Proof by Mathematical Induction

1 Let P(n) be the statement $1 + 3 + 5 + ... + (2n - 1) = n^2$. Show that P(1) and P(2) are both true. **[2]**

2 Let P(n) be the statement $\sum_{r=1}^{n} r^2 = \frac{n(n + 1)(2n + 1)}{6}$.

Assuming that P(k) is true, write down the statement P(k + 1) that has to be proved in the inductive step. **[2]**

3 A sequence is defined by $u_1 = 3$ and $u_{n+1} = u_n + 4$. Find u_2, u_3 and u_4, and hence suggest a formula for u_n. **[3]**

4 A student is proving a result by induction. Write down the final sentence of the proof, once the basis case and the inductive step have both been completed. **[2]**

5 It is given that $\sum_{r=1}^{k} r^3 = \frac{k^2(k + 1)^2}{4}$.

Write down an expression for $\sum_{r=1}^{k+1} r^3$ in terms of k, which includes the term $(k + 1)^3$. Do not simplify. **[2]**

6 The sequence u_n is defined by $u_1 = 2$ and $u_{n+1} = 2u_n$. Show that the formula $u_n = 2^n$ is correct for n = 1, 2 and 3. **[3]**

7 Prove by induction that $\sum_{r=1}^{n} (2r - 1) = n^2$ for all integers $n \geq 1$. **[4]**

8 Prove by induction that $\sum_{r=1}^{n} r = \frac{n(n + 1)}{2}$ for all integers $n \geq 1$. **[4]**

9 Prove by induction that $1 + 2 + 4 + ... + 2^{n-1} = 2^n - 1$ for all integers $n \geq 1$. **[4]**

10 A sequence is defined by $u_1 = 5$ and $u_{n+1} = u_n + 3$ for $n \geq 1$. Prove by induction that $u_n = 3n + 2$. **[4]**

11 A sequence is defined by $u_1 = 4$ and $u_{n+1} = 3u_n - 2$. Prove by induction that $u_n = 3^n + 1$ for all $n \geq 1$. **[4]**

12 Prove by induction that $\sum_{r=1}^{n} \frac{1}{r(r + 1)} = \frac{n}{n + 1}$ for all integers $n \geq 1$. **[5]**

13 Prove by induction that $\sum_{r=1}^{n} r \times r! = (n + 1)! - 1$ for all integers $n \geq 1$. **[5]**

14 Prove by induction that $\sum_{r=1}^{n} r \times 2^r = (n - 1)2^{n+1} + 2$ for all integers $n \geq 1$. **[6]**

15 Prove by induction that, for all integers $n \geq 2$,
$(1 - \frac{1}{2^2})(1 - \frac{1}{3^2})(1 - \frac{1}{4^2}) ... (1 - \frac{1}{n^2}) = \frac{n + 1}{2n}$. **[6]**

16 Prove by induction that the nth derivative of $x e^x$ is $(x + n)e^x$, for all integers $n \geq 1$. **[5]**

17 A sequence is defined by $u_1 = 1$ and $u_{n+1} = u_n + n^2$. Prove by induction that $u_n = 1 + \frac{n(n-1)(2n-1)}{6}$ for all integers $n \geq 1$. **[7]**

18 A student makes the following attempt to prove that $n^2 + n$ is odd for all positive integers n.
'Assume that $k^2 + k$ is odd. Then $(k + 1)^2 + (k + 1) = k^2 + k + 2(k + 1)$. Since $2(k + 1)$ is even and $k^2 + k$ is odd, the sum is odd. So if the result is true for $n = k$, it is true for $n = k + 1$. Therefore it is true for all positive integers n.'
Explain why the 'proof' is not valid, and state with a reason whether the result itself is true. **[4]**

19 A sequence is defined by $u_1 = \frac{3}{2}$ and $u_{n+1} = \frac{2u_n - 1}{u_n}$ for $n \geq 1$. Prove by induction that $u_n = \frac{n + 2}{n + 1}$ for all integers $n \geq 1$. **[6]**

20 Prove by induction that $\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)} = \frac{n}{2n+1}$ for all integers $n \geq 1$. **[5]**

21 The Fibonacci numbers are defined by $F_1 = 1$, $F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$.
Prove by induction that $F_1^2 + F_2^2 + ... + F_n^2 = F_nF_{n+1}$ for all integers $n \geq 1$. **[7]**

22 The function f is defined by $f(x) = 2 - \frac{1}{x}$, $x \neq 0$. The function f^n is f applied n times, so $f^1 = f$, $f^2(x) = f(f(x))$, and so on.
Prove by induction that $f^n(x) = \frac{(n+1)x - n}{nx - (n-1)}$ for all integers $n \geq 1$. **[6]**

23 Let P(n) be the statement that the nth derivative of $x e^{-x}$ is $(-1)^n(x - n)e^{-x}$.
(a) Prove by induction that P(n) is true for all integers $n \geq 1$. **[6]**
(b) Hence find the 10th derivative of $x e^{-x}$ at $x = 3$. **[2]**

24 Prove by induction that, for all integers $n \geq 1$ and x not a multiple of π,
$\cos x \cos 2x \cos 4x ... \cos (2^{n-1}x) = \frac{\sin (2^n x)}{2^n \sin x}$. **[7]**

25 Prove by induction that, for all integers $n \geq 1$ and θ not a multiple of π,
$\cos \theta + \cos 3\theta + \cos 5\theta + ... + \cos (2n - 1)\theta = \frac{\sin 2n\theta}{2 \sin \theta}$. **[8]**

26 Prove by induction that $(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$ for all positive integers n. **[7]**

27 A sequence is defined by $u_1 = 1$, $u_2 = 4$ and $u_{n+2} = 4u_{n+1} - 3u_n$ for $n \geq 1$.
Prove by induction that $u_n = \frac{3^n - 1}{2}$ for all integers $n \geq 1$. **[8]**

28 Prove by induction that the nth derivative of $e^x \cos x$ is $(\sqrt{2})^n e^x \cos (x + \frac{n\pi}{4})$ for all integers $n \geq 1$. **[8]**

29 Prove by induction that $\sum_{r=1}^{n} r^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$ for all integers $n \geq 1$. **[9]**

30 Strong induction assumes that the statement is true for every integer from the base case up to k, and uses this to prove it for k + 1.

(a) Use strong induction to prove that every integer $n \geq 2$ can be written as a product of one or more prime numbers. **[6]**

(b) Explain why the ordinary form of induction (assuming only the case n = k) would not be enough for this proof. **[2]**

Induction with Series

1 Let P(n) be the statement $\sum_{r=1}^{n} r(3r + 1) = n(n + 1)^2$.

Show that P(n) is true for n = 1, 2 and 3. **[3]**

2 Show that $\frac{k(2k + 1)}{6} + (k + 1) = \frac{(k + 2)(2k + 3)}{6}$. **[2]**

3 Let P(n) be the statement $\sum_{r=1}^{n} r(r + 3) = \frac{n(n + 1)(n + 5)}{3}$.

Write down the statement P(k + 1). **[2]**

4 Show that the formula $\sum_{r=1}^{n} (3r - 1) = \frac{n(3n + 1)}{2}$ is correct for n = 1 and n = 2. **[2]**

5 Show that $1^3 + 2^3 + 3^3 = (1 + 2 + 3)^2$. **[2]**

6 Let $S_n = 2 + 4 + 8 + ... + 2^n$. Find S_1, S_2 and S_3, and hence suggest a formula for S_n. **[3]**

7 Prove by induction that $\sum_{r=1}^{n} (3r - 1) = \frac{n(3n + 1)}{2}$ for all integers $n \geq 1$. **[4]**

8 Prove by induction that $\sum_{r=1}^{n} r(r + 1) = \frac{n(n + 1)(n + 2)}{3}$ for all integers $n \geq 1$. **[5]**

9 Prove by induction that $\sum_{r=1}^{n} r^2 = \frac{n(n + 1)(2n + 1)}{6}$ for all integers $n \geq 1$. **[5]**

10 Prove by induction that $1 + 3 + 9 + ... + 3^{n-1} = \frac{3^n - 1}{2}$ for all integers $n \geq 1$. **[4]**

11 Prove by induction that $\sum_{r=1}^{n} r(r + 3) = \frac{n(n + 1)(n + 5)}{3}$ for all integers $n \geq 1$. **[5]**

12 Prove by induction that $\sum_{r=1}^{n} (r + 1) \times 2^r = n \times 2^{n+1}$ for all integers $n \geq 1$. **[5]**

13 Prove by induction that $1 + 4 + 12 + ... + n \times 2^{n-1} = (n - 1)2^n + 1$ for all integers $n \geq 1$. **[5]**

14 Prove by induction that $\sum_{r=1}^{n} r^3 = \frac{n^2(n + 1)^2}{4}$ for all integers $n \geq 1$. **[6]**

15 Prove by induction that $\sum_{r=1}^{n} \frac{1}{r(r+2)} = \frac{n(3n+5)}{4(n+1)(n+2)}$ for all integers $n \geq 1$. **[7]**

16 Prove by induction that $\sum_{r=1}^{n} r(r+1)(r+2) = \frac{n(n+1)(n+2)(n+3)}{4}$ for all integers $n \geq 1$. **[6]**

17 Prove by induction that $1^2 + 3^2 + 5^2 + ... + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$ for all integers $n \geq 1$. **[6]**

18 Prove by induction that $1^2 - 2^2 + 3^2 - 4^2 + ... + (-1)^{n+1}n^2 = (-1)^{n+1}\frac{n(n+1)}{2}$ for all integers $n \geq 1$. **[7]**

19 Prove by induction that $\sum_{r=1}^{n} \frac{r}{2^r} = 2 - \frac{n+2}{2^n}$ for all integers $n \geq 1$. **[6]**

20 Prove by induction that $\sum_{r=2}^{n} (r-1)r(r+1) = \frac{(n-1)n(n+1)(n+2)}{4}$ for all integers $n \geq 2$. **[7]**

21 Prove by induction that $\sum_{r=1}^{n} r \times 3^r = \frac{(2n-1)3^{n+1}+3}{4}$ for all integers $n \geq 1$. **[7]**

22 Prove by induction that $\sum_{r=1}^{n} (r^2+1) \times r! = n \times (n+1)!$ for all integers $n \geq 1$. **[7]**

23 Let $f(n) = \sum_{r=1}^{n} r(2r+1)$.

(a) Prove by induction that $f(n) = \frac{n(n+1)(4n+5)}{6}$ for all integers $n \geq 1$. **[5]**

(b) Hence show that $\sum_{r=n+1}^{2n} r(2r+1) = \frac{n(4n+1)(7n+5)}{6}$. **[3]**

24 Prove by induction that $1^3 + 3^3 + 5^3 + ... + (2n-1)^3 = n^2(2n^2-1)$ for all integers $n \geq 1$. **[6]**

25 A student attempts to prove that $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$.

'The result is true for $n = 1$. Assume it is true for $n = k$. Then the sum to $k + 1$ is $\frac{k(k+1)(2k+1)}{6} + (k+1) = \frac{(k+1)(2k^2+k+6)}{6}$, and this equals $\frac{(k+1)(k+2)(2k+3)}{6}$. So the result is true for $n = k + 1$.'

Identify two errors in the student's inductive step, and write a correct inductive step. **[6]**

26 Prove by induction that $\sum_{r=1}^{n} \frac{2r+1}{r^2(r+1)^2} = 1 - \frac{1}{(n+1)^2}$ for all integers $n \geq 1$. **[7]**

27 Prove by induction that $\sum_{r=1}^{n} r^2 \times 2^r = (n^2-2n+3)2^{n+1} - 6$ for all integers $n \geq 1$. **[8]**

28 Prove by induction that, for all integers $n \geq 1$ and θ not a multiple of 2π,

$$\sin\theta + \sin 2\theta + \sin 3\theta + ... + \sin n\theta = \frac{\sin\left(\frac{n\theta}{2}\right)\sin\left(\frac{(n+1)\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}.$$ **[10]**

29 Let $S_n = \sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)(r+3)}$.

(a) Prove by induction that $\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)(r+3)} = \frac{1}{18} - \frac{1}{3(n+1)(n+2)(n+3)}$ for all integers $n \geq 1$. **[7]**

(b) Hence write down the value of $\sum_{r=1}^{\infty} \frac{1}{r(r+1)(r+2)(r+3)}$. **[1]**

30 Prove by induction that $\sum_{r=1}^{n} r \times C(n, r) = n \times 2^{n-1}$ for all integers $n \geq 1$, where $C(n, r) = \frac{n!}{r!(n-r)!}$.

You may use the identity $C(k + 1, r) = C(k, r) + C(k, r - 1)$ and the result $\sum_{r=0}^{k} C(k, r) = 2^k$. **[9]**

Induction with Divisibility

1 Let $f(n) = 7^n - 1$. Show that f(1) and f(2) are both divisible by 6. **[2]**

2 Let $f(n) = 5^n - 1$. Show that $f(k + 1) - f(k) = 4 \times 5^k$. **[2]**

3 Let $f(n) = 2^{2n} - 1$. Show that $f(k + 1) = 4f(k) + 3$. **[2]**

4 Let $f(n) = n^3 - n$. Evaluate f(1), f(2) and f(3), and state the largest integer that divides all three values. **[3]**

5 Let $f(n) = 6^n + 4$. Show that f(1) is divisible by 5, and that $f(k + 1) - f(k) = 5 \times 6^k$. **[3]**

6 A proof by induction shows that f(n) is divisible by 8 for n = 1 and that if f(k) is divisible by 8 then f(k + 1) is divisible by 8. Write down the conclusion of the proof. **[2]**

7 Prove by induction that $5^n - 1$ is divisible by 4 for all integers $n \geq 1$. **[4]**

8 Prove by induction that $n(n + 1)(n + 2)$ is divisible by 6 for all integers $n \geq 1$. **[5]**

9 A sequence is defined by $u_1 = 2$ and $u_{n+1} = 3u_n + 4$ for $n \geq 1$. Prove by induction that $u_n + 2$ is divisible by 4 for all integers $n \geq 1$. **[4]**

10 Prove by induction that $6^n + 4$ is divisible by 5 for all integers $n \geq 1$. **[4]**

11 A sequence has terms $u_n = 10^{n+1} + 4 \times 3^n$ for $n \geq 1$. Prove by induction that every term of the sequence is a multiple of 7. **[4]**

12 Prove by induction that $n^3 + 2n$ is divisible by 3 for all integers $n \geq 1$. **[5]**

13 Prove by induction that $5^n + 2 \times 11^n$ is divisible by 3 for all integers $n \geq 1$. **[4]**

14 Prove by induction that $2^{n+2} + 3^{2n+1}$ is divisible by 7 for all integers $n \geq 1$. **[6]**

15 Prove by induction that $5^{n+1} + 2 \times 3^n + 1$ is divisible by 8 for all integers $n \geq 1$. **[7]**

16 Prove by induction that $3^{2n+2} - 8n - 9$ is divisible by 64 for all integers $n \geq 1$. **[7]**

17 Prove by induction that $n(n^2 + 5)$ is divisible by 6 for all integers $n \geq 1$. **[7]**

18 Prove by induction that $(6n + 1) \times 7^n - 1$ is divisible by 6 for all integers $n \geq 1$. **[5]**

19 Prove by induction that $n^3 + (n + 1)^3 + (n + 2)^3$ is divisible by 9 for all integers $n \geq 1$. **[6]**

20 The Fibonacci numbers are defined by $F_1 = 1$, $F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$. Prove by induction that F_{3n} is even for all integers $n \geq 1$. **[5]**

21 Prove by induction that $2^{3n+3} - 7n - 8$ is divisible by 49 for all integers $n \geq 1$. **[8]**

22 A student attempts to prove that $5^n + 3$ is divisible by 4 for all integers $n \geq 1$.
'n = 1: 5 + 3 = 8, which is divisible by 4. Assume $5^k + 3 = 4m$. Then $5^{k+1} + 3 = 5(5^k + 3) + 12 = 20m + 12 = 4(5m + 3)$, which is divisible by 4. So by induction the result is true.'
There is an error in the algebra of the inductive step. Identify it, show with k = 1 that the student's identity is false, and write a correct inductive step. **[6]**

23 Prove by induction that $4^{2n+1} + 3^{n+2}$ is divisible by 13 for all integers $n \geq 1$. **[8]**

24 Prove by induction that $4^n + 6n - 1$ is divisible by 9 for all integers $n \geq 1$. **[7]**

25 Prove by induction that $10^n + 3 \times 4^{n+2} + 5$ is divisible by 9 for all integers $n \geq 1$. **[8]**

26 Prove by induction that $\frac{(2n)!}{2^n\, n!}$ is an integer for all integers $n \geq 1$. **[6]**

27 Prove by induction that 3^{n+1} divides $2^{3^n} + 1$ for all integers $n \geq 0$. **[10]**

28 Let $f(n) = n^5 - n$.
(a) Prove by induction that f(n) is divisible by 5 for all integers $n \geq 1$. **[5]**
(b) Factorise f(n) completely, and hence prove that f(n) is divisible by 30 for all integers $n \geq 1$. **[4]**

29 The Fibonacci numbers are defined by $F_1 = 1$, $F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$.
(a) Show that $F_{m+5} = 5F_{m+1} + 3F_m$ for all integers $m \geq 1$. **[3]**
(b) Hence prove by induction that F_{5n} is divisible by 5 for all integers $n \geq 1$. **[5]**

30 Let $u_n = (3 + \sqrt{5})^n + (3 - \sqrt{5})^n$ for $n \geq 0$.
(a) Show that $u_{n+2} = 6u_{n+1} - 4u_n$. **[3]**
(b) Prove by induction that u_n is divisible by 2^n for all integers $n \geq 1$. **[6]**

Induction with Matrices

1 Given that $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, find A^2 and A^3, and hence suggest a formula for A^n. **[3]**

2 Given that $A^k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ and $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, find $A^{k+1} = A^kA$. **[2]**

3 Let P(n) be the statement $\begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ 5n & 1 \end{pmatrix}$.
Show that P(1) is true, and write down the statement P(k + 1). **[2]**

4 Given that $B = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, find B^2 and B^3, and hence suggest a formula for B^n. **[3]**

5 Let $A = \begin{pmatrix} 5 & 8 \\ -2 & -3 \end{pmatrix}$. Show that the formula $A^n = \begin{pmatrix} 1 + 4n & 8n \\ -2n & 1 - 4n \end{pmatrix}$ is correct for n = 1 and n = 2. **[2]**

6 Let $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Show that $J^2 = -I$ and $J^4 = I$, where I is the 2 × 2 identity matrix. **[2]**

7 Prove by induction that $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ for all integers $n \geq 1$. **[4]**

8 Prove by induction that $\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ 3n & 1 \end{pmatrix}$ for all integers $n \geq 1$. **[4]**

9 Let $A = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$ and let I be the 2 × 2 identity matrix. Show that $A^2 = 2A - I$, and hence prove by induction that $A^n = nA - (n - 1)I$ for all integers $n \geq 1$. **[5]**

10 Prove by induction that $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}^n = \begin{pmatrix} 2^n & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (-1)^n \end{pmatrix}$ for all integers $n \geq 1$. **[4]**

11 Prove by induction that $\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}^n = \begin{pmatrix} 1 & 2^n - 1 \\ 0 & 2^n \end{pmatrix}$ for all integers $n \geq 1$. **[5]**

12 Prove by induction that $\begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 2^n & 3(2^n - 1) \\ 0 & 1 \end{pmatrix}$ for all integers $n \geq 1$. **[5]**

13 Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Prove by induction that $A^n = 2^{n-1}A$ for all integers $n \geq 1$. **[4]**

14 Prove by induction that $\begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}^n = \begin{pmatrix} 2n + 1 & -2n \\ 2n & 1 - 2n \end{pmatrix}$ for all integers $n \geq 1$. **[6]**

15 Prove by induction that $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}^n = \begin{pmatrix} 2^n & n2^{n-1} \\ 0 & 2^n \end{pmatrix}$ for all integers $n \geq 1$. **[6]**

16 Prove by induction that $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^n = \frac{1}{2}\begin{pmatrix} 3^n + 1 & 3^n - 1 \\ 3^n - 1 & 3^n + 1 \end{pmatrix}$ for all integers $n \geq 1$. **[7]**

17 Prove by induction that $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n & \frac{n(n-1)}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}$ for all integers $n \geq 1$. **[7]**

18 Prove by induction that, for all integers $n \geq 1$,
$\left(\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\right)^n = \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix}$. **[7]**

19 Let $A = \begin{pmatrix} 4 & -6 \\ 1 & -1 \end{pmatrix}$.
Show that $A^2 = 3A - 2I$, and hence prove by induction that $A^n = (2^n - 1)A - (2^n - 2)I$ for all integers $n \geq 1$. **[6]**

20 Prove by induction that $\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}^n = \begin{pmatrix} 2^{n-1}(2-n) & -n2^{n-1} \\ n2^{n-1} & 2^{n-1}(2+n) \end{pmatrix}$ for all integers $n \geq 1$. **[7]**

21 The Fibonacci numbers are defined by $F_0 = 0$, $F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$. Let $M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

(a) Prove by induction that $M^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$ for all integers $n \geq 1$. **[6]**

(b) By considering determinants, deduce that $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$. **[3]**

22 Let $A = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$.
Prove by induction that $A^n = \begin{pmatrix} 2 \times 3^n - 2^n & 2^{n+1} - 2 \times 3^n \\ 3^n - 2^n & 2^{n+1} - 3^n \end{pmatrix}$ for all integers $n \geq 1$. **[8]**

23 Let $T = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$, where a, b and c are constants.
Prove by induction that $T^n = \begin{pmatrix} 1 & na & nb + \frac{n(n-1)ac}{2} \\ 0 & 1 & nc \\ 0 & 0 & 1 \end{pmatrix}$ for all integers $n \geq 1$. **[7]**

24 The 2×2 matrix A satisfies $A^2 = A + I$, where I is the identity matrix. The Fibonacci numbers are $F_0 = 0$, $F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$.
Prove by induction that $A^n = F_nA + F_{n-1}I$ for all integers $n \geq 2$. **[6]**

25 Let $A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$. A student claims that $A^n = \begin{pmatrix} 1 & 2^n \\ 0 & 3^n \end{pmatrix}$ for all integers $n \geq 1$, and 'proves' it by multiplying each entry of A^k by the corresponding entry of A.
Show that the claim is false, explain the student's error, and find and prove by induction the correct formula for A^n. **[7]**

26 A and B are square matrices of the same size such that $AB = BA$.
Prove by induction that $AB^n = B^nA$ for all integers $n \geq 1$, and hence prove by induction that $(AB)^n = A^nB^n$ for all integers $n \geq 1$. **[8]**

27 On any day a shop's website is either 'up' or 'down'. If it is up one day, the probability that it is up the next day is 0.7. If it is down one day, the probability that it is up the next day is 0.4. This is modelled by the matrix $P = \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix}$, where the entry in row i and column j is the probability of moving from state i to state j (state 1 is up, state 2 is down).

(a) Prove by induction that $P^n = \frac{1}{7}\begin{pmatrix} 4 & 3 \\ 4 & 3 \end{pmatrix} + \frac{1}{7}(0.3)^n \begin{pmatrix} 3 & -3 \\ -4 & 4 \end{pmatrix}$ for all integers $n \geq 1$. **[7]**

(b) Hence find the limit of P^n as $n \to \infty$, and interpret it: in the long run, what proportion of days is the website up? **[3]**

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28 Let $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$.

Prove by induction that $A^n = \begin{pmatrix} 2^n & n2^{n-1} & n(n-1)2^{n-3} \\ 0 & 2^n & n2^{n-1} \\ 0 & 0 & 2^n \end{pmatrix}$ for all integers $n \geq 1$. **[9]**

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29 For $n \geq 1$, let D_n be the determinant of the $n \times n$ matrix with 2 on the leading diagonal, -1 on the diagonals immediately above and below it, and 0 elsewhere. For example $D_1 = |2|$ and $D_3 = \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix}$.

Show that $D_{n+2} = 2D_{n+1} - D_n$ for $n \geq 1$, and hence prove by strong induction that $D_n = n + 1$ for all integers $n \geq 1$. **[9]**

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30 Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$.

(a) Prove by induction that $A^n = \begin{pmatrix} 1 & 2^n - 1 & 3^n - 2^{n+1} + 1 \\ 0 & 2^n & 2(3^n - 2^n) \\ 0 & 0 & 3^n \end{pmatrix}$ for all integers $n \geq 1$. **[8]**

(b) Hence find the top right entry of A^6. **[2]**

Induction with Inequalities

1 Show that $2^n > n$ for n = 1, 2 and 3. **[3]**

2 It is given that $2^k > k^2$ for a positive integer k. Show that $2^{k+1} > 2k^2$. **[2]**

3 Show that $n! > 2^n$ holds for n = 4 but not for n = 3. **[3]**

4 Show that $k^2 + 2k + 1 < 2k^2$ for all integers $k \geq 3$. **[2]**

5 Verify that $4^n \geq 1 + 3n$ for n = 1, 2 and 3. **[3]**

6 By evaluating 2^n and n^2 for n = 1, 2, ..., 5, find the smallest integer N such that $2^n > n^2$ for n = N, N + 1, ... up to 5. **[2]**

7 Prove by induction that $2^n \geq n + 1$ for all integers $n \geq 1$. **[4]**

8 Prove by induction that $3^n \geq 2n + 1$ for all integers $n \geq 1$. **[4]**

9 Prove by induction that $n! \geq 2^{n-1}$ for all integers $n \geq 1$. **[4]**

10 Prove by induction that $n! < n^n$ for all integers $n \geq 2$. **[5]**

11 Prove by induction that $n! > 2^n$ for all integers $n \geq 4$. **[4]**

12 Prove by induction that $2^n > n^2$ for all integers $n \geq 5$. **[5]**

13 Prove by induction that $3^n > (n + 1)^2$ for all integers $n \geq 3$. **[5]**

14 Prove by induction that $(1 + x)^n \geq 1 + nx$ for all integers $n \geq 1$, where x is a real constant with $x > -1$. **[6]**

15 Prove by induction that $\sum_{r=1}^{n} \frac{1}{r^2} \leq 2 - \frac{1}{n}$ for all integers $n \geq 1$. **[6]**

16 Prove by induction that $\sum_{r=1}^{n} \frac{1}{\sqrt{r}} \geq \sqrt{n}$ for all integers $n \geq 1$. **[6]**

17 Prove by induction that $2^n > n^3$ for all integers $n \geq 10$. **[6]**

18 Prove by induction that $(2n)! \geq 2^n \times n!$ for all integers $n \geq 1$. **[5]**

19 Prove by induction that $\sum_{r=1}^{n} r^2 > \frac{n^3}{3}$ for all integers $n \geq 1$. **[5]**

20 Prove by induction that $|\sin n\theta| \leq n|\sin \theta|$ for all integers $n \geq 1$ and all real θ. **[6]**

21 The sequence u_n is defined by $u_1 = 1$ and $u_{n+1} = \sqrt{2 + u_n}$ for $n \geq 1$.

(a) Prove by induction that $u_n < 2$ for all integers $n \geq 1$. **[4]**

(b) Hence prove by induction that $u_{n+1} > u_n$ for all integers $n \geq 1$. **[5]**

22 Prove by induction that $\sum_{r=1}^{n} \frac{1}{\sqrt{r}} < 2\sqrt{n}$ for all integers $n \geq 1$. **[7]**

23 A student attempts to prove that $2^n > n^2$ for all integers $n \geq 1$.
'n = 1: 2 > 1, true. Assume $2^k > k^2$. Then $2^{k+1} > 2k^2 = k^2 + k^2 \geq k^2 + 2k + 1 = (k + 1)^2$, so the result is true for n = k + 1. So true for all $n \geq 1$.'
The statement is false. Find the values of $n \leq 5$ for which it fails, identify the step in the student's argument that is not valid for all k, and state the values of n for which the result can be proved from this argument. **[6]**

24 Prove by induction that $(1 + x)^n \geq 1 + nx + \frac{n(n-1)}{2}x^2$ for all integers $n \geq 1$, where $x \geq 0$. **[7]**

25 The Fibonacci numbers are defined by $F_1 = 1$, $F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$. Prove by strong induction that $F_n \geq (\frac{3}{2})^{n-2}$ for all integers $n \geq 1$. **[8]**

26 Let $S_n = \frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{2n}$. Prove by induction that $S_n \geq \frac{1}{2}$ for all integers $n \geq 1$. **[7]**

27 Use the result $(1 + x)^m \geq 1 + mx$ for $x > -1$ and integers $m \geq 1$ to prove by induction that $n! \leq (\frac{n + 1}{2})^n$ for all integers $n \geq 1$. **[10]**

28 Prove by induction that $\sum_{r=1}^{n} \frac{1}{r^3} \leq \frac{5}{4} - \frac{1}{2n^2}$ for all integers $n \geq 2$. **[9]**

29 Prove by induction that $C(2n, n) \geq \frac{4^n}{2n}$ for all integers $n \geq 1$, where $C(2n, n) = \frac{(2n)!}{(n!)^2}$. **[10]**

30 The sequence u_n is defined by $u_1 = 1$, $u_2 = 3$ and $u_{n+2} = u_{n+1} + 2u_n$ for $n \geq 1$.
Prove by strong induction that $2^{n-1} \leq u_n \leq 2^n$ for all integers $n \geq 1$. **[9]**

Further Proof Techniques

1 Prove that $n^2 + 3n$ is even for every positive integer n. **[3]**

2 A student claims that $n^2 + n + 11$ is a prime number for every positive integer n. Disprove this claim by finding a counterexample. **[2]**

3 Show by a counterexample that the statement 'if $x^2 > 9$ then $x > 3$' is false. **[2]**

4 Given that $x + y = 1$, prove that $x^2 + y = y^2 + x$. **[3]**

5 Prove that the product of two odd integers is always odd. **[3]**

6 You are asked to prove by contradiction that $\sqrt{3}$ is irrational. Write down the assumption you would make at the start, and the first deduction from it. **[2]**

7 Prove that the sum of two consecutive even powers of 3, that is $3^{2k} + 3^{2k+2}$ where k is a positive integer, is always a multiple of 10. **[3]**

8 Prove that the square of any odd integer leaves remainder 1 when divided by 4. **[4]**

9 Prove by exhaustion that if n is an integer that is not divisible by 3, then $n^2 - 1$ is divisible by 3. **[4]**

10 Prove by contradiction that there are no integers m and n such that $6m + 9n = 100$. **[4]**

11 Prove by contradiction that $\sqrt{2}$ is irrational. **[5]**

12 Prove that for all positive real numbers p and q, $\frac{p}{q} + \frac{q}{p} \geq 2$. **[4]**

13 Prove that, for all real $k > 0$, $\frac{3k + 1}{3k + 2} > \frac{3k}{3k + 1}$. **[4]**

14 Prove by contradiction that $\sqrt{3}$ is irrational. You may assume that if p^2 is divisible by 3, where p is an integer, then p is divisible by 3. **[6]**

15 Prove that, if p is a prime number greater than 3, then $p^2 - 1$ is divisible by 24. **[6]**

16 Prove by contradiction that, for all real θ, $3 \cos \theta + 4 \sin \theta \leq 5$. **[5]**

17 Prove that the difference of the squares of two integers can never be of the form $4k + 2$, where k is an integer. **[6]**

18 Prove that, if a and b are distinct positive integers, then $2(a^2 + b^2)$ can be written as the sum of the squares of two distinct positive integers. **[5]**

19 Prove by contradiction that $\log_2 3$ is irrational. **[5]**

20 Prove that, for all positive real numbers a and b, $a^3 + b^3 \geq a^2b + ab^2$. **[5]**

21 Given that $\sqrt{3}$ is irrational, prove by contradiction that $\sqrt{2} + \sqrt{3}$ is irrational. **[7]**

22 Prove that there are infinitely many prime numbers. **[7]**

23 Consider the following 'proof' that 1 = 2.

'Let a = b. Then $a^2 = ab$. So $a^2 - b^2 = ab - b^2$, and factorising, $(a + b)(a - b) = b(a - b)$. Dividing by $(a - b)$ gives $a + b = b$. Since a = b, this gives $2b = b$, so $2 = 1$.'

A second student tests the formula $n^2 + n + 41$ for n = 1, 2, ..., 10, finds a prime every time, and concludes that the formula always gives a prime.

(a) Identify the error in the first 'proof'. **[3]**

(b) Explain why the second student's argument is not a proof, and find a value of n that disproves the conclusion. **[3]**

24 Prove that, for all real numbers a, b and c, $a^2 + b^2 + c^2 \geq ab + bc + ca$. **[6]**

25 The product of four consecutive positive integers has 1 added to it.

(a) Prove that the result is always a perfect square. **[5]**

(b) Hence write $10 \times 11 \times 12 \times 13 + 1$ as a perfect square. **[1]**

26 The positive integers a, b and c satisfy $a^2 + b^2 = c^2$. Prove by contradiction that a and b cannot both be odd. **[6]**

27 It is given that $\sqrt{2}$ is irrational. Let $x = \sqrt{2}^{\sqrt{2}}$.

By considering x, and $x^{\sqrt{2}}$, prove that an irrational number raised to an irrational power can be rational. **[8]**

28 Prove, without using induction, that the sum of the cubes of any three consecutive positive integers is divisible by 9. **[8]**

29 Prove by contradiction that there are infinitely many prime numbers of the form $4k + 3$, where k is a non-negative integer. **[9]**

30 The rational root theorem states that if a polynomial with integer coefficients has a rational root $\frac{p}{q}$ in lowest terms, then p divides the constant term and q divides the leading coefficient.

(a) Show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, and hence show that $x = \cos 20°$ is a root of $8x^3 - 6x - 1 = 0$. **[4]**

(b) Use the theorem to prove that $\cos 20°$ is irrational. **[5]**

Induction with Derivatives

1 Given that $y = e^{3x}$, prove by induction that $\frac{d^n y}{d x^n} = 3^n e^{3x}$ for all integers $n \geq 1$. **[3]**

2 Given that $y = e^{-2x}$, find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$. Hence conjecture a formula for $\frac{d^n y}{dx^n}$. **[3]**

3 Given that $y = \ln(1 + x)$, find the first three derivatives of y with respect to x. Hence conjecture a formula for $\frac{d^n y}{dx^n}$. **[3]**

4 Given that $y = \cos 2x$, find $\frac{d^4 y}{dx^4}$ in terms of y. Hence find the value of $\frac{d^{20} y}{dx^{20}}$ at $x = 0$. **[3]**

5 Given that $y = x^4$, find the value of $\frac{d^7 y}{dx^7}$, and explain your answer. **[2]**

6 Let $y = (1 - x)^{-1}$. It is given that, for some integer $k \geq 1$, $\frac{d^k y}{dx^k} = k!(1 - x)^{-(k+1)}$.

By differentiating, show that $\frac{d^{k+1} y}{dx^{k+1}} = (k + 1)!(1 - x)^{-(k+2)}$. **[2]**

7 Prove by induction that $\frac{d^n(\sin x)}{dx^n} = \sin\left(x + \frac{n\pi}{2}\right)$ for all integers $n \geq 1$. **[4]**

8 Given that $y = \cos 3x$, prove by induction that $\frac{d^n y}{dx^n} = 3^n \cos\left(3x + \frac{n\pi}{2}\right)$ for all integers $n \geq 1$. **[5]**

9 Given that $y = x e^x$, prove by induction that $\frac{d^n y}{dx^n} = (x + n)e^x$ for all integers $n \geq 1$. **[4]**

10 Given that $y = \frac{1}{1 + x}$, prove by induction that $\frac{d^n y}{dx^n} = \frac{(-1)^n n!}{(1 + x)^{n+1}}$ for all integers $n \geq 1$. **[5]**

11 Given that $y = \ln(1 - 2x)$ for $x < \frac{1}{2}$, prove by induction that $\frac{d^n y}{dx^n} = -\frac{2^n(n - 1)!}{(1 - 2x)^n}$ for all integers $n \geq 1$. **[5]**

12 Given that $y = 3x^3 + 2x - 5$, prove by induction that $\frac{d^n y}{dx^n} = 0$ for all integers $n \geq 4$.

Hence write down the value of $\frac{d^{10} y}{dx^{10}}$. **[4]**

13 Let $y = e^{-x/2}$.

(a) Prove by induction that $\frac{d^n y}{dx^n} = \left(-\frac{1}{2}\right)^n e^{-x/2}$ for all integers $n \geq 1$. **[4]**

(b) Hence find the exact value of $\frac{d^6 y}{dx^6}$ when $x = 2 \ln 2$. **[1]**

14 Given that $y = x e^{2x}$, prove by induction that $\frac{d^n y}{dx^n} = 2^{n-1}(2x + n)e^{2x}$ for all integers $n \geq 1$. **[6]**

15 Let $y = \frac{1}{(1 + x)(2 + x)}$.

(a) Express y in partial fractions. **[2]**

(b) Hence prove by induction that $\frac{d^n y}{dx^n} = (-1)^n n! \left[\frac{1}{(1 + x)^{n+1}} - \frac{1}{(2 + x)^{n+1}}\right]$ for all integers $n \geq 1$. **[4]**

16 Given that $y = e^{-x} \sin x$, prove by induction that $\frac{d^n y}{d x^n} = (\sqrt{2})^n e^{-x} \sin (x + \frac{3n\pi}{4})$ for all integers $n \geq 1$. **[7]**

17 Given that $y = x^2 e^x$, prove by induction that $\frac{d^n y}{d x^n} = [x^2 + 2nx + n(n - 1)]e^x$ for all integers $n \geq 1$. **[6]**

18 Given that $y = \sin^2 x$, show that $\frac{d^n y}{d x^n} = 2^{n-1} \sin (2x + \frac{(n - 1)\pi}{2})$ for all integers $n \geq 1$. **[5]**

19 A student attempts to prove that, if $y = x e^{3x}$, then $\frac{d^n y}{d x^n} = 3^n x e^{3x}$ for all integers $n \geq 1$. The student writes:

'Assume $\frac{d^k y}{d x^k} = 3^k x e^{3x}$. Differentiating, $\frac{d^{k+1} y}{d x^{k+1}} = 3^k \times 3x e^{3x} = 3^{k+1} x e^{3x}$, which is the formula with $n = k + 1$, so it is true for all n.'

(a) Show that the formula is false when $n = 1$. **[2]**

(b) Identify two errors in the student's argument. **[2]**

(c) A correct formula is $\frac{d^n y}{d x^n} = 3^{n-1}(3x + n)e^{3x}$. Show that, if this is true for $n = k$, then it is true for $n = k + 1$. **[2]**

20 Let $y = \sin (2x + \frac{\pi}{3})$.

(a) Prove by induction that $\frac{d^n y}{d x^n} = 2^n \sin (2x + \frac{\pi}{3} + \frac{n\pi}{2})$ for all integers $n \geq 1$. **[5]**

(b) Hence find the exact value of $\frac{d^{10} y}{d x^{10}}$ when $x = 0$. **[2]**

21 Let $y = x \sin x$.

(a) Prove by induction that $\frac{d^n y}{d x^n} = x \sin (x + \frac{n\pi}{2}) + n \sin (x + \frac{(n - 1)\pi}{2})$ for all integers $n \geq 1$. **[7]**

(b) Hence find the exact value of $\frac{d^{10} y}{d x^{10}}$ when $x = \frac{\pi}{2}$. **[2]**

22 Let $y = \arctan x$.

(a) Prove by induction that $(1 + x^2) \frac{d^{n+1} y}{d x^{n+1}} + 2nx \frac{d^n y}{d x^n} + n(n - 1) \frac{d^{n-1} y}{d x^{n-1}} = 0$ for all integers $n \geq 1$, where $\frac{d^0 y}{d x^0}$ means y. **[5]**

(b) Hence find the values of $\frac{d^3 y}{d x^3}$ and $\frac{d^5 y}{d x^5}$ when $x = 0$. **[3]**

(c) Hence write down the Maclaurin series of $\arctan x$ up to and including the term in x^5. **[1]**

23 Let $y = \sin^3 x$.

(a) By expanding $\sin (2x + x)$, show that $\sin 3x = 3 \sin x - 4 \sin^3 x$. Hence express $\sin^3 x$ in terms of $\sin x$ and $\sin 3x$. **[3]**

(b) Prove by induction that $\frac{d^n y}{d x^n} = \frac{3}{4} \sin (x + \frac{n\pi}{2}) - \frac{3^n}{4} \sin (3x + \frac{n\pi}{2})$ for all integers $n \geq 1$. **[5]**

24 Let $y = (x + 1)e^{-x}$.

(a) Find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$, simplifying each. **[2]**

(b) Conjecture a formula for $\frac{d^n y}{dx^n}$. **[1]**

(c) Prove your conjecture by induction. **[4]**

(d) Hence find the value of $\frac{d^{10}y}{dx^{10}}$ when $x = 0$. **[1]**

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25 A student attempts to prove that, if $y = 5e^{4x}$, then $\frac{d^n y}{dx^n} = 5 \times 4^{n+1} e^{4x}$ for all integers $n \geq 1$. The student writes:

'Assume $\frac{d^k y}{dx^k} = 5 \times 4^{k+1} e^{4x}$. Differentiating, $\frac{d^{k+1} y}{dx^{k+1}} = 5 \times 4^{k+2} e^{4x}$, which is the formula with $n = k + 1$. So the formula is true for all n by induction.'

(a) Explain the error in the student's argument, and show that the conclusion is false. **[3]**

(b) State the correct formula for $\frac{d^n y}{dx^n}$ and prove it by induction. **[4]**

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26 Let $y = (1 + x)^p$, where p is a real constant and $x > -1$.

(a) Prove by induction that $\frac{d^n y}{dx^n} = p(p - 1)(p - 2) \ldots (p - n + 1)(1 + x)^{p-n}$ for all integers $n \geq 1$. **[5]**

(b) Hence show that the coefficient of x^n in the Maclaurin series of y is $\frac{p(p - 1) \ldots (p - n + 1)}{n!}$. **[2]**

(c) Use this to find the coefficient of x^3 in the expansion of $(1 + x)^{1/2}$. **[1]**

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27 Let $y = e^{2 \arcsin x}$ for $-1 < x < 1$.

(a) Show that $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 4y = 0$. **[3]**

(b) Prove by induction that, for all integers $n \geq 0$, $(1 - x^2)\frac{d^{n+2}y}{dx^{n+2}} - (2n + 1)x\frac{d^{n+1}y}{dx^{n+1}} - (n^2 + 4)\frac{d^n y}{dx^n} = 0$, where $\frac{d^0 y}{dx^0}$ means y. **[6]**

(c) Hence find the Maclaurin series of y up to and including the term in x^4. **[3]**

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28 Let $y = \tan x$.

(a) Show that $\frac{dy}{dx} = 1 + y^2$, and hence that $\frac{d^2y}{dx^2} = 2y(1 + y^2)$. **[3]**

(b) Prove by induction that, for all integers $n \geq 1$, $\frac{d^n y}{dx^n}$ is a polynomial in y of degree $n + 1$ whose leading coefficient is n!. **[6]**

(c) Find $\frac{d^3y}{dx^3}$ as a polynomial in y, and hence show that $\tan x = x + \frac{x^3}{3} + \ldots$ near $x = 0$. **[2]**

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29 Let $y = e^{x^2}$, and let $\frac{d^0 y}{dx^0}$ mean y.

(a) Prove by induction that, for all integers $n \geq 1$, $\frac{d^{n+1}y}{dx^{n+1}} = 2x\frac{d^n y}{dx^n} + 2n\frac{d^{n-1}y}{dx^{n-1}}$. **[5]**

(b) Hence find the value of $\frac{d^6 y}{dx^6}$ at $x = 0$. **[3]**

(c) Prove by induction that $\frac{d^{2m}y}{dx^{2m}} = \frac{(2m)!}{m!}$ at $x = 0$ for all integers $m \geq 1$. **[4]**

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30 Let $y = e^{ax} \sin bx$, where a and b are positive constants. Let $r = \sqrt{a^2 + b^2}$ and let φ be the acute angle such that $\cos \varphi = \frac{a}{r}$ and $\sin \varphi = \frac{b}{r}$.

(a) Prove by induction that $\frac{d^n y}{d x^n} = r^n e^{ax} \sin (bx + n\varphi)$ for all integers $n \geq 1$. **[7]**

(b) Now let $y = \exp(\sqrt{3}x) \sin x$. Write down the values of r and φ. **[2]**

(c) Hence show that $\frac{d^{12} y}{d x^{12}} = 4096y$, and find the exact value of $\frac{d^6 y}{d x^6}$ when $x = \frac{\pi}{2}$. **[3]**