

Deriving the t-Formulae

- 1 Given that $t = \tan(x/2)$, write down expressions, in terms of t , for $\sin x$ and $\cos x$. [2]
- 2 Given that $\tan(x/2) = \frac{3}{4}$, find the values of $\sin x$ and $\cos x$. [2]
- 3 Given that $\tan(x/2) = \frac{1}{3}$, use $\tan x = \frac{2t}{1-t^2}$ to find the value of $\tan x$. [2]
- 4 Given that $t = \tan(x/2)$, show that $1 + \cos x = \frac{2}{1+t^2}$. [2]
- 5 Given that $t = \tan(x/2)$, express $\sec x$ in terms of t . [2]
- 6 Given that $\tan(x/2) = -2$, find the value of $\cos x$. [2]
- 7 Starting from $\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2}$, show that $\sin x = \frac{2t}{1+t^2}$, where $t = \tan \frac{x}{2}$. [3]
- 8 Starting from $\cos x = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}$, show that $\cos x = \frac{1-t^2}{1+t^2}$, where $t = \tan \frac{x}{2}$. [3]
- 9 Use the identity $\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$ to show that $\tan x = \frac{2t}{1-t^2}$, where $t = \tan \frac{x}{2}$. [2]
- 10 Given that $t = \tan(x/2)$, express $3 \sin x + 4 \cos x$ as a single fraction in terms of t . [3]
- 11 The angle x is acute and $\cos x = \frac{3}{5}$. Use a t-formula to find the exact value of $\tan(x/2)$. [3]
- 12 Use the formula for $\tan x$ in terms of $t = \tan(x/2)$, with $x = 45^\circ$, to find the exact value of $\tan 22.5^\circ$. [4]
- 13 Given that $t = \tan(x/2)$, use $\cos x = \frac{1-t^2}{1+t^2}$ to show that $t^2 = \frac{1 - \cos x}{1 + \cos x}$. [3]
- 14 Given that $t = \tan(x/2)$, show that
 - (a) $1 + \sin x = \frac{(1+t)^2}{1+t^2}$ [3]
 - (b) $1 + \sin x + \cos x = \frac{2(1+t)}{1+t^2}$ [3]
- 15 Given that $t = \tan(x/2)$:
 - (a) show that $4 - 3 \cos x = \frac{1+7t^2}{1+t^2}$. [3]
 - (b) Hence find the least value of $4 - 3 \cos x$, and state the value of t at which it occurs. [2]
- 16 The angle x is obtuse and $\sin x = \frac{12}{13}$. Find the exact value of $\tan(x/2)$. [4]
- 17 Given that $u = \tan 2\theta$, show that $\sin 4\theta + 2 \cos 4\theta - 1 = \frac{(1-u)(1+3u)}{1+u^2}$. [4]

18 Given that $t = \tan(x/2)$:

(a) show that $\sec x - 1 = \frac{2t^2}{1 - t^2}$. [3]

(b) Hence find the value of $\sec x - 1$ when $\tan(x/2) = \frac{1}{2}$. [1]

19 Given that $t = \tan(x/2)$:

(a) show that $t + \frac{1}{t} = \frac{2}{\sin x}$. [3]

(b) Hence prove that $\tan(x/2) + \cot(x/2) \geq 2$ for $0 < x < \pi$. [2]

20 Given that $t = \tan(x/2)$, show that $\cot x + t = \operatorname{cosec} x$. [4]

21 Given that $t = \tan(x/2)$, prove that $\frac{1 + \sin x - \cos x}{1 + \sin x + \cos x} = t$. [6]

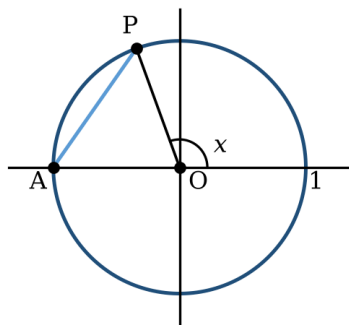
22 Let $t = \tan(x/2)$.

(a) Use the formula for $\tan(A + B)$ to show that $\tan(\frac{\pi}{4} + \frac{x}{2}) = \frac{1 + t}{1 - t}$. [3]

(b) Hence show that $\sec x + \tan x = \tan(\frac{\pi}{4} + \frac{x}{2})$. [4]

23 Given that $t = \tan(x/2)$, show that $\cos x + \cos 2x = \frac{2(1 - 3t^2)}{(1 + t^2)^2}$. [6]

24 The diagram shows the unit circle with centre O. The point A is $(-1, 0)$ and the point P($\cos x$, $\sin x$) lies on the circle with $0 < x < \pi$. The line AP has gradient t .



(a) The line AP meets the circle again at P. By solving the equation of AP with the equation of the circle, show that P has coordinates $(\frac{1 - t^2}{1 + t^2}, \frac{2t}{1 + t^2})$. [4]

(b) Hence, using a circle theorem, explain why $t = \tan(x/2)$. [2]

25 Let $t = \tan(x/2)$.

(a) Show that $t = \frac{\sin x}{1 + \cos x} = \frac{1 - \cos x}{\sin x}$. [3]

(b) Use $x = 30^\circ$ to find the exact value of $\tan 15^\circ$. [2]

(c) Use $x = 225^\circ$ to find the exact value of $\tan 112.5^\circ$. [3]

26 Let $t = \tan(x/2)$, where t can take any real value.

(a) Show that $1 - \sin x = \frac{(1 - t)^2}{1 + t^2}$ and write down a similar expression for $1 + \sin x$. Hence explain why $-1 \leq \sin x \leq 1$. [3]

(b) State the value of x in the interval $0 \leq x \leq 2\pi$ for which t is not defined, and give the values of $\sin x$ and $\cos x$ there. [2]

(c) Show that there is no real value of t for which $\cos x = -1$, and explain how this is consistent with part (b). [2]

- 27** A, B and C are the angles of a triangle, and $a = \tan(A/2)$, $b = \tan(B/2)$, $c = \tan(C/2)$.
- (a) Show that $ab + bc + ca = 1$. [7]
- (b) In one such triangle $a = \frac{1}{2}$ and $b = \frac{1}{3}$. Find c , and hence find the size of angle C. [2]
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- 28** Let $t = \tan(x/2)$, where $\cos(x/2) \neq 0$.
- (a) Show that $\frac{(1 + it)^2}{1 + t^2} = \cos x + i \sin x$. [4]
- (b) Hence deduce the expressions for $\cos x$ and $\sin x$ in terms of t . [2]
- (c) Use de Moivre's theorem to find an expression for $\sin 3x$ in terms of t . [4]
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- 29** Let $t = \tan(x/2)$, where $t = \frac{p}{q}$ and p and q are positive integers with $p < q$.
- (a) Show that $\sin x = \frac{2pq}{p^2 + q^2}$ and $\cos x = \frac{q^2 - p^2}{p^2 + q^2}$. [2]
- (b) Given that $\sin x = \frac{21}{29}$ and $\cos x = \frac{20}{29}$, find t as a fraction in its lowest terms. [2]
- (c) Find the three integers ($q^2 - p^2$, $2pq$, $p^2 + q^2$) obtained when $t = \frac{4}{9}$, and verify that they form a Pythagorean triple. [2]
- (d) Explain why, if $\sin x$ and $\cos x$ are both rational, then $\tan(x/2)$ is rational. [2]
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- 30** Let $t = \tan(x/2)$ and $s = \tan(x/4)$, where $0 < x < \pi$.
- (a) Show that $t = \frac{2s}{1 - s^2}$. [1]
- (b) Hence show that $s = \frac{\sqrt{1 + t^2} - 1}{t}$. [4]
- (c) Use $x = \pi/2$ to find the exact value of $\tan(\pi/8)$. [2]
- (d) Verify the formula in part (b) when $x = 2\pi/3$. [2]

Solving Trigonometric Equations with t

- 1** Use the substitution $t = \tan(x/2)$ to show that the equation $\sin x + \cos x = 1$ can be written as $t(1 - t) = 0$. [3]
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- 2** Solve $\tan(x/2) = -\sqrt{3}$ for $0 \leq x < 2\pi$, giving your answer as a multiple of π . [2]
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- 3** Given that $t = \tan(x/2)$, solve $3t^2 - 4t + 1 = 0$ and hence find the values of x , in radians, in the interval $0 \leq x < 2\pi$. Give each answer exactly or to 3 significant figures. [3]
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- 4** A student uses $t = \tan(x/2)$ to solve a trigonometric equation for $0 \leq x < 2\pi$. Explain why the student must also test $x = \pi$ in the original equation. [2]
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- 5** Use the substitution $t = \tan(x/2)$ to show that $\cos x - 3 \sin x + 2 = 0$ can be written as $t^2 - 6t + 3 = 0$. [3]
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- 6** Given that $\tan(x/2) = 2$ and $0^\circ \leq x < 360^\circ$, find x to 1 decimal place. [2]
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- 7** Use the substitution $t = \tan(x/2)$ to solve $2 \sin x + \cos x = 1$ for $0^\circ \leq x < 360^\circ$. Give your answers to 1 decimal place. [4]
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- 8** Solve $3 \sin x - \cos x = 2$ for $0^\circ \leq x < 360^\circ$, giving your answers to 1 decimal place. [5]
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- 9** Solve $3 \cos x + \sin x = 2$ for $0^\circ \leq x < 360^\circ$, giving your answers to 1 decimal place. [5]

- 10** Solve $2 \sin x - 3 \cos x = 1$ for $0 \leq x < 2\pi$, giving your answers in radians to 3 significant figures. [5]
- 11** Solve $2 \cos x + 3 \sin x + 1 = 0$ for $-180^\circ < x < 180^\circ$, giving your answers to 1 decimal place. [5]
- 12** Solve $\sin x + 7 \cos x = 5$ for $0^\circ \leq x < 360^\circ$, giving your answers to 1 decimal place. [5]
- 13** Use $t = \tan(x/2)$ to solve $\tan(x/2) = 2 \sin x$ for $0^\circ \leq x < 360^\circ$. [4]
- 14** Solve $4 \cos x - 3 \sin x = 2$ for $0^\circ \leq x < 360^\circ$, giving your answers to 1 decimal place. [6]
- 15** Show that the equation $3 \cos x + 4 \sin x = 5$ has exactly one solution in the interval $0^\circ \leq x < 360^\circ$, and find it to 1 decimal place. [5]
- 16** A student solves $2 \sin x + 3 \cos x + 3 = 0$ for $0^\circ \leq x < 360^\circ$ using $t = \tan(x/2)$ and finds only $x = 247.4^\circ$.
 (a) Show that the substitution leads to a linear equation in t and find its solution. [3]
 (b) Find the solution that the student missed, and explain why it was missed. [3]
- 17** Use the substitution $t = \tan(x/2)$ to solve $\sec x + \tan x = 3$ for $0 \leq x < 2\pi$. Give your answer in radians to 3 significant figures. [5]
- 18** Show that the equation $2 \sin x + 3 \cos x = 6$ has no solutions. [4]
- 19** Use $t = \tan(x/2)$ to find all solutions of the equation $\sin x = 3 \tan(x/2)$ in the interval $0 \leq x < 2\pi$. [5]
- 20** The height, h metres, above the ground of a passenger on a ride is modelled by $h = 20 + 5 \sin \phi + 12 \cos \phi$, where ϕ radians is the angle through which the ride has turned and $0 \leq \phi < 2\pi$.
 Use the substitution $t = \tan(\phi/2)$ to find the values of ϕ for which the passenger is at a height of 28 m. Give your answers to 3 significant figures. [6]
- 21** Solve $3 \sin x + 4 \cos x = 4$ for $0^\circ \leq x < 360^\circ$ using two different methods:
 (a) the substitution $t = \tan(x/2)$, [4]
 (b) writing $3 \sin x + 4 \cos x$ in the form $R \sin(x + \alpha)$. [3]
- 22** Use the substitution $t = \tan x$ to solve $\sin 2x + 3 \cos 2x = 1$ for $0^\circ \leq x < 180^\circ$. Give non-exact answers to 1 decimal place. [6]
- 23** Solve $2 \sin 3x + \cos 3x = 2$ for $0^\circ \leq x < 180^\circ$, giving your answers to 1 decimal place. [6]
- 24** Use $t = \tan(x/2)$ to solve the equation $\cos x + \tan(x/2) = 1$ for $0 \leq x < 2\pi$, $x \neq \pi$. Give your answers exactly. [6]
- 25** The equation $2 \sin x + \cos x = k$, where k is a constant, is to be solved for $0 \leq x < 2\pi$.
 (a) Show that, with $t = \tan(x/2)$, the equation becomes $(k + 1)t^2 - 4t + (k - 1) = 0$. [2]
 (b) Hence show that, for $k \neq -1$, the equation has real solutions only if $-\sqrt{5} \leq k \leq \sqrt{5}$. [3]
 (c) Show that when $k = -1$ there are two solutions in the interval, and find them in radians to 3 significant figures. [3]
- 26** Solve $\operatorname{cosec} x + 2 \cot x = 3$ for $0^\circ < x < 360^\circ$, $x \neq 180^\circ$. Give your answers to 1 decimal place. [6]

- 27** The equation $a \sin x + b \cos x = c$, where $b + c \neq 0$, has two solutions α and β in the interval $0^\circ \leq x < 360^\circ$.
- (a) Show that $\tan\left(\frac{\alpha + \beta}{2}\right) = \frac{a}{b}$. [6]
- (b) For $5 \sin x + 12 \cos x = 6$, use the result of part (a) to show that $\alpha + \beta = 360^\circ + 2 \arctan \frac{5}{12}$, where $\alpha < \beta$, and check this against the values of α and β . [3]
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- 28** Let $t = \tan(x/2)$.
- (a) Show that $\sin 2x = \frac{4t(1 - t^2)}{(1 + t^2)^2}$. [3]
- (b) Hence show that $\sin x + \sin 2x = \frac{2t(3 - t^2)}{(1 + t^2)^2}$. [3]
- (c) Hence solve $\sin x + \sin 2x = 0$ for $0^\circ \leq x < 360^\circ$, explaining how you deal with $x = 180^\circ$. [3]
- (d) Verify your answers by solving $\sin x(1 + 2 \cos x) = 0$. [1]
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- 29** Use the substitution $t = \tan(x/2)$ to solve $2 \sin x + 3 \cos x = 2 \tan(x/2)$ for $0^\circ \leq x < 360^\circ$, $x \neq 180^\circ$. Give your answers to 1 decimal place where necessary.
- (a) Show that the equation becomes $2t^3 + 3t^2 - 2t - 3 = 0$. [3]
- (b) Hence solve the equation. [5]
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- 30** Find the general solution, in radians, of the equation $4 \sin x - 3 \cos x = 3$.
- (a) Show that the substitution $t = \tan(x/2)$ gives a linear equation in t , and solve it. [4]
- (b) Hence show that the general solution is $x = 2 \arctan \frac{3}{4} + 2n\pi$ or $x = (2n + 1)\pi$, where n is an integer, and explain why the second family must be checked separately. [4]

Integration Using $t = \tan(x/2)$

- 1** Given that $t = \tan(x/2)$, show that $dx = \frac{2}{1 + t^2} dt$. [2]
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- 2** The substitution $t = \tan(x/2)$ is used in an integral with limits $x = \frac{\pi}{3}$ and $x = \frac{2\pi}{3}$. Find the corresponding limits for t , giving exact values. [2]
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- 3** Use the substitution $t = \tan(x/2)$ to show that $\int \frac{1}{1 + \cos x} dx = \tan \frac{x}{2} + c$. [3]
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- 4** Use the substitution $t = \tan(x/2)$ to find $\int \frac{1}{\sin x} dx$. [3]
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- 5** Use the substitution $t = \tan(x/2)$ to evaluate $\int_0^{\pi/2} \frac{1}{1 + \sin x} dx$. [3]
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- 6** Use the substitution $t = \tan(x/2)$ to show that $\int \frac{1}{1 - \cos x} dx = -\cot \frac{x}{2} + c$. [3]
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- 7** Use the substitution $t = \tan(x/2)$ to find $\int \frac{1}{2 + \cos x} dx$. [4]
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- 8** Use the substitution $t = \tan(x/2)$ to show that $\int_0^{\pi/2} \frac{1}{1 + \sin x + \cos x} dx = \ln 2$. [4]
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- 9** Use the substitution $t = \tan(x/2)$ to show that $\int_0^{\pi/2} \frac{1}{5 + 3 \cos x} dx = \frac{1}{2} \arctan \frac{1}{2}$. [5]

- 10** Use the substitution $t = \tan(x/2)$ to find $\int \sec x \, dx$, giving your answer in terms of t . [4]
- 11** Use the substitution $t = \tan(x/2)$ to show that $\int_0^{\pi/2} \frac{1}{2 - \cos x} \, dx = \frac{2\pi}{3\sqrt{3}}$. [5]
- 12** Use the substitution $t = \tan(x/2)$ to find the exact value of $\int_0^{\pi/4} \frac{1}{1 - \sin x} \, dx$. [5]
- 13** Use the substitution $t = \tan(x/2)$ to show that $\int_0^{\pi/3} \frac{1 - \cos x}{\sin x} \, dx = \ln \frac{4}{3}$. [4]
- 14** Use the substitution $t = \tan(x/2)$ to evaluate $\int_0^{\pi/2} \frac{1}{3 + \sin x} \, dx$, giving your answer to 3 significant figures. [6]
- 15** Use the substitution $t = \tan(x/2)$ to show that $\int_0^{\pi/2} \frac{1}{2 + 2 \sin x + \cos x} \, dx = \ln \frac{3}{2}$. [6]
- 16** Use the substitution $t = \tan(x/2)$ to show that $\int_{\pi/3}^{\pi/2} \frac{1}{1 + \sin x - \cos x} \, dx = \ln\left(\frac{\sqrt{3} + 1}{2}\right)$. [6]
- 17** Use the substitution $t = \tan(x/2)$ to show that $\int_0^{\pi} \frac{1}{3 + 2 \cos x} \, dx = \frac{\pi}{\sqrt{5}}$. State what happens to t as $x \rightarrow \pi$. [6]
- 18** Use the substitution $t = \tan(x/2)$ to show that $\int_0^{\pi/2} \frac{1}{1 + 2 \cos x} \, dx = \frac{1}{\sqrt{3}} \ln(2 + \sqrt{3})$. [6]
- 19** Use the substitution $t = \tan(x/2)$ to show that $\int_0^{\pi/2} \frac{1}{4 + 5 \sin x} \, dx = \frac{1}{3} \ln 2$. [6]
- 20** Show that $\int_0^{\pi/2} \frac{1}{3 \cos x + 4 \sin x} \, dx = \frac{1}{5} \ln 6$. [7]
- 21** Use the substitution $t = \tan(x/2)$ to show that $\int_0^{\pi/2} \frac{1}{(1 + \sin x + \cos x)^2} \, dx = 1 - \ln 2$. [7]
- 22** Use the substitution $t = \tan(x/2)$ to show that $\int_0^{\pi/2} \frac{1}{2 \sin x + \cos x + 3} \, dx = \arctan \frac{1}{3}$. [6]
- 23** Use the substitution $t = \tan(x/2)$ to find the exact value of $\int_0^{\pi/2} \frac{1}{3 \sin x + 4 \cos x + 5} \, dx$. [6]
- 24** Use the substitution $t = \tan x$ to show that $\int_0^{\pi/2} \frac{1}{5 + 4 \cos 2x} \, dx = \frac{\pi}{6}$. [6]
- 25** The region R is bounded by the curve $y = \frac{1}{1 + \cos x}$, the x -axis, the y -axis and the line $x = \frac{\pi}{2}$. The region R is rotated through 2π radians about the x -axis. Use the substitution $t = \tan(x/2)$ to show that the volume of the solid formed is $\frac{2\pi}{3}$. [6]
- 26** A student evaluates $\int_0^{2\pi} \frac{1}{2 + \cos x} \, dx$ using $t = \tan(x/2)$. The limits for t are $\tan 0 = 0$ and $\tan \pi = 0$, so the student concludes that the integral is 0.
 (a) Explain what is wrong with this method, and why the answer 0 cannot be correct. [2]
 (b) Find the correct value of the integral. [4]

27 Throughout this question a and b are constants with $a > b > 0$.

(a) Use the substitution $t = \tan(x/2)$ to prove that $\int_0^\pi \frac{1}{a + b \cos x} dx = \frac{\pi}{\sqrt{a^2 - b^2}}$. [7]

(b) Hence write down the value of $\int_0^\pi \frac{1}{13 + 5 \cos x} dx$, and deduce the value of $\int_0^{2\pi} \frac{1}{13 + 5 \cos x} dx$. [2]

28 Let $J = \int_0^{\pi/2} \frac{\sin x}{1 + \sin x + \cos x} dx$ and $K = \int_0^{\pi/2} \frac{\cos x}{1 + \sin x + \cos x} dx$.

(a) Use the substitution $t = \tan(x/2)$ to show that $J = \int_0^1 \frac{2t}{(1+t)(1+t^2)} dt$, and hence find J exactly. [6]

(b) Use the substitution $u = \frac{\pi}{2} - x$ to show that $K = J$. [2]

(c) By considering $J + K$, and using the result $\int_0^{\pi/2} \frac{1}{1 + \sin x + \cos x} dx = \ln 2$, verify your answer to part (a). [2]

29 For positive integers n , let $I_n = \int_0^{\pi/2} \frac{1}{(1 + \cos x)^n} dx$ and $J_n = \int_0^1 (1 + t^2)^n dt$.

(a) Use the substitution $t = \tan(x/2)$ to show that $I_n = 2^{1-n} J_{n-1}$. [3]

(b) Find I_1 and I_2 . [2]

(c) By integrating by parts, or otherwise, show that $(2n + 1)J_n = 2^n + 2n J_{n-1}$. [4]

(d) Hence find the exact value of I_3 . [2]

30 Use the substitution $t = \tan(x/2)$ to show that $\int_0^{\pi/2} \frac{1}{2 + 3 \cos x} dx = \frac{2}{\sqrt{5}} \ln\left(\frac{1 + \sqrt{5}}{2}\right)$. [8]