple of 8}$$

4 marks

19 (6 marks)

$$(a) (x + 1)^2 = x^2 + 2x + 1$$

M1

$$(x^2 + 2x + 1)^2 = x^4 + 4x^3 + 6x^2 + 4x + 1$$

M1 A1

$$\text{Answer: } x^4 + 4x^3 + 6x^2 + 4x + 1$$

3 marks

$$(b) \text{ Take } x = 0.01: 1.01^4 = 0.01^4 + 4(0.01)^3 + 6(0.01)^2 + 4(0.01) + 1$$

M1 A1

$$= 0.00000001 + 0.000004 + 0.0006 + 0.04 + 1 = 1.04060401$$

A1

$$\text{Answer: } 1.04060401$$

3 marks

20 (7 marks)

$$(a) \text{ Area of the rectangle} = (3x + 2)(2x + 1) = 6x^2 + 7x + 2$$

M1 A1

$$\text{Subtract the square } (x - 1)^2 = x^2 - 2x + 1: 5x^2 + 9x + 1$$

A1

$$\text{Answer: Shown}$$

3 marks

$$(b) 5x^2 + 9x + 1 = 39, \text{ so } 5x^2 + 9x - 38 = 0$$

M1 A1

$$(5x + 19)(x - 2) = 0$$

M1

$$x = 2 \text{ (} x = -\frac{19}{5} \text{ is rejected as a length cannot be negative)}$$

A1

$$\text{Answer: } x = 2$$

4 marks

21 (8 marks)

$$(a) \text{ Outer rectangle measures } (x + 2) \text{ m by } (x + 6) \text{ m}$$

B1

$$\text{Area of path} = (x + 2)(x + 6) - x(x + 4) = x^2 + 8x + 12 - x^2 - 4x$$

M1

$$= 4x + 12$$

A1

$$\text{Answer: Shown}$$

3 marks

$$(b) 4x + 12 = 60, \text{ so } x = 12$$

M1

The lawn is 12 m by 16 m

A1

$$\text{Answer: } 12 \text{ m by } 16 \text{ m}$$

2 marks

$$(c) \text{ Outer rectangle measures } (x + 3) \text{ m by } (x + 7) \text{ m}$$

B1

$$\text{Area} = (x + 3)(x + 7) - x(x + 4) = x^2 + 10x + 21 - x^2 - 4x$$

M1

$$= 6x + 21$$

A1

$$\text{Answer: Shown}$$

3 marks

22 (7 marks)

$$(a) a^3 + a^2b + ab^2 - a^2b - ab^2 - b^3$$

M1

$$= a^3 - b^3$$

A1

$$\text{Answer: } a^3 - b^3$$

2 marks

$$(b) x^6 - 64 = (x^3)^2 - 8^2 = (x^3 - 8)(x^3 + 8)$$

M1

$$\text{From (a) with } a = x, b = 2: x^3 - 8 = (x - 2)(x^2 + 2x + 4)$$

M1 A1

$$\text{Replacing } b \text{ by } -2 \text{ in (a): } x^3 + 8 = (x + 2)(x^2 - 2x + 4)$$

M1

$$x^6 - 64 = (x - 2)(x + 2)(x^2 + 2x + 4)(x^2 - 2x + 4)$$

A1

$$\text{Answer: } (x - 2)(x + 2)(x^2 + 2x + 4)(x^2 - 2x + 4)$$

5 marks

23 (7 marks)

(a) $(x^2 + 2)^2 = x^4 + 4x^2 + 4$ M1

Subtract $4x^2$: $x^4 + 4$ A1**Answer: Shown** 2 marks

(b) Difference of two squares: $(x^2 + 2 - 2x)(x^2 + 2 + 2x)$ M1 A1

$(x^2 - 2x + 2)(x^2 + 2x + 2)$ A1

Answer: $(x^2 - 2x + 2)(x^2 + 2x + 2)$ 3 marks

(c) Take $x = 5$: $(25 - 10 + 2)(25 + 10 + 2) = 17 \times 37$ M1

Both factors exceed 1, so 629 is composite E1

Answer: $629 = 17 \times 37$ 2 marks

24 (8 marks)

(a) $(x - 1)(x + 4) = x^2 + 3x - 4$ and $(x - 2)(x + 5) = x^2 + 3x - 10$ M1 A1

Let $m = x^2 + 3x$. The product is $(m - 4)(m - 10) = m^2 - 14m + 40$ M1

$= (x^2 + 3x)^2 - 14(x^2 + 3x) + 40 = x^4 + 6x^3 - 5x^2 - 42x + 40$ A1

Answer: Shown 4 marks

(b) $(m - 4)(m - 10) = 40$ gives $m^2 - 14m = 0$ M1

$m = 0$ or $m = 14$ A1

$x^2 + 3x = 0$ gives $x = 0$ or $x = -3$ A1

$x^2 + 3x - 14 = 0$ gives $x = \frac{-3 \pm \sqrt{65}}{2}$ A1

Answer: $x = 0, -3, \frac{-3 \pm \sqrt{65}}{2}$ 4 marks

25 (7 marks)

(a) $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ M1

$3a^2b + 3ab^2 = 3ab(a + b)$ M1

Rearranging gives the result A1

Answer: Shown 3 marks

(b) $125 - 3 \times 4 \times 5$ M1

$= 65$ A1

Answer: 65 2 marks

(c) a and b are the roots of $t^2 - 5t + 4 = 0$, so they are 1 and 4 M1

$1^3 + 4^3 = 65$ A1

Answer: 65 2 marks**26 (6 marks)**

(a) The student has squared each term and missed the middle term E1

$(2x - 3)(2x - 3) = 4x^2 - 6x - 6x + 9$ M1

$= 4x^2 - 12x + 9$ A1

Answer: $4x^2 - 12x + 9$ 3 marks

(b) $(x - 2)(x - 3) = x^2 - 5x + 6$, which does not match M1 A1

The correct factors have product -6 and sum -5 : $(x - 6)(x + 1)$ A1

Answer: $(x - 6)(x + 1)$ 3 marks

27 (8 marks)

(a) Pair the outer and inner factors: $n(n + 3) = n^2 + 3n$ M1

$(n + 1)(n + 2) = n^2 + 3n + 2$ A1

Answer: Shown 2 marks

(b) Let $m = n^2 + 3n$; the expression is $m(m + 2) + 1$ M1

$= m^2 + 2m + 1$ M1

$= (m + 1)^2 = (n^2 + 3n + 1)^2$ A1

 n is an integer, so $n^2 + 3n + 1$ is an integer and the expression is a perfect square E1**Answer: Proved** 4 marks

$$(c) n = 20: n^2 + 3n + 1 = 400 + 60 + 1 = 461$$

M1

$$20 \times 21 \times 22 \times 23 + 1 = 461^2 (= 212\,521)$$

A1

Answer: 461^2

2 marks

28 (7 marks)

$$(a) \text{ Put } x = y: f(y, y) = (6 + 7 - 12 - 3 + 2)y^4$$

M1

$$= 0, \text{ so } (x - y) \text{ is a factor}$$

A1

Answer: Shown

2 marks

$$(b) \text{ Dividing by } (x - y): f(x, y) = (x - y)(6x^3 + 13x^2y + xy^2 - 2y^3)$$

M1 A1

$$\text{Put } x = -2y \text{ in the cubic: } 6(-8) + 13(4) + (-2) - 2 = 0, \text{ so } (x + 2y) \text{ is a factor}$$

M1

$$6x^3 + 13x^2y + xy^2 - 2y^3 = (x + 2y)(6x^2 + xy - y^2)$$

A1

$$6x^2 + xy - y^2 = (2x + y)(3x - y), \text{ so } f(x, y) = (x - y)(x + 2y)(2x + y)(3x - y)$$

A1

Answer: $(x - y)(x + 2y)(2x + y)(3x - y)$

5 marks

29 (9 marks)

$$(a) n(n^2 - 1)$$

M1

$$= n(n - 1)(n + 1)$$

A1

Answer: $(n - 1)n(n + 1)$

2 marks

$$(b) (n - 1), n, (n + 1) \text{ are three consecutive integers}$$

B1

$$\text{At least one of them is even, so the product is divisible by 2}$$

M1 E1

$$\text{Exactly one of them is a multiple of 3, so the product is divisible by 3}$$

E1

Answer: Proved

4 marks

$$(c) \text{ By (b), } n^3 - n \text{ is always divisible by 6, so it is divisible by 30 exactly when it is divisible by 5}$$

M1

$$n = 2: 6 \text{ and } n = 3: 24 \text{ are not multiples of 5}$$

M1

$$n = 4: 4^3 - 4 = 60 = 2 \times 30$$

A1

Answer: $n = 4$

3 marks

30 (10 marks)

$$(a) (x + \frac{1}{x})^2 = x^2 + 2 + \frac{1}{x^2}$$

M1 A1

$$\text{So } x^2 + \frac{1}{x^2} = 25 - 2 = 23$$

A1

Answer: Shown

3 marks

$$(b) (x + \frac{1}{x})^3 = x^3 + 3x + \frac{3}{x} + \frac{1}{x^3}$$

M1 A1

$$= x^3 + \frac{1}{x^3} + 3(x + \frac{1}{x})$$

M1

$$x^3 + \frac{1}{x^3} = 125 - 15 = 110$$

A1

Answer: 110

4 marks

$$(c) (x^2 + \frac{1}{x^2})^2 = x^4 + 2 + \frac{1}{x^4}$$

M1 A1

$$x^4 + \frac{1}{x^4} = 23^2 - 2 = 527$$

A1

Answer: 527

3 marks

Quadratics and Completing the Square**1 (2 marks)**

$$x^2 + 6x = (x + 3)^2 - 9$$

M1

$$x^2 + 6x + 2 = (x + 3)^2 - 7$$

A1

Answer: $(x + 3)^2 - 7$

2 marks

2 (3 marks)

$$x^2 - 8x + 3 = (x - 4)^2 - 16 + 3$$

M1

$$= (x - 4)^2 - 13$$

A1

Minimum value is -13 when $x = 4$

A1

Answer: Minimum -13 at $x = 4$

3 marks

3 (2 marks)

$$(x + 2)(x - 5) = 0 \text{ gives } x^2 - 3x - 10 = 0$$

M1

$$b = -3 \text{ and } c = -10$$

A1

Answer: $b = -3, c = -10$

2 marks

4 (2 marks)

$$(x - 5)(x + 2) = 0$$

M1

$$x = 5 \text{ or } x = -2$$

A1

Answer: $x = 5, x = -2$

2 marks

5 (2 marks)A square is zero when $x = 3$

M1

Minimum at $(3, 4)$

A1

Answer: $(3, 4)$

2 marks

6 (2 marks)

$$(2x - 1)(x - 3) = 0$$

M1

$$x = \frac{1}{2} \text{ or } x = 3$$

A1

Answer: $x = \frac{1}{2}, x = 3$

2 marks

7 (3 marks)

$$2x^2 - 12x + 5 = 2(x^2 - 6x) + 5$$

M1

$$= 2[(x - 3)^2 - 9] + 5$$

M1

$$= 2(x - 3)^2 - 13$$

A1

Answer: $2(x - 3)^2 - 13$

3 marks

8 (5 marks)

$$(a) x^2 - 4x = (x - 2)^2 - 4$$

M1

$$x^2 - 4x - 5 = (x - 2)^2 - 9$$

A1

Answer: $(x - 2)^2 - 9$

2 marks

(b) Turning point (minimum) at $(2, -9)$, U-shaped

B1

y-intercept: $x = 0$ gives $(0, -5)$

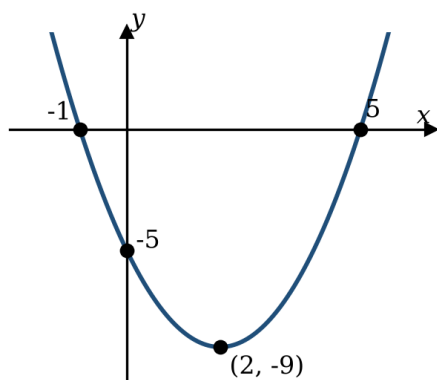
B1

 $x^2 - 4x - 5 = (x - 5)(x + 1) = 0$, so crosses the x-axis at $(-1, 0)$ and $(5, 0)$

B1

Answer: See sketch

3 marks



9 (4 marks)

$$\text{Let } y = x^2: y^2 + 4y - 12 = 0$$

M1

$$(y + 6)(y - 2) = 0$$

A1

$$y = -6 \text{ gives no real } x, \text{ because } x^2 \text{ cannot be negative; } y = 2$$

B1

$$x = \pm\sqrt{2}$$

A1

$$\text{Answer: } x = \pm\sqrt{2}$$

4 marks

10 (3 marks)

$$-2(x^2 - 3x) - 4 = -2\left[\left(x - \frac{3}{2}\right)^2 - \frac{9}{4}\right] - 4$$

M1 A1

$$= \frac{1}{2} - 2\left(x - \frac{3}{2}\right)^2, \text{ maximum value } \frac{1}{2}$$

A1

$$\text{Answer: } \frac{1}{2} - 2\left(x - \frac{3}{2}\right)^2; \text{ max } \frac{1}{2}$$

3 marks

11 (3 marks)

$$(x - 2)^2 - 4 - 3 = 0, \text{ so } (x - 2)^2 = 7$$

M1 A1

$$x = 2 \pm \sqrt{7}$$

A1

$$\text{Answer: } x = 2 \pm \sqrt{7}$$

3 marks

12 (3 marks)

$$x = \frac{-3 \pm \sqrt{9 + 4}}{2} = \frac{-3 \pm \sqrt{13}}{2}$$

M1 A1

$$x = 0.303 \text{ or } x = -3.30$$

A1

$$\text{Answer: } x = 0.303, x = -3.30$$

3 marks

13 (4 marks)

$$\text{y-axis: } x = 0 \text{ gives } y = 5, \text{ so C is } (0, 5)$$

B1

$$\text{x-axis: } x^2 - 6x + 5 = 0$$

M1

$$(x - 1)(x - 5) = 0$$

A1

$$A(1, 0) \text{ and } B(5, 0)$$

A1

$$\text{Answer: } A(1, 0), B(5, 0), C(0, 5)$$

4 marks

14 (4 marks)

$$(x + 3)^2 - 9 - 2 = 0$$

M1

$$(x + 3)^2 = 11$$

A1

$$x + 3 = \pm\sqrt{11}$$

M1

$$x = -3 \pm \sqrt{11}$$

A1

$$\text{Answer: } x = -3 \pm \sqrt{11}$$

4 marks

15 (7 marks)

$$(a) f(x) = -(x^2 - 6x) + 7$$

M1

$$= -[(x - 3)^2 - 9] + 7$$

M1

$$= 16 - (x - 3)^2$$

A1

$$\text{Answer: } 16 - (x - 3)^2$$

3 marks

$$(b) \text{ Maximum value } 16 \text{ when } x = 3$$

B1

$$\text{Answer: } 16 \text{ at } x = 3$$

1 mark

$$(c) \text{ Inverted U shape with maximum at } (3, 16)$$

B1

$$\text{y-intercept } (0, 7)$$

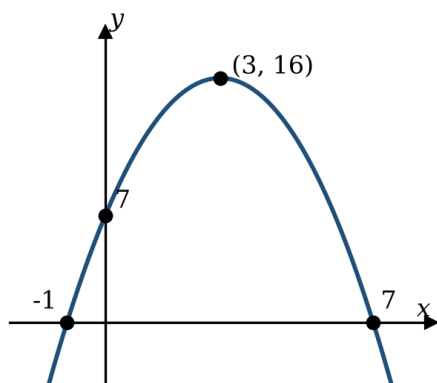
B1

$$-x^2 + 6x + 7 = -(x - 7)(x + 1) = 0, \text{ so crosses the x-axis at } (-1, 0) \text{ and } (7, 0)$$

B1

$$\text{Answer: See sketch}$$

3 marks



16 (4 marks)

(a) $(x - 4)^2 - 16 + 19$

M1

$= (x - 4)^2 + 3$

A1

Answer: $(x - 4)^2 + 3$

2 marks

(b) A square is at least 0, so $f(x) \geq 3 > 0$

E1

Minimum value 3, at $x = 4$

B1

Answer: Minimum 3

2 marks

17 (4 marks)

Expand the right-hand side: $Bx^2 + 4Bx + 4B + C$

M1

Compare the x^2 terms: $B = 5$

A1

Compare the x terms: $A = 4B = 20$

A1

Compare the constants: $4B + C = -7$, so $C = -27$

A1

Answer: $A = 20$, $B = 5$, $C = -27$

4 marks

18 (6 marks)

(a) $y = a(x - 1)(x + 5)$

M1

$x = 0$: $-5a = 15$, so $a = -3$

M1

$y = -3(x - 1)(x + 5) = -3x^2 - 12x + 15$

A1

Answer: $y = -3x^2 - 12x + 15$

3 marks

(b) Axis of symmetry halfway between the roots: $x = -2$

M1

$y = -3(-3)(3) = 27$

M1

$(-2, 27)$

A1

Answer: $(-2, 27)$

3 marks

19 (6 marks)

(a) $x^2 - 4x + 1 = (x - 2)^2 - 3$

M1 A1

T is $(2, -3)$

A1

Answer: T(2, -3)

3 marks

(b) $(x - 2)^2 = 3$

M1

$x - 2 = \pm\sqrt{3}$

A1

$x = 2 + \sqrt{3}$ or $x = 2 - \sqrt{3}$

A1

Answer: $x = 2 \pm \sqrt{3}$

3 marks

20 (5 marks)

$x^2 - 3x + 4 = 2x - 2$

M1

$x^2 - 5x + 6 = 0$

A1

$(x - 2)(x - 3) = 0$, so $x = 2$ or $x = 3$

M1 A1

A(2, 2) and B(3, 4)

A1

Answer: A(2, 2), B(3, 4)

5 marks

21 (8 marks)

(a) $y = (x - 3)^2 + 2$ M1

$= x^2 - 6x + 9 + 2$ M1

So $b = -6$ and $c = 11$ A1

Answer: $b = -6, c = 11$ 3 marks

(b) $(x - 3)^2 \geq 0$, so $y \geq 2$ for all x M1

The minimum value of y is $2 > 0$, so $y = 0$ has no solution A1

Answer: Shown 2 marks

(c) $(x - 3)^2 + 2 = 6$, so $(x - 3)^2 = 4$ M1

$x = 1$ or $x = 5$ A1

$AB = 5 - 1 = 4$ A1

Answer: 4 3 marks

22 (8 marks)

(a) $h = -5(t^2 - \frac{8}{5}t) + 2$ M1

$= -5[(t - \frac{4}{5})^2 - \frac{16}{25}] + 2$ M1

$= \frac{26}{5} - 5(t - \frac{4}{5})^2$ ($A = 5.2, B = 5, C = 0.8$) A1

Answer: $5.2 - 5(t - 0.8)^2$ 3 marks

(b) $(t - 0.8)^2 \geq 0$, so the greatest value of h is 5.2 B1

This occurs when $t = 0.8$ B1

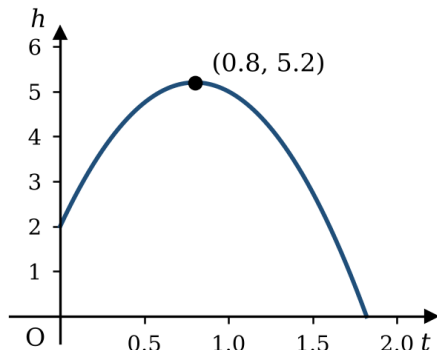
Answer: 5.2 m at $t = 0.8$ s 2 marks

(c) $h = 0$: $5(t - 0.8)^2 = 5.2$, so $(t - 0.8)^2 = 1.04$ M1

$t = 0.8 \pm \sqrt{1.04}$ A1

The positive root is $t = 1.82$ s (the other root is negative) A1

Answer: 1.82 s 3 marks

**23 (6 marks)**

(a) Original share $\frac{840}{n}$; new share $\frac{840}{n - 6}$ B1

$\frac{840}{n - 6} - \frac{840}{n} = 7$ M1

Multiply by $n(n - 6)$: $840n - 840(n - 6) = 7n(n - 6)$, so $5040 = 7n^2 - 42n$ and $n^2 - 6n - 720 = 0$ A1

Answer: Shown 3 marks

(b) $(n - 30)(n + 24) = 0$ M1

n cannot be negative, so $n = 30$ A1

Answer: $n = 30$ 2 marks

(c) $\frac{840}{24} = £35$ B1

Answer: £35 1 mark

24 (8 marks)

$$(a) x^2 + kx + 4 = \left(x + \frac{k}{2}\right)^2 - \frac{k^2}{4} + 4$$

M1 A1

$$\text{Minimum at } x = -\frac{k}{2}, y = 4 - \frac{k^2}{4}$$

A1

Answer: Shown

3 marks

$$(b) 4 - \frac{k^2}{4} = 3\left(-\frac{k}{2}\right) - 5$$

M1

$$\text{Multiply by 4: } 16 - k^2 = -6k - 20$$

M1

$$k^2 - 6k - 36 = 0$$

A1

$$k = \frac{6 \pm \sqrt{36 + 144}}{2} = 3 \pm 3\sqrt{5}$$

M1 A1

$$\text{Answer: } k = 3 \pm 3\sqrt{5}$$

5 marks

25 (8 marks)

$$(a) \text{ Sum of roots} = 7$$

B1

$$\text{Product of roots} = 3$$

B1

$$\text{Answer: } \alpha + \beta = 7, \alpha\beta = 3$$

2 marks

$$(b) \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$$

M1

$$= 49 - 6 = 43$$

A1

$$\text{Answer: } 43$$

2 marks

$$(c) \text{ Sum} = \frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta} = \frac{7}{3}$$

M1 A1

$$\text{Product} = \frac{1}{\alpha\beta} = \frac{1}{3}$$

A1

$$x^2 - \frac{7}{3}x + \frac{1}{3} = 0, \text{ so } 3x^2 - 7x + 1 = 0$$

A1

$$\text{Answer: } 3x^2 - 7x + 1 = 0$$

4 marks

26 (7 marks)

$$(a) C: x = 0 \text{ gives } y = 3, \text{ so } C(0, 3)$$

B1

$$3 + 2x - x^2 = 0, \text{ so } (3 - x)(1 + x) = 0, \text{ giving } A(-1, 0) \text{ and } B(3, 0)$$

M1 A1

$$y = 4 - (x - 1)^2$$

M1

$$P \text{ is } (1, 4)$$

A1

$$\text{Answer: } A(-1, 0), B(3, 0), C(0, 3), P(1, 4)$$

5 marks

$$(b) \text{ Base } AB = 4, \text{ height } OC = 3$$

M1

$$\text{Area} = \frac{1}{2} \times 4 \times 3 = 6$$

A1

$$\text{Answer: } 6$$

2 marks

27 (9 marks)

$$(a) x^2 - 4kx = (x - 2k)^2 - 4k^2$$

M1

$$f(x) = (x - 2k)^2 - k^2 + 6k + 10, \text{ minimum value } -k^2 + 6k + 10$$

A1

$$-k^2 + 6k + 10 = -(k - 3)^2 + 9 + 10 = 19 - (k - 3)^2$$

A1

$$\text{Answer: Shown}$$

3 marks

$$(b) (k - 3)^2 \geq 0, \text{ so the minimum value of } f \text{ is at most } 19$$

M1

$$\text{The greatest value is } 19, \text{ when } k = 3$$

A1

$$\text{Answer: } 19 \text{ when } k = 3$$

2 marks

$$(c) \text{ Need the minimum value positive: } 19 - (k - 3)^2 > 0$$

M1

$$(k - 3)^2 < 19$$

A1

$$-\sqrt{19} < k - 3 < \sqrt{19}$$

M1

$$3 - \sqrt{19} < k < 3 + \sqrt{19}$$

A1

$$\text{Answer: } 3 - \sqrt{19} < k < 3 + \sqrt{19}$$

4 marks

28 (8 marks)

$$(a) \left(x - \frac{4}{x}\right)^2 = x^2 - 2 \times x \times \frac{4}{x} + \frac{16}{x^2} \quad \text{M1}$$

$$= x^2 - 8 + \frac{16}{x^2}, \text{ so } g(x) = \left(x - \frac{4}{x}\right)^2 + 8 \quad \text{A1}$$

Answer: Shown

2 marks

$$(b) \left(x - \frac{4}{x}\right)^2 \geq 0, \text{ so } g(x) \geq 8 \quad \text{B1}$$

$$\text{Equality when } x = \frac{4}{x}, \text{ i.e. } x^2 = 4 \quad \text{M1}$$

$$\text{Minimum value 8 at } x = 2 \text{ and } x = -2 \quad \text{A1}$$

Answer: 8 at $x = \pm 2$

3 marks

$$(c) x = 0 \text{ is not a root, so divide by } x^2: x^2 + \frac{16}{x^2} = k, \text{ i.e. } g(x) = k \quad \text{M1}$$

$$g(x) \geq 8 \text{ for all real } x \neq 0 \text{ by (b)} \quad \text{E1}$$

$$\text{So } g(x) = k \text{ has no solution when } k < 8 \quad \text{E1}$$

Answer: Shown

3 marks

29 (9 marks)

$$(a) \alpha + \beta = -\frac{q}{p} \text{ and } \alpha\beta = \frac{r}{p} \quad \text{B1}$$

$$(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta \quad \text{M1}$$

$$k^2 = \frac{q^2}{p^2} - \frac{4r}{p} = \frac{q^2 - 4pr}{p^2} \quad \text{M1}$$

$$\text{So } q^2 - 4pr = p^2k^2 \quad \text{A1}$$

Answer: Shown

4 marks

$$(b) p = 3, q = -14, r = c, k = 2$$

B1

$$196 - 12c = 9 \times 4 = 36$$

M1

$$c = \frac{160}{12} = \frac{40}{3} \quad \text{A1}$$

$$\text{Answer: } c = \frac{40}{3}$$

3 marks

$$(c) \text{ Sum of roots } \frac{14}{3}, \text{ difference 2: } 2\alpha = \frac{14}{3} + 2 = \frac{20}{3} \quad \text{M1}$$

$$x = \frac{10}{3} \text{ or } x = \frac{4}{3} \quad \text{A1}$$

$$\text{Answer: } x = \frac{10}{3}, x = \frac{4}{3} \quad \text{2 marks}$$

30 (9 marks)

$$(a) \text{ Stationary at } x = 2 \text{ with value 9: } g(x) = a(x - 2)^2 + 9 \quad \text{M1}$$

$$(0, 5): 4a + 9 = 5, \text{ so } a = -1 \quad \text{M1 A1}$$

$$g(x) = -(x^2 - 4x + 4) + 9 = -x^2 + 4x + 5 \quad \text{M1}$$

$$a = -1, b = 4, c = 5 \quad \text{A1}$$

Answer: $a = -1, b = 4, c = 5$

5 marks

$$(b) -x^2 + 4x + 5 \geq 0 \text{ gives } x^2 - 4x - 5 \leq 0, \text{ so } (x - 5)(x + 1) \leq 0, \text{ i.e. } -1 \leq x \leq 5 \quad \text{M1 A1}$$

$$g(6) = -36 + 24 + 5 = -7 \quad \text{M1}$$

$$\text{Range } -7 \leq g(x) \leq 9 \quad \text{A1}$$

$$\text{Answer: } -1 \leq x \leq 5; -7 \leq g(x) \leq 9 \quad \text{4 marks}$$

Algebraic Fractions

1 (2 marks)

Numerator = $(x - 3)(x + 3)$; denominator = $x(x + 3)$ M1

Cancelling $(x + 3)$: $\frac{x - 3}{x}$ A1

Answer: $\frac{x - 3}{x}$ 2 marks

2 (2 marks)

Common denominator $x(x + 1)$: $\frac{2(x + 1) + 3x}{x(x + 1)}$ M1

= $\frac{5x + 2}{x(x + 1)}$ A1

Answer: $\frac{5x + 2}{x(x + 1)}$ 2 marks

3 (2 marks)

Divide numbers: $\frac{12}{8} = \frac{3}{2}$; subtract indices of x and y M1

= $\frac{3x^2}{2y}$ A1

Answer: $\frac{3x^2}{2y}$ 2 marks

4 (2 marks)

Factorise the numerator: $\frac{x(x - 5)}{x}$ M1

= $x - 5$ A1

Answer: $x - 5$ 2 marks

5 (2 marks)

Common denominator 12: $\frac{4x}{12} + \frac{3x}{12}$ M1

= $\frac{7x}{12}$ A1

Answer: $\frac{7x}{12}$ 2 marks

6 (2 marks)

$\frac{8x}{4x^2}$ M1

= $\frac{2}{x}$ A1

Answer: $\frac{2}{x}$ 2 marks

7 (3 marks)

Numerator = $(x + 2)(x + 3)$ M1

Denominator = $(x - 4)(x + 3)$ M1

Cancelling $(x + 3)$: $\frac{x + 2}{x - 4}$ A1

Answer: $\frac{x + 2}{x - 4}$ 3 marks

8 (3 marks)

Common denominator $(x - 2)(x + 1)$: $\frac{3(x + 1) - 2(x - 2)}{(x - 2)(x + 1)}$ M1 A1

$$= \frac{x + 7}{(x - 2)(x + 1)}$$
 A1

Answer: $\frac{x + 7}{(x - 2)(x + 1)}$ 3 marks

9 (3 marks)

Factorise: $\frac{x + 2}{x - 3} \times \frac{(x - 3)(x + 3)}{(x - 2)(x + 2)}$ M1 A1

Cancel $(x + 2)$ and $(x - 3)$: $\frac{x + 3}{x - 2}$ A1

Answer: $\frac{x + 3}{x - 2}$ 3 marks

10 (3 marks)

$y^3 - 8 = (y - 2)(y^2 + 2y + 4)$ and $y^2 - 4 = (y - 2)(y + 2)$ M1 A1

Cancel the common factor $(y - 2)$: the result is $\frac{y^2 + 2y + 4}{y + 2}$ A1

Answer: $\frac{y^2 + 2y + 4}{y + 2}$ 3 marks

11 (3 marks)

$2x^2 - x - 6 = (2x + 3)(x - 2)$ M1

$2x^2 + 7x + 6 = (2x + 3)(x + 2)$ M1

Cancelling $(2x + 3)$: $\frac{x - 2}{x + 2}$ A1

Answer: $\frac{x - 2}{x + 2}$ 3 marks

12 (3 marks)

Multiply by 12: $4x - 3(x - 1) = 24$ M1 A1

$x + 3 = 24$, so $x = 21$ A1

Answer: $x = 21$ 3 marks

13 (3 marks)

Cross-multiply: $3x = 2(x - 1)$ M1 A1

$x = -2$ A1

Answer: $x = -2$ 3 marks

14 (4 marks)

Factorise: $\frac{(x - 2)(x + 2)}{2(x + 3)} \div \frac{x - 2}{(x - 3)(x + 3)}$ M1

$= \frac{(x - 2)(x + 2)}{2(x + 3)} \times \frac{(x - 3)(x + 3)}{x - 2}$ M1

Cancel the common factors $(x - 2)$ and $(x + 3)$ M1

$= \frac{(x + 2)(x - 3)}{2}$ A1

Answer: $\frac{(x + 2)(x - 3)}{2}$ 4 marks

15 (5 marks)Multiply through by $x(x + 1)$: $2(x + 1) + 3x = 2x(x + 1)$

M1

$$5x + 2 = 2x^2 + 2x$$

A1

$$2x^2 - 3x - 2 = 0$$

A1

$$(2x + 1)(x - 2) = 0$$

M1

$$x = 2 \text{ or } x = -\frac{1}{2}$$

A1

Answer: $x = 2$ or $x = -\frac{1}{2}$

5 marks

16 (5 marks) $x^2 - 1 = (x - 1)(x + 1)$; multiply through by $(x - 1)(x + 1)$

M1

$$x(x + 1) - 2(x - 1) = 8$$

A1

$$x^2 - x - 6 = 0$$

A1

$$(x - 3)(x + 2) = 0$$

M1

$$x = 3 \text{ or } x = -2 \text{ (neither is } \pm 1)$$

A1

Answer: $x = 3$ or $x = -2$

5 marks

17 (4 marks)

$$x^2(x + 1) = x^3 + x^2$$
; subtract: $-3x^2 + 7x - 1$

M1

$$-3x(x + 1) = -3x^2 - 3x$$
; subtract: $10x - 1$

M1

$$10(x + 1) = 10x + 10$$
; subtract: -11

A1

$$a = 1, b = -3, c = 10, d = -11$$

A1

Answer: $a = 1, b = -3, c = 10, d = -11$

4 marks

18 (4 marks)

Factorise: $\frac{x(x + 3)}{(x - 4)(x + 3)} \times \frac{(x - 4)(x + 4)}{x^2}$

M1 A1

Cancel x , $(x + 3)$, $(x - 4)$

M1

$$= \frac{x + 4}{x}$$

A1

Answer: $\frac{x + 4}{x}$

4 marks

19 (4 marks)

$$x^2 + 3x - 1 = (x + 2)(x + 1) - 3$$

M1 A1

$$\frac{(x + 2)(x + 1) - 3}{x + 2} = x + 1 - \frac{3}{x + 2}$$

M1

$$p = 1, q = -3$$

A1

Answer: $p = 1, q = -3$

4 marks

20 (6 marks)

(a) Times: $\frac{24}{x}$ and $\frac{24}{x + 4}$

B1

$$\frac{24}{x} - \frac{24}{x + 4} = \frac{1}{2}$$

M1

$$48(x + 4) - 48x = x(x + 4), \text{ so } 192 = x^2 + 4x$$

M1 A1

Answer: Shown

4 marks

(b) $(x - 12)(x + 16) = 0$

M1

$$x = 12 \text{ km/h (the root } x = -16 \text{ is rejected)}$$

A1

Answer: 12 km/h

2 marks

21 (7 marks)

(a) In one hour pipe A fills $\frac{1}{x}$ of the pool and pipe B fills $\frac{1}{x+3}$ of it

B1

$$\frac{1}{x} + \frac{1}{x+3} = \frac{1}{2}$$

M1

Multiply by $2x(x+3)$: $2(x+3) + 2x = x(x+3)$

M1

$$4x + 6 = x^2 + 3x, \text{ so } x^2 - x - 6 = 0$$

A1

Answer: Shown

4 marks

(b) $(x-3)(x+2) = 0$

M1

$x = 3$ ($x = -2$ is rejected as a time cannot be negative)

A1

Pipe A takes 3 hours and pipe B takes 6 hours

A1

Answer: A: 3 hours, B: 6 hours

3 marks

22 (7 marks)

(a) Only common factors can be cancelled; x is a term in each sum, not a factor of the whole numerator or denominator

E1

Answer: Terms cannot be cancelled, only factors

1 mark

(b) Numerator: $x^2 + x - 12 = (x+4)(x-3)$

M1

Denominator: $x^2 - 16 = (x-4)(x+4)$

M1

Cancelling $(x+4)$: $\frac{x-3}{x-4}$

A1

Answer: $\frac{x-3}{x-4}$

3 marks

(c) $\frac{x-3}{x-4} = 2$, so $x-3 = 2(x-4)$

M1

$x = 5$

A1

$x = 5$ is not one of the excluded values ($x \neq \pm 4$), so it is valid

B1

Answer: $x = 5$

3 marks

23 (8 marks)

(a) $ff(x) = \frac{\frac{x+1}{x-1} + 1}{\frac{x+1}{x-1} - 1}$

M1

Multiply top and bottom by $(x-1)$: $\frac{(x+1) + (x-1)}{(x+1) - (x-1)}$

M1 A1

$= \frac{2x}{2} = x$

A1

Answer: Shown

4 marks

(b) $x+1 = 3(x-1)$

M1

$x = 2$

A1

Answer: $x = 2$

2 marks

(c) Since $ff(x) = x$, f is its own inverse

E1

$f^{-1}(x) = \frac{x+1}{x-1}$

B1

Answer: $f^{-1}(x) = f(x)$

2 marks

24 (6 marks)

(a) $\frac{1}{x-3} + \frac{1}{x+3} = \frac{(x+3) + (x-3)}{(x-3)(x+3)} = \frac{2x}{x^2-9}$

M1 A1

$\frac{1}{x-3} - \frac{1}{x+3} = \frac{(x+3) - (x-3)}{(x-3)(x+3)} = \frac{6}{x^2-9}$

M1 A1

$E = \frac{2x}{x^2-9} \div \frac{6}{x^2-9} = \frac{2x}{6} = \frac{x}{3}$

A1

Answer: Shown

5 marks

(b) $\frac{x}{3} = 4$, so $x = 12$

B1

Answer: $x = 12$

1 mark

25 (6 marks)

$$x^2 - x - 6 = (x - 3)(x + 2)$$

B1

Multiply through by $(x - 3)(x + 2)$: $2x - (x + 2) = x(x - 3)$

M1 A1

$$x - 2 = x^2 - 3x, \text{ so } x^2 - 4x + 2 = 0$$

M1

$$x = \frac{4 \pm \sqrt{16 - 8}}{2}$$

A1

$$x = 2 \pm \sqrt{2}, \text{ and neither value is 3 or } -2$$

A1

Answer: $x = 2 \pm \sqrt{2}$

6 marks

26 (6 marks)

$$(a) \frac{(x+1) + (x-1)}{(x-1)(x+1)}$$

M1

$$= \frac{2x}{x^2 - 1}$$

A1

Answer: Shown

2 marks

$$(b) \frac{2x}{x^2 - 1} = \frac{3}{4}, \text{ so } 8x = 3x^2 - 3$$

M1 A1

$$3x^2 - 8x - 3 = 0$$

A1

$$(3x + 1)(x - 3) = 0, \text{ so } x = -\frac{1}{3} \text{ or } x = 3$$

A1

Answer: $x = 3$ or $x = -\frac{1}{3}$

4 marks

27 (8 marks)

$$(a) 1 - \frac{1}{x+1} = \frac{x+1-1}{x+1} = \frac{x}{x+1}$$

M1 A1

$$1 + \frac{1}{x-1} = \frac{x-1+1}{x-1} = \frac{x}{x-1}$$

M1

$$h(x) = \frac{x}{x+1} \times \frac{x-1}{x} = \frac{x-1}{x+1}$$

A1

Answer: Shown

4 marks

$$(b) 3(x - 1) = x + 1$$

M1

$$x = 2$$

A1

Answer: $x = 2$

2 marks

$$(c) x - 1 = x + 1$$

M1

$-1 = 1$, which is impossible, so there is no solution

E1

Answer: No solution

2 marks

28 (9 marks)

$$(a) \text{ Common denominator } ab: \frac{a^2 + b^2 - 2ab}{ab}$$

M1 A1

$$= \frac{(a-b)^2}{ab}$$

A1

Answer: Shown

3 marks

$$(b) (a-b)^2 \geq 0 \text{ and } ab > 0, \text{ so } \frac{(a-b)^2}{ab} \geq 0$$

E1

$$\text{Hence } \frac{a}{b} + \frac{b}{a} - 2 \geq 0, \text{ i.e. } \frac{a}{b} + \frac{b}{a} \geq 2$$

E1

Answer: Proved

2 marks

$$(c) \text{ Equality holds when } a = b$$

B1

Answer: $a = b$

1 mark

$$(d) \frac{x^2 + 9}{3x} = \frac{x}{3} + \frac{3}{x}$$

M1

This is of the form $\frac{a}{b} + \frac{b}{a}$ with $a = x$, $b = 3$, so it is ≥ 2 : minimum value 2

A1

Equality when $x = 3$

A1

Answer: Minimum 2 at $x = 3$

3 marks

29 (8 marks)

$$(a) \frac{(1+x) - (1-x)}{(1-x)(1+x)}$$

$$= \frac{2x}{1-x^2}$$

M1

A1

Answer: Shown

2 marks

$$(b) \frac{2x}{1-x^2} - \frac{2x}{1+x^2} = \frac{2x(1+x^2) - 2x(1-x^2)}{(1-x^2)(1+x^2)}$$

$$= \frac{4x^3}{1-x^4}$$

M1

A1 A1

Answer: Shown

3 marks

$$(c) \text{ Use (b): } \frac{4x^3}{1-x^4} - \frac{4x^3}{1+x^4}$$

$$= \frac{4x^3(1+x^4) - 4x^3(1-x^4)}{(1-x^4)(1+x^4)}$$

$$= \frac{8x^7}{1-x^8}$$

M1

M1

A1

$$\text{Answer: } \frac{8x^7}{1-x^8}$$

3 marks

30 (10 marks)

$$(a) \frac{1}{k(k+1)} - \frac{1}{(k+1)(k+2)} = \frac{(k+2) - k}{k(k+1)(k+2)}$$

$$= \frac{2}{k(k+1)(k+2)}, \text{ so the identity holds}$$

M1 A1

A1

Answer: Shown

3 marks

$$(b) \text{ Sum} = \frac{1}{2} \left[\left(\frac{1}{1 \cdot 2} - \frac{1}{2 \cdot 3} \right) + \left(\frac{1}{2 \cdot 3} - \frac{1}{3 \cdot 4} \right) + \dots \right]$$

M1 A1

$$\text{Terms cancel in pairs, leaving } \frac{1}{2} \left(\frac{1}{2} - \frac{1}{(N+1)(N+2)} \right)$$

M1

$$= \frac{1}{4} - \frac{1}{2(N+1)(N+2)}$$

A1

Answer: Shown

4 marks

$$(c) \text{ As } N \rightarrow \infty \text{ the second term tends to 0, so the sum to infinity is } \frac{1}{4}$$

B1

$$\text{Need } \frac{1}{2(N+1)(N+2)} < 0.001, \text{ so } (N+1)(N+2) > 500$$

M1

$$21 \times 22 = 462 < 500 \text{ and } 22 \times 23 = 506 > 500, \text{ so } N = 21$$

A1

$$\text{Answer: } \frac{1}{4}; N = 21$$

3 marks

Polynomial Division and Factor Theorem**1 (2 marks)**

$$\text{Remainder theorem: remainder} = f(2)$$

M1

$$= 8 + 4 - 5 = 7$$

A1

Answer: 7

2 marks

2 (2 marks)

$$f(3) = 27 - 36 + 3 + 6$$

M1

$$= 0, \text{ so } (x - 3) \text{ is a factor by the factor theorem}$$

A1

Answer: Shown

2 marks

3 (1 mark)

$$f(2) = 8 - 6 + 1 = 3$$

A1

Answer: 3

1 mark

4 (2 marks)

$$f(1) = 0: 1 + k - 8 + 3 = 0$$

M1

$$k = 4$$

A1

Answer: $k = 4$

2 marks

5 (2 marks)

$$x^2 + 5x + 6 = (x + 2)(x + 3)$$

M1

Quotient $x + 3$, remainder 0

A1

Answer: $x + 3$

2 marks

6 (2 marks)

$$\text{Remainder} = f(-1) = -2 - 1 + 4$$

M1

$$= 1$$

A1

Answer: 1

2 marks

7 (4 marks)

$$(a) f(2) = 16 + 12 - 22 - 6 = 0, \text{ so } (x - 2) \text{ is a factor}$$

B1

Answer: Shown

1 mark

$$(b) \text{ Divide by } (x - 2): f(x) = (x - 2)(2x^2 + 7x + 3)$$

M1 A1

$$= (x - 2)(2x + 1)(x + 3)$$

A1

Answer: $(x - 2)(2x + 1)(x + 3)$

3 marks

8 (3 marks)

$$\text{Let } f(x) = x^3 + ax^2 - 4x + 3; \text{ remainder} = f(-2)$$

M1

$$f(-2) = -8 + 4a + 8 + 3 = 4a + 3$$

A1

$$4a + 3 = 11, \text{ so } a = 2$$

A1

Answer: $a = 2$

3 marks

9 (4 marks)

$$f(2) = 16 + 4 - 26 + 6 = 0, \text{ so } (x - 2) \text{ is a factor}$$

B1

$$2x^3 + x^2 - 13x + 6 = (x - 2)(2x^2 + 5x - 3)$$

M1 A1

$$= (x - 2)(2x - 1)(x + 3)$$

A1

Answer: $(x - 2)(2x - 1)(x + 3)$

4 marks

10 (4 marks)

$$f(-1) = -1 + 7 - 6 = 0, \text{ so } (x + 1) \text{ is a factor}$$

M1 A1

$$x^3 - 7x - 6 = (x + 1)(x^2 - x - 6)$$

M1

$$= (x + 1)(x - 3)(x + 2)$$

A1

Answer: $(x + 1)(x - 3)(x + 2)$

4 marks

11 (4 marks)

$$f(4) = 6 \times 64 - 25 \times 16 + 8 + 8 = 384 - 400 + 16 = 0$$

M1

$$\text{So } (x - 4) \text{ is a factor and } 6x^3 - 25x^2 + 2x + 8 = (x - 4)(6x^2 - x - 2)$$

A1

$$6x^2 - x - 2 = (3x - 2)(2x + 1)$$

M1

$$x = 4, x = \frac{2}{3} \text{ or } x = -\frac{1}{2}$$

A1

Answer: $x = 4, \frac{2}{3}, -\frac{1}{2}$

4 marks

12 (4 marks)

$$f(2) = 24 - 4 - 24 + 4 = 0$$

M1 A1

$$3x^3 - x^2 - 12x + 4 = (x - 2)(3x^2 + 5x - 2)$$

M1

$$= (x - 2)(3x - 1)(x + 2)$$

A1

Answer: $(x - 2)(3x - 1)(x + 2)$

4 marks

13 (3 marks)

$$\text{Remainder} = f\left(\frac{1}{2}\right)$$

M1

$$= 4 \times \frac{1}{8} - \frac{3}{2} + 5$$

M1

$$= 4$$

A1

Answer: 4

3 marks

14 (4 marks)

$$2x^3 \div x^2 = 2x, \text{ and } 2x(x^2 + x - 1) = 2x^3 + 2x^2 - 2x$$

M1

$$\text{Subtracting: } -5x^2 + 6x - 5$$

A1

$$-5x^2 \div x^2 = -5, \text{ and } -5(x^2 + x - 1) = -5x^2 - 5x + 5$$

M1

$$\text{Subtracting: } 11x - 10. \text{ Quotient } 2x - 5, \text{ remainder } 11x - 10$$

A1

Answer: Quotient $2x - 5$, remainder $11x - 10$

4 marks

15 (6 marks)

$$f(2) = 0: 16 + 4a + 2b - 6 = 0$$

M1

$$2a + b = -5$$

A1

$$f(-1) = 12: -2 + a - b - 6 = 12$$

M1

$$a - b = 20$$

A1

$$\text{Adding: } 3a = 15$$

M1

$$a = 5, b = -15$$

A1

Answer: $a = 5, b = -15$

6 marks

16 (5 marks)

$$f(x) = (x - 1)(x + 2)Q(x) + ax + b$$

M1

$$x = 1: f(1) = 1 \text{ so } a + b = 1$$

M1 A1

$$x = -2: f(-2) = -8 + 8 + 10 + 3 = 13, \text{ so } -2a + b = 13$$

A1

$$\text{Subtracting: } 3a = -12, \text{ so } a = -4 \text{ and } b = 5$$

A1

Answer: $a = -4, b = 5$

5 marks

17 (7 marks)

$$\text{(a) } f(1) = 0: 1 + p + q - 6 = 0, \text{ so } p + q = 5$$

M1 A1

$$f(-3) = 0: -27 + 9p - 3q - 6 = 0, \text{ so } 3p - q = 11$$

M1 A1

$$\text{Adding: } 4p = 16, \text{ so } p = 4 \text{ and } q = 1$$

A1

Answer: $p = 4, q = 1$

5 marks

$$\text{(b) The third factor is } (x + k) \text{ with } (-1)(3)k = -6, \text{ so } k = 2$$

M1

$$f(x) = (x - 1)(x + 3)(x + 2)$$

A1

Answer: $(x - 1)(x + 2)(x + 3)$

2 marks

18 (4 marks)

$$f(2) = 8 - 8 + 8 - 8 = 0, \text{ so } (x - 2) \text{ is a factor}$$

M1 A1

$$x^3 - 2x^2 + 4x - 8 = (x - 2)(x^2 + 4)$$

M1

$$x^2 + 4 = 0 \text{ has no real solutions, so } x = 2 \text{ is the only real root}$$

E1

Answer: Shown

4 marks

19 (5 marks)

$$\text{(a) } f(x) = (x^2 - 2)(2x + 3) + (x - 4)$$

M1

$$= 2x^3 + 3x^2 - 4x - 6 + x - 4$$

A1

$$= 2x^3 + 3x^2 - 3x - 10$$

A1

Answer: $2x^3 + 3x^2 - 3x - 10$

3 marks

$$\text{(b) Remainder} = f(2)$$

M1

$$= 16 + 12 - 6 - 10 = 12$$

A1

Answer: 12

2 marks

20 (4 marks)

Roots: $x = -2$ (single root, crosses) and $x = 1$ (repeated root, touches the axis)

M1 A1

y-intercept: $x = 0$ gives $y = 2 \times 1 = 2$

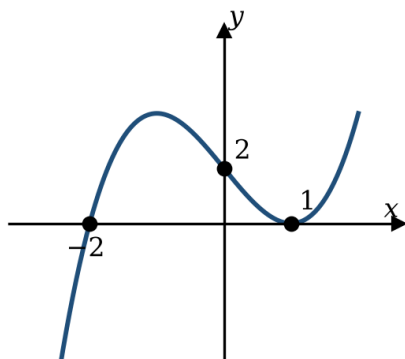
B1

Positive cubic: starts bottom left and ends top right

B1

Answer: See sketch

4 marks

**21 (8 marks)**

(a) $f(-2) = 6(-8) + 4 + 38 + 6$

M1

$= 0$, so $(x + 2)$ is a factor

A1

Answer: Shown

2 marks

(b) $f(x) = (x + 2)(6x^2 - 11x + 3)$

M1 A1

$= (x + 2)(2x - 3)(3x - 1)$

A1

Answer: $(x + 2)(2x - 3)(3x - 1)$

3 marks

(c) Positive cubic shape (rising from bottom left to top right)

B1

x-intercepts -2 , $\frac{1}{3}$ and $\frac{3}{2}$

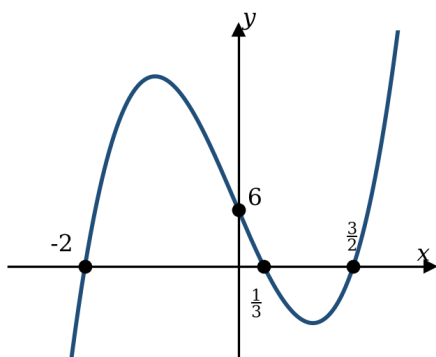
B1

y-intercept $f(0) = 6$

B1

Answer: See sketch

3 marks

**22 (7 marks)**

(a) $f(x) = k(x + 1)(x - 2)(x - 4)$

M1

$f(0) = 8k = 16$, so $k = 2$

M1

$f(x) = 2(x^3 - 5x^2 + 2x + 8) = 2x^3 - 10x^2 + 4x + 16$

A1

Answer: $2x^3 - 10x^2 + 4x + 16$

3 marks

(b) Remainder $= f(1)$

M1

$= 2 - 10 + 4 + 16 = 12$

A1

Answer: 12

2 marks

(c) $f(3) + c = 0$, with $f(3) = 54 - 90 + 12 + 16 = -8$

M1

$c = 8$

A1

Answer: $c = 8$

2 marks

23 (9 marks)

(a) $f(1) = 1 - 1 - 7 + 1 + 6 = 0$

M1 A1

$f(-1) = 1 + 1 - 7 - 1 + 6 = 0$, so both are factors

A1

Answer: Shown

3 marks

(b) $(x - 1)(x + 1) = x^2 - 1$, and $f(x) = (x^2 - 1)(x^2 - x - 6)$
 $= (x - 1)(x + 1)(x - 3)(x + 2)$

M1 A1

A1

Answer: $(x - 1)(x + 1)(x - 3)(x + 2)$

3 marks

(c) Roots at $x = -2, -1, 1, 3$

B1

$f(0) = 6$, so the y-intercept is 6

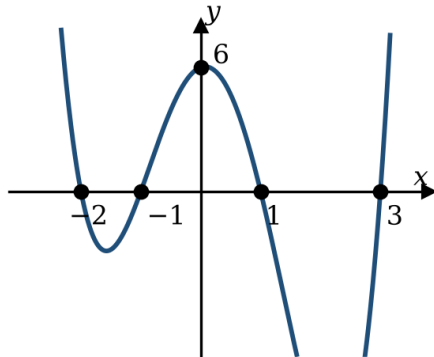
B1

Positive quartic, W-shaped, crossing at each root

B1

Answer: See sketch

3 marks



24 (7 marks)

(a) $g(-1) = -1 + 1$

M1

$= 0$

A1

Answer: Shown

2 marks

(b) Divide: $x^5 + 1 = (x + 1)(x^4 + ax^3 + bx^2 + cx + d)$

M1

Matching coefficients gives $a = -1$, $b = 1$, $c = -1$, $d = 1$

M1

$(x + 1)(x^4 - x^3 + x^2 - x + 1)$

A1

Answer: $(x + 1)(x^4 - x^3 + x^2 - x + 1)$

3 marks

(c) Put $x = 2$: $33 = 3 \times (16 - 8 + 4 - 2 + 1)$

M1

$= 3 \times 11$

A1

Answer: Shown

2 marks

25 (7 marks)

(a) $f(2) = 0$: $8 + 4(a - 1) - 8 + b = 0$, so $b = 4 - 4a$

M1 A1

$f(-a) = 0$: $-a^3 + (a - 1)a^2 + 4a + b = 0$, so $b = a^2 - 4a$

M1 A1

$a^2 - 4a = 4 - 4a$ gives $a^2 = 4$, so $a = 2$ (as $a > 0$) and $b = -4$

A1

Answer: $a = 2$, $b = -4$

5 marks

(b) $x^3 + x^2 - 4x - 4 = x^2(x + 1) - 4(x + 1) = (x + 1)(x - 2)(x + 2)$

M1

$x = -1$, $x = 2$, $x = -2$

A1

Answer: $x = -2, -1, 2$

2 marks

26 (8 marks)

(a) $f(-2) = -32 + 48 - 18 + 2$

M1

$= 0$

A1

Answer: Shown

2 marks

(b) $f(x) = (x + 2)(4x^2 + 4x + 1)$

M1 A1

$= (x + 2)(2x + 1)^2$

A1

Answer: $(x + 2)(2x + 1)^2$

3 marks

(c) $x = -2$ or $x = -\frac{1}{2}$

B1

$x = -\frac{1}{2}$ is repeated

B1

The graph touches the x-axis (turning point) at $x = -\frac{1}{2}$

E1

Answer: $x = -2$, $x = -\frac{1}{2}$ (repeated)

3 marks

27 (9 marks)

(a) Let $g(x) = x^n - 1$. $g(1) = 0$, so $(x - 1)$ is a factor: $g(x) = (x - 1)q(x)$, where q has integer coefficients **B1**

Put $x = 7$: $7^n - 1 = 6 \times q(7)$ **M1**

$q(7)$ is an integer, so $7^n - 1$ is a multiple of 6 **E1**

Answer: Shown **3 marks**

(b) Let $h(x) = x^n - 2^n$. $h(2) = 0$, so $h(x) = (x - 2)q(x)$, where q has integer coefficients **B1**

Put $x = 5$: $5^n - 2^n = 3 \times q(5)$ **M1**

$q(5)$ is an integer, so the expression is a multiple of 3 **E1**

Answer: Proved **3 marks**

(c) Let $f(x) = x^n + a^n$; $(x + a)$ is a factor if $f(-a) = 0$ **M1**

n odd: $(-a)^n + a^n = -a^n + a^n = 0$, so $(x + a)$ is a factor **A1**

n even: $(-a)^n + a^n = 2a^n \neq 0$, so it is not a factor **E1**

Answer: Shown **3 marks**

28 (9 marks)

(a) $x^2 - x - 6 = (x - 3)(x + 2)$, so $f(3) = 0$ and $f(-2) = 0$ **M1**

$f(3) = 81 + 27a + 3b - 12 = 0$, so $9a + b = -23$ **A1**

$f(-2) = 16 - 8a - 2b - 12 = 0$, so $4a + b = 2$ **M1 A1**

Subtracting: $5a = -25$, so $a = -5$ and $b = 22$ **A1**

Answer: $a = -5$, $b = 22$ **5 marks**

(b) Dividing $f(x) = x^4 - 5x^3 + 22x - 12$ by $x^2 - x - 6$ gives quotient $x^2 - 4x + 2$ **M1 A1**

$x^2 - 4x + 2 = (x - 2)^2 - 2 = (x - 2 - \sqrt{2})(x - 2 + \sqrt{2})$ **A1**

Answer: $(x - 3)(x + 2)(x - 2 - \sqrt{2})(x - 2 + \sqrt{2})$ **3 marks**

(c) $x = 3$, $x = -2$, $x = 2 + \sqrt{2}$, $x = 2 - \sqrt{2}$ **A1**

Answer: $x = 3, -2, 2 \pm \sqrt{2}$ **1 mark**

29 (9 marks)

(a) The remainder has degree less than the divisor, which has degree 3 **E1**

So $R(x) = ax^2 + bx + c$ **B1**

Answer: $R(x) = ax^2 + bx + c$ **2 marks**

(b) $f(x) = (x - 1)(x - 2)(x - 3)Q(x) + R(x)$, and the bracket is zero at $x = 1, 2, 3$ **M1**

so $R(1) = f(1) = 1$, $R(2) = f(2) = 4$, $R(3) = f(3) = 9$ **A1**

$a + b + c = 1$, $4a + 2b + c = 4$, $9a + 3b + c = 9$ **A1**

Answer: Shown **3 marks**

(c) Subtract the first from the second: $3a + b = 3$ **M1**

Subtract the second from the third: $5a + b = 5$ **M1**

Hence $2a = 2$, $a = 1$, $b = 0$, $c = 0$ **A1**

$R(x) = x^2$ **A1**

Answer: $R(x) = x^2$ **4 marks**

30 (11 marks)

(a) A repeated root means the curve touches the x -axis there **E1**

So the x -axis is a tangent, giving $f(x) = 0$ **B1**

and the gradient is zero there: $f'(x) = 0$ **B1**

Answer: Explained **3 marks**

(b) $f'(x) = 3x^2 - 12 = 0$, so $x = 2$ or $x = -2$ **M1 A1**

$x = 2$: $8 - 24 + q = 0$, so $q = 16$ **M1 A1**

$x = -2$: $-8 + 24 + q = 0$, so $q = -16$ **A1**

Answer: $q = 16$ or $q = -16$ **5 marks**

(c) $f(x) = x^3 - 12x + 16$ has factor $(x - 2)^2$ **M1**

$f(x) = (x - 2)^2(x + 4)$ **A1**

Roots $x = 2$ (repeated) and $x = -4$ **A1**

Answer: $(x - 2)^2(x + 4)$ **3 marks**

Partial Fractions

1 (3 marks)

$$\frac{x+7}{(x+1)(x+3)} \equiv \frac{A}{x+1} + \frac{B}{x+3}, \text{ so } x+7 \equiv A(x+3) + B(x+1)$$

M1

$$x = -1: 6 = 2A, \text{ so } A = 3$$

A1

$$x = -3: 4 = -2B, \text{ so } B = -2. \text{ Result: } \frac{3}{x+1} - \frac{2}{x+3}$$

A1

$$\text{Answer: } \frac{3}{x+1} - \frac{2}{x+3}$$

3 marks

2 (3 marks)

$$x^2 - x - 6 = (x - 3)(x + 2)$$

B1

$$3x + 1 \equiv A(x + 2) + B(x - 3); x = 3 \text{ gives } 10 = 5A, x = -2 \text{ gives } -5 = -5B$$

M1

$$A = 2, B = 1: \frac{2}{x-3} + \frac{1}{x+2}$$

A1

$$\text{Answer: } \frac{2}{x-3} + \frac{1}{x+2}$$

3 marks

3 (3 marks)

$$\frac{5}{(x-1)(x+4)} \equiv \frac{A}{x-1} + \frac{B}{x+4}, \text{ so } 5 \equiv A(x+4) + B(x-1)$$

M1

$$x = 1: 5 = 5A, A = 1. x = -4: 5 = -5B, B = -1$$

A1

$$\frac{1}{x-1} - \frac{1}{x+4}$$

A1

$$\text{Answer: } \frac{1}{x-1} - \frac{1}{x+4}$$

3 marks

4 (3 marks)

$$2x + 1 \equiv A(x + 1) + Bx$$

M1

$$x = 0: A = 1. x = -1: -1 = -B, B = 1$$

A1

$$\frac{1}{x} + \frac{1}{x+1}$$

A1

$$\text{Answer: } \frac{1}{x} + \frac{1}{x+1}$$

3 marks

5 (3 marks)

$$3x - 4 \equiv A(x - 3) + B(x + 2)$$

M1

$$x = -2: -10 = -5A, A = 2. x = 3: 5 = 5B, B = 1$$

A1 A1

$$\text{Answer: } \frac{2}{x+2} + \frac{1}{x-3}$$

3 marks

6 (3 marks)

$$7 \equiv A(x - 5) + B(x + 2)$$

M1

$$x = -2: 7 = -7A, A = -1. x = 5: 7 = 7B, B = 1$$

A1

$$\frac{1}{x-5} - \frac{1}{x+2}$$

A1

$$\text{Answer: } \frac{1}{x-5} - \frac{1}{x+2}$$

3 marks

7 (4 marks)

$$x + 5 \equiv A(x - 1)^2 + B(x + 1)(x - 1) + C(x + 1)$$

M1

$$x = -1: 4 = 4A, \text{ so } A = 1. x = 1: 6 = 2C, \text{ so } C = 3$$

M1 A1

$$\text{Coefficient of } x^2: 0 = A + B, \text{ so } B = -1$$

A1

$$\text{Answer: } A = 1, B = -1, C = 3$$

4 marks

8 (3 marks)

$$\frac{5x+5}{(2x-1)(x+2)} \equiv \frac{A}{2x-1} + \frac{B}{x+2}, \text{ so } 5x+5 \equiv A(x+2) + B(2x-1) \quad \text{M1}$$

$$x = \frac{1}{2}: \frac{15}{2} = \frac{5}{2}A, \text{ so } A = 3 \quad \text{A1}$$

$$x = -2: -5 = -5B, \text{ so } B = 1: \frac{3}{2x-1} + \frac{1}{x+2} \quad \text{A1}$$

$$\text{Answer: } \frac{3}{2x-1} + \frac{1}{x+2} \quad \text{3 marks}$$

9 (4 marks)

$$3x+2 \equiv Ax(x+1) + B(x+1) + Cx^2 \quad \text{M1}$$

$$x = 0: B = 2 \quad \text{A1}$$

$$x = -1: -1 = C \quad \text{A1}$$

$$\text{Coefficient of } x^2: 0 = A + C, \text{ so } A = 1 \quad \text{A1}$$

$$\text{Answer: } A = 1, B = 2, C = -1 \quad \text{4 marks}$$

10 (3 marks)

$$2x \equiv A(x-3) + B \quad \text{M1}$$

$$\text{Coefficient of } x: A = 2; \text{ constant: } 0 = -3A + B, \text{ so } B = 6 \quad \text{M1 A1}$$

$$\text{Answer: } \frac{2}{x-3} + \frac{6}{(x-3)^2} \quad \text{3 marks}$$

11 (6 marks)

$$\text{(a) } 3x+5 \equiv A(x+3) + B(x+1) \quad \text{M1}$$

$$x = -1: 2 = 2A, \text{ so } A = 1. \quad x = -3: -4 = -2B, \text{ so } B = 2 \quad \text{A1}$$

$$\frac{1}{x+1} + \frac{2}{x+3} \quad \text{A1}$$

$$\text{Answer: } \frac{1}{x+1} + \frac{2}{x+3} \quad \text{3 marks}$$

$$\text{(b) Integral} = [\ln|x+1| + 2 \ln|x+3|] \text{ from } 0 \text{ to } 1 \quad \text{M1}$$

$$= (\ln 2 + 2 \ln 4) - (0 + 2 \ln 3) \quad \text{A1}$$

$$= \ln\left(\frac{2 \times 16}{9}\right) = \ln\left(\frac{32}{9}\right) \quad \text{A1}$$

$$\text{Answer: } \ln\left(\frac{32}{9}\right) \quad \text{3 marks}$$

12 (4 marks)

$$x^2 + 3x + 3 = (x+1)(x+2) + 1, \text{ so the fraction is } 1 + \frac{1}{(x+1)(x+2)} \quad \text{M1 A1}$$

$$1 \equiv A(x+2) + B(x+1): x = -1 \text{ gives } A = 1, x = -2 \text{ gives } B = -1 \quad \text{M1 A1}$$

$$\text{Answer: } 1 + \frac{1}{x+1} - \frac{1}{x+2} \quad \text{4 marks}$$

13 (3 marks)

$$x^2 - 1 = (x-1)(x+1), \text{ so } 4 \equiv A(x+1) + B(x-1) \quad \text{M1}$$

$$x = 1: 4 = 2A, A = 2. \quad x = -1: 4 = -2B, B = -2 \quad \text{A1}$$

$$\frac{2}{x-1} - \frac{2}{x+1} \quad \text{A1}$$

$$\text{Answer: } \frac{2}{x-1} - \frac{2}{x+1} \quad \text{3 marks}$$

14 (4 marks)

$$\text{The numerator and denominator have the same degree, so divide: } x^2 + 4x + 1 = (x^2 + x - 2) + (3x + 3) \quad \text{M1 A1}$$

$$\frac{3x+3}{(x-1)(x+2)} \equiv \frac{A}{x-1} + \frac{B}{x+2}, \text{ so } 3x+3 \equiv A(x+2) + B(x-1) \quad \text{M1}$$

$$x = 1: 6 = 3A, A = 2. \quad x = -2: -3 = -3B, B = 1. \text{ So } P = 1, A = 2, B = 1 \quad \text{A1}$$

$$\text{Answer: } 1 + \frac{2}{x-1} + \frac{1}{x+2} \quad \text{4 marks}$$

15 (7 marks)

(a) $4x \equiv A(x + 3) + B(x - 1)$ M1

$x = 1: 4 = 4A, \text{ so } A = 1. x = -3: -12 = -4B, \text{ so } B = 3$ A1

$f(x) = \frac{1}{x-1} + \frac{3}{x+3}$ A1

Answer: $\frac{1}{x-1} + \frac{3}{x+3}$ 3 marks

(b) $\int_2^5 f(x) dx = [\ln|x-1| + 3 \ln|x+3|]$ from 2 to 5 M1 A1

$= (\ln 4 + 3 \ln 8) - (\ln 1 + 3 \ln 5)$ M1

$= \ln\left(\frac{4 \times 8^3}{5^3}\right) = \ln\left(\frac{2048}{125}\right)$ A1

Answer: Shown 4 marks**16 (5 marks)**

Write as $\frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{x-4}$, so $6x^2 - 9x - 6 \equiv A(x+2)(x-4) + B(x-1)(x-4) + C(x-1)(x+2)$ M1 A1

$x = 1: -9 = -9A, \text{ so } A = 1$ A1

$x = -2: 36 = 18B, \text{ so } B = 2$ A1

$x = 4: 54 = 18C, \text{ so } C = 3. \text{ Result } \frac{1}{x-1} + \frac{2}{x+2} + \frac{3}{x-4}$ A1

Answer: $\frac{1}{x-1} + \frac{2}{x+2} + \frac{3}{x-4}$ 5 marks

17 (8 marks)

(a) $5 \equiv A(2-x) + B(1+x)$ M1

$x = 2: B = \frac{5}{3}; x = -1: A = \frac{5}{3}$ A1

$f(x) = \frac{5}{3(1+x)} + \frac{5}{3(2-x)}$ A1

Answer: $\frac{5}{3(1+x)} + \frac{5}{3(2-x)}$ 3 marks

(b) $\frac{5}{3}(1+x)^{-1} = \frac{5}{3}(1-x+x^2-\dots)$ M1 A1

$\frac{5}{3(2-x)} = \frac{5}{6}\left(1-\frac{x}{2}\right)^{-1} = \frac{5}{6}\left(1+\frac{x}{2}+\frac{x^2}{4}+\dots\right)$ M1 A1

Sum: $\frac{5}{2} - \frac{5}{4}x + \frac{15}{8}x^2$, valid for $|x| < 1$ A1

Answer: $\frac{5}{2} - \frac{5}{4}x + \frac{15}{8}x^2; |x| < 1$ 5 marks

18 (8 marks)(a) The degrees are equal, so subtract $2(x-3)(x+1) = 2x^2 - 4x - 6$ from the numerator: remainder $12x - 4$ M1 A1

$\frac{12x-4}{(x-3)(x+1)} \equiv \frac{A}{x-3} + \frac{B}{x+1}; 12x-4 = A(x+1) + B(x-3)$ M1

$x = 3: 32 = 4A, A = 8. x = -1: -16 = -4B, B = 4. \text{ So } P = 2$ A1

Answer: $2 + \frac{8}{x-3} + \frac{4}{x+1}$ 4 marks

(b) $\int_4^6 g(x) dx = [2x + 8 \ln|x-3| + 4 \ln|x+1|]$ from 4 to 6 M1 A1

$= (12 + 8 \ln 3 + 4 \ln 7) - (8 + 8 \ln 1 + 4 \ln 5)$ M1

$= 4 + 8 \ln 3 + 4 \ln 7 - 4 \ln 5 = 4 + 8 \ln 3 + 4 \ln\left(\frac{7}{5}\right)$ A1

Answer: Shown 4 marks

19 (5 marks)

$$x^3 + 2 = x(x^2 - 1) + x + 2$$

M1 A1

$$\frac{x+2}{x^2-1} = \frac{x+2}{(x-1)(x+1)} \equiv \frac{A}{x-1} + \frac{B}{x+1}$$

M1

$$x+2 \equiv A(x+1) + B(x-1); x=1 \text{ gives } A = \frac{3}{2}, x=-1 \text{ gives } B = -\frac{1}{2}$$

A1

$$\frac{x^3+2}{x^2-1} = x + \frac{3}{2(x-1)} - \frac{1}{2(x+1)}$$

A1

$$\text{Answer: } x + \frac{3}{2(x-1)} - \frac{1}{2(x+1)}$$

5 marks

20 (6 marks)

$$x^2 - 2x - 8 = (x-4)(x+2), \text{ so } x+14 \equiv A(x+2) + B(x-4)$$

B1 M1

$$x=4: 18 = 6A, A=3. x=-2: 12 = -6B, B=-2$$

A1

$$\int_5^7 = [3 \ln|x-4| - 2 \ln|x+2|] \text{ from 5 to 7}$$

M1

$$= (3 \ln 3 - 2 \ln 9) - (3 \ln 1 - 2 \ln 7)$$

A1

$$= 3 \ln 3 - 4 \ln 3 + 2 \ln 7 = \ln\left(\frac{49}{3}\right)$$

A1

$$\text{Answer: } \ln\left(\frac{49}{3}\right)$$

6 marks

21 (8 marks)

$$(a) 4 + 5x \equiv A(1 + 2x) + B(1 - x)$$

M1

$$x=1: 9 = 3A, A=3. x=-\frac{1}{2}: \frac{3}{2} = \frac{3}{2}B, B=1$$

A1

$$g(x) = \frac{3}{1-x} + \frac{1}{1+2x}$$

A1

$$\text{Answer: } \frac{3}{1-x} + \frac{1}{1+2x}$$

3 marks

$$(b) 3(1-x)^{-1} = 3 + 3x + 3x^2 + 3x^3 + \dots$$

M1

$$(1+2x)^{-1} = 1 - 2x + 4x^2 - 8x^3 + \dots$$

A1

$$g(x) = 4 + x + 7x^2 - 5x^3 + \dots$$

A1

$$\text{Answer: } 4 + x + 7x^2 - 5x^3$$

3 marks

$$(c) (1-x)^{-1} \text{ needs } |x| < 1 \text{ and } (1+2x)^{-1} \text{ needs } |2x| < 1$$

B1

$$\text{Both hold when } |x| < \frac{1}{2}$$

B1

$$\text{Answer: } |x| < \frac{1}{2}$$

2 marks

22 (8 marks)

$$(a) 3x^2 + x + 8 \equiv A(x^2 + 4) + (Bx + C)(x + 1)$$

M1

$$x=-1: 10 = 5A, \text{ so } A=2$$

A1

$$\text{Coefficient of } x^2: 3 = A + B, \text{ so } B=1$$

M1

$$\text{Constant terms: } 8 = 4A + C, \text{ so } C=0$$

A1

$$\text{Answer: } \frac{2}{x+1} + \frac{x}{x^2+4}$$

4 marks

$$(b) \int_0^2 \frac{2}{x+1} dx = [2 \ln(x+1)] = 2 \ln 3$$

M1 A1

$$\int_0^2 \frac{x}{x^2+4} dx = \left[\frac{1}{2} \ln(x^2+4) \right] = \frac{1}{2} (\ln 8 - \ln 4) = \frac{1}{2} \ln 2$$

M1

$$\text{Total: } 2 \ln 3 + \frac{1}{2} \ln 2$$

A1

$$\text{Answer: } 2 \ln 3 + \frac{1}{2} \ln 2$$

4 marks

23 (9 marks)

$$(a) 5x^2 - 6x + 2 \equiv A(x^2 + 1) + (Bx + C)(x - 2)$$

M1 A1

$$x = 2: 10 = 5A, \text{ so } A = 2$$

A1

$$\text{Coefficient of } x^2: 5 = A + B, \text{ so } B = 3$$

M1

$$\text{Constant: } 2 = A - 2C, \text{ so } C = 0$$

A1

$$\text{Answer: } A = 2, B = 3, C = 0$$

5 marks

$$(b) \int_3^5 \frac{2}{x-2} dx = [2 \ln(x-2)] = 2 \ln 3$$

M1 A1

$$\int_3^5 \frac{3x}{x^2+1} dx = \left[\frac{3}{2} \ln(x^2+1) \right] = \frac{3}{2}(\ln 26 - \ln 10)$$

M1

$$= 2 \ln 3 + \frac{3}{2} \ln\left(\frac{13}{5}\right)$$

A1

$$\text{Answer: } 2 \ln 3 + \frac{3}{2} \ln\left(\frac{13}{5}\right)$$

4 marks

24 (9 marks)

$$(a) 1 \equiv A(y+2) + B(y-1)$$

M1

$$y = 1: A = \frac{1}{3}, y = -2: B = -\frac{1}{3}$$

A1

$$\frac{1}{3}\left(\frac{1}{y-1} - \frac{1}{y+2}\right)$$

A1

$$\text{Answer: } \frac{1}{3}\left(\frac{1}{y-1} - \frac{1}{y+2}\right)$$

3 marks

$$(b) \text{ Separate variables: } \int \frac{1}{(y-1)(y+2)} dy = \int 2 dx$$

M1

$$\frac{1}{3}[\ln|y-1| - \ln|y+2|] = 2x + c$$

M1 A1

$$\text{When } x = 0, y = 2: \frac{1}{3} \ln\left(\frac{1}{4}\right) = c$$

M1

$$\ln\left(\frac{y-1}{y+2}\right) = 6x + \ln\left(\frac{1}{4}\right), \text{ so } \frac{y-1}{y+2} = \frac{1}{4}e^{6x}$$

A1

$$4y - 4 = e^{6x}y + 2e^{6x}, \text{ so } y(4 - e^{6x}) = 4 + 2e^{6x}, \text{ giving the result}$$

A1

$$\text{Answer: Shown}$$

6 marks

25 (8 marks)

$$(a) 6 \equiv A(x+2) + B(x+1)$$

M1

$$x = -1: A = 6, x = -2: 6 = -B, \text{ so } B = -6$$

A1

$$\frac{6}{x+1} - \frac{6}{x+2}$$

A1

$$\text{Answer: } \frac{6}{x+1} - \frac{6}{x+2}$$

3 marks

$$(b) \text{ Area} = \int_1^3 \left(\frac{6}{x+1} - \frac{6}{x+2}\right) dx$$

M1

$$= [6 \ln(x+1) - 6 \ln(x+2)] \text{ from } 1 \text{ to } 3$$

M1 A1

$$= 6(\ln 4 - \ln 5) - 6(\ln 2 - \ln 3)$$

M1

$$= 6 \ln\left(\frac{4 \times 3}{5 \times 2}\right) = 6 \ln\left(\frac{6}{5}\right)$$

A1

$$\text{Answer: Shown}$$

5 marks

26 (10 marks)

$$(a) x + 3 \equiv A(1-2x)^2 + B(1+x)(1-2x) + C(1+x)$$

M1 A1

$$x = -1: 2 = 9A, \text{ so } A = \frac{2}{9}$$

A1

$$x = \frac{1}{2}: \frac{7}{2} = \frac{3}{2}C, \text{ so } C = \frac{7}{3}$$

A1

$$\text{Coefficient of } x^2: 0 = 4A - 2B, \text{ so } B = \frac{4}{9}$$

A1

$$\text{Answer: } A = \frac{2}{9}, B = \frac{4}{9}, C = \frac{7}{3}$$

5 marks

$$(b) (1+x)^{-1} = 1 - x + x^2 - x^3, (1-2x)^{-1} = 1 + 2x + 4x^2 + 8x^3, (1-2x)^{-2} = 1 + 4x + 12x^2 +$$

M1 A1

$$32x^3$$

Combine with A, B, C: constant $\frac{2}{9} + \frac{4}{9} + \frac{7}{3} = 3$

M1

$$f(x) = 3 + 10x + 30x^2 + 78x^3 + \dots$$

A1

Valid for $|x| < \frac{1}{2}$

B1

Answer: $3 + 10x + 30x^2 + 78x^3$; $|x| < \frac{1}{2}$

5 marks

27 (9 marks)

(a) $\frac{1}{r(r+2)} \equiv \frac{A}{r} + \frac{B}{r+2}$, so $1 \equiv A(r+2) + Br$

M1

$r = 0$: $A = \frac{1}{2}$. $r = -2$: $B = -\frac{1}{2}$. Hence the result

A1

Answer: Shown

2 marks

(b) $\frac{4}{r(r+2)} = 2\left(\frac{1}{r} - \frac{1}{r+2}\right)$

M1

Sum = $2\left[(1 - \frac{1}{3}) + (\frac{1}{2} - \frac{1}{4}) + (\frac{1}{3} - \frac{1}{5}) + \dots + (\frac{1}{n-1} - \frac{1}{n+1}) + (\frac{1}{n} - \frac{1}{n+2})\right]$

M1

All terms cancel except $1, \frac{1}{2}, -\frac{1}{n+1}$ and $-\frac{1}{n+2}$

A1

Sum = $2\left[1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right] = 3 - \frac{2}{n+1} - \frac{2}{n+2}$

A1

Answer: Shown

4 marks

(c) As $n \rightarrow \infty$, $\frac{2}{n+1} \rightarrow 0$ and $\frac{2}{n+2} \rightarrow 0$, so the sum is 3

B1

Answer: 3

1 mark

(d) Need $\frac{2}{n+1} + \frac{2}{n+2} < 0.04$

M1

$n = 98$: 0.0402 (too big). $n = 99$: $\frac{2}{100} + \frac{2}{101} = 0.0398 < 0.04$. Least n is 99

A1

Answer: $n = 99$

2 marks

28 (12 marks)

(a) $\frac{1}{p(3-p)} \equiv \frac{A}{p} + \frac{B}{3-p}$, so $1 \equiv A(3-p) + Bp$

M1

$p = 0$: $A = \frac{1}{3}$. $p = 3$: $B = \frac{1}{3}$

A1

Answer: Shown

2 marks

(b) Separate: $\int \frac{1}{p(3-p)} dp = \int \frac{1}{3-p} dp$

M1

$\frac{1}{3}[\ln p - \ln(3-p)] = t + c$

A1

$t = 0, p = 1$: $c = \frac{1}{3} \ln \frac{1}{2}$

M1

$\ln\left(\frac{p}{3-p}\right) = 3t - \ln 2$, so $\frac{p}{3-p} = \frac{1}{2}e^{3t}$

M1 A1

$p = \frac{3e^{3t}}{2 + e^{3t}} = \frac{3}{1 + 2e^{-3t}}$

A1

Answer: Shown

6 marks

(c) $\frac{3}{1 + 2e^{-3t}} = 2.4$, so $1 + 2e^{-3t} = 1.25$

M1

$e^{-3t} = \frac{1}{8}$, so $-3t = -\ln 8$

A1

$t = \frac{\ln 8}{3} = \ln 2$ days

A1

Answer: $t = \ln 2$

3 marks

(d) As $t \rightarrow \infty$, $e^{-3t} \rightarrow 0$ so $p \rightarrow 3$: the population approaches 3 thousand

B1

Answer: Tends to 3 thousand

1 mark

29 (10 marks)

$$(a) \frac{1}{r^2} - \frac{1}{(r+1)^2} = \frac{(r+1)^2 - r^2}{r^2(r+1)^2}$$

M1 A1

$$(r+1)^2 - r^2 = 2r + 1$$

A1

Answer: Shown

3 marks

$$(b) \text{ Using (a), the terms telescope: } \left(\frac{1}{1^2} - \frac{1}{2^2}\right) + \left(\frac{1}{2^2} - \frac{1}{3^2}\right) + \dots$$

M1 A1

$$= 1 - \frac{1}{(N+1)^2}$$

A1

Answer: Shown

3 marks

$$(c) \text{ As } N \rightarrow \infty, \frac{1}{(N+1)^2} \rightarrow 0, \text{ so the sum is } 1$$

B1

Answer: 1

1 mark

$$(d) 1 - \frac{1}{(N+1)^2} > 0.99, \text{ so } (N+1)^2 > 100$$

M1

N = 9 gives exactly 100, which is not greater

M1

N = 10

A1

Answer: N = 10

3 marks

30 (11 marks)

$$(a) 1 \equiv A(x+1)(x+2)(x+3) + Bx(x+2)(x+3) + Cx(x+1)(x+3) + Dx(x+1)(x+2)$$

M1 A1

$$x = 0: 1 = 6A, \text{ so } A = \frac{1}{6}$$

A1

$$x = -1: 1 = -2B, \text{ so } B = -\frac{1}{2}$$

A1

$$x = -2: 1 = 2C, \text{ so } C = \frac{1}{2}$$

A1

$$x = -3: 1 = -6D, \text{ so } D = -\frac{1}{6}$$

A1

$$\text{Answer: } A = \frac{1}{6}, B = -\frac{1}{2}, C = \frac{1}{2}, D = -\frac{1}{6}$$

6 marks

$$(b) \int_1^2 f(x) dx = \left[\frac{1}{6} \ln x - \frac{1}{2} \ln(x+1) + \frac{1}{2} \ln(x+2) - \frac{1}{6} \ln(x+3) \right] \text{ from } 1 \text{ to } 2$$

M1 A1

$$\text{At } x = 2: \frac{1}{6} \ln 2 - \frac{1}{2} \ln 3 + \frac{1}{2} \ln 4 - \frac{1}{6} \ln 5$$

M1

$$\text{At } x = 1: 0 - \frac{1}{2} \ln 2 + \frac{1}{2} \ln 3 - \frac{1}{6} \ln 4$$

A1

$$\text{Difference (with } \ln 4 = 2 \ln 2): 2 \ln 2 - \ln 3 - \frac{1}{6} \ln 5$$

A1

$$\text{Answer: } 2 \ln 2 - \ln 3 - \frac{1}{6} \ln 5$$

5 marks