

Deriving the t-Formulae

1 (2 marks)

$$\sin x = \frac{2t}{1+t^2}$$

B1

$$\cos x = \frac{1-t^2}{1+t^2}$$

B1

$$\text{Answer: } \sin x = \frac{2t}{1+t^2}, \cos x = \frac{1-t^2}{1+t^2}$$

2 marks

2 (2 marks)

$$\sin x = \frac{\frac{3}{2}}{\frac{25}{16}} = \frac{24}{25}$$

B1

$$\cos x = \frac{1 - \frac{9}{16}}{1 + \frac{9}{16}} = \frac{7}{25}$$

B1

$$\text{Answer: } \sin x = \frac{24}{25}, \cos x = \frac{7}{25}$$

2 marks

3 (2 marks)

$$\tan x = \frac{\frac{2}{3}}{1 - \frac{1}{9}}$$

M1

$$= \frac{\frac{2}{3}}{\frac{8}{9}} = \frac{3}{4}$$

A1

$$\text{Answer: } \frac{3}{4}$$

2 marks

4 (2 marks)

$$1 + \cos x = 1 + \frac{1-t^2}{1+t^2} = \frac{1+t^2+1-t^2}{1+t^2}$$

M1

$$= \frac{2}{1+t^2}$$

A1

$$\text{Answer: } 1 + \cos x = \frac{2}{1+t^2}$$

2 marks

5 (2 marks)

$$\sec x = \frac{1}{\cos x}$$

M1

$$= \frac{1+t^2}{1-t^2}$$

A1

$$\text{Answer: } \sec x = \frac{1+t^2}{1-t^2}$$

2 marks

6 (2 marks)

$$\cos x = \frac{1 - (-2)^2}{1 + (-2)^2}$$

M1

$$= \frac{-3}{5}$$

A1

$$\text{Answer: } -\frac{3}{5}$$

2 marks

7 (3 marks)

$$\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2} = \frac{2 \sin \frac{x}{2} \cos \frac{x}{2}}{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2}}, \text{ using } \sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} = 1 \quad \text{M1}$$

Divide numerator and denominator by $\cos^2 \frac{x}{2}$ M1

$$\sin x = \frac{2 \tan \frac{x}{2}}{\tan^2 \frac{x}{2} + 1} = \frac{2t}{1 + t^2} \quad \text{A1}$$

Answer: $\sin x = \frac{2t}{1 + t^2}$ 3 marks

8 (3 marks)

$$\cos x = \frac{\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}}{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2}}, \text{ using } \sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} = 1 \quad \text{M1}$$

Divide numerator and denominator by $\cos^2 \frac{x}{2}$ M1

$$\cos x = \frac{1 - t^2}{1 + t^2} \quad \text{A1}$$

Answer: $\cos x = \frac{1 - t^2}{1 + t^2}$ 3 marks

9 (2 marks)

Put $A = \frac{x}{2}$, so $2A = x$ M1

$$\tan x = \frac{2t}{1 - t^2} \quad \text{A1}$$

Answer: $\tan x = \frac{2t}{1 - t^2}$ 2 marks

10 (3 marks)

$$3 \sin x + 4 \cos x = \frac{6t}{1 + t^2} + \frac{4(1 - t^2)}{1 + t^2} \quad \text{M1}$$

$$= \frac{6t + 4 - 4t^2}{1 + t^2} \quad \text{M1}$$

$$= \frac{4 + 6t - 4t^2}{1 + t^2} \quad \text{A1}$$

Answer: $\frac{4 + 6t - 4t^2}{1 + t^2}$ 3 marks

11 (3 marks)

$$\frac{3}{5} = \frac{1 - t^2}{1 + t^2} \text{ so } 3 + 3t^2 = 5 - 5t^2 \quad \text{M1}$$

$$8t^2 = 2 \text{ so } t^2 = \frac{1}{4} \quad \text{A1}$$

$x/2$ is acute so $t > 0$, giving $t = \frac{1}{2}$ A1

Answer: $\frac{1}{2}$ 3 marks

12 (4 marks)

$$\tan 45^\circ = 1 = \frac{2t}{1-t^2}$$

M1

$$t^2 + 2t - 1 = 0$$

A1

$$t = -1 \pm \sqrt{2}$$

M1

$$22.5^\circ \text{ is acute so } t > 0: \tan 22.5^\circ = \sqrt{2} - 1$$

A1

$$\text{Answer: } \sqrt{2} - 1$$

4 marks

13 (3 marks)

$$\cos x (1 + t^2) = 1 - t^2$$

M1

$$\cos x + t^2 \cos x = 1 - t^2, \text{ so } t^2 + t^2 \cos x = 1 - \cos x$$

M1

$$t^2(1 + \cos x) = 1 - \cos x, \text{ so } t^2 = \frac{1 - \cos x}{1 + \cos x}$$

A1

$$\text{Answer: } t^2 = \frac{1 - \cos x}{1 + \cos x}$$

3 marks

14 (6 marks)

$$\text{(a) } 1 + \sin x = 1 + \frac{2t}{1+t^2}$$

M1

$$= \frac{1+t^2+2t}{1+t^2}$$

A1

$$= \frac{(1+t)^2}{1+t^2}$$

A1

$$\text{Answer: } 1 + \sin x = \frac{(1+t)^2}{1+t^2}$$

3 marks

$$\text{(b) } 1 + \sin x + \cos x = \frac{1+t^2+2t+1-t^2}{1+t^2}$$

M1

$$= \frac{2+2t}{1+t^2}$$

A1

$$= \frac{2(1+t)}{1+t^2}$$

A1

$$\text{Answer: } 1 + \sin x + \cos x = \frac{2(1+t)}{1+t^2}$$

3 marks

15 (5 marks)

$$\text{(a) } 4 - 3 \cos x = \frac{4(1+t^2) - 3(1-t^2)}{1+t^2}$$

M1

$$= \frac{4+4t^2-3+3t^2}{1+t^2}$$

A1

$$= \frac{1+7t^2}{1+t^2}$$

A1

$$\text{Answer: } 4 - 3 \cos x = \frac{1+7t^2}{1+t^2}$$

3 marks

$$\text{(b) } \frac{1+7t^2}{1+t^2} = 7 - \frac{6}{1+t^2}, \text{ which is smallest when } t^2 \text{ is smallest}$$

M1

Least value 1, at $t = 0$

A1

Answer: least value 1, when $t = 0$

2 marks

16 (4 marks)

$$\cos x = -\frac{5}{13} \text{ since } x \text{ is obtuse}$$

B1

$$t^2 = \frac{1 - \cos x}{1 + \cos x} = \frac{18}{8} = \frac{9}{4}$$

M1 A1

$$x/2 \text{ lies between } 45^\circ \text{ and } 90^\circ, \text{ so } t > 0 \text{ and } t = \frac{3}{2}$$

A1

$$\text{Answer: } \frac{3}{2}$$

4 marks

17 (4 marks)

$$\text{With } x = 4\theta, \tan(x/2) = u, \text{ so } \sin 4\theta = \frac{2u}{1+u^2} \quad \text{B1}$$

$$\cos 4\theta = \frac{1-u^2}{1+u^2} \quad \text{B1}$$

$$\sin 4\theta + 2 \cos 4\theta - 1 = \frac{2u + 2 - 2u^2 - 1 - u^2}{1+u^2} = \frac{1+2u-3u^2}{1+u^2} \quad \text{M1}$$

$$= \frac{(1-u)(1+3u)}{1+u^2} \quad \text{A1}$$

$$\text{Answer: } \sin 4\theta + 2 \cos 4\theta - 1 = \frac{(1-u)(1+3u)}{1+u^2} \quad \text{4 marks}$$

18 (4 marks)

$$\text{(a) } \sec x = \frac{1+t^2}{1-t^2} \quad \text{B1}$$

$$\sec x - 1 = \frac{1+t^2 - (1-t^2)}{1-t^2} \quad \text{M1}$$

$$= \frac{2t^2}{1-t^2} \quad \text{A1}$$

$$\text{Answer: } \sec x - 1 = \frac{2t^2}{1-t^2} \quad \text{3 marks}$$

$$\text{(b) } \frac{2 \times \frac{1}{4}}{1 - \frac{1}{4}} = \frac{2}{3} \quad \text{B1}$$

$$\text{Answer: } \frac{2}{3} \quad \text{1 mark}$$

19 (5 marks)

$$\text{(a) } t + \frac{1}{t} = \frac{t^2+1}{t} \quad \text{M1}$$

$$\sin x = \frac{2t}{1+t^2} \text{ so } \frac{2}{\sin x} = \frac{1+t^2}{t} \quad \text{M1}$$

$$\text{Hence } t + \frac{1}{t} = \frac{2}{\sin x} \quad \text{A1}$$

$$\text{Answer: } t + \frac{1}{t} = \frac{2}{\sin x} \quad \text{3 marks}$$

$$\text{(b) For } 0 < x < \pi, 0 < \sin x \leq 1, \text{ so } \frac{2}{\sin x} \geq 2 \quad \text{M1}$$

$$\text{Therefore } \tan(x/2) + \cot(x/2) \geq 2 \quad \text{A1}$$

$$\text{Answer: } \tan(x/2) + \cot(x/2) \geq 2 \text{ for } 0 < x < \pi \quad \text{2 marks}$$

20 (4 marks)

$$\cot x = \frac{1}{\tan x} = \frac{1-t^2}{2t} \quad \text{B1}$$

$$\cot x + t = \frac{1-t^2}{2t} + t = \frac{1-t^2+2t^2}{2t} \quad \text{M1}$$

$$= \frac{1+t^2}{2t} \quad \text{A1}$$

$$\sin x = \frac{2t}{1+t^2} \text{ so } \operatorname{cosec} x = \frac{1+t^2}{2t} \quad \text{A1}$$

$$\text{Answer: } \cot x + t = \operatorname{cosec} x \quad \text{4 marks}$$

21 (6 marks)

$$\text{Numerator: } 1 + \frac{2t}{1+t^2} - \frac{1-t^2}{1+t^2} = \frac{1+t^2+2t-1+t^2}{1+t^2} = \frac{2t^2+2t}{1+t^2}$$

M1 A1

$$\text{Denominator: } \frac{2+2t}{1+t^2}$$

B1

$$\text{So the fraction is } \frac{2t(t+1)}{2(1+t)}$$

M1

$$= t, \text{ valid provided } 1 + \sin x + \cos x \neq 0, \text{ i.e. } x \neq \pi$$

A1 E1

$$\text{Answer: } \frac{1 + \sin x - \cos x}{1 + \sin x + \cos x} = t$$

6 marks

22 (7 marks)

$$(a) \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

M1

$$\tan\left(\frac{\pi}{4} + \frac{x}{2}\right) = \frac{1+t}{1-1 \times t}$$

M1 A1

$$\text{Answer: } \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) = \frac{1+t}{1-t}$$

3 marks

$$(b) \sec x + \tan x = \frac{1 + \sin x}{\cos x}$$

M1

$$= \frac{(1+t)^2}{1+t^2} \div \frac{1-t^2}{1+t^2} = \frac{(1+t)^2}{1-t^2}$$

M1 A1

$$= \frac{(1+t)^2}{(1-t)(1+t)} = \frac{1+t}{1-t} = \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$$

A1

$$\text{Answer: } \sec x + \tan x = \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$$

4 marks

23 (6 marks)

$$\cos 2x = 2 \cos^2 x - 1, \text{ so } \cos x + \cos 2x = c + 2c^2 - 1 \text{ with } c = \frac{1-t^2}{1+t^2}$$

M1 B1

$$= \frac{(1-t^2)(1+t^2) + 2(1-t^2)^2 - (1+t^2)^2}{(1+t^2)^2}$$

M1

$$\text{Numerator: } 1 - t^4 + 2 - 4t^2 + 2t^4 - 1 - 2t^2 - t^4$$

A1

$$= 2 - 6t^2 = 2(1 - 3t^2), \text{ giving the required result}$$

A1

$$\text{Check at } x = \pi/2 \text{ (} t = 1 \text{): } \cos x + \cos 2x = -1 \text{ and } \frac{2(1-3)}{4} = -1$$

B1

$$\text{Answer: } \cos x + \cos 2x = \frac{2(1-3t^2)}{(1+t^2)^2}$$

6 marks

24 (6 marks)

$$(a) \text{ Line AP: } y = t(X+1); \text{ circle: } X^2 + y^2 = 1$$

B1

$$X^2 + t^2(X+1)^2 = 1, \text{ so } (X+1)[(X-1) + t^2(X+1)] = 0$$

M1

$$X \neq -1 \text{ at P, so } X(1+t^2) = 1-t^2 \text{ and } X = \frac{1-t^2}{1+t^2}$$

A1

$$y = t(X+1) = \frac{2t}{1+t^2}$$

A1

$$\text{Answer: } P = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$$

4 marks

$$(b) \text{ Angle POB, where B is (1, 0), is } x, \text{ and it is an exterior angle of triangle AOP}$$

M1

$$OA = OP, \text{ so the triangle is isosceles and angle PAO} = x/2; \text{ the gradient of AP is } \tan(\text{angle PAO}),$$

A1

$$\text{so } t = \tan(x/2)$$

$$\text{Answer: } t = \tan(x/2)$$

2 marks

25 (8 marks)

$$(a) t = \frac{\sin x}{1 + \cos x} \cdot \frac{2t}{1 + t^2} \div \frac{2}{1 + t^2} = t$$

M1 A1

$$t = \frac{1 - \cos x}{\sin x} \cdot \frac{2t^2}{1 + t^2} \div \frac{2t}{1 + t^2} = t$$

B1

$$\text{Answer: } t = \frac{\sin x}{1 + \cos x} = \frac{1 - \cos x}{\sin x}$$

3 marks

$$(b) \tan 15^\circ = \frac{1 - \cos 30^\circ}{\sin 30^\circ} = \frac{1 - \frac{\sqrt{3}}{2}}{\frac{1}{2}}$$

M1

$$= 2 - \sqrt{3}$$

A1

$$\text{Answer: } 2 - \sqrt{3}$$

2 marks

$$(c) \sin 225^\circ = \cos 225^\circ = -\frac{\sqrt{2}}{2}$$

B1

$$\tan 112.5^\circ = \frac{\sin 225^\circ}{1 + \cos 225^\circ} = \frac{-\sqrt{2}}{2 - \sqrt{2}}$$

M1

$$= -\frac{\sqrt{2}(2 + \sqrt{2})}{2} = -(1 + \sqrt{2})$$

A1

$$\text{Answer: } -(1 + \sqrt{2})$$

3 marks

26 (7 marks)

$$(a) 1 - \sin x = \frac{1 + t^2 - 2t}{1 + t^2} = \frac{(1 - t)^2}{1 + t^2}$$

M1

$$1 + \sin x = \frac{(1 + t)^2}{1 + t^2}$$

B1

Both are squares divided by a positive quantity, so ≥ 0 , giving $-1 \leq \sin x \leq 1$

E1

$$\text{Answer: } 1 + \sin x = \frac{(1 + t)^2}{1 + t^2}, \text{ so } -1 \leq \sin x \leq 1$$

3 marks

$$(b) x = \pi$$

B1

$$\sin \pi = 0 \text{ and } \cos \pi = -1$$

B1

$$\text{Answer: } x = \pi; \sin x = 0, \cos x = -1$$

2 marks

$$(c) \frac{1 - t^2}{1 + t^2} = -1 \text{ gives } 1 - t^2 = -1 - t^2, \text{ i.e. } 1 = -1, \text{ impossible}$$

M1

$\cos x = -1$ occurs only at $x = \pi \pmod{2\pi}$, where t does not exist

E1

$$\text{Answer: no real } t; x = \pi \text{ gives no } t$$

2 marks

27 (9 marks)

$$(a) \frac{C}{2} = \frac{\pi}{2} - \frac{A+B}{2}, \text{ so } c = \cot\left(\frac{A+B}{2}\right)$$

B1

$$c = \frac{1}{\tan\left(\frac{A+B}{2}\right)}$$

M1

$$\tan\left(\frac{A+B}{2}\right) = \tan\left(\frac{A}{2} + \frac{B}{2}\right) = \frac{a+b}{1-ab}$$

M1 A1

$$\text{So } c = \frac{1-ab}{a+b}$$

A1

$$c(a+b) = 1 - ab$$

M1

$$ab + bc + ca = 1$$

A1

$$\text{Answer: } ab + bc + ca = 1$$

7 marks

$$(b) c = \frac{1 - \frac{1}{6}}{\frac{1}{2} + \frac{1}{3}} = \frac{5}{6} \div \frac{5}{6} = 1$$

M1

$$\tan(C/2) = 1 \text{ so } C/2 = 45^\circ \text{ and } C = 90^\circ$$

A1

$$\text{Answer: } c = 1, C = 90^\circ$$

2 marks

28 (10 marks)

$$(a) 1 + it = \frac{\cos \frac{x}{2} + i \sin \frac{x}{2}}{\cos \frac{x}{2}} \quad \text{B1}$$

$$(1 + it)^2 = \frac{(\cos \frac{x}{2} + i \sin \frac{x}{2})^2}{\cos^2 \frac{x}{2}} = \frac{\cos x + i \sin x}{\cos^2 \frac{x}{2}}, \text{ by de Moivre's theorem} \quad \text{M1}$$

$$1 + t^2 = \sec^2 \frac{x}{2} = \frac{1}{\cos^2 \frac{x}{2}} \quad \text{M1}$$

Dividing gives $\cos x + i \sin x$ A1

Answer: $\frac{(1 + it)^2}{1 + t^2} = \cos x + i \sin x$ 4 marks

$$(b) (1 + it)^2 = 1 - t^2 + 2it \quad \text{M1}$$

Equating real and imaginary parts: $\cos x = \frac{1 - t^2}{1 + t^2}$, $\sin x = \frac{2t}{1 + t^2}$ A1

Answer: $\cos x = \frac{1 - t^2}{1 + t^2}$, $\sin x = \frac{2t}{1 + t^2}$ 2 marks

$$(c) \cos 3x + i \sin 3x = (\cos x + i \sin x)^3 = \frac{(1 + it)^6}{(1 + t^2)^3} \quad \text{M1}$$

Expand $(1 + it)^6$ using the binomial theorem M1

Imaginary part: $6t - 20t^3 + 6t^5$ A1

$$\sin 3x = \frac{6t - 20t^3 + 6t^5}{(1 + t^2)^3} \quad \text{A1}$$

Answer: $\frac{6t - 20t^3 + 6t^5}{(1 + t^2)^3}$ 4 marks

29 (8 marks)

$$(a) \sin x = \frac{2\frac{p}{q}}{1 + \frac{p^2}{q^2}} \quad \text{M1}$$

Multiply numerator and denominator by q^2 : $\sin x = \frac{2pq}{q^2 + p^2}$, $\cos x = \frac{q^2 - p^2}{q^2 + p^2}$ A1

Answer: $\sin x = \frac{2pq}{p^2 + q^2}$ and $\cos x = \frac{q^2 - p^2}{p^2 + q^2}$ 2 marks

$$(b) t = \frac{\sin x}{1 + \cos x} = \frac{\frac{21}{29}}{\frac{49}{29}} \quad \text{M1}$$

$$= \frac{3}{7} \quad \text{A1}$$

Answer: $\frac{3}{7}$ 2 marks

$$(c) q^2 - p^2 = 65, 2pq = 72, p^2 + q^2 = 97 \quad \text{B1}$$

$$65^2 + 72^2 = 4225 + 5184 = 9409 = 97^2 \quad \text{B1}$$

Answer: (65, 72, 97) 2 marks

$$(d) \tan(x/2) = \frac{\sin x}{1 + \cos x} \text{ (from the half-angle forms)} \quad \text{M1}$$

The sum and quotient of rational numbers is rational (denominator $1 + \cos x$ is non-zero) E1

Answer: $t = \sin x / (1 + \cos x)$ is rational 2 marks

30 (9 marks)

$$(a) \text{Apply } \tan 2A = \frac{2 \tan A}{1 - \tan^2 A} \text{ with } A = x/4 \quad \text{B1}$$

Answer: $t = \frac{2s}{1 - s^2}$ 1 mark

$$(b) t(1 - s^2) = 2s, \text{ so } ts^2 + 2s - t = 0 \quad \text{M1}$$

$$s = \frac{-2 \pm \sqrt{4 + 4t^2}}{2t} = \frac{-1 \pm \sqrt{1 + t^2}}{t}$$

M1 A1

$0 < x/4 < \pi/4$ so $s > 0$; the root with the minus sign is negative, so $s = \frac{\sqrt{1 + t^2} - 1}{t}$

E1

Answer: $s = \frac{\sqrt{1 + t^2} - 1}{t}$

4 marks

(c) $t = \tan(\pi/4) = 1$

B1

$s = \sqrt{2} - 1$

A1

Answer: $\sqrt{2} - 1$

2 marks

(d) $t = \tan(\pi/3) = \sqrt{3}$, so $s = \frac{2 - 1}{\sqrt{3}} = \frac{1}{\sqrt{3}}$

M1

$\tan(\pi/6) = \frac{1}{\sqrt{3}}$, as required

A1

Answer: $s = \frac{1}{\sqrt{3}} = \tan \frac{\pi}{6}$

2 marks

Solving Trigonometric Equations with t

1 (3 marks)

$$\frac{2t}{1 + t^2} + \frac{1 - t^2}{1 + t^2} = 1$$

M1

$$2t + 1 - t^2 = 1 + t^2$$

M1

$$2t - 2t^2 = 0, \text{ so } t(1 - t) = 0$$

A1

Answer: $t(1 - t) = 0$

3 marks

2 (2 marks)

$\tan(\pi/3) = \sqrt{3}$; for $0 \leq x/2 < \pi$, $x/2 = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

M1

$$x = \frac{4\pi}{3}$$

A1

Answer: $x = \frac{4\pi}{3}$

2 marks

3 (3 marks)

$(3t - 1)(t - 1) = 0$, so $t = \frac{1}{3}$ or $t = 1$

M1

$t = 1$: $x = 2 \times \frac{\pi}{4} = \frac{\pi}{2}$

A1

$t = \frac{1}{3}$: $x = 2 \arctan \frac{1}{3} = 0.644$

A1

Answer: $x = \frac{\pi}{2}$ or $x = 0.644$

3 marks

4 (2 marks)

$\tan(\pi/2)$ is undefined, so $x = \pi$ can never appear as a root t of the quadratic

E1

So $x = \pi$ must be substituted directly into the original equation

B1

Answer: t is undefined at $x = \pi$, so test it separately

2 marks

5 (3 marks)

$$\frac{1 - t^2}{1 + t^2} - \frac{6t}{1 + t^2} + 2 = 0$$

M1

$$1 - t^2 - 6t + 2 + 2t^2 = 0$$

M1

$$t^2 - 6t + 3 = 0$$

A1

Answer: $t^2 - 6t + 3 = 0$

3 marks

6 (2 marks)

$$x/2 = \arctan 2 = 63.43^\circ$$

M1

$$x = 126.9^\circ$$

A1

Answer: $x = 126.9^\circ$

2 marks

7 (4 marks)

$$\frac{4t}{1+t^2} + \frac{1-t^2}{1+t^2} = 1, \text{ so } 4t + 1 - t^2 = 1 + t^2$$

M1

$$2t^2 - 4t = 0, \text{ so } t = 0 \text{ or } t = 2$$

A1

$$t = 0: x = 0^\circ; t = 2: x = 2 \arctan 2 = 126.9^\circ$$

M1

Check $x = 180^\circ$: $0 - 1 \neq 1$, so it is not a solution. $x = 0^\circ$ or 126.9°

A1

Answer: $x = 0^\circ$ or $x = 126.9^\circ$

4 marks

8 (5 marks)

$$\text{With } t = \tan(x/2): \frac{6t}{1+t^2} - \frac{1-t^2}{1+t^2} = 2$$

M1

$$6t - 1 + t^2 = 2 + 2t^2, \text{ so } t^2 - 6t + 3 = 0$$

A1

$$t = 3 \pm \sqrt{6} = 5.45 \text{ or } 0.551$$

M1

$$x = 2 \arctan(5.45) = 159.2^\circ$$

A1

$$x = 2 \arctan(0.551) = 57.7^\circ$$

A1

Answer: $x = 57.7^\circ$ or $x = 159.2^\circ$

5 marks

9 (5 marks)

$$\text{With } t = \tan(x/2): \frac{3(1-t^2)}{1+t^2} + \frac{2t}{1+t^2} = 2$$

M1

$$3 - 3t^2 + 2t = 2 + 2t^2, \text{ so } 5t^2 - 2t - 1 = 0$$

A1

$$t = \frac{1 \pm \sqrt{6}}{5} = 0.69 \text{ or } -0.29$$

M1

$$x = 2 \arctan(0.69) = 69.2^\circ$$

A1

$$x = 2 \arctan(-0.29) + 360^\circ = 327.7^\circ$$

A1

Answer: $x = 69.2^\circ$ or $x = 327.7^\circ$

5 marks

10 (5 marks)

$$\text{With } t = \tan(x/2): \frac{4t}{1+t^2} - \frac{3(1-t^2)}{1+t^2} = 1$$

M1

$$4t - 3 + 3t^2 = 1 + t^2, \text{ so } t^2 + 2t - 2 = 0$$

A1

$$t = -1 \pm \sqrt{3}, \text{ so } t = 0.732 \text{ or } t = -2.73$$

M1

$$x = 2 \arctan(0.732) = 1.26$$

A1

$$x = 2(\arctan(-2.73) + \pi) = 3.84$$

A1

Answer: $x = 1.26$ or $x = 3.84$

5 marks

11 (5 marks)

$$\text{With } t = \tan(x/2): \frac{2(1-t^2)}{1+t^2} + \frac{6t}{1+t^2} + 1 = 0$$

M1

$$2 - 2t^2 + 6t + 1 + t^2 = 0, \text{ so } t^2 - 6t - 3 = 0$$

A1

$$t = 3 \pm 2\sqrt{3} = 6.46 \text{ or } -0.464$$

M1

$$x = 2 \arctan(6.46) = 162.4^\circ$$

A1

$$x = 2 \arctan(-0.464) = -49.8^\circ$$

A1

Answer: $x = -49.8^\circ$ or $x = 162.4^\circ$

5 marks

12 (5 marks)

$$\text{With } t = \tan(x/2): \frac{2t}{1+t^2} + \frac{7(1-t^2)}{1+t^2} = 5$$

M1

$$2t + 7 - 7t^2 = 5 + 5t^2, \text{ so } 12t^2 - 2t - 2 = 0$$

A1

$$6t^2 - t - 1 = 0, \text{ so } (2t - 1)(3t + 1) = 0$$

M1

$$t = \frac{1}{2}: x = 2 \arctan \frac{1}{2} = 53.1^\circ$$

A1

$$t = -\frac{1}{3}: x = 360^\circ - 2 \arctan \frac{1}{3} = 323.1^\circ$$

A1

Answer: $x = 53.1^\circ$ or $x = 323.1^\circ$

5 marks

13 (4 marks)

$$t = \frac{4t}{1+t^2}, \text{ so } t(1+t^2) = 4t$$

M1

$$t(t^2 - 3) = 0, \text{ so } t = 0 \text{ or } t = \pm\sqrt{3}$$

A1

$$x = 0^\circ, 2 \times 60^\circ = 120^\circ, 2 \times 120^\circ = 240^\circ$$

A1

$x = 180^\circ$ is excluded because $\tan 90^\circ$ is undefined

B1

Answer: $x = 0^\circ, 120^\circ$ or 240°

4 marks

14 (6 marks)

$$\text{With } t = \tan(x/2): \frac{4(1-t^2)}{1+t^2} - \frac{6t}{1+t^2} = 2$$

M1

$$4 - 4t^2 - 6t = 2 + 2t^2$$

M1

$$6t^2 + 6t - 2 = 0, \text{ so } 3t^2 + 3t - 1 = 0$$

A1

$$t = \frac{-3 \pm \sqrt{21}}{6}$$

M1

$$t = 0.264: x = 29.6^\circ$$

A1

$$t = -1.26: x = 2(180^\circ + \arctan(-1.26)) = 256.7^\circ$$

A1

Answer: $x = 29.6^\circ$ or $x = 256.7^\circ$

6 marks

15 (5 marks)

$$\text{With } t = \tan(x/2): 3(1-t^2) + 8t = 5(1+t^2)$$

M1

$$8t^2 - 8t + 2 = 0, \text{ i.e. } 2(2t - 1)^2 = 0$$

A1

A repeated root $t = \frac{1}{2}$, so there is one value of $x/2$ in $[0^\circ, 180^\circ)$

E1

$$x = 2 \arctan \frac{1}{2} = 53.1^\circ$$

A1

Check $x = 180^\circ$: $-3 \neq 5$, so no further solution

B1

Answer: $x = 53.1^\circ$ (only)

5 marks

16 (6 marks)

$$\text{(a) } \frac{4t}{1+t^2} + \frac{3(1-t^2)}{1+t^2} + 3 = 0$$

M1

$$4t + 3 - 3t^2 + 3 + 3t^2 = 0, \text{ so } 4t + 6 = 0 \text{ (the } t^2 \text{ terms cancel)}$$

A1

$$t = -\frac{3}{2}$$

A1

$$\text{Answer: } t = -\frac{3}{2}$$

3 marks

$$\text{(b) } x = 180^\circ \text{ gives } 2 \sin 180^\circ + 3 \cos 180^\circ + 3 = 0 - 3 + 3 = 0, \text{ so } x = 180^\circ \text{ is a solution}$$

M1 A1

The t^2 terms cancelled because $t = \tan 90^\circ$ is undefined: $x = 180^\circ$ can never be produced by the substitution

E1

Answer: $x = 180^\circ$ (t is undefined there)

3 marks

17 (5 marks)

$$\sec x + \tan x = \frac{1 + \sin x}{\cos x} \quad \text{M1}$$

$$= \frac{(1+t)^2}{1+t^2} \div \frac{1-t^2}{1+t^2} \quad \text{M1}$$

$$= \frac{1+t}{1-t} \quad \text{A1}$$

$$\frac{1+t}{1-t} = 3 \text{ gives } 1+t = 3 - 3t, \text{ so } t = \frac{1}{2} \quad \text{M1}$$

$$x = 2 \arctan \frac{1}{2} = 0.927 \quad \text{A1}$$

Answer: $x = 0.927$ 5 marks

18 (4 marks)

With $t = \tan(x/2)$: $4t + 3(1 - t^2) = 6(1 + t^2)$ M1

$$9t^2 - 4t + 3 = 0 \quad \text{A1}$$

$$\text{Discriminant} = 16 - 108 = -92 < 0 \quad \text{M1}$$

No real t , and $x = 180^\circ$ gives $-3 \neq 6$, so there are no solutions A1

Answer: no solutions 4 marks

19 (5 marks)

$$\frac{2t}{1+t^2} = 3t \quad \text{M1}$$

$$2t = 3t + 3t^3, \text{ so } t(1 + 3t^2) = 0 \quad \text{A1}$$

$$1 + 3t^2 = 0 \text{ has no real solutions, so } t = 0 \quad \text{M1}$$

$$x = 0 \quad \text{A1}$$

$x = \pi$ is not valid since $\tan(\pi/2)$ is undefined B1

Answer: $x = 0$ only 5 marks

20 (6 marks)

$$5 \sin \varphi + 12 \cos \varphi = 8 \quad \text{B1}$$

$$\frac{10t}{1+t^2} + \frac{12(1-t^2)}{1+t^2} = 8 \quad \text{M1}$$

$$10t + 12 - 12t^2 = 8 + 8t^2, \text{ so } 10t^2 - 5t - 2 = 0 \quad \text{A1}$$

$$t = \frac{5 \pm \sqrt{105}}{20} = 0.762 \text{ or } -0.262 \quad \text{M1}$$

$$\varphi = 2 \arctan(0.762) = 1.30 \quad \text{A1}$$

$$\varphi = 2(\pi + \arctan(-0.262)) = 5.77 \quad \text{A1}$$

Answer: $\varphi = 1.30$ or $\varphi = 5.77$ 6 marks

21 (7 marks)

$$\text{(a)} \frac{6t}{1+t^2} + \frac{4(1-t^2)}{1+t^2} = 4, \text{ so } 6t + 4 - 4t^2 = 4 + 4t^2 \quad \text{M1}$$

$$8t^2 - 6t = 0, \text{ so } t(4t - 3) = 0 \quad \text{A1}$$

$$t = 0 \text{ or } t = \frac{3}{4} \quad \text{M1}$$

$$x = 0^\circ \text{ or } x = 2 \arctan \frac{3}{4} = 73.74^\circ \quad \text{A1}$$

Answer: $x = 0^\circ$ or 73.74° 4 marks

$$\text{(b)} 3 \sin x + 4 \cos x = 5 \sin(x + 53.13^\circ), \text{ so } \sin(x + 53.13^\circ) = \frac{4}{5} \quad \text{M1}$$

$$x + 53.13^\circ = 53.13^\circ \text{ or } 126.87^\circ \quad \text{M1}$$

$$x = 0^\circ \text{ or } x = 73.74^\circ, \text{ agreeing with part (a)} \quad \text{A1}$$

Answer: $x = 0^\circ$ or 73.74° 3 marks

22 (6 marks)

$$\sin 2x = \frac{2t}{1+t^2}, \cos 2x = \frac{1-t^2}{1+t^2}$$

B1

$$\frac{2t}{1+t^2} + \frac{3(1-t^2)}{1+t^2} = 1$$

M1

$$2t + 3 - 3t^2 = 1 + t^2, \text{ so } 4t^2 - 2t - 2 = 0$$

A1

$$2t^2 - t - 1 = 0, \text{ so } (2t+1)(t-1) = 0$$

M1

$$\tan x = 1: x = 45^\circ$$

A1

$$\tan x = -\frac{1}{2}: x = 180^\circ - 26.6^\circ = 153.4^\circ$$

A1**Answer: $x = 45^\circ$ or $x = 153.4^\circ$** **6 marks****23 (6 marks)**

$$\text{Let } y = 3x, \text{ so } 0^\circ \leq y < 540^\circ, \text{ and } t = \tan(y/2): \frac{4t}{1+t^2} + \frac{1-t^2}{1+t^2} = 2$$

M1

$$4t + 1 - t^2 = 2 + 2t^2, \text{ so } 3t^2 - 4t + 1 = 0$$

A1

$$(3t-1)(t-1) = 0, \text{ so } t = \frac{1}{3} \text{ or } t = 1$$

M1

$$t = 1: y = 90^\circ, 450^\circ; t = \frac{1}{3}: y = 36.9^\circ, 396.9^\circ (y = 180^\circ \text{ gives } -1 \neq 2)$$

A1

$$x = \frac{y}{3}: 30^\circ, 150^\circ, 12.3^\circ, 132.3^\circ$$

M1

$$x = 12.3^\circ, 30.0^\circ, 132.3^\circ, 150.0^\circ$$

A1**Answer: $x = 12.3^\circ, 30.0^\circ, 132.3^\circ, 150.0^\circ$** **6 marks****24 (6 marks)**

$$\frac{1-t^2}{1+t^2} + t = 1$$

M1

$$1 - t^2 + t(1+t^2) = 1 + t^2$$

M1

$$t^3 - 2t^2 + t = 0$$

A1

$$t(t-1)^2 = 0$$

M1

$$t = 0 \text{ or } t = 1$$

A1

$$x = 0 \text{ or } x = \frac{\pi}{2}$$

A1**Answer: $x = 0$ or $x = \frac{\pi}{2}$** **6 marks****25 (8 marks)**

$$(a) \frac{4t}{1+t^2} + \frac{1-t^2}{1+t^2} = k$$

M1

$$4t + 1 - t^2 = k + kt^2, \text{ so } (k+1)t^2 - 4t + (k-1) = 0$$

A1**Answer: $(k+1)t^2 - 4t + (k-1) = 0$** **2 marks****(b) Discriminant: $16 - 4(k+1)(k-1) \geq 0$** **M1**

$$16 - 4k^2 + 4 \geq 0, \text{ so } k^2 \leq 5$$

A1

$$-\sqrt{5} \leq k \leq \sqrt{5}$$

A1**Answer: $-\sqrt{5} \leq k \leq \sqrt{5}$** **3 marks**

$$(c) k = -1: -4t - 2 = 0, \text{ so } t = -\frac{1}{2}$$

M1

$$x = 2(\pi - \arctan \frac{1}{2}) = 5.36$$

A1Also $x = \pi$ gives $0 - 1 = -1$, so $x = \pi$ is a second solution**B1****Answer: $x = 5.36$ and $x = \pi$** **3 marks**

26 (6 marks)

$$\operatorname{cosec} x + 2 \cot x = \frac{1 + 2 \cos x}{\sin x}, \text{ so } 1 + 2 \cos x = 3 \sin x \quad \text{M1}$$

$$\frac{1 + t^2 + 2(1 - t^2)}{1 + t^2} = \frac{6t}{1 + t^2} \quad \text{M1}$$

$$3 - t^2 = 6t, \text{ so } t^2 + 6t - 3 = 0 \quad \text{A1}$$

$$t = -3 \pm 2\sqrt{3} \quad \text{M1}$$

$$t = 0.464: x = 49.8^\circ \quad \text{A1}$$

$$t = -6.46: x = 2(180^\circ + \arctan(-6.46)) = 197.6^\circ \quad \text{A1}$$

Answer: $x = 49.8^\circ$ or $x = 197.6^\circ$

6 marks

27 (9 marks)

(a) With $t = \tan(x/2)$: $2at + b(1 - t^2) = c(1 + t^2)$, so $(b + c)t^2 - 2at + (c - b) = 0$ M1 A1

$\tan(\alpha/2)$ and $\tan(\beta/2)$ are its roots, so $\tan \frac{\alpha}{2} + \tan \frac{\beta}{2} = \frac{2a}{b + c}$ M1

and $\tan \frac{\alpha}{2} \tan \frac{\beta}{2} = \frac{c - b}{c + b}$ A1

$$\tan\left(\frac{\alpha + \beta}{2}\right) = \frac{\tan \frac{\alpha}{2} + \tan \frac{\beta}{2}}{1 - \tan \frac{\alpha}{2} \tan \frac{\beta}{2}} = \frac{2a}{b + c} \div \frac{2b}{b + c}$$

M1

$$= \frac{a}{b}$$

A1

Answer: $\tan\left(\frac{\alpha + \beta}{2}\right) = \frac{a}{b}$

6 marks

(b) $\tan\left(\frac{\alpha + \beta}{2}\right) = \frac{5}{12}$, so $\frac{\alpha + \beta}{2} = \arctan \frac{5}{12} + 180k^\circ$ M1

Solving gives $\alpha = 85.1^\circ$, $\beta = 320.1^\circ$, so $\frac{\alpha + \beta}{2} = 202.62^\circ = 180^\circ + 22.62^\circ$ M1

Hence $\alpha + \beta = 360^\circ + 2 \arctan \frac{5}{12} = 405.2^\circ$, as required A1

Answer: $\alpha + \beta = 360^\circ + 2 \arctan \frac{5}{12} = 405.2^\circ$

3 marks

28 (10 marks)

(a) $\sin 2x = 2 \sin x \cos x$ M1

$$= 2 \times \frac{2t}{1 + t^2} \times \frac{1 - t^2}{1 + t^2}$$

M1

$$= \frac{4t(1 - t^2)}{(1 + t^2)^2}$$

A1

Answer: $\sin 2x = \frac{4t(1 - t^2)}{(1 + t^2)^2}$

3 marks

(b) $\sin x = \frac{2t(1 + t^2)}{(1 + t^2)^2}$ M1

$$\sin x + \sin 2x = \frac{2t + 2t^3 + 4t - 4t^3}{(1 + t^2)^2}$$

M1

$$= \frac{6t - 2t^3}{(1 + t^2)^2} = \frac{2t(3 - t^2)}{(1 + t^2)^2}$$

A1

Answer: $\sin x + \sin 2x = \frac{2t(3 - t^2)}{(1 + t^2)^2}$

3 marks

(c) $t = 0$ or $t^2 = 3$, so $t = 0$ or $t = \pm\sqrt{3}$ M1

$x = 0^\circ, 120^\circ, 240^\circ$ (since $\tan 60^\circ = \sqrt{3}$ and $\tan 120^\circ = -\sqrt{3}$) A1

t is undefined at 180° , but $\sin 180^\circ + \sin 360^\circ = 0$, so $x = 180^\circ$ is also a solution B1

Answer: $x = 0^\circ, 120^\circ, 180^\circ, 240^\circ$

3 marks

(d) $\sin x = 0$ gives $0^\circ, 180^\circ$; $\cos x = -\frac{1}{2}$ gives $120^\circ, 240^\circ$ B1

Answer: $0^\circ, 120^\circ, 180^\circ, 240^\circ$

1 mark

29 (8 marks)

$$(a) \frac{4t}{1+t^2} + \frac{3(1-t^2)}{1+t^2} = 2t \quad \text{M1}$$

$$4t + 3 - 3t^2 = 2t(1+t^2) \quad \text{M1}$$

$$2t^3 + 3t^2 - 2t - 3 = 0 \quad \text{A1}$$

Answer: the equation becomes $2t^3 + 3t^2 - 2t - 3 = 0$ 3 marks

$$(b) t^2(2t+3) - (2t+3) = 0, \text{ so } (t^2-1)(2t+3) = 0 \quad \text{M1 A1}$$

$$t = 1, t = -1 \text{ or } t = -\frac{3}{2} \quad \text{A1}$$

$$t = 1: x = 90^\circ; t = -1: x = 270^\circ \quad \text{A1}$$

$$t = -\frac{3}{2}: x = 360^\circ - 2 \arctan \frac{3}{2} = 247.4^\circ \quad \text{A1}$$

Answer: $x = 90^\circ, 270^\circ$ or 247.4° 5 marks

30 (8 marks)

$$(a) \frac{8t}{1+t^2} - \frac{3(1-t^2)}{1+t^2} = 3 \quad \text{M1}$$

$$8t - 3 + 3t^2 = 3 + 3t^2 \quad \text{M1}$$

$$\text{The } t^2 \text{ terms cancel: } 8t = 6 \quad \text{A1}$$

$$t = \frac{3}{4} \quad \text{A1}$$

Answer: $t = \frac{3}{4}$ 4 marks

$$(b) \tan(x/2) = \frac{3}{4} \text{ gives } x/2 = \arctan \frac{3}{4} + n\pi \quad \text{M1}$$

$$\text{so } x = 2 \arctan \frac{3}{4} + 2n\pi = 1.29 + 2n\pi \quad \text{A1}$$

$$\text{For } x = (2n+1)\pi: 4 \sin x - 3 \cos x = 0 - 3(-1) = 3 \quad \text{M1}$$

$t = \tan(x/2)$ is undefined at these values, so they cannot arise from the quadratic and must be tested directly E1

Answer: $x = 1.29 + 2n\pi$ or $x = (2n+1)\pi$ 4 marks

Integration Using $t = \tan(x/2)$ **1 (2 marks)**

$$\frac{dt}{dx} = \frac{1}{2} \sec^2 \frac{x}{2} = \frac{1}{2}(1 + \tan^2 \frac{x}{2}) = \frac{1}{2}(1 + t^2) \quad \text{M1}$$

$$\text{So } dx = \frac{2}{1+t^2} dt \quad \text{A1}$$

Answer: $dx = \frac{2}{1+t^2} dt$ 2 marks

2 (2 marks)

$$x = \frac{\pi}{3}: t = \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}} \quad \text{B1}$$

$$x = \frac{2\pi}{3}: t = \tan \frac{\pi}{3} = \sqrt{3} \quad \text{B1}$$

Answer: $t = \frac{1}{\sqrt{3}}$ to $t = \sqrt{3}$ 2 marks

3 (3 marks)

$$1 + \cos x = \frac{2}{1+t^2} \text{ and } dx = \frac{2}{1+t^2} dt \quad \text{M1}$$

$$\int \frac{1}{1+\cos x} dx = \int \frac{1+t^2}{2} \times \frac{2}{1+t^2} dt = \int 1 dt \quad \text{A1}$$

$$= t + c = \tan \frac{x}{2} + c \quad \text{A1}$$

Answer: $\tan \frac{x}{2} + c$ 3 marks

4 (3 marks)

$$\sin x = \frac{2t}{1+t^2} \text{ and } dx = \frac{2}{1+t^2} dt \quad \text{M1}$$

$$\int \frac{1+t^2}{2t} \times \frac{2}{1+t^2} dt = \int \frac{1}{t} dt \quad \text{A1}$$

$$= \ln|t| + c = \ln\left|\tan \frac{x}{2}\right| + c \quad \text{A1}$$

Answer: $\ln\left|\tan \frac{x}{2}\right| + c$ 3 marks

5 (3 marks)

$$1 + \sin x = \frac{(1+t)^2}{1+t^2}, \text{ so the integrand becomes } \frac{2}{(1+t)^2} dt, \text{ with limits } t = 0 \text{ to } t = 1 \quad \text{M1 A1}$$

$$\left[-\frac{2}{1+t}\right]_0^1 = -1 + 2 = 1 \quad \text{A1}$$

Answer: 1 3 marks

6 (3 marks)

$$1 - \cos x = \frac{2t^2}{1+t^2}, \text{ so } \int \frac{1+t^2}{2t^2} \times \frac{2}{1+t^2} dt = \int \frac{1}{t^2} dt \quad \text{M1 A1}$$

$$= -\frac{1}{t} + c = -\cot \frac{x}{2} + c \quad \text{A1}$$

Answer: $-\cot \frac{x}{2} + c$ 3 marks

7 (4 marks)

$$2 + \cos x = \frac{2+2t^2+1-t^2}{1+t^2} = \frac{3+t^2}{1+t^2} \quad \text{M1 A1}$$

$$\int \frac{2}{3+t^2} dt \quad \text{M1}$$

$$= \frac{2}{\sqrt{3}} \arctan \frac{t}{\sqrt{3}} + c = \frac{2}{\sqrt{3}} \arctan\left(\frac{1}{\sqrt{3}} \tan \frac{x}{2}\right) + c \quad \text{A1}$$

Answer: $\frac{2}{\sqrt{3}} \arctan\left(\frac{1}{\sqrt{3}} \tan \frac{x}{2}\right) + c$ 4 marks

8 (4 marks)

$$1 + \sin x + \cos x = \frac{2(1+t)}{1+t^2}, \text{ so the integrand is } \frac{1+t^2}{2(1+t)} \times \frac{2}{1+t^2} = \frac{1}{1+t} \quad \text{M1 A1}$$

$$\text{Limits } t = 0 \text{ to } 1: [\ln(1+t)]_0^1 \quad \text{M1}$$

$$= \ln 2 - \ln 1 = \ln 2 \quad \text{A1}$$

Answer: $\ln 2$ 4 marks

9 (5 marks)

$$5 + 3 \cos x = \frac{5 + 5t^2 + 3 - 3t^2}{1 + t^2} = \frac{8 + 2t^2}{1 + t^2}$$

M1 A1

$$\text{Integrand: } \frac{1+t^2}{8+2t^2} \times \frac{2}{1+t^2} dt = \frac{1}{4+t^2} dt, \text{ limits 0 to 1}$$

M1 A1

$$\left[\frac{1}{2} \arctan \frac{t}{2} \right]_0^1 = \frac{1}{2} \arctan \frac{1}{2}$$

A1

$$\text{Answer: } \frac{1}{2} \arctan \frac{1}{2}$$

5 marks

10 (4 marks)

$$\sec x = \frac{1+t^2}{1-t^2}, \text{ so } \int \sec x dx = \int \frac{2}{1-t^2} dt$$

M1 A1

$$\frac{2}{1-t^2} = \frac{1}{1-t} + \frac{1}{1+t}$$

M1

$$= \ln \left| \frac{1+t}{1-t} \right| + c, \text{ where } t = \tan \frac{x}{2}$$

A1

$$\text{Answer: } \ln \left| \frac{1+t}{1-t} \right| + c, t = \tan \frac{x}{2}$$

4 marks

11 (5 marks)

$$2 - \cos x = \frac{2 + 2t^2 - 1 + t^2}{1 + t^2} = \frac{1 + 3t^2}{1 + t^2}$$

M1 A1

$$\text{Integrand: } \frac{2}{1+3t^2} dt, \text{ limits 0 to 1}$$

M1

$$\left[\frac{2}{\sqrt{3}} \arctan(\sqrt{3} t) \right]_0^1$$

A1

$$= \frac{2}{\sqrt{3}} \times \frac{\pi}{3} = \frac{2\pi}{3\sqrt{3}}$$

A1

$$\text{Answer: } \frac{2\pi}{3\sqrt{3}}$$

5 marks

12 (5 marks)

$$1 - \sin x = \frac{(1-t)^2}{1+t^2}, \text{ so the integrand is } \frac{2}{(1-t)^2} dt$$

M1 A1

$$\text{Upper limit } t = \tan \frac{\pi}{8} = \sqrt{2} - 1 \text{ (from } \tan \frac{\pi}{4} = 1 = \frac{2t}{1-t^2} \text{)}$$

B1

$$\left[\frac{2}{1-t} \right]_0^{\sqrt{2}-1} = \frac{2}{2-\sqrt{2}} - 2$$

M1

$$= 2 + \sqrt{2} - 2 = \sqrt{2}$$

A1

$$\text{Answer: } \sqrt{2}$$

5 marks

13 (4 marks)

$$\frac{1 - \cos x}{\sin x} = t, \text{ so the integral is } \int \frac{2t}{1+t^2} dt \text{ with limits 0 to } \frac{1}{\sqrt{3}}$$

M1 A1

$$[\ln(1+t^2)]_0^{1/\sqrt{3}} = \ln \frac{4}{3}$$

M1 A1

$$\text{Answer: } \ln \frac{4}{3}$$

4 marks

14 (6 marks)

$$3 + \sin x = \frac{3t^2 + 2t + 3}{1 + t^2}, \text{ so the integral is } \int \frac{2}{3t^2 + 2t + 3} dt, \text{ limits 0 to 1}$$

M1 A1

$$3t^2 + 2t + 3 = 3\left[\left(t + \frac{1}{3}\right)^2 + \frac{8}{9}\right]$$

M1

$$\int = \frac{1}{\sqrt{2}} \arctan \frac{3t + 1}{2\sqrt{2}}$$

A1

$$= \frac{1}{\sqrt{2}} \left[\arctan \sqrt{2} - \arctan \frac{1}{2\sqrt{2}} \right]$$

M1

$$= 0.435$$

A1

Answer: 0.435

6 marks

15 (6 marks)

$$2 + 2 \sin x + \cos x = \frac{2 + 2t^2 + 4t + 1 - t^2}{1 + t^2} = \frac{t^2 + 4t + 3}{1 + t^2}$$

M1 A1

$$\text{Integrand: } \frac{2}{(t+1)(t+3)} dt$$

M1

$$\frac{2}{(t+1)(t+3)} = \frac{1}{t+1} - \frac{1}{t+3}$$

A1

$$\left[\ln \frac{t+1}{t+3} \right]_0^1 = \ln \frac{2}{4} - \ln \frac{1}{3}$$

M1

$$= \ln \frac{3}{2}$$

A1

Answer: $\ln \frac{3}{2}$

6 marks

16 (6 marks)

$$1 + \sin x - \cos x = \frac{1 + t^2 + 2t - 1 + t^2}{1 + t^2} = \frac{2t(1+t)}{1+t^2}$$

M1 A1

$$\text{Integrand: } \frac{1}{t(1+t)} dt, \text{ with limits } t = \frac{1}{\sqrt{3}} \text{ to } 1$$

M1 A1

$$\frac{1}{t(1+t)} = \frac{1}{t} - \frac{1}{1+t}, \text{ so } \left[\ln \frac{t}{1+t} \right]$$

M1

$$\ln \frac{1}{2} - \ln \frac{1}{\sqrt{3}+1} = \ln \left(\frac{\sqrt{3}+1}{2} \right)$$

A1

Answer: $\ln \left(\frac{\sqrt{3}+1}{2} \right)$

6 marks

17 (6 marks)

$$3 + 2 \cos x = \frac{3 + 3t^2 + 2 - 2t^2}{1 + t^2} = \frac{5 + t^2}{1 + t^2}$$

M1 A1

$$\text{Integrand: } \frac{2}{5 + t^2} dt$$

A1

$$\text{As } x \rightarrow \pi, t \rightarrow \infty, \text{ so the limits are 0 and } \infty$$

B1

$$\left[\frac{2}{\sqrt{5}} \arctan \frac{t}{\sqrt{5}} \right]_0^\infty$$

M1

$$= \frac{2}{\sqrt{5}} \times \frac{\pi}{2} = \frac{\pi}{\sqrt{5}}$$

A1

Answer: $\frac{\pi}{\sqrt{5}}$; $t \rightarrow \infty$ as $x \rightarrow \pi$

6 marks

18 (6 marks)

$$1 + 2 \cos x = \frac{1 + t^2 + 2 - 2t^2}{1 + t^2} = \frac{3 - t^2}{1 + t^2}$$

M1 A1

Integrand: $\frac{2}{3 - t^2} dt$, limits 0 to 1

A1

$$\frac{2}{3 - t^2} = \frac{1}{\sqrt{3}} \left(\frac{1}{\sqrt{3} + t} + \frac{1}{\sqrt{3} - t} \right)$$

M1

$$\left[\frac{1}{\sqrt{3}} \ln \frac{\sqrt{3} + t}{\sqrt{3} - t} \right]_0^1 = \frac{1}{\sqrt{3}} \ln \frac{\sqrt{3} + 1}{\sqrt{3} - 1}$$

A1

$$\frac{\sqrt{3} + 1}{\sqrt{3} - 1} = \frac{(\sqrt{3} + 1)^2}{2} = 2 + \sqrt{3}$$

A1

Answer: $\frac{1}{\sqrt{3}} \ln(2 + \sqrt{3})$

6 marks

19 (6 marks)

$$4 + 5 \sin x = \frac{4t^2 + 10t + 4}{1 + t^2}, \text{ so the integrand is } \frac{2}{4t^2 + 10t + 4} dt = \frac{1}{(2t + 1)(t + 2)} dt$$

M1 A1

$$\frac{1}{(2t + 1)(t + 2)} = \frac{1}{3} \left(\frac{2}{2t + 1} - \frac{1}{t + 2} \right)$$

M1 A1

$$\frac{1}{3} [\ln(2t + 1) - \ln(t + 2)]_0^1$$

M1

$$= \frac{1}{3} [(\ln 3 - \ln 3) - (0 - \ln 2)] = \frac{1}{3} \ln 2$$

A1

Answer: $\frac{1}{3} \ln 2$

6 marks

20 (7 marks)

$$\text{With } t = \tan \frac{x}{2}: 3 \cos x + 4 \sin x = \frac{3 - 3t^2 + 8t}{1 + t^2}$$

M1 A1

$$3 + 8t - 3t^2 = (3t + 1)(3 - t), \text{ so the integrand is } \frac{2}{(3t + 1)(3 - t)} dt, \text{ limits 0 to 1}$$

B1 M1

$$\frac{2}{(3t + 1)(3 - t)} = \frac{1}{5} \left(\frac{3}{3t + 1} + \frac{1}{3 - t} \right)$$

A1

$$\frac{1}{5} [\ln(3t + 1) - \ln(3 - t)]_0^1$$

M1

$$= \frac{1}{5} [(\ln 4 - \ln 2) - (0 - \ln 3)] = \frac{1}{5} \ln 6$$

A1

Answer: $\frac{1}{5} \ln 6$

7 marks

21 (7 marks)

$$1 + \sin x + \cos x = \frac{2(1 + t)}{1 + t^2}; \text{ limits } t = 0 \text{ to } 1$$

B1

Integrand: $\frac{(1 + t^2)^2}{4(1 + t)^2} \times \frac{2}{1 + t^2} = \frac{1 + t^2}{2(1 + t)^2}$

M1 A1

$$\frac{1 + t^2}{(1 + t)^2} = 1 - \frac{2t}{(1 + t)^2} = 1 - \frac{2}{1 + t} + \frac{2}{(1 + t)^2}$$

M1

$$\frac{1}{2} \left[t - 2 \ln(1 + t) - \frac{2}{1 + t} \right]_0^1$$

A1

$$= \frac{1}{2} [(1 - 2 \ln 2 - 1) - (0 - 0 - 2)]$$

M1

$$= \frac{1}{2} (2 - 2 \ln 2) = 1 - \ln 2$$

A1

Answer: $1 - \ln 2$

7 marks

22 (6 marks)

$$2 \sin x + \cos x + 3 = \frac{4t + 1 - t^2 + 3 + 3t^2}{1 + t^2} = \frac{2t^2 + 4t + 4}{1 + t^2}$$

M1 A1

$$\text{Integrand: } \frac{1}{t^2 + 2t + 2} dt = \frac{1}{(t+1)^2 + 1} dt$$

M1

$$[\arctan(t+1)]_0^1 = \arctan 2 - \arctan 1$$

A1

$$\tan(\arctan 2 - \arctan 1) = \frac{2-1}{1+2 \times 1} = \frac{1}{3}$$

M1

$$\text{So the value is } \arctan \frac{1}{3}$$

A1

$$\text{Answer: } \arctan \frac{1}{3}$$

6 marks

23 (6 marks)

$$3 \sin x + 4 \cos x + 5 = \frac{6t + 4 - 4t^2 + 5 + 5t^2}{1 + t^2} = \frac{t^2 + 6t + 9}{1 + t^2}$$

M1 A1

$$= \frac{(t+3)^2}{1 + t^2}$$

B1

$$\text{Integrand: } \frac{2}{(t+3)^2} dt, \text{ limits 0 to 1}$$

M1

$$\left[-\frac{2}{t+3}\right]_0^1 = -\frac{1}{2} + \frac{2}{3}$$

A1

$$= \frac{1}{6}$$

A1

$$\text{Answer: } \frac{1}{6}$$

6 marks

24 (6 marks)

$$dx = \frac{1}{1+t^2} dt \text{ and } \cos 2x = \frac{1-t^2}{1+t^2}$$

B1 B1

$$5 + 4 \cos 2x = \frac{5 + 5t^2 + 4 - 4t^2}{1 + t^2} = \frac{9 + t^2}{1 + t^2}$$

M1

$$\text{Integrand: } \frac{1}{9+t^2} dt, \text{ limits } t = 0 \text{ to } \infty \text{ (as } x \rightarrow \frac{\pi}{2}, \tan x \rightarrow \infty)$$

A1

$$\left[\frac{1}{3} \arctan \frac{t}{3}\right]_0^\infty = \frac{1}{3} \times \frac{\pi}{2}$$

M1

$$= \frac{\pi}{6}$$

A1

$$\text{Answer: } \frac{\pi}{6}$$

6 marks

25 (6 marks)

$$V = \pi \int y^2 dx = \pi \int_0^{\pi/2} \frac{1}{(1 + \cos x)^2} dx$$

M1

$$1 + \cos x = \frac{2}{1+t^2}, dx = \frac{2}{1+t^2} dt; \text{ limits } t = 0 \text{ to } 1$$

M1

$$\text{Integrand: } \frac{(1+t^2)^2}{4} \times \frac{2}{1+t^2} = \frac{1+t^2}{2}$$

A1

$$V = \pi \times \frac{1}{2} \left[t + \frac{t^3}{3}\right]_0^1$$

M1

$$= \pi \times \frac{1}{2} \times \frac{4}{3}$$

M1

$$= \frac{2\pi}{3}$$

A1

$$\text{Answer: } \frac{2\pi}{3}$$

6 marks

26 (6 marks)

(a) $\tan(x/2)$ is undefined at $x = \pi$, so the substitution is not valid over the whole interval 0 to 2π E1

The integrand is positive throughout the interval, so its integral must be positive E1

Answer: t is undefined at $x = \pi$; the integrand is positive 2 marks

(b) Split at π : for 0 to π , t runs from 0 to ∞ and $f = \int_0^\infty \frac{2}{3+t^2} dt$ M1

$$= \left[\frac{2}{\sqrt{3}} \arctan \frac{t}{\sqrt{3}} \right]_0^\infty = \frac{2}{\sqrt{3}} \times \frac{\pi}{2} = \frac{\pi}{\sqrt{3}}$$
A1

$\cos(2\pi - x) = \cos x$, so the integral from π to 2π is equal M1

$$\text{Total} = \frac{2\pi}{\sqrt{3}}$$
A1

Answer: $\frac{2\pi}{\sqrt{3}}$ 4 marks

27 (9 marks)

$$(a) a + b \cos x = \frac{a(1+t^2) + b(1-t^2)}{1+t^2} = \frac{(a+b) + (a-b)t^2}{1+t^2}$$
M1 A1

$$\text{Integrand: } \frac{2}{(a+b) + (a-b)t^2} dt$$
A1

$x = 0$ gives $t = 0$; $x \rightarrow \pi$ gives $t \rightarrow \infty$ B1

Write as $\frac{2}{a-b} \times \frac{1}{k^2+t^2}$ with $k^2 = \frac{a+b}{a-b}$ M1

$$\left[\frac{2}{(a-b)k} \arctan \frac{t}{k} \right]_0^\infty = \frac{2}{(a-b)k} \times \frac{\pi}{2}$$
M1

$$(a-b)k = (a-b)\sqrt{\frac{a+b}{a-b}} = \sqrt{(a+b)(a-b)}, \text{ so the value is } \frac{\pi}{\sqrt{a^2-b^2}}$$
A1

Answer: $\frac{\pi}{\sqrt{a^2-b^2}}$ 7 marks

$$(b) \frac{\pi}{\sqrt{169-25}} = \frac{\pi}{12}$$
B1

The integrand has period 2π and is symmetric about π , so the integral over 0 to 2π is $\frac{\pi}{6}$ B1

Answer: $\frac{\pi}{12}$ and $\frac{\pi}{6}$ 2 marks

28 (10 marks)

$$(a) \sin x = \frac{2t}{1+t^2}, 1 + \sin x + \cos x = \frac{2(1+t)}{1+t^2}, dx = \frac{2}{1+t^2} dt$$
M1

$$\text{Integrand: } \frac{2t}{1+t^2} \times \frac{1+t^2}{2(1+t)} \times \frac{2}{1+t^2} = \frac{2t}{(1+t)(1+t^2)}$$
A1

$$\frac{2t}{(1+t)(1+t^2)} = -\frac{1}{1+t} + \frac{t+1}{1+t^2}$$
M1 A1

$$f = -\ln(1+t) + \frac{1}{2} \ln(1+t^2) + \arctan t$$
M1

$$\text{Limits 0 to 1: } -\ln 2 + \frac{1}{2} \ln 2 + \frac{\pi}{4} = \frac{\pi}{4} - \frac{1}{2} \ln 2$$
A1

Answer: $J = \frac{\pi}{4} - \frac{1}{2} \ln 2$ 6 marks

(b) Under $u = \frac{\pi}{2} - x$, $\sin x$ becomes $\cos u$, $\cos x$ becomes $\sin u$, $dx = -du$, and the limits swap M1

$$K = \int^{\pi/2} \frac{\sin u}{1 + \sin u + \cos u} du = J$$
A1

Answer: $K = J$ 2 marks

$$(c) J + K = \int^{\pi/2} \frac{(1 + \sin x + \cos x) - 1}{1 + \sin x + \cos x} dx = \frac{\pi}{2} - \ln 2$$
M1

So $2J = \frac{\pi}{2} - \ln 2$ and $J = \frac{\pi}{4} - \frac{1}{2} \ln 2$, as in part (a) A1

Answer: $J = \frac{\pi}{4} - \frac{1}{2} \ln 2$ 2 marks

29 (11 marks)

$$(a) 1 + \cos x = \frac{2}{1+t^2}, \text{ so } \frac{1}{(1+\cos x)^n} = \frac{(1+t^2)^n}{2^n} \quad \text{B1}$$

$$dx = \frac{2}{1+t^2} dt, \text{ limits } t = 0 \text{ to } 1 \quad \text{M1}$$

$$I_n = 2^{1-n} \int_0^1 (1+t^2)^{n-1} dt = 2^{1-n} J_{n-1} \quad \text{A1}$$

$$\text{Answer: } I_n = 2^{1-n} J_{n-1} \quad \text{3 marks}$$

$$(b) I_1 = J_0 = 1 \quad \text{B1}$$

$$I_2 = \frac{1}{2} J_1 = \frac{1}{2} \left(1 + \frac{1}{3}\right) = \frac{2}{3} \quad \text{B1}$$

$$\text{Answer: } I_1 = 1, I_2 = \frac{2}{3} \quad \text{2 marks}$$

$$(c) J_n = [t(1+t^2)^n]_0^1 - \int_0^1 t \times 2nt(1+t^2)^{n-1} dt \quad \text{M1 A1}$$

$$t^2 = (1+t^2) - 1, \text{ so the last integral is } 2n(J_n - J_{n-1}) \quad \text{M1}$$

$$J_n = 2^n - 2n J_n + 2n J_{n-1}, \text{ so } (2n+1)J_n = 2^n + 2n J_{n-1} \quad \text{A1}$$

$$\text{Answer: } (2n+1)J_n = 2^n + 2n J_{n-1} \quad \text{4 marks}$$

$$(d) J_1 = \frac{4}{3}; 5J_2 = 4 + 4 \times \frac{4}{3}, \text{ so } J_2 = \frac{28}{15} \quad \text{M1}$$

$$I_3 = 2^{-2} J_2 = \frac{7}{15} \quad \text{A1}$$

$$\text{Answer: } \frac{7}{15} \quad \text{2 marks}$$

30 (8 marks)

$$2 + 3 \cos x = \frac{2 + 2t^2 + 3 - 3t^2}{1+t^2} = \frac{5-t^2}{1+t^2} \quad \text{M1 A1}$$

$$\text{Integrand: } \frac{2}{5-t^2} dt, \text{ limits } 0 \text{ to } 1 \quad \text{A1}$$

$$\frac{2}{5-t^2} = \frac{1}{\sqrt{5}} \left(\frac{1}{\sqrt{5}+t} + \frac{1}{\sqrt{5}-t} \right) \quad \text{M1}$$

$$f = \frac{1}{\sqrt{5}} \ln \frac{\sqrt{5}+t}{\sqrt{5}-t} \quad \text{A1}$$

$$\text{Limits: } \frac{1}{\sqrt{5}} \ln \frac{\sqrt{5}+1}{\sqrt{5}-1} \quad \text{M1}$$

$$\frac{\sqrt{5}+1}{\sqrt{5}-1} = \frac{(\sqrt{5}+1)^2}{4} = \left(\frac{1+\sqrt{5}}{2} \right)^2 \quad \text{M1}$$

$$\text{So the value is } \frac{2}{\sqrt{5}} \ln \left(\frac{1+\sqrt{5}}{2} \right) \quad \text{A1}$$

$$\text{Answer: } \frac{2}{\sqrt{5}} \ln \left(\frac{1+\sqrt{5}}{2} \right) \quad \text{8 marks}$$