Proof by Mathematical Induction

1 (2 marks)

$n = 1$: LHS = 1, RHS = $1^2 = 1$, so true — B1

$n = 2$: LHS = $1 + 3 = 4$, RHS = $2^2 = 4$, so true — B1

Answer: P(1) and P(2) are both true — 2 marks

2 (2 marks)

Replace n by k + 1 on both sides — M1

$\sum_{r=1}^{k+1} r^2 = \frac{(k + 1)(k + 2)(2k + 3)}{6}$ — A1

Answer: $\sum_{r=1}^{k+1} r^2 = \frac{(k + 1)(k + 2)(2k + 3)}{6}$ — 2 marks

3 (3 marks)

$u_2 = 7$, $u_3 = 11$, $u_4 = 15$ — B1 B1

The terms go up by 4 from 3, so $u_n = 4n - 1$ — B1

Answer: $u_n = 4n - 1$ — 3 marks

4 (2 marks)

The statement is true for n = 1, and if it is true for n = k then it is true for n = k + 1 — E1

Therefore, by mathematical induction, it is true for all positive integers n — E1

Answer: True for n = 1, and if true for n = k then true for n = k + 1; therefore, by mathematical induction, true for all — 2 marks

5 (2 marks)

$\sum_{r=1}^{k+1} r^3 = \sum_{r=1}^{k} r^3 + (k + 1)^3$ — M1

$= \frac{k^2(k + 1)^2}{4} + (k + 1)^3$ — A1

Answer: $\frac{k^2(k + 1)^2}{4} + (k + 1)^3$ — 2 marks

6 (3 marks)

$u_2 = 4 = 2^2$ — B1

$u_3 = 8 = 2^3$ — B1

$u_1 = 2 = 2^1$, so the formula works for n = 1, 2, 3 — B1

Answer: the formula $u_n = 2^n$ is correct for n = 1, 2 and 3 — 3 marks

7 (4 marks)

$n = 1$: LHS = $1 = 1^2$, true — B1

Assume true for n = k: $\sum_{r=1}^{k} (2r - 1) = k^2$. Then $\sum_{r=1}^{k+1} (2r - 1) = k^2 + 2(k + 1) - 1$ — M1

$= k^2 + 2k + 1 = (k + 1)^2$ — A1

True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers $n \geq 1$ by induction — A1

Answer: $\sum_{r=1}^{n} (2r - 1) = n^2$ for all integers $n \geq 1$ — 4 marks

8 (4 marks)

n = 1: LHS = 1, RHS = $\frac{1 \times 2}{2}$ = 1, true **B1**

Assume true for n = k. Then $\sum_{r=1}^{k+1} r = \frac{k(k+1)}{2} + (k+1)$ **M1**

$= \frac{(k+1)(k+2)}{2}$, which is the formula with n = k + 1 **A1**

True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers n ≥ 1 by induction **A1**

Answer: $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$ for all integers n ≥ 1 **4 marks**

9 (4 marks)

n = 1: LHS = 1, RHS = 2 − 1 = 1, true **B1**

Assume true for n = k. Sum of k + 1 terms = $(2^k - 1) + 2^k$ **M1**

$= 2 \times 2^k - 1 = 2^{k+1} - 1$ **A1**

True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers n ≥ 1 by induction **A1**

Answer: $1 + 2 + 4 + ... + 2^{n-1} = 2^n - 1$ for all integers n ≥ 1 **4 marks**

10 (4 marks)

n = 1: $u_1 = 5 = 3 \times 1 + 2$, true **B1**

Assume $u_k = 3k + 2$. Then $u_{k+1} = u_k + 3 = 3k + 2 + 3$ **M1**

$= 3(k + 1) + 2$ **A1**

True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers n ≥ 1 by induction **A1**

Answer: $u_n = 3n + 2$ **4 marks**

11 (4 marks)

n = 1: $u_1 = 4 = 3^1 + 1$, true **B1**

Assume $u_k = 3^k + 1$. Then $u_{k+1} = 3(3^k + 1) - 2$ **M1**

$= 3^{k+1} + 3 - 2 = 3^{k+1} + 1$ **A1**

True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers n ≥ 1 by induction **A1**

Answer: $u_n = 3^n + 1$ for all n ≥ 1 **4 marks**

12 (5 marks)

n = 1: LHS = $\frac{1}{1 \times 2} = \frac{1}{2}$, RHS = $\frac{1}{2}$, true **B1**

Assume true for n = k. Then sum to k + 1 = $\frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$ **M1**

$= \frac{k(k+2)+1}{(k+1)(k+2)}$ **M1**

$= \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}$ **A1**

True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers n ≥ 1 by induction **A1**

Answer: $\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$ for all integers n ≥ 1 **5 marks**

13 (5 marks)

n = 1: LHS = 1 × 1! = 1, RHS = 2! − 1 = 1, true **B1**

Assume true for n = k. Then sum to k + 1 = (k + 1)! − 1 + (k + 1)(k + 1)! **M1**

= (k + 1)![1 + (k + 1)] − 1 **M1**

= (k + 1)!(k + 2) − 1 = (k + 2)! − 1 **A1**

True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers n ≥ 1 by induction **A1**

Answer: $\sum_{r=1}^{n} r \times r! = (n + 1)! - 1$ for all integers n ≥ 1 **5 marks**

14 (6 marks)

n = 1: LHS = 2, RHS = 0 + 2 = 2, true **B1**

Assume true for n = k. Then sum to k + 1 = $(k - 1)2^{k+1} + 2 + (k + 1)2^{k+1}$ **M1 A1**

= $2^{k+1}[(k - 1) + (k + 1)] + 2$ **M1**

= $2k \times 2^{k+1} + 2 = k \times 2^{k+2} + 2$, which is the formula with n = k + 1 **A1**

True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers n ≥ 1 by induction **A1**

Answer: $\sum_{r=1}^{n} r \times 2^r = (n - 1)2^{n+1} + 2$ for all integers n ≥ 1 **6 marks**

15 (6 marks)

n = 2: LHS = $1 - \frac{1}{4} = \frac{3}{4}$, RHS = $\frac{3}{4}$, true **B1**

Assume true for n = k. Then the product to k + 1 = $\frac{k + 1}{2k} \times (1 - \frac{1}{(k + 1)^2})$ **M1**

$1 - \frac{1}{(k + 1)^2} = \frac{k(k + 2)}{(k + 1)^2}$ **M1**

Product = $\frac{k + 1}{2k} \times \frac{k(k + 2)}{(k + 1)^2}$ **A1**

= $\frac{k + 2}{2(k + 1)}$, which is the formula with n = k + 1 **A1**

True for n = 2, and true for n = k implies true for n = k + 1, so true for all integers n ≥ 2 **A1**

Answer: $(1 - \frac{1}{2^2})(1 - \frac{1}{3^2}) \ldots (1 - \frac{1}{n^2}) = \frac{n + 1}{2n}$ for all integers n ≥ 2 **6 marks**

16 (5 marks)

n = 1: derivative of $x e^x$ is $e^x + x e^x = (x + 1)e^x$, true **B1**

Assume the kth derivative is $(x + k)e^x$ **M1**

Differentiate: $(1 + x + k)e^x$ using the product rule **M1 A1**

= $(x + (k + 1))e^x$, the result for n = k + 1. True for n = 1 and true for n = k implies true for n = k + 1, so true for all n ≥ 1 **A1**

Answer: the nth derivative of $x e^x$ is $(x + n)e^x$, for all integers n ≥ 1 **5 marks**

17 (7 marks)

n = 1: $u_1 = 1$ and $1 + \frac{1 \times 0 \times 1}{6} = 1$, true **B1**

Assume $u_k = 1 + \frac{k(k - 1)(2k - 1)}{6}$. Then $u_{k+1} = 1 + \frac{k(k - 1)(2k - 1)}{6} + k^2$ **M1 A1**

= $1 + \frac{k[(k - 1)(2k - 1) + 6k]}{6}$ **M1**

$(k - 1)(2k - 1) + 6k = 2k^2 + 3k + 1 = (2k + 1)(k + 1)$ **M1**

So $u_{k+1} = 1 + \frac{(k + 1)k(2k + 1)}{6}$, the formula with n = k + 1 **A1**

True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers n ≥ 1 by induction **A1**

Answer: $u_n = 1 + \frac{n(n - 1)(2n - 1)}{6}$ for all integers n ≥ 1 **7 marks**

18 (4 marks)

The proof has no basis case: it never checks a starting value — **E1**

n = 1: $1^2 + 1 = 2$ is even, so the statement fails at n = 1 — **B1**

In fact $n^2 + n = n(n + 1)$ is a product of consecutive integers, so it is always even — **M1**

So the result is false for every n, although the inductive step itself is valid — **A1**

Answer: No basis case; the result is false (always even) — **4 marks**

19 (6 marks)

n = 1: $u_1 = \frac{3}{2} = \frac{1+2}{1+1}$, true — **B1**

Assume $u_k = \frac{k+2}{k+1}$. Then $2u_k - 1 = \frac{2(k+2)-(k+1)}{k+1} = \frac{k+3}{k+1}$ — **M1 A1**

$u_{k+1} = \frac{k+3}{k+1} \div \frac{k+2}{k+1}$ — **M1**

$= \frac{k+3}{k+2} = \frac{(k+1)+2}{(k+1)+1}$, the formula with n = k + 1 — **A1**

True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers $n \geq 1$ by induction — **A1**

Answer: $u_n = \frac{n+2}{n+1}$ — **6 marks**

20 (5 marks)

n = 1: LHS = $\frac{1}{1 \times 3} = \frac{1}{3}$, RHS = $\frac{1}{3}$, true — **B1**

Assume true for n = k. Sum to k + 1 = $\frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)}$ — **M1**

$= \frac{k(2k+3)+1}{(2k+1)(2k+3)} = \frac{2k^2+3k+1}{(2k+1)(2k+3)}$ — **M1**

$= \frac{(2k+1)(k+1)}{(2k+1)(2k+3)} = \frac{k+1}{2(k+1)+1}$ — **A1**

True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers $n \geq 1$ by induction — **A1**

Answer: $\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)} = \frac{n}{2n+1}$ for all integers $n \geq 1$ — **5 marks**

21 (7 marks)

n = 1: LHS = 1, RHS = $F_1F_2 = 1$, true — **B1**

Assume the sum to F_k^2 equals F_kF_{k+1} — **M1**

Sum to k + 1 terms = $F_kF_{k+1} + F_{k+1}^2$ — **M1 A1**

$= F_{k+1}(F_k + F_{k+1})$ — **M1**

$= F_{k+1}F_{k+2}$ by the definition, which is the statement for n = k + 1 — **A1**

True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers $n \geq 1$ by induction — **A1**

Answer: $F_1^2 + F_2^2 + \ldots + F_n^2 = F_nF_{n+1}$ for all integers $n \geq 1$ — **7 marks**

22 (6 marks)

n = 1: $\frac{2x-1}{x-0} = 2 - \frac{1}{x} = f(x)$, true **B1**

Assume $f^k(x) = \frac{(k+1)x-k}{kx-(k-1)}$. Then $f^{k+1}(x) = f(f^k(x)) = 2 - \frac{kx-(k-1)}{(k+1)x-k}$ **M1 A1**

$= \frac{2(k+1)x-2k-kx+(k-1)}{(k+1)x-k}$ **M1**

$= \frac{(k+2)x-(k+1)}{(k+1)x-k}$ **A1**

This is the formula with n = k + 1, since (k + 1) + 1 = k + 2 and (k + 1) − 1 = k. True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers n ≥ 1 by induction **A1**

Answer: $f^n(x) = \frac{(n+1)x-n}{nx-(n-1)}$ **6 marks**

23 (8 marks)

(a) n = 1: derivative = $e^{-x} - x\,e^{-x} = -(x-1)e^{-x} = (-1)^1(x-1)e^{-x}$, true **B1**

Assume the kth derivative is $(-1)^k(x-k)e^{-x}$ **M1**

Differentiate: $(-1)^k[e^{-x} - (x-k)e^{-x}]$ **M1 A1**

$= (-1)^k(1+k-x)e^{-x} = (-1)^{k+1}(x-(k+1))e^{-x}$, which is P(k + 1) **A1**

True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers n ≥ 1 by induction **A1**

Answer: the nth derivative of $x\,e^{-x}$ is $(-1)^n(x-n)e^{-x}$ for all integers n ≥ 1 **6 marks**

(b) $(-1)^{10}(3-10)e^{-3}$ **M1**

$= -7e^{-3}$ **A1**

Answer: $-7e^{-3}$ **2 marks**

24 (7 marks)

n = 1: RHS = $\frac{\sin 2x}{2\sin x} = \frac{2\sin x\cos x}{2\sin x} = \cos x$ = LHS, true **B1**

Assume the product of the first k factors is $\frac{\sin(2^k x)}{2^k \sin x}$. Then the product of k + 1 factors is $\frac{\sin(2^k x)}{2^k \sin x}$ **M1 A1**

$\times \cos(2^k x)$

$\sin(2^k x)\cos(2^k x) = \frac{1}{2}\sin(2 \times 2^k x) = \frac{1}{2}\sin(2^{k+1}x)$ **M1 A1**

So the product = $\frac{\sin(2^{k+1}x)}{2 \times 2^k \sin x} = \frac{\sin(2^{k+1}x)}{2^{k+1}\sin x}$, the formula with n = k + 1 **A1**

True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers n ≥ 1 by induction **A1**

Answer: $\cos x \cos 2x \cos 4x \ldots \cos(2^{n-1}x) = \frac{\sin(2^n x)}{2^n \sin x}$ **7 marks**

25 (8 marks)

n = 1: LHS = $\cos\theta$, RHS = $\frac{\sin 2\theta}{2\sin\theta} = \frac{2\sin\theta\cos\theta}{2\sin\theta} = \cos\theta$, true **B1**

Assume true for n = k. Sum to k + 1 terms = $\frac{\sin 2k\theta}{2\sin\theta} + \cos(2k+1)\theta$ **M1 A1**

$= \frac{\sin 2k\theta + 2\sin\theta\cos(2k+1)\theta}{2\sin\theta}$ **M1**

$2\sin\theta\cos(2k+1)\theta = \sin(2k+2)\theta - \sin 2k\theta$ (product to sum) **M1 A1**

So the sum = $\frac{\sin(2k+2)\theta}{2\sin\theta} = \frac{\sin 2(k+1)\theta}{2\sin\theta}$ **A1**

True for n = 1, true for n = k implies true for n = k + 1, so true for all integers n ≥ 1 **A1**

Answer: $\cos\theta + \cos 3\theta + \ldots + \cos(2n-1)\theta = \frac{\sin 2n\theta}{2\sin\theta}$ for all integers n ≥ 1 **8 marks**

26 (7 marks)

n = 1: both sides are $\cos\theta + i\sin\theta$, true — **B1**

Assume $(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$ — **M1**

$(\cos\theta + i\sin\theta)^{k+1} = (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$ — **M1**

$= (\cos k\theta\cos\theta - \sin k\theta\sin\theta) + i(\sin k\theta\cos\theta + \cos k\theta\sin\theta)$ — **M1 A1**

$= \cos(k+1)\theta + i\sin(k+1)\theta$ using the compound angle formulae, which is the result for n = k + 1 — **A1**

True for n = 1, and true for n = k implies true for n = k + 1, so true for all positive integers n by induction — **A1**

Answer: $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ for all positive integers n — **7 marks**

27 (8 marks)

Check n = 1: $\frac{3-1}{2} = 1$ and n = 2: $\frac{9-1}{2} = 4$, both correct (two base cases are needed) — **B1 B1**

Assume the formula holds for n = k and n = k + 1 — **M1**

$u_{k+2} = 4 \times \frac{3^{k+1}-1}{2} - 3 \times \frac{3^k - 1}{2}$ — **M1 A1**

$= \frac{4\times 3^{k+1} - 4 - 3^{k+1} + 3}{2}$ — **M1**

$= \frac{3\times 3^{k+1} - 1}{2} = \frac{3^{k+2}-1}{2}$ — **A1**

True for n = 1, 2, and true for n = k, k + 1 implies true for n = k + 2, so true for all $n \geq 1$ — **A1**

Answer: $u_n = \frac{3^n - 1}{2}$ for all integers $n \geq 1$ — **8 marks**

28 (8 marks)

n = 1: $\frac{d}{dx}(e^x\cos x) = e^x(\cos x - \sin x)$ — **B1**

$\cos x - \sin x = \sqrt{2}\cos(x + \frac{\pi}{4})$, so the derivative is $\sqrt{2}\,e^x\cos(x + \frac{\pi}{4})$, true — **M1 A1**

Assume the kth derivative is $(\sqrt{2})^k e^x\cos(x + \frac{k\pi}{4})$ — **M1**

Differentiate: $(\sqrt{2})^k e^x[\cos(x + \frac{k\pi}{4}) - \sin(x + \frac{k\pi}{4})]$ — **M1 A1**

Using $\cos A - \sin A = \sqrt{2}\cos(A + \frac{\pi}{4})$ with $A = x + \frac{k\pi}{4}$: this is $(\sqrt{2})^{k+1} e^x\cos(x + \frac{(k+1)\pi}{4})$, the result for n = k + 1 — **A1**

True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers $n \geq 1$ by induction — **B1**

Answer: the nth derivative is $(\sqrt{2})^n e^x\cos(x + \frac{n\pi}{4})$ — **8 marks**

29 (9 marks)

n = 1: LHS = 1, RHS = $\frac{1\times 2\times 3\times 5}{30} = 1$, true — **B1**

Assume true for n = k. Sum to k + 1 = $\frac{k(k+1)(2k+1)(3k^2+3k-1)}{30} + (k+1)^4$ — **M1 A1**

Take out the factor $(k+1)$: $\frac{(k+1)}{30}[k(2k+1)(3k^2+3k-1) + 30(k+1)^3]$ — **M1**

$k(2k+1)(3k^2+3k-1) = 6k^4 + 9k^3 + k^2 - k$ — **M1**

Bracket = $6k^4 + 39k^3 + 91k^2 + 89k + 30$ — **A1**

$= (k+2)(2k+3)(3k^2+9k+5)$, checked by expanding — **M1**

Since $3(k+1)^2 + 3(k+1) - 1 = 3k^2 + 9k + 5$, this is the formula with n = k + 1 — **A1**

True for n = 1 and true for n = k implies true for n = k + 1, so true for all $n \geq 1$ — **A1**

Answer: $\sum_{r=1}^{n} r^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$ for all integers $n \geq 1$ — **9 marks**

30 (8 marks)

(a) Base case: n = 2 is itself prime, so it is a product of one prime — **B1**

Assume that every integer m with $2 \leq m \leq k$ is a product of primes — **M1**

Consider k + 1. If k + 1 is prime, it is a product of one prime — **B1**

Otherwise k + 1 = ab with $2 \leq a, b \leq k$ — **M1**

By the assumption a and b are each products of primes — **A1**

So k + 1 = ab is a product of primes. By strong induction, true for all $n \geq 2$ — **A1**

Answer: every integer $n \geq 2$ is a product of one or more primes — **6 marks**

(b) The factors a and b of k + 1 are generally smaller than k, not equal to k — **E1**

So the assumption about k alone does not say anything about a and b: all smaller cases are needed — **E1**

Answer: k + 1 = ab with a and b generally much smaller than k, so the case n = k alone says nothing about a or b — **2 marks**

Induction with Series

1 (3 marks)

$n = 1$: LHS = 4, RHS = $1 \times 2^2 = 4$ — **B1**

$n = 2$: LHS = 4 + 14 = 18, RHS = $2 \times 3^2 = 18$ — **B1**

$n = 3$: LHS = 18 + 30 = 48, RHS = $3 \times 4^2 = 48$ — **B1**

Answer: P(1), P(2) and P(3) are all true — **3 marks**

2 (2 marks)

$$\frac{k(2k+1)}{6} + (k+1) = \frac{2k^2 + k + 6k + 6}{6} = \frac{2k^2 + 7k + 6}{6}$$ — **M1**

$$= \frac{(k+2)(2k+3)}{6}$$ — **A1**

Answer: $\mathbf{\frac{k(2k+1)}{6} + (k+1) = \frac{(k+2)(2k+3)}{6}}$ — **2 marks**

3 (2 marks)

Replace n by k + 1 throughout — **M1**

$$\sum_{r=1}^{k+1} r(r+3) = \frac{(k+1)(k+2)(k+6)}{3}$$ — **A1**

Answer: $\mathbf{\sum_{r=1}^{k+1} r(r+3) = \frac{(k+1)(k+2)(k+6)}{3}}$ — **2 marks**

4 (2 marks)

$n = 1$: LHS = 2, RHS = $\frac{1 \times 4}{2} = 2$ — **B1**

$n = 2$: LHS = 2 + 5 = 7, RHS = $\frac{2 \times 7}{2} = 7$ — **B1**

Answer: the formula $\sum_{r=1}^{n} (3r - 1) = \frac{n(3n+1)}{2}$ is correct for n = 1 and n = 2 — **2 marks**

5 (2 marks)

LHS = 1 + 8 + 27 = 36 — **B1**

RHS = $6^2 = 36$, so they are equal — **B1**

Answer: $\mathbf{1^3 + 2^3 + 3^3 = (1 + 2 + 3)^2}$ — **2 marks**

6 (3 marks)

$S_1 = 2$, $S_2 = 6$, $S_3 = 14$ — **B1**

Each is 2 less than a power of 2: $4 - 2$, $8 - 2$, $16 - 2$ — **M1**

$S_n = 2^{n+1} - 2$ — **A1**

Answer: $\mathbf{S_n = 2^{n+1} - 2}$ — **3 marks**

7 (4 marks)

n = 1: LHS = 2, RHS = $\frac{1 \times 4}{2}$ = 2, true **B1**

Assume true for n = k. Sum to k + 1 = $\frac{k(3k + 1)}{2} + 3(k + 1) - 1$ **M1**

= $\frac{3k^2 + k + 6k + 4}{2} = \frac{3k^2 + 7k + 4}{2} = \frac{(k + 1)(3k + 4)}{2}$ **A1**

This is the formula with n = k + 1 [(k + 1)(3(k + 1) + 1)/2]. Conclude: true for n = 1, and true for n = k implies true for n = k + 1, so true for all integers n ≥ 1 by induction **A1**

Answer: $\sum_{r=1}^{n} (3r - 1) = \frac{n(3n + 1)}{2}$ for all integers n ≥ 1 **4 marks**

8 (5 marks)

n = 1: LHS = 2, RHS = $\frac{1 \times 2 \times 3}{3}$ = 2, true **B1**

Assume true for n = k. Sum to k + 1 = $\frac{k(k + 1)(k + 2)}{3} + (k + 1)(k + 2)$ **M1**

= $\frac{(k + 1)(k + 2)}{3}[k + 3]$ **M1 A1**

= $\frac{(k + 1)(k + 2)(k + 3)}{3}$, the formula for n = k + 1. True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers n ≥ 1 by induction **A1**

Answer: $\sum_{r=1}^{n} r(r + 1) = \frac{n(n + 1)(n + 2)}{3}$ for all integers n ≥ 1 **5 marks**

9 (5 marks)

n = 1: LHS = 1, RHS = $\frac{1 \times 2 \times 3}{6}$ = 1, true **B1**

Assume true for n = k. Sum to k + 1 = $\frac{k(k + 1)(2k + 1)}{6} + (k + 1)^2$ **M1**

= $\frac{(k + 1)}{6}[k(2k + 1) + 6(k + 1)] = \frac{(k + 1)(2k^2 + 7k + 6)}{6}$ **M1 A1**

= $\frac{(k + 1)(k + 2)(2k + 3)}{6}$, the formula for n = k + 1. True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers n ≥ 1 by induction **A1**

Answer: $\sum_{r=1}^{n} r^2 = \frac{n(n + 1)(2n + 1)}{6}$ for all integers n ≥ 1 **5 marks**

10 (4 marks)

n = 1: LHS = 1, RHS = $\frac{3 - 1}{2}$ = 1, true **B1**

Assume true for n = k. Sum of k + 1 terms = $\frac{3^k - 1}{2} + 3^k$ **M1**

= $\frac{3^k - 1 + 2 \times 3^k}{2} = \frac{3^{k+1} - 1}{2}$ **A1**

True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers n ≥ 1 by induction **A1**

Answer: $1 + 3 + 9 + ... + 3^{n-1} = \frac{3^n - 1}{2}$ for all integers n ≥ 1 **4 marks**

11 (5 marks)

n = 1: LHS = 4, RHS = $\frac{1 \times 2 \times 6}{3}$ = 4, true **B1**

Assume true for n = k. Sum to k + 1 = $\frac{k(k+1)(k+5)}{3} + (k+1)(k+4)$ **M1**

= $\frac{(k+1)}{3}[k(k+5) + 3(k+4)]$ **M1**

= $\frac{(k+1)(k^2+8k+12)}{3} = \frac{(k+1)(k+2)(k+6)}{3}$ **A1**

True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers n ≥ 1 by induction **A1**

Answer: $\sum_{r=1}^{n} r(r+3) = \frac{n(n+1)(n+5)}{3}$ for all integers n ≥ 1 **5 marks**

12 (5 marks)

n = 1: LHS = 2 × 2 = 4, RHS = 1 × 2^2 = 4, true **B1**

Assume true for n = k. Sum to k + 1 = k × 2^{k+1} + (k + 2) × 2^{k+1} **M1**

= $2^{k+1}[k + (k+2)] = 2^{k+1} \times 2(k+1)$ **M1 A1**

= (k + 1) × 2^{k+2}, which is the formula with n = k + 1. True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers n ≥ 1 by induction **A1**

Answer: $\sum_{r=1}^{n} (r+1) \times 2^r = n \times 2^{n+1}$ for all integers n ≥ 1 **5 marks**

13 (5 marks)

n = 1: LHS = 1, RHS = 0 + 1 = 1, true **B1**

Assume true for n = k. Sum to k + 1 terms = $(k-1)2^k + 1 + (k+1)2^k$ **M1**

= $2^k[(k-1) + (k+1)] + 1 = 2k \times 2^k + 1$ **M1 A1**

= k × 2^{k+1} + 1, the formula for n = k + 1. True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers n ≥ 1 by induction **A1**

Answer: 1 + 4 + 12 + ... + n × 2^{n-1} = $(n-1)2^n + 1$ for all integers n ≥ 1 **5 marks**

14 (6 marks)

n = 1: LHS = 1, RHS = $\frac{1 \times 4}{4}$ = 1, true **B1**

Assume true for n = k. Sum to k + 1 = $\frac{k^2(k+1)^2}{4} + (k+1)^3$ **M1 A1**

= $\frac{(k+1)^2}{4}[k^2 + 4(k+1)]$ **M1**

= $\frac{(k+1)^2(k+2)^2}{4}$ **A1**

This is the formula for n = k + 1. True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers n ≥ 1 by induction **A1**

Answer: $\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$ for all integers n ≥ 1 **6 marks**

15 (7 marks)

n = 1: LHS = $\frac{1}{3}$, RHS = $\frac{1 \times 8}{4 \times 2 \times 3} = \frac{1}{3}$, true **B1**

Assume true for n = k. Sum to k + 1 = $\frac{k(3k + 5)}{4(k + 1)(k + 2)} + \frac{1}{(k + 1)(k + 3)}$ **M1 A1**

Common denominator 4(k + 1)(k + 2)(k + 3): numerator k(3k + 5)(k + 3) + 4(k + 2) **M1**

$= 3k^3 + 14k^2 + 19k + 8$ **A1**

$= (k + 1)(3k^2 + 11k + 8) = (k + 1)^2(3k + 8)$ **M1**

So the sum = $\frac{(k + 1)(3k + 8)}{4(k + 2)(k + 3)}$, the formula for n = k + 1. True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers n ≥ 1 by induction **A1**

Answer: $\sum_{r=1}^{n} \frac{1}{r(r + 2)} = \frac{n(3n + 5)}{4(n + 1)(n + 2)}$ for all integers n ≥ 1 **7 marks**

16 (6 marks)

n = 1: LHS = 6, RHS = $\frac{1 \times 2 \times 3 \times 4}{4} = 6$, true **B1**

Assume true for n = k. Sum to k + 1 = $\frac{k(k + 1)(k + 2)(k + 3)}{4} + (k + 1)(k + 2)(k + 3)$ **M1 A1**

$= \frac{(k + 1)(k + 2)(k + 3)}{4}[k + 4]$ **M1 A1**

$= \frac{(k + 1)(k + 2)(k + 3)(k + 4)}{4}$, the formula for n = k + 1. True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers n ≥ 1 by induction **A1**

Answer: $\sum_{r=1}^{n} r(r + 1)(r + 2) = \frac{n(n + 1)(n + 2)(n + 3)}{4}$ for all integers n ≥ 1 **6 marks**

17 (6 marks)

n = 1: LHS = 1, RHS = $\frac{1 \times 1 \times 3}{3} = 1$, true **B1**

Assume true for n = k. Sum of k + 1 terms = $\frac{k(2k - 1)(2k + 1)}{3} + (2k + 1)^2$ **M1 A1**

$= \frac{(2k + 1)}{3}[k(2k - 1) + 3(2k + 1)]$ **M1**

$= \frac{(2k + 1)(2k^2 + 5k + 3)}{3} = \frac{(2k + 1)(2k + 3)(k + 1)}{3}$ **A1**

This is the formula for n = k + 1. True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers n ≥ 1 by induction **A1**

Answer: $1^2 + 3^2 + 5^2 + ... + (2n - 1)^2 = \frac{n(2n - 1)(2n + 1)}{3}$ for all integers n ≥ 1 **6 marks**

18 (7 marks)

n = 1: LHS = 1, RHS = (+1) × $\frac{1 \times 2}{2} = 1$, true **B1**

Assume true for n = k. Sum to k + 1 terms = $(-1)^{k+1}\frac{k(k + 1)}{2} + (-1)^{k+2}(k + 1)^2$ **M1 A1**

Factor $(-1)^{k+1}$(k + 1): the sum = $(-1)^{k+1}(k + 1)[\frac{k}{2} - (k + 1)]$ **M1 A1**

$\frac{k}{2} - (k + 1) = -\frac{k + 2}{2}$ **M1**

So the sum = $(-1)^{k+2}\frac{(k + 1)(k + 2)}{2}$, the formula for n = k + 1. True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers n ≥ 1 by induction **A1**

Answer: $1^2 - 2^2 + 3^2 - 4^2 + ... + (-1)^{n+1}n^2 = (-1)^{n+1}\frac{n(n + 1)}{2}$ for all integers n ≥ 1 **7 marks**

19 (6 marks)

$n = 1$: LHS $= \frac{1}{2}$, RHS $= 2 - \frac{3}{2} = \frac{1}{2}$, true **B1**

Assume true for $n = k$. Sum to $k + 1 = 2 - \frac{k + 2}{2^k} + \frac{k + 1}{2^{k+1}}$ **M1 A1**

$= 2 - \frac{2(k + 2) - (k + 1)}{2^{k+1}}$ **M1**

$= 2 - \frac{k + 3}{2^{k+1}}$ **A1**

This is the formula for $n = k + 1$. True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction **A1**

Answer: $\sum_{r=1}^{n} \frac{r}{2^r} = 2 - \frac{n + 2}{2^n}$ for all integers $n \geq 1$ **6 marks**

20 (7 marks)

$n = 2$: LHS $= 1 \times 2 \times 3 = 6$, RHS $= \frac{1 \times 2 \times 3 \times 4}{4} = 6$, true **B1**

Assume true for $n = k$, where $k \geq 2$. Sum to $k + 1 = \frac{(k - 1)k(k + 1)(k + 2)}{4} + k(k + 1)(k + 2)$ **M1 A1**

$= \frac{k(k + 1)(k + 2)}{4}[(k - 1) + 4] = \frac{k(k + 1)(k + 2)(k + 3)}{4}$ **M1 A1**

This is the formula with $n = k + 1$, as $((k + 1) - 1)(k + 1)((k + 1) + 1)((k + 1) + 2) = k(k + 1)(k + 2)(k + 3)$ **A1**

True for $n = 2$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 2$ by induction **A1**

Answer: $\sum_{r=2}^{n} (r - 1)r(r + 1) = \frac{(n - 1)n(n + 1)(n + 2)}{4}$ for all integers $n \geq 2$ **7 marks**

21 (7 marks)

$n = 1$: LHS $= 3$, RHS $= \frac{1 \times 9 + 3}{4} = 3$, true **B1**

Assume true for $n = k$. Sum to $k + 1 = \frac{(2k - 1)3^{k+1} + 3}{4} + (k + 1)3^{k+1}$ **M1 A1**

$= \frac{3^{k+1}[(2k - 1) + 4(k + 1)] + 3}{4}$ **M1**

$= \frac{3^{k+1}(6k + 3) + 3}{4} = \frac{3^{k+2}(2k + 1) + 3}{4}$ **M1 A1**

This is the formula for $n = k + 1$, as $(2(k + 1) - 1) = 2k + 1$. True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction **A1**

Answer: $\sum_{r=1}^{n} r \times 3^r = \frac{(2n - 1)3^{n+1} + 3}{4}$ for all integers $n \geq 1$ **7 marks**

22 (7 marks)

$n = 1$: LHS $= 2 \times 1 = 2$, RHS $= 1 \times 2! = 2$, true **B1**

Assume true for $n = k$. Sum to $k + 1 = k \times (k + 1)! + ((k + 1)^2 + 1)(k + 1)!$ **M1 A1**

$= (k + 1)![k + k^2 + 2k + 2]$ **M1**

$= (k + 1)!(k^2 + 3k + 2) = (k + 1)!(k + 1)(k + 2)$ **M1 A1**

$= (k + 1) \times (k + 2)!$, the formula for $n = k + 1$. True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction **A1**

Answer: $\sum_{r=1}^{n} (r^2 + 1) \times r! = n \times (n + 1)!$ for all integers $n \geq 1$ **7 marks**

23 (8 marks)

(a) n = 1: f(1) = 3 and $\frac{1 \times 2 \times 9}{6} = 3$, true **B1**

Assume f(k) = $\frac{k(k+1)(4k+5)}{6}$. Then f(k + 1) = $\frac{k(k+1)(4k+5)}{6} + (k+1)(2k+3)$ **M1**

= $\frac{(k+1)}{6}[k(4k+5) + 6(2k+3)] = \frac{(k+1)(4k^2+17k+18)}{6}$ **M1 A1**

= $\frac{(k+1)(k+2)(4k+9)}{6}$, which is the formula with n = k + 1. True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers n ≥ 1 by induction **A1**

Answer: f(n) = $\frac{n(n+1)(4n+5)}{6}$ for all integers n ≥ 1 **5 marks**

(b) $\sum_{r=n+1}^{2n} r(2r+1) = f(2n) - f(n)$ **M1**

= $\frac{2n(2n+1)(8n+5)}{6} - \frac{n(n+1)(4n+5)}{6} = \frac{n}{6}[2(2n+1)(8n+5) - (n+1)(4n+5)]$ **A1**

= $\frac{n}{6}(28n^2+27n+5) = \frac{n(4n+1)(7n+5)}{6}$ **A1**

Answer: $\sum_{r=n+1}^{2n} r(2r+1) = \frac{n(4n+1)(7n+5)}{6}$ **3 marks**

24 (6 marks)

n = 1: LHS = 1, RHS = 1 × 1 = 1, true **B1**

Assume true for n = k. Sum of k + 1 terms = $k^2(2k^2 - 1) + (2k+1)^3$ **M1 A1**

= $2k^4 - k^2 + 8k^3 + 12k^2 + 6k + 1 = 2k^4 + 8k^3 + 11k^2 + 6k + 1$ **M1**

$(k+1)^2(2(k+1)^2 - 1) = (k^2+2k+1)(2k^2+4k+1) = 2k^4 + 8k^3 + 11k^2 + 6k + 1$ **A1**

The two agree, so the formula holds for n = k + 1. True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers n ≥ 1 by induction **A1**

Answer: $1^3 + 3^3 + 5^3 + ... + (2n-1)^3 = n^2(2n^2 - 1)$ for all integers n ≥ 1 **6 marks**

25 (6 marks)

Error 1: the term added should be the (k + 1)th term, $(k+1)^2$, not (k + 1) **E1**

Error 2: $(k+1)(2k^2 + k + 6)$ does not equal $(k+1)(k+2)(2k+3) = (k+1)(2k^2 + 7k + 6)$, so the claimed equality is false **E1**

Correct step: sum to k + 1 = $\frac{k(k+1)(2k+1)}{6} + (k+1)^2$ **M1**

= $\frac{(k+1)}{6}[k(2k+1) + 6(k+1)]$ **M1**

= $\frac{(k+1)(2k^2+7k+6)}{6}$ **A1**

= $\frac{(k+1)(k+2)(2k+3)}{6}$, the formula with n = k + 1 **A1**

Answer: Added (k + 1) instead of $(k+1)^2$; false equality; corrected step gives $\frac{(k+1)(k+2)(2k+3)}{6}$ **6 marks**

26 (7 marks)

n = 1: LHS = $\frac{3}{4}$, RHS = $1 - \frac{1}{4} = \frac{3}{4}$, true **B1**

Assume true for n = k. Sum to k + 1 = $1 - \frac{1}{(k+1)^2} + \frac{2k+3}{(k+1)^2(k+2)^2}$ **M1 A1**

$= 1 - \frac{(k+2)^2 - (2k+3)}{(k+1)^2(k+2)^2}$ **M1 A1**

$(k+2)^2 - (2k+3) = k^2 + 2k + 1 = (k+1)^2$ **M1**

So the sum = $1 - \frac{1}{(k+2)^2}$, the formula with n = k + 1. True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers n ≥ 1 by induction **A1**

Answer: $\sum_{r=1}^{n} \frac{2r+1}{r^2(r+1)^2} = 1 - \frac{1}{(n+1)^2}$ for all integers n ≥ 1 **7 marks**

27 (8 marks)

n = 1: LHS = 2, RHS = $(1 - 2 + 3) \times 4 - 6 = 2$, true **B1**

Assume true for n = k. Sum to k + 1 = $(k^2 - 2k + 3)2^{k+1} - 6 + (k+1)^2 2^{k+1}$ **M1 A1**

$= 2^{k+1}[k^2 - 2k + 3 + k^2 + 2k + 1] - 6$ **M1**

$= 2^{k+1}(2k^2 + 4) - 6$ **A1**

$= 2^{k+2}(k^2 + 2) - 6$ **M1**

For n = k + 1 the formula gives $((k+1)^2 - 2(k+1) + 3)2^{k+2} - 6 = (k^2 + 2)2^{k+2} - 6$, which matches **A1**

Conclude: true for n = 1 and true for n = k implies true for n = k + 1, so true for all n ≥ 1 **A1**

Answer: $\sum_{r=1}^{n} r^2 \times 2^r = (n^2 - 2n + 3)2^{n+1} - 6$ for all integers n ≥ 1 **8 marks**

28 (10 marks)

n = 1: RHS = $\frac{\sin(\frac{\theta}{2})\sin\theta}{\sin(\frac{\theta}{2})} = \sin\theta$ = LHS, true **B1**

Assume true for n = k. Sum to k + 1 terms = $\frac{\sin(\frac{k\theta}{2})\sin(\frac{(k+1)\theta}{2})}{\sin(\frac{\theta}{2})} + \sin(k+1)\theta$ **M1 A1**

$\sin(k+1)\theta = 2\sin(\frac{(k+1)\theta}{2})\cos(\frac{(k+1)\theta}{2})$ **M1**

Sum = $\frac{\sin(\frac{(k+1)\theta}{2})}{\sin(\frac{\theta}{2})}[\sin(\frac{k\theta}{2}) + 2\sin(\frac{\theta}{2})\cos(\frac{(k+1)\theta}{2})]$ **M1 A1**

$2\sin(\frac{\theta}{2})\cos(\frac{(k+1)\theta}{2}) = \sin(\frac{(k+2)\theta}{2}) - \sin(\frac{k\theta}{2})$ **M1 A1**

So the bracket is $\sin(\frac{(k+2)\theta}{2})$, and the sum = $\frac{\sin(\frac{(k+1)\theta}{2})\sin(\frac{(k+2)\theta}{2})}{\sin(\frac{\theta}{2})}$, the formula for n = k + 1 **A1**

True for n = 1 and true for n = k implies true for n = k + 1, so true for all n ≥ 1 **A1**

Answer: for all integers n ≥ 1 and θ not a multiple of 2π, $\sin\theta + \sin 2\theta + \sin 3\theta + ... + \sin n\theta = \frac{\sin(\frac{n\theta}{2})\sin(\frac{(n+1)\theta}{2})}{\sin(\frac{\theta}{2})}$ **10 marks**

29 (8 marks)

(a) n = 1: LHS = $\frac{1}{24}$, RHS = $\frac{1}{18} - \frac{1}{72} = \frac{3}{72} = \frac{1}{24}$, true **B1**

Assume true for n = k. Sum to k + 1 = $\frac{1}{18} - \frac{1}{3(k+1)(k+2)(k+3)} + \frac{1}{(k+1)(k+2)(k+3)(k+4)}$ **M1 A1**

The last two terms combine to $-\frac{(k+4)-3}{3(k+1)(k+2)(k+3)(k+4)}$ **M1 A1**

$= -\frac{k+1}{3(k+1)(k+2)(k+3)(k+4)} = -\frac{1}{3(k+2)(k+3)(k+4)}$ **A1**

This gives the formula with n = k + 1. True for n = 1 and n = k implies n = k + 1, so true for all n ≥ 1 **A1**

Answer: $S_n = \frac{1}{18} - \frac{1}{3(n+1)(n+2)(n+3)}$ for all integers n ≥ 1 **7 marks**

(b) As n → ∞ the term $\frac{1}{3(n+1)(n+2)(n+3)} \to 0$, so the sum is $\frac{1}{18}$ **B1**

Answer: $\frac{1}{18}$ **1 mark**

30 (9 marks)

n = 1: LHS = 1 × C(1, 1) = 1, RHS = $1 \times 2^0 = 1$, true **B1**

Assume true for n = k. Then $\sum_{r=1}^{k+1} r\,C(k+1, r) = \sum_{r=1}^{k+1} r\,C(k, r) + \sum_{r=1}^{k+1} r\,C(k, r-1)$ **M1 A1**

First sum: C(k, k + 1) = 0, so it equals $k \times 2^{k-1}$ by the assumption **M1**

Second sum: write r = (r − 1) + 1, so it is $\sum_{r=1}^{k+1} (r-1)C(k, r-1) + \sum_{r=1}^{k+1} C(k, r-1)$ **M1**

$= k \times 2^{k-1} + 2^k$ (using the assumption and the sum of the binomial coefficients) **A1**

Total = $2k \times 2^{k-1} + 2^k = k \times 2^k + 2^k = (k+1)2^k$ **M1**

This is the formula with n = k + 1 **A1**

True for n = 1 and true for n = k implies n = k + 1, so true for all n ≥ 1 **A1**

Answer: $\sum_{r=1}^{n} r \times C(n, r) = n \times 2^{n-1}$ for all integers n ≥ 1, where $C(n, r) = \frac{n!}{r!(n-r)!}$ **9 marks**

Induction with Divisibility

1 (2 marks)

f(1) = 6 = 6 × 1 **B1**

f(2) = 48 = 6 × 8 **B1**

Answer: f(1) = 6 = 6 × 1 and f(2) = 48 = 6 × 8 **2 marks**

2 (2 marks)

$f(k + 1) - f(k) = (5^{k+1} - 1) - (5^k - 1)$ **M1**

$= 5 \times 5^k - 5^k = 4 \times 5^k$ **A1**

Answer: $f(k + 1) - f(k) = 4 \times 5^k$ **2 marks**

3 (2 marks)

$f(k + 1) = 2^{2k+2} - 1 = 4 \times 2^{2k} - 1$ **M1**

$4f(k) + 3 = 4 \times 2^{2k} - 4 + 3 = 4 \times 2^{2k} - 1$, so f(k + 1) = 4f(k) + 3 **A1**

Answer: f(k + 1) = 4f(k) + 3 **2 marks**

4 (3 marks)

f(1) = 0, f(2) = 6, f(3) = 24 **B1 B1**

The largest integer dividing 0, 6 and 24 is 6 **B1**

Answer: 6 **3 marks**

5 (3 marks)

$f(1) = 10 = 5 \times 2$ **B1**

$f(k + 1) - f(k) = 6 \times 6^k + 4 - 6^k - 4$ **M1**

$= 5 \times 6^k$ **A1**

Answer: f(1) = 10 = 5 × 2 and f(k + 1) − f(k) = 5 × 6^k **3 marks**

6 (2 marks)

The statement is true for n = 1 and true for n = k + 1 whenever it is true for n = k **E1**

So, by induction, f(n) is divisible by 8 for all positive integers n **E1**

Answer: Divisible by 8 for all positive integers n **2 marks**

7 (4 marks)

n = 1: 5 − 1 = 4, which is divisible by 4, true **B1**

Assume $f(k) = 5^k - 1 = 4m$ for some integer m. Then $f(k + 1) = 5 \times 5^k - 1 = 5(4m + 1) - 1$ **M1**

$= 20m + 4 = 4(5m + 1)$, divisible by 4 **A1**

True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers n ≥ 1 by induction **A1**

Answer: $5^n - 1$ is divisible by 4 for all integers n ≥ 1 **4 marks**

8 (5 marks)

n = 1: 1 × 2 × 3 = 6, which is divisible by 6, true **B1**

Assume f(k) = k(k + 1)(k + 2) is divisible by 6. Then f(k + 1) − f(k) = (k + 1)(k + 2)(k + 3) − k(k + 1)(k + 2) **M1**

= 3(k + 1)(k + 2) **A1**

(k + 1)(k + 2) is a product of consecutive integers so is even, hence 3(k + 1)(k + 2) is a multiple of 6 **E1**

So f(k + 1) = f(k) + (a multiple of 6) is divisible by 6. True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers n ≥ 1 by induction **A1**

Answer: n(n + 1)(n + 2) is divisible by 6 for all integers n ≥ 1 **5 marks**

9 (4 marks)

n = 1: $u_1 + 2 = 4$, which is divisible by 4, true **B1**

Assume $u_k + 2 = 4m$ for some integer m. Then $u_{k+1} + 2 = 3u_k + 4 + 2 = 3u_k + 6$ **M1**

$= 3(u_k + 2) = 12m = 4(3m)$, divisible by 4 **A1**

True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers n ≥ 1 by induction **A1**

Answer: $u_n + 2$ is divisible by 4 for all integers n ≥ 1 **4 marks**

10 (4 marks)

n = 1: 6 + 4 = 10, which is divisible by 5, true **B1**

Assume $f(k) = 6^k + 4 = 5m$. Then $f(k + 1) = 6 \times 6^k + 4 = 6(5m - 4) + 4$ **M1**

$= 30m - 20 = 5(6m - 4)$, divisible by 5 **A1**

True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers n ≥ 1 by induction **A1**

Answer: $6^n + 4$ is divisible by 5 for all integers n ≥ 1 **4 marks**

11 (4 marks)

n = 1: $u_1 = 100 + 12 = 112 = 7 \times 16$, true **B1**

Assume $u_k = 10^{k+1} + 4 \times 3^k$ is a multiple of 7. Then $u_{k+1} = 10 \times 10^{k+1} + 12 \times 3^k$ **M1**

$= 3(10^{k+1} + 4 \times 3^k) + 7 \times 10^{k+1} = 3u_k + 7 \times 10^{k+1}$ **A1**

Both terms are multiples of 7, so u_{k+1} is a multiple of 7. True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers n ≥ 1 by induction **A1**

Answer: every term $u_n = 10^{n+1} + 4 \times 3^n$ is a multiple of 7 **4 marks**

12 (5 marks)

n = 1: 1 + 2 = 3, which is divisible by 3, true — **B1**

Assume $f(k) = k^3 + 2k = 3m$. Then $f(k + 1) = (k + 1)^3 + 2(k + 1) = (k^3 + 2k) + 3k^2 + 3k + 3$ — **M1 A1**

$= 3m + 3(k^2 + k + 1) = 3(m + k^2 + k + 1)$, divisible by 3 — **A1**

True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers $n \geq 1$ by induction — **A1**

Answer: $n^3 + 2n$ is divisible by 3 for all integers $n \geq 1$ — **5 marks**

13 (4 marks)

n = 1: 5 + 22 = 27, which is divisible by 3, true — **B1**

Assume $f(k) = 5^k + 2 \times 11^k = 3m$. Then $f(k + 1) = 5 \times 5^k + 22 \times 11^k = 5(5^k + 2 \times 11^k) + 12 \times 11^k$ — **M1**

$= 15m + 12 \times 11^k = 3(5m + 4 \times 11^k)$, divisible by 3 — **A1**

True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers $n \geq 1$ by induction — **A1**

Answer: $5^n + 2 \times 11^n$ is divisible by 3 for all integers $n \geq 1$ — **4 marks**

14 (6 marks)

n = 1: 8 + 27 = 35, which is divisible by 7, true — **B1**

Assume $f(k) = 2^{k+2} + 3^{2k+1}$ is divisible by 7 — **M1**

$f(k + 1) = 2 \times 2^{k+2} + 9 \times 3^{2k+1} = 2f(k) + 7 \times 3^{2k+1}$ — **M1 A1**

Both terms are multiples of 7, so f(k + 1) is divisible by 7 — **A1**

True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers $n \geq 1$ by induction — **A1**

Answer: $2^{n+2} + 3^{2n+1}$ is divisible by 7 for all integers $n \geq 1$ — **6 marks**

15 (7 marks)

n = 1: 25 + 6 + 1 = 32, which is divisible by 8, true — **B1**

Assume $f(k) = 5^{k+1} + 2 \times 3^k + 1$ is divisible by 8 — **M1**

$f(k + 1) - 5f(k) = (2 \times 3^{k+1} + 1) - (10 \times 3^k + 5)$ — **M1 A1**

$= 6 \times 3^k - 10 \times 3^k - 4 = -4(3^k + 1)$ — **A1**

$3^k + 1$ is even, so $4(3^k + 1)$ is a multiple of 8, and $f(k + 1) = 5f(k) - 4(3^k + 1)$ is divisible by 8 — **E1**

True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers $n \geq 1$ by induction — **A1**

Answer: $5^{n+1} + 2 \times 3^n + 1$ is divisible by 8 for all integers $n \geq 1$ — **7 marks**

16 (7 marks)

n = 1: 81 − 8 − 9 = 64, which is divisible by 64, true — **B1**

Assume $f(k) = 3^{2k+2} - 8k - 9$ is divisible by 64. Then $f(k + 1) = 3^{2k+4} - 8(k + 1) - 9 = 9 \times 3^{2k+2} - 8k - 17$ — **M1 A1**

$f(k + 1) - 9f(k) = (9 \times 3^{2k+2} - 8k - 17) - (9 \times 3^{2k+2} - 72k - 81)$ — **M1**

$= 64k + 64 = 64(k + 1)$ — **A1**

So $f(k + 1) = 9f(k) + 64(k + 1)$, divisible by 64 — **A1**

True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers $n \geq 1$ by induction — **A1**

Answer: $3^{2n+2} - 8n - 9$ is divisible by 64 for all integers $n \geq 1$ — **7 marks**

17 (7 marks)

n = 1: $1 \times 6 = 6$, which is divisible by 6, true — **B1**

Assume $f(k) = k^3 + 5k$ is divisible by 6 — **M1**

$f(k + 1) = (k + 1)^3 + 5(k + 1) = f(k) + 3k^2 + 3k + 6$ — **M1 A1**

$3k^2 + 3k = 3k(k + 1)$, and $k(k + 1)$ is a product of consecutive integers so is even — **E1**

So $3k(k + 1) + 6$ is a multiple of 6, and $f(k + 1)$ is divisible by 6 — **A1**

True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers $n \geq 1$ by induction — **A1**

Answer: $n(n^2 + 5)$ is divisible by 6 for all integers $n \geq 1$ — **7 marks**

18 (5 marks)

n = 1: $7 \times 7 - 1 = 48 = 6 \times 8$, true — **B1**

Assume $f(k) = (6k + 1)7^k - 1$ is divisible by 6. Then $f(k + 1) = (6k + 7)7^{k+1} - 1$ — **M1**

$f(k + 1) - 7f(k) = (6k + 7)7^{k+1} - 1 - (6k + 1)7^{k+1} + 7 = 6 \times 7^{k+1} + 6$ — **M1 A1**

So $f(k + 1) = 7f(k) + 6(7^{k+1} + 1)$, a sum of multiples of 6. True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers $n \geq 1$ by induction — **A1**

Answer: $(6n + 1) \times 7^n - 1$ is divisible by 6 for all integers $n \geq 1$ — **5 marks**

19 (6 marks)

n = 1: $1 + 8 + 27 = 36 = 9 \times 4$, true — **B1**

Assume $f(k) = k^3 + (k + 1)^3 + (k + 2)^3$ is divisible by 9 — **M1**

$f(k + 1) - f(k) = (k + 3)^3 - k^3$ — **M1 A1**

$= 9k^2 + 27k + 27 = 9(k^2 + 3k + 3)$ — **A1**

So $f(k + 1) = f(k) + 9(k^2 + 3k + 3)$, a sum of multiples of 9. True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers $n \geq 1$ by induction — **A1**

Answer: $n^3 + (n + 1)^3 + (n + 2)^3$ is divisible by 9 for all integers $n \geq 1$ — **6 marks**

20 (5 marks)

n = 1: $F_3 = F_2 + F_1 = 2$, which is even, true — **B1**

Assume F_{3k} is even — **M1**

$F_{3k+3} = F_{3k+2} + F_{3k+1} = (F_{3k+1} + F_{3k}) + F_{3k+1}$ — **M1**

$= 2F_{3k+1} + F_{3k}$ — **A1**

Both terms are even, so F_{3k+3} is even. True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers $n \geq 1$ by induction — **A1**

Answer: F_{3n} is even for all integers $n \geq 1$ — **5 marks**

21 (8 marks)

n = 1: $64 - 7 - 8 = 49$, which is divisible by 49, true — **B1**

Assume $f(k) = 2^{3k+3} - 7k - 8$ is divisible by 49 — **M1**

$f(k + 1) = 2^{3k+6} - 7(k + 1) - 8 = 8 \times 2^{3k+3} - 7k - 15$ — **M1 A1**

$8f(k) = 8 \times 2^{3k+3} - 56k - 64$, so $f(k + 1) - 8f(k) = 49k + 49$ — **M1 A1**

So $f(k + 1) = 8f(k) + 49(k + 1)$, divisible by 49 — **A1**

True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers $n \geq 1$ by induction — **A1**

Answer: $2^{3n+3} - 7n - 8$ is divisible by 49 for all integers $n \geq 1$ — **8 marks**

22 (6 marks)

Error: $5(5^k + 3) = 5^{k+1} + 15$, so $5^{k+1} + 3 = 5(5^k + 3) - 12$, not + 12 — **E1**

Check k = 1: $5^2 + 3 = 28$ but $5(5 + 3) + 12 = 52$, so the identity is false — **B1**

Correct step: assume $5^k + 3 = 4m$, so $5^k = 4m - 3$ — **M1**

$5^{k+1} + 3 = 5(4m - 3) + 3$ — **M1**

$= 20m - 12 = 4(5m - 3)$ — **A1**

This is divisible by 4, and with the basis case the result follows by induction — **A1**

Answer: Sign error: $5^{k+1} + 3 = 5(5^k + 3) - 12$; the correct step gives $4(5m - 3)$ — **6 marks**

23 (8 marks)

n = 1: 64 + 27 = 91 = 13 × 7, which is divisible by 13, true **B1**

Assume $f(k) = 4^{2k+1} + 3^{k+2}$ is divisible by 13 **M1**

$f(k + 1) = 16 \times 4^{2k+1} + 3 \times 3^{k+2}$ **M1 A1**

$= 3(4^{2k+1} + 3^{k+2}) + 13 \times 4^{2k+1} = 3f(k) + 13 \times 4^{2k+1}$ **M1 A1**

Both terms are multiples of 13, so f(k + 1) is divisible by 13 **A1**

True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers $n \geq 1$ by induction **A1**

Answer: $4^{2n+1} + 3^{n+2}$ is divisible by 13 for all integers $n \geq 1$ **8 marks**

24 (7 marks)

n = 1: 4 + 6 − 1 = 9, which is divisible by 9, true **B1**

Assume $f(k) = 4^k + 6k - 1 = 9m$ **M1**

$f(k + 1) = 4 \times 4^k + 6k + 5$ **M1**

Substituting $4^k = 9m - 6k + 1$ gives $f(k + 1) = 4(9m - 6k + 1) + 6k + 5$ **A1**

$= 36m - 18k + 9 = 9(4m - 2k + 1)$ **M1**

This is a multiple of 9 **A1**

True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers $n \geq 1$ by induction **A1**

Answer: $4^n + 6n - 1$ is divisible by 9 for all integers $n \geq 1$ **7 marks**

25 (8 marks)

n = 1: 10 + 192 + 5 = 207 = 9 × 23, which is divisible by 9, true **B1**

Assume $f(k) = 10^k + 3 \times 4^{k+2} + 5$ is divisible by 9 **M1**

$f(k + 1) - f(k) = (10^{k+1} - 10^k) + 3(4^{k+3} - 4^{k+2})$ **M1 A1**

$= 9 \times 10^k + 9 \times 4^{k+2}$ **M1 A1**

So $f(k + 1) = f(k) + 9(10^k + 4^{k+2})$, divisible by 9 **A1**

True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers $n \geq 1$ by induction **A1**

Answer: $10^n + 3 \times 4^{n+2} + 5$ is divisible by 9 for all integers $n \geq 1$ **8 marks**

26 (6 marks)

n = 1: $\frac{2!}{2 \times 1!} = 1$, an integer, true **B1**

Assume $\frac{(2k)!}{2^k\, k!} = m$, an integer **M1**

$\frac{(2k + 2)!}{2^{k+1}(k + 1)!} = \frac{(2k + 2)(2k + 1)(2k)!}{2 \times 2^k(k + 1)\, k!}$ **M1 A1**

$= \frac{2(k + 1)(2k + 1)}{2(k + 1)} \times \frac{(2k)!}{2^k\, k!} = (2k + 1)m$ **M1**

which is an integer. True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers $n \geq 1$ by induction **A1**

Answer: $\frac{(2n)!}{2^n\, n!}$ is an integer for all integers $n \geq 1$ **6 marks**

27 (10 marks)

n = 0: $2^1 + 1 = 3$, which is divisible by 3^1, true — **B1**

Assume $2^{3^k} + 1 = 3^{k+1}m$ for an integer m. Let $a = 2^{3^k}$, so $a + 1 = 3^{k+1}m$ — **M1**

$2^{3^{k+1}} = (2^{3^k})^3 = a^3$ — **M1**

$a^3 + 1 = (a + 1)(a^2 - a + 1)$ — **M1 A1**

Since $a = 3^{k+1}m - 1$: $a^2 - a + 1 = (3^{k+1}m - 1)^2 - (3^{k+1}m - 1) + 1$ — **M1**

Expanding, $a^2 - a + 1 = 3^{2k+2}m^2 - 3 \times 3^{k+1}m + 3 = 3(3^{2k+1}m^2 - 3^{k+1}m + 1)$ — **A1**

So $a^3 + 1 = 3^{k+1}m \times 3 \times$ (integer), which is divisible by 3^{k+2} — **M1 A1**

True for n = 0, and true for n = k implies true for n = k + 1, so true for all $n \geq 0$ — **A1**

Answer: 3^{n+1} divides $2^{3^n} + 1$ for all integers $n \geq 0$ — **10 marks**

28 (9 marks)

(a) n = 1: f(1) = 0, divisible by 5, true — **B1**

Assume $f(k) = k^5 - k = 5m$ — **M1**

$f(k + 1) = k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - k - 1$ — **M1**

$= (k^5 - k) + 5(k^4 + 2k^3 + 2k^2 + k) = 5(m + k^4 + 2k^3 + 2k^2 + k)$ — **A1**

True for n = 1, true for n = k implies n = k + 1, so true for all $n \geq 1$ — **A1**

Answer: f(n) is divisible by 5 for all integers $n \geq 1$ — **5 marks**

(b) $f(n) = n(n^4 - 1) = n(n^2 - 1)(n^2 + 1) = (n - 1)n(n + 1)(n^2 + 1)$ — **M1 A1**

$(n - 1)n(n + 1)$ is a product of three consecutive integers so is divisible by 2 and by 3, hence by 6 — **E1**

6 and 5 have no common factor, so f(n) is divisible by 30 — **E1**

Answer: $f(n) = (n - 1)n(n + 1)(n^2 + 1)$ is divisible by 5 and by 6, hence by 30 — **4 marks**

29 (8 marks)

(a) $F_{m+5} = F_{m+4} + F_{m+3} = 2F_{m+3} + F_{m+2}$ — **M1**

$= 3F_{m+2} + 2F_{m+1} = 3F_{m+1} + 3F_m + 2F_{m+1}$ — **M1**

$= 5F_{m+1} + 3F_m$ — **A1**

Answer: $F_{m+5} = 5F_{m+1} + 3F_m$ for all integers $m \geq 1$ — **3 marks**

(b) n = 1: $F_5 = 5$, divisible by 5, true — **B1**

Assume F_{5k} is divisible by 5 — **M1**

$F_{5k+5} = 5F_{5k+1} + 3F_{5k}$ using part (a) with m = 5k — **M1 A1**

Both terms are multiples of 5; true for n = 1 and n = k implies n = k + 1, so true for all $n \geq 1$ — **A1**

Answer: F_{5n} is divisible by 5 for all integers $n \geq 1$ — **5 marks**

30 (9 marks)

(a) $\alpha = 3 + \sqrt{5}$ and $\beta = 3 - \sqrt{5}$ have sum 6 and product 4, so both are roots of $x^2 - 6x + 4 = 0$ — **M1 A1**

So $\alpha^{n+2} = 6\alpha^{n+1} - 4\alpha^n$ and similarly for β; adding gives $u_{n+2} = 6u_{n+1} - 4u_n$ — **A1**

Answer: $u_{n+2} = 6u_{n+1} - 4u_n$ — **3 marks**

(b) $u_1 = 6$, divisible by 2; $u_2 = (14 + 6\sqrt{5}) + (14 - 6\sqrt{5}) = 28$, divisible by 4 — **B1 B1**

Assume $u_k = 2^k a$ and $u_{k+1} = 2^{k+1}b$ for integers a, b — **M1**

$u_{k+2} = 6 \times 2^{k+1}b - 4 \times 2^k a$ — **M1**

$= 2^{k+2}(3b - a)$ — **A1**

True for n = 1, 2 and true for n = k, k + 1 implies n = k + 2, so true for all $n \geq 1$ — **A1**

Answer: u_n is divisible by 2^n for all integers $n \geq 1$ — **6 marks**

Induction with Matrices

1 (3 marks)

$A^2 = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}$ **B1**

$A^3 = \begin{pmatrix} 8 & 0 \\ 0 & 27 \end{pmatrix}$ **B1**

$A^n = \begin{pmatrix} 2^n & 0 \\ 0 & 3^n \end{pmatrix}$ **B1**

Answer: $\begin{pmatrix} \mathbf{2^n} & \mathbf{0} \\ \mathbf{0} & \mathbf{3^n} \end{pmatrix}$ **3 marks**

2 (2 marks)

$A^{k+1} = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ **M1**

$= \begin{pmatrix} 1 & k+1 \\ 0 & 1 \end{pmatrix}$ **A1**

Answer: $\begin{pmatrix} \mathbf{1} & \mathbf{k+1} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}$ **2 marks**

3 (2 marks)

$n = 1$: both sides equal $\begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}$, true **B1**

$P(k+1)$: $\begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & 0 \\ 5(k+1) & 1 \end{pmatrix}$ **B1**

Answer: $\begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{5} & \mathbf{1} \end{pmatrix}^{\mathbf{k+1}} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{5(k+1)} & \mathbf{1} \end{pmatrix}$ **2 marks**

4 (3 marks)

$B^2 = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$ **B1**

$B^3 = \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}$ **B1**

$B^n = \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix}$ **B1**

Answer: $\begin{pmatrix} \mathbf{1} & \mathbf{2n} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}$ **3 marks**

5 (2 marks)

$n = 1$: $\begin{pmatrix} 5 & 8 \\ -2 & -3 \end{pmatrix}$ matches A **B1**

$n = 2$: $A^2 = \begin{pmatrix} 25-16 & 40-24 \\ -10+6 & -16+9 \end{pmatrix} = \begin{pmatrix} 9 & 16 \\ -4 & -7 \end{pmatrix}$, and the formula gives $\begin{pmatrix} 9 & 16 \\ -4 & -7 \end{pmatrix}$ **B1**

Answer: the formula holds for n = 1 and n = 2 **2 marks**

6 (2 marks)

$J^2 = \begin{pmatrix} 0-1 & 0 \\ 0 & -1+0 \end{pmatrix} = -I$ **B1**

$J^4 = (-I)^2 = I$ **B1**

Answer: $J^2 = -I$ and $J^4 = I$, where I is the 2×2 identity matrix **2 marks**

7 (4 marks)

n = 1: LHS = $\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}$ = RHS, true **B1**

Assume true for n = k. Then $\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}^{k+1} = \begin{pmatrix}1 & k\\0 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}$ **M1**

$= \begin{pmatrix}1 & k+1\\0 & 1\end{pmatrix}$, the result for n = k + 1 **A1**

True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers $n \geq 1$ by induction **A1**

Answer: $\begin{pmatrix}\mathbf{1} & \mathbf{1}\\\mathbf{0} & \mathbf{1}\end{pmatrix}^{\mathbf{n}} = \begin{pmatrix}\mathbf{1} & \mathbf{n}\\\mathbf{0} & \mathbf{1}\end{pmatrix}$ **for all integers $n \geq 1$** **4 marks**

8 (4 marks)

n = 1: LHS = $\begin{pmatrix}1 & 0\\3 & 1\end{pmatrix}$ = RHS, true **B1**

Assume true for n = k. Then $\begin{pmatrix}1 & 0\\3 & 1\end{pmatrix}^{k+1} = \begin{pmatrix}1 & 0\\3k & 1\end{pmatrix}\begin{pmatrix}1 & 0\\3 & 1\end{pmatrix}$ **M1**

$= \begin{pmatrix}1 & 0\\3k+3 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\3(k+1) & 1\end{pmatrix}$ **A1**

True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers $n \geq 1$ by induction **A1**

Answer: $\begin{pmatrix}\mathbf{1} & \mathbf{0}\\\mathbf{3} & \mathbf{1}\end{pmatrix}^{\mathbf{n}} = \begin{pmatrix}\mathbf{1} & \mathbf{0}\\\mathbf{3n} & \mathbf{1}\end{pmatrix}$ **for all integers $n \geq 1$** **4 marks**

9 (5 marks)

$A^2 = \begin{pmatrix}3 & -2\\2 & -1\end{pmatrix}$ and $2A - I = \begin{pmatrix}3 & -2\\2 & -1\end{pmatrix}$, so $A^2 = 2A - I$ **B1**

n = 1: $1 \times A - 0 \times I = A$, true **B1**

Assume $A^k = kA - (k-1)I$. Then $A^{k+1} = A^kA = kA^2 - (k-1)A$ **M1**

$= k(2A - I) - (k-1)A = (k+1)A - kI$ **A1**

This is the result with n = k + 1. True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers $n \geq 1$ by induction **A1**

Answer: $A^2 = 2A - I$ and $A^n = nA - (n-1)I$ for all integers $n \geq 1$ **5 marks**

10 (4 marks)

n = 1: both sides equal the given matrix, true **B1**

Assume true for n = k. Then $A^{k+1} = A^kA = \begin{pmatrix}2^k & 0 & 0\\0 & 1 & 0\\0 & 0 & (-1)^k\end{pmatrix}\begin{pmatrix}2 & 0 & 0\\0 & 1 & 0\\0 & 0 & -1\end{pmatrix}$ **M1**

$= \begin{pmatrix}2^{k+1} & 0 & 0\\0 & 1 & 0\\0 & 0 & (-1)^{k+1}\end{pmatrix}$ **A1**

True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers $n \geq 1$ by induction **A1**

Answer: $\begin{pmatrix}\mathbf{2} & \mathbf{0} & \mathbf{0}\\\mathbf{0} & \mathbf{1} & \mathbf{0}\\\mathbf{0} & \mathbf{0} & \mathbf{-1}\end{pmatrix}^{\mathbf{n}} = \begin{pmatrix}\mathbf{2^n} & \mathbf{0} & \mathbf{0}\\\mathbf{0} & \mathbf{1} & \mathbf{0}\\\mathbf{0} & \mathbf{0} & \mathbf{(-1)^n}\end{pmatrix}$ **for all integers $n \geq 1$** **4 marks**

11 (5 marks)

n = 1: RHS = $\begin{pmatrix} 1 & 2-1 \\ 0 & 2 \end{pmatrix}$ = LHS, true **B1**

Assume true for n = k. Then $A^{k+1} = A^kA = \begin{pmatrix} 1 & 2^k-1 \\ 0 & 2^k \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$

$= \begin{pmatrix} 1 & 1+2(2^k-1) \\ 0 & 2\times 2^k \end{pmatrix}$ **M1 A1**

$= \begin{pmatrix} 1 & 2^{k+1}-1 \\ 0 & 2^{k+1} \end{pmatrix}$ **A1**

True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers $n \geq 1$ by induction **A1**

Answer: $\begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{2} \end{pmatrix}^{\mathbf{n}} = \begin{pmatrix} \mathbf{1} & \mathbf{2^n-1} \\ \mathbf{0} & \mathbf{2^n} \end{pmatrix}$ **for all integers $\mathbf{n \geq 1}$** **5 marks**

12 (5 marks)

n = 1: RHS = $\begin{pmatrix} 2 & 3\times 1 \\ 0 & 1 \end{pmatrix}$ = LHS, true **B1**

Assume true for n = k. Then $A^{k+1} = A^kA = \begin{pmatrix} 2^k & 3(2^k-1) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 2^{k+1} & 3\times 2^k+3(2^k-1) \\ 0 & 1 \end{pmatrix}$ **M1 A1**

$= \begin{pmatrix} 2^{k+1} & 3(2^{k+1}-1) \\ 0 & 1 \end{pmatrix}$ **A1**

True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers $n \geq 1$ by induction **A1**

Answer: $\begin{pmatrix} \mathbf{2} & \mathbf{3} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}^{\mathbf{n}} = \begin{pmatrix} \mathbf{2^n} & \mathbf{3(2^n-1)} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}$ **for all integers $\mathbf{n \geq 1}$** **5 marks**

13 (4 marks)

n = 1: $A^1 = A = 2^0A$, true **B1**

Assume $A^k = 2^{k-1}A$. Then $A^{k+1} = A^kA = 2^{k-1}A^2$ **M1**

$A^2 = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = 2A$, so $A^{k+1} = 2^{k-1}\times 2A = 2^kA$ **A1**

True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers $n \geq 1$ by induction **A1**

Answer: $\mathbf{A^n = 2^{n-1}A}$ for all integers $\mathbf{n \geq 1}$ **4 marks**

14 (6 marks)

n = 1: RHS = $\begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}$ = LHS, true **B1**

Assume true for n = k. Then $A^{k+1} = A^kA = \begin{pmatrix} 2k+1 & -2k \\ 2k & 1-2k \end{pmatrix}\begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}$ **M1**

$= \begin{pmatrix} 3(2k+1)-4k & -2(2k+1)+2k \\ 6k+2(1-2k) & -4k-(1-2k) \end{pmatrix}$ **M1 A1**

$= \begin{pmatrix} 2k+3 & -2k-2 \\ 2k+2 & -2k-1 \end{pmatrix} = \begin{pmatrix} 2(k+1)+1 & -2(k+1) \\ 2(k+1) & 1-2(k+1) \end{pmatrix}$ **A1**

True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers $n \geq 1$ by induction **A1**

Answer: $\begin{pmatrix} \mathbf{3} & \mathbf{-2} \\ \mathbf{2} & \mathbf{-1} \end{pmatrix}^{\mathbf{n}} = \begin{pmatrix} \mathbf{2n+1} & \mathbf{-2n} \\ \mathbf{2n} & \mathbf{1-2n} \end{pmatrix}$ **for all integers $\mathbf{n \geq 1}$** **6 marks**

15 (6 marks)

n = 1: RHS = $\begin{pmatrix} 2 & 1 \times 2^0 \\ 0 & 2 \end{pmatrix}$ = LHS, true **B1**

Assume true for n = k. Then $A^{k+1} = A^kA = \begin{pmatrix} 2^k & k2^{k-1} \\ 0 & 2^k \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ **M1**

Top right: $2^k \times 1 + k2^{k-1} \times 2 = 2^k + k2^k = (k + 1)2^k$ **M1 A1**

So $A^{k+1} = \begin{pmatrix} 2^{k+1} & (k + 1)2^k \\ 0 & 2^{k+1} \end{pmatrix}$, the result for n = k + 1 **A1**

True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers $n \geq 1$ by induction **A1**

Answer: $\begin{pmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{0} & \mathbf{2} \end{pmatrix}^{\mathbf{n}} = \begin{pmatrix} \mathbf{2^n} & \mathbf{n2^{n-1}} \\ \mathbf{0} & \mathbf{2^n} \end{pmatrix}$ **for all integers $\mathbf{n \geq 1}$** **6 marks**

16 (7 marks)

n = 1: RHS = $\frac{1}{2}\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ = LHS, true **B1**

Assume true for n = k. Then $A^{k+1} = \frac{1}{2}\begin{pmatrix} 3^k + 1 & 3^k - 1 \\ 3^k - 1 & 3^k + 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ **M1**

Top left: $\frac{1}{2}[2(3^k + 1) + (3^k - 1)] = \frac{1}{2}(3^{k+1} + 1)$ **M1 A1**

Top right: $\frac{1}{2}[(3^k + 1) + 2(3^k - 1)] = \frac{1}{2}(3^{k+1} - 1)$ **A1**

Bottom row: $\frac{1}{2}[2(3^k - 1) + (3^k + 1)] = \frac{1}{2}(3^{k+1} - 1)$ and $\frac{1}{2}[(3^k - 1) + 2(3^k + 1)] = \frac{1}{2}(3^{k+1} + 1)$, **A1**

so A^{k+1} has the form of the formula with n = k + 1

True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers $n \geq 1$ by induction **A1**

Answer: $\begin{pmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} \end{pmatrix}^{\mathbf{n}} = \frac{\mathbf{1}}{\mathbf{2}}\begin{pmatrix} \mathbf{3^n + 1} & \mathbf{3^n - 1} \\ \mathbf{3^n - 1} & \mathbf{3^n + 1} \end{pmatrix}$ **for all integers $\mathbf{n \geq 1}$** **7 marks**

17 (7 marks)

n = 1: RHS = $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ = LHS, true **B1**

Assume true for n = k. Then $A^{k+1} = A^kA = \begin{pmatrix} 1 & k & \frac{k(k-1)}{2} \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ **M1**

Row 1: $(1, 1 + k, k + \frac{k(k-1)}{2})$ and row 2: (0, 1, 1 + k), row 3: (0, 0, 1) **M1 A1**

$k + \frac{k(k-1)}{2} = \frac{k(k+1)}{2} = \frac{(k+1)((k+1)-1)}{2}$ **M1**

So A^{k+1} has the required form with n = k + 1 **A1**

True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers $n \geq 1$ by induction **A1**

Answer: $\begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix}^{\mathbf{n}} = \begin{pmatrix} \mathbf{1} & \mathbf{n} & \frac{\mathbf{n(n-1)}}{\mathbf{2}} \\ \mathbf{0} & \mathbf{1} & \mathbf{n} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix}$ **for all integers $\mathbf{n \geq 1}$** **7 marks**

18 (7 marks)

n = 1: both sides equal the rotation matrix, true **B1**

Assume true for n = k. Then $R^{k+1} = R^kR = \begin{pmatrix}\cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta\end{pmatrix}\begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}$ **M1**

Top left: $\cos k\theta\cos\theta - \sin k\theta\sin\theta = \cos(k+1)\theta$ **M1 A1**

Top right: $-\cos k\theta\sin\theta - \sin k\theta\cos\theta = -\sin(k+1)\theta$ **A1**

Bottom row similarly gives $\sin(k+1)\theta$ and $\cos(k+1)\theta$, the result for n = k + 1 **A1**

True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers $n \geq 1$ by induction **A1**

Answer: for all integers $n \geq 1$, $\left(\begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}\right)^n = \begin{pmatrix}\cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta\end{pmatrix}$ **7 marks**

19 (6 marks)

$A^2 = \begin{pmatrix}10 & -18 \\ 3 & -5\end{pmatrix}$ and $3A - 2I = \begin{pmatrix}10 & -18 \\ 3 & -5\end{pmatrix}$, so $A^2 = 3A - 2I$ **B1**

n = 1: $(2-1)A - 0 \times I = A$, true **B1**

Assume $A^k = (2^k - 1)A - (2^k - 2)I$. Then $A^{k+1} = (2^k - 1)A^2 - (2^k - 2)A$ **M1**

$= (2^k - 1)(3A - 2I) - (2^k - 2)A$ **M1**

$= (3 \times 2^k - 3 - 2^k + 2)A - 2(2^k - 1)I$ **A1**

$= (2^{k+1} - 1)A - (2^{k+1} - 2)I$, the result for n = k + 1. True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers $n \geq 1$ by induction **A1**

Answer: $A^2 = 3A - 2I$, and hence prove by induction that $A^n = (2^n - 1)A - (2^n - 2)I$ for all integers $n \geq 1$ **6 marks**

20 (7 marks)

n = 1: RHS $= \begin{pmatrix}1 \times 1 & -1 \\ 1 & 1 \times 3\end{pmatrix} = A = $ LHS, true **B1**

Assume true for n = k. Then $A^{k+1} = A^kA = \begin{pmatrix}2^{k-1}(2-k) & -k2^{k-1} \\ k2^{k-1} & 2^{k-1}(2+k)\end{pmatrix}\begin{pmatrix}1 & -1 \\ 1 & 3\end{pmatrix}$ **M1**

Top left: $2^{k-1}(2-k) - k2^{k-1} = 2^{k-1}(2-2k) = 2^k(1-k)$ **M1 A1**

Top right: $-2^{k-1}(2-k) - 3k2^{k-1} = -2^{k-1}(2+2k) = -(k+1)2^k$ **A1**

Bottom row: $k2^{k-1} + 2^{k-1}(2+k) = 2^k(k+1)$ and $-k2^{k-1} + 3 \times 2^{k-1}(2+k) = 2^k(k+3)$

These equal the entries of the formula with n = k + 1: $2^k(2-(k+1))$, $-(k+1)2^k$, $(k+1)2^k$, $2^k(2+k+1)$ **A1**

True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers $n \geq 1$ by induction **A1**

Answer: $\begin{pmatrix}1 & -1 \\ 1 & 3\end{pmatrix}^n = \begin{pmatrix}2^{n-1}(2-n) & -n2^{n-1} \\ n2^{n-1} & 2^{n-1}(2+n)\end{pmatrix}$ for all integers $n \geq 1$ **7 marks**

21 (9 marks)

(a) n = 1: RHS $= \begin{pmatrix}F_2 & F_1 \\ F_1 & F_0\end{pmatrix} = \begin{pmatrix}1 & 1 \\ 1 & 0\end{pmatrix} = M$, true **B1**

Assume $M^k = \begin{pmatrix}F_{k+1} & F_k \\ F_k & F_{k-1}\end{pmatrix}$. Then $M^{k+1} = M^kM$ **M1**

$= \begin{pmatrix}F_{k+1} + F_k & F_{k+1} \\ F_k + F_{k-1} & F_k\end{pmatrix}$ **M1 A1**

$= \begin{pmatrix}F_{k+2} & F_{k+1} \\ F_{k+1} & F_k\end{pmatrix}$ using the definition, the result for n = k + 1 **A1**

True for n = 1 and true for n = k implies true for n = k + 1, so true for all $n \geq 1$ **A1**

Answer: $M^n = \begin{pmatrix}F_{n+1} & F_n \\ F_n & F_{n-1}\end{pmatrix}$ for all integers $n \geq 1$ **6 marks**

(b) $\det M = -1$, so $\det(M^n) = (-1)^n$ **M1 A1**

$\det(M^n) = F_{n+1}F_{n-1} - F_n^2$, so $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$ **A1**

Answer: $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$ **3 marks**

22 (8 marks)

n = 1: $\begin{pmatrix} 6-2 & 4-6 \\ 3-2 & 4-3 \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$ = A, true **B1**

Assume the formula for n = k. Then $A^{k+1} = A^kA$, with $A = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$ **M1**

Top left: $4(2 \times 3^k - 2^k) + (2^{k+1} - 2 \times 3^k) = 6 \times 3^k - 2 \times 2^k = 2 \times 3^{k+1} - 2^{k+1}$ **M1 A1**

Top right: $-2(2 \times 3^k - 2^k) + (2^{k+1} - 2 \times 3^k) = 2^{k+2} - 6 \times 3^k = 2^{k+2} - 2 \times 3^{k+1}$ **M1 A1**

Bottom left: $4(3^k - 2^k) + (2^{k+1} - 3^k) = 3^{k+1} - 2^{k+1}$

Bottom right: $-2(3^k - 2^k) + (2^{k+1} - 3^k) = 2^{k+2} - 3^{k+1}$ **A1**

These are the entries of the formula with n = k + 1. True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers $n \geq 1$ by induction **A1**

Answer: $A^n = \begin{pmatrix} 2 \times 3^n - 2^n & 2^{n+1} - 2 \times 3^n \\ 3^n - 2^n & 2^{n+1} - 3^n \end{pmatrix}$ for all integers $n \geq 1$ **8 marks**

23 (7 marks)

n = 1: the formula gives $\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$ = T, true **B1**

Assume true for n = k. Then $T^{k+1} = T^kT = \begin{pmatrix} 1 & ka & kb + \frac{k(k-1)ac}{2} \\ 0 & 1 & kc \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$ **M1**

Row 1: $(1, a + ka, b + kac + kb + \frac{k(k-1)ac}{2})$ **M1 A1**

Row 2: (0, 1, c + kc), row 3: (0, 0, 1) **A1**

$b + kb = (k + 1)b$ and $kac + \frac{k(k-1)ac}{2} = \frac{k(k+1)ac}{2} = \frac{(k+1)((k+1)-1)ac}{2}$ **M1**

So T^{k+1} has the required form with n = k + 1. True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers $n \geq 1$ by induction **A1**

Answer: $T^n = \begin{pmatrix} 1 & na & nb + \frac{n(n-1)ac}{2} \\ 0 & 1 & nc \\ 0 & 0 & 1 \end{pmatrix}$ for all integers $n \geq 1$ **7 marks**

24 (6 marks)

n = 2: $F_2A + F_1I = A + I = A^2$, true **B1**

Assume $A^k = F_kA + F_{k-1}I$. Then $A^{k+1} = A \times A^k = F_kA^2 + F_{k-1}A$ **M1**

$= F_k(A + I) + F_{k-1}A$ **M1**

$= (F_k + F_{k-1})A + F_kI$ **A1**

$= F_{k+1}A + F_kI$, the result for n = k + 1 **A1**

True for n = 2 and true for n = k implies true for n = k + 1, so true for all $n \geq 2$ **A1**

Answer: $A^n = F_nA + F_{n-1}I$ for all integers $n \geq 2$ **6 marks**

25 (7 marks)

$A^2 = \begin{pmatrix} 1 & 8 \\ 0 & 9 \end{pmatrix}$, but the claim gives $\begin{pmatrix} 1 & 4 \\ 0 & 9 \end{pmatrix}$, so the claim is false **B1**

Error: matrix multiplication is row by column, not entry by entry; the top right entry of A^2 is $1 \times 2 + 2 \times 3 = 8$ **E1**

Conjecture: $A^n = \begin{pmatrix} 1 & 3^n - 1 \\ 0 & 3^n \end{pmatrix}$ ($n = 1$: $\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$; $n = 2$: $\begin{pmatrix} 1 & 8 \\ 0 & 9 \end{pmatrix}$) **B1**

Assume $A^k = \begin{pmatrix} 1 & 3^k - 1 \\ 0 & 3^k \end{pmatrix}$. Then $A^{k+1} = A^k A$ **M1**

$= \begin{pmatrix} 1 & 2 + 3(3^k - 1) \\ 0 & 3 \times 3^k \end{pmatrix}$ **M1 A1**

$= \begin{pmatrix} 1 & 3^{k+1} - 1 \\ 0 & 3^{k+1} \end{pmatrix}$, the formula with $n = k + 1$. True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction **A1**

Answer: $A^n = \begin{pmatrix} \mathbf{1} & \mathbf{3^n - 1} \\ \mathbf{0} & \mathbf{3^n} \end{pmatrix}$ **7 marks**

26 (8 marks)

Part 1: $n = 1$ is the given condition $AB = BA$ **B1**

Assume $AB^k = B^k A$. Then $AB^{k+1} = (AB^k)B = B^k AB = B^k BA = B^{k+1} A$ **M1 A1**

So $AB^n = B^n A$ for all $n \geq 1$ by induction **A1**

Part 2: $n = 1$: $(AB)^1 = A^1 B^1$, true **B1**

Assume $(AB)^k = A^k B^k$. Then $(AB)^{k+1} = (AB)^k AB = A^k B^k AB$ **M1**

Using part 1, $B^k A = AB^k$, so this is $A^k AB^k B$ **A1**

$= A^{k+1} B^{k+1}$. True for $n = 1$, and $n = k$ implies $n = k + 1$, so true for all $n \geq 1$ **A1**

Answer: $AB^n = B^n A$ for all integers $n \geq 1$, and hence prove by induction that $(AB)^n = A^n B^n$ for all integers $n \geq 1$ **8 marks**

27 (10 marks)

(a) $n = 1$: $\frac{1}{7}\begin{pmatrix} 4 & 3 \\ 4 & 3 \end{pmatrix} + \frac{0.3}{7}\begin{pmatrix} 3 & -3 \\ -4 & 4 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 4.9 & 2.1 \\ 2.8 & 4.2 \end{pmatrix} = P$, true **B1**

Assume the formula for $n = k$. Then $P^{k+1} = P^k P$ **M1**

$\frac{1}{7}\begin{pmatrix} 4 & 3 \\ 4 & 3 \end{pmatrix} P = \frac{1}{7}\begin{pmatrix} 4 & 3 \\ 4 & 3 \end{pmatrix}$ (each row (4, 3) times P gives (4, 3)) **M1 A1**

$\begin{pmatrix} 3 & -3 \\ -4 & 4 \end{pmatrix} P = \begin{pmatrix} 0.9 & -0.9 \\ -1.2 & 1.2 \end{pmatrix} = 0.3\begin{pmatrix} 3 & -3 \\ -4 & 4 \end{pmatrix}$ **M1 A1**

So $P^{k+1} = \frac{1}{7}\begin{pmatrix} 4 & 3 \\ 4 & 3 \end{pmatrix} + \frac{1}{7}(0.3)^{k+1}\begin{pmatrix} 3 & -3 \\ -4 & 4 \end{pmatrix}$. True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction **A1**

Answer: $P^n = \frac{1}{7}\begin{pmatrix} \mathbf{4} & \mathbf{3} \\ \mathbf{4} & \mathbf{3} \end{pmatrix} + \frac{1}{7}\mathbf{(0.3)^n}\begin{pmatrix} \mathbf{3} & \mathbf{-3} \\ \mathbf{-4} & \mathbf{4} \end{pmatrix}$ for all integers $\mathbf{n \geq 1}$ **7 marks**

(b) As $n \to \infty$, $(0.3)^n \to 0$ **B1**

$P^n \to \frac{1}{7}\begin{pmatrix} 4 & 3 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} \frac{4}{7} & \frac{3}{7} \\ \frac{4}{7} & \frac{3}{7} \end{pmatrix}$ **B1**

Whatever the starting state, the website is up on $\frac{4}{7}$ of days in the long run **E1**

Answer: $\frac{4}{7}$ **3 marks**

28 (9 marks)

n = 1: the formula gives $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ = A (the top right entry is $1 \times 0 \times 2^{-2} = 0$), true **B1**

Assume the formula for n = k. Then $A^{k+1} = A^kA$, where A has columns (2, 0, 0), (1, 2, 0), (0, 1, 2) **M1**

Row 1 of A^kA: $(2^{k+1}, 2^k + 2k2^{k-1}, k2^{k-1} + 2k(k-1)2^{k-3})$ **M1 A1**

Middle entry: $2^k + k2^k = (k + 1)2^k$, as required for n = k + 1 **A1**

Right entry: $k2^{k-1} + k(k-1)2^{k-2} = k2^{k-2}(2 + k - 1) = k(k + 1)2^{k-2}$, which is $(k + 1)((k + 1) - 1)2^{(k+1)-3}$ **M1 A1**

Row 2 of A^kA: $(0, 2^{k+1}, 2^k + 2k2^{k-1}) = (0, 2^{k+1}, (k + 1)2^k)$; row 3: $(0, 0, 2^{k+1})$ **A1**

All entries match the formula with n = k + 1. True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers $n \geq 1$ by induction **A1**

Answer: $\mathbf{A^n = \begin{pmatrix} 2^n & n2^{n-1} & n(n-1)2^{n-3} \\ 0 & 2^n & n2^{n-1} \\ 0 & 0 & 2^n \end{pmatrix}}$ **for all integers $\mathbf{n \geq 1}$** **9 marks**

29 (9 marks)

$D_1 = 2$ and $D_2 = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3$ **B1 B1**

Expand the (n + 2) × (n + 2) determinant along the first row: $D_{n+2} = 2 \times D_{n+1} - (-1) \times M$, where M is the minor obtained by deleting row 1 and column 2 **M1**

The first column of M is $(-1, 0, ..., 0)^T$, so expanding M down that column gives $M = -D_n$ **M1**

Hence $D_{n+2} = 2D_{n+1} + (-D_n) = 2D_{n+1} - D_n$ **A1**

Base cases: $D_1 = 2 = 1 + 1$ and $D_2 = 3 = 2 + 1$ **B1**

Assume $D_k = k + 1$ and $D_{k+1} = k + 2$. Then $D_{k+2} = 2(k + 2) - (k + 1)$ **M1**

$= k + 3 = (k + 2) + 1$, the result for n = k + 2 **A1**

True for n = 1, 2, and n = k, k + 1 implies n = k + 2, so $D_n = n + 1$ for all integers $n \geq 1$ by strong induction **A1**

Answer: $\mathbf{D_{n+2} = 2D_{n+1} - D_n}$**, and** $\mathbf{D_n = n + 1}$ **for all integers $\mathbf{n \geq 1}$** **9 marks**

30 (10 marks)

(a) n = 1: the formula gives $\begin{pmatrix} 1 & 1 & 3-4+1 \\ 0 & 2 & 2(3-2) \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$ = A, true **B1**

Assume the formula for n = k. Then $A^{k+1} = A^kA$, where the columns of A are (1, 0, 0), (1, 2, 0), (0, 2, 3) **M1**

Entry (1, 2): $1 + 2(2^k - 1) = 2^{k+1} - 1$ **M1 A1**

Entry (1, 3): $2(2^k - 1) + 3(3^k - 2^{k+1} + 1) = 3^{k+1} - 2^{k+2} + 1$ **M1 A1**

Entry (2, 3): $2 \times 2^k + 6(3^k - 2^k) = 2(3^{k+1} - 2^{k+1})$; entries (2, 2) = 2^{k+1} and (3, 3) = 3^{k+1} **A1**

These match the formula with n = k + 1. True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers $n \geq 1$ by induction **A1**

Answer: $\mathbf{A^n = \begin{pmatrix} 1 & 2^n - 1 & 3^n - 2^{n+1} + 1 \\ 0 & 2^n & 2(3^n - 2^n) \\ 0 & 0 & 3^n \end{pmatrix}}$ **for all integers $\mathbf{n \geq 1}$** **8 marks**

(b) $3^6 - 2^7 + 1$ **M1**

$= 729 - 128 + 1 = 602$ **A1**

Answer: 602 **2 marks**

Induction with Inequalities

1 (3 marks)

$n = 1$: $2 > 1$ **B1**

$n = 2$: $4 > 2$ **B1**

$n = 3$: $8 > 3$ **B1**

Answer: $2^n > n$ for $n = 1$, 2 and 3 **3 marks**

2 (2 marks)

$2^{k+1} = 2 \times 2^k$ **M1**

$> 2 \times k^2 = 2k^2$, since multiplying both sides of the inequality by 2 preserves it **A1**

Answer: $2^{k+1} > 2k^2$ **2 marks**

3 (3 marks)

$n = 4$: $4! = 24$ and $2^4 = 16$, and $24 > 16$ **B1 B1**

$n = 3$: $3! = 6$ and $2^3 = 8$, and $6 < 8$, so the inequality fails **B1**

Answer: $n! > 2^n$ holds for $n = 4$ but not for $n = 3$ **3 marks**

4 (2 marks)

The inequality is equivalent to $k^2 - 2k - 1 > 0$, that is $(k - 1)^2 > 2$ **M1**

For $k \geq 3$, $(k - 1)^2 \geq 4 > 2$, so it is true **A1**

Answer: $k^2 + 2k + 1 < 2k^2$ for all integers $k \geq 3$ **2 marks**

5 (3 marks)

$n = 1$: $4 \geq 4$ **B1**

$n = 2$: $16 \geq 7$ **B1**

$n = 3$: $64 \geq 10$ **B1**

Answer: $4^n \geq 1 + 3n$ for $n = 1$, 2 and 3 **3 marks**

6 (2 marks)

$n = 2$: 4 and 4; $n = 3$: 8 and 9; $n = 4$: 16 and 16 (so the inequality fails for $n = 2, 3, 4$) **M1**

$n = 5$: $32 > 25$, so $N = 5$ **A1**

Answer: $N = 5$ **2 marks**

7 (4 marks)

$n = 1$: $2 \geq 2$, true **B1**

Assume $2^k \geq k + 1$. Then $2^{k+1} = 2 \times 2^k \geq 2(k + 1)$ **M1**

$2(k + 1) = 2k + 2 \geq k + 2 = (k + 1) + 1$, so $2^{k+1} \geq (k + 1) + 1$ **A1**

True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction **A1**

Answer: $2^n \geq n + 1$ for all integers $n \geq 1$ **4 marks**

8 (4 marks)

$n = 1$: $3 \geq 3$, true **B1**

Assume $3^k \geq 2k + 1$. Then $3^{k+1} \geq 3(2k + 1) = 6k + 3$ **M1**

$6k + 3 \geq 2k + 3 = 2(k + 1) + 1$, so $3^{k+1} \geq 2(k + 1) + 1$ **A1**

True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction **A1**

Answer: $3^n \geq 2n + 1$ for all integers $n \geq 1$ **4 marks**

9 (4 marks)

$n = 1$: $1! = 1 \geq 2^0 = 1$, true — **B1**

Assume $k! \geq 2^{k-1}$. Then $(k + 1)! = (k + 1) \times k! \geq (k + 1) \times 2^{k-1}$ — **M1**

Since $k + 1 \geq 2$, $(k + 1)2^{k-1} \geq 2 \times 2^{k-1} = 2^k$ — **A1**

True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction — **A1**

Answer: $n! \geq 2^{n-1}$ for all integers $n \geq 1$ — **4 marks**

10 (5 marks)

$n = 2$: $2! = 2 < 4 = 2^2$, true — **B1**

Assume $k! < k^k$ for some $k \geq 2$. Then $(k + 1)! = (k + 1)k! < (k + 1)k^k$ — **M1**

Since $k < k + 1$, $k^k < (k + 1)^k$ — **M1**

So $(k + 1)k^k < (k + 1)(k + 1)^k = (k + 1)^{k+1}$, hence $(k + 1)! < (k + 1)^{k+1}$ — **A1**

True for $n = 2$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 2$ by induction — **A1**

Answer: $n! < n^n$ for all integers $n \geq 2$ — **5 marks**

11 (4 marks)

$n = 4$: $4! = 24 > 16 = 2^4$, true — **B1**

Assume $k! > 2^k$ for some $k \geq 4$. Then $(k + 1)! = (k + 1)k! > (k + 1)2^k$ — **M1**

Since $k + 1 \geq 5 > 2$, $(k + 1)2^k > 2 \times 2^k = 2^{k+1}$ — **A1**

True for $n = 4$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 4$ by induction — **A1**

Answer: $n! > 2^n$ for all integers $n \geq 4$ — **4 marks**

12 (5 marks)

$n = 5$: $2^5 = 32 > 25 = 5^2$, true — **B1**

Assume $2^k > k^2$ for some $k \geq 5$. Then $2^{k+1} > 2k^2$ — **M1**

$2k^2 - (k + 1)^2 = k^2 - 2k - 1 = (k - 1)^2 - 2 > 0$ for $k \geq 5$ — **M1**

Hence $2^{k+1} > 2k^2 > (k + 1)^2$ — **A1**

True for $n = 5$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 5$ by induction — **A1**

Answer: $2^n > n^2$ for all integers $n \geq 5$ — **5 marks**

13 (5 marks)

$n = 3$: $3^3 = 27 > 16 = 4^2$, true — **B1**

Assume $3^k > (k + 1)^2$ for some $k \geq 3$. Then $3^{k+1} > 3(k + 1)^2$ — **M1**

$3(k + 1)^2 - (k + 2)^2 = 2k^2 + 2k - 1$ — **M1**

For $k \geq 3$ this is positive, so $3^{k+1} > (k + 2)^2 = ((k + 1) + 1)^2$ — **A1**

True for $n = 3$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 3$ by induction — **A1**

Answer: $3^n > (n + 1)^2$ for all integers $n \geq 3$ — **5 marks**

14 (6 marks)

$n = 1$: $(1 + x)^1 = 1 + x \geq 1 + x$, true — **B1**

Assume $(1 + x)^k \geq 1 + kx$ — **M1**

Since $1 + x > 0$, multiplying by $(1 + x)$ keeps the inequality: $(1 + x)^{k+1} \geq (1 + kx)(1 + x)$ — **M1 A1**

$(1 + kx)(1 + x) = 1 + (k + 1)x + kx^2 \geq 1 + (k + 1)x$ since $kx^2 \geq 0$ — **A1**

True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction — **A1**

Answer: $(1 + x)^n \geq 1 + nx$ for all integers $n \geq 1$, where x is a real constant with $x > -1$ — **6 marks**

15 (6 marks)

n = 1: LHS = 1, RHS = 2 − 1 = 1, so true **B1**

Assume the sum to k is at most $2 - \frac{1}{k}$. Sum to k + 1 ≤ $2 - \frac{1}{k} + \frac{1}{(k+1)^2}$ **M1**

$\frac{1}{(k+1)^2} < \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$ **M1 A1**

So the sum to k + 1 < $2 - \frac{1}{k} + \frac{1}{k} - \frac{1}{k+1} = 2 - \frac{1}{k+1}$ **A1**

True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers n ≥ 1 by induction **A1**

Answer: $\sum_{r=1}^{n} \frac{1}{r^2} \leq 2 - \frac{1}{n}$ for all integers n ≥ 1 **6 marks**

16 (6 marks)

n = 1: LHS = 1, RHS = 1, true **B1**

Assume the sum to k is at least $\sqrt{k}$. Sum to k + 1 ≥ $\sqrt{k} + \frac{1}{\sqrt{k+1}}$ **M1**

It is enough to show $\sqrt{k} + \frac{1}{\sqrt{k+1}} \geq \sqrt{k+1}$; multiplying by $\sqrt{k+1}$ this is $\sqrt{k(k+1)} + 1 \geq k + 1$ **M1 A1**

$\sqrt{k(k+1)} \geq \sqrt{k^2} = k$, so it is true **A1**

True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers n ≥ 1 by induction **A1**

Answer: $\sum_{r=1}^{n} \frac{1}{\sqrt{r}} \geq \sqrt{n}$ for all integers n ≥ 1 **6 marks**

17 (6 marks)

n = 10: $2^{10} = 1024 > 1000 = 10^3$, true **B1**

Assume $2^k > k^3$ for some k ≥ 10. Then $2^{k+1} > 2k^3$ **M1**

Show $2k^3 \geq (k+1)^3$, that is $(1 + \frac{1}{k})^3 \leq 2$ **M1**

For k ≥ 10, $(1 + \frac{1}{k})^3 \leq (1.1)^3 = 1.331 < 2$ **A1 A1**

Hence $2^{k+1} > (k+1)^3$. True for n = 10, and true for n = k implies true for n = k + 1, so true for all integers n ≥ 10 by induction **A1**

Answer: $2^n > n^3$ for all integers n ≥ 10 **6 marks**

18 (5 marks)

n = 1: 2! = 2 and $2^1 \times 1! = 2$, true **B1**

Assume (2k)! ≥ $2^k \times k!$. Then (2k + 2)! = (2k + 2)(2k + 1)(2k)! ≥ (2k + 2)(2k + 1) × $2^k \times k!$ **M1**

Now (2k + 2)(2k + 1) = 2(k + 1)(2k + 1) ≥ 2(k + 1) × 2 since 2k + 1 ≥ 2 **M1 A1**

So (2k + 2)! ≥ 2(k + 1) × 2 × $2^k \times k!$ ≥ $2^{k+1}(k+1)k! = 2^{k+1}(k+1)!$, the result for n = k + 1. True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers n ≥ 1 by induction **A1**

Answer: (2n)! ≥ $2^n \times n!$ for all integers n ≥ 1 **5 marks**

19 (5 marks)

$n = 1$: LHS $= 1 > \frac{1}{3} =$ RHS, true **B1**

Assume $\sum_{r=1}^{k} r^2 > \frac{k^3}{3}$. Then $\sum_{r=1}^{k+1} r^2 > \frac{k^3}{3} + (k+1)^2$ **M1**

$\frac{k^3}{3} + (k+1)^2 - \frac{(k+1)^3}{3} = \frac{k^3 + 3(k+1)^2 - (k+1)^3}{3} = \frac{3k^2 + 6k + 3 - 3k^2 - 3k - 1}{3} = \frac{3k+2}{3} > 0$ **M1 A1**

So $\sum_{r=1}^{k+1} r^2 > \frac{(k+1)^3}{3}$. True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction **A1**

Answer: $\sum_{r=1}^{n} r^2 > \frac{n^3}{3}$ for all integers $n \geq 1$ **5 marks**

20 (6 marks)

$n = 1$: $|\sin\theta| \leq |\sin\theta|$, true **B1**

Assume $|\sin k\theta| \leq k|\sin\theta|$. Then $\sin(k+1)\theta = \sin k\theta \cos\theta + \cos k\theta \sin\theta$ **M1**

$|\sin(k+1)\theta| \leq |\sin k\theta||\cos\theta| + |\cos k\theta||\sin\theta|$ **M1 A1**

Since $|\cos\theta| \leq 1$ and $|\cos k\theta| \leq 1$, this is $\leq |\sin k\theta| + |\sin\theta| \leq k|\sin\theta| + |\sin\theta|$ **M1**

$= (k+1)|\sin\theta|$. True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction **A1**

Answer: $|\sin n\theta| \leq n|\sin\theta|$ for all integers $n \geq 1$ and all real θ **6 marks**

21 (9 marks)

(a) $n = 1$: $u_1 = 1 < 2$, true **B1**

Assume $u_k < 2$. Then $2 + u_k < 4$, so $u_{k+1} = \sqrt{2 + u_k} < 2$ **M1 A1**

True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction **A1**

Answer: $u_n < 2$ for all integers $n \geq 1$ **4 marks**

(b) $n = 1$: $u_2 = \sqrt{3} > 1 = u_1$, true **B1**

Assume $u_{k+1} > u_k$. Then $2 + u_{k+1} > 2 + u_k$ **M1**

So $\sqrt{2 + u_{k+1}} > \sqrt{2 + u_k}$, that is $u_{k+2} > u_{k+1}$ **M1 A1**

True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction **A1**

Answer: $u_{n+1} > u_n$ for all integers $n \geq 1$ **5 marks**

22 (7 marks)

$n = 1$: LHS $= 1 < 2 =$ RHS, true **B1**

Assume the sum to k is less than $2\sqrt{k}$. Sum to $k + 1 < 2\sqrt{k} + \frac{1}{\sqrt{k+1}}$ **M1**

It is enough to show $\frac{1}{\sqrt{k+1}} \leq 2(\sqrt{k+1} - \sqrt{k})$ **M1**

$2(\sqrt{k+1} - \sqrt{k}) = \frac{2}{\sqrt{k+1} + \sqrt{k}}$ using $(a - b)(a + b) = a^2 - b^2$ **M1 A1**

Since $\sqrt{k} < \sqrt{k+1}$, $\sqrt{k+1} + \sqrt{k} < 2\sqrt{k+1}$, so $\frac{2}{\sqrt{k+1} + \sqrt{k}} > \frac{1}{\sqrt{k+1}}$ **A1**

So the sum to $k + 1 < 2\sqrt{k+1}$. True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction **A1**

Answer: $\sum_{r=1}^{n} \frac{1}{\sqrt{r}} < 2\sqrt{n}$ for all integers $n \geq 1$ **7 marks**

23 (6 marks)

n = 2: 4 = 4, n = 3: 8 < 9 and n = 4: 16 = 16, so it fails for n = 2, 3, 4 **B1 B1**

The step $k^2 \geq 2k + 1$ is the invalid one: it needs $(k - 1)^2 \geq 2$ **E1**

$k^2 \geq 2k + 1$ fails for k = 1 and k = 2 (1 < 3, 4 < 5) but holds for $k \geq 3$ **M1**

So the step is valid only for $k \geq 3$, but the base case n = 1 cannot lead there. A base case where the statement holds and $k \geq 3$ is needed: n = 5 (32 > 25) **E1**

So the argument proves the result for all $n \geq 5$ **A1**

Answer: Fails for n = 2, 3, 4; step needs $k \geq 3$; valid for $n \geq 5$ **6 marks**

24 (7 marks)

n = 1: LHS = 1 + x, RHS = 1 + x + 0, true **B1**

Assume $(1 + x)^k \geq 1 + kx + \frac{k(k-1)}{2}x^2$ **M1**

Since 1 + x > 0, $(1 + x)^{k+1} \geq (1 + x)(1 + kx + \frac{k(k-1)}{2}x^2)$ **M1 A1**

Expand: $1 + (k + 1)x + [k + \frac{k(k-1)}{2}]x^2 + \frac{k(k-1)}{2}x^3$ **M1**

$k + \frac{k(k-1)}{2} = \frac{k(k+1)}{2}$ **A1**

The x^3 term is non-negative, so $(1 + x)^{k+1} \geq 1 + (k + 1)x + \frac{(k+1)k}{2}x^2$. True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers $n \geq 1$ by induction **A1**

Answer: $(1 + x)^n \geq 1 + nx + \frac{n(n-1)}{2}x^2$ for all integers $n \geq 1$, where $x \geq 0$ **7 marks**

25 (8 marks)

n = 1: $F_1 = 1 \geq (\frac{3}{2})^{-1} = \frac{2}{3}$, true; n = 2: $F_2 = 1 \geq 1$, true **B1 B1**

Assume $F_k \geq (\frac{3}{2})^{k-2}$ and $F_{k+1} \geq (\frac{3}{2})^{k-1}$ **M1**

$F_{k+2} = F_{k+1} + F_k \geq (\frac{3}{2})^{k-1} + (\frac{3}{2})^{k-2}$ **M1 A1**

$= (\frac{3}{2})^{k-2}(\frac{3}{2} + 1) = (\frac{3}{2})^{k-2} \times \frac{5}{2}$ **M1**

$\frac{5}{2} > \frac{9}{4} = (\frac{3}{2})^2$, so $F_{k+2} \geq (\frac{3}{2})^k$ **A1**

True for n = 1, 2, and n = k, k + 1 implies n = k + 2, so true for all $n \geq 1$ **A1**

Answer: $F_n \geq (\frac{3}{2})^{n-2}$ for all integers $n \geq 1$ **8 marks**

26 (7 marks)

n = 1: $S_1 = \frac{1}{2} \geq \frac{1}{2}$, true **B1**

Assume $S_k \geq \frac{1}{2}$ **M1**

$S_{k+1} = \frac{1}{k+2} + \ldots + \frac{1}{2k} + \frac{1}{2k+1} + \frac{1}{2k+2}$ **M1**

$S_{k+1} - S_k = \frac{1}{2k+1} + \frac{1}{2k+2} - \frac{1}{k+1}$ **M1 A1**

$= \frac{1}{2k+1} - \frac{1}{2k+2} > 0$ **M1**

So $S_{k+1} > S_k \geq \frac{1}{2}$. True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers $n \geq 1$ by induction **A1**

Answer: $S_n \geq \frac{1}{2}$ for all integers $n \geq 1$ **7 marks**

27 (10 marks)

$n = 1$: $1! = 1 \leq 1^1 = 1$, true **B1**

Assume $k! \leq (\frac{k+1}{2})^k$. Then $(k+1)! = (k+1)k! \leq (k+1)(\frac{k+1}{2})^k$ **M1 A1**

Write $k + 1 = 2 \times (\frac{k+1}{2})$, so $(k+1)! \leq 2(\frac{k+1}{2})^{k+1}$ **M1**

We need $2(\frac{k+1}{2})^{k+1} \leq (\frac{k+2}{2})^{k+1}$, that is $(\frac{k+2}{k+1})^{k+1} \geq 2$ **M1 A1**

$\frac{k+2}{k+1} = 1 + \frac{1}{k+1}$ **B1**

By the given result with $x = \frac{1}{k+1}$ and $m = k + 1$: $(1 + \frac{1}{k+1})^{k+1} \geq 1 + (k+1) \times \frac{1}{k+1} = 2$ **M1 A1**

True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction **A1**

Answer: $n! \leq (\frac{n+1}{2})^n$ for all integers $n \geq 1$ **10 marks**

28 (9 marks)

$n = 2$: LHS $= 1 + \frac{1}{8} = \frac{9}{8}$, RHS $= \frac{5}{4} - \frac{1}{8} = \frac{9}{8}$, true **B1**

Assume the sum to k is at most $\frac{5}{4} - \frac{1}{2k^2}$ for some $k \geq 2$. Then sum to $k + 1 \leq \frac{5}{4} - \frac{1}{2k^2} + \frac{1}{(k+1)^3}$ **M1 A1**

It is enough to show $\frac{1}{(k+1)^3} \leq \frac{1}{2k^2} - \frac{1}{2(k+1)^2}$ **M1**

RHS $= \frac{(k+1)^2 - k^2}{2k^2(k+1)^2} = \frac{2k+1}{2k^2(k+1)^2}$ **M1 A1**

The inequality becomes $\frac{1}{k+1} \leq \frac{2k+1}{2k^2}$, that is $2k^2 \leq (2k+1)(k+1) = 2k^2 + 3k + 1$, which is true **M1 A1**

True for $n = 2$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 2$ by induction **A1**

Answer: $\sum_{r=1}^{n} \frac{1}{r^3} \leq \frac{5}{4} - \frac{1}{2n^2}$ for all integers $n \geq 2$ **9 marks**

29 (10 marks)

$n = 1$: $C(2, 1) = 2$ and $\frac{4}{2} = 2$, true **B1**

Assume $C(2k, k) \geq \frac{4^k}{2k}$ **M1**

$C(2k+2, k+1) = \frac{(2k+2)!}{((k+1)!)^2} = \frac{(2k+2)(2k+1)}{(k+1)^2} C(2k, k) = \frac{2(2k+1)}{k+1} C(2k, k)$ **M1 A1**

So $C(2k+2, k+1) \geq \frac{2(2k+1)}{k+1} \times \frac{4^k}{2k} = \frac{(2k+1)4^k}{k(k+1)}$ **M1 A1**

The target is $\frac{4^{k+1}}{2(k+1)} = \frac{2 \times 4^k}{k+1}$ **B1**

$\frac{(2k+1)4^k}{k(k+1)} \geq \frac{2 \times 4^k}{k+1}$ because $\frac{2k+1}{k} \geq 2$ **M1 A1**

True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction **A1**

Answer: $C(2n, n) \geq \frac{4^n}{2n}$ for all integers $n \geq 1$, where $C(2n, n) = \frac{(2n)!}{(n!)^2}$ **10 marks**

30 (9 marks)

$n = 1$: $1 \leq 1 \leq 2$, true — B1

$n = 2$: $2 \leq 3 \leq 4$, true — B1

Assume $2^{k-1} \leq u_k \leq 2^k$ and $2^k \leq u_{k+1} \leq 2^{k+1}$ — M1

Upper bound: $u_{k+2} = u_{k+1} + 2u_k \leq 2^{k+1} + 2 \times 2^k = 2^{k+2}$ — M1 A1

Lower bound: $u_{k+2} \geq 2^k + 2 \times 2^{k-1} = 2^{k+1}$ — M1 A1

So $2^{k+1} \leq u_{k+2} \leq 2^{k+2}$, the result for $n = k + 2$ — A1

True for $n = 1$, 2, and $n = k$, $k + 1$ implies $n = k + 2$, so true for all $n \geq 1$ — A1

Answer: $2^{n-1} \leq u_n \leq 2^n$ for all integers $n \geq 1$ — 9 marks

Further Proof Techniques

1 (3 marks)

$n^2 + 3n = n(n + 3)$ — M1

If n is even, the product is even — B1

If n is odd, $n + 3$ is even, so the product is even — B1

Answer: $n^2 + 3n$ is even for every positive integer n — 3 marks

2 (2 marks)

Try $n = 10$: $100 + 10 + 11 = 121$ — M1

$121 = 11 \times 11$ is not prime, so the claim is false — A1

Answer: $n = 10$ gives $121 = 11^2$, which is not prime — 2 marks

3 (2 marks)

Take $x = -4$ — B1

$x^2 = 16 > 9$ but $x = -4$ is not greater than 3 — B1

Answer: $x = -4$: $x^2 = 16 > 9$ but $x < 3$ — 2 marks

4 (3 marks)

$x^2 + y - (y^2 + x) = x^2 - y^2 - (x - y)$ — M1

$= (x - y)(x + y) - (x - y) = (x - y)(x + y - 1)$ — M1

Since $x + y = 1$, the second factor is 0, so the difference is 0 and $x^2 + y = y^2 + x$ — A1

Answer: $x^2 + y = y^2 + x$ — 3 marks

5 (3 marks)

Let the integers be $2a + 1$ and $2b + 1$, where a and b are integers — B1

$(2a + 1)(2b + 1) = 4ab + 2a + 2b + 1$ — M1

$= 2(2ab + a + b) + 1$, which is odd — A1

Answer: the product of two odd integers is always odd — 3 marks

6 (2 marks)

Assume $\sqrt{3}$ is rational: $\sqrt{3} = \frac{p}{q}$, where p and q are integers with no common factor — B1

Squaring: $3 = \frac{p^2}{q^2}$, so $p^2 = 3q^2$ — B1

Answer: $\sqrt{3} = \frac{p}{q}$ in lowest terms, so $p^2 = 3q^2$ — 2 marks

7 (3 marks)

$3^{2k+2} = 9 \times 3^{2k}$ — B1

$3^{2k} + 3^{2k+2} = 3^{2k}(1 + 9) = 10 \times 3^{2k}$ — M1

3^{2k} is an integer, so the sum is a multiple of 10 — A1

Answer: $3^{2k} + 3^{2k+2} = 10 \times 3^{2k}$, a multiple of 10 — 3 marks

8 (4 marks)

Let the odd integer be $2n + 1$ — B1

$(2n + 1)^2 = 4n^2 + 4n + 1$ — M1

$= 4n(n + 1) + 1$ — A1

This is a multiple of 4 plus 1, so the remainder is 1 — E1

Answer: the square of any odd integer leaves remainder 1 when divided by 4 — 4 marks

9 (4 marks)

Not divisible by 3 means $n = 3m + 1$ or $n = 3m + 2$ (two cases) — B1

Case 1: $n^2 - 1 = 9m^2 + 6m = 3m(3m + 2)$, divisible by 3 — M1 A1

Case 2: $n^2 - 1 = 9m^2 + 12m + 3 = 3(3m^2 + 4m + 1)$, divisible by 3 — A1

Answer: if n is an integer that is not divisible by 3, then $n^2 - 1$ is divisible by 3 — 4 marks

10 (4 marks)

Assume there are integers m and n with $6m + 9n = 100$ — B1

Then $3(2m + 3n) = 100$, so 100 is divisible by 3 — M1

But $100 = 3 \times 33 + 1$ is not divisible by 3, a contradiction — A1

So no such integers exist — E1

Answer: there are no integers m and n such that $6m + 9n = 100$ — 4 marks

11 (5 marks)

Assume $\sqrt{2} = \frac{p}{q}$, where p and q are integers with no common factor — B1

Then $2q^2 = p^2$, so p^2 is even, hence p is even; write $p = 2r$ — M1 A1

Then $2q^2 = 4r^2$, so $q^2 = 2r^2$, hence q is even — M1

p and q are both even, contradicting that they have no common factor, so $\sqrt{2}$ is irrational — E1

Answer: $\sqrt{2}$ is irrational — 5 marks

12 (4 marks)

$\frac{p}{q} + \frac{q}{p} - 2 = \frac{p^2 + q^2 - 2pq}{pq}$ — M1 A1

$= \frac{(p - q)^2}{pq}$ — M1

$(p - q)^2 \geq 0$ and $pq > 0$, so the expression is ≥ 0, hence $\frac{p}{q} + \frac{q}{p} \geq 2$ — A1

Answer: for all positive real numbers p and q, $\frac{p}{q} + \frac{q}{p} \geq 2$ — 4 marks

13 (4 marks)

$\frac{3k + 1}{3k + 2} - \frac{3k}{3k + 1} = \frac{(3k + 1)^2 - 3k(3k + 2)}{(3k + 2)(3k + 1)}$ — M1 A1

$(3k + 1)^2 - 3k(3k + 2) = 9k^2 + 6k + 1 - 9k^2 - 6k = 1$ — M1

The difference is $\frac{1}{(3k + 2)(3k + 1)}$, which is positive for $k > 0$, so the inequality holds — A1

Answer: $\frac{3k + 1}{3k + 2} > \frac{3k}{3k + 1}$ for all real $k > 0$ — 4 marks

14 (6 marks)

Assume $\sqrt{3} = \frac{p}{q}$, p and q integers with no common factor — B1

$3q^2 = p^2$, so p^2 is divisible by 3 — M1

So p is divisible by 3: $p = 3r$ — A1

$3q^2 = 9r^2$, so $q^2 = 3r^2$ — M1

So q^2 is divisible by 3, hence q is divisible by 3 — A1

p and q share the factor 3, a contradiction, so $\sqrt{3}$ is irrational — E1

Answer: $\sqrt{3}$ is irrational — 6 marks

15 (6 marks)

$p^2 - 1 = (p - 1)(p + 1)$ **B1**

p is odd, so $p - 1$ and $p + 1$ are consecutive even numbers; one of them is a multiple of 4 **M1**

So $(p - 1)(p + 1)$ is divisible by $2 \times 4 = 8$ **A1**

Of $p - 1$, p, $p + 1$ (consecutive), one is a multiple of 3; it is not p (a prime greater than 3) **M1**

So $(p - 1)(p + 1)$ is divisible by 3 **A1**

8 and 3 have no common factor, so $p^2 - 1$ is divisible by 24 **A1**

Answer: if p is a prime number greater than 3, then $p^2 - 1$ is divisible by 24 **6 marks**

16 (5 marks)

Assume that there is a real θ with $3 \cos \theta + 4 \sin \theta > 5$ **B1**

Then $(3 \cos \theta + 4 \sin \theta)^2 > 25$, since both sides are positive **M1**

$(3 \cos \theta + 4 \sin \theta)^2 + (4 \cos \theta - 3 \sin \theta)^2 = 25(\cos^2\theta + \sin^2\theta) = 25$ **M1 A1**

So $(3 \cos \theta + 4 \sin \theta)^2 = 25 - (4 \cos \theta - 3 \sin \theta)^2 \leq 25$, which contradicts the assumption, so $3 \cos \theta + 4 \sin \theta \leq 5$ for all θ **E1**

Answer: $3 \cos \theta + 4 \sin \theta \leq 5$ for all real θ **5 marks**

17 (6 marks)

$a^2 - b^2 = (a - b)(a + b)$ **B1**

$a - b$ and $a + b$ have the same parity, since their difference is $2b$ **M1 A1**

If both are odd, the product is odd, so not of the form $4k + 2$ **M1**

If both are even, the product is divisible by $2 \times 2 = 4$, so not of the form $4k + 2$ **M1**

Hence $a^2 - b^2$ is never of the form $4k + 2$ **A1**

Answer: the difference of the squares of two integers can never be of the form $4k + 2$, where k is an integer **6 marks**

18 (5 marks)

$(a + b)^2 + (a - b)^2 = a^2 + 2ab + b^2 + a^2 - 2ab + b^2 = 2(a^2 + b^2)$ **M1 A1**

$a + b$ is a positive integer, and $a - b$ is a non-zero integer since $a \neq b$, so $(a - b)^2 = |a - b|^2$ with $|a - b|$ a positive integer **B1**

$|a - b| < a + b$ because a and b are positive, so the two integers $a + b$ and $|a - b|$ are distinct **E1**

So $2(a^2 + b^2) = (a + b)^2 + |a - b|^2$, a sum of two squares of distinct positive integers **A1**

Answer: $2(a^2 + b^2) = (a + b)^2 + (a - b)^2$ **5 marks**

19 (5 marks)

Assume $\log_2 3 = \frac{p}{q}$, where p and q are positive integers **B1**

Then $2^{p/q} = 3$, so $2^p = 3^q$ **M1 A1**

2^p is even ($p \geq 1$) but 3^q is odd **M1**

This is a contradiction, so $\log_2 3$ is irrational **E1**

Answer: $\log_2 3$ is irrational **5 marks**

20 (5 marks)

$a^3 + b^3 - a^2b - ab^2 = a^2(a - b) - b^2(a - b)$ **M1 A1**

$= (a - b)(a^2 - b^2) = (a - b)^2(a + b)$ **M1 A1**

$(a - b)^2 \geq 0$ and $a + b > 0$, so the expression is ≥ 0, which proves the result **A1**

Answer: for all positive real numbers a and b, $a^3 + b^3 \geq a^2b + ab^2$ **5 marks**

21 (7 marks)

Assume $\sqrt{2} + \sqrt{3} = r$, where r is rational and $r \neq 0$ **B1**

$(\sqrt{3} + \sqrt{2})(\sqrt{3} - \sqrt{2}) = 3 - 2 = 1$ **M1 A1**

So $\sqrt{3} - \sqrt{2} = \frac{1}{r}$, which is rational **A1**

Adding: $2\sqrt{3} = r + \frac{1}{r}$, so $\sqrt{3} = \frac{1}{2}(r + \frac{1}{r})$ **M1 A1**

The right side is rational, contradicting that $\sqrt{3}$ is irrational, so $\sqrt{2} + \sqrt{3}$ is irrational **E1**

Answer: $\sqrt{2} + \sqrt{3}$ is irrational **7 marks**

22 (7 marks)

Assume there are finitely many primes $p_1, p_2, ..., p_m$ **B1**

Let $N = p_1p_2 ... p_m + 1$ **M1**

$N > 1$, so N has a prime factor; this must be one of the p_i **M1 A1**

But N divided by p_i leaves remainder 1 for every i, so no p_i divides N **M1 A1**

This is a contradiction, so there are infinitely many primes **E1**

Answer: there are infinitely many prime numbers **7 marks**

23 (6 marks)

(a) $a = b$, so $a - b = 0$ **B1**

The step 'dividing by $(a - b)$' is a division by zero **E1**

which is not allowed, so the deduction $a + b = b$ is invalid **E1**

Answer: the step divides by $a - b = 0$ **3 marks**

(b) Checking particular cases does not prove a statement for all n **E1**

Take $n = 41$: $41^2 + 41 + 41 = 41(41 + 1 + 1) = 41 \times 43$ **M1**

which is not prime, so the conclusion is false **A1**

Answer: $n = 41$ gives 41×43 **3 marks**

24 (6 marks)

Consider $2(a^2 + b^2 + c^2) - 2(ab + bc + ca)$ **M1**

$= (a^2 - 2ab + b^2) + (b^2 - 2bc + c^2) + (c^2 - 2ca + a^2)$ **M1 A1**

$= (a - b)^2 + (b - c)^2 + (c - a)^2$ **A1**

Each square is non-negative, so the sum is ≥ 0 **E1**

Dividing by 2 gives $a^2 + b^2 + c^2 \geq ab + bc + ca$ **A1**

Answer: for all real numbers a, b and c, $a^2 + b^2 + c^2 \geq ab + bc + ca$ **6 marks**

25 (6 marks)

(a) Let the integers be n, n + 1, n + 2, n + 3. Pair the outer and inner: $n(n + 3) = n^2 + 3n$ and $(n + 1)(n + 2) = n^2 + 3n + 2$ **M1 A1**

Let $t = n^2 + 3n$. Product + 1 = $t(t + 2) + 1$ **M1**

$= t^2 + 2t + 1 = (t + 1)^2$ **A1**

$= (n^2 + 3n + 1)^2$, a perfect square **A1**

Answer: $n(n + 1)(n + 2)(n + 3) + 1 = (n^2 + 3n + 1)^2$ **5 marks**

(b) $n = 10$: $100 + 30 + 1 = 131$, so the number is $131^2 = 17161$ **B1**

Answer: $131^2 = 17161$ **1 mark**

26 (6 marks)

Assume a and b are both odd: $a = 2m + 1$, $b = 2n + 1$ **B1**

$a^2 + b^2 = 4m^2 + 4m + 1 + 4n^2 + 4n + 1 = 4(m^2 + m + n^2 + n) + 2$ **M1 A1**

So c^2 leaves remainder 2 when divided by 4 **A1**

If c is even, $c^2 = 4t$ has remainder 0; if c is odd, c^2 has remainder 1 **M1**

Neither gives remainder 2, a contradiction, so a and b are not both odd **E1**

Answer: a and b cannot both be odd **6 marks**

27 (8 marks)

Case 1: x is rational. Then $\sqrt{2}$ raised to the power $\sqrt{2}$, both irrational, gives a rational number **M1 A1**

Case 2: x is irrational. Consider $x^{\sqrt{2}} = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}$ **M1**

$= \sqrt{2}^{\sqrt{2} \times \sqrt{2}} = \sqrt{2}^{2}$ **M1 A1**

$= 2$, which is rational **A1**

In case 2, x and $\sqrt{2}$ are both irrational and $x^{\sqrt{2}} = 2$ is rational **E1**

Either way there are irrational a, b with a^b rational, which proves the statement **E1**

Answer: $\sqrt{2}^{\sqrt{2}}$ is rational, or irrational with $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = 2$ rational **8 marks**

28 (8 marks)

Let the integers be n, n + 1, n + 2. Sum $= n^3 + (n + 1)^3 + (n + 2)^3$ **B1**

$= n^3 + n^3 + 3n^2 + 3n + 1 + n^3 + 6n^2 + 12n + 8$ **M1**

$= 3n^3 + 9n^2 + 15n + 9$ **A1**

$= 3(n^3 + 5n) + 9n^2 + 9$ **M1**

It remains to show 3 divides $n^3 + 5n = n(n^2 + 5)$ **M1**

If n = 3m, n is divisible by 3. If $n = 3m \pm 1$, $n^2 + 5 = 9m^2 \pm 6m + 6$ is divisible by 3 **M1 A1**

So $n^3 + 5n = 3t$, and the sum $= 9t + 9n^2 + 9 = 9(t + n^2 + 1)$, divisible by 9 **A1**

Answer: the sum of the cubes of any three consecutive positive integers is divisible by 9 **8 marks**

29 (9 marks)

Assume there are finitely many such primes, $p_1 = 3, p_2, ..., p_m$ **B1**

Let $N = 4p_1p_2 ... p_m - 1$. Then $N = 4(p_1 ... p_m - 1) + 3$, so N leaves remainder 3 on division by 4 **M1 A1**

N is odd, so all its prime factors are odd, so each is of the form 4j + 1 or 4j + 3 **M1**

The product of numbers of the form 4j + 1 is of the form 4j + 1: $(4a + 1)(4b + 1) = 4(4ab + a + b) + 1$ **M1 A1**

So N cannot have only prime factors of the form 4j + 1, hence it has a prime factor of the form 4j + 3 **A1**

Each p_i divides $4p_1 ... p_m$, so p_i divides N only if p_i divides 1; so this prime factor is not in the list **M1**

This contradicts the assumption that the list is complete, so there are infinitely many **E1**

Answer: there are infinitely many prime numbers of the form 4k + 3, where k is a non-negative integer **9 marks**

30 (9 marks)

(a) $\cos 3\theta = \cos(2\theta + \theta) = \cos 2\theta \cos\theta - \sin 2\theta \sin\theta$ **M1**

$= (2\cos^2\theta - 1)\cos\theta - 2\sin^2\theta \cos\theta = 2\cos^3\theta - \cos\theta - 2(1 - \cos^2\theta)\cos\theta = 4\cos^3\theta - 3\cos\theta$ **M1 A1**

$\theta = 20°$: $\cos 60° = \frac{1}{2} = 4x^3 - 3x$, so $8x^3 - 6x - 1 = 0$ **A1**

Answer: $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, and hence show that $x = \cos 20°$ is a root of $8x^3 - 6x - 1 = 0$ **4 marks**

(b) Suppose $\cos 20° = \frac{p}{q}$ in lowest terms, with $q > 0$ **B1**

p divides 1 and q divides 8, so $\frac{p}{q}$ is one of $\pm 1, \pm\frac{1}{2}, \pm\frac{1}{4}, \pm\frac{1}{8}$ **M1 A1**

Substituting into $8x^3 - 6x - 1$: $x = 1$ gives 1, $x = -1$ gives -3, $x = \frac{1}{2}$ gives -3, $x = -\frac{1}{2}$ gives 1, $x = \frac{1}{4}$ gives $-\frac{19}{8}$, $x = -\frac{1}{4}$ gives $\frac{3}{8}$, $x = \frac{1}{8}$ gives $-\frac{111}{64}$, $x = -\frac{1}{8}$ gives $-\frac{17}{64}$ **M1**

None of the candidates is a root, a contradiction, so cos 20° is irrational **E1**

Answer: no rational candidate is a root, so cos 20° is irrational **5 marks**

Induction with Derivatives

1 (3 marks)

$n = 1$: $\frac{dy}{dx} = 3e^{3x} = 3^1e^{3x}$, true **B1**

Assume $\frac{d^k y}{dx^k} = 3^k e^{3x}$. Then $\frac{d^{k+1} y}{dx^{k+1}} = 3^k \times 3e^{3x} = 3^{k+1}e^{3x}$ **M1**

True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction **A1**

Answer: $\frac{d^n y}{dx^n} = 3^n e^{3x}$ **3 marks**

2 (3 marks)

$\frac{dy}{dx} = -2e^{-2x}$, $\frac{d^2 y}{dx^2} = 4e^{-2x}$, $\frac{d^3 y}{dx^3} = -8e^{-2x}$ **M1 A1**

Each differentiation multiplies by -2, so $\frac{d^n y}{dx^n} = (-2)^n e^{-2x}$ **B1**

Answer: $\frac{d^n y}{dx^n} = (-2)^n e^{-2x}$ **3 marks**

3 (3 marks)

$\frac{dy}{dx} = \frac{1}{1+x}$, $\frac{d^2 y}{dx^2} = -\frac{1}{(1+x)^2}$, $\frac{d^3 y}{dx^3} = \frac{2}{(1+x)^3}$ **M1 A1**

Conjecture: $\frac{d^n y}{dx^n} = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$ **B1**

Answer: $\frac{d^n y}{dx^n} = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$ **3 marks**

4 (3 marks)

$\frac{dy}{dx} = -2\sin 2x$, $\frac{d^2 y}{dx^2} = -4\cos 2x$, $\frac{d^3 y}{dx^3} = 8\sin 2x$, $\frac{d^4 y}{dx^4} = 16\cos 2x$ **M1**

So $\frac{d^4 y}{dx^4} = 16y$ **A1**

$20 = 4 \times 5$, so $\frac{d^{20} y}{dx^{20}} = 16^5 y = 2^{20}\cos 2x$, which is $2^{20} = 1048576$ at $x = 0$ **B1**

Answer: $\frac{d^4 y}{dx^4} = 16y$; the value is 1048576 **3 marks**

5 (2 marks)

$\frac{d^4 y}{dx^4} = 24$, a constant **B1**

so $\frac{d^5 y}{dx^5} = 0$ and every higher derivative is also 0, hence $\frac{d^7 y}{dx^7} = 0$ **B1**

Answer: 0 **2 marks**

6 (2 marks)

$\frac{d}{dx}[k!(1-x)^{-(k+1)}] = k! \times (-(k+1))(1-x)^{-(k+2)} \times (-1)$ **M1**

$= (k+1)\,k!\,(1-x)^{-(k+2)} = (k+1)!(1-x)^{-(k+2)}$ **A1**

Answer: $\frac{d^{k+1} y}{dx^{k+1}} = (k+1)!(1-x)^{-(k+2)}$ **2 marks**

7 (4 marks)

n = 1: $\frac{d}{dx}(\sin x) = \cos x = \sin\left(x + \frac{\pi}{2}\right)$, true **B1**

Assume true for n = k: the kth derivative of sin x is $\sin\left(x + \frac{k\pi}{2}\right)$. Differentiating gives $\cos\left(x + \frac{k\pi}{2}\right)$ **M1**

Since $\cos\theta = \sin\left(\theta + \frac{\pi}{2}\right)$, this is $\sin\left(x + \frac{k\pi}{2} + \frac{\pi}{2}\right) = \sin\left(x + \frac{(k+1)\pi}{2}\right)$, the result for n = k + 1 **A1**

True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers $n \geq 1$ by induction **E1**

Answer: the nth derivative of sin x is $\sin\left(x + \frac{n\pi}{2}\right)$ **4 marks**

8 (5 marks)

n = 1: $\frac{dy}{dx} = -3\sin 3x = 3\cos\left(3x + \frac{\pi}{2}\right)$, true **B1**

Assume $\frac{d^k y}{dx^k} = 3^k \cos\left(3x + \frac{k\pi}{2}\right)$. Differentiate: $\frac{d^{k+1} y}{dx^{k+1}} = -3^k \times 3\sin\left(3x + \frac{k\pi}{2}\right)$ **M1 A1**

Since $-\sin\theta = \cos\left(\theta + \frac{\pi}{2}\right)$, this is $3^{k+1}\cos\left(3x + \frac{k\pi}{2} + \frac{\pi}{2}\right) = 3^{k+1}\cos\left(3x + \frac{(k+1)\pi}{2}\right)$ **A1**

True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers $n \geq 1$ by induction **E1**

Answer: $\frac{d^n y}{dx^n} = 3^n \cos\left(3x + \frac{n\pi}{2}\right)$ **5 marks**

9 (4 marks)

n = 1: $\frac{dy}{dx} = e^x + x e^x = (x + 1)e^x$, true **B1**

Assume $\frac{d^k y}{dx^k} = (x + k)e^x$. By the product rule, $\frac{d^{k+1} y}{dx^{k+1}} = e^x + (x + k)e^x$ **M1**

$= (x + k + 1)e^x$, the result for n = k + 1 **A1**

True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers $n \geq 1$ by induction **E1**

Answer: $\frac{d^n y}{dx^n} = (x + n)e^x$ **4 marks**

10 (5 marks)

n = 1: $\frac{dy}{dx} = -(1 + x)^{-2} = \frac{(-1)^1 1!}{(1 + x)^2}$, true **B1**

Assume $\frac{d^k y}{dx^k} = (-1)^k k!(1 + x)^{-(k+1)}$. Differentiating: $\frac{d^{k+1} y}{dx^{k+1}} = (-1)^k k! \times (-(k + 1))(1 + x)^{-(k+2)}$ **M1 A1**

$= (-1)^{k+1}(k + 1)!(1 + x)^{-(k+2)}$, the result for n = k + 1 **A1**

True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers $n \geq 1$ by induction **E1**

Answer: $\frac{d^n y}{dx^n} = \frac{(-1)^n n!}{(1 + x)^{n+1}}$ **5 marks**

11 (5 marks)

n = 1: $\frac{dy}{dx} = \frac{-2}{1-2x} = -\frac{2^1\,0!}{(1-2x)^1}$, true **B1**

Assume $\frac{d^k y}{dx^k} = -2^k(k-1)!(1-2x)^{-k}$. Differentiating: $\frac{d^{k+1} y}{dx^{k+1}} = -2^k(k-1)! \times (-k)(1-2x)^{-(k+1)} \times (-2)$ **M1 A1**

$= -2^{k+1}\,k!\,(1-2x)^{-(k+1)}$, the result for n = k + 1 **A1**

True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers $n \geq 1$ by induction **E1**

Answer: $\frac{d^n y}{dx^n} = -\frac{2^n(n-1)!}{(1-2x)^n}$ **5 marks**

12 (4 marks)

n = 4: $y' = 9x^2 + 2$, $y'' = 18x$, $y''' = 18$, so $\frac{d^4 y}{dx^4} = 0$, true **B1**

Assume $\frac{d^k y}{dx^k} = 0$ for some $k \geq 4$. Then $\frac{d^{k+1} y}{dx^{k+1}} = \frac{d}{dx}(0) = 0$, the result for n = k + 1 **M1**

True for n = 4, and true for n = k implies true for n = k + 1, so true for all integers $n \geq 4$ by induction **A1**

Hence $\frac{d^{10} y}{dx^{10}} = 0$ **B1**

Answer: $\frac{d^n y}{dx^n} = 0$ for $n \geq 4$, so $\frac{d^{10} y}{dx^{10}} = 0$ **4 marks**

13 (5 marks)

(a) n = 1: $\frac{dy}{dx} = -\frac{1}{2}e^{-x/2}$, true **B1**

Assume $\frac{d^k y}{dx^k} = (-\frac{1}{2})^k e^{-x/2}$. Differentiating: $\frac{d^{k+1} y}{dx^{k+1}} = (-\frac{1}{2})^k \times (-\frac{1}{2})e^{-x/2}$ **M1**

$= (-\frac{1}{2})^{k+1} e^{-x/2}$, the result for n = k + 1 **A1**

True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers $n \geq 1$ by induction **E1**

Answer: $\frac{d^n y}{dx^n} = (-\frac{1}{2})^n e^{-x/2}$ **4 marks**

(b) $(-\frac{1}{2})^6 e^{-\ln 2} = \frac{1}{64} \times \frac{1}{2} = \frac{1}{128}$ **B1**

Answer: $\frac{1}{128}$ **1 mark**

14 (6 marks)

n = 1: $\frac{dy}{dx} = e^{2x} + 2x\,e^{2x} = (2x+1)e^{2x} = 2^0(2x+1)e^{2x}$, true **B1**

Assume $\frac{d^k y}{dx^k} = 2^{k-1}(2x+k)e^{2x}$. By the product rule, $\frac{d^{k+1} y}{dx^{k+1}} = 2^{k-1}[2e^{2x} + 2(2x+k)e^{2x}]$ **M1 A1**

$= 2^{k-1} \times 2(1+2x+k)e^{2x}$ **M1**

$= 2^k(2x+k+1)e^{2x}$, the result for n = k + 1 **A1**

True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers $n \geq 1$ by induction **E1**

Answer: $\frac{d^n y}{dx^n} = 2^{n-1}(2x+n)e^{2x}$ **6 marks**

15 (6 marks)

(a) $\frac{1}{(1+x)(2+x)} = \frac{A}{1+x} + \frac{B}{2+x}$, so $1 = A(2+x) + B(1+x)$ **M1**

$A = 1$, $B = -1$, so $y = \frac{1}{1+x} - \frac{1}{2+x}$ **A1**

Answer: $y = \frac{1}{1+x} - \frac{1}{2+x}$ **2 marks**

(b) $n = 1$: $\frac{dy}{dx} = -(1+x)^{-2} + (2+x)^{-2} = (-1)^1\, 1![(1+x)^{-2} - (2+x)^{-2}]$, true **B1**

Assume $\frac{d^k y}{dx^k} = (-1)^k\, k![(1+x)^{-(k+1)} - (2+x)^{-(k+1)}]$. Differentiating: $\frac{d^{k+1} y}{dx^{k+1}} = (-1)^k\, k![-(k+1)(1+x)^{-(k+2)} + (k+1)(2+x)^{-(k+2)}]$ **M1**

$= (-1)^{k+1}(k+1)![(1+x)^{-(k+2)} - (2+x)^{-(k+2)}]$, the result for $n = k+1$ **A1**

True for $n = 1$, and true for $n = k$ implies true for $n = k+1$, so true for all integers $n \geq 1$ by induction **E1**

Answer: $\frac{d^n y}{dx^n} = (-1)^n\, n!\, [\frac{1}{(1+x)^{n+1}} - \frac{1}{(2+x)^{n+1}}]$ **4 marks**

16 (7 marks)

$n = 1$: $\frac{dy}{dx} = e^{-x}(\cos x - \sin x)$ **M1**

$\sin(x + \frac{3\pi}{4}) = -\frac{1}{\sqrt{2}} \sin x + \frac{1}{\sqrt{2}} \cos x$, so $\cos x - \sin x = \sqrt{2} \sin(x + \frac{3\pi}{4})$, true for $n = 1$ **B1**

Assume $\frac{d^k y}{dx^k} = (\sqrt{2})^k\, e^{-x} \sin(A)$, where $A = x + \frac{3k\pi}{4}$. Differentiating: $\frac{d^{k+1} y}{dx^{k+1}} = (\sqrt{2})^k\, e^{-x}(\cos A - \sin A)$ **M1 A1**

$\cos A - \sin A = \sqrt{2} \sin(A + \frac{3\pi}{4})$ by the same identity **M1**

So $\frac{d^{k+1} y}{dx^{k+1}} = (\sqrt{2})^{k+1}\, e^{-x} \sin(x + \frac{3(k+1)\pi}{4})$, the result for $n = k+1$ **A1**

True for $n = 1$, and true for $n = k$ implies true for $n = k+1$, so true for all integers $n \geq 1$ by induction **E1**

Answer: $\frac{d^n y}{dx^n} = (\sqrt{2})^n\, e^{-x} \sin(x + \frac{3n\pi}{4})$ **7 marks**

17 (6 marks)

$n = 1$: $\frac{dy}{dx} = (2x + x^2)e^x = [x^2 + 2x + 0]e^x$, true **B1**

Assume $\frac{d^k y}{dx^k} = [x^2 + 2kx + k(k-1)]e^x$. By the product rule, $\frac{d^{k+1} y}{dx^{k+1}} = [2x + 2k]e^x + [x^2 + 2kx + k(k-1)]e^x$ **M1 A1**

$= [x^2 + (2k+2)x + k^2 + k]e^x$ **M1**

$= [x^2 + 2(k+1)x + (k+1)k]e^x$, the result for $n = k+1$ **A1**

True for $n = 1$, and true for $n = k$ implies true for $n = k+1$, so true for all integers $n \geq 1$ by induction **E1**

Answer: $\frac{d^n y}{dx^n} = [x^2 + 2nx + n(n-1)]e^x$ **6 marks**

18 (5 marks)

$n = 1$: $\frac{dy}{dx} = 2 \sin x \cos x = \sin 2x = 2^0 \sin(2x + 0)$, true **B1**

Assume $\frac{d^k y}{dx^k} = 2^{k-1} \sin(2x + \frac{(k-1)\pi}{2})$. Differentiating: $\frac{d^{k+1} y}{dx^{k+1}} = 2^{k-1} \times 2 \cos(2x + \frac{(k-1)\pi}{2})$ **M1 A1**

$\cos\theta = \sin(\theta + \frac{\pi}{2})$, so $\frac{d^{k+1} y}{dx^{k+1}} = 2^k \sin(2x + \frac{k\pi}{2})$, the result for $n = k + 1$ **A1**

True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction **E1**

Answer: $\frac{d^n y}{dx^n} = 2^{n-1} \sin(2x + \frac{(n-1)\pi}{2})$ **5 marks**

19 (6 marks)

(a) $\frac{dy}{dx} = e^{3x} + 3x e^{3x} = (1 + 3x)e^{3x}$ **M1**

The formula gives $3x e^{3x}$, which is different, so the formula is false for $n = 1$ **A1**

Answer: $\frac{dy}{dx} = (1 + 3x)e^{3x}$, not $3x e^{3x}$ **2 marks**

(b) The product rule has not been used: the derivative of x has been ignored, treating x as a constant **E1**

No basis case was checked; with $n = 1$ the argument fails, so the induction is invalid **E1**

Answer: the product rule was ignored, and the basis case was not checked **2 marks**

(c) $\frac{d}{dx}[3^{k-1}(3x + k)e^{3x}] = 3^{k-1}[3e^{3x} + 3(3x + k)e^{3x}]$ **M1**

$= 3^{k-1} \times 3(3x + k + 1)e^{3x} = 3^k(3x + (k + 1))e^{3x}$, the formula with $n = k + 1$ **A1**

Answer: $3^k(3x + k + 1)e^{3x}$ **2 marks**

20 (7 marks)

(a) $n = 1$: $\frac{dy}{dx} = 2 \cos(2x + \frac{\pi}{3}) = 2 \sin(2x + \frac{\pi}{3} + \frac{\pi}{2})$, true **B1**

Assume $\frac{d^k y}{dx^k} = 2^k \sin(2x + \frac{\pi}{3} + \frac{k\pi}{2})$. Differentiating: $\frac{d^{k+1} y}{dx^{k+1}} = 2^k \times 2 \cos(2x + \frac{\pi}{3} + \frac{k\pi}{2})$ **M1 A1**

$= 2^{k+1} \sin(2x + \frac{\pi}{3} + \frac{k\pi}{2} + \frac{\pi}{2}) = 2^{k+1} \sin(2x + \frac{\pi}{3} + \frac{(k+1)\pi}{2})$ **A1**

True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for all integers $n \geq 1$ by induction **E1**

Answer: $\frac{d^n y}{dx^n} = 2^n \sin(2x + \frac{\pi}{3} + \frac{n\pi}{2})$ **5 marks**

(b) $2^{10} \sin(\frac{\pi}{3} + 5\pi) = 1024 \sin(\frac{\pi}{3} + \pi)$ **M1**

$= -1024 \sin\frac{\pi}{3} = -512\sqrt{3}$ **A1**

Answer: $-512\sqrt{3}$ **2 marks**

21 (9 marks)

(a) n = 1: $\frac{dy}{dx} = \sin x + x\cos x = x\sin\left(x + \frac{\pi}{2}\right) + 1 \times \sin(x + 0)$, true **B1**

Assume $\frac{d^k y}{dx^k} = x\sin\left(x + \frac{k\pi}{2}\right) + k\sin\left(x + \frac{(k-1)\pi}{2}\right)$. Differentiating by the product rule: $\frac{d^{k+1} y}{dx^{k+1}}$ **M1 A1**

$= \sin\left(x + \frac{k\pi}{2}\right) + x\cos\left(x + \frac{k\pi}{2}\right) + k\cos\left(x + \frac{(k-1)\pi}{2}\right)$

Using $\cos\theta = \sin\left(\theta + \frac{\pi}{2}\right)$: $x\cos\left(x + \frac{k\pi}{2}\right) = x\sin\left(x + \frac{(k+1)\pi}{2}\right)$ and $k\cos\left(x + \frac{(k-1)\pi}{2}\right) = k\sin$ **M1 A1**

$\left(x + \frac{k\pi}{2}\right)$

So $\frac{d^{k+1} y}{dx^{k+1}} = x\sin\left(x + \frac{(k+1)\pi}{2}\right) + (k+1)\sin\left(x + \frac{k\pi}{2}\right)$, the result for n = k + 1 **A1**

True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers $n \geq 1$ by induction **E1**

Answer: $\frac{d^n y}{dx^n} = x\sin\left(x + \frac{n\pi}{2}\right) + n\sin\left(x + \frac{(n-1)\pi}{2}\right)$ **7 marks**

(b) n = 10: $x\sin(x + 5\pi) + 10\sin\left(x + \frac{9\pi}{2}\right) = -x\sin x + 10\cos x$ **M1**

At $x = \frac{\pi}{2}$: $-\frac{\pi}{2} \times 1 + 10 \times 0 = -\frac{\pi}{2}$ **A1**

Answer: $-\frac{\pi}{2}$ **2 marks**

22 (9 marks)

(a) n = 1: $\frac{dy}{dx} = \frac{1}{1+x^2}$, so $(1+x^2)\frac{dy}{dx} = 1$. Differentiating, $(1+x^2)\frac{d^2 y}{dx^2} + 2x\frac{dy}{dx} = 0$, which is the **B1**

statement for n = 1

Assume $(1+x^2)\frac{d^{k+1} y}{dx^{k+1}} + 2kx\frac{d^k y}{dx^k} + k(k-1)\frac{d^{k-1} y}{dx^{k-1}} = 0$. Differentiate each term, using the **M1**

product rule on the first two

$(1+x^2)\frac{d^{k+2} y}{dx^{k+2}} + 2x\frac{d^{k+1} y}{dx^{k+1}} + 2kx\frac{d^{k+1} y}{dx^{k+1}} + 2k\frac{d^k y}{dx^k} + k(k-1)\frac{d^k y}{dx^k} = 0$ **A1**

$= (1+x^2)\frac{d^{k+2} y}{dx^{k+2}} + 2(k+1)x\frac{d^{k+1} y}{dx^{k+1}} + (k^2+k)\frac{d^k y}{dx^k} = 0$, the statement for n = k + 1 **A1**

True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers $n \geq 1$ by induction **E1**

Answer: $(1+x^2)\frac{d^{n+1} y}{dx^{n+1}} + 2nx\frac{d^n y}{dx^n} + n(n-1)\frac{d^{n-1} y}{dx^{n-1}} = 0$ **5 marks**

(b) At x = 0: $\frac{d^{n+1} y}{dx^{n+1}} = -n(n-1)\frac{d^{n-1} y}{dx^{n-1}}$. Also $\frac{dy}{dx} = 1$ at x = 0 **M1**

n = 2: $\frac{d^3 y}{dx^3} = -2 \times 1 \times 1 = -2$ **A1**

n = 4: $\frac{d^5 y}{dx^5} = -4 \times 3 \times (-2) = 24$ **A1**

Answer: $\frac{d^3 y}{dx^3} = -2$, $\frac{d^5 y}{dx^5} = 24$ **3 marks**

(c) The even derivatives at 0 are all 0, so $\arctan x = x - \frac{2}{3!}x^3 + \frac{24}{5!}x^5 = x - \frac{x^3}{3} + \frac{x^5}{5}$ **B1**

Answer: $x - \frac{x^3}{3} + \frac{x^5}{5}$ **1 mark**

23 (8 marks)

(a) $\sin 3x = \sin 2x \cos x + \cos 2x \sin x = 2 \sin x \cos^2 x + (1 - 2\sin^2 x) \sin x$ **M1**

$= 2 \sin x (1 - \sin^2 x) + \sin x - 2 \sin^3 x = 3 \sin x - 4 \sin^3 x$ **A1**

So $\sin^3 x = \frac{1}{4}(3 \sin x - \sin 3x)$ **A1**

Answer: $\sin^3 x = \frac{1}{4}(3 \sin x - \sin 3x)$ **3 marks**

(b) n = 1: $\frac{dy}{dx} = \frac{1}{4}(3 \cos x - 3 \cos 3x) = \frac{3}{4} \sin (x + \frac{\pi}{2}) - \frac{3}{4} \sin (3x + \frac{\pi}{2})$, true **B1**

Assume $\frac{d^k y}{dx^k} = \frac{3}{4} \sin (x + \frac{k\pi}{2}) - \frac{3^k}{4} \sin (3x + \frac{k\pi}{2})$. Differentiating: $\frac{d^{k+1} y}{dx^{k+1}} = \frac{3}{4} \cos (x + \frac{k\pi}{2}) - \frac{3^{k+1}}{4} \cos (3x + \frac{k\pi}{2})$ **M1 A1**

Using $\cos \theta = \sin (\theta + \frac{\pi}{2})$ on each term: $\frac{d^{k+1} y}{dx^{k+1}} = \frac{3}{4} \sin (x + \frac{(k+1)\pi}{2}) - \frac{3^{k+1}}{4} \sin (3x + \frac{(k+1)\pi}{2})$ **A1**

True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers $n \geq 1$ by induction **E1**

Answer: $\frac{d^n y}{dx^n} = \frac{3}{4} \sin (x + \frac{n\pi}{2}) - \frac{3^n}{4} \sin (3x + \frac{n\pi}{2})$ **5 marks**

24 (8 marks)

(a) $\frac{dy}{dx} = e^{-x} - (x + 1)e^{-x} = -x e^{-x}$ **M1**

$\frac{d^2 y}{dx^2} = (x - 1)e^{-x}$, $\frac{d^3 y}{dx^3} = (2 - x)e^{-x}$ **A1**

Answer: $-x e^{-x}$, $(x - 1)e^{-x}$, $(2 - x)e^{-x}$ **2 marks**

(b) The signs alternate and the constant in the bracket drops by 1 each time: $\frac{d^n y}{dx^n} = (-1)^n(x + 1 - n)e^{-x}$ **B1**

Answer: $\frac{d^n y}{dx^n} = (-1)^n(x + 1 - n)e^{-x}$ **1 mark**

(c) n = 1 is true by part (a): $(-1)^1(x + 1 - 1)e^{-x} = -x e^{-x}$ **B1**

Assume $\frac{d^k y}{dx^k} = (-1)^k(x + 1 - k)e^{-x}$. Differentiating: $\frac{d^{k+1} y}{dx^{k+1}} = (-1)^k e^{-x}[1 - (x + 1 - k)] = (-1)^k(k - x)e^{-x}$ **M1**

$= (-1)^{k+1}(x - k)e^{-x} = (-1)^{k+1}(x + 1 - (k + 1))e^{-x}$, the result for n = k + 1 **A1**

True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers $n \geq 1$ by induction **E1**

Answer: $\frac{d^n y}{dx^n} = (-1)^n(x + 1 - n)e^{-x}$ **4 marks**

(d) $(-1)^{10}(0 + 1 - 10)e^0 = -9$ **B1**

Answer: -9 **1 mark**

25 (7 marks)

(a) The student has not verified the formula for n = 1 (the basis case), so the induction is incomplete **E1**

$\frac{dy}{dx} = 20e^{4x}$ **M1**

but the formula gives $5 \times 4^2 e^{4x} = 80e^{4x}$, so the conclusion is false **A1**

Answer: no basis case; the formula fails at n = 1 ($20e^{4x}$ against $80e^{4x}$) **3 marks**

(b) Correct formula: $\frac{d^n y}{dx^n} = 5 \times 4^n e^{4x}$. n = 1: $\frac{dy}{dx} = 20e^{4x} = 5 \times 4^1 e^{4x}$, true **B1**

Assume $\frac{d^k y}{dx^k} = 5 \times 4^k e^{4x}$. Differentiating: $\frac{d^{k+1} y}{dx^{k+1}} = 5 \times 4^k \times 4e^{4x}$ **M1**

$= 5 \times 4^{k+1} e^{4x}$, the result for n = k + 1 **A1**

True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers $n \geq 1$ by induction **E1**

Answer: $\frac{d^n y}{d x^n} = 5 \times 4^n e^{4x}$ **4 marks**

26 (8 marks)

(a) n = 1: $\frac{d y}{d x} = p(1 + x)^{p-1}$, which is the formula with n = 1 **B1**

Assume $\frac{d^k y}{d x^k} = p(p - 1) \ldots (p - k + 1)(1 + x)^{p-k}$. Differentiating: $\frac{d^{k+1} y}{d x^{k+1}} = p(p - 1) \ldots (p - k + 1) \times (p - k)(1 + x)^{p-k-1}$ **M1 A1**

$= p(p - 1) \ldots (p - k + 1)(p - k)(1 + x)^{p-(k+1)}$, which has k + 1 factors, the result for n = k + 1 **A1**

True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers $n \geq 1$ by induction **E1**

Answer: $\frac{d^n y}{d x^n} = p(p - 1) \ldots (p - n + 1)(1 + x)^{p-n}$ **5 marks**

(b) At x = 0: $\frac{d^n y}{d x^n} = p(p - 1) \ldots (p - n + 1)$ **M1**

The coefficient of x^n is $\frac{1}{n!}$ times this, which gives the stated result **A1**

Answer: $\frac{p(p - 1) \ldots (p - n + 1)}{n!}$ **2 marks**

(c) $\frac{1}{2} \times (-\frac{1}{2}) \times (-\frac{3}{2}) \div 3! = \frac{3}{8} \div 6 = \frac{1}{16}$ **B1**

Answer: $\frac{1}{16}$ **1 mark**

27 (12 marks)

(a) $\frac{d y}{d x} = \frac{2e^{2 \arcsin x}}{\sqrt{1 - x^2}}$, so $\sqrt{1 - x^2} \frac{d y}{d x} = 2y$ **M1**

Differentiating: $\sqrt{1 - x^2} \frac{d^2 y}{d x^2} - \frac{x}{\sqrt{1 - x^2}} \frac{d y}{d x} = 2\frac{d y}{d x}$ **M1**

Multiplying by $\sqrt{1 - x^2}$: $(1 - x^2)\frac{d^2 y}{d x^2} - x\frac{d y}{d x} = 2\sqrt{1 - x^2}\frac{d y}{d x} = 4y$, as required **A1**

Answer: $(1 - x^2) \frac{d^2 y}{d x^2} - x \frac{d y}{d x} - 4y = 0$ **3 marks**

(b) n = 0 is the result of part (a) **B1**

Assume $(1 - x^2) \frac{d^{k+2} y}{d x^{k+2}} - (2k + 1)x \frac{d^{k+1} y}{d x^{k+1}} - (k^2 + 4) \frac{d^k y}{d x^k} = 0$. Differentiate, using the product rule **M1**

$(1 - x^2) \frac{d^{k+3} y}{d x^{k+3}} - 2x \frac{d^{k+2} y}{d x^{k+2}} - (2k + 1)x \frac{d^{k+2} y}{d x^{k+2}} - (2k + 1) \frac{d^{k+1} y}{d x^{k+1}} - (k^2 + 4) \frac{d^{k+1} y}{d x^{k+1}} = 0$ **A1**

Collect terms: $(1 - x^2) \frac{d^{k+3} y}{d x^{k+3}} - (2k + 3)x \frac{d^{k+2} y}{d x^{k+2}} - (k^2 + 2k + 5) \frac{d^{k+1} y}{d x^{k+1}} = 0$ **M1**

$k^2 + 2k + 5 = (k + 1)^2 + 4$, so this is the statement for n = k + 1 **A1**

True for n = 0, and true for n = k implies true for n = k + 1, so true for all integers $n \geq 0$ by induction **E1**

Answer: $(1 - x^2) \frac{d^{n+2} y}{d x^{n+2}} - (2n + 1)x \frac{d^{n+1} y}{d x^{n+1}} - (n^2 + 4) \frac{d^n y}{d x^n} = 0$ **6 marks**

(c) At x = 0: $\frac{d^{n+2} y}{d x^{n+2}} = (n^2 + 4) \frac{d^n y}{d x^n}$. y(0) = 1 and $\frac{d y}{d x} = 2$ at x = 0 **M1**

$\frac{d^2 y}{d x^2} = 4 \times 1 = 4$, $\frac{d^3 y}{d x^3} = 5 \times 2 = 10$, $\frac{d^4 y}{d x^4} = 8 \times 4 = 32$ **A1**

$y = 1 + 2x + \frac{4}{2!}x^2 + \frac{10}{3!}x^3 + \frac{32}{4!}x^4 = 1 + 2x + 2x^2 + \frac{5}{3}x^3 + \frac{4}{3}x^4$ **A1**

Answer: $1 + 2x + 2x^2 + \frac{5}{3}x^3 + \frac{4}{3}x^4$ **3 marks**

28 (11 marks)

(a) $\frac{dy}{dx} = \sec^2 x = 1 + \tan^2 x = 1 + y^2$ **B1**

$\frac{d^2y}{dx^2} = 2y\frac{dy}{dx}$ **M1**

$= 2y(1 + y^2)$ **A1**

Answer: $\frac{dy}{dx} = 1 + y^2$, $\frac{d^2y}{dx^2} = 2y(1 + y^2)$ **3 marks**

(b) n = 1: $\frac{dy}{dx} = 1 + y^2$, a polynomial of degree 2 with leading coefficient 1 = 1! **B1**

Assume $\frac{d^k y}{dx^k} = P(y)$, a polynomial of degree k + 1 with leading coefficient k! **M1**

Differentiating with respect to x by the chain rule: $\frac{d^{k+1}y}{dx^{k+1}} = P'(y)\frac{dy}{dx} = (1 + y^2)\,P'(y)$ **A1**

P'(y) has degree k and leading coefficient (k + 1) × k! = (k + 1)! **A1**

Multiplying by $(1 + y^2)$ gives degree k + 2 with leading coefficient (k + 1)!, the statement for n = k + 1 **A1**

True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers n ≥ 1 by induction **E1**

Answer: $\frac{d^n y}{dx^n}$ is a polynomial in y of degree n + 1 with leading coefficient n! **6 marks**

(c) $\frac{d^3y}{dx^3} = (1 + y^2)(2 + 6y^2) = 2 + 8y^2 + 6y^4$ **M1**

At x = 0: y = 0, $\frac{dy}{dx} = 1$, $\frac{d^2y}{dx^2} = 0$, $\frac{d^3y}{dx^3} = 2$, so $\tan x = x + \frac{2}{3!}x^3 + \ldots = x + \frac{x^3}{3} + \ldots$ **A1**

Answer: $\frac{d^3y}{dx^3} = 2 + 8y^2 + 6y^4$ **2 marks**

29 (12 marks)

(a) n = 1: $\frac{dy}{dx} = 2xy$, and differentiating, $\frac{d^2y}{dx^2} = 2y + 2x\frac{dy}{dx} = 2x\frac{dy}{dx} + 2 \times 1 \times y$, true **B1**

Assume $\frac{d^{k+1}y}{dx^{k+1}} = 2x\,\frac{d^k y}{dx^k} + 2k\,\frac{d^{k-1}y}{dx^{k-1}}$. Differentiate: $\frac{d^{k+2}y}{dx^{k+2}} = 2\,\frac{d^k y}{dx^k} + 2x\,\frac{d^{k+1}y}{dx^{k+1}} + 2k\,\frac{d^k y}{dx^k}$ **M1 A1**

$= 2x\,\frac{d^{k+1}y}{dx^{k+1}} + 2(k + 1)\,\frac{d^k y}{dx^k}$, the statement for n = k + 1 **A1**

True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers n ≥ 1 by induction **E1**

Answer: $\frac{d^{n+1}y}{dx^{n+1}} = 2x\,\frac{d^n y}{dx^n} + 2n\,\frac{d^{n-1}y}{dx^{n-1}}$ **5 marks**

(b) At x = 0: $\frac{d^{n+1}y}{dx^{n+1}} = 2n\,\frac{d^{n-1}y}{dx^{n-1}}$, and y(0) = 1 **M1**

n = 1: $\frac{d^2y}{dx^2} = 2$; n = 3: $\frac{d^4y}{dx^4} = 6 \times 2 = 12$ **A1**

n = 5: $\frac{d^6y}{dx^6} = 10 \times 12 = 120$ **A1**

Answer: 120 **3 marks**

(c) m = 1: $\frac{d^2y}{dx^2} = 2 = \frac{2!}{1!}$, true **B1**

Assume $\frac{d^{2k}y}{dx^{2k}} = \frac{(2k)!}{k!}$ at x = 0. By part (b) with n = 2k + 1, $\frac{d^{2k+2}y}{dx^{2k+2}} = 2(2k + 1) \times \frac{(2k)!}{k!}$ **M1**

$= \frac{(2k + 2)(2k + 1)(2k)!}{(k + 1)k!} = \frac{(2k + 2)!}{(k + 1)!}$, the result for m = k + 1 **A1**

True for m = 1, and true for m = k implies true for m = k + 1, so true for all integers m ≥ 1 by **E1**

induction

Answer: $\mathbf{\frac{d^{2m}y}{dx^{2m}} = \frac{(2m)!}{m!}}$ **at x = 0** **4 marks**

30 (12 marks)

(a) n = 1: $\frac{dy}{dx} = e^{ax}(a \sin bx + b \cos bx)$ **M1**

$= r\,e^{ax}(\cos \varphi \sin bx + \sin \varphi \cos bx) = r\,e^{ax} \sin (bx + \varphi)$, true **B1**

Assume $\frac{d^k y}{dx^k} = r^k e^{ax} \sin \theta$, where $\theta = bx + k\varphi$. Differentiating: $\frac{d^{k+1}y}{dx^{k+1}} = r^k e^{ax}(a \sin \theta + b \cos \theta)$ **M1 A1**

$a \sin \theta + b \cos \theta = r(\cos \varphi \sin \theta + \sin \varphi \cos \theta) = r \sin (\theta + \varphi)$ **M1**

So $\frac{d^{k+1}y}{dx^{k+1}} = r^{k+1} e^{ax} \sin (bx + (k + 1)\varphi)$, the result for n = k + 1 **A1**

True for n = 1, and true for n = k implies true for n = k + 1, so true for all integers $n \geq 1$ by induction **E1**

Answer: $\mathbf{\frac{d^n y}{dx^n} = r^n e^{ax} \sin (bx + n\varphi)}$ **7 marks**

(b) $r = \sqrt{3 + 1} = 2$ **M1**

$\cos \varphi = \frac{\sqrt{3}}{2}$, $\sin \varphi = \frac{1}{2}$, so $\varphi = \frac{\pi}{6}$ **A1**

Answer: $\mathbf{r = 2, \varphi = \frac{\pi}{6}}$ **2 marks**

(c) $\frac{d^{12}y}{dx^{12}} = 2^{12} \exp(\sqrt{3}x) \sin (x + 2\pi) = 4096 \exp(\sqrt{3}x) \sin x = 4096y$ **B1**

$\frac{d^6 y}{dx^6} = 2^6 \exp(\sqrt{3}x) \sin (x + \pi) = -64 \exp(\sqrt{3}x) \sin x$ **M1**

At $x = \frac{\pi}{2}$: $-64 \exp(\frac{\sqrt{3}\pi}{2})$ **A1**

Answer: $\mathbf{\frac{d^{12}y}{dx^{12}} = 4096y;\ \frac{d^6 y}{dx^6} = -64 \exp(\frac{\sqrt{3}\pi}{2})}$ **3 marks**