

Proof by Mathematical Induction

1 (2 marks)

$n = 1$: LHS = 1, RHS = $1^2 = 1$, so true

B1

$n = 2$: LHS = $1 + 3 = 4$, RHS = $2^2 = 4$, so true

B1

Answer: P(1) and P(2) are both true

2 marks

2 (2 marks)

Replace n by $k + 1$ on both sides

M1

$$\sum_{r=1}^{k+1} r^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

A1

Answer: $\sum_{r=1}^{k+1} r^2 = \frac{(k+1)(k+2)(2k+3)}{6}$

2 marks

3 (3 marks)

$u_2 = 7$, $u_3 = 11$, $u_4 = 15$

B1 B1

The terms go up by 4 from 3, so $u_n = 4n - 1$

B1

Answer: $u_n = 4n - 1$

3 marks

4 (2 marks)

The statement is true for $n = 1$, and if it is true for $n = k$ then it is true for $n = k + 1$

E1

Therefore, by mathematical induction, it is true for all positive integers n

E1

Answer: True for $n = 1$, and if true for $n = k$ then true for $n = k + 1$; therefore, by mathematical induction, true for all

5 (2 marks)

$$\begin{aligned} \sum_{r=1}^{k+1} r^3 &= \sum_{r=1}^k r^3 + (k+1)^3 \\ &= \frac{k^2(k+1)^2}{4} + (k+1)^3 \end{aligned}$$

M1

A1

Answer: $\frac{k^2(k+1)^2}{4} + (k+1)^3$

2 marks

6 (3 marks)

$$u_2 = 4 = 2^2$$

B1

$$u_3 = 8 = 2^3$$

B1

$u_1 = 2 = 2^1$, so the formula works for $n = 1, 2, 3$

B1

Answer: the formula $u_n = 2^n$ is correct for $n = 1, 2$ and 3

3 marks

7 (4 marks)

$n = 1$: LHS = $1 = 1^2$, true

B1

Assume true for $n = k$: $\sum_{r=1}^k (2r - 1) = k^2$. Then $\sum_{r=1}^{k+1} (2r - 1) = k^2 + 2(k+1) - 1$

M1

$$= k^2 + 2k + 1 = (k+1)^2$$

A1

True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction

A1

Answer: $\sum_{r=1}^n (2r - 1) = n^2$ for all integers $n \geq 1$

4 marks

8 (4 marks)

$$n = 1: \text{LHS} = 1, \text{RHS} = \frac{1 \times 2}{2} = 1, \text{true}$$

B1

$$\text{Assume true for } n = k. \text{ Then } \sum_{r=1}^{k+1} r = \frac{k(k+1)}{2} + (k+1)$$

M1

$$= \frac{(k+1)(k+2)}{2}, \text{ which is the formula with } n = k+1$$

A1

True for $n = 1$, and true for $n = k$ implies true for $n = k+1$, so true for all integers $n \geq 1$ by induction

A1

$$\text{Answer: } \sum_{r=1}^n r = \frac{n(n+1)}{2} \text{ for all integers } n \geq 1$$

4 marks**9 (4 marks)**

$$n = 1: \text{LHS} = 1, \text{RHS} = 2 - 1 = 1, \text{true}$$

B1

$$\text{Assume true for } n = k. \text{ Sum of } k+1 \text{ terms} = (2^k - 1) + 2^k$$

M1

$$= 2 \times 2^k - 1 = 2^{k+1} - 1$$

A1

True for $n = 1$, and true for $n = k$ implies true for $n = k+1$, so true for all integers $n \geq 1$ by induction

A1

$$\text{Answer: } 1 + 2 + 4 + \dots + 2^{n-1} = 2^n - 1 \text{ for all integers } n \geq 1$$

4 marks**10 (4 marks)**

$$n = 1: u_1 = 5 = 3 \times 1 + 2, \text{true}$$

B1

$$\text{Assume } u_k = 3k + 2. \text{ Then } u_{k+1} = u_k + 3 = 3k + 2 + 3$$

M1

$$= 3(k+1) + 2$$

A1

True for $n = 1$, and true for $n = k$ implies true for $n = k+1$, so true for all integers $n \geq 1$ by induction

A1

$$\text{Answer: } u_n = 3n + 2$$

4 marks**11 (4 marks)**

$$n = 1: u_1 = 4 = 3^1 + 1, \text{true}$$

B1

$$\text{Assume } u_k = 3^k + 1. \text{ Then } u_{k+1} = 3(3^k + 1) - 2$$

M1

$$= 3^{k+1} + 3 - 2 = 3^{k+1} + 1$$

A1

True for $n = 1$, and true for $n = k$ implies true for $n = k+1$, so true for all integers $n \geq 1$ by induction

A1

$$\text{Answer: } u_n = 3^n + 1 \text{ for all } n \geq 1$$

4 marks**12 (5 marks)**

$$n = 1: \text{LHS} = \frac{1}{1 \times 2} = \frac{1}{2}, \text{RHS} = \frac{1}{2}, \text{true}$$

B1

$$\text{Assume true for } n = k. \text{ Then sum to } k+1 = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$

M1

$$= \frac{k(k+2) + 1}{(k+1)(k+2)}$$

M1

$$= \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

A1

True for $n = 1$, and true for $n = k$ implies true for $n = k+1$, so true for all integers $n \geq 1$ by induction

A1

$$\text{Answer: } \sum_{r=1}^n \frac{1}{r(r+1)} = \frac{n}{n+1} \text{ for all integers } n \geq 1$$

5 marks

13 (5 marks)

$n = 1$: LHS = $1 \times 1! = 1$, RHS = $2! - 1 = 1$, true

B1

Assume true for $n = k$. Then sum to $k + 1 = (k + 1)! - 1 + (k + 1)(k + 1)!$

M1

$$= (k + 1)![1 + (k + 1)] - 1$$

M1

$$= (k + 1)!(k + 2) - 1 = (k + 2)! - 1$$

A1

True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction

A1

Answer: $\sum_{r=1}^n r \times r! = (n + 1)! - 1$ for all integers $n \geq 1$

5 marks

14 (6 marks)

$n = 1$: LHS = 2, RHS = $0 + 2 = 2$, true

B1

Assume true for $n = k$. Then sum to $k + 1 = (k - 1)2^{k+1} + 2 + (k + 1)2^{k+1}$

M1 A1

$$= 2^{k+1}[(k - 1) + (k + 1)] + 2$$

M1

$$= 2k \times 2^{k+1} + 2 = k \times 2^{k+2} + 2, \text{ which is the formula with } n = k + 1$$

A1

True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction

A1

Answer: $\sum_{r=1}^n r \times 2^r = (n - 1)2^{n+1} + 2$ for all integers $n \geq 1$

6 marks

15 (6 marks)

$n = 2$: LHS = $1 - \frac{1}{4} = \frac{3}{4}$, RHS = $\frac{3}{4}$, true

B1

Assume true for $n = k$. Then the product to $k + 1 = \frac{k + 1}{2k} \times (1 - \frac{1}{(k + 1)^2})$

M1

$$1 - \frac{1}{(k + 1)^2} = \frac{k(k + 2)}{(k + 1)^2}$$

M1

$$\text{Product} = \frac{k + 1}{2k} \times \frac{k(k + 2)}{(k + 1)^2}$$

A1

$$= \frac{k + 2}{2(k + 1)}, \text{ which is the formula with } n = k + 1$$

A1

True for $n = 2$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 2$

A1

Answer: $(1 - \frac{1}{2^2})(1 - \frac{1}{3^2}) \dots (1 - \frac{1}{n^2}) = \frac{n + 1}{2n}$ for all integers $n \geq 2$

6 marks

16 (5 marks)

$n = 1$: derivative of $x e^x$ is $e^x + x e^x = (x + 1)e^x$, true

B1

Assume the k th derivative is $(x + k)e^x$

M1

Differentiate: $(1 + x + k)e^x$ using the product rule

M1 A1

$= (x + (k + 1))e^x$, the result for $n = k + 1$. True for $n = 1$ and true for $n = k$ implies true for $n = k + 1$, so true for all $n \geq 1$

A1

Answer: the n th derivative of $x e^x$ is $(x + n)e^x$, for all integers $n \geq 1$

5 marks

17 (7 marks)

$n = 1$: $u_1 = 1$ and $1 + \frac{1 \times 0 \times 1}{6} = 1$, true

B1

Assume $u_k = 1 + \frac{k(k - 1)(2k - 1)}{6}$. Then $u_{k+1} = 1 + \frac{k(k - 1)(2k - 1)}{6} + k^2$

M1 A1

$$= 1 + \frac{k[(k - 1)(2k - 1) + 6k]}{6}$$

M1

$$(k - 1)(2k - 1) + 6k = 2k^2 + 3k + 1 = (2k + 1)(k + 1)$$

M1

$$\text{So } u_{k+1} = 1 + \frac{(k + 1)k(2k + 1)}{6}, \text{ the formula with } n = k + 1$$

A1

True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction

A1

Answer: $u_n = 1 + \frac{n(n - 1)(2n - 1)}{6}$ for all integers $n \geq 1$

7 marks

18 (4 marks)

The proof has no basis case: it never checks a starting value

E1

$n = 1$: $1^2 + 1 = 2$ is even, so the statement fails at $n = 1$

B1

In fact $n^2 + n = n(n + 1)$ is a product of consecutive integers, so it is always even

M1

So the result is false for every n , although the inductive step itself is valid

A1

Answer: No basis case; the result is false (always even)

4 marks

19 (6 marks)

$$n = 1: u_1 = \frac{3}{2} = \frac{1+2}{1+1}, \text{ true}$$

B1

$$\text{Assume } u_k = \frac{k+2}{k+1}. \text{ Then } 2u_k - 1 = \frac{2(k+2) - (k+1)}{k+1} = \frac{k+3}{k+1}$$

M1 A1

$$u_{k+1} = \frac{k+3}{k+1} \div \frac{k+2}{k+1}$$

M1

$$= \frac{k+3}{k+2} = \frac{(k+1)+2}{(k+1)+1}, \text{ the formula with } n = k+1$$

A1

True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction

A1

$$\text{Answer: } u_n = \frac{n+2}{n+1}$$

6 marks

20 (5 marks)

$$n = 1: \text{LHS} = \frac{1}{1 \times 3} = \frac{1}{3}, \text{ RHS} = \frac{1}{3}, \text{ true}$$

B1

$$\text{Assume true for } n = k. \text{ Sum to } k+1 = \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)}$$

M1

$$= \frac{k(2k+3) + 1}{(2k+1)(2k+3)} = \frac{2k^2 + 3k + 1}{(2k+1)(2k+3)}$$

M1

$$= \frac{(2k+1)(k+1)}{(2k+1)(2k+3)} = \frac{k+1}{2(k+1)+1}$$

A1

True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction

A1

$$\text{Answer: } \sum_{r=1}^n \frac{1}{(2r-1)(2r+1)} = \frac{n}{2n+1} \text{ for all integers } n \geq 1$$

5 marks

21 (7 marks)

$$n = 1: \text{LHS} = 1, \text{ RHS} = F_1 F_2 = 1, \text{ true}$$

B1

$$\text{Assume the sum to } F_k^2 \text{ equals } F_k F_{k+1}$$

M1

$$\text{Sum to } k+1 \text{ terms} = F_k F_{k+1} + F_{k+1}^2$$

M1 A1

$$= F_{k+1}(F_k + F_{k+1})$$

M1

$$= F_{k+1} F_{k+2} \text{ by the definition, which is the statement for } n = k+1$$

A1

True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction

A1

$$\text{Answer: } F_1^2 + F_2^2 + \dots + F_n^2 = F_n F_{n+1} \text{ for all integers } n \geq 1$$

7 marks

22 (6 marks)

$$n = 1: \frac{2x - 1}{x - 0} = 2 - \frac{1}{x} = f(x), \text{ true} \quad \text{B1}$$

$$\text{Assume } f^k(x) = \frac{(k+1)x - k}{kx - (k-1)}. \text{ Then } f^{k+1}(x) = f(f^k(x)) = 2 - \frac{kx - (k-1)}{(k+1)x - k} \quad \text{M1 A1}$$

$$= \frac{2(k+1)x - 2k - kx + (k-1)}{(k+1)x - k} \quad \text{M1}$$

$$= \frac{(k+2)x - (k+1)}{(k+1)x - k} \quad \text{A1}$$

This is the formula with $n = k + 1$, since $(k + 1) + 1 = k + 2$ and $(k + 1) - 1 = k$. True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction A1

$$\text{Answer: } f^n(x) = \frac{(n+1)x - n}{nx - (n-1)} \quad \text{6 marks}$$

23 (8 marks)

$$(a) n = 1: \text{derivative} = e^{-x} - x e^{-x} = -(x - 1)e^{-x} = (-1)^1(x - 1)e^{-x}, \text{ true} \quad \text{B1}$$

$$\text{Assume the } k\text{th derivative is } (-1)^k(x - k)e^{-x} \quad \text{M1}$$

$$\text{Differentiate: } (-1)^k[e^{-x} - (x - k)e^{-x}] \quad \text{M1 A1}$$

$$= (-1)^k(1 + k - x)e^{-x} = (-1)^{k+1}(x - (k + 1))e^{-x}, \text{ which is } P(k + 1) \quad \text{A1}$$

True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction A1

$$\text{Answer: the } n\text{th derivative of } x e^{-x} \text{ is } (-1)^n(x - n)e^{-x} \text{ for all integers } n \geq 1 \quad \text{6 marks}$$

$$(b) (-1)^{10}(3 - 10)e^{-3} \quad \text{M1}$$

$$= -7e^{-3} \quad \text{A1}$$

$$\text{Answer: } -7e^{-3} \quad \text{2 marks}$$

24 (7 marks)

$$n = 1: \text{RHS} = \frac{\sin 2x}{2 \sin x} = \frac{2 \sin x \cos x}{2 \sin x} = \cos x = \text{LHS}, \text{ true} \quad \text{B1}$$

$$\text{Assume the product of the first } k \text{ factors is } \frac{\sin(2^k x)}{2^k \sin x}. \text{ Then the product of } k + 1 \text{ factors is } \frac{\sin(2^k x)}{2^k \sin x} \quad \text{M1 A1}$$

$$\times \cos(2^k x)$$

$$\sin(2^k x) \cos(2^k x) = \frac{1}{2} \sin(2 \times 2^k x) = \frac{1}{2} \sin(2^{k+1} x) \quad \text{M1 A1}$$

$$\text{So the product} = \frac{\sin(2^{k+1} x)}{2 \times 2^k \sin x} = \frac{\sin(2^{k+1} x)}{2^{k+1} \sin x}, \text{ the formula with } n = k + 1 \quad \text{A1}$$

True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction A1

$$\text{Answer: } \cos x \cos 2x \cos 4x \dots \cos(2^{n-1}x) = \frac{\sin(2^n x)}{2^n \sin x} \quad \text{7 marks}$$

25 (8 marks)

$$n = 1: \text{LHS} = \cos \theta, \text{ RHS} = \frac{\sin 2\theta}{2 \sin \theta} = \frac{2 \sin \theta \cos \theta}{2 \sin \theta} = \cos \theta, \text{ true} \quad \text{B1}$$

$$\text{Assume true for } n = k. \text{ Sum to } k + 1 \text{ terms} = \frac{\sin 2k\theta}{2 \sin \theta} + \cos(2k + 1)\theta \quad \text{M1 A1}$$

$$= \frac{\sin 2k\theta + 2 \sin \theta \cos(2k + 1)\theta}{2 \sin \theta} \quad \text{M1}$$

$$2 \sin \theta \cos(2k + 1)\theta = \sin(2k + 2)\theta - \sin 2k\theta \text{ (product to sum)} \quad \text{M1 A1}$$

$$\text{So the sum} = \frac{\sin(2k + 2)\theta}{2 \sin \theta} = \frac{\sin 2(k + 1)\theta}{2 \sin \theta} \quad \text{A1}$$

True for $n = 1$, true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ A1

$$\text{Answer: } \cos \theta + \cos 3\theta + \dots + \cos(2n - 1)\theta = \frac{\sin 2n\theta}{2 \sin \theta} \text{ for all integers } n \geq 1 \quad \text{8 marks}$$

26 (7 marks)

$n = 1$: both sides are $\cos \theta + i \sin \theta$, true

B1

Assume $(\cos \theta + i \sin \theta)^k = \cos k\theta + i \sin k\theta$

M1

$(\cos \theta + i \sin \theta)^{k+1} = (\cos k\theta + i \sin k\theta)(\cos \theta + i \sin \theta)$

M1

$= (\cos k\theta \cos \theta - \sin k\theta \sin \theta) + i(\sin k\theta \cos \theta + \cos k\theta \sin \theta)$

M1 A1

$= \cos(k+1)\theta + i \sin(k+1)\theta$ using the compound angle formulae, which is the result for $n = k + 1$

A1

True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all positive integers n by induction

A1

Answer: $(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$ for all positive integers n

7 marks

27 (8 marks)

Check $n = 1$: $\frac{3-1}{2} = 1$ and $n = 2$: $\frac{9-1}{2} = 4$, both correct (two base cases are needed)

B1 B1

Assume the formula holds for $n = k$ and $n = k + 1$

M1

$$u_{k+2} = 4 \times \frac{3^{k+1} - 1}{2} - 3 \times \frac{3^k - 1}{2}$$

M1 A1

$$= \frac{4 \times 3^{k+1} - 4 - 3^{k+1} + 3}{2}$$

M1

$$= \frac{3 \times 3^{k+1} - 1}{2} = \frac{3^{k+2} - 1}{2}$$

A1

True for $n = 1, 2$, and true for $n = k, k + 1$ implies true for $n = k + 2$, so true for all $n \geq 1$

A1

Answer: $u_n = \frac{3^n - 1}{2}$ for all integers $n \geq 1$

8 marks

28 (8 marks)

$n = 1$: $\frac{d}{dx}(e^x \cos x) = e^x(\cos x - \sin x)$

B1

$\cos x - \sin x = \sqrt{2} \cos(x + \frac{\pi}{4})$, so the derivative is $\sqrt{2} e^x \cos(x + \frac{\pi}{4})$, true

M1 A1

Assume the k th derivative is $(\sqrt{2})^k e^x \cos(x + \frac{k\pi}{4})$

M1

Differentiate: $(\sqrt{2})^k e^x [\cos(x + \frac{k\pi}{4}) - \sin(x + \frac{k\pi}{4})]$

M1 A1

Using $\cos A - \sin A = \sqrt{2} \cos(A + \frac{\pi}{4})$ with $A = x + \frac{k\pi}{4}$: this is $(\sqrt{2})^{k+1} e^x \cos(x + \frac{(k+1)\pi}{4})$, the result for $n = k + 1$

A1

True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction

B1

Answer: the n th derivative is $(\sqrt{2})^n e^x \cos(x + \frac{n\pi}{4})$

8 marks

29 (9 marks)

$n = 1$: LHS = 1, RHS = $\frac{1 \times 2 \times 3 \times 5}{30} = 1$, true

B1

Assume true for $n = k$. Sum to $k + 1 = \frac{k(k+1)(2k+1)(3k^2+3k-1)}{30} + (k+1)^4$

M1 A1

Take out the factor $(k+1)$: $\frac{(k+1)}{30} [k(2k+1)(3k^2+3k-1) + 30(k+1)^3]$

M1

$$k(2k+1)(3k^2+3k-1) = 6k^4 + 9k^3 + k^2 - k$$

M1

$$\text{Bracket} = 6k^4 + 39k^3 + 91k^2 + 89k + 30$$

A1

$= (k+2)(2k+3)(3k^2+9k+5)$, checked by expanding

M1

Since $3(k+1)^2 + 3(k+1) - 1 = 3k^2 + 9k + 5$, this is the formula with $n = k + 1$

A1

True for $n = 1$ and true for $n = k$ implies true for $n = k + 1$, so true for all $n \geq 1$

A1

Answer: $\sum_{r=1}^n r^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$ for all integers $n \geq 1$

9 marks

30 (8 marks)

(a) Base case: $n = 2$ is itself prime, so it is a product of one prime

B1

Assume that every integer m with $2 \leq m \leq k$ is a product of primes

M1

Consider $k + 1$. If $k + 1$ is prime, it is a product of one prime

B1

Otherwise $k + 1 = ab$ with $2 \leq a, b \leq k$

M1

By the assumption a and b are each products of primes

A1

So $k + 1 = ab$ is a product of primes. By strong induction, true for all $n \geq 2$

A1

Answer: every integer $n \geq 2$ is a product of one or more primes

6 marks

(b) The factors a and b of $k + 1$ are generally smaller than k , not equal to k

E1

So the assumption about k alone does not say anything about a and b : all smaller cases are needed

E1

Answer: $k + 1 = ab$ with a and b generally much smaller than k , so the case $n = k$ alone says nothing about a or b

Induction with Series**1 (3 marks)**

$n = 1$: LHS = 4, RHS = $1 \times 2^2 = 4$

B1

$n = 2$: LHS = $4 + 14 = 18$, RHS = $2 \times 3^2 = 18$

B1

$n = 3$: LHS = $18 + 30 = 48$, RHS = $3 \times 4^2 = 48$

B1

Answer: P(1), P(2) and P(3) are all true

3 marks

2 (2 marks)

$$\frac{k(2k+1)}{6} + (k+1) = \frac{2k^2 + k + 6k + 6}{6} = \frac{2k^2 + 7k + 6}{6}$$

M1

$$= \frac{(k+2)(2k+3)}{6}$$

A1

Answer: $\frac{k(2k+1)}{6} + (k+1) = \frac{(k+2)(2k+3)}{6}$

2 marks

3 (2 marks)

Replace n by $k + 1$ throughout

M1

$$\sum_{r=1}^{k+1} r(r+3) = \frac{(k+1)(k+2)(k+6)}{3}$$

A1

Answer: $\sum_{r=1}^{k+1} r(r+3) = \frac{(k+1)(k+2)(k+6)}{3}$

2 marks

4 (2 marks)

$n = 1$: LHS = 2, RHS = $\frac{1 \times 4}{2} = 2$

B1

$n = 2$: LHS = $2 + 5 = 7$, RHS = $\frac{2 \times 7}{2} = 7$

B1

Answer: the formula $\sum_{r=1}^n (3r - 1) = \frac{n(3n+1)}{2}$ is correct for $n = 1$ and $n = 2$

2 marks

5 (2 marks)

LHS = $1 + 8 + 27 = 36$

B1

RHS = $6^2 = 36$, so they are equal

B1

Answer: $1^3 + 2^3 + 3^3 = (1 + 2 + 3)^2$

2 marks

6 (3 marks)

$S_1 = 2, S_2 = 6, S_3 = 14$

B1

Each is 2 less than a power of 2: $4 - 2, 8 - 2, 16 - 2$

M1

$S_n = 2^{n+1} - 2$

A1

Answer: $S_n = 2^{n+1} - 2$

3 marks

7 (4 marks)

$$n = 1: \text{LHS} = 2, \text{RHS} = \frac{1 \times 4}{2} = 2, \text{true}$$

B1

$$\text{Assume true for } n = k. \text{ Sum to } k + 1 = \frac{k(3k + 1)}{2} + 3(k + 1) - 1$$

M1

$$= \frac{3k^2 + k + 6k + 4}{2} = \frac{3k^2 + 7k + 4}{2} = \frac{(k + 1)(3k + 4)}{2}$$

A1

This is the formula with $n = k + 1$ $[(k + 1)(3(k + 1) + 1)/2]$. Conclude: true for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction

A1

$$\text{Answer: } \sum_{r=1}^n (3r - 1) = \frac{n(3n + 1)}{2} \text{ for all integers } n \geq 1$$

4 marks**8 (5 marks)**

$$n = 1: \text{LHS} = 2, \text{RHS} = \frac{1 \times 2 \times 3}{3} = 2, \text{true}$$

B1

$$\text{Assume true for } n = k. \text{ Sum to } k + 1 = \frac{k(k + 1)(k + 2)}{3} + (k + 1)(k + 2)$$

M1

$$= \frac{(k + 1)(k + 2)}{3} [k + 3]$$

M1 A1

$$= \frac{(k + 1)(k + 2)(k + 3)}{3}, \text{ the formula for } n = k + 1. \text{ True for } n = 1, \text{ and true for } n = k \text{ implies true for}$$

A1

$n = k + 1$, so true for all integers $n \geq 1$ by induction

$$\text{Answer: } \sum_{r=1}^n r(r + 1) = \frac{n(n + 1)(n + 2)}{3} \text{ for all integers } n \geq 1$$

5 marks**9 (5 marks)**

$$n = 1: \text{LHS} = 1, \text{RHS} = \frac{1 \times 2 \times 3}{6} = 1, \text{true}$$

B1

$$\text{Assume true for } n = k. \text{ Sum to } k + 1 = \frac{k(k + 1)(2k + 1)}{6} + (k + 1)^2$$

M1

$$= \frac{(k + 1)}{6} [k(2k + 1) + 6(k + 1)] = \frac{(k + 1)(2k^2 + 7k + 6)}{6}$$

M1 A1

$$= \frac{(k + 1)(k + 2)(2k + 3)}{6}, \text{ the formula for } n = k + 1. \text{ True for } n = 1, \text{ and true for } n = k \text{ implies true}$$

A1

for $n = k + 1$, so true for all integers $n \geq 1$ by induction

$$\text{Answer: } \sum_{r=1}^n r^2 = \frac{n(n + 1)(2n + 1)}{6} \text{ for all integers } n \geq 1$$

5 marks**10 (4 marks)**

$$n = 1: \text{LHS} = 1, \text{RHS} = \frac{3 - 1}{2} = 1, \text{true}$$

B1

$$\text{Assume true for } n = k. \text{ Sum of } k + 1 \text{ terms} = \frac{3^k - 1}{2} + 3^k$$

M1

$$= \frac{3^k - 1 + 2 \times 3^k}{2} = \frac{3^{k+1} - 1}{2}$$

A1

True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction

A1

$$\text{Answer: } 1 + 3 + 9 + \dots + 3^{n-1} = \frac{3^n - 1}{2} \text{ for all integers } n \geq 1$$

4 marks

11 (5 marks)

$$n = 1: \text{LHS} = 4, \text{RHS} = \frac{1 \times 2 \times 6}{3} = 4, \text{true}$$

B1

$$\text{Assume true for } n = k. \text{ Sum to } k + 1 = \frac{k(k+1)(k+5)}{3} + (k+1)(k+4)$$

M1

$$= \frac{(k+1)}{3} [k(k+5) + 3(k+4)]$$

M1

$$= \frac{(k+1)(k^2 + 8k + 12)}{3} = \frac{(k+1)(k+2)(k+6)}{3}$$

A1

True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction

A1

$$\text{Answer: } \sum_{r=1}^n r(r+3) = \frac{n(n+1)(n+5)}{3} \text{ for all integers } n \geq 1$$

5 marks**12 (5 marks)**

$$n = 1: \text{LHS} = 2 \times 2 = 4, \text{RHS} = 1 \times 2^2 = 4, \text{true}$$

B1

$$\text{Assume true for } n = k. \text{ Sum to } k + 1 = k \times 2^{k+1} + (k+2) \times 2^{k+1}$$

M1

$$= 2^{k+1} [k + (k+2)] = 2^{k+1} \times 2(k+1)$$

M1 A1

$= (k+1) \times 2^{k+2}$, which is the formula with $n = k + 1$. True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction

A1

$$\text{Answer: } \sum_{r=1}^n (r+1) \times 2^r = n \times 2^{n+1} \text{ for all integers } n \geq 1$$

5 marks**13 (5 marks)**

$$n = 1: \text{LHS} = 1, \text{RHS} = 0 + 1 = 1, \text{true}$$

B1

$$\text{Assume true for } n = k. \text{ Sum to } k + 1 \text{ terms} = (k-1)2^k + 1 + (k+1)2^k$$

M1

$$= 2^k [(k-1) + (k+1)] + 1 = 2k \times 2^k + 1$$

M1 A1

$= k \times 2^{k+1} + 1$, the formula for $n = k + 1$. True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction

A1

$$\text{Answer: } 1 + 4 + 12 + \dots + n \times 2^{n-1} = (n-1)2^n + 1 \text{ for all integers } n \geq 1$$

5 marks**14 (6 marks)**

$$n = 1: \text{LHS} = 1, \text{RHS} = \frac{1 \times 4}{4} = 1, \text{true}$$

B1

$$\text{Assume true for } n = k. \text{ Sum to } k + 1 = \frac{k^2(k+1)^2}{4} + (k+1)^3$$

M1 A1

$$= \frac{(k+1)^2}{4} [k^2 + 4(k+1)]$$

M1

$$= \frac{(k+1)^2(k+2)^2}{4}$$

A1

This is the formula for $n = k + 1$. True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction

A1

$$\text{Answer: } \sum_{r=1}^n r^3 = \frac{n^2(n+1)^2}{4} \text{ for all integers } n \geq 1$$

6 marks

15 (7 marks)

$$n = 1: \text{LHS} = \frac{1}{3}, \text{RHS} = \frac{1 \times 8}{4 \times 2 \times 3} = \frac{1}{3}, \text{true}$$

B1

$$\text{Assume true for } n = k. \text{ Sum to } k + 1 = \frac{k(3k + 5)}{4(k + 1)(k + 2)} + \frac{1}{(k + 1)(k + 3)}$$

M1 A1

$$\text{Common denominator } 4(k + 1)(k + 2)(k + 3): \text{ numerator } k(3k + 5)(k + 3) + 4(k + 2)$$

M1

$$= 3k^3 + 14k^2 + 19k + 8$$

A1

$$= (k + 1)(3k^2 + 11k + 8) = (k + 1)^2(3k + 8)$$

M1

$$\text{So the sum} = \frac{(k + 1)(3k + 8)}{4(k + 2)(k + 3)}, \text{ the formula for } n = k + 1. \text{ True for } n = 1, \text{ and true for } n = k \text{ implies}$$

A1

true for $n = k + 1$, so true for all integers $n \geq 1$ by induction

$$\text{Answer: } \sum_{r=1}^n \frac{1}{r(r+2)} = \frac{n(3n+5)}{4(n+1)(n+2)} \text{ for all integers } n \geq 1$$

7 marks**16 (6 marks)**

$$n = 1: \text{LHS} = 6, \text{RHS} = \frac{1 \times 2 \times 3 \times 4}{4} = 6, \text{true}$$

B1

$$\text{Assume true for } n = k. \text{ Sum to } k + 1 = \frac{k(k+1)(k+2)(k+3)}{4} + (k+1)(k+2)(k+3)$$

M1 A1

$$= \frac{(k+1)(k+2)(k+3)}{4} [k+4]$$

M1 A1

$$= \frac{(k+1)(k+2)(k+3)(k+4)}{4}, \text{ the formula for } n = k + 1. \text{ True for } n = 1, \text{ and true for } n = k \text{ implies}$$

A1

true for $n = k + 1$, so true for all integers $n \geq 1$ by induction

$$\text{Answer: } \sum_{r=1}^n r(r+1)(r+2) = \frac{n(n+1)(n+2)(n+3)}{4} \text{ for all integers } n \geq 1$$

6 marks**17 (6 marks)**

$$n = 1: \text{LHS} = 1, \text{RHS} = \frac{1 \times 1 \times 3}{3} = 1, \text{true}$$

B1

$$\text{Assume true for } n = k. \text{ Sum of } k + 1 \text{ terms} = \frac{k(2k-1)(2k+1)}{3} + (2k+1)^2$$

M1 A1

$$= \frac{(2k+1)}{3} [k(2k-1) + 3(2k+1)]$$

M1

$$= \frac{(2k+1)(2k^2+5k+3)}{3} = \frac{(2k+1)(2k+3)(k+1)}{3}$$

A1

This is the formula for $n = k + 1$. True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction

A1

$$\text{Answer: } 1^2 + 3^2 + 5^2 + \dots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3} \text{ for all integers } n \geq 1$$

6 marks**18 (7 marks)**

$$n = 1: \text{LHS} = 1, \text{RHS} = (+1) \times \frac{1 \times 2}{2} = 1, \text{true}$$

B1

$$\text{Assume true for } n = k. \text{ Sum to } k + 1 \text{ terms} = (-1)^{k+1} \frac{k(k+1)}{2} + (-1)^{k+2} (k+1)^2$$

M1 A1

$$\text{Factor } (-1)^{k+1} (k+1): \text{ the sum} = (-1)^{k+1} (k+1) \left[\frac{k}{2} - (k+1) \right]$$

M1 A1

$$\frac{k}{2} - (k+1) = -\frac{k+2}{2}$$

M1

$$\text{So the sum} = (-1)^{k+2} \frac{(k+1)(k+2)}{2}, \text{ the formula for } n = k + 1. \text{ True for } n = 1, \text{ and true for } n = k$$

A1

implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction

$$\text{Answer: } 1^2 - 2^2 + 3^2 - 4^2 + \dots + (-1)^{n+1} n^2 = (-1)^{n+1} \frac{n(n+1)}{2} \text{ for all integers } n \geq 1$$

7 marks

19 (6 marks)

$$n = 1: \text{LHS} = \frac{1}{2}, \text{RHS} = 2 - \frac{3}{2} = \frac{1}{2}, \text{true}$$

B1

$$\text{Assume true for } n = k. \text{ Sum to } k + 1 = 2 - \frac{k+2}{2^k} + \frac{k+1}{2^{k+1}}$$

M1 A1

$$= 2 - \frac{2(k+2) - (k+1)}{2^{k+1}}$$

M1

$$= 2 - \frac{k+3}{2^{k+1}}$$

A1

This is the formula for $n = k + 1$. True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction

A1

$$\text{Answer: } \sum_{r=1}^n \frac{r}{2^r} = 2 - \frac{n+2}{2^n} \text{ for all integers } n \geq 1$$

6 marks**20 (7 marks)**

$$n = 2: \text{LHS} = 1 \times 2 \times 3 = 6, \text{RHS} = \frac{1 \times 2 \times 3 \times 4}{4} = 6, \text{true}$$

B1

$$\text{Assume true for } n = k, \text{ where } k \geq 2. \text{ Sum to } k + 1 = \frac{(k-1)k(k+1)(k+2)}{4} + k(k+1)(k+2)$$

M1 A1

$$= \frac{k(k+1)(k+2)}{4} [(k-1) + 4] = \frac{k(k+1)(k+2)(k+3)}{4}$$

M1 A1

This is the formula with $n = k + 1$, as $((k+1) - 1)(k+1)((k+1) + 1)((k+1) + 2) = k(k+1)(k+2)(k+3)$

A1

True for $n = 2$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 2$ by induction

A1

$$\text{Answer: } \sum_{r=2}^n (r-1)r(r+1) = \frac{(n-1)n(n+1)(n+2)}{4} \text{ for all integers } n \geq 2$$

7 marks**21 (7 marks)**

$$n = 1: \text{LHS} = 3, \text{RHS} = \frac{1 \times 9 + 3}{4} = 3, \text{true}$$

B1

$$\text{Assume true for } n = k. \text{ Sum to } k + 1 = \frac{(2k-1)3^{k+1} + 3}{4} + (k+1)3^{k+1}$$

M1 A1

$$= \frac{3^{k+1}[(2k-1) + 4(k+1)] + 3}{4}$$

M1

$$= \frac{3^{k+1}(6k+3) + 3}{4} = \frac{3^{k+2}(2k+1) + 3}{4}$$

M1 A1

This is the formula for $n = k + 1$, as $(2(k+1) - 1) = 2k + 1$. True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction

A1

$$\text{Answer: } \sum_{r=1}^n r \times 3^r = \frac{(2n-1)3^{n+1} + 3}{4} \text{ for all integers } n \geq 1$$

7 marks**22 (7 marks)**

$$n = 1: \text{LHS} = 2 \times 1 = 2, \text{RHS} = 1 \times 2! = 2, \text{true}$$

B1

$$\text{Assume true for } n = k. \text{ Sum to } k + 1 = k \times (k+1)! + ((k+1)^2 + 1)(k+1)!$$

M1 A1

$$= (k+1)! [k + k^2 + 2k + 2]$$

M1

$$= (k+1)!(k^2 + 3k + 2) = (k+1)!(k+1)(k+2)$$

M1 A1

$= (k+1) \times (k+2)!$, the formula for $n = k + 1$. True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction

A1

$$\text{Answer: } \sum_{r=1}^n (r^2 + 1) \times r! = n \times (n+1)! \text{ for all integers } n \geq 1$$

7 marks

23 (8 marks)

(a) $n = 1$: $f(1) = 3$ and $\frac{1 \times 2 \times 9}{6} = 3$, true

B1

Assume $f(k) = \frac{k(k+1)(4k+5)}{6}$. Then $f(k+1) = \frac{k(k+1)(4k+5)}{6} + (k+1)(2k+3)$

M1

$$= \frac{(k+1)}{6} [k(4k+5) + 6(2k+3)] = \frac{(k+1)(4k^2 + 17k + 18)}{6}$$

M1 A1

$$= \frac{(k+1)(k+2)(4k+9)}{6}, \text{ which is the formula with } n = k+1. \text{ True for } n = 1, \text{ and true for } n = k$$

A1

implies true for $n = k+1$, so true for all integers $n \geq 1$ by induction

Answer: $f(n) = \frac{n(n+1)(4n+5)}{6}$ for all integers $n \geq 1$

5 marks

(b) $\sum_{r=n+1}^{2n} r(2r+1) = f(2n) - f(n)$

M1

$$= \frac{2n(2n+1)(8n+5)}{6} - \frac{n(n+1)(4n+5)}{6} = \frac{n}{6} [2(2n+1)(8n+5) - (n+1)(4n+5)]$$

A1

$$= \frac{n}{6} (28n^2 + 27n + 5) = \frac{n(4n+1)(7n+5)}{6}$$

A1

Answer: $\sum_{r=n+1}^{2n} r(2r+1) = \frac{n(4n+1)(7n+5)}{6}$

3 marks

24 (6 marks)

$n = 1$: LHS = 1, RHS = $1 \times 1 = 1$, true

B1

Assume true for $n = k$. Sum of $k+1$ terms = $k^2(2k^2 - 1) + (2k+1)^3$

M1 A1

$$= 2k^4 - k^2 + 8k^3 + 12k^2 + 6k + 1 = 2k^4 + 8k^3 + 11k^2 + 6k + 1$$

M1

$$(k+1)^2(2(k+1)^2 - 1) = (k^2 + 2k + 1)(2k^2 + 4k + 1) = 2k^4 + 8k^3 + 11k^2 + 6k + 1$$

A1

The two agree, so the formula holds for $n = k+1$. True for $n = 1$, and true for $n = k$ implies true for $n = k+1$, so true for all integers $n \geq 1$ by induction

A1

Answer: $1^3 + 3^3 + 5^3 + \dots + (2n-1)^3 = n^2(2n^2 - 1)$ for all integers $n \geq 1$

6 marks

25 (6 marks)

Error 1: the term added should be the $(k+1)$ th term, $(k+1)^2$, not $(k+1)$

E1

Error 2: $(k+1)(2k^2 + k + 6)$ does not equal $(k+1)(k+2)(2k+3) = (k+1)(2k^2 + 7k + 6)$, so the claimed equality is false

E1

Correct step: sum to $k+1 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$

M1

$$= \frac{(k+1)}{6} [k(2k+1) + 6(k+1)]$$

M1

$$= \frac{(k+1)(2k^2 + 7k + 6)}{6}$$

A1

$$= \frac{(k+1)(k+2)(2k+3)}{6}, \text{ the formula with } n = k+1$$

A1

Answer: Added $(k+1)$ instead of $(k+1)^2$; false equality; corrected step gives $\frac{(k+1)(k+2)(2k+3)}{6}$

6 marks

26 (7 marks)

$$n = 1: \text{LHS} = \frac{3}{4}, \text{RHS} = 1 - \frac{1}{4} = \frac{3}{4}, \text{true}$$

B1

$$\text{Assume true for } n = k. \text{ Sum to } k + 1 = 1 - \frac{1}{(k+1)^2} + \frac{2k+3}{(k+1)^2(k+2)^2}$$

M1 A1

$$= 1 - \frac{(k+2)^2 - (2k+3)}{(k+1)^2(k+2)^2}$$

M1 A1

$$(k+2)^2 - (2k+3) = k^2 + 2k + 1 = (k+1)^2$$

M1

So the sum $= 1 - \frac{1}{(k+2)^2}$, the formula with $n = k + 1$. True for $n = 1$, and true for $n = k$ implies

A1

true for $n = k + 1$, so true for all integers $n \geq 1$ by induction

$$\text{Answer: } \sum_{r=1}^n \frac{2r+1}{r^2(r+1)^2} = 1 - \frac{1}{(n+1)^2} \text{ for all integers } n \geq 1$$

7 marks**27 (8 marks)**

$$n = 1: \text{LHS} = 2, \text{RHS} = (1 - 2 + 3) \times 4 - 6 = 2, \text{true}$$

B1

$$\text{Assume true for } n = k. \text{ Sum to } k + 1 = (k^2 - 2k + 3)2^{k+1} - 6 + (k+1)2^{k+1}$$

M1 A1

$$= 2^{k+1}[k^2 - 2k + 3 + k^2 + 2k + 1] - 6$$

M1

$$= 2^{k+1}(2k^2 + 4) - 6$$

A1

$$= 2^{k+2}(k^2 + 2) - 6$$

M1

For $n = k + 1$ the formula gives $((k+1)^2 - 2(k+1) + 3)2^{k+2} - 6 = (k^2 + 2)2^{k+2} - 6$, which matches

A1

Conclude: true for $n = 1$ and true for $n = k$ implies true for $n = k + 1$, so true for all $n \geq 1$

A1

$$\text{Answer: } \sum_{r=1}^n r^2 \times 2^r = (n^2 - 2n + 3)2^{n+1} - 6 \text{ for all integers } n \geq 1$$

8 marks**28 (10 marks)**

$$n = 1: \text{RHS} = \frac{\sin(\frac{\theta}{2}) \sin \theta}{\sin(\frac{\theta}{2})} = \sin \theta = \text{LHS}, \text{true}$$

B1

$$\text{Assume true for } n = k. \text{ Sum to } k + 1 \text{ terms} = \frac{\sin(\frac{k\theta}{2}) \sin(\frac{(k+1)\theta}{2})}{\sin(\frac{\theta}{2})} + \sin(k+1)\theta$$

M1 A1

$$\sin(k+1)\theta = 2 \sin(\frac{(k+1)\theta}{2}) \cos(\frac{(k+1)\theta}{2})$$

M1

$$\text{Sum} = \frac{\sin(\frac{(k+1)\theta}{2})}{\sin(\frac{\theta}{2})} [\sin(\frac{k\theta}{2}) + 2 \sin(\frac{\theta}{2}) \cos(\frac{(k+1)\theta}{2})]$$

M1 A1

$$2 \sin(\frac{\theta}{2}) \cos(\frac{(k+1)\theta}{2}) = \sin(\frac{(k+2)\theta}{2}) - \sin(\frac{k\theta}{2})$$

M1 A1

$$\text{So the bracket is } \sin(\frac{(k+2)\theta}{2}), \text{ and the sum} = \frac{\sin(\frac{(k+1)\theta}{2}) \sin(\frac{(k+2)\theta}{2})}{\sin(\frac{\theta}{2})}, \text{ the formula for } n = k +$$

A1

1

True for $n = 1$ and true for $n = k$ implies true for $n = k + 1$, so true for all $n \geq 1$

A1

$$\text{Answer: for all integers } n \geq 1 \text{ and } \theta \text{ not a multiple of } 2\pi, \sin \theta + \sin 2\theta + \sin 3\theta + \dots + \sin n\theta = \frac{\sin(\frac{n\theta}{2}) \sin(\frac{(n+1)\theta}{2})}{\sin(\frac{\theta}{2})}$$

10 marks

29 (8 marks)

(a) $n = 1$: LHS = $\frac{1}{24}$, RHS = $\frac{1}{18} - \frac{1}{72} = \frac{3}{72} = \frac{1}{24}$, true B1

Assume true for $n = k$. Sum to $k + 1 = \frac{1}{18} - \frac{1}{3(k+1)(k+2)(k+3)} + \frac{1}{(k+1)(k+2)(k+3)(k+4)}$ M1 A1

The last two terms combine to $-\frac{(k+4) - 3}{3(k+1)(k+2)(k+3)(k+4)}$ M1 A1

$= -\frac{k+1}{3(k+1)(k+2)(k+3)(k+4)} = -\frac{1}{3(k+2)(k+3)(k+4)}$ A1

This gives the formula with $n = k + 1$. True for $n = 1$ and $n = k$ implies $n = k + 1$, so true for all $n \geq 1$ A1

Answer: $S_n = \frac{1}{18} - \frac{1}{3(n+1)(n+2)(n+3)}$ for all integers $n \geq 1$ 7 marks

(b) As $n \rightarrow \infty$ the term $\frac{1}{3(n+1)(n+2)(n+3)} \rightarrow 0$, so the sum is $\frac{1}{18}$ B1

Answer: $\frac{1}{18}$ 1 mark

30 (9 marks)

$n = 1$: LHS = $1 \times C(1, 1) = 1$, RHS = $1 \times 2^0 = 1$, true B1

Assume true for $n = k$. Then $\sum_{r=1}^{k+1} r C(k+1, r) = \sum_{r=1}^{k+1} r C(k, r) + \sum_{r=1}^{k+1} r C(k, r-1)$ M1 A1

First sum: $C(k, k+1) = 0$, so it equals $k \times 2^{k-1}$ by the assumption M1

Second sum: write $r = (r-1) + 1$, so it is $\sum_{r=1}^{k+1} (r-1)C(k, r-1) + \sum_{r=1}^{k+1} C(k, r-1)$ M1

$= k \times 2^{k-1} + 2^k$ (using the assumption and the sum of the binomial coefficients) A1

Total = $2k \times 2^{k-1} + 2^k = k \times 2^k + 2^k = (k+1)2^k$ M1

This is the formula with $n = k + 1$ A1

True for $n = 1$ and true for $n = k$ implies $n = k + 1$, so true for all $n \geq 1$ A1

Answer: $\sum_{r=1}^n r \times C(n, r) = n \times 2^{n-1}$ for all integers $n \geq 1$, where $C(n, r) = \frac{n!}{r!(n-r)!}$ 9 marks

Induction with Divisibility**1 (2 marks)**

$f(1) = 6 = 6 \times 1$ B1

$f(2) = 48 = 6 \times 8$ B1

Answer: $f(1) = 6 = 6 \times 1$ and $f(2) = 48 = 6 \times 8$ 2 marks

2 (2 marks)

$f(k+1) - f(k) = (5^{k+1} - 1) - (5^k - 1)$ M1

$= 5 \times 5^k - 5^k = 4 \times 5^k$ A1

Answer: $f(k+1) - f(k) = 4 \times 5^k$ 2 marks

3 (2 marks)

$f(k+1) = 2^{2k+2} - 1 = 4 \times 2^{2k} - 1$ M1

$4f(k) + 3 = 4 \times 2^{2k} - 4 + 3 = 4 \times 2^{2k} - 1$, so $f(k+1) = 4f(k) + 3$ A1

Answer: $f(k+1) = 4f(k) + 3$ 2 marks

4 (3 marks)

$f(1) = 0$, $f(2) = 6$, $f(3) = 24$ B1 B1

The largest integer dividing 0, 6 and 24 is 6 B1

Answer: 6 3 marks

5 (3 marks)

$$f(1) = 10 = 5 \times 2$$

B1

$$f(k+1) - f(k) = 6 \times 6^k + 4 - 6^k - 4$$

M1

$$= 5 \times 6^k$$

A1

Answer: $f(1) = 10 = 5 \times 2$ and $f(k+1) - f(k) = 5 \times 6^k$

3 marks

6 (2 marks)

The statement is true for $n = 1$ and true for $n = k + 1$ whenever it is true for $n = k$

E1

So, by induction, $f(n)$ is divisible by 8 for all positive integers n

E1

Answer: Divisible by 8 for all positive integers n

2 marks

7 (4 marks)

$n = 1$: $5 - 1 = 4$, which is divisible by 4, true

B1

Assume $f(k) = 5^k - 1 = 4m$ for some integer m . Then $f(k+1) = 5 \times 5^k - 1 = 5(4m+1) - 1$

M1

$$= 20m + 4 = 4(5m + 1), \text{ divisible by 4}$$

A1

True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction

A1

Answer: $5^n - 1$ is divisible by 4 for all integers $n \geq 1$

4 marks

8 (5 marks)

$n = 1$: $1 \times 2 \times 3 = 6$, which is divisible by 6, true

B1

Assume $f(k) = k(k+1)(k+2)$ is divisible by 6. Then $f(k+1) - f(k) = (k+1)(k+2)(k+3) - k(k+1)(k+2)$

M1

$$= 3(k+1)(k+2)$$

A1

$(k+1)(k+2)$ is a product of consecutive integers so is even, hence $3(k+1)(k+2)$ is a multiple of 6

E1

So $f(k+1) = f(k) + (\text{a multiple of 6})$ is divisible by 6. True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction

A1

Answer: $n(n+1)(n+2)$ is divisible by 6 for all integers $n \geq 1$

5 marks

9 (4 marks)

$n = 1$: $u_1 + 2 = 4$, which is divisible by 4, true

B1

Assume $u_k + 2 = 4m$ for some integer m . Then $u_{k+1} + 2 = 3u_k + 4 + 2 = 3u_k + 6$

M1

$$= 3(u_k + 2) = 12m = 4(3m), \text{ divisible by 4}$$

A1

True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction

A1

Answer: $u_n + 2$ is divisible by 4 for all integers $n \geq 1$

4 marks

10 (4 marks)

$n = 1$: $6 + 4 = 10$, which is divisible by 5, true

B1

Assume $f(k) = 6^k + 4 = 5m$. Then $f(k+1) = 6 \times 6^k + 4 = 6(5m - 4) + 4$

M1

$$= 30m - 20 = 5(6m - 4), \text{ divisible by 5}$$

A1

True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction

A1

Answer: $6^n + 4$ is divisible by 5 for all integers $n \geq 1$

4 marks

11 (4 marks)

$n = 1$: $u_1 = 100 + 12 = 112 = 7 \times 16$, true

B1

Assume $u_k = 10^{k+1} + 4 \times 3^k$ is a multiple of 7. Then $u_{k+1} = 10 \times 10^{k+1} + 12 \times 3^k$

M1

$$= 3(10^{k+1} + 4 \times 3^k) + 7 \times 10^{k+1} = 3u_k + 7 \times 10^{k+1}$$

A1

Both terms are multiples of 7, so u_{k+1} is a multiple of 7. True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction

A1

Answer: every term $u_n = 10^{n+1} + 4 \times 3^n$ is a multiple of 7

4 marks

12 (5 marks)

$n = 1$: $1 + 2 = 3$, which is divisible by 3, true

B1

Assume $f(k) = k^3 + 2k = 3m$. Then $f(k + 1) = (k + 1)^3 + 2(k + 1) = (k^3 + 2k) + 3k^2 + 3k + 3$

M1 A1

$= 3m + 3(k^2 + k + 1) = 3(m + k^2 + k + 1)$, divisible by 3

A1

True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction

A1

Answer: $n^3 + 2n$ is divisible by 3 for all integers $n \geq 1$

5 marks

13 (4 marks)

$n = 1$: $5 + 22 = 27$, which is divisible by 3, true

B1

Assume $f(k) = 5^k + 2 \times 11^k = 3m$. Then $f(k + 1) = 5 \times 5^k + 22 \times 11^k = 5(5^k + 2 \times 11^k) + 12 \times 11^k$

M1

$= 15m + 12 \times 11^k = 3(5m + 4 \times 11^k)$, divisible by 3

A1

True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction

A1

Answer: $5^n + 2 \times 11^n$ is divisible by 3 for all integers $n \geq 1$

4 marks

14 (6 marks)

$n = 1$: $8 + 27 = 35$, which is divisible by 7, true

B1

Assume $f(k) = 2^{k+2} + 3^{2k+1}$ is divisible by 7

M1

$f(k + 1) = 2 \times 2^{k+2} + 9 \times 3^{2k+1} = 2f(k) + 7 \times 3^{2k+1}$

M1 A1

Both terms are multiples of 7, so $f(k + 1)$ is divisible by 7

A1

True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction

A1

Answer: $2^{n+2} + 3^{2n+1}$ is divisible by 7 for all integers $n \geq 1$

6 marks

15 (7 marks)

$n = 1$: $25 + 6 + 1 = 32$, which is divisible by 8, true

B1

Assume $f(k) = 5^{k+1} + 2 \times 3^k + 1$ is divisible by 8

M1

$f(k + 1) - 5f(k) = (2 \times 3^{k+1} + 1) - (10 \times 3^k + 5)$

M1 A1

$= 6 \times 3^k - 10 \times 3^k - 4 = -4(3^k + 1)$

A1

$3^k + 1$ is even, so $4(3^k + 1)$ is a multiple of 8, and $f(k + 1) = 5f(k) - 4(3^k + 1)$ is divisible by 8

E1

True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction

A1

Answer: $5^{n+1} + 2 \times 3^n + 1$ is divisible by 8 for all integers $n \geq 1$

7 marks

16 (7 marks)

$n = 1$: $81 - 8 - 9 = 64$, which is divisible by 64, true

B1

Assume $f(k) = 3^{2k+2} - 8k - 9$ is divisible by 64. Then $f(k + 1) = 3^{2k+4} - 8(k + 1) - 9 = 9 \times 3^{2k+2} - 8k - 17$

M1 A1

$f(k + 1) - 9f(k) = (9 \times 3^{2k+2} - 8k - 17) - (9 \times 3^{2k+2} - 72k - 81)$

M1

$= 64k + 64 = 64(k + 1)$

A1

So $f(k + 1) = 9f(k) + 64(k + 1)$, divisible by 64

A1

True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction

A1

Answer: $3^{2n+2} - 8n - 9$ is divisible by 64 for all integers $n \geq 1$

7 marks

17 (7 marks)

$n = 1$: $1 \times 6 = 6$, which is divisible by 6, true

B1

Assume $f(k) = k^3 + 5k$ is divisible by 6

M1

$$f(k+1) = (k+1)^3 + 5(k+1) = f(k) + 3k^2 + 3k + 6$$

M1 A1

$3k^2 + 3k = 3k(k+1)$, and $k(k+1)$ is a product of consecutive integers so is even

E1

So $3k(k+1) + 6$ is a multiple of 6, and $f(k+1)$ is divisible by 6

A1

True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction

A1

Answer: $n(n^2 + 5)$ is divisible by 6 for all integers $n \geq 1$

7 marks

18 (5 marks)

$n = 1$: $7 \times 7 - 1 = 48 = 6 \times 8$, true

B1

Assume $f(k) = (6k+1)7^k - 1$ is divisible by 6. Then $f(k+1) = (6k+7)7^{k+1} - 1$

M1

$$f(k+1) - 7f(k) = (6k+7)7^{k+1} - 1 - (6k+1)7^{k+1} + 7 = 6 \times 7^{k+1} + 6$$

M1 A1

So $f(k+1) = 7f(k) + 6(7^{k+1} + 1)$, a sum of multiples of 6. True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction

A1

Answer: $(6n+1) \times 7^n - 1$ is divisible by 6 for all integers $n \geq 1$

5 marks

19 (6 marks)

$n = 1$: $1 + 8 + 27 = 36 = 9 \times 4$, true

B1

Assume $f(k) = k^3 + (k+1)^3 + (k+2)^3$ is divisible by 9

M1

$$f(k+1) - f(k) = (k+3)^3 - k^3$$

M1 A1

$$= 9k^2 + 27k + 27 = 9(k^2 + 3k + 3)$$

A1

So $f(k+1) = f(k) + 9(k^2 + 3k + 3)$, a sum of multiples of 9. True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction

A1

Answer: $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9 for all integers $n \geq 1$

6 marks

20 (5 marks)

$n = 1$: $F_3 = F_2 + F_1 = 2$, which is even, true

B1

Assume F_{3k} is even

M1

$$F_{3k+3} = F_{3k+2} + F_{3k+1} = (F_{3k+1} + F_{3k}) + F_{3k+1}$$

M1

$$= 2F_{3k+1} + F_{3k}$$

A1

Both terms are even, so F_{3k+3} is even. True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction

A1

Answer: F_{3n} is even for all integers $n \geq 1$

5 marks

21 (8 marks)

$n = 1$: $64 - 7 - 8 = 49$, which is divisible by 49, true

B1

Assume $f(k) = 2^{3k+3} - 7k - 8$ is divisible by 49

M1

$$f(k+1) = 2^{3k+6} - 7(k+1) - 8 = 8 \times 2^{3k+3} - 7k - 15$$

M1 A1

$$8f(k) = 8 \times 2^{3k+3} - 56k - 64, \text{ so } f(k+1) - 8f(k) = 49k + 49$$

M1 A1

So $f(k+1) = 8f(k) + 49(k+1)$, divisible by 49

A1

True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction

A1

Answer: $2^{3n+3} - 7n - 8$ is divisible by 49 for all integers $n \geq 1$

8 marks

22 (6 marks)

Error: $5(5^k + 3) = 5^{k+1} + 15$, so $5^{k+1} + 3 = 5(5^k + 3) - 12$, not $+ 12$

E1

Check $k = 1$: $5^2 + 3 = 28$ but $5(5 + 3) + 12 = 52$, so the identity is false

B1

Correct step: assume $5^k + 3 = 4m$, so $5^k = 4m - 3$

M1

$$5^{k+1} + 3 = 5(4m - 3) + 3$$

M1

$$= 20m - 12 = 4(5m - 3)$$

A1

This is divisible by 4, and with the basis case the result follows by induction

A1

Answer: Sign error: $5^{k+1} + 3 = 5(5^k + 3) - 12$; the correct step gives $4(5m - 3)$

6 marks

23 (8 marks)

$n = 1$: $64 + 27 = 91 = 13 \times 7$, which is divisible by 13, true

B1

Assume $f(k) = 4^{2k+1} + 3^{k+2}$ is divisible by 13

M1

$$f(k+1) = 16 \times 4^{2k+1} + 3 \times 3^{k+2}$$

M1 A1

$$= 3(4^{2k+1} + 3^{k+2}) + 13 \times 4^{2k+1} = 3f(k) + 13 \times 4^{2k+1}$$

M1 A1

Both terms are multiples of 13, so $f(k+1)$ is divisible by 13

A1

True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction

A1

Answer: $4^{2n+1} + 3^{n+2}$ is divisible by 13 for all integers $n \geq 1$

8 marks

24 (7 marks)

$n = 1$: $4 + 6 - 1 = 9$, which is divisible by 9, true

B1

Assume $f(k) = 4^k + 6k - 1 = 9m$

M1

$$f(k+1) = 4 \times 4^k + 6k + 5$$

M1

Substituting $4^k = 9m - 6k + 1$ gives $f(k+1) = 4(9m - 6k + 1) + 6k + 5$

A1

$$= 36m - 18k + 9 = 9(4m - 2k + 1)$$

M1

This is a multiple of 9

A1

True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction

A1

Answer: $4^n + 6n - 1$ is divisible by 9 for all integers $n \geq 1$

7 marks

25 (8 marks)

$n = 1$: $10 + 192 + 5 = 207 = 9 \times 23$, which is divisible by 9, true

B1

Assume $f(k) = 10^k + 3 \times 4^{k+2} + 5$ is divisible by 9

M1

$$f(k+1) - f(k) = (10^{k+1} - 10^k) + 3(4^{k+3} - 4^{k+2})$$

M1 A1

$$= 9 \times 10^k + 9 \times 4^{k+2}$$

M1 A1

So $f(k+1) = f(k) + 9(10^k + 4^{k+2})$, divisible by 9

A1

True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction

A1

Answer: $10^n + 3 \times 4^{n+2} + 5$ is divisible by 9 for all integers $n \geq 1$

8 marks

26 (6 marks)

$n = 1$: $\frac{2!}{2 \times 1!} = 1$, an integer, true

B1

Assume $\frac{(2k)!}{2^k k!} = m$, an integer

M1

$$\frac{(2k+2)!}{2^{k+1}(k+1)!} = \frac{(2k+2)(2k+1)(2k)!}{2 \times 2^k(k+1)k!}$$

M1 A1

$$= \frac{2(k+1)(2k+1)}{2(k+1)} \times \frac{(2k)!}{2^k k!} = (2k+1)m$$

M1

which is an integer. True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction

A1

Answer: $\frac{(2n)!}{2^n n!}$ is an integer for all integers $n \geq 1$

6 marks

27 (10 marks)

$n = 0$: $2^1 + 1 = 3$, which is divisible by 3^1 , true

B1

Assume $2^{3^k} + 1 = 3^{k+1}m$ for an integer m . Let $a = 2^{3^k}$, so $a + 1 = 3^{k+1}m$

M1

$$2^{3^{k+1}} = (2^{3^k})^3 = a^3$$

M1

$$a^3 + 1 = (a + 1)(a^2 - a + 1)$$

M1 A1

$$\text{Since } a = 3^{k+1}m - 1: a^2 - a + 1 = (3^{k+1}m - 1)^2 - (3^{k+1}m - 1) + 1$$

M1

$$\text{Expanding, } a^2 - a + 1 = 3^{2k+2}m^2 - 3 \times 3^{k+1}m + 3 = 3(3^{2k+1}m^2 - 3^{k+1}m + 1)$$

A1

So $a^3 + 1 = 3^{k+1}m \times 3 \times (\text{integer})$, which is divisible by 3^{k+2}

M1 A1

True for $n = 0$, and true for $n = k$ implies true for $n = k + 1$, so true for all $n \geq 0$

A1

Answer: 3^{n+1} divides $2^{3^n} + 1$ for all integers $n \geq 0$

10 marks

28 (9 marks)

(a) $n = 1$: $f(1) = 0$, divisible by 5, true

B1

$$\text{Assume } f(k) = k^5 - k = 5m$$

M1

$$f(k+1) = k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - k - 1$$

M1

$$= (k^5 - k) + 5(k^4 + 2k^3 + 2k^2 + k) = 5(m + k^4 + 2k^3 + 2k^2 + k)$$

A1

True for $n = 1$, true for $n = k$ implies $n = k + 1$, so true for all $n \geq 1$

A1

Answer: $f(n)$ is divisible by 5 for all integers $n \geq 1$

5 marks

$$(b) f(n) = n(n^4 - 1) = n(n^2 - 1)(n^2 + 1) = (n - 1)n(n + 1)(n^2 + 1)$$

M1 A1

$(n - 1)n(n + 1)$ is a product of three consecutive integers so is divisible by 2 and by 3, hence by 6

E1

6 and 5 have no common factor, so $f(n)$ is divisible by 30

E1

Answer: $f(n) = (n - 1)n(n + 1)(n^2 + 1)$ is divisible by 5 and by 6, hence by 30

4 marks

29 (8 marks)

$$(a) F_{m+5} = F_{m+4} + F_{m+3} = 2F_{m+3} + F_{m+2}$$

M1

$$= 3F_{m+2} + 2F_{m+1} = 3F_{m+1} + 3F_m + 2F_{m+1}$$

M1

$$= 5F_{m+1} + 3F_m$$

A1

Answer: $F_{m+5} = 5F_{m+1} + 3F_m$ for all integers $m \geq 1$

3 marks

(b) $n = 1$: $F_5 = 5$, divisible by 5, true

B1

Assume F_{5k} is divisible by 5

M1

$$F_{5k+5} = 5F_{5k+1} + 3F_{5k} \text{ using part (a) with } m = 5k$$

M1 A1

Both terms are multiples of 5; true for $n = 1$ and $n = k$ implies $n = k + 1$, so true for all $n \geq 1$

A1

Answer: F_{5n} is divisible by 5 for all integers $n \geq 1$

5 marks

30 (9 marks)

(a) $\alpha = 3 + \sqrt{5}$ and $\beta = 3 - \sqrt{5}$ have sum 6 and product 4, so both are roots of $x^2 - 6x + 4 = 0$

M1 A1

So $\alpha^{n+2} = 6\alpha^{n+1} - 4\alpha^n$ and similarly for β ; adding gives $u_{n+2} = 6u_{n+1} - 4u_n$

A1

Answer: $u_{n+2} = 6u_{n+1} - 4u_n$

3 marks

(b) $u_1 = 6$, divisible by 2; $u_2 = (14 + 6\sqrt{5}) + (14 - 6\sqrt{5}) = 28$, divisible by 4

B1 B1

Assume $u_k = 2^k a$ and $u_{k+1} = 2^{k+1} b$ for integers a, b

M1

$$u_{k+2} = 6 \times 2^{k+1} b - 4 \times 2^k a$$

M1

$$= 2^{k+2}(3b - a)$$

A1

True for $n = 1, 2$ and true for $n = k, k + 1$ implies $n = k + 2$, so true for all $n \geq 1$

A1

Answer: u_n is divisible by 2^n for all integers $n \geq 1$

6 marks

Induction with Matrices

1 (3 marks)

$$A^2 = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix} \quad \text{B1}$$

$$A^3 = \begin{pmatrix} 8 & 0 \\ 0 & 27 \end{pmatrix} \quad \text{B1}$$

$$A^n = \begin{pmatrix} 2^n & 0 \\ 0 & 3^n \end{pmatrix} \quad \text{B1}$$

$$\text{Answer: } \begin{pmatrix} 2^n & 0 \\ 0 & 3^n \end{pmatrix} \quad \text{3 marks}$$

2 (2 marks)

$$A^{k+1} = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{M1}$$

$$= \begin{pmatrix} 1 & k+1 \\ 0 & 1 \end{pmatrix} \quad \text{A1}$$

$$\text{Answer: } \begin{pmatrix} 1 & k+1 \\ 0 & 1 \end{pmatrix} \quad \text{2 marks}$$

3 (2 marks)

$$n = 1: \text{ both sides equal } \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}, \text{ true} \quad \text{B1}$$

$$P(k+1): \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & 0 \\ 5(k+1) & 1 \end{pmatrix} \quad \text{B1}$$

$$\text{Answer: } \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & 0 \\ 5(k+1) & 1 \end{pmatrix} \quad \text{2 marks}$$

4 (3 marks)

$$B^2 = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} \quad \text{B1}$$

$$B^3 = \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix} \quad \text{B1}$$

$$B^n = \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix} \quad \text{B1}$$

$$\text{Answer: } \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix} \quad \text{3 marks}$$

5 (2 marks)

$$n = 1: \begin{pmatrix} 5 & 8 \\ -2 & -3 \end{pmatrix} \text{ matches A} \quad \text{B1}$$

$$n = 2: A^2 = \begin{pmatrix} 25 - 16 & 40 - 24 \\ -10 + 6 & -16 + 9 \end{pmatrix} = \begin{pmatrix} 9 & 16 \\ -4 & -7 \end{pmatrix}, \text{ and the formula gives } \begin{pmatrix} 9 & 16 \\ -4 & -7 \end{pmatrix} \quad \text{B1}$$

$$\text{Answer: the formula holds for } n = 1 \text{ and } n = 2 \quad \text{2 marks}$$

6 (2 marks)

$$J^2 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & & -1 + 0 \end{pmatrix} = -I \quad \text{B1}$$

$$J^4 = (-I)^2 = I \quad \text{B1}$$

$$\text{Answer: } J^2 = -I \text{ and } J^4 = I, \text{ where } I \text{ is the } 2 \times 2 \text{ identity matrix} \quad \text{2 marks}$$

7 (4 marks)

$$n = 1: \text{LHS} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \text{RHS, true}$$

B1

$$\text{Assume true for } n = k. \text{ Then } \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

M1

$$= \begin{pmatrix} 1 & k+1 \\ 0 & 1 \end{pmatrix}, \text{ the result for } n = k+1$$

A1

True for $n = 1$, and true for $n = k$ implies true for $n = k+1$, so true for all integers $n \geq 1$ by induction

A1

$$\text{Answer: } \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \text{ for all integers } n \geq 1$$

4 marks**8 (4 marks)**

$$n = 1: \text{LHS} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} = \text{RHS, true}$$

B1

$$\text{Assume true for } n = k. \text{ Then } \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & 0 \\ 3k & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$$

M1

$$= \begin{pmatrix} 1 & 0 \\ 3k+3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3(k+1) & 1 \end{pmatrix}$$

A1

True for $n = 1$, and true for $n = k$ implies true for $n = k+1$, so true for all integers $n \geq 1$ by induction

A1

$$\text{Answer: } \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ 3n & 1 \end{pmatrix} \text{ for all integers } n \geq 1$$

4 marks**9 (5 marks)**

$$A^2 = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix} \text{ and } 2A - I = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}, \text{ so } A^2 = 2A - I$$

B1

$$n = 1: 1 \times A - 0 \times I = A, \text{ true}$$

B1

$$\text{Assume } A^k = kA - (k-1)I. \text{ Then } A^{k+1} = A^k A = kA^2 - (k-1)A$$

M1

$$= k(2A - I) - (k-1)A = (k+1)A - kI$$

A1

This is the result with $n = k+1$. True for $n = 1$, and true for $n = k$ implies true for $n = k+1$, so true for all integers $n \geq 1$ by induction

A1

$$\text{Answer: } A^2 = 2A - I \text{ and } A^n = nA - (n-1)I \text{ for all integers } n \geq 1$$

5 marks**10 (4 marks)**

$$n = 1: \text{both sides equal the given matrix, true}$$

B1

$$\text{Assume true for } n = k. \text{ Then } A^{k+1} = A^k A = \begin{pmatrix} 2^k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (-1)^k \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

M1

$$= \begin{pmatrix} 2^{k+1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (-1)^{k+1} \end{pmatrix}$$

A1

True for $n = 1$, and true for $n = k$ implies true for $n = k+1$, so true for all integers $n \geq 1$ by induction

A1

$$\text{Answer: } \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}^n = \begin{pmatrix} 2^n & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (-1)^n \end{pmatrix} \text{ for all integers } n \geq 1$$

4 marks

11 (5 marks)

$$n = 1: \text{RHS} = \begin{pmatrix} 1 & 2 - 1 \\ 0 & 2 \end{pmatrix} = \text{LHS, true}$$

B1

$$\text{Assume true for } n = k. \text{ Then } A^{k+1} = A^k A = \begin{pmatrix} 1 & 2^k - 1 \\ 0 & 2^k \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 + 2(2^k - 1) \\ 0 & 2 \times 2^k \end{pmatrix}$$

M1 A1

$$= \begin{pmatrix} 1 & 2^{k+1} - 1 \\ 0 & 2^{k+1} \end{pmatrix}$$

A1

True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction

A1

$$\text{Answer: } \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}^n = \begin{pmatrix} 1 & 2^n - 1 \\ 0 & 2^n \end{pmatrix} \text{ for all integers } n \geq 1$$

5 marks**12 (5 marks)**

$$n = 1: \text{RHS} = \begin{pmatrix} 2 & 3 \times 1 \\ 0 & 1 \end{pmatrix} = \text{LHS, true}$$

B1

$$\text{Assume true for } n = k. \text{ Then } A^{k+1} = A^k A = \begin{pmatrix} 2^k & 3(2^k - 1) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2^{k+1} & 3 \times 2^k + 3(2^k - 1) \\ 0 & 1 \end{pmatrix}$$

M1 A1

$$= \begin{pmatrix} 2^{k+1} & 3(2^{k+1} - 1) \\ 0 & 1 \end{pmatrix}$$

A1

True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction

A1

$$\text{Answer: } \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 2^n & 3(2^n - 1) \\ 0 & 1 \end{pmatrix} \text{ for all integers } n \geq 1$$

5 marks**13 (4 marks)**

$$n = 1: A^1 = A = 2^0 A, \text{ true}$$

B1

$$\text{Assume } A^k = 2^{k-1} A. \text{ Then } A^{k+1} = A^k A = 2^{k-1} A^2$$

M1

$$A^2 = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = 2A, \text{ so } A^{k+1} = 2^{k-1} \times 2A = 2^k A$$

A1

True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction

A1

$$\text{Answer: } A^n = 2^{n-1} A \text{ for all integers } n \geq 1$$

4 marks**14 (6 marks)**

$$n = 1: \text{RHS} = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix} = \text{LHS, true}$$

B1

$$\text{Assume true for } n = k. \text{ Then } A^{k+1} = A^k A = \begin{pmatrix} 2k + 1 & -2k \\ 2k & 1 - 2k \end{pmatrix} \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}$$

M1

$$= \begin{pmatrix} 3(2k + 1) - 4k & -2(2k + 1) + 2k \\ 6k + 2(1 - 2k) & -4k - (1 - 2k) \end{pmatrix}$$

M1 A1

$$= \begin{pmatrix} 2k + 3 & -2k - 2 \\ 2k + 2 & -2k - 1 \end{pmatrix} = \begin{pmatrix} 2(k + 1) + 1 & -2(k + 1) \\ 2(k + 1) & 1 - 2(k + 1) \end{pmatrix}$$

A1

True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction

A1

$$\text{Answer: } \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}^n = \begin{pmatrix} 2n + 1 & -2n \\ 2n & 1 - 2n \end{pmatrix} \text{ for all integers } n \geq 1$$

6 marks

15 (6 marks)

$$n = 1: \text{RHS} = \begin{pmatrix} 2 & 1 \times 2^0 \\ 0 & 2 \end{pmatrix} = \text{LHS, true}$$

B1

$$\text{Assume true for } n = k. \text{ Then } A^{k+1} = A^k A = \begin{pmatrix} 2^k & k2^{k-1} \\ 0 & 2^k \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

M1

$$\text{Top right: } 2^k \times 1 + k2^{k-1} \times 2 = 2^k + k2^k = (k+1)2^k$$

M1 A1

$$\text{So } A^{k+1} = \begin{pmatrix} 2^{k+1} & (k+1)2^k \\ 0 & 2^{k+1} \end{pmatrix}, \text{ the result for } n = k+1$$

A1

True for $n = 1$, and true for $n = k$ implies true for $n = k+1$, so true for all integers $n \geq 1$ by induction

A1

$$\text{Answer: } \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}^n = \begin{pmatrix} 2^n & n2^{n-1} \\ 0 & 2^n \end{pmatrix} \text{ for all integers } n \geq 1$$

6 marks**16 (7 marks)**

$$n = 1: \text{RHS} = \frac{1}{2} \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \text{LHS, true}$$

B1

$$\text{Assume true for } n = k. \text{ Then } A^{k+1} = \frac{1}{2} \begin{pmatrix} 3^k + 1 & 3^k - 1 \\ 3^k - 1 & 3^k + 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

M1

$$\text{Top left: } \frac{1}{2}[2(3^k + 1) + (3^k - 1)] = \frac{1}{2}(3^{k+1} + 1)$$

M1 A1

$$\text{Top right: } \frac{1}{2}[(3^k + 1) + 2(3^k - 1)] = \frac{1}{2}(3^{k+1} - 1)$$

A1

$$\text{Bottom row: } \frac{1}{2}[2(3^k - 1) + (3^k + 1)] = \frac{1}{2}(3^{k+1} - 1) \text{ and } \frac{1}{2}[(3^k - 1) + 2(3^k + 1)] = \frac{1}{2}(3^{k+1} + 1),$$

A1

so A^{k+1} has the form of the formula with $n = k+1$

True for $n = 1$, and true for $n = k$ implies true for $n = k+1$, so true for all integers $n \geq 1$ by induction

A1

$$\text{Answer: } \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^n = \frac{1}{2} \begin{pmatrix} 3^n + 1 & 3^n - 1 \\ 3^n - 1 & 3^n + 1 \end{pmatrix} \text{ for all integers } n \geq 1$$

7 marks**17 (7 marks)**

$$n = 1: \text{RHS} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \text{LHS, true}$$

B1

$$\text{Assume true for } n = k. \text{ Then } A^{k+1} = A^k A = \begin{pmatrix} 1 & k & \frac{k(k-1)}{2} \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

M1

$$\text{Row 1: } (1, 1+k, k + \frac{k(k-1)}{2}) \text{ and row 2: } (0, 1, 1+k), \text{ row 3: } (0, 0, 1)$$

M1 A1

$$k + \frac{k(k-1)}{2} = \frac{k(k+1)}{2} = \frac{(k+1)((k+1)-1)}{2}$$

M1

So A^{k+1} has the required form with $n = k+1$

A1

True for $n = 1$, and true for $n = k$ implies true for $n = k+1$, so true for all integers $n \geq 1$ by induction

A1

$$\text{Answer: } \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n & \frac{n(n-1)}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix} \text{ for all integers } n \geq 1$$

7 marks

18 (7 marks)

$n = 1$: both sides equal the rotation matrix, true

B1

Assume true for $n = k$. Then $R^{k+1} = R^k R = \begin{pmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$

M1

Top left: $\cos k\theta \cos \theta - \sin k\theta \sin \theta = \cos (k+1)\theta$

M1 A1

Top right: $-\cos k\theta \sin \theta - \sin k\theta \cos \theta = -\sin (k+1)\theta$

A1

Bottom row similarly gives $\sin (k+1)\theta$ and $\cos (k+1)\theta$, the result for $n = k+1$

A1

True for $n = 1$, and true for $n = k$ implies true for $n = k+1$, so true for all integers $n \geq 1$ by induction

A1

Answer: for all integers $n \geq 1$, $\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}^n = \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix}$

7 marks

19 (6 marks)

$A^2 = \begin{pmatrix} 10 & -18 \\ 3 & -5 \end{pmatrix}$ and $3A - 2I = \begin{pmatrix} 10 & -18 \\ 3 & -5 \end{pmatrix}$, so $A^2 = 3A - 2I$

B1

$n = 1$: $(2 - 1)A - 0 \times I = A$, true

B1

Assume $A^k = (2^k - 1)A - (2^k - 2)I$. Then $A^{k+1} = (2^k - 1)A^2 - (2^k - 2)A$

M1

$= (2^k - 1)(3A - 2I) - (2^k - 2)A$

M1

$= (3 \times 2^k - 3 - 2^k + 2)A - 2(2^k - 1)I$

A1

$= (2^{k+1} - 1)A - (2^{k+1} - 2)I$, the result for $n = k+1$. True for $n = 1$, and true for $n = k$ implies true for $n = k+1$, so true for all integers $n \geq 1$ by induction

A1

Answer: $A^2 = 3A - 2I$, and hence prove by induction that $A^n = (2^n - 1)A - (2^n - 2)I$ for all integers $n \geq 1$

6 marks

20 (7 marks)

$n = 1$: $RHS = \begin{pmatrix} 1 \times 1 & -1 \\ 1 & 1 \times 3 \end{pmatrix} = A = LHS$, true

B1

Assume true for $n = k$. Then $A^{k+1} = A^k A = \begin{pmatrix} 2^{k-1}(2-k) & -k2^{k-1} \\ k2^{k-1} & 2^{k-1}(2+k) \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$

M1

Top left: $2^{k-1}(2-k) - k2^{k-1} = 2^{k-1}(2-2k) = 2^k(1-k)$

M1 A1

Top right: $-2^{k-1}(2-k) - 3k2^{k-1} = -2^{k-1}(2+2k) = -(k+1)2^k$

A1

Bottom row: $k2^{k-1} + 2^{k-1}(2+k) = 2^k(k+1)$ and $-k2^{k-1} + 3 \times 2^{k-1}(2+k) = 2^k(k+3)$

These equal the entries of the formula with $n = k+1$: $2^k(2 - (k+1))$, $-(k+1)2^k$, $(k+1)2^k$, $2^k(2 + k + 1)$

A1

True for $n = 1$, and true for $n = k$ implies true for $n = k+1$, so true for all integers $n \geq 1$ by induction

A1

Answer: $\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}^n = \begin{pmatrix} 2^{n-1}(2-n) & -n2^{n-1} \\ n2^{n-1} & 2^{n-1}(2+n) \end{pmatrix}$ for all integers $n \geq 1$

7 marks

21 (9 marks)

(a) $n = 1$: $RHS = \begin{pmatrix} F_2 & F_1 \\ F_1 & F_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = M$, true

B1

Assume $M^k = \begin{pmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{pmatrix}$. Then $M^{k+1} = M^k M$

M1

$= \begin{pmatrix} F_{k+1} + F_k & F_{k+1} \\ F_k + F_{k-1} & F_k \end{pmatrix}$

M1 A1

$= \begin{pmatrix} F_{k+2} & F_{k+1} \\ F_{k+1} & F_k \end{pmatrix}$ using the definition, the result for $n = k+1$

A1

True for $n = 1$ and true for $n = k$ implies true for $n = k+1$, so true for all $n \geq 1$

A1

Answer: $M^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$ for all integers $n \geq 1$

6 marks

(b) $\det M = -1$, so $\det (M^n) = (-1)^n$

M1 A1

$$\det(M^n) = F_{n+1}F_{n-1} - F_n^2, \text{ so } F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

A1

$$\text{Answer: } F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

3 marks

22 (8 marks)

$$n = 1: \begin{pmatrix} 6-2 & 4-6 \\ 3-2 & 4-3 \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix} = A, \text{ true}$$

B1

$$\text{Assume the formula for } n = k. \text{ Then } A^{k+1} = A^k A, \text{ with } A = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$$

M1

$$\text{Top left: } 4(2 \times 3^k - 2^k) + (2^{k+1} - 2 \times 3^k) = 6 \times 3^k - 2 \times 2^k = 2 \times 3^{k+1} - 2^{k+1}$$

M1 A1

$$\text{Top right: } -2(2 \times 3^k - 2^k) + (2^{k+1} - 2 \times 3^k) = 2^{k+2} - 6 \times 3^k = 2^{k+2} - 2 \times 3^{k+1}$$

M1 A1

$$\text{Bottom left: } 4(3^k - 2^k) + (2^{k+1} - 3^k) = 3^{k+1} - 2^{k+1}$$

$$\text{Bottom right: } -2(3^k - 2^k) + (2^{k+1} - 3^k) = 2^{k+2} - 3^{k+1}$$

A1

These are the entries of the formula with $n = k + 1$. True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction

A1

$$\text{Answer: } A^n = \begin{pmatrix} 2 \times 3^n - 2^n & 2^{n+1} - 2 \times 3^n \\ 3^n - 2^n & 2^{n+1} - 3^n \end{pmatrix} \text{ for all integers } n \geq 1$$

8 marks

23 (7 marks)

$$n = 1: \text{ the formula gives } \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} = T, \text{ true}$$

B1

$$\text{Assume true for } n = k. \text{ Then } T^{k+1} = T^k T = \begin{pmatrix} 1 & ka & kb + \frac{k(k-1)ac}{2} \\ 0 & 1 & kc \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

M1

$$\text{Row 1: } (1, a + ka, b + kac + kb + \frac{k(k-1)ac}{2})$$

M1 A1

$$\text{Row 2: } (0, 1, c + kc), \text{ row 3: } (0, 0, 1)$$

A1

$$b + kb = (k+1)b \text{ and } kac + \frac{k(k-1)ac}{2} = \frac{k(k+1)ac}{2} = \frac{(k+1)((k+1)-1)ac}{2}$$

M1

So T^{k+1} has the required form with $n = k + 1$. True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction

A1

$$\text{Answer: } T^n = \begin{pmatrix} 1 & na & nb + \frac{n(n-1)ac}{2} \\ 0 & 1 & nc \\ 0 & 0 & 1 \end{pmatrix} \text{ for all integers } n \geq 1$$

7 marks

24 (6 marks)

$$n = 2: F_2A + F_1I = A + I = A^2, \text{ true}$$

B1

$$\text{Assume } A^k = F_kA + F_{k-1}I. \text{ Then } A^{k+1} = A \times A^k = F_kA^2 + F_{k-1}A$$

M1

$$= F_k(A + I) + F_{k-1}A$$

M1

$$= (F_k + F_{k-1})A + F_kI$$

A1

$$= F_{k+1}A + F_kI, \text{ the result for } n = k + 1$$

A1

True for $n = 2$ and true for $n = k$ implies true for $n = k + 1$, so true for all $n \geq 2$

A1

$$\text{Answer: } A^n = F_nA + F_{n-1}I \text{ for all integers } n \geq 2$$

6 marks

25 (7 marks)

$$A^2 = \begin{pmatrix} 1 & 8 \\ 0 & 9 \end{pmatrix}, \text{ but the claim gives } \begin{pmatrix} 1 & 4 \\ 0 & 9 \end{pmatrix}, \text{ so the claim is false}$$

B1

Error: matrix multiplication is row by column, not entry by entry; the top right entry of A^2 is $1 \times 2 + 2 \times 3 = 8$

E1

$$\text{Conjecture: } A^n = \begin{pmatrix} 1 & 3^n - 1 \\ 0 & 3^n \end{pmatrix} \quad (n = 1: \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}; n = 2: \begin{pmatrix} 1 & 8 \\ 0 & 9 \end{pmatrix})$$

B1

$$\text{Assume } A^k = \begin{pmatrix} 1 & 3^k - 1 \\ 0 & 3^k \end{pmatrix}. \text{ Then } A^{k+1} = A^k A$$

M1

$$= \begin{pmatrix} 1 & 2 + 3(3^k - 1) \\ 0 & 3 \times 3^k \end{pmatrix}$$

M1 A1

$$= \begin{pmatrix} 1 & 3^{k+1} - 1 \\ 0 & 3^{k+1} \end{pmatrix}, \text{ the formula with } n = k + 1. \text{ True for } n = 1, \text{ and true for } n = k \text{ implies true for } n$$

A1

$= k + 1$, so true for all integers $n \geq 1$ by induction

$$\text{Answer: } A^n = \begin{pmatrix} 1 & 3^n - 1 \\ 0 & 3^n \end{pmatrix}$$

7 marks

26 (8 marks)

Part 1: $n = 1$ is the given condition $AB = BA$

B1

Assume $AB^k = B^k A$. Then $AB^{k+1} = (AB^k)B = B^k AB = B^k BA = B^{k+1} A$

M1 A1

So $AB^n = B^n A$ for all $n \geq 1$ by induction

A1

Part 2: $n = 1$: $(AB)^1 = A^1 B^1$, true

B1

Assume $(AB)^k = A^k B^k$. Then $(AB)^{k+1} = (AB)^k AB = A^k B^k AB$

M1

Using part 1, $B^k A = AB^k$, so this is $A^k AB^k B$

A1

$= A^{k+1} B^{k+1}$. True for $n = 1$, and $n = k$ implies $n = k + 1$, so true for all $n \geq 1$

A1

Answer: $AB^n = B^n A$ for all integers $n \geq 1$, and hence prove by induction that $(AB)^n = A^n B^n$ for all integers $n \geq 1$

27 (10 marks)

$$(a) n = 1: \frac{1}{7} \begin{pmatrix} 4 & 3 \\ 4 & 3 \end{pmatrix} + \frac{0.3}{7} \begin{pmatrix} 3 & -3 \\ -4 & 4 \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 4.9 & 2.1 \\ 2.8 & 4.2 \end{pmatrix} = P, \text{ true}$$

B1

Assume the formula for $n = k$. Then $P^{k+1} = P^k P$

M1

$$\frac{1}{7} \begin{pmatrix} 4 & 3 \\ 4 & 3 \end{pmatrix} P = \frac{1}{7} \begin{pmatrix} 4 & 3 \\ 4 & 3 \end{pmatrix} \text{ (each row } (4, 3) \text{ times } P \text{ gives } (4, 3))$$

M1 A1

$$\begin{pmatrix} 3 & -3 \\ -4 & 4 \end{pmatrix} P = \begin{pmatrix} 0.9 & -0.9 \\ -1.2 & 1.2 \end{pmatrix} = 0.3 \begin{pmatrix} 3 & -3 \\ -4 & 4 \end{pmatrix}$$

M1 A1

$$\text{So } P^{k+1} = \frac{1}{7} \begin{pmatrix} 4 & 3 \\ 4 & 3 \end{pmatrix} + \frac{1}{7} (0.3)^{k+1} \begin{pmatrix} 3 & -3 \\ -4 & 4 \end{pmatrix}. \text{ True for } n = 1, \text{ and true for } n = k \text{ implies true for } n =$$

A1

$k + 1$, so true for all integers $n \geq 1$ by induction

$$\text{Answer: } P^n = \frac{1}{7} \begin{pmatrix} 4 & 3 \\ 4 & 3 \end{pmatrix} + \frac{1}{7} (0.3)^n \begin{pmatrix} 3 & -3 \\ -4 & 4 \end{pmatrix} \text{ for all integers } n \geq 1$$

7 marks

(b) As $n \rightarrow \infty$, $(0.3)^n \rightarrow 0$

B1

$$P^n \rightarrow \frac{1}{7} \begin{pmatrix} 4 & 3 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} \frac{4}{7} & \frac{3}{7} \\ \frac{4}{7} & \frac{3}{7} \end{pmatrix}$$

B1

Whatever the starting state, the website is up on $\frac{4}{7}$ of days in the long run

E1

$$\text{Answer: } \frac{4}{7}$$

3 marks

28 (9 marks)

$n = 1$: the formula gives $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} = A$ (the top right entry is $1 \times 0 \times 2^{-2} = 0$), true B1

Assume the formula for $n = k$. Then $A^{k+1} = A^k A$, where A has columns $(2, 0, 0)$, $(1, 2, 0)$, $(0, 1, 2)$ M1

Row 1 of $A^k A$: $(2^{k+1}, 2^k + 2k2^{k-1}, k2^{k-1} + 2k(k-1)2^{k-3})$ M1 A1

Middle entry: $2^k + k2^k = (k+1)2^k$, as required for $n = k+1$ A1

Right entry: $k2^{k-1} + k(k-1)2^{k-2} = k2^{k-2}(2 + k - 1) = k(k+1)2^{k-2}$, which is $(k+1)((k+1) - 1)2^{(k+1)-3}$ M1 A1

Row 2 of $A^k A$: $(0, 2^{k+1}, 2^k + 2k2^{k-1}) = (0, 2^{k+1}, (k+1)2^k)$; row 3: $(0, 0, 2^{k+1})$ A1

All entries match the formula with $n = k+1$. True for $n = 1$, and true for $n = k$ implies true for $n = k+1$, so true for all integers $n \geq 1$ by induction A1

Answer: $A^n = \begin{pmatrix} 2^n & n2^{n-1} & n(n-1)2^{n-3} \\ 0 & 2^n & n2^{n-1} \\ 0 & 0 & 2^n \end{pmatrix}$ for all integers $n \geq 1$ 9 marks

29 (9 marks)

$D_1 = 2$ and $D_2 = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3$ B1 B1

Expand the $(n+2) \times (n+2)$ determinant along the first row: $D_{n+2} = 2 \times D_{n+1} - (-1) \times M$, where M is the minor obtained by deleting row 1 and column 2 M1

The first column of M is $(-1, 0, \dots, 0)^T$, so expanding M down that column gives $M = -D_n$ M1

Hence $D_{n+2} = 2D_{n+1} + (-D_n) = 2D_{n+1} - D_n$ A1

Base cases: $D_1 = 2 = 1 + 1$ and $D_2 = 3 = 2 + 1$ B1

Assume $D_k = k + 1$ and $D_{k+1} = k + 2$. Then $D_{k+2} = 2(k+2) - (k+1)$ M1

$= k + 3 = (k+2) + 1$, the result for $n = k+2$ A1

True for $n = 1, 2$, and $n = k, k+1$ implies $n = k+2$, so $D_n = n + 1$ for all integers $n \geq 1$ by strong induction A1

Answer: $D_{n+2} = 2D_{n+1} - D_n$, and $D_n = n + 1$ for all integers $n \geq 1$ 9 marks

30 (10 marks)

(a) $n = 1$: the formula gives $\begin{pmatrix} 1 & 1 & 3 - 4 + 1 \\ 0 & 2 & 2(3 - 2) \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix} = A$, true B1

Assume the formula for $n = k$. Then $A^{k+1} = A^k A$, where the columns of A are $(1, 0, 0)$, $(1, 2, 0)$, $(0, 2, 3)$ M1

Entry $(1, 2)$: $1 + 2(2^k - 1) = 2^{k+1} - 1$ M1 A1

Entry $(1, 3)$: $2(2^k - 1) + 3(3^k - 2^{k+1} + 1) = 3^{k+1} - 2^{k+2} + 1$ M1 A1

Entry $(2, 3)$: $2 \times 2^k + 6(3^k - 2^k) = 2(3^{k+1} - 2^{k+1})$; entries $(2, 2) = 2^{k+1}$ and $(3, 3) = 3^{k+1}$ A1

These match the formula with $n = k+1$. True for $n = 1$, and true for $n = k$ implies true for $n = k+1$, so true for all integers $n \geq 1$ by induction A1

Answer: $A^n = \begin{pmatrix} 1 & 2^n - 1 & 3^n - 2^{n+1} + 1 \\ 0 & 2^n & 2(3^n - 2^n) \\ 0 & 0 & 3^n \end{pmatrix}$ for all integers $n \geq 1$ 8 marks

(b) $3^6 - 2^7 + 1$ M1

$= 729 - 128 + 1 = 602$ A1

Answer: 602 2 marks

Induction with Inequalities

1 (3 marks)

$$n = 1: 2 > 1$$

B1

$$n = 2: 4 > 2$$

B1

$$n = 3: 8 > 3$$

B1

Answer: $2^n > n$ for $n = 1, 2$ and 3

3 marks

2 (2 marks)

$$2^{k+1} = 2 \times 2^k$$

M1

$> 2 \times k^2 = 2k^2$, since multiplying both sides of the inequality by 2 preserves it

A1

Answer: $2^{k+1} > 2k^2$

2 marks

3 (3 marks)

$$n = 4: 4! = 24 \text{ and } 2^4 = 16, \text{ and } 24 > 16$$

B1 B1

$$n = 3: 3! = 6 \text{ and } 2^3 = 8, \text{ and } 6 < 8, \text{ so the inequality fails}$$

B1

Answer: $n! > 2^n$ holds for $n = 4$ but not for $n = 3$

3 marks

4 (2 marks)

The inequality is equivalent to $k^2 - 2k - 1 > 0$, that is $(k - 1)^2 > 2$

M1

For $k \geq 3$, $(k - 1)^2 \geq 4 > 2$, so it is true

A1

Answer: $k^2 + 2k + 1 < 2k^2$ for all integers $k \geq 3$

2 marks

5 (3 marks)

$$n = 1: 4 \geq 4$$

B1

$$n = 2: 16 \geq 7$$

B1

$$n = 3: 64 \geq 10$$

B1

Answer: $4^n \geq 1 + 3n$ for $n = 1, 2$ and 3

3 marks

6 (2 marks)

$$n = 2: 4 \text{ and } 4; n = 3: 8 \text{ and } 9; n = 4: 16 \text{ and } 16 \text{ (so the inequality fails for } n = 2, 3, 4)$$

M1

$$n = 5: 32 > 25, \text{ so } N = 5$$

A1

Answer: $N = 5$

2 marks

7 (4 marks)

$$n = 1: 2 \geq 2, \text{ true}$$

B1

$$\text{Assume } 2^k \geq k + 1. \text{ Then } 2^{k+1} = 2 \times 2^k \geq 2(k + 1)$$

M1

$$2(k + 1) = 2k + 2 \geq k + 2 = (k + 1) + 1, \text{ so } 2^{k+1} \geq (k + 1) + 1$$

A1

True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction

A1

Answer: $2^n \geq n + 1$ for all integers $n \geq 1$

4 marks

8 (4 marks)

$$n = 1: 3 \geq 3, \text{ true}$$

B1

$$\text{Assume } 3^k \geq 2k + 1. \text{ Then } 3^{k+1} \geq 3(2k + 1) = 6k + 3$$

M1

$$6k + 3 \geq 2k + 3 = 2(k + 1) + 1, \text{ so } 3^{k+1} \geq 2(k + 1) + 1$$

A1

True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction

A1

Answer: $3^n \geq 2n + 1$ for all integers $n \geq 1$

4 marks

9 (4 marks)

$n = 1$: $1! = 1 \geq 2^0 = 1$, true

B1

Assume $k! \geq 2^{k-1}$. Then $(k+1)! = (k+1) \times k! \geq (k+1) \times 2^{k-1}$

M1

Since $k+1 \geq 2$, $(k+1)2^{k-1} \geq 2 \times 2^{k-1} = 2^k$

A1

True for $n = 1$, and true for $n = k$ implies true for $n = k+1$, so true for all integers $n \geq 1$ by induction

A1

Answer: $n! \geq 2^{n-1}$ for all integers $n \geq 1$

4 marks

10 (5 marks)

$n = 2$: $2! = 2 < 4 = 2^2$, true

B1

Assume $k! < k^k$ for some $k \geq 2$. Then $(k+1)! = (k+1)k! < (k+1)k^k$

M1

Since $k < k+1$, $k^k < (k+1)^k$

M1

So $(k+1)k^k < (k+1)(k+1)^k = (k+1)^{k+1}$, hence $(k+1)! < (k+1)^{k+1}$

A1

True for $n = 2$, and true for $n = k$ implies true for $n = k+1$, so true for all integers $n \geq 2$ by induction

A1

Answer: $n! < n^n$ for all integers $n \geq 2$

5 marks

11 (4 marks)

$n = 4$: $4! = 24 > 16 = 2^4$, true

B1

Assume $k! > 2^k$ for some $k \geq 4$. Then $(k+1)! = (k+1)k! > (k+1)2^k$

M1

Since $k+1 \geq 5 > 2$, $(k+1)2^k > 2 \times 2^k = 2^{k+1}$

A1

True for $n = 4$, and true for $n = k$ implies true for $n = k+1$, so true for all integers $n \geq 4$ by induction

A1

Answer: $n! > 2^n$ for all integers $n \geq 4$

4 marks

12 (5 marks)

$n = 5$: $2^5 = 32 > 25 = 5^2$, true

B1

Assume $2^k > k^2$ for some $k \geq 5$. Then $2^{k+1} > 2k^2$

M1

$2k^2 - (k+1)^2 = k^2 - 2k - 1 = (k-1)^2 - 2 > 0$ for $k \geq 5$

M1

Hence $2^{k+1} > 2k^2 > (k+1)^2$

A1

True for $n = 5$, and true for $n = k$ implies true for $n = k+1$, so true for all integers $n \geq 5$ by induction

A1

Answer: $2^n > n^2$ for all integers $n \geq 5$

5 marks

13 (5 marks)

$n = 3$: $3^3 = 27 > 16 = 4^2$, true

B1

Assume $3^k > (k+1)^2$ for some $k \geq 3$. Then $3^{k+1} > 3(k+1)^2$

M1

$3(k+1)^2 - (k+2)^2 = 2k^2 + 2k - 1$

M1

For $k \geq 3$ this is positive, so $3^{k+1} > (k+2)^2 = ((k+1)+1)^2$

A1

True for $n = 3$, and true for $n = k$ implies true for $n = k+1$, so true for all integers $n \geq 3$ by induction

A1

Answer: $3^n > (n+1)^2$ for all integers $n \geq 3$

5 marks

14 (6 marks)

$n = 1$: $(1+x)^1 = 1+x \geq 1+x$, true

B1

Assume $(1+x)^k \geq 1+kx$

M1

Since $1+x > 0$, multiplying by $(1+x)$ keeps the inequality: $(1+x)^{k+1} \geq (1+kx)(1+x)$

M1 A1

$(1+kx)(1+x) = 1 + (k+1)x + kx^2 \geq 1 + (k+1)x$ since $kx^2 \geq 0$

A1

True for $n = 1$, and true for $n = k$ implies true for $n = k+1$, so true for all integers $n \geq 1$ by induction

A1

Answer: $(1+x)^n \geq 1+nx$ for all integers $n \geq 1$, where x is a real constant with $x > -1$

6 marks

15 (6 marks)

$n = 1$: LHS = 1, RHS = $2 - 1 = 1$, so true

B1

Assume the sum to k is at most $2 - \frac{1}{k}$. Sum to $k + 1 \leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2}$

M1

$$\frac{1}{(k+1)^2} < \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

M1 A1

So the sum to $k + 1 < 2 - \frac{1}{k} + \frac{1}{k} - \frac{1}{k+1} = 2 - \frac{1}{k+1}$

A1

True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction

A1

Answer: $\sum_{r=1}^n \frac{1}{r^2} \leq 2 - \frac{1}{n}$ for all integers $n \geq 1$

6 marks

16 (6 marks)

$n = 1$: LHS = 1, RHS = 1, true

B1

Assume the sum to k is at least $\sqrt{k}$. Sum to $k + 1 \geq \sqrt{k} + \frac{1}{\sqrt{k+1}}$

M1

It is enough to show $\sqrt{k} + \frac{1}{\sqrt{k+1}} \geq \sqrt{k+1}$; multiplying by $\sqrt{k+1}$ this is $\sqrt{k(k+1)} + 1 \geq k + 1$

M1 A1

$\sqrt{k(k+1)} \geq \sqrt{k^2} = k$, so it is true

A1

True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction

A1

Answer: $\sum_{r=1}^n \frac{1}{\sqrt{r}} \geq \sqrt{n}$ for all integers $n \geq 1$

6 marks

17 (6 marks)

$n = 10$: $2^{10} = 1024 > 1000 = 10^3$, true

B1

Assume $2^k > k^3$ for some $k \geq 10$. Then $2^{k+1} > 2k^3$

M1

Show $2k^3 \geq (k+1)^3$, that is $(1 + \frac{1}{k})^3 \leq 2$

M1

For $k \geq 10$, $(1 + \frac{1}{k})^3 \leq (1.1)^3 = 1.331 < 2$

A1 A1

Hence $2^{k+1} > (k+1)^3$. True for $n = 10$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 10$ by induction

A1

Answer: $2^n > n^3$ for all integers $n \geq 10$

6 marks

18 (5 marks)

$n = 1$: $2! = 2$ and $2^1 \times 1! = 2$, true

B1

Assume $(2k)! \geq 2^k \times k!$. Then $(2k+2)! = (2k+2)(2k+1)(2k)! \geq (2k+2)(2k+1) \times 2^k \times k!$

M1

Now $(2k+2)(2k+1) = 2(k+1)(2k+1) \geq 2(k+1) \times 2$ since $2k+1 \geq 2$

M1 A1

So $(2k+2)! \geq 2(k+1) \times 2 \times 2^k \times k! \geq 2^{k+1}(k+1)k! = 2^{k+1}(k+1)!$, the result for $n = k + 1$. True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction

A1

Answer: $(2n)! \geq 2^n \times n!$ for all integers $n \geq 1$

5 marks

19 (5 marks)

$$n = 1: \text{LHS} = 1 > \frac{1}{3} = \text{RHS, true}$$

B1

$$\text{Assume } \sum_{r=1}^k r^2 > \frac{k^3}{3}. \text{ Then } \sum_{r=1}^{k+1} r^2 > \frac{k^3}{3} + (k+1)^2$$

M1

$$\frac{k^3}{3} + (k+1)^2 - \frac{(k+1)^3}{3} = \frac{k^3 + 3(k+1)^2 - (k+1)^3}{3} = \frac{3k^2 + 6k + 3 - 3k^2 - 3k - 1}{3} = \frac{3k + 2}{3} > 0$$

M1 A1

$$\text{So } \sum_{r=1}^{k+1} r^2 > \frac{(k+1)^3}{3}. \text{ True for } n = 1, \text{ and true for } n = k \text{ implies true for } n = k + 1, \text{ so true for all}$$

A1integers $n \geq 1$ by induction

$$\text{Answer: } \sum_{r=1}^n r^2 > \frac{n^3}{3} \text{ for all integers } n \geq 1$$

5 marks**20 (6 marks)**

$$n = 1: |\sin \theta| \leq |\sin \theta|, \text{ true}$$

B1

$$\text{Assume } |\sin k\theta| \leq k|\sin \theta|. \text{ Then } \sin(k+1)\theta = \sin k\theta \cos \theta + \cos k\theta \sin \theta$$

M1

$$|\sin(k+1)\theta| \leq |\sin k\theta| |\cos \theta| + |\cos k\theta| |\sin \theta|$$

M1 A1

$$\text{Since } |\cos \theta| \leq 1 \text{ and } |\cos k\theta| \leq 1, \text{ this is } \leq |\sin k\theta| + |\sin \theta| \leq k|\sin \theta| + |\sin \theta|$$

M1

$$= (k+1)|\sin \theta|. \text{ True for } n = 1, \text{ and true for } n = k \text{ implies true for } n = k + 1, \text{ so true for all}$$

A1integers $n \geq 1$ by induction

$$\text{Answer: } |\sin n\theta| \leq n|\sin \theta| \text{ for all integers } n \geq 1 \text{ and all real } \theta$$

6 marks**21 (9 marks)**

$$(a) n = 1: u_1 = 1 < 2, \text{ true}$$

B1

$$\text{Assume } u_k < 2. \text{ Then } 2 + u_k < 4, \text{ so } u_{k+1} = \sqrt{2 + u_k} < 2$$

M1 A1True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction**A1**

$$\text{Answer: } u_n < 2 \text{ for all integers } n \geq 1$$

4 marks

$$(b) n = 1: u_2 = \sqrt{3} > 1 = u_1, \text{ true}$$

B1

$$\text{Assume } u_{k+1} > u_k. \text{ Then } 2 + u_{k+1} > 2 + u_k$$

M1

$$\text{So } \sqrt{2 + u_{k+1}} > \sqrt{2 + u_k}, \text{ that is } u_{k+2} > u_{k+1}$$

M1 A1True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction**A1**

$$\text{Answer: } u_{n+1} > u_n \text{ for all integers } n \geq 1$$

5 marks**22 (7 marks)**

$$n = 1: \text{LHS} = 1 < 2 = \text{RHS, true}$$

B1

$$\text{Assume the sum to } k \text{ is less than } 2\sqrt{k}. \text{ Sum to } k+1 < 2\sqrt{k} + \frac{1}{\sqrt{k+1}}$$

M1

$$\text{It is enough to show } \frac{1}{\sqrt{k+1}} \leq 2(\sqrt{k+1} - \sqrt{k})$$

M1

$$2(\sqrt{k+1} - \sqrt{k}) = \frac{2}{\sqrt{k+1} + \sqrt{k}} \text{ using } (a-b)(a+b) = a^2 - b^2$$

M1 A1

$$\text{Since } \sqrt{k} < \sqrt{k+1}, \sqrt{k+1} + \sqrt{k} < 2\sqrt{k+1}, \text{ so } \frac{2}{\sqrt{k+1} + \sqrt{k}} > \frac{1}{\sqrt{k+1}}$$

A1So the sum to $k+1 < 2\sqrt{k+1}$. True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction**A1**

$$\text{Answer: } \sum_{r=1}^n \frac{1}{\sqrt{r}} < 2\sqrt{n} \text{ for all integers } n \geq 1$$

7 marks

23 (6 marks)

$n = 2: 4 = 4$, $n = 3: 8 < 9$ and $n = 4: 16 = 16$, so it fails for $n = 2, 3, 4$

B1 B1

The step $k^2 \geq 2k + 1$ is the invalid one: it needs $(k - 1)^2 \geq 2$

E1

$k^2 \geq 2k + 1$ fails for $k = 1$ and $k = 2$ ($1 < 3$, $4 < 5$) but holds for $k \geq 3$

M1

So the step is valid only for $k \geq 3$, but the base case $n = 1$ cannot lead there. A base case where the statement holds and $k \geq 3$ is needed: $n = 5$ ($32 > 25$)

E1

So the argument proves the result for all $n \geq 5$

A1

Answer: Fails for $n = 2, 3, 4$; step needs $k \geq 3$; valid for $n \geq 5$

6 marks

24 (7 marks)

$n = 1$: LHS = $1 + x$, RHS = $1 + x + 0$, true

B1

Assume $(1 + x)^k \geq 1 + kx + \frac{k(k-1)}{2}x^2$

M1

Since $1 + x > 0$, $(1 + x)^{k+1} \geq (1 + x)(1 + kx + \frac{k(k-1)}{2}x^2)$

M1 A1

Expand: $1 + (k+1)x + [k + \frac{k(k-1)}{2}]x^2 + \frac{k(k-1)}{2}x^3$

M1

$k + \frac{k(k-1)}{2} = \frac{k(k+1)}{2}$

A1

The x^3 term is non-negative, so $(1 + x)^{k+1} \geq 1 + (k+1)x + \frac{(k+1)k}{2}x^2$. True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction

A1

Answer: $(1 + x)^n \geq 1 + nx + \frac{n(n-1)}{2}x^2$ for all integers $n \geq 1$, where $x \geq 0$

7 marks

25 (8 marks)

$n = 1$: $F_1 = 1 \geq (\frac{3}{2})^{-1} = \frac{2}{3}$, true; $n = 2$: $F_2 = 1 \geq 1$, true

B1 B1

Assume $F_k \geq (\frac{3}{2})^{k-2}$ and $F_{k+1} \geq (\frac{3}{2})^{k-1}$

M1

$F_{k+2} = F_{k+1} + F_k \geq (\frac{3}{2})^{k-1} + (\frac{3}{2})^{k-2}$

M1 A1

$= (\frac{3}{2})^{k-2}(\frac{3}{2} + 1) = (\frac{3}{2})^{k-2} \times \frac{5}{2}$

M1

$\frac{5}{2} > \frac{9}{4} = (\frac{3}{2})^2$, so $F_{k+2} \geq (\frac{3}{2})^k$

A1

True for $n = 1, 2$, and $n = k$, $k + 1$ implies $n = k + 2$, so true for all $n \geq 1$

A1

Answer: $F_n \geq (\frac{3}{2})^{n-2}$ for all integers $n \geq 1$

8 marks

26 (7 marks)

$n = 1$: $S_1 = \frac{1}{2} \geq \frac{1}{2}$, true

B1

Assume $S_k \geq \frac{1}{2}$

M1

$S_{k+1} = \frac{1}{k+2} + \dots + \frac{1}{2k} + \frac{1}{2k+1} + \frac{1}{2k+2}$

M1

$S_{k+1} - S_k = \frac{1}{2k+1} + \frac{1}{2k+2} - \frac{1}{k+1}$

M1 A1

$= \frac{1}{2k+1} - \frac{1}{2k+2} > 0$

M1

So $S_{k+1} > S_k \geq \frac{1}{2}$. True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction

A1

Answer: $S_n \geq \frac{1}{2}$ for all integers $n \geq 1$

7 marks

27 (10 marks)

$$n = 1: 1! = 1 \leq 1^1 = 1, \text{ true}$$

B1

$$\text{Assume } k! \leq \left(\frac{k+1}{2}\right)^k. \text{ Then } (k+1)! = (k+1)k! \leq (k+1)\left(\frac{k+1}{2}\right)^k$$

M1 A1

$$\text{Write } k+1 = 2 \times \left(\frac{k+1}{2}\right), \text{ so } (k+1)! \leq 2\left(\frac{k+1}{2}\right)^{k+1}$$

M1

$$\text{We need } 2\left(\frac{k+1}{2}\right)^{k+1} \leq \left(\frac{k+2}{2}\right)^{k+1}, \text{ that is } \left(\frac{k+2}{k+1}\right)^{k+1} \geq 2$$

M1 A1

$$\frac{k+2}{k+1} = 1 + \frac{1}{k+1}$$

B1

$$\text{By the given result with } x = \frac{1}{k+1} \text{ and } m = k+1: \left(1 + \frac{1}{k+1}\right)^{k+1} \geq 1 + (k+1) \times \frac{1}{k+1} = 2$$

M1 A1

True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction

A1

$$\text{Answer: } n! \leq \left(\frac{n+1}{2}\right)^n \text{ for all integers } n \geq 1$$

10 marks

28 (9 marks)

$$n = 2: \text{LHS} = 1 + \frac{1}{8} = \frac{9}{8}, \text{ RHS} = \frac{5}{4} - \frac{1}{8} = \frac{9}{8}, \text{ true}$$

B1

$$\text{Assume the sum to } k \text{ is at most } \frac{5}{4} - \frac{1}{2k^2} \text{ for some } k \geq 2. \text{ Then sum to } k+1 \leq \frac{5}{4} - \frac{1}{2k^2} +$$

M1 A1

$$\frac{1}{(k+1)^3}$$

$$\text{It is enough to show } \frac{1}{(k+1)^3} \leq \frac{1}{2k^2} - \frac{1}{2(k+1)^2}$$

M1

$$\text{RHS} = \frac{(k+1)^2 - k^2}{2k^2(k+1)^2} = \frac{2k+1}{2k^2(k+1)^2}$$

M1 A1

$$\text{The inequality becomes } \frac{1}{k+1} \leq \frac{2k+1}{2k^2}, \text{ that is } 2k^2 \leq (2k+1)(k+1) = 2k^2 + 3k + 1, \text{ which is}$$

M1 A1

true

True for $n = 2$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 2$ by induction

A1

$$\text{Answer: } \sum_{r=1}^n \frac{1}{r^3} \leq \frac{5}{4} - \frac{1}{2n^2} \text{ for all integers } n \geq 2$$

9 marks

29 (10 marks)

$$n = 1: C(2, 1) = 2 \text{ and } \frac{4}{2} = 2, \text{ true}$$

B1

$$\text{Assume } C(2k, k) \geq \frac{4^k}{2k}$$

M1

$$C(2k+2, k+1) = \frac{(2k+2)!}{((k+1)!)^2} = \frac{(2k+2)(2k+1)}{(k+1)^2} C(2k, k) = \frac{2(2k+1)}{k+1} C(2k, k)$$

M1 A1

$$\text{So } C(2k+2, k+1) \geq \frac{2(2k+1)}{k+1} \times \frac{4^k}{2k} = \frac{(2k+1)4^k}{k(k+1)}$$

M1 A1

$$\text{The target is } \frac{4^{k+1}}{2(k+1)} = \frac{2 \times 4^k}{k+1}$$

B1

$$\frac{(2k+1)4^k}{k(k+1)} \geq \frac{2 \times 4^k}{k+1} \text{ because } \frac{2k+1}{k} \geq 2$$

M1 A1

True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction

A1

$$\text{Answer: } C(2n, n) \geq \frac{4^n}{2n} \text{ for all integers } n \geq 1, \text{ where } C(2n, n) = \frac{(2n)!}{(n!)^2}$$

10 marks

30 (9 marks) $n = 1: 1 \leq 1 \leq 2$, true**B1** $n = 2: 2 \leq 3 \leq 4$, true**B1**Assume $2^{k-1} \leq u_k \leq 2^k$ and $2^k \leq u_{k+1} \leq 2^{k+1}$ **M1**Upper bound: $u_{k+2} = u_{k+1} + 2u_k \leq 2^{k+1} + 2 \times 2^k = 2^{k+2}$ **M1 A1**Lower bound: $u_{k+2} \geq 2^k + 2 \times 2^{k-1} = 2^{k+1}$ **M1 A1**So $2^{k+1} \leq u_{k+2} \leq 2^{k+2}$, the result for $n = k + 2$ **A1**True for $n = 1, 2$, and $n = k, k + 1$ implies $n = k + 2$, so true for all $n \geq 1$ **A1****Answer: $2^{n-1} \leq u_n \leq 2^n$ for all integers $n \geq 1$** **9 marks****Further Proof Techniques****1 (3 marks)** $n^2 + 3n = n(n + 3)$ **M1**If n is even, the product is even**B1**If n is odd, $n + 3$ is even, so the product is even**B1****Answer: $n^2 + 3n$ is even for every positive integer n** **3 marks****2 (2 marks)**Try $n = 10: 100 + 10 + 11 = 121$ **M1** $121 = 11 \times 11$ is not prime, so the claim is false**A1****Answer: $n = 10$ gives $121 = 11^2$, which is not prime****2 marks****3 (2 marks)**Take $x = -4$ **B1** $x^2 = 16 > 9$ but $x = -4$ is not greater than 3**B1****Answer: $x = -4: x^2 = 16 > 9$ but $x < 3$** **2 marks****4 (3 marks)** $x^2 + y - (y^2 + x) = x^2 - y^2 - (x - y)$ **M1** $= (x - y)(x + y) - (x - y) = (x - y)(x + y - 1)$ **M1**Since $x + y = 1$, the second factor is 0, so the difference is 0 and $x^2 + y = y^2 + x$ **A1****Answer: $x^2 + y = y^2 + x$** **3 marks****5 (3 marks)**Let the integers be $2a + 1$ and $2b + 1$, where a and b are integers**B1** $(2a + 1)(2b + 1) = 4ab + 2a + 2b + 1$ **M1** $= 2(2ab + a + b) + 1$, which is odd**A1****Answer: the product of two odd integers is always odd****3 marks****6 (2 marks)**Assume $\sqrt{3}$ is rational: $\sqrt{3} = \frac{p}{q}$, where p and q are integers with no common factor**B1**Squaring: $3 = \frac{p^2}{q^2}$, so $p^2 = 3q^2$ **B1****Answer: $\sqrt{3} = \frac{p}{q}$ in lowest terms, so $p^2 = 3q^2$** **2 marks****7 (3 marks)** $3^{2k+2} = 9 \times 3^{2k}$ **B1** $3^{2k} + 3^{2k+2} = 3^{2k}(1 + 9) = 10 \times 3^{2k}$ **M1** 3^{2k} is an integer, so the sum is a multiple of 10**A1****Answer: $3^{2k} + 3^{2k+2} = 10 \times 3^{2k}$, a multiple of 10****3 marks**

8 (4 marks)

Let the odd integer be $2n + 1$

B1

$$(2n + 1)^2 = 4n^2 + 4n + 1$$

M1

$$= 4n(n + 1) + 1$$

A1

This is a multiple of 4 plus 1, so the remainder is 1

E1

Answer: the square of any odd integer leaves remainder 1 when divided by 4

4 marks

9 (4 marks)

Not divisible by 3 means $n = 3m + 1$ or $n = 3m + 2$ (two cases)

B1

Case 1: $n^2 - 1 = 9m^2 + 6m = 3m(3m + 2)$, divisible by 3

M1 A1

Case 2: $n^2 - 1 = 9m^2 + 12m + 3 = 3(3m^2 + 4m + 1)$, divisible by 3

A1

Answer: if n is an integer that is not divisible by 3, then $n^2 - 1$ is divisible by 3

4 marks

10 (4 marks)

Assume there are integers m and n with $6m + 9n = 100$

B1

Then $3(2m + 3n) = 100$, so 100 is divisible by 3

M1

But $100 = 3 \times 33 + 1$ is not divisible by 3, a contradiction

A1

So no such integers exist

E1

Answer: there are no integers m and n such that $6m + 9n = 100$

4 marks

11 (5 marks)

Assume $\sqrt{2} = \frac{p}{q}$, where p and q are integers with no common factor

B1

Then $2q^2 = p^2$, so p^2 is even, hence p is even; write $p = 2r$

M1 A1

Then $2q^2 = 4r^2$, so $q^2 = 2r^2$, hence q is even

M1

p and q are both even, contradicting that they have no common factor, so $\sqrt{2}$ is irrational

E1

Answer: $\sqrt{2}$ is irrational

5 marks

12 (4 marks)

$$\frac{p}{q} + \frac{q}{p} - 2 = \frac{p^2 + q^2 - 2pq}{pq}$$

M1 A1

$$= \frac{(p - q)^2}{pq}$$

M1

$(p - q)^2 \geq 0$ and $pq > 0$, so the expression is ≥ 0 , hence $\frac{p}{q} + \frac{q}{p} \geq 2$

A1

Answer: for all positive real numbers p and q , $\frac{p}{q} + \frac{q}{p} \geq 2$

4 marks

13 (4 marks)

$$\frac{3k + 1}{3k + 2} - \frac{3k}{3k + 1} = \frac{(3k + 1)^2 - 3k(3k + 2)}{(3k + 2)(3k + 1)}$$

M1 A1

$$(3k + 1)^2 - 3k(3k + 2) = 9k^2 + 6k + 1 - 9k^2 - 6k = 1$$

M1

The difference is $\frac{1}{(3k + 2)(3k + 1)}$, which is positive for $k > 0$, so the inequality holds

A1

Answer: $\frac{3k + 1}{3k + 2} > \frac{3k}{3k + 1}$ for all real $k > 0$

4 marks

14 (6 marks)

Assume $\sqrt{3} = \frac{p}{q}$, p and q integers with no common factor

B1

$3q^2 = p^2$, so p^2 is divisible by 3

M1

So p is divisible by 3: $p = 3r$

A1

$3q^2 = 9r^2$, so $q^2 = 3r^2$

M1

So q^2 is divisible by 3, hence q is divisible by 3

A1

p and q share the factor 3, a contradiction, so $\sqrt{3}$ is irrational

E1

Answer: $\sqrt{3}$ is irrational

6 marks

15 (6 marks)

$$p^2 - 1 = (p - 1)(p + 1)$$

B1

p is odd, so $p - 1$ and $p + 1$ are consecutive even numbers; one of them is a multiple of 4

M1

So $(p - 1)(p + 1)$ is divisible by $2 \times 4 = 8$

A1

Of $p - 1$, p , $p + 1$ (consecutive), one is a multiple of 3; it is not p (a prime greater than 3)

M1

So $(p - 1)(p + 1)$ is divisible by 3

A1

8 and 3 have no common factor, so $p^2 - 1$ is divisible by 24

A1

Answer: if p is a prime number greater than 3, then $p^2 - 1$ is divisible by 24

6 marks

16 (5 marks)

Assume that there is a real θ with $3 \cos \theta + 4 \sin \theta > 5$

B1

Then $(3 \cos \theta + 4 \sin \theta)^2 > 25$, since both sides are positive

M1

$$(3 \cos \theta + 4 \sin \theta)^2 + (4 \cos \theta - 3 \sin \theta)^2 = 25(\cos^2 \theta + \sin^2 \theta) = 25$$

M1 A1

So $(3 \cos \theta + 4 \sin \theta)^2 = 25 - (4 \cos \theta - 3 \sin \theta)^2 \leq 25$, which contradicts the assumption, so $3 \cos \theta + 4 \sin \theta \leq 5$ for all θ

E1

Answer: $3 \cos \theta + 4 \sin \theta \leq 5$ for all real θ

5 marks

17 (6 marks)

$$a^2 - b^2 = (a - b)(a + b)$$

B1

$a - b$ and $a + b$ have the same parity, since their difference is $2b$

M1 A1

If both are odd, the product is odd, so not of the form $4k + 2$

M1

If both are even, the product is divisible by $2 \times 2 = 4$, so not of the form $4k + 2$

M1

Hence $a^2 - b^2$ is never of the form $4k + 2$

A1

Answer: the difference of the squares of two integers can never be of the form $4k + 2$, where k is an integer

18 (5 marks)

$$(a + b)^2 + (a - b)^2 = a^2 + 2ab + b^2 + a^2 - 2ab + b^2 = 2(a^2 + b^2)$$

M1 A1

$a + b$ is a positive integer, and $a - b$ is a non-zero integer since $a \neq b$, so $(a - b)^2 = |a - b|^2$ with $|a - b|$ a positive integer

B1

$|a - b| < a + b$ because a and b are positive, so the two integers $a + b$ and $|a - b|$ are distinct

E1

So $2(a^2 + b^2) = (a + b)^2 + |a - b|^2$, a sum of two squares of distinct positive integers

A1

Answer: $2(a^2 + b^2) = (a + b)^2 + (a - b)^2$

5 marks

19 (5 marks)

Assume $\log_2 3 = \frac{p}{q}$, where p and q are positive integers

B1

Then $2^{p/q} = 3$, so $2^p = 3^q$

M1 A1

2^p is even ($p \geq 1$) but 3^q is odd

M1

This is a contradiction, so $\log_2 3$ is irrational

E1

Answer: $\log_2 3$ is irrational

5 marks

20 (5 marks)

$$a^3 + b^3 - a^2b - ab^2 = a^2(a - b) - b^2(a - b)$$

M1 A1

$$= (a - b)(a^2 - b^2) = (a - b)^2(a + b)$$

M1 A1

$(a - b)^2 \geq 0$ and $a + b > 0$, so the expression is ≥ 0 , which proves the result

A1

Answer: for all positive real numbers a and b , $a^3 + b^3 \geq a^2b + ab^2$

5 marks

21 (7 marks)

Assume $\sqrt{2} + \sqrt{3} = r$, where r is rational and $r \neq 0$

B1

$$(\sqrt{3} + \sqrt{2})(\sqrt{3} - \sqrt{2}) = 3 - 2 = 1$$

M1 A1

So $\sqrt{3} - \sqrt{2} = \frac{1}{r}$, which is rational

A1

$$\text{Adding: } 2\sqrt{3} = r + \frac{1}{r}, \text{ so } \sqrt{3} = \frac{1}{2}\left(r + \frac{1}{r}\right)$$

M1 A1

The right side is rational, contradicting that $\sqrt{3}$ is irrational, so $\sqrt{2} + \sqrt{3}$ is irrational

E1

Answer: $\sqrt{2} + \sqrt{3}$ is irrational

7 marks

22 (7 marks)

Assume there are finitely many primes $p_1, p_2, \dots, p_m$

B1

$$\text{Let } N = p_1 p_2 \dots p_m + 1$$

M1

$N > 1$, so N has a prime factor; this must be one of the p_i

M1 A1

But N divided by p_i leaves remainder 1 for every i , so no p_i divides N

M1 A1

This is a contradiction, so there are infinitely many primes

E1

Answer: there are infinitely many prime numbers

7 marks

23 (6 marks)

(a) $a = b$, so $a - b = 0$

B1

The step 'dividing by $(a - b)$ ' is a division by zero

E1

which is not allowed, so the deduction $a + b = b$ is invalid

E1

Answer: the step divides by $a - b = 0$

3 marks

(b) Checking particular cases does not prove a statement for all n

E1

$$\text{Take } n = 41: 41^2 + 41 + 41 = 41(41 + 1 + 1) = 41 \times 43$$

M1

which is not prime, so the conclusion is false

A1

Answer: $n = 41$ gives 41×43

3 marks

24 (6 marks)

$$\text{Consider } 2(a^2 + b^2 + c^2) - 2(ab + bc + ca)$$

M1

$$= (a^2 - 2ab + b^2) + (b^2 - 2bc + c^2) + (c^2 - 2ca + a^2)$$

M1 A1

$$= (a - b)^2 + (b - c)^2 + (c - a)^2$$

A1

Each square is non-negative, so the sum is ≥ 0

E1

$$\text{Dividing by 2 gives } a^2 + b^2 + c^2 \geq ab + bc + ca$$

A1

Answer: for all real numbers a, b and c , $a^2 + b^2 + c^2 \geq ab + bc + ca$

6 marks

25 (6 marks)

(a) Let the integers be $n, n + 1, n + 2, n + 3$. Pair the outer and inner: $n(n + 3) = n^2 + 3n$ and $(n + 1)(n + 2) = n^2 + 3n + 2$

M1 A1

$$\text{Let } t = n^2 + 3n. \text{ Product } + 1 = t(t + 2) + 1$$

M1

$$= t^2 + 2t + 1 = (t + 1)^2$$

A1

$$= (n^2 + 3n + 1)^2, \text{ a perfect square}$$

A1

$$\text{Answer: } n(n + 1)(n + 2)(n + 3) + 1 = (n^2 + 3n + 1)^2$$

5 marks

$$(b) n = 10: 100 + 30 + 1 = 131, \text{ so the number is } 131^2 = 17161$$

B1

$$\text{Answer: } 131^2 = 17161$$

1 mark

26 (6 marks)

Assume a and b are both odd: $a = 2m + 1, b = 2n + 1$

B1

$$a^2 + b^2 = 4m^2 + 4m + 1 + 4n^2 + 4n + 1 = 4(m^2 + m + n^2 + n) + 2$$

M1 A1

So c^2 leaves remainder 2 when divided by 4

A1

If c is even, $c^2 = 4t$ has remainder 0; if c is odd, c^2 has remainder 1

M1

Neither gives remainder 2, a contradiction, so a and b are not both odd

E1

Answer: a and b cannot both be odd

6 marks

27 (8 marks)

Case 1: x is rational. Then $\sqrt{2}$ raised to the power $\sqrt{2}$, both irrational, gives a rational number

M1 A1

Case 2: x is irrational. Consider $x^{\sqrt{2}} = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}$

M1

$$= \sqrt{2}^{\sqrt{2} \times \sqrt{2}} = \sqrt{2}^2$$

M1 A1

$= 2$, which is rational

A1

In case 2, x and $\sqrt{2}$ are both irrational and $x^{\sqrt{2}} = 2$ is rational

E1

Either way there are irrational a, b with a^b rational, which proves the statement

E1

Answer: $\sqrt{2}^{\sqrt{2}}$ is rational, or irrational with $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = 2$ rational

8 marks

28 (8 marks)

Let the integers be $n, n + 1, n + 2$. Sum $= n^3 + (n + 1)^3 + (n + 2)^3$

B1

$$= n^3 + n^3 + 3n^2 + 3n + 1 + n^3 + 6n^2 + 12n + 8$$

M1

$$= 3n^3 + 9n^2 + 15n + 9$$

A1

$$= 3(n^3 + 5n) + 9n^2 + 9$$

M1

It remains to show 3 divides $n^3 + 5n = n(n^2 + 5)$

M1

If $n = 3m$, n is divisible by 3. If $n = 3m \pm 1$, $n^2 + 5 = 9m^2 \pm 6m + 6$ is divisible by 3

M1 A1

So $n^3 + 5n = 3t$, and the sum $= 9t + 9n^2 + 9 = 9(t + n^2 + 1)$, divisible by 9

A1

Answer: the sum of the cubes of any three consecutive positive integers is divisible by 9

8 marks

29 (9 marks)

Assume there are finitely many such primes, $p_1 = 3, p_2, \dots, p_m$

B1

Let $N = 4p_1p_2 \dots p_m - 1$. Then $N = 4(p_1 \dots p_m - 1) + 3$, so N leaves remainder 3 on division by 4

M1 A1

N is odd, so all its prime factors are odd, so each is of the form $4j + 1$ or $4j + 3$

M1

The product of numbers of the form $4j + 1$ is of the form $4j + 1$: $(4a + 1)(4b + 1) = 4(4ab + a + b) + 1$

M1 A1

So N cannot have only prime factors of the form $4j + 1$, hence it has a prime factor of the form $4j + 3$

A1

Each p_i divides $4p_1 \dots p_m$, so p_i divides N only if p_i divides 1; so this prime factor is not in the list

M1

This contradicts the assumption that the list is complete, so there are infinitely many

E1

Answer: there are infinitely many prime numbers of the form $4k + 3$, where k is a non-negative integer

marks

30 (9 marks)

(a) $\cos 3\theta = \cos (2\theta + \theta) = \cos 2\theta \cos \theta - \sin 2\theta \sin \theta$

M1

$$= (2\cos^2\theta - 1)\cos \theta - 2\sin^2\theta \cos \theta = 2\cos^3\theta - \cos \theta - 2(1 - \cos^2\theta)\cos \theta = 4\cos^3\theta - 3\cos \theta$$

M1 A1

$$\theta = 20^\circ: \cos 60^\circ = \frac{1}{2} = 4x^3 - 3x, \text{ so } 8x^3 - 6x - 1 = 0$$

A1

Answer: $\cos 3\theta = 4\cos^3\theta - 3\cos \theta$, and hence show that $x = \cos 20^\circ$ is a root of $8x^3 - 6x - 1 = 0$

4 marks

(b) Suppose $\cos 20^\circ = \frac{p}{q}$ in lowest terms, with $q > 0$

B1

p divides 1 and q divides 8, so $\frac{p}{q}$ is one of $\pm 1, \pm \frac{1}{2}, \pm \frac{1}{4}, \pm \frac{1}{8}$

M1 A1

Substituting into $8x^3 - 6x - 1$: $x = 1$ gives 1, $x = -1$ gives -3 , $x = \frac{1}{2}$ gives -3 , $x = -\frac{1}{2}$ gives

M1

$$1, x = \frac{1}{4} \text{ gives } -\frac{19}{8}, x = -\frac{1}{4} \text{ gives } \frac{3}{8}, x = \frac{1}{8} \text{ gives } -\frac{111}{64}, x = -\frac{1}{8} \text{ gives } -\frac{17}{64}$$

None of the candidates is a root, a contradiction, so $\cos 20^\circ$ is irrational

E1

Answer: no rational candidate is a root, so $\cos 20^\circ$ is irrational

5 marks