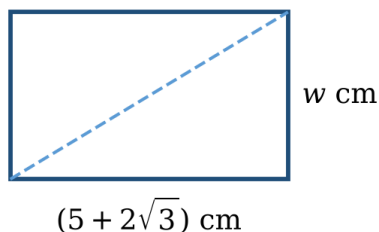


Indices and Surds

- 1 Find the exact value of $81^{3/4}$. [2]
- 2 Express $\sqrt{75} + \sqrt{12} - \sqrt{27}$ in the form $k\sqrt{3}$, where k is an integer. [2]
- 3 Evaluate $8^{-2/3}$. [2]
- 4 Write $\frac{4}{\sqrt{x}} + 5x\sqrt{x}$ as the sum of two terms of the form kx^n , where k and n are constants. [3]
- 5 Expand and simplify $(3 + \sqrt{5})(3 - \sqrt{5})$. [2]
- 6 Simplify $(4x^3y^{-2})^2$. [2]
- 7 Simplify fully $\frac{(2x^3)^2 \times 3x^{-1}}{6x^4}$. [3]
- 8 Rationalise the denominator and simplify $\frac{6 + \sqrt{2}}{3 - \sqrt{2}}$, giving your answer in the form $\frac{a + b\sqrt{2}}{c}$, where a , b and c are integers. [3]
- 9 Solve $4^x = 32$. [3]
- 10 Expand and simplify $(2 - 3\sqrt{5})^2$, giving your answer in the form $a + b\sqrt{5}$. [3]
- 11 Simplify $\frac{2x^3\sqrt{x}}{4x^{-2}}$, giving your answer in the form kx^p . [3]
- 12 Solve the equation $x\sqrt{12} - 5 = \frac{3x}{\sqrt{3}}$, giving your answer in the form $k\sqrt{3}$, where k is a fraction. [3]
- 13 Given that $2^x = a$, express 8^{x+1} in terms of a . [3]
- 14 Solve the equation $2^{2x+1} - 5 \times 2^x + 2 = 0$. [5]
- 15 Find the values of x and y given that $2^{x+y} = 32$ and $9^x \div 3^y = 3$. [5]
- 16 The curve with equation $y = ax^n$, where a and n are constants, passes through the points (4, 12) and (9, 18). Find the values of n and a . [5]
- 17 Show that $\frac{1}{2 - \sqrt{3}} - \frac{1}{2 + \sqrt{3}} = 2\sqrt{3}$. [4]
- 18 Solve the equation $\sqrt{x} - \frac{12}{\sqrt{x}} = 4$. [5]
- 19 This question is about powers of 3.
 - (a) Show that $3^{n+2} - 3^n = 8 \times 3^n$. [2]
 - (b) Hence find the exact value of $\frac{3^{101} - 3^{99}}{3^{97}}$. [3]
- 20 Solve $x^{2/3} - 5x^{1/3} + 4 = 0$. [5]

- 21** A rectangle has length $(5 + 2\sqrt{3})$ cm and area $(21 + 11\sqrt{3})$ cm². All lengths are exact.



- (a) Show that the width of the rectangle is $(3 + \sqrt{3})$ cm. [3]
 (b) Find the perimeter of the rectangle, in the form $a + b\sqrt{3}$. [2]
 (c) Show that the square of the length of a diagonal is $(49 + 26\sqrt{3})$ cm². [3]

- 22** A student writes: $(2 + \sqrt{3})^2 = 2^2 + (\sqrt{3})^2 = 4 + 3 = 7$.

- (a) Explain the error in the student's working. [1]
 (b) Expand $(2 + \sqrt{3})^2$ correctly, giving your answer in the form $p + q\sqrt{3}$. [2]
 (c) Hence show that $(2 + \sqrt{3})^2 + (2 - \sqrt{3})^2 = 14$. [2]
 (d) Hence find $(2 + \sqrt{3})^3$ in the form $a + b\sqrt{3}$, where a and b are integers. [3]

- 23** a and b are positive integers such that $(a + b\sqrt{3})^2 = 19 + 8\sqrt{3}$.

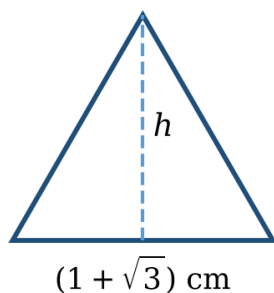
- (a) Show that $a^2 + 3b^2 = 19$ and $ab = 4$, and hence find a and b . [4]
 (b) Hence write $\sqrt{19 + 8\sqrt{3}}$ in the form $p + q\sqrt{3}$. [2]

- 24** The mass M grams of a radioactive sample at time t hours is modelled by $M = 96 \times 2^{-t/4}$.

- (a) Find the exact mass after 10 hours, in the form $k\sqrt{2}$. [3]
 (b) Find the time at which the mass is 12 grams. [3]

- 25** Given that $\frac{6\sqrt{2}}{\sqrt{3} - 1} - \frac{\sqrt{6}}{\sqrt{2} + 1} = a\sqrt{2} + b\sqrt{3} + c\sqrt{6}$, where a , b and c are integers, find a , b and c . [5]

- 26** The diagram shows an equilateral triangle of side $(1 + \sqrt{3})$ cm and height h cm.



- (a) Show that $h = \frac{3 + \sqrt{3}}{2}$. [3]
 (b) Find the exact area of the triangle, in the form $a + b\sqrt{3}$. [3]

- 27** For positive integers n , let $u_n = \frac{1}{\sqrt{n+1} + \sqrt{n}}$.

- (a) Show that $u_n = \sqrt{n+1} - \sqrt{n}$. [3]
 (b) Hence show that $\sum_{n=1}^N u_n = \sqrt{N+1} - 1$. [3]
 (c) Find the least value of N for which $\sum_{n=1}^N u_n > 9$. [3]

- 28** In this question you may use the fact that a rational number can be written as $\frac{p}{q}$ with p and q integers, $q \neq 0$.
- (a) Prove by contradiction that $\sqrt{2}$ is irrational. [4]
- (b) Hence prove that $3 + 5\sqrt{2}$ is irrational. [3]
- (c) Give a counterexample to the statement: 'the product of two irrational numbers is irrational'. [2]
-
- 29** Solve the simultaneous equations
 $(x + y\sqrt{2})^2 = 22 + 12\sqrt{2}$ and $3x = 2y$,
 where x and y are real numbers. [7]
-
- 30** For integers $n \geq 0$, let $u_n = (3 + \sqrt{5})^n + (3 - \sqrt{5})^n$.
- (a) Find u_0 , u_1 and u_2 . [3]
- (b) Show that $3 + \sqrt{5}$ and $3 - \sqrt{5}$ are the roots of $t^2 - 6t + 4 = 0$. [2]
- (c) Hence show that $u_{n+1} = 6u_n - 4u_{n-1}$ for $n \geq 1$. [3]
- (d) Find u_3 , and explain why u_n is an even integer for every $n \geq 1$. [3]

Expanding and Factorising

- 1** Expand and simplify $(x + 3)(x - 5)$. [2]
-
- 2** Factorise fully $6x^2y - 9xy^2$. [2]
-
- 3** Simplify $4(x^2 - 3x) - (x + 1)(x + 4)$. [3]
-
- 4** Factorise $15x^3 - 10x^2$ fully. [2]
-
- 5** Factorise $x^2 - 49$. [1]
-
- 6** Factorise $x^2 + x - 20$. [2]
-
- 7** Expand and simplify $(2x - 1)(x + 4)(x - 3)$. [3]
-
- 8** Factorise completely $4x^3 - 25x$. [3]
-
- 9** Factorise $6x^2 + x - 12$. [3]
-
- 10** Factorise $2x^2 - xy - y^2$. [3]
-
- 11** Factorise $x^3 - x^2 - 6x$ completely. [3]
-
- 12** Show that $(x + 3)^2 - (x - 3)^2 \equiv 12x$. [3]
-
- 13** Factorise completely $2x^3 + 3x^2 - 8x - 12$. [4]
-
- 14** Show that $(x + 2)^3 - (x - 2)^3 \equiv 12x^2 + 16$. [4]
-
- 15** Given that $f(x) = x^4 - 13x^2 + 36$
- (a) factorise $f(x)$ completely. [3]
- (b) Hence solve $f(x) = 0$. [1]
-
- 16** Expand $(3x - 2y)^3$. [4]

17 Factorise $81x^4 - 16$ completely over the integers. [4]

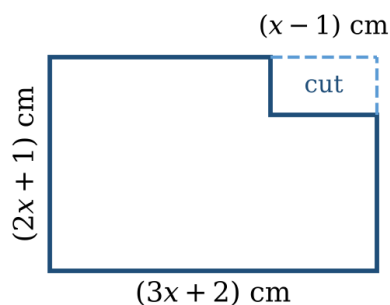
18 Prove that the difference between the squares of two consecutive odd integers is always a multiple of 8. [4]

19 This question is about powers of a binomial.

(a) Expand $(x + 1)^4$. [3]

(b) Hence, by choosing a suitable value of x , find the exact value of 1.01^4 . [3]

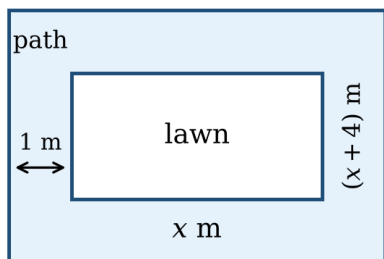
20 The diagram shows a rectangular metal sheet of length $(3x + 2)$ cm and width $(2x + 1)$ cm. A square of side $(x - 1)$ cm is cut from one corner.



(a) Show that the area of the remaining shape is $(5x^2 + 9x + 1)$ cm². [3]

(b) The area of the remaining shape is 39 cm². Find the value of x . [4]

21 A rectangular lawn measures x metres by $(x + 4)$ metres. A path of constant width surrounds the lawn, as shown.



(a) The path is 1 m wide. Show that the area of the path is $(4x + 12)$ m². [3]

(b) The area of the path is 60 m². Find the dimensions of the lawn. [2]

(c) A second path of width 1.5 m surrounds a lawn of the same size. Show that its area is $(6x + 21)$ m². [3]

22 This question is about factorising a difference of cubes.

(a) Expand and simplify $(a - b)(a^2 + ab + b^2)$. [2]

(b) Hence factorise $x^6 - 64$ completely into two linear factors and two quadratic factors. [5]

23 This question is about a quartic expression with no real factors of degree one.

(a) Show that $x^4 + 4 \equiv (x^2 + 2)^2 - 4x^2$. [2]

(b) Hence factorise $x^4 + 4$ into two quadratic factors with integer coefficients. [3]

(c) Hence show that $5^4 + 4$ is not a prime number. [2]

24 Consider the expression $(x - 1)(x - 2)(x + 4)(x + 5)$.

(a) Show that $(x - 1)(x - 2)(x + 4)(x + 5) \equiv x^4 + 6x^3 - 5x^2 - 42x + 40$. [4]

(b) Hence solve $(x - 1)(x - 2)(x + 4)(x + 5) = 40$, giving exact answers. [4]

- 25** This question is about sums of cubes. The real numbers a and b satisfy $a + b = 5$ and $ab = 4$.
- (a) Show that $a^3 + b^3 = (a + b)^3 - 3ab(a + b)$. [3]
- (b) Hence find $a^3 + b^3$. [2]
- (c) Verify your answer to (b) by finding a and b . [2]
-
- 26** A student's homework contains two errors:
- Line 1: $(2x - 3)^2 = 4x^2 - 9$
- Line 2: $x^2 - 5x - 6 = (x - 2)(x - 3)$
- (a) Explain the error in Line 1 and write down the correct expansion. [3]
- (b) Show that Line 2 is wrong by expanding the student's answer, and factorise $x^2 - 5x - 6$ correctly. [3]
-
- 27** Let n be a positive integer.
- (a) Show that $n(n + 1)(n + 2)(n + 3) = (n^2 + 3n)(n^2 + 3n + 2)$. [2]
- (b) Hence prove that $n(n + 1)(n + 2)(n + 3) + 1$ is a perfect square. [4]
- (c) Hence write $20 \times 21 \times 22 \times 23 + 1$ as the square of an integer. [2]
-
- 28** Let $f(x, y) = 6x^4 + 7x^3y - 12x^2y^2 - 3xy^3 + 2y^4$.
- (a) Show that $(x - y)$ is a factor of $f(x, y)$. [2]
- (b) Hence express $f(x, y)$ as the product of four linear factors. [5]
-
- 29** n is an integer.
- (a) Factorise $n^3 - n$ completely. [2]
- (b) Hence prove that $n^3 - n$ is divisible by 6 for every integer n . [4]
- (c) Hence find the smallest integer $n > 1$ for which $n^3 - n$ is divisible by 30. [3]
-
- 30** x is a non-zero real number such that $x + \frac{1}{x} = 5$.
- (a) Show that $x^2 + \frac{1}{x^2} = 23$. [3]
- (b) Find the value of $x^3 + \frac{1}{x^3}$. [4]
- (c) Hence find the value of $x^4 + \frac{1}{x^4}$. [3]

Quadratics and Completing the Square

- 1** Write $x^2 + 6x + 2$ in the form $(x + a)^2 + b$, where a and b are constants. [2]
-
- 2** Find the minimum value of $x^2 - 8x + 3$ and the value of x for which it occurs. [3]
-
- 3** The equation $x^2 + bx + c = 0$, where b and c are constants, has roots -2 and 5 . Find the values of b and c . [2]
-
- 4** Solve $x^2 - 3x - 10 = 0$. [2]
-
- 5** Write down the coordinates of the turning point of the curve $y = (x - 3)^2 + 4$. [2]
-
- 6** Solve $2x^2 - 7x + 3 = 0$. [2]
-
- 7** Write $2x^2 - 12x + 5$ in the form $a(x + b)^2 + c$. [3]

- 8** The curve C has equation $y = x^2 - 4x - 5$.
- (a) Write $x^2 - 4x - 5$ in the form $(x + p)^2 + q$. [2]
- (b) Sketch C, showing the coordinates of its turning point and of any points where it meets the axes. [3]

- 9** Solve $x^4 + 4x^2 - 12 = 0$. [4]

- 10** Express $-2x^2 + 6x - 4$ in the form $p - q(x - r)^2$, and state the maximum value of the expression. [3]

- 11** Solve $x^2 - 4x - 3 = 0$, giving your answers as exact surds. [3]

- 12** Use the quadratic formula to solve $x^2 + 3x - 1 = 0$. Give your answers to 3 significant figures. [3]

- 13** The curve $y = x^2 - 6x + 5$ meets the x-axis at A and B and the y-axis at C. Find the coordinates of A, B and C. [4]

- 14** Solve $x^2 + 6x - 2 = 0$ by completing the square. Give your answers as exact values. [4]

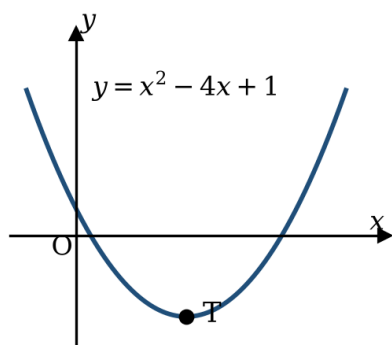
- 15** $f(x) = -x^2 + 6x + 7$.
- (a) Express $f(x)$ in the form $a - (x - b)^2$, where a and b are constants. [3]
- (b) Write down the maximum value of $f(x)$ and the value of x at which it occurs. [1]
- (c) Sketch the graph of $y = f(x)$, showing the coordinates of the turning point and of the points where the curve crosses the axes. [3]

- 16** $f(x) = x^2 - 8x + 19$.
- (a) Write $f(x)$ in the form $(x + a)^2 + b$. [2]
- (b) Hence explain why $f(x) > 0$ for all real x, and state the minimum value of $f(x)$. [2]

- 17** It is given that $5x^2 + Ax - 7 \equiv B(x + 2)^2 + C$ for all values of x, where A, B and C are constants. Find A, B and C. [4]

- 18** A quadratic curve passes through (1, 0) and (-5, 0) and has y-intercept 15.
- (a) Find the equation of the curve. [3]
- (b) Find the coordinates of the turning point. [3]

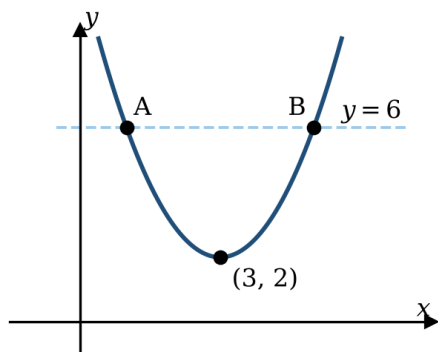
- 19** The diagram shows the curve with equation $y = x^2 - 4x + 1$, with minimum point T.



- (a) By completing the square, find the coordinates of T. [3]
- (b) Find the exact x-coordinates of the points where the curve crosses the x-axis. [3]

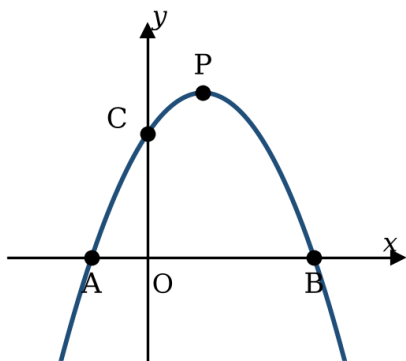
- 20** The line $y = 2x - 2$ meets the curve $y = x^2 - 3x + 4$ at the points A and B. Find the coordinates of A and B. [5]

- 21** The diagram shows the curve with equation $y = x^2 + bx + c$, which has a turning point at $(3, 2)$. The line $y = 6$ meets the curve at the points A and B.



- (a) Find the values of b and c . [3]
 (b) Hence show that the curve does not meet the x -axis. [2]
 (c) Find the exact length of AB . [3]
-
- 22** A ball is thrown upwards from a point 2 m above horizontal ground. At time t seconds after being thrown, the height of the ball above the ground is h metres, where $h = 2 + 8t - 5t^2$.
- (a) Write h in the form $A - B(t - C)^2$, where A , B and C are constants. [3]
 (b) State the maximum height reached by the ball and the time at which this occurs. [2]
 (c) Find the time at which the ball hits the ground, giving your answer to 3 significant figures. [3]
-
- 23** A prize of £840 is shared equally among n winners. Later, 6 of the winners are disqualified, and each of the remaining winners then receives £7 more.
- (a) Show that $n^2 - 6n - 720 = 0$. [3]
 (b) Hence find the value of n . [2]
 (c) Find the amount that each of the remaining winners receives. [1]
-
- 24** The quadratic curve C has equation $y = x^2 + kx + 4$, where k is a constant. The minimum point of C lies on the line $y = 3x - 5$.
- (a) Show that the minimum point of C has coordinates $(-\frac{k}{2}, 4 - \frac{k^2}{4})$. [3]
 (b) Find the possible values of k . [5]
-
- 25** The roots of the equation $x^2 - 7x + 3 = 0$ are α and β .
- (a) Write down the values of $\alpha + \beta$ and $\alpha\beta$. [2]
 (b) Find the value of $\alpha^2 + \beta^2$. [2]
 (c) Find a quadratic equation with integer coefficients whose roots are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$. [4]

- 26** The diagram shows the curve $y = 3 + 2x - x^2$. It meets the x -axis at A and B, the y -axis at C and has a maximum point P.



- (a) Find the coordinates of A, B, C and P. [5]
 (b) Find the area of triangle ABC. [2]
-
- 27** The function f is defined by $f(x) = x^2 - 4kx + 3k^2 + 6k + 10$, where k is a constant.
- (a) By completing the square in x , show that the minimum value of $f(x)$ is $19 - (k - 3)^2$. [3]
 (b) Hence find the greatest possible value of the minimum value of $f(x)$, and the value of k for which it occurs. [2]
 (c) Find the set of values of k for which $f(x) > 0$ for all real x . [4]
-
- 28** For $x \neq 0$, let $g(x) = x^2 + \frac{16}{x^2}$.
- (a) Show that $g(x) \equiv (x - \frac{4}{x})^2 + 8$. [2]
 (b) Hence find the minimum value of $g(x)$ and the values of x for which it occurs. [3]
 (c) Hence show that the equation $x^4 - kx^2 + 16 = 0$ has no real solutions when $k < 8$. [3]
-
- 29** The roots of the quadratic equation $px^2 + qx + r = 0$ are α and β , where $\alpha > \beta$, and $\alpha - \beta = k$.
- (a) Show that $q^2 - 4pr = p^2k^2$. [4]
 (b) The roots of $3x^2 - 14x + c = 0$ differ by 2. Find the value of c . [3]
 (c) Hence solve $3x^2 - 14x + c = 0$ for this value of c . [2]
-
- 30** The function $g(x) = ax^2 + bx + c$ has a stationary value of 9 at $x = 2$. The graph of $y = g(x)$ passes through the point $(0, 5)$.
- (a) Find the values of a , b and c . [5]
 (b) Solve $g(x) \geq 0$, and hence state the range of $g(x)$ for $0 \leq x \leq 6$. [4]

Algebraic Fractions

- 1** Simplify $\frac{x^2 - 9}{x^2 + 3x}$. [2]
-
- 2** Write $\frac{2}{x} + \frac{3}{x+1}$ as a single fraction. [2]
-
- 3** Simplify $\frac{12x^3y}{8xy^2}$. [2]
-
- 4** Simplify $\frac{x^2 - 5x}{x}$. [2]
-

- 5** Write $\frac{x}{3} + \frac{x}{4}$ as a single fraction. [2]
-
- 6** Simplify $\frac{x}{4} \times \frac{8}{x^2}$. [2]
-
- 7** Simplify fully $\frac{x^2 + 5x + 6}{x^2 - x - 12}$. [3]
-
- 8** Express $\frac{3}{x-2} - \frac{2}{x+1}$ as a single fraction in its simplest form. [3]
-
- 9** Simplify $\frac{x+2}{x-3} \times \frac{x^2-9}{x^2-4}$. [3]
-
- 10** Simplify $\frac{y^3-8}{y^2-4}$. [3]
-
- 11** Simplify fully $\frac{2x^2-x-6}{2x^2+7x+6}$. [3]
-
- 12** Solve $\frac{x}{3} - \frac{x-1}{4} = 2$. [3]
-
- 13** Solve $\frac{3}{x-1} = \frac{2}{x}$. [3]
-
- 14** Simplify $\frac{x^2-4}{2x+6} \div \frac{x-2}{x^2-9}$, giving your answer as a single fraction. [4]
-
- 15** Solve the equation $\frac{2}{x} + \frac{3}{x+1} = 2$. [5]
-
- 16** Solve $\frac{x}{x-1} - \frac{2}{x+1} = \frac{8}{x^2-1}$. [5]
-
- 17** Find the constants a, b, c and d such that $\frac{x^3 - 2x^2 + 7x - 1}{x+1} \equiv ax^2 + bx + c + \frac{d}{x+1}$. [4]
-
- 18** Simplify $\frac{x^2+3x}{x^2-x-12} \div \frac{x^2}{x^2-16}$. [4]
-
- 19** Write $\frac{x^2+3x-1}{x+2}$ in the form $x + p + \frac{q}{x+2}$, where p and q are constants. [4]
-
- 20** A cyclist rides 24 km at a steady speed of x km/h. A second cyclist rides the same 24 km at (x + 4) km/h and takes 30 minutes less.
 (a) Show that $x^2 + 4x - 192 = 0$. [4]
 (b) Hence find the speed of the first cyclist. [2]
-
- 21** A swimming pool is filled by two pipes. Pipe A alone would fill the pool in x hours. Pipe B alone would fill the pool in (x + 3) hours. When both pipes are used together the pool is filled in 2 hours.
 (a) Show that $x^2 - x - 6 = 0$. [4]
 (b) Solve the equation and hence state how long each pipe takes to fill the pool alone. [3]
-

- 22** A student writes: ' $\frac{x+3}{x+5} = \frac{3}{5}$, because the x on the top and the x on the bottom cancel.'
- (a) Explain why the student's statement is wrong. [1]
- (b) Simplify fully $\frac{x^2 + x - 12}{x^2 - 16}$. [3]
- (c) Hence solve $\frac{x^2 + x - 12}{x^2 - 16} = 2$. [3]
-
- 23** The function f is defined by $f(x) = \frac{x+1}{x-1}$, $x \in \mathbb{R}$, $x \neq 1$.
- (a) Show that $ff(x) = x$. [4]
- (b) Solve $f(x) = 3$. [2]
- (c) Write down $f^{-1}(x)$, giving a reason. [2]
-
- 24** Consider the expression $E = \frac{\frac{1}{x-3} + \frac{1}{x+3}}{\frac{1}{x-3} - \frac{1}{x+3}}$, where $x \neq \pm 3$ and $x \neq 0$.
- (a) Show that $E = \frac{x}{3}$. [5]
- (b) Hence solve $E = 4$. [1]
-
- 25** Solve the equation $\frac{2x}{x^2 - x - 6} - \frac{1}{x-3} = \frac{x}{x+2}$, giving your answers as exact surds. [6]
-
- 26** This question is about combining two fractions with related denominators.
- (a) Show that $\frac{1}{x-1} + \frac{1}{x+1} \equiv \frac{2x}{x^2-1}$. [2]
- (b) Hence solve $\frac{1}{x-1} + \frac{1}{x+1} = \frac{3}{4}$. [4]
-
- 27** Let $h(x) = \frac{1 - \frac{1}{x+1}}{1 + \frac{1}{x-1}}$, where $x \neq 0, 1, -1$.
- (a) Show that $h(x) = \frac{x-1}{x+1}$. [4]
- (b) Solve $h(x) = \frac{1}{3}$. [2]
- (c) Show that $h(x) = 1$ has no solution. [2]
-
- 28** Throughout this question a and b are positive real numbers.
- (a) Show that $\frac{a}{b} + \frac{b}{a} - 2 \equiv \frac{(a-b)^2}{ab}$. [3]
- (b) Hence prove that $\frac{a}{b} + \frac{b}{a} \geq 2$ for all positive real numbers a and b . [2]
- (c) State the condition for equality. [1]
- (d) Hence find the minimum value of $\frac{x^2+9}{3x}$ for $x > 0$, and the value of x at which it occurs. [3]
-
- 29** Throughout this question $x \neq 0, 1$ and -1 .
- (a) Show that $\frac{1}{1-x} - \frac{1}{1+x} \equiv \frac{2x}{1-x^2}$. [2]
- (b) Hence show that $\frac{1}{1-x} - \frac{1}{1+x} - \frac{2x}{1+x^2} \equiv \frac{4x^3}{1-x^4}$. [3]
- (c) Hence simplify $\frac{1}{1-x} - \frac{1}{1+x} - \frac{2x}{1+x^2} - \frac{4x^3}{1+x^4}$, giving your answer as a single fraction. [3]

- 30**
- (a) Show that $\frac{1}{k(k+1)(k+2)} \equiv \frac{1}{2} \left(\frac{1}{k(k+1)} - \frac{1}{(k+1)(k+2)} \right)$. [3]
- (b) Hence show that $\sum_{k=1}^N \frac{1}{k(k+1)(k+2)} = \frac{1}{4} - \frac{1}{2(N+1)(N+2)}$. [4]
- (c) Write down the sum to infinity, and find the least N for which the sum to N terms is within 0.001 of this value. [3]

Polynomial Division and Factor Theorem

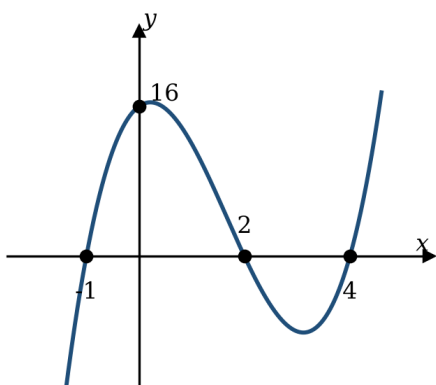
- 1** Find the remainder when $x^3 + 2x - 5$ is divided by $(x - 2)$. [2]
- 2** Show that $(x - 3)$ is a factor of $x^3 - 4x^2 + x + 6$. [2]
- 3** Find $f(2)$ where $f(x) = x^3 - 3x + 1$. [1]
- 4** Given that $(x - 1)$ is a factor of $x^3 + kx^2 - 8x + 3$, find the value of the constant k . [2]
- 5** Divide $x^2 + 5x + 6$ by $(x + 2)$. [2]
- 6** Find the remainder when $2x^3 - x^2 + 4$ is divided by $(x + 1)$. [2]
- 7** $f(x) = 2x^3 + 3x^2 - 11x - 6$.
 (a) Show that $(x - 2)$ is a factor of $f(x)$. [1]
 (b) Hence factorise $f(x)$ completely. [3]
- 8** When $x^3 + ax^2 - 4x + 3$ is divided by $(x + 2)$ the remainder is 11. Find the value of the constant a . [3]
- 9** Divide $2x^3 + x^2 - 13x + 6$ by $(x - 2)$ and hence factorise $2x^3 + x^2 - 13x + 6$ completely. [4]
- 10** Factorise $x^3 - 7x - 6$ completely. [4]
- 11** Solve the equation $6x^3 - 25x^2 + 2x + 8 = 0$. [4]
- 12** Show that $(x - 2)$ is a factor of $3x^3 - x^2 - 12x + 4$ and factorise the expression completely. [4]
- 13** Find the remainder when $4x^3 - 3x + 5$ is divided by $(2x - 1)$. [3]
- 14** Find the quotient and the remainder when $2x^3 - 3x^2 + 4x - 5$ is divided by $(x^2 + x - 1)$. [4]
- 15** $f(x) = 2x^3 + ax^2 + bx - 6$, where a and b are constants. $(x - 2)$ is a factor of $f(x)$, and when $f(x)$ is divided by $(x + 1)$ the remainder is 12. Find the values of a and b . [6]
- 16** When $f(x) = x^3 + 2x^2 - 5x + 3$ is divided by $(x - 1)(x + 2)$ the remainder is $ax + b$. Find a and b . [5]
- 17** $f(x) = x^3 + px^2 + qx - 6$, where p and q are constants. $(x - 1)$ and $(x + 3)$ are factors of $f(x)$.
 (a) Find the values of p and q . [5]
 (b) Hence factorise $f(x)$ completely. [2]
- 18** Show that the equation $x^3 - 2x^2 + 4x - 8 = 0$ has exactly one real root. [4]

- 19** When a polynomial $f(x)$ is divided by $(x^2 - 2)$ the quotient is $(2x + 3)$ and the remainder is $(x - 4)$.
- (a) Find $f(x)$, giving your answer in the form $ax^3 + bx^2 + cx + d$. [3]
- (b) Find the remainder when $f(x)$ is divided by $(x - 2)$. [2]

- 20** Sketch the curve $y = (x + 2)(x - 1)^2$, showing the coordinates of the points where it meets the axes. [4]

- 21** $f(x) = 6x^3 + x^2 - 19x + 6$.
- (a) Show that $(x + 2)$ is a factor of $f(x)$. [2]
- (b) Hence factorise $f(x)$ completely. [3]
- (c) Sketch the curve $y = f(x)$, showing the coordinates of the points where it meets the axes. [3]

- 22** The diagram shows the curve $y = f(x)$, where $f(x)$ is a cubic. The curve crosses the x -axis at $x = -1$, $x = 2$ and $x = 4$, and crosses the y -axis at $(0, 16)$.



- (a) Find $f(x)$, giving your answer in the form $px^3 + qx^2 + rx + s$. [3]
- (b) Find the remainder when $f(x)$ is divided by $(x - 1)$. [2]
- (c) Find the constant c such that $(x - 3)$ is a factor of $f(x) + c$. [2]
- 23** $f(x) = x^4 - x^3 - 7x^2 + x + 6$.
- (a) Show that $(x - 1)$ and $(x + 1)$ are factors of $f(x)$. [3]
- (b) Hence factorise $f(x)$ completely. [3]
- (c) Sketch the curve $y = f(x)$, labelling the intercepts with the axes. [3]
- 24** $g(x) = x^5 + 1$.
- (a) Use the factor theorem to show that $(x + 1)$ is a factor of $g(x)$. [2]
- (b) Hence express $g(x)$ as the product of $(x + 1)$ and a quartic factor. [3]
- (c) Hence show that $2^5 + 1 = 3 \times 11$. [2]
- 25** $f(x) = x^3 + (a - 1)x^2 - 4x + b$, where a and b are non-zero constants and $a > 0$. $(x - 2)$ and $(x + a)$ are both factors of $f(x)$.
- (a) By forming two equations, show that $a = 2$ and find the value of b . [5]
- (b) Hence solve $f(x) = 0$. [2]
- 26** $f(x) = 4x^3 + 12x^2 + 9x + 2$.
- (a) Show that $(x + 2)$ is a factor of $f(x)$. [2]
- (b) Factorise $f(x)$ completely. [3]
- (c) Solve $f(x) = 0$ and state which root is repeated, explaining what this means for the graph of $y = f(x)$. [3]

- 27** In this question n is a positive integer.
- (a) Use the factor theorem to show that $7^n - 1$ is divisible by 6. [3]
- (b) Prove that $5^n - 2^n$ is divisible by 3. [3]
- (c) Let a be a non-zero constant. Show that $(x + a)$ is a factor of $x^n + a^n$ when n is odd, and explain why it is not a factor when n is even. [3]
-
- 28** $f(x) = x^4 + ax^3 + bx - 12$, where a and b are constants. $f(x)$ is exactly divisible by $(x^2 - x - 6)$.
- (a) Find the values of a and b . [5]
- (b) Hence factorise $f(x)$ completely, giving any irrational factors in exact form. [3]
- (c) Write down all the real solutions of $f(x) = 0$. [1]
-
- 29** The polynomial $f(x)$ leaves remainders 1, 4 and 9 when divided by $(x - 1)$, $(x - 2)$ and $(x - 3)$ respectively. When $f(x)$ is divided by $(x - 1)(x - 2)(x - 3)$ the remainder is $R(x)$.
- (a) Explain why $R(x)$ has degree at most 2, and write $R(x) = ax^2 + bx + c$. [2]
- (b) Show that $R(1) = 1$, $R(2) = 4$ and $R(3) = 9$, and form three equations in a , b , c . [3]
- (c) Solve the equations and state $R(x)$. [4]
-
- 30** The equation $x^3 - 12x + q = 0$, where q is a constant, has a repeated root.
- (a) By considering $f(x) = x^3 - 12x + q$ and its derivative, explain why a repeated root satisfies $f(x) = 0$ and $f'(x) = 0$. [3]
- (b) Find the two possible values of q . [5]
- (c) For the larger value of q , factorise $f(x)$ completely and write down all the roots. [3]

Partial Fractions

- 1** Express $\frac{x+7}{(x+1)(x+3)}$ in partial fractions. [3]
-
- 2** Express $\frac{3x+1}{x^2-x-6}$ in partial fractions. [3]
-
- 3** Express $\frac{5}{(x-1)(x+4)}$ in partial fractions. [3]
-
- 4** Express $\frac{2x+1}{x(x+1)}$ in partial fractions. [3]
-
- 5** Express $\frac{3x-4}{(x+2)(x-3)}$ in the form $\frac{A}{x+2} + \frac{B}{x-3}$. [3]
-
- 6** Express $\frac{7}{(x+2)(x-5)}$ in partial fractions. [3]
-
- 7** Express $\frac{x+5}{(x+1)(x-1)^2}$ in the form $\frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$, where A , B and C are constants to be found. [4]
-
- 8** Write $\frac{5x+5}{(2x-1)(x+2)}$ as the sum of two partial fractions. [3]
-
- 9** Express $\frac{3x+2}{x^2(x+1)}$ in the form $\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$. [4]
-
- 10** Express $\frac{2x}{(x-3)^2}$ in the form $\frac{A}{x-3} + \frac{B}{(x-3)^2}$. [3]

- 11** This question is about integrating a rational function.
- (a) Express $\frac{3x+5}{(x+1)(x+3)}$ in partial fractions. [3]
- (b) Hence find the exact value of $\int_0^1 \frac{3x+5}{(x+1)(x+3)} dx$, giving your answer as a single logarithm. [3]
-
- 12** Express $\frac{x^2+3x+3}{(x+1)(x+2)}$ in the form $1 + \frac{A}{x+1} + \frac{B}{x+2}$. [4]
-
- 13** Express $\frac{4}{x^2-1}$ in partial fractions. [3]
-
- 14** Express $\frac{x^2+4x+1}{(x-1)(x+2)}$ in the form $P + \frac{A}{x-1} + \frac{B}{x+2}$, where P, A and B are constants. [4]
-
- 15** $f(x) = \frac{4x}{(x-1)(x+3)}$.
- (a) Express $f(x)$ in partial fractions. [3]
- (b) Hence show that $\int_2^5 f(x) dx = \ln\left(\frac{2048}{125}\right)$. [4]
-
- 16** Express $\frac{6x^2-9x-6}{(x-1)(x+2)(x-4)}$ in partial fractions. [5]
-
- 17** $f(x) = \frac{5}{(1+x)(2-x)}$.
- (a) Express $f(x)$ in partial fractions. [3]
- (b) Hence expand $f(x)$ in ascending powers of x up to and including x^2 , and state the values of x for which the expansion is valid. [5]
-
- 18** $g(x) = \frac{2x^2+8x-10}{(x-3)(x+1)}$.
- (a) Express $g(x)$ in the form $P + \frac{A}{x-3} + \frac{B}{x+1}$. [4]
- (b) Hence show that $\int_4^6 g(x) dx = 4 + 8 \ln 3 + 4 \ln\left(\frac{7}{5}\right)$. [4]
-
- 19** Express $\frac{x^3+2}{x^2-1}$ in the form $Px + \frac{A}{x-1} + \frac{B}{x+1}$, where P, A and B are constants. [5]
-
- 20** Express $\frac{x+14}{x^2-2x-8}$ in partial fractions and hence show that $\int_5^7 \frac{x+14}{x^2-2x-8} dx = \ln\left(\frac{49}{3}\right)$. [6]
-
- 21** $g(x) = \frac{4+5x}{(1-x)(1+2x)}$.
- (a) Express $g(x)$ in partial fractions. [3]
- (b) Hence find the series expansion of $g(x)$ in ascending powers of x up to and including the term in x^3 . [3]
- (c) State the range of values of x for which the expansion in (b) is valid. [2]
-
- 22** This question is about a rational function with a quadratic factor in the denominator.
- (a) Express $\frac{3x^2+x+8}{(x+1)(x^2+4)}$ in the form $\frac{A}{x+1} + \frac{Bx+C}{x^2+4}$. [4]
- (b) Hence find the exact value of $\int_0^2 \frac{3x^2+x+8}{(x+1)(x^2+4)} dx$, giving your answer in the form $p \ln 3 + q \ln 2$. [4]

23

$$h(x) = \frac{5x^2 - 6x + 2}{(x - 2)(x^2 + 1)}.$$

(a) Find constants A, B and C such that $h(x) \equiv \frac{A}{x - 2} + \frac{Bx + C}{x^2 + 1}$. [5]

(b) Hence find $\int_3^5 h(x) \, dx$, giving your answer in the form $p \ln 3 + q \ln\left(\frac{13}{5}\right)$. [4]

24

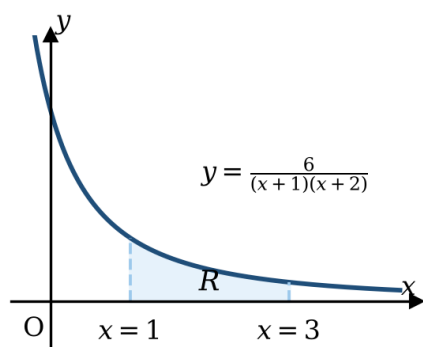
A quantity y satisfies the differential equation $\frac{dy}{dx} = 2(y - 1)(y + 2)$, and $y = 2$ when $x = 0$.

(a) Express $\frac{1}{(y - 1)(y + 2)}$ in partial fractions. [3]

(b) Solve the differential equation to show that $y = \frac{4 + 2e^{6x}}{4 - e^{6x}}$. [6]

25

The diagram shows part of the curve with equation $y = \frac{6}{(x + 1)(x + 2)}$. The shaded region R is bounded by the curve, the x-axis and the lines $x = 1$ and $x = 3$.



(a) Express $\frac{6}{(x + 1)(x + 2)}$ in partial fractions. [3]

(b) Hence show that the area of R is $6 \ln\left(\frac{6}{5}\right)$. [5]

26

$$f(x) = \frac{x + 3}{(1 + x)(1 - 2x)^2}.$$

(a) Express $f(x)$ in the form $\frac{A}{1 + x} + \frac{B}{1 - 2x} + \frac{C}{(1 - 2x)^2}$. [5]

(b) Hence find the first four terms of the expansion of $f(x)$ in ascending powers of x , and state the range of validity. [5]

27

This question is about summing a series using partial fractions.

(a) Show that $\frac{1}{r(r + 2)} \equiv \frac{1}{2}\left(\frac{1}{r} - \frac{1}{r + 2}\right)$. [2]

(b) Hence show that $\sum_{r=1}^n \frac{4}{r(r + 2)} = 3 - \frac{2}{n + 1} - \frac{2}{n + 2}$. [4]

(c) Write down the value of $\sum_{r=1}^{\infty} \frac{4}{r(r + 2)}$. [1]

(d) Find the least value of n for which the sum in (b) exceeds 2.96. [2]

28 The population P , in thousands, of a colony of bacteria at time t days satisfies the differential equation

$$\frac{dP}{dt} = P(3 - P), \quad 0 < P < 3.$$

Initially the population is 1 thousand.

- (a) Show that $\frac{1}{P(3-P)} \equiv \frac{1}{3}\left(\frac{1}{P} + \frac{1}{3-P}\right)$. [2]
- (b) Solve the differential equation to show that $P = \frac{3}{1 + 2e^{-3t}}$. [6]
- (c) Find the exact time at which the population reaches 2.4 thousand. [3]
- (d) Describe what happens to the population in the long term. [1]

29

- (a) Show that $\frac{2r+1}{r^2(r+1)^2} \equiv \frac{1}{r^2} - \frac{1}{(r+1)^2}$. [3]
- (b) Hence show that $\sum_{r=1}^N \frac{2r+1}{r^2(r+1)^2} = 1 - \frac{1}{(N+1)^2}$. [3]
- (c) Write down the sum to infinity of the series. [1]
- (d) Find the least value of N for which the sum to N terms exceeds 0.99. [3]

30

$$f(x) = \frac{1}{x(x+1)(x+2)(x+3)}.$$

- (a) Find constants A , B , C and D such that $f(x) \equiv \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2} + \frac{D}{x+3}$. [6]
- (b) Hence find $\int_1^2 f(x) dx$, giving your answer in the form $p \ln 2 + q \ln 3 + s \ln 5$. [5]