

Polar Coordinate Systems

1 (2 marks)

$$x = 4\cos \frac{\pi}{3} = 2, y = 4\sin \frac{\pi}{3} = 2\sqrt{3}$$

M1 A1

Answer: (2, $2\sqrt{3}$)

2 marks

2 (2 marks)

$$r = \sqrt{3^2 + 3^2} = 3\sqrt{2}, \tan \theta = \frac{3}{3} = 1 \text{ with B in the first quadrant, so } \theta = \frac{\pi}{4}$$

M1 A1

Answer: ($3\sqrt{2}, \frac{\pi}{4}$)

2 marks

3 (1 mark)

$$\frac{7\pi}{4} - 2\pi = -\frac{\pi}{4}$$

B1

Answer: ($5, -\frac{\pi}{4}$)

1 mark

4 (2 marks)

(a) Every point is 3 units from the pole: a circle with centre O and radius 3

B1

Answer: Circle, centre O, radius 3

1 mark

(b) A half-line from O making an angle of $\frac{\pi}{4}$ with the initial line

B1

Answer: Half-line from O at angle $\frac{\pi}{4}$

1 mark

5 (2 marks)

OA is along the initial line and OB is perpendicular to it, so $AB^2 = 3^2 + 4^2 = 25$

M1

AB = 5

A1

Answer: 5

2 marks

6 (2 marks)

$$x = -3\cos \frac{\pi}{2} = 0, y = -3\sin \frac{\pi}{2} = -3$$

M1 A1

Answer: (0, -3)

2 marks

7 (3 marks)

$$\text{Angle AOB} = \frac{5\pi}{6} - \frac{\pi}{6} = \frac{2\pi}{3}$$

B1

$$AB^2 = 6^2 + 6^2 - 2(6)(6)\cos \frac{2\pi}{3} = 108$$

M1

$$AB = 6\sqrt{3}$$

A1

Answer: $6\sqrt{3}$

3 marks

8 (3 marks)

$$\text{Angle AOB} = \frac{5\pi}{12} - \frac{\pi}{12} = \frac{\pi}{3}$$

B1

$$\text{Area} = \frac{1}{2}(5)(8)\sin \frac{\pi}{3}$$

M1

$$= 10\sqrt{3}$$

A1

Answer: $10\sqrt{3}$

3 marks

9 (2 marks)

$$x = -2\cos \frac{2\pi}{3} = -2(-\frac{1}{2}) = 1$$

M1

$$y = -2\sin \frac{2\pi}{3} = -\sqrt{3}, \text{ so Q is } (1, -\sqrt{3})$$

A1

Answer: (1, $-\sqrt{3}$)

2 marks

10 (3 marks)

$$r = \sqrt{4 + 12} = 4$$

B1

$\tan \theta = -\sqrt{3}$ and R is in the second quadrant

M1

$$\theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

A1

Answer: $(4, \frac{2\pi}{3})$

3 marks

11 (3 marks)

$$\text{Angle POQ} = \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$$

B1

$$PQ^2 = 4^2 + 2^2 - 2(4)(2)\cos \frac{\pi}{3} = 12$$

M1

$$PQ = 2\sqrt{3}$$

A1

Answer: $2\sqrt{3}$

3 marks

12 (2 marks)

A negative r means moving a distance |r| in the direction opposite to θ , which is the same as adding π to the angle

E1

$(-3, \frac{\pi}{4}) = (3, \frac{\pi}{4} + \pi) = (3, \frac{5\pi}{4})$, so they are the same point

E1

Answer: They are the same point

2 marks

13 (3 marks)

Subtract 2π from the angle: $(3, -\frac{4\pi}{3})$

B1

Negative radius, add or subtract π to the angle: $(-3, -\frac{\pi}{3})$

B1

Another correct pair, e.g. $(-3, \frac{5\pi}{3})$

B1

Answer: e.g. $(3, -\frac{4\pi}{3})$, $(-3, -\frac{\pi}{3})$, $(-3, \frac{5\pi}{3})$

3 marks

14 (4 marks)

R is a quarter of the annulus between the circles $r = 2$ and $r = 4$, in the first quadrant (see sketch)

B1

$$\text{Area} = \frac{1}{4}(\pi(4)^2 - \pi(2)^2)$$

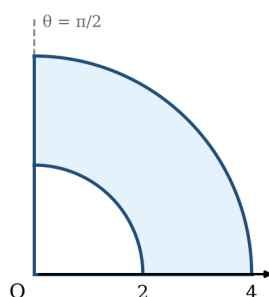
M1 A1

$$= 3\pi$$

A1

Answer: 3π

4 marks



15 (4 marks)

$$\text{Area} = \frac{1}{2}(4)(6)\sin(\theta - \frac{\pi}{12}) = 12 \sin(\theta - \frac{\pi}{12})$$

M1

$$12 \sin(\theta - \frac{\pi}{12}) = 6\sqrt{3}, \text{ so } \sin(\theta - \frac{\pi}{12}) = \frac{\sqrt{3}}{2}$$

M1

$$\theta - \frac{\pi}{12} = \frac{\pi}{3} \text{ (the other solution } \frac{2\pi}{3} \text{ gives } \theta > \frac{\pi}{2})$$

A1

$$\theta = \frac{5\pi}{12}$$

A1

Answer: $\theta = \frac{5\pi}{12}$

4 marks

16 (4 marks)

Bearing 060° is $\frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$ from East, so A is $(12, \frac{\pi}{6})$ M1

Bearing 150° is $\frac{\pi}{2} - \frac{5\pi}{6} = -\frac{\pi}{3}$ from East, so B is $(9, -\frac{\pi}{3})$ M1

Angle AOB = $\frac{\pi}{6} + \frac{\pi}{3} = \frac{\pi}{2}$, so $AB^2 = 12^2 + 9^2 = 225$ M1

$AB = 15$ km A1

Answer: 15 km 4 marks

17 (3 marks)

$y = 4 \sin \theta = -2\sqrt{2}$, so $\sin \theta = -\frac{\sqrt{2}}{2}$ M1

$\sin \theta$ is negative so θ is in the third or fourth quadrant, with reference angle $\frac{\pi}{4}$ M1

$\theta = \frac{5\pi}{4}$ or $\frac{7\pi}{4}$ A1

Answer: $\theta = \frac{5\pi}{4}, \frac{7\pi}{4}$ 3 marks

18 (4 marks)

$OA = OB = 2$, so a rhombus has OD as the diagonal bisecting angle AOB M1

Angle AOD = $\frac{\pi}{3}$ so $\theta = \frac{\pi}{3}$ A1

Triangle OAD is equilateral ($OA = AD = 2$ and angle AOD = $\frac{\pi}{3}$) or $OD = 2(2)\cos \frac{\pi}{3}$ M1

$OD = 2$, so D is $(2, \frac{\pi}{3})$ A1

Answer: $(2, \frac{\pi}{3})$ 4 marks

19 (5 marks)

Area = $\frac{1}{2}ab = 30$, so $ab = 60$ M1 A1

$AB^2 = a^2 + b^2 = 169$ M1

$(a + b)^2 = 169 + 120 = 289$ so $a + b = 17$ and $(b - a)^2 = 169 - 120 = 49$ so $b - a = 7$ M1

$a = 5, b = 12$ A1

Answer: $a = 5, b = 12$ 5 marks

20 (5 marks)

Angle AOB = $\frac{2\pi}{3} - \frac{\pi}{6} = \frac{\pi}{2}$, so OA is perpendicular to OB and the rectangle is well defined B1

OC is the diagonal: $OC^2 = 3^2 + 4^2 = 25$, $r = 5$ M1 A1

Angle AOC = $\arctan \frac{4}{3} = 0.9273$ M1

$\theta = \frac{\pi}{6} + 0.9273 = 1.45$ (3 s.f.), so C is $(5, 1.45)$ A1

Answer: (5, 1.45) 5 marks

21 (6 marks)

Angle AOB = $\frac{5\pi}{6} - \frac{\pi}{6} = \frac{2\pi}{3}$ B1

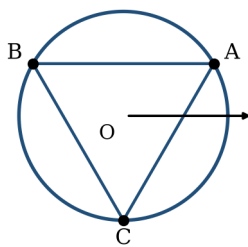
Angle BOC = $\frac{3\pi}{2} - \frac{5\pi}{6} = \frac{2\pi}{3}$ and angle COA = $2\pi - \frac{3\pi}{2} + \frac{\pi}{6} = \frac{2\pi}{3}$ B1

$AB^2 = 2^2 + 2^2 - 2(2)(2)\cos \frac{2\pi}{3} = 12$, so $AB = 2\sqrt{3}$; likewise $BC = CA = 2\sqrt{3}$ M1 A1

Area = $\frac{1}{2}(2\sqrt{3})^2 \sin \frac{\pi}{3}$ or $3 \times \frac{1}{2}(2)(2)\sin \frac{2\pi}{3}$ M1

$= 3\sqrt{3}$ A1

Answer: $3\sqrt{3}$ 6 marks

**22 (7 marks)**

(a) Adjacent vertices are $\frac{2\pi}{6} = \frac{\pi}{3}$ apart in angle

$$C: \frac{\pi}{12} + \frac{2\pi}{3} = \frac{3\pi}{4}, E: \frac{\pi}{12} + \frac{4\pi}{3} = \frac{17\pi}{12}$$

M1 A1

Answer: $C(4, \frac{3\pi}{4}), E(4, \frac{17\pi}{12})$

2 marks

(b) Six congruent triangles, each with area $\frac{1}{2}(4)(4)\sin \frac{\pi}{3}$

M1 A1

$$\text{Area} = 6 \times 4\sqrt{3} = 24\sqrt{3}$$

A1

Answer: $24\sqrt{3}$

3 marks

(c) Angle AOC = $\frac{2\pi}{3}$, $AC^2 = 16 + 16 - 32\cos \frac{2\pi}{3} = 48$

M1

$$AC = 4\sqrt{3}$$

A1

Answer: $4\sqrt{3}$

2 marks

23 (6 marks)

$$[OAB] = \frac{1}{2}(3)(5)\sin \frac{\pi}{3} = \frac{15\sqrt{3}}{4}$$

M1 A1

$$[OBC] = \frac{1}{2}(5)(4)\sin \frac{\pi}{3} = 5\sqrt{3}$$

A1

$$[OAC] = \frac{1}{2}(3)(4)\sin \frac{2\pi}{3} = 3\sqrt{3}$$

A1

B lies beyond AC as seen from O, so $[ABC] = [OAB] + [OBC] - [OAC]$

M1

$$= \frac{15\sqrt{3}}{4} + 5\sqrt{3} - 3\sqrt{3} = \frac{23\sqrt{3}}{4}$$

A1

Answer: $\frac{23\sqrt{3}}{4}$

6 marks

24 (4 marks)

Let M be the midpoint of OA. Since $OB = AB$, BM is perpendicular to OA and $OM = 3$

M1 A1

In right-angled triangle OMB, $\cos \theta = \frac{OM}{OB} = \frac{3}{r}$

M1

$$\text{So } r = \frac{3}{\cos \theta} = 3 \sec \theta$$

A1

Answer: $r = 3 \sec \theta$

4 marks

25 (5 marks)

Cartesian coordinates: $A(a \cos \alpha, a \sin \alpha), B(b \cos \beta, b \sin \beta)$

B1

$$AB^2 = (b \cos \beta - a \cos \alpha)^2 + (b \sin \beta - a \sin \alpha)^2$$

M1

$$= b^2(\cos^2 \beta + \sin^2 \beta) + a^2(\cos^2 \alpha + \sin^2 \alpha) - 2ab(\cos \alpha \cos \beta + \sin \alpha \sin \beta)$$

M1 A1

$$= a^2 + b^2 - 2ab \cos(\alpha - \beta), \text{ using } \cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

A1

Answer: $AB = \sqrt{a^2 + b^2 - 2ab \cos(\alpha - \beta)}$

5 marks

26 (6 marks)

Angle AOB = $\frac{\pi}{2}$, so AB is a diameter of C (angle in a semicircle)

M1 A1

$AB = \sqrt{6^2 + 8^2} = 10$, so the radius is 5

A1

Centre is the midpoint of AB: Cartesian (3, 4)

M1

$r = \sqrt{3^2 + 4^2} = 5$, $\theta = \arctan \frac{4}{3} = 0.927$

M1 A1

Answer: Centre (5, 0.927), radius 5

6 marks

27 (9 marks)

(a) $AB^2 = 5^2 + 3^2 - 2(5)(3)\cos \theta$

M1

$= 34 - 30 \cos \theta$

A1

Answer: $AB^2 = 34 - 30 \cos \theta$

2 marks

(b) AB^2 is least when $\cos \theta = 1$: $AB^2 = 4$, $AB = 2$

M1 A1

AB^2 is greatest when $\cos \theta = -1$: $AB^2 = 64$, $AB = 8$

A1

Answer: Least 2, greatest 8

3 marks

(c) $34 - 30 \cos \theta = 49$, so $\cos \theta = -\frac{1}{2}$

M1 A1

$\theta = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$

M1

$\theta = 2.09$ or 4.19 (3 s.f.)

A1

Answer: $\theta = 2.09, 4.19$

4 marks

28 (8 marks)

$PA^2 = 2^2 + 2^2 - 2(2)(2)\cos \theta = 8 - 8 \cos \theta$

M1 A1

$PB^2 = 8 - 8 \cos(\theta - \frac{2\pi}{3})$, $PC^2 = 8 - 8 \cos(\theta - \frac{4\pi}{3})$

M1 A1

Sum = $24 - 8[\cos \theta + \cos(\theta - \frac{2\pi}{3}) + \cos(\theta - \frac{4\pi}{3})]$

M1

$\cos(\theta - \frac{2\pi}{3}) = \cos \theta \cos \frac{2\pi}{3} + \sin \theta \sin \frac{2\pi}{3} = -\frac{1}{2}\cos \theta + \frac{\sqrt{3}}{2}\sin \theta$

M1

$\cos(\theta - \frac{4\pi}{3}) = -\frac{1}{2}\cos \theta - \frac{\sqrt{3}}{2}\sin \theta$

A1

The bracket is $\cos \theta - \cos \theta = 0$, so the sum is 24 whatever θ

A1

Answer: 24

8 marks

29 (8 marks)

The statement that O lies inside ABC means the three angles AOB, BOC, COA at O add to 2π

E1

Hence ABC is split into triangles OAB, OBC and OCA, so $[ABC] = [OAB] + [OBC] + [OCA]$

M1 A1

Angle AOB = $\theta_2 - \theta_1$, so $[OAB] = \frac{1}{2}r_1r_2 \sin(\theta_2 - \theta_1)$

M1 A1

Angle BOC = $\theta_3 - \theta_2$, so $[OBC] = \frac{1}{2}r_2r_3 \sin(\theta_3 - \theta_2)$

M1 A1

Angle COA = $2\pi - (\theta_3 - \theta_1)$, so $[OCA] = \frac{1}{2}r_3r_1 \sin(\theta_1 + 2\pi - \theta_3)$

A1

Answer: Area = $\frac{1}{2}[r_1r_2 \sin(\theta_2 - \theta_1) + r_2r_3 \sin(\theta_3 - \theta_2) + r_3r_1 \sin(\theta_1 + 2\pi - \theta_3)]$

8 marks

30 (8 marks)

(a) Triangle OAB is isosceles with $OA = OB$, so OM bisects angle AOB and is perpendicular to AB

M1 A1

Hence the angle of M is $\frac{\varphi}{2}$

A1

In right-angled triangle OMA, $OM = OA \cos \frac{\varphi}{2} = 6 \cos \frac{\varphi}{2}$

A1

Answer: $M = (6 \cos \frac{\varphi}{2}, \frac{\varphi}{2})$

4 marks

(b) Area of OAM = $\frac{1}{2}(6)(6 \cos \frac{\varphi}{2})\sin \frac{\varphi}{2}$

M1

$$= 9 \sin \varphi \text{ (using } 2 \sin \frac{\varphi}{2} \cos \frac{\varphi}{2} = \sin \varphi \text{)}$$

A1

Maximum when $\sin \varphi = 1$

M1

Maximum area 9 when $\varphi = \frac{\pi}{2}$

A1

Answer: Maximum 9 at $\varphi = \frac{\pi}{2}$

4 marks

Converting Between Polar and Cartesian Coordinates

1 (1 mark)

$$x^2 + y^2 = r^2, \text{ so } r^2 = 25, r = 5$$

B1

Answer: $r = 5$

1 mark

2 (1 mark)

$$r^2 = x^2 + y^2 = 16$$

B1

Answer: $x^2 + y^2 = 16$

1 mark

3 (2 marks)

$$\tan \theta = \frac{y}{x} = 1$$

M1

$$\theta = \frac{\pi}{4}$$

A1

Answer: $\theta = \frac{\pi}{4}$

2 marks

4 (2 marks)

$$x = r \cos \theta, \text{ so } r \cos \theta = 3$$

M1

$$r = 3 \sec \theta$$

A1

Answer: $r = 3 \sec \theta$

2 marks

5 (1 mark)

$$y = r \sin \theta, \text{ so } y = 2$$

B1

Answer: $y = 2$

1 mark

6 (2 marks)

$$\text{Multiply by } r: r^2 = 6r \cos \theta$$

M1

$$x^2 + y^2 = 6x$$

A1

Answer: $x^2 + y^2 = 6x$

2 marks

7 (3 marks)

$$\text{Substitute } x = r \cos \theta, y = r \sin \theta: r^2 \sin \theta \cos \theta = 8$$

M1 A1

$$2 \sin \theta \cos \theta = \sin 2\theta, \text{ so } r^2 \sin 2\theta = 16$$

A1

Answer: $r^2 \sin 2\theta = 16$

3 marks

8 (3 marks)

$$r^2 \sin^2 \theta = 4r \cos \theta$$

M1

$$r \sin^2 \theta = 4 \cos \theta \text{ (} r \neq 0 \text{)}$$

A1

$$r = \frac{4 \cos \theta}{\sin^2 \theta}$$

A1

Answer: $r = \frac{4 \cos \theta}{\sin^2 \theta}$

3 marks

9 (3 marks)

Substitute: $r \sin \theta = 2r \cos \theta + 3$

M1

$r(\sin \theta - 2 \cos \theta) = 3$

M1

$$r = \frac{3}{\sin \theta - 2 \cos \theta}$$

A1

Answer: $r = \frac{3}{\sin \theta - 2 \cos \theta}$

3 marks

10 (3 marks)

Expand: $x^2 + y^2 - 6x = 0$

M1

$r^2 - 6r \cos \theta = 0$

M1

$r = 6 \cos \theta$

A1

Answer: $r = 6 \cos \theta$

3 marks

11 (3 marks)

$r^2 \cos^2 \theta - r^2 \sin^2 \theta = 4$

M1

$r^2 \cos 2\theta = 4$

M1

$r^2 = 4 \sec 2\theta$

A1

Answer: $r^2 = 4 \sec 2\theta$

3 marks

12 (3 marks)

$r \cos \theta + r \sin \theta = 2$

M1

$x + y = 2$

A1

A straight line with gradient -1

B1

Answer: $x + y = 2$, a straight line

3 marks

13 (4 marks)

When $\theta = \frac{\pi}{3}$: $r = 4 \cos \frac{\pi}{3} = 2$, so the point lies on the curve

B1

$r^2 = 4r \cos \theta$

M1

$x^2 + y^2 = 4x$

A1

(equivalently $(x - 2)^2 + y^2 = 4$)

B1

Answer: $x^2 + y^2 = 4x$

4 marks

14 (5 marks)

Multiply by r : $r^2 = 4r \cos \theta + 2r \sin \theta$

M1

$x^2 + y^2 = 4x + 2y$

A1

Complete the square: $(x - 2)^2 + (y - 1)^2 = 5$

M1 A1

Centre $(2, 1)$, radius $\sqrt{5}$

A1

Answer: Circle, centre $(2, 1)$, radius $\sqrt{5}$

5 marks

15 (4 marks)

$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$

B1

$r^2 \times r^2 = 9(r^2 \cos^2 \theta - r^2 \sin^2 \theta)$

M1

$r^4 = 9(x^2 - y^2)$

M1

$(x^2 + y^2)^2 = 9(x^2 - y^2)$

A1

Answer: $(x^2 + y^2)^2 = 9(x^2 - y^2)$

4 marks

16 (4 marks)

$$r \cos\left(\theta - \frac{\pi}{6}\right) = 4 \quad \text{M1}$$

$$r(\cos \theta \cos \frac{\pi}{6} + \sin \theta \sin \frac{\pi}{6}) = 4 \quad \text{M1}$$

$$\frac{\sqrt{3}}{2}x + \frac{1}{2}y = 4 \quad \text{A1}$$

$$\sqrt{3}x + y = 8 \quad \text{A1}$$

$$\text{Answer: } \sqrt{3}x + y = 8 \quad \text{4 marks}$$

17 (4 marks)

$$r - r \cos \theta = 2, \text{ so } r = x + 2 \quad \text{M1 A1}$$

$$x^2 + y^2 = (x + 2)^2 = x^2 + 4x + 4 \quad \text{M1}$$

$$y^2 = 4x + 4 = 4(x + 1) \quad \text{A1}$$

$$\text{Answer: } y^2 = 4(x + 1) \quad \text{4 marks}$$

18 (3 marks)

$$r^2 - 6r \cos \theta + 8r \sin \theta = 0 \quad \text{M1}$$

$$r = 6 \cos \theta - 8 \sin \theta \quad \text{M1}$$

$$a = 6, b = -8 \quad \text{A1}$$

$$\text{Answer: } r = 6 \cos \theta - 8 \sin \theta \quad \text{3 marks}$$

19 (5 marks)

$$\sin 2\theta = 2 \sin \theta \cos \theta, \text{ so } r = 2 \sin \theta \cos \theta \quad \text{B1}$$

$$\text{Multiply by } r^2: r^3 = 2(r \sin \theta)(r \cos \theta) \quad \text{M1 A1}$$

$$r^3 = 2xy, \text{ and squaring gives } r^6 = 4x^2y^2 \quad \text{M1}$$

$$(x^2 + y^2)^3 = 4x^2y^2 \quad \text{A1}$$

$$\text{Answer: } (x^2 + y^2)^3 = 4x^2y^2 \quad \text{5 marks}$$

20 (3 marks)

$$r \sin \theta = r^2 \cos^2 \theta \quad \text{M1}$$

$$r \cos^2 \theta = \sin \theta \quad (r \neq 0) \quad \text{M1}$$

$$r = \frac{\sin \theta}{\cos^2 \theta} \times \frac{1}{\cos \theta} = \tan \theta \sec \theta \quad \text{A1}$$

$$\text{Answer: } r = \tan \theta \sec \theta \quad \text{3 marks}$$

21 (4 marks)

$$r^4 = 8r^2 \sin \theta \cos \theta \quad \text{M1 A1}$$

$$r^2 = 8 \sin \theta \cos \theta \quad (r \neq 0) \quad \text{M1}$$

$$= 4 \sin 2\theta \quad \text{A1}$$

$$\text{Answer: } r^2 = 4 \sin 2\theta \quad \text{4 marks}$$

22 (5 marks)

$$AP^2 \times BP^2 = 16: [(x + 2)^2 + y^2][(x - 2)^2 + y^2] = 16 \quad \text{M1 A1}$$

$$\text{Let } u = x^2 + y^2 + 4; \text{ the factors are } u + 4x \text{ and } u - 4x, \text{ so } u^2 - 16x^2 = 16 \quad \text{M1}$$

$$(r^2 + 4)^2 - 16r^2 \cos^2 \theta = 16, \text{ so } r^4 + 8r^2 - 16r^2 \cos^2 \theta = 0 \quad \text{M1}$$

$$r^2 = 16 \cos^2 \theta - 8 = 8(2 \cos^2 \theta - 1) = 8 \cos 2\theta \quad \text{A1}$$

$$\text{Answer: } r^2 = 8 \cos 2\theta \quad \text{5 marks}$$

23 (5 marks)

$$r \cos \theta = 1 + 2 \cos \theta, \text{ so } x = 1 + 2 \frac{x}{r} \quad \text{M1 A1}$$

$$r(x - 1) = 2x \quad \text{M1}$$

$$r^2(x - 1)^2 = 4x^2 \quad \text{M1}$$

$$(x^2 + y^2)(x - 1)^2 = 4x^2 \quad \text{A1}$$

$$\text{Answer: } (x^2 + y^2)(x - 1)^2 = 4x^2 \quad \text{5 marks}$$

24 (5 marks)

$$r^6 = r^4(\cos \theta - \sin \theta)^4$$

M1 A1

$$r^2 = (\cos \theta - \sin \theta)^4 \quad (r \neq 0)$$

M1

$$(\cos \theta - \sin \theta)^2 = \cos^2 \theta - 2 \sin \theta \cos \theta + \sin^2 \theta = 1 - \sin 2\theta$$

M1

Since $r \geq 0$, $r = (1 - \sin 2\theta)$ (this is never negative)

A1

$$\text{Answer: } r = 1 - \sin 2\theta$$

5 marks

25 (5 marks)

$$2r - r \cos \theta = 4, \text{ so } 2r = x + 4$$

M1 A1

$$4r^2 = (x + 4)^2$$

M1

$$4x^2 + 4y^2 = x^2 + 8x + 16$$

M1

$$3x^2 - 8x + 4y^2 = 16$$

A1

$$\text{Answer: } 3x^2 - 8x + 4y^2 = 16$$

5 marks

26 (4 marks)

The right-hand side of $(x - 1)^2 + (y - 1)^2 = 2$ is the radius squared, so the radius is $\sqrt{2}$, not 2

E1 B1

$$r = 2 \cos \theta + 2 \sin \theta = 2\sqrt{2} \cos(\theta - \frac{\pi}{4})$$

M1

The greatest r is $2\sqrt{2}$, which is a diameter since the circle passes through O, so the radius is $\sqrt{2}$

A1

Answer: Radius is $\sqrt{2}$, not 2

4 marks

27 (9 marks)

(a) $x^2 + y^2 = r^2$ and $x = r \cos \theta$, so $(r^2 - 2ar \cos \theta)^2 = 4a^2 r^2$

M1 A1

$$r^2(r - 2a \cos \theta)^2 = 4a^2 r^2, \text{ so } r = 0 \text{ or } (r - 2a \cos \theta)^2 = 4a^2$$

M1 A1

$$r - 2a \cos \theta = \pm 2a, \text{ so } r = 2a(1 + \cos \theta) \text{ or } r = -2a(1 - \cos \theta)$$

M1 A1

The point $(-2a(1 - \cos \theta), \theta)$ is the point $(2a(1 - \cos \theta), \theta + \pi)$, and $\cos(\theta + \pi) = -\cos \theta$, so it equals $(2a(1 + \cos \varphi), \varphi)$ with $\varphi = \theta + \pi$, which lies on the first curve ($r = 0$ is the pole, also on it)

E1

$$\text{Answer: } r = 2a(1 + \cos \theta)$$

7 marks

(b) $r = 4(1 + \cos \frac{\pi}{3}) = 6$

M1

$$x = 6 \cos \frac{\pi}{3} = 3, y = 6 \sin \frac{\pi}{3} = 3\sqrt{3}, \text{ so } (3, 3\sqrt{3})$$

A1

$$\text{Answer: } (3, 3\sqrt{3})$$

2 marks

28 (7 marks)

$$\cos 3\theta = 4\cos^3 \theta - 3\cos \theta$$

B1

$$r = 4\cos^3 \theta - 3\cos \theta, \text{ so } r^4 = 4(r \cos \theta)^3 - 3(r \cos \theta)r^2$$

M1 A1

$$r^4 = 4x^3 - 3x(x^2 + y^2)$$

M1 A1

$$r^4 = 4x^3 - 3x^3 - 3xy^2$$

M1

$$(x^2 + y^2)^2 = x^3 - 3xy^2$$

A1

$$\text{Answer: } (x^2 + y^2)^2 = x^3 - 3xy^2$$

7 marks

29 (9 marks)

(a) $r + 2r \cos \theta = 6$, so $r = 6 - 2x$

M1 A1

$$x^2 + y^2 = (6 - 2x)^2 = 36 - 24x + 4x^2$$

M1

$$3x^2 - 24x - y^2 + 36 = 0$$

A1

$$3(x - 4)^2 - 48 - y^2 + 36 = 0, \text{ so } 3(x - 4)^2 - y^2 = 12$$

M1

$$\text{Divide by 12: } \frac{(x - 4)^2}{4} - \frac{y^2}{12} = 1$$

A1

$$\text{Answer: } \frac{(x - 4)^2}{4} - \frac{y^2}{12} = 1$$

6 marks

(b) Centre (4, 0)

B1

Asymptotes: $\frac{(x-4)^2}{4} = \frac{y^2}{12}, y = \pm\sqrt{3}(x-4)$

M1

Gradients $\pm\sqrt{3}$

A1

Answer: Centre (4, 0), gradients $\pm\sqrt{3}$

3 marks

30 (9 marks)

(a) The point $P(8, \frac{\pi}{3})$ is the other end of the diameter through O

B1

For a general point $Q(r, \theta)$ on C, angle $OQP = \frac{\pi}{2}$ (angle in a semicircle)

M1

In right-angled triangle OQP, $r = OP \cos(\theta - \frac{\pi}{3}) = 8 \cos(\theta - \frac{\pi}{3})$

A1

Answer: $r = 8 \cos(\theta - \frac{\pi}{3})$

3 marks

(b) $r^2 = 8r(\cos \theta \cos \frac{\pi}{3} + \sin \theta \sin \frac{\pi}{3})$

M1

$x^2 + y^2 = 4x + 4\sqrt{3}y$

M1 A1

Answer: $x^2 + y^2 = 4x + 4\sqrt{3}y$

3 marks

(c) $\theta = 0$: $r = 8 \cos(-\frac{\pi}{3}) = 4$, giving (4, 0)

M1 A1

$\theta = \frac{\pi}{2}$: $r = 8 \cos \frac{\pi}{6} = 4\sqrt{3}$, giving $(4\sqrt{3}, \frac{\pi}{2})$

A1

Answer: (4, 0) and $(4\sqrt{3}, \frac{\pi}{2})$

3 marks

Sketching Polar Curves

1 (2 marks)

Every point is at distance 3 from O

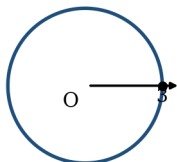
M1

A circle with centre O and radius 3

A1

Answer: Circle, centre O, radius 3

2 marks



2 (2 marks)

θ is constant, so the points lie on a straight line through O

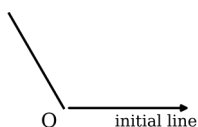
M1

A half-line from O making an angle of $\frac{2\pi}{3}$ with the initial line

A1

Answer: Half-line from O at $\frac{2\pi}{3}$ to the initial line

2 marks



3 (2 marks)

Greatest $r = 5$ when $\cos \theta = 1, \theta = 0$

B1

Least $r = 1$ when $\cos \theta = -1, \theta = \pi$

B1

Answer: Greatest 5 at $\theta = 0$; least 1 at $\theta = \pi$

2 marks

4 (2 marks)

$$\theta = \frac{\pi}{3}: r = 2 + 2\left(\frac{1}{2}\right) = 3$$

B1

$$\theta = \frac{2\pi}{3}: r = 2 + 2\left(-\frac{1}{2}\right) = 1$$

B1

Answer: 3 and 1

2 marks

5 (2 marks)The initial line ($\theta = 0$)

B1

 $\cos(-\theta) = \cos \theta$, so (r, θ) and $(r, -\theta)$ are both on C

E1

Answer: The initial line

2 marks

6 (2 marks)

$$r = \frac{\pi}{2}, \pi, 2\pi \text{ respectively}$$

B1

 r increases steadily as θ increases, so S spirals anticlockwise away from O

B1

Answer: $\frac{\pi}{2}, \pi, 2\pi$; an outward spiral

2 marks

7 (3 marks)

The curve is a circle

B1

It passes through O, where $\theta = \pm \frac{\pi}{2}$, and its centre lies on the initial line

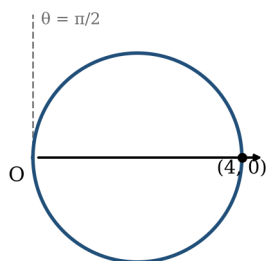
B1

Crosses the initial line at $(4, 0)$, the diameter being 4

B1

Answer: Circle with diameter 4 along the initial line, through O

3 marks

**8 (3 marks)**

The curve is a circle through O

B1

Its diameter lies along the line $\theta = \frac{\pi}{2}$

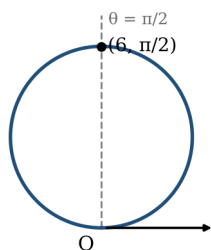
B1

Greatest distance 6 at $\theta = \frac{\pi}{2}$

B1

Answer: Circle with diameter 6 along $\theta = \frac{\pi}{2}$

3 marks

**9 (4 marks)** $r = 4$ at $\theta = 0$, giving $(4, 0)$

B1

 $r = 2$ at $\theta = \frac{\pi}{2}$ and at $\theta = \frac{3\pi}{2}$

B1

 $r = 0$ at $\theta = \pi$: the curve reaches O with a cusp

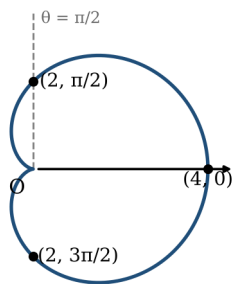
B1

Heart-shaped (cardioid), symmetrical about the initial line

B1

Answer: Cardioid, with $(4, 0)$, $(2, \frac{\pi}{2})$, $(2, \frac{3\pi}{2})$ and a cusp at O

4 marks



10 (4 marks)

$r = 5$ at $\theta = 0$, giving $(5, 0)$

B1

$r = 1$ at $\theta = \pi$, giving $(1, \pi)$

B1

r never reaches 0 (least value 1), so the curve is a smooth closed curve not passing through O

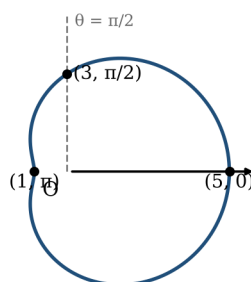
B1

Symmetrical about the initial line, larger at the right than left

B1

Answer: Smooth closed curve (a limaçon without a loop), r from 1 to 5

4 marks



11 (3 marks)

Starts at O ($\theta = 0$), then r increases with θ

B1

Passes through $(\frac{\pi}{2}, \frac{\pi}{2})$, (π, π) , $(\frac{3\pi}{2}, \frac{3\pi}{2})$

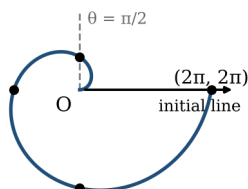
B1

Ends at $(2\pi, 2\pi)$ on the initial line: one anticlockwise turn of a spiral

B1

Answer: Spiral from O, ending at $(2\pi, 2\pi)$

3 marks



12 (3 marks)

$r \cos \theta = 2$, so $x = 2$: a vertical line

B1

It crosses the initial line at $(2, 0)$

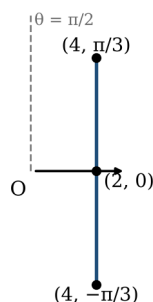
B1

Endpoints $(4, \pm\frac{\pi}{3})$, i.e. $(2, \pm 2\sqrt{3})$ in Cartesian coordinates: a segment of $x = 2$

B1

Answer: Segment of the line $x = 2$ from $(2, -2\sqrt{3})$ to $(2, 2\sqrt{3})$

3 marks



13 (4 marks)

$r^2 = 4r \cos \theta + 3r \sin \theta$, so $x^2 + y^2 = 4x + 3y$: a circle

B1

Through O; meets the initial line at (4, 0)

B1

Meets $\theta = \frac{\pi}{2}$ at $(3, \frac{\pi}{2})$

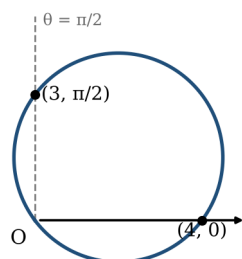
B1

Diameter = $\sqrt{4^2 + 3^2} = 5$ (greatest r)

B1

Answer: Circle through O of diameter 5

4 marks



14 (5 marks)

Greatest $|r| = 4$ when $\sin 2\theta = \pm 1$, e.g. at $\theta = \frac{\pi}{4}$

B1

$r = 0$ when $2\theta = 0, \pi, 2\pi, 3\pi$, so $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ (and 2π)

B1

For $0 < \theta < \frac{\pi}{2}$, $r > 0$ gives a petal in the first quadrant

M1

Negative r for $\frac{\pi}{2} < \theta < \pi$ gives a petal in the fourth quadrant, and so on

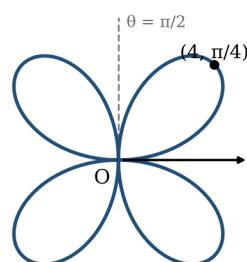
M1

Four petals of length 4, centred on $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

A1

Answer: Four-petalled rose, petals of length 4 along $y = \pm x$

5 marks



15 (5 marks)

$r^2 \geq 0$ requires $\cos 2\theta \geq 0$

M1

So $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$ and $\frac{3\pi}{4} \leq \theta \leq \frac{5\pi}{4}$

A1

Greatest $r = 2$ when $\theta = 0$ and $\theta = \pi$

B1

$r = 0$ at $\theta = \pm \frac{\pi}{4}$ so the curve passes through O at angles $\pm \frac{\pi}{4}$

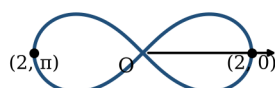
B1

Two loops, a figure of eight, symmetrical about both axes

B1

Answer: Figure of eight (lemniscate) with $r_{\max} = 2$ at $\theta = 0$ and π

5 marks



16 (5 marks)

$r = 3$ at $\theta = 0$, giving $(3, 0)$

B1

$r = 0$ when $\cos \theta = -\frac{1}{2}$: $\theta = \frac{2\pi}{3}$ and $\frac{4\pi}{3}$

M1 A1

For $\frac{2\pi}{3} < \theta < \frac{4\pi}{3}$, $r < 0$, so these points form an inner loop on the positive side of O, reaching $(-1, \pi)$, i.e. $(1, 0)$

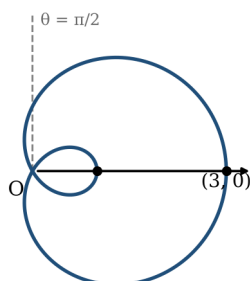
M1

Large outer loop with a smaller inner loop, symmetrical about the initial line

A1

Answer: Limaçon with inner loop; $r = 0$ at $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

5 marks

**17 (5 marks)**

$r = 3$ at $\theta = 0$, giving a maximum at $(3, 0)$

B1

$r = 0$ when $3\theta = \pm\frac{\pi}{2}$, so at $\theta = \pm\frac{\pi}{6}$

B1

This range gives one petal, symmetrical about the initial line, of length 3

M1

Rotation symmetry: the full curve has petals centred on $\theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$

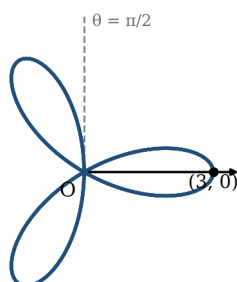
A1

Three petals

A1

Answer: One petal along the initial line of length 3; 3 petals in total

5 marks

**18 (4 marks)**

$\theta = 0$: $r = 1$, so the start is $(1, 0)$

B1

$\theta = 2\pi$: $r = e^\pi$, so the end is $(e^\pi, 2\pi)$ on the initial line

B1

r increases rapidly with θ and never reaches O, so the curve is a spiral that grows in each turn

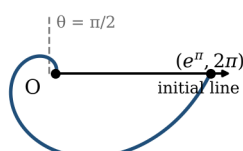
M1

One anticlockwise turn, with the gap between successive crossings of the initial line widening

A1

Answer: Spiral from $(1, 0)$ to $(e^\pi, 2\pi)$

4 marks



19 (5 marks)

$\theta = 0$: $r = 1$, giving $(1, 0)$, and $\theta = \pm \frac{\pi}{2}$: $r = 2$

B1 B1

As $\theta \rightarrow \pm\pi$, $1 + \cos \theta \rightarrow 0$ so $r \rightarrow \infty$: the curve opens to the left

B1

Cartesian: $r + r \cos \theta = 2$, $r = 2 - x$, so $x^2 + y^2 = (2 - x)^2$

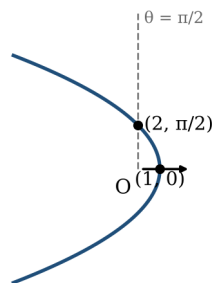
M1

$y^2 = 4 - 4x$, a parabola

A1

Answer: A parabola opening in the negative x direction, $y^2 = 4 - 4x$

5 marks

**20 (3 marks)**

$\theta = 0$: $a + b = 6$

M1

$\theta = \pi$: $a - b = 2$

M1

$a = 4$, $b = 2$

A1

Answer: $a = 4$, $b = 2$

3 marks

21 (5 marks)

Rotating by $\frac{\pi}{2}$ anticlockwise replaces θ by $\theta - \frac{\pi}{2}$

M1

$r = 2(1 + \cos(\theta - \frac{\pi}{2})) = 2(1 + \sin \theta)$

A1

C_1 : cardioid with cusp at O, pointing along the initial line

B1

C_2 : the same cardioid rotated, with maximum $r = 4$ at $\theta = \frac{\pi}{2}$

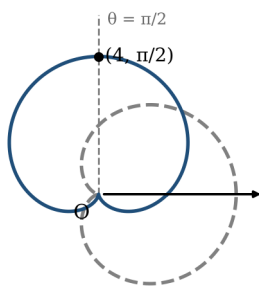
B1

Cusp of C_2 at O, on $\theta = \frac{3\pi}{2}$; C_2 symmetrical about $\theta = \frac{\pi}{2}$

B1

Answer: $r = 2(1 + \sin \theta)$

5 marks

**22 (6 marks)**

C crosses the initial line when $\theta = 0, 2\pi, 4\pi$

M1

$r = 1, 3, 5$ respectively: $(1, 0)$, $(3, 2\pi)$, $(5, 4\pi)$

A1

Consecutive points are on the same ray, so the distances are $3 - 1 = 2$ and $5 - 3 = 2$

M1 A1

Sketch: an anticlockwise spiral starting at $(1, 0)$

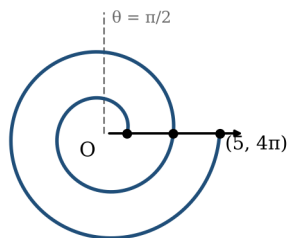
B1

Crossing the initial line at distances 1, 3, 5, the gaps between turns equal

B1

Answer: $(1, 0)$, $(3, 2\pi)$, $(5, 4\pi)$; gap 2

6 marks

**23 (5 marks)**

$$y = r \sin \theta = \frac{\sin \theta}{\theta}$$

M1

$$\text{As } \theta \rightarrow 0, \frac{\sin \theta}{\theta} \rightarrow 1, \text{ so } y \rightarrow 1$$

A1

$$x = r \cos \theta = \frac{\cos \theta}{\theta} \rightarrow \infty, \text{ so the curve approaches } y = 1 \text{ from below as } x \rightarrow \infty$$

M1

$$\text{As } \theta \rightarrow \infty, r \rightarrow 0, \text{ so } C \text{ spirals into } O$$

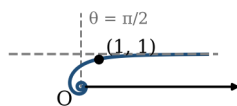
B1

Sketch: branch asymptotic to $y = 1$, then spiralling inwards to O

B1

Answer: Asymptote $y = 1$; spirals into O

5 marks

**24 (6 marks)**

$$x = r \cos \theta = 2 \cos \theta + 1$$

M1 A1

$$\text{As } \theta \rightarrow \pm \frac{\pi}{2}, \cos \theta \rightarrow 0, \text{ so } x \rightarrow 1$$

A1

$$y = r \sin \theta = 2 \sin \theta + \tan \theta \rightarrow \pm \infty, \text{ so } x = 1 \text{ is an asymptote}$$

M1

$$r = 3 \text{ at } \theta = 0, \text{ giving } (3, 0)$$

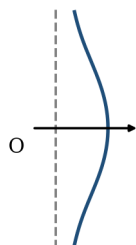
B1

Sketch: symmetrical about the initial line, with both arms approaching $x = 1$

B1

Answer: Asymptote $x = 1$

6 marks

**25 (5 marks)**

$$\text{Greatest } r = a, \text{ so } a = 4$$

B1

$$\text{For } r = a \cos n\theta \text{ with } n \text{ odd, the curve has } n \text{ petals, so } n = 5$$

M1 A1

$$\text{Tips where } \cos 5\theta = \pm 1: 5\theta = k\pi, \theta = \frac{k\pi}{5}, \text{ with } r = 4 \text{ when } k \text{ is even}$$

M1

$$r = 4: \theta = 0, \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5}; \text{ the odd } k \text{ give } r = -4, \text{ which are the same petals}$$

A1

$$\text{Answer: } a = 4, n = 5; \text{ tips } (4, \frac{2k\pi}{5}), k = 0, 1, 2, 3, 4$$

5 marks

26 (4 marks)

For θ in a range where $r < 0$, the point (r, θ) is the same as $(-r, \theta + \pi)$ M1

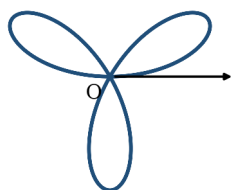
$-r = -2 \sin 3\theta = 2 \sin(3\theta + 3\pi) = 2 \sin 3(\theta + \pi)$, so $(-r, \theta + \pi)$ is the point of the curve at angle $\theta + \pi$ with a positive radius E1

So each negative- r petal coincides with a positive- r petal already drawn: only three distinct petals E1

Tips of petals at $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$ and $\frac{3\pi}{2}$; the lines of symmetry through them are $\theta = \frac{\pi}{6}, \theta = \frac{5\pi}{6}$ and A1

$\theta = \frac{\pi}{2}$ (the line through $(2, \frac{3\pi}{2})$)

Answer: 3 petals; lines of symmetry $\theta = \frac{\pi}{6}, \theta = \frac{5\pi}{6}, \theta = \frac{3\pi}{2}$ 4 marks

**27 (9 marks)**

(a) $\cos \theta = -\frac{2}{3}$ M1

$\theta = \arccos(-\frac{2}{3}) = 2.30$ A1

and $\theta = 2\pi - 2.30 = 3.98$ A1

Answer: $\theta = 2.30, 3.98$ 3 marks

(b) $\theta = 0$: $r = 5$, giving $(5, 0)$ B1

$\theta = \pi$: $r = -1$, so the point is $(1, 0)$, the inner loop reaching the positive x axis B1

Outer loop for $\theta < 2.30$ and $\theta > 3.98$, symmetrical about the initial line B1

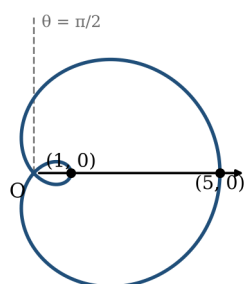
Inner loop for $2.30 < \theta < 3.98$ (where $r < 0$), lying inside the outer loop, with its tip at $(1, 0)$ B1

Answer: Limaçon with inner loop crossing the initial line at $(5, 0)$ and $(1, 0)$ 4 marks

(c) Loop needs r to change sign: $2 + k \cos \theta = 0$ has a solution if $|k| > 2$ M1

$k > 2$ or $k < -2$ A1

Answer: $|k| > 2$ 2 marks

**28 (10 marks)**

(a) $1 + 2 \cos \theta = 0$, so $\cos \theta = -\frac{1}{2}$ M1

$\theta = \frac{2\pi}{3}$ and $\frac{4\pi}{3}$ A1

As θ approaches these values, $|r| \rightarrow \infty$, so the curve has asymptotes in these directions B1

Answer: $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$ 3 marks

(b) $r + 2r \cos \theta = 2$, so $r = 2 - 2x$ M1

$x^2 + y^2 = 4(1 - x)^2$ M1

$3x^2 - 8x + 4 - y^2 = 0$ A1

$3(x - \frac{4}{3})^2 - \frac{16}{3} + 4 - y^2 = 0$, so $3(x - \frac{4}{3})^2 - y^2 = \frac{4}{3}$ A1

Answer: $3(x - \frac{4}{3})^2 - y^2 = \frac{4}{3}$

4 marks

(c) A hyperbola with centre $(\frac{4}{3}, 0)$

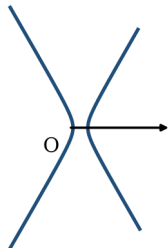
B1

$\theta = 0$: $r = \frac{2}{3}$, giving $(\frac{2}{3}, 0)$; $\theta = \pi$: $r = -2$, giving $(2, 0)$

B1 B1

Answer: Hyperbola; crosses at $(\frac{2}{3}, 0)$ and $(2, 0)$

3 marks



29 (9 marks)

(a) $r \cos \theta = 2 \sin^2 \theta$, so $x = 2 \frac{y^2}{r^2}$

M1 A1

$x(x^2 + y^2) = 2y^2$

A1

Answer: $x(x^2 + y^2) = 2y^2$

3 marks

(b) $x = r \cos \theta = 2 \sin^2 \theta \rightarrow 2$ as $\theta \rightarrow \pm \frac{\pi}{2}$

M1

$y = r \sin \theta = 2 \sin^2 \theta \tan \theta \rightarrow \pm \infty$

M1

so the curve approaches $x = 2$ as $|y| \rightarrow \infty$

A1

Answer: $x \rightarrow 2$ as $\theta \rightarrow \frac{\pi}{2}$, so $x = 2$ is an asymptote

3 marks

(c) Passes through O with $r \approx 2\theta^2$ near $\theta = 0$, so there is a cusp at O (tangent along the initial line)

B1

Symmetrical about the initial line, since r is even in θ

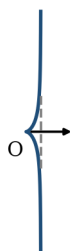
B1

Two branches (above and below) rising to the asymptote $x = 2$

B1

Answer: Cusp at O; asymptote $x = 2$

3 marks



30 (10 marks)

(a) r is least when $\cos \theta$ is greatest: $\theta = 0$, $r = \frac{12}{1.6} = 7.5$ million km

M1 A1

r is greatest when $\cos \theta = -1$: $\theta = \pi$, $r = \frac{12}{0.4} = 30$ million km

M1 A1

Answer: P: 7.5 at $\theta = 0$; A: 30 at $\theta = \pi$

4 marks

(b) $r + 0.6x = 12$, so $r = 12 - 0.6x$

M1

$x^2 + y^2 = 144 - 14.4x + 0.36x^2$

M1

$0.64x^2 + 14.4x + y^2 = 144$

A1

$0.64(x + 11.25)^2 - 81 + y^2 = 144$, so $0.64(x + 11.25)^2 + y^2 = 225$

M1

(centre $(-11.25, 0)$, semi-axes 18.75 and 15)

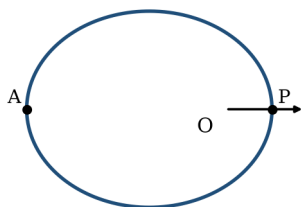
A1

$PA = 7.5 + 30 = 37.5$ million km

B1

Answer: PA = 37.5 million km

6 marks



Intersections of Polar Curves

1 (2 marks)

$$2 = 1 + \cos \theta, \text{ so } \cos \theta = 1$$

M1

$$\theta = 0, \text{ so the point is } (2, 0)$$

A1

Answer: (2, 0)

2 marks

2 (2 marks)

$$3 + \cos \theta = 5 - \cos \theta, \text{ so } \cos \theta = 1$$

M1

$$\theta = 0 \text{ and } r = 4, \text{ so P is } (4, 0)$$

A1

Answer: (4, 0)

2 marks

3 (2 marks)

$$r = 4 \cos \frac{\pi}{3}$$

M1

$$= 2, \text{ so the point is } (2, \frac{\pi}{3})$$

A1

Answer: (2, $\frac{\pi}{3}$)

2 marks

4 (2 marks)

$$\text{The curve passes through O when } r = 0: 1 + \cos \theta = 0$$

M1

$$\cos \theta = -1, \theta = \pi$$

A1

Answer: $\theta = \pi$

2 marks

5 (2 marks)

$$2 \sin \theta = 2 \cos \theta, \text{ so } \tan \theta = 1 \text{ and } \theta = \frac{\pi}{4}$$

M1

$$r = 2 \sin \frac{\pi}{4} = \sqrt{2}, \text{ so P is } (\sqrt{2}, \frac{\pi}{4})$$

A1

Answer: $(\sqrt{2}, \frac{\pi}{4})$

2 marks

6 (3 marks)

$$2 \sin \theta = 1, \text{ so } \sin \theta = \frac{1}{2}$$

M1

$$\theta = \frac{\pi}{6} \text{ or } \frac{5\pi}{6}$$

A1

$$\text{The points are } (1, \frac{\pi}{6}) \text{ and } (1, \frac{5\pi}{6})$$

A1

Answer: $(1, \frac{\pi}{6}), (1, \frac{5\pi}{6})$

3 marks

7 (4 marks)

From C_2 , $r = \sec \theta$. Equate: $4 \cos \theta = \frac{1}{\cos \theta}$

M1

$\cos^2 \theta = \frac{1}{4}$, so $\cos \theta = \frac{1}{2}$ (since $r > 0$ needs $\cos \theta > 0$)

M1

$$\theta = \pm \frac{\pi}{3}$$

A1

$r = 2$: the points are $(2, \frac{\pi}{3})$ and $(2, -\frac{\pi}{3})$

A1

Answer: $(2, \frac{\pi}{3}), (2, -\frac{\pi}{3})$

4 marks

8 (4 marks)

$$2(1 + \cos \theta) = 3$$

M1

$$\cos \theta = \frac{1}{2}$$

A1

$$\theta = \pm \frac{\pi}{3}$$

A1

Points $(3, \frac{\pi}{3})$ and $(3, -\frac{\pi}{3})$

A1

Answer: $(3, \frac{\pi}{3}), (3, -\frac{\pi}{3})$

4 marks

9 (5 marks)

$$1 + \sin \theta = 3 \sin \theta, \text{ so } \sin \theta = \frac{1}{2}$$

M1 A1

$$\theta = \frac{\pi}{6} \text{ or } \frac{5\pi}{6}, \text{ with } r = \frac{3}{2} \text{ in each case}$$

A1

C_2 passes through O ($\theta = 0$), and C_1 passes through O when $\sin \theta = -1$ ($\theta = \frac{3\pi}{2}$)

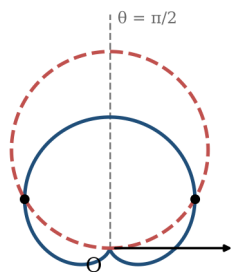
M1

So O is also a point of intersection, giving $(1.5, \frac{\pi}{6}), (1.5, \frac{5\pi}{6})$ and the pole

A1

Answer: $(\frac{3}{2}, \frac{\pi}{6}), (\frac{3}{2}, \frac{5\pi}{6})$ and the pole

5 marks

**10 (4 marks)**

$$r = \frac{2}{\sin \theta}, \text{ so } 4 \sin \theta = \frac{2}{\sin \theta}$$

M1

$$\sin^2 \theta = \frac{1}{2}, \text{ so } \sin \theta = \frac{\sqrt{2}}{2}$$

A1

$$\theta = \frac{\pi}{4} \text{ or } \frac{3\pi}{4}$$

A1

$r = 2\sqrt{2}$: the points are $(2\sqrt{2}, \frac{\pi}{4})$ and $(2\sqrt{2}, \frac{3\pi}{4})$

A1

Answer: $(2\sqrt{2}, \frac{\pi}{4}), (2\sqrt{2}, \frac{3\pi}{4})$

4 marks

11 (4 marks)

$$6 \cos \theta = 2, \text{ so } \cos \theta = \frac{1}{3}$$

M1 A1

$$\sin \theta = \pm \frac{2\sqrt{2}}{3}$$

M1

$$x = 2 \cos \theta = \frac{2}{3}, y = \pm \frac{4\sqrt{2}}{3}, \text{ so } P\left(\frac{2}{3}, \frac{4\sqrt{2}}{3}\right) \text{ and } Q\left(\frac{2}{3}, -\frac{4\sqrt{2}}{3}\right)$$

A1

$$\text{Answer: } \left(\frac{2}{3}, \pm \frac{4\sqrt{2}}{3}\right)$$

4 marks

12 (3 marks)

$$4 \sin \theta = 2, \text{ so } \sin \theta = \frac{1}{2}$$

M1

$$\theta = \frac{\pi}{6} \text{ or } \frac{5\pi}{6}$$

A1

$$\text{Points } \left(4, \frac{\pi}{6}\right) \text{ and } \left(4, \frac{5\pi}{6}\right)$$

A1

$$\text{Answer: } \left(4, \frac{\pi}{6}\right), \left(4, \frac{5\pi}{6}\right)$$

3 marks

13 (4 marks)

$$4(1 + \cos \theta)\cos \theta = 3, \text{ so } 4\cos^2 \theta + 4 \cos \theta - 3 = 0$$

M1 A1

$$(2 \cos \theta - 1)(2 \cos \theta + 3) = 0, \text{ so } \cos \theta = \frac{1}{2} \text{ (} \cos \theta = -\frac{3}{2} \text{ is impossible)}$$

M1

$$\theta = \pm \frac{\pi}{3}, r = 6: \text{ points } \left(6, \frac{\pi}{3}\right) \text{ and } \left(6, -\frac{\pi}{3}\right)$$

A1

$$\text{Answer: } \left(6, \frac{\pi}{3}\right), \left(6, -\frac{\pi}{3}\right)$$

4 marks

14 (5 marks)

$$\text{Intersection when } \theta = 2 \sin \theta, \text{ i.e. } f(\theta) = \theta - 2 \sin \theta = 0$$

M1

$$f(1.8) = 1.8 - 2 \sin 1.8 = -0.15 < 0$$

M1

$$f(2.0) = 2 - 2 \sin 2 = 0.18 > 0$$

A1

Sign change in a continuous function, so a root lies between 1.8 and 2.0

E1

Solving numerically: $\theta = 1.90$ (3 s.f.)

A1

Answer: $\theta = 1.90$

5 marks

15 (4 marks)

$$\text{Equating: } 2 \cos \theta = 2 + \cos \theta, \text{ so } \cos \theta = 2, \text{ which has no solution}$$

M1 A1

$$C_1 \text{ passes through O at } \theta = \frac{\pi}{2} \text{ and } \frac{3\pi}{2}, \text{ where } C_2 \text{ has } r = 2 \neq 0$$

M1

$$\text{A point } (r, \theta) \text{ of } C_1 \text{ with } r < 0 \text{ corresponds to } (-r, \theta + \pi); \text{ then } C_2 \text{ at } \theta + \pi \text{ requires } -2 \cos \theta = 2$$

E1

$$- \cos \theta, \text{ so } \cos \theta = -2, \text{ impossible. Hence no intersections}$$

Answer: No intersections

4 marks

16 (5 marks)

$$3 \cos \theta = 1 + \cos \theta, \text{ so } \cos \theta = \frac{1}{2}$$

M1 A1

$$\theta = \pm \frac{\pi}{3}, r = \frac{3}{2}: \text{ points } \left(\frac{3}{2}, \frac{\pi}{3}\right) \text{ and } \left(\frac{3}{2}, -\frac{\pi}{3}\right)$$

A1

$$C_1 \text{ is at O when } \theta = \pm \frac{\pi}{2}, \text{ and } C_2 \text{ is at O when } \theta = \pi$$

M1

So the pole is also a common point

A1

$$\text{Answer: } \left(\frac{3}{2}, \pm \frac{\pi}{3}\right) \text{ and the pole}$$

5 marks

17 (4 marks)

$$2 \cos 2\theta = 1, \text{ so } \cos 2\theta = \frac{1}{2}$$

M1

$$2\theta = \pm \frac{\pi}{3} + 2k\pi, \text{ so } \theta = \pm \frac{\pi}{6} \text{ and } \pi \pm \frac{\pi}{6}$$

M1 A1

$$\text{Points } (1, \frac{\pi}{6}), (1, \frac{5\pi}{6}), (1, \frac{7\pi}{6}), (1, \frac{11\pi}{6})$$

A1

$$\text{Answer: } (1, \frac{\pi}{6}), (1, \frac{5\pi}{6}), (1, \frac{7\pi}{6}), (1, \frac{11\pi}{6})$$

4 marks

18 (5 marks)

$$1 + \cos \theta = 1 - \cos \theta, \text{ so } \cos \theta = 0$$

M1 A1

$$\theta = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}, \text{ with } r = 1$$

A1

$$C_1 \text{ reaches O when } \theta = \pi \text{ and } C_2 \text{ reaches O when } \theta = 0$$

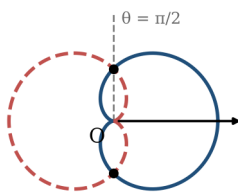
M1

$$\text{So the pole is also an intersection, giving } (1, \frac{\pi}{2}), (1, \frac{3\pi}{2}) \text{ and the pole}$$

A1

$$\text{Answer: } (1, \frac{\pi}{2}), (1, \frac{3\pi}{2}) \text{ and the pole}$$

5 marks

**19 (5 marks)**

$$2 \sin \theta = \tan \theta = \frac{\sin \theta}{\cos \theta}$$

M1

$$\sin \theta (2 \cos \theta - 1) = 0$$

M1 A1

$$\sin \theta = 0 \text{ gives } \theta = 0, r = 0 \text{ (the pole). } \cos \theta = \frac{1}{2} \text{ gives } \theta = \frac{\pi}{3}$$

A1

$$r = 2 \sin \frac{\pi}{3} = \sqrt{3}: \text{ the points are } (\sqrt{3}, \frac{\pi}{3}) \text{ and the pole}$$

A1

$$\text{Answer: } (\sqrt{3}, \frac{\pi}{3}) \text{ and the pole}$$

5 marks

20 (5 marks)

$$100(1 + \cos \theta) = 150, \text{ so } \cos \theta = \frac{1}{2}$$

M1 A1

$$\theta = \pm \frac{\pi}{3}: \text{ points } (150, \frac{\pi}{3}) \text{ and } (150, -\frac{\pi}{3})$$

A1

$$\text{Angle between them } \frac{2\pi}{3}, \text{ chord} = 2(150) \sin \frac{\pi}{3}$$

M1

$$= 150\sqrt{3} \approx 260 \text{ m}$$

A1

$$\text{Answer: } (150, \pm \frac{\pi}{3}); \text{ distance } 150\sqrt{3} \approx 260 \text{ m}$$

5 marks

21 (6 marks)

$$r = \frac{3}{\sin \theta}, \text{ so } 2(1 + \sin \theta) \sin \theta = 3$$

M1

$$2 \sin^2 \theta + 2 \sin \theta - 3 = 0$$

A1

$$\sin \theta = \frac{-1 + \sqrt{7}}{2} \text{ (the other root is below } -1)$$

M1 A1

$$r = \frac{3}{\sin \theta} = \frac{6}{\sqrt{7} - 1} = 1 + \sqrt{7}$$

M1

$$\theta = 0.966 \text{ or } \pi - 0.966 = 2.18 \text{ (3 s.f.)}$$

A1

$$\text{Answer: } r = 1 + \sqrt{7}; \theta = 0.966, 2.18$$

6 marks

22 (6 marks)

Points of C_1 with $r = 1$: $\sin 2\theta = \frac{1}{2}$

M1

$$2\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \text{ so } \theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$$

A1

Points of C_1 with $r = -1$: $\sin 2\theta = -\frac{1}{2}$

M1

$$2\theta = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}, \text{ so } \theta = \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$$

A1

The point $(-1, \theta)$ is the point $(1, \theta + \pi)$, giving angles $\frac{19\pi}{12}, \frac{23\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}$

M1

Eight points $(1, \theta)$ with $\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$

A1

Answer: Eight points $(1, \theta)$, $\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$

6 marks

23 (3 marks)

At $\theta = \frac{3\pi}{4}$: $r = 4 \cos \frac{3\pi}{4} = -2\sqrt{2}$ on the first curve and $r = 2 \sec \frac{3\pi}{4} = -2\sqrt{2}$ on the second

M1

$(-2\sqrt{2}, \frac{3\pi}{4}) = (2\sqrt{2}, \frac{7\pi}{4})$, which is the same point as the solution $\theta = \frac{7\pi}{4}$; similarly $\theta = \frac{5\pi}{4}$ gives

E1

the same point as $\theta = \frac{\pi}{4}$

There are only two distinct points: $(2\sqrt{2}, \frac{\pi}{4})$ and $(2\sqrt{2}, \frac{7\pi}{4})$

A1

Answer: Two points of intersection

3 marks

24 (6 marks)

$2(1 + \sin \theta) = 2(1 + \cos \theta)$, so $\sin \theta = \cos \theta$, $\tan \theta = 1$

M1 A1

$$\theta = \frac{\pi}{4} \text{ or } \frac{5\pi}{4}$$

A1

$$\theta = \frac{\pi}{4}: r = 2(1 + \frac{\sqrt{2}}{2}) = 2 + \sqrt{2}; \theta = \frac{5\pi}{4}: r = 2 - \sqrt{2}$$

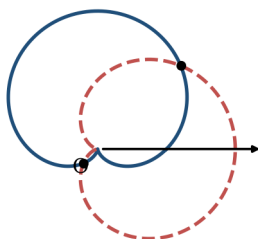
M1 A1

C_1 passes through O at $\theta = \frac{3\pi}{2}$ and C_2 at $\theta = \pi$, so the pole is also common

A1

Answer: $(2 + \sqrt{2}, \frac{\pi}{4})$, $(2 - \sqrt{2}, \frac{5\pi}{4})$ and the pole

6 marks



25 (7 marks)

(a) $\frac{4}{1 + \cos \theta} = 4$, so $1 + \cos \theta = 1$

M1

$$\cos \theta = 0, \theta = \pm \frac{\pi}{2}$$

A1

Points $(4, \frac{\pi}{2})$ and $(4, -\frac{\pi}{2})$

A1

Answer: $(4, \frac{\pi}{2})$, $(4, -\frac{\pi}{2})$

3 marks

(b) $r + r \cos \theta = 4$, so $r = 4 - x$

M1

$$x^2 + y^2 = 16 - 8x + x^2, \text{ so } y^2 = 16 - 8x$$

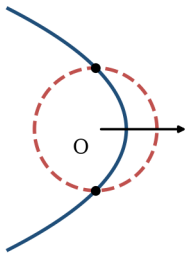
A1

$$\text{With } x^2 + y^2 = 16: x^2 + 16 - 8x = 16, \text{ so } x(x - 8) = 0$$

M1

$$x = 0 \text{ gives } y = \pm 4, \text{ i.e. } (4, \pm \frac{\pi}{2}) \text{ (} x = 8 \text{ gives } y^2 < 0, \text{ rejected)}$$

A1

Answer: $y^2 = 16 - 8x$; points $(0, \pm 4)$ **4 marks****26 (5 marks)**

$$3 + 2 \cos \theta = k, \text{ so } \cos \theta = \frac{k-3}{2}$$

M1 A1

Solutions exist only if $-1 \leq \frac{k-3}{2} \leq 1$, i.e. $1 \leq k \leq 5$

M1

$k = 1$ or $k = 5$: $\cos \theta = -1$ or 1 , one solution each, so 1 point

A1

$1 < k < 5$: two values of θ , so 2 points; $k < 1$ or $k > 5$: no points

A1

Answer: 0 points if $k < 1$ or $k > 5$; 1 point if $k = 1$ or 5 ; 2 points if $1 < k < 5$

5 marks**27 (8 marks)**

$$2 + 2 \cos \theta = 3 \sin \theta, \text{ so } 3 \sin \theta - 2 \cos \theta = 2$$

M1

$$3 \sin \theta - 2 \cos \theta = \sqrt{13} \sin(\theta - \alpha), \text{ where } \tan \alpha = \frac{2}{3}$$

M1 A1

$$\sin(\theta - \alpha) = \frac{2}{\sqrt{13}}, \text{ and } \sin \alpha = \frac{2}{\sqrt{13}}, \text{ so } \theta - \alpha = \alpha \text{ or } \pi - \alpha$$

M1 A1

$$\theta = 2\alpha = 1.18 \text{ (3 s.f.) or } \theta = \pi$$

A1

$$\theta = 2\alpha: r = 3 \sin 2\alpha = 3(2)\left(\frac{2}{\sqrt{13}}\right)\left(\frac{3}{\sqrt{13}}\right) = \frac{36}{13}$$

M1

$\theta = \pi$: $r = 0$ on both curves, i.e. the pole. Intersections: $(\frac{36}{13}, 1.18)$ and the pole

A1

Answer: $(\frac{36}{13}, 1.18)$ and the pole

8 marks**28 (8 marks)**

$$1 + 2 \cos \theta = 1, \text{ so } \cos \theta = 0, \theta = \frac{\pi}{2}, \frac{3\pi}{2}: \text{ points } (1, \frac{\pi}{2}) \text{ and } (1, \frac{3\pi}{2})$$

M1 A1

Points with $r = -1$ on C_1 : $1 + 2 \cos \theta = -1$, so $\cos \theta = -1$

M1

$\theta = \pi$: the point $(-1, \pi)$ is the point $(1, 0)$

A1

Three points: $(1, \frac{\pi}{2})$, $(1, \frac{3\pi}{2})$ and $(1, 0)$

A1

For C : $1 + 2 \cos \theta = 2$ gives $\cos \theta = \frac{1}{2}$, $\theta = \pm \frac{\pi}{3}$: two points

M1

$1 + 2 \cos \theta = -2$ gives $\cos \theta = -\frac{3}{2}$, impossible, so no extra points from negative r (inner loop has $|r| \leq 1$)

E1

Hence exactly two points $(2, \pm \frac{\pi}{3})$

A1

Answer: $(1, \frac{\pi}{2})$, $(1, \frac{3\pi}{2})$, $(1, 0)$; C meets C_1 at $(2, \pm \frac{\pi}{3})$ only

8 marks

29 (10 marks)

$$(a) 2p \cos \theta = 2q \sin \theta, \text{ so } \tan \theta = \frac{p}{q} \quad \text{M1 A1}$$

$$\cos \theta = \frac{q}{\sqrt{p^2 + q^2}} \text{ (right-angled triangle, } 0 < \theta < \frac{\pi}{2} \text{)} \quad \text{M1}$$

$$r = 2p \cos \theta = \frac{2pq}{\sqrt{p^2 + q^2}} \quad \text{A1}$$

$$\text{Answer: } P = \left(\frac{2pq}{\sqrt{p^2 + q^2}}, \theta \right) \text{ with } \tan \theta = \frac{p}{q} \quad \text{4 marks}$$

$$(b) C_1: x^2 + y^2 = 2px, \text{ so centre } (p, 0), \text{ radius } p \quad \text{M1 A1}$$

$$C_2: x^2 + y^2 = 2qy, \text{ so centre } (0, q), \text{ radius } q \quad \text{A1}$$

$$\text{Distance between centres: } d^2 = p^2 + q^2 \quad \text{M1}$$

$$\text{Sum of the squares of the radii: } p^2 + q^2 = d^2 \quad \text{A1}$$

So the radii to P are perpendicular (Pythagoras converse), i.e. the circles are orthogonal at P A1

Answer: $x^2 + y^2 = 2px$ and $x^2 + y^2 = 2qy$; radii perpendicular at P 6 marks

30 (10 marks)

$$\text{Equate the } r \text{ values at the same } \theta: 4(1 + 2 \cos \theta) = 3(2 - \cos \theta) \quad \text{M1}$$

$$11 \cos \theta = 2, \text{ so } \cos \theta = \frac{2}{11}, \theta = \pm 1.39 \quad \text{M1 A1}$$

$$r = \frac{4}{2 - 2/11} = \frac{11}{5} \text{ (check: } \frac{3}{1 + 4/11} = \frac{11}{5} \text{): points } \left(\frac{11}{5}, \pm 1.39 \right) \quad \text{A1}$$

$$\text{A point of H with } r < 0 \text{ is } (r, \theta) = (-r, \theta + \pi) \text{ and must lie on E: so } -\frac{3}{1 + 2 \cos \theta} = \quad \text{M1}$$

$$\frac{4}{2 - \cos(\theta + \pi)} = \frac{4}{2 + \cos \theta}$$

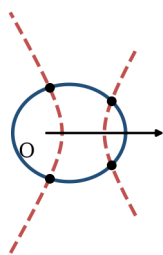
$$-3(2 + \cos \theta) = 4(1 + 2 \cos \theta), \text{ so } \cos \theta = -\frac{10}{11} \quad \text{M1}$$

$$\text{Then } 1 + 2 \cos \theta = -\frac{9}{11} < 0, \text{ consistent with } r < 0, \text{ and } r = -\frac{11}{3} \quad \text{A1}$$

$$\text{The point is } \left(\frac{11}{3}, \theta + \pi \right), \text{ where } \cos(\theta + \pi) = \frac{10}{11}: \text{ angle } \pm 0.430 \quad \text{M1}$$

$$\text{Four points: } \left(\frac{11}{5}, \pm 1.39 \right) \text{ and } \left(\frac{11}{3}, \pm 0.430 \right) \quad \text{A1 A1}$$

$$\text{Answer: } \left(\frac{11}{5}, \pm 1.39 \right) \text{ and } \left(\frac{11}{3}, \pm 0.430 \right) \quad \text{10 marks}$$

**Tangents to Polar Curves****1 (3 marks)**

$$y = (2 + \cos \theta) \sin \theta = 2 \sin \theta + \sin \theta \cos \theta \quad \text{M1}$$

$$\frac{dy}{d\theta} = 2 \cos \theta + \cos^2 \theta - \sin^2 \theta \quad \text{M1}$$

$$= 2 \cos \theta + \cos 2\theta \quad \text{A1}$$

$$\text{Answer: } 2 \cos \theta + \cos 2\theta \quad \text{3 marks}$$

2 (2 marks)

$r = 0$ when $\sin \theta = 0$, so $\theta = 0$ (or π)

M1

The tangent at O is the line $\theta = 0$, the initial line

A1

Answer: $\theta = 0$

2 marks

3 (2 marks)

$r = 0$ when $\cos 2\theta = 0$, so $2\theta = \frac{\pi}{2}, \frac{3\pi}{2}$

M1

The tangents at O are $\theta = \frac{\pi}{4}$ and $\theta = \frac{3\pi}{4}$

A1

Answer: $\theta = \frac{\pi}{4}, \theta = \frac{3\pi}{4}$

2 marks

4 (2 marks)

$x = \theta \cos \theta$

B1

$\frac{dx}{d\theta} = \cos \theta - \theta \sin \theta$

B1

Answer: $\cos \theta - \theta \sin \theta$

2 marks

5 (2 marks)

$y = f(\theta)\sin \theta$, so by the product rule

M1

$\frac{dy}{d\theta} = \frac{dr}{d\theta} \sin \theta + r \cos \theta$

A1

Answer: $\frac{dy}{d\theta} = \frac{dr}{d\theta} \sin \theta + r \cos \theta$

2 marks

6 (2 marks)

The tangent is perpendicular to OP, i.e. the horizontal line $y = 5$

B1

$r \sin \theta = 5$, so $r = 5 \operatorname{cosec} \theta$

B1

Answer: $r = 5 \operatorname{cosec} \theta$

2 marks

7 (4 marks)

$y = r \sin \theta = 4 \sin \theta \cos \theta = 2 \sin 2\theta$

M1

$\frac{dy}{d\theta} = 4 \cos 2\theta = 0$

M1

$\theta = \pm \frac{\pi}{4}$

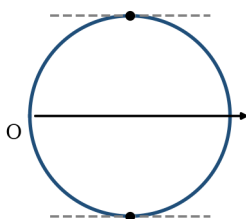
A1

$r = 4 \cos \frac{\pi}{4} = 2\sqrt{2}$: the points are $(2\sqrt{2}, \frac{\pi}{4})$ and $(2\sqrt{2}, -\frac{\pi}{4})$

A1

Answer: $(2\sqrt{2}, \frac{\pi}{4}), (2\sqrt{2}, -\frac{\pi}{4})$

4 marks

**8 (4 marks)**

$x = r \cos \theta = 3 \sin \theta \cos \theta = \frac{3}{2} \sin 2\theta$

M1

$\frac{dx}{d\theta} = 3 \cos 2\theta = 0$

M1

$\theta = \frac{\pi}{4}$ or $\frac{3\pi}{4}$

A1

$r = 3 \sin \frac{\pi}{4} = \frac{3\sqrt{2}}{2}$: the points are $(\frac{3\sqrt{2}}{2}, \frac{\pi}{4})$ and $(\frac{3\sqrt{2}}{2}, \frac{3\pi}{4})$

A1

Answer: $(\frac{3\sqrt{2}}{2}, \frac{\pi}{4}), (\frac{3\sqrt{2}}{2}, \frac{3\pi}{4})$

4 marks

9 (3 marks)

$$r = 0: \cos 3\theta = 0$$

M1

$$3\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$$

A1

$$\text{The tangents at O are } \theta = \frac{\pi}{6}, \theta = \frac{\pi}{2} \text{ and } \theta = \frac{5\pi}{6}$$

A1

$$\text{Answer: } \theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$$

3 marks

10 (4 marks)

$$x = \theta \cos \theta, y = \theta \sin \theta$$

B1

$$\frac{dy}{d\theta} = \sin \theta + \theta \cos \theta = 1 \text{ and } \frac{dx}{d\theta} = \cos \theta - \theta \sin \theta = -\frac{\pi}{2} \text{ at } \theta = \frac{\pi}{2}$$

M1 A1

$$\frac{dy}{dx} = -\frac{2}{\pi}$$

A1

$$\text{Answer: } -\frac{2}{\pi}$$

4 marks

11 (3 marks)

$$x = a(1 + \cos \theta)\cos \theta, y = a(1 + \cos \theta)\sin \theta$$

B1

$$\text{At } \theta = \frac{\pi}{2}: \frac{dy}{d\theta} = a(\cos \theta + \cos 2\theta) = -a \text{ and } \frac{dx}{d\theta} = -a(\sin \theta + \sin 2\theta) = -a$$

M1

$$\frac{dy}{dx} = \frac{-a}{-a} = 1$$

A1

$$\text{Answer: Gradient} = 1$$

3 marks

12 (3 marks)

$$r = 0 \text{ when } 3 \sin \theta + 4 \cos \theta = 0$$

M1

$$\tan \theta = -\frac{4}{3}$$

A1

$$\text{The tangent is the line } y = -\frac{4}{3}x$$

A1

$$\text{Answer: } y = -\frac{4}{3}x$$

3 marks

13 (4 marks)

$$\frac{dr}{d\theta} = -6 \sin \theta = -3 \text{ at } \theta = \frac{\pi}{6}; r = 3\sqrt{3}$$

B1

$$\frac{dy}{d\theta} = \frac{dr}{d\theta} \sin \theta + r \cos \theta = -\frac{3}{2} + \frac{9}{2} = 3$$

M1

$$\frac{dx}{d\theta} = \frac{dr}{d\theta} \cos \theta - r \sin \theta = -\frac{3\sqrt{3}}{2} - \frac{3\sqrt{3}}{2} = -3\sqrt{3}$$

M1

$$\frac{dy}{dx} = -\frac{1}{\sqrt{3}}$$

A1

$$\text{Answer: } -\frac{1}{\sqrt{3}}$$

4 marks

14 (4 marks)

$$\text{At } \theta = \frac{\pi}{2}, r = 1, \text{ so the point is } (0, 1)$$

B1

$$\frac{dy}{d\theta} = \cos \theta + \cos 2\theta = -1, \frac{dx}{d\theta} = -(\sin \theta + \sin 2\theta) = -1, \text{ so } \frac{dy}{dx} = 1$$

M1

$$\text{Gradient of normal} = -1$$

M1

$$y = -x + 1$$

A1

$$\text{Answer: } y = 1 - x$$

4 marks

15 (5 marks)

P has Cartesian coordinates $(\frac{1}{2}, \frac{\sqrt{3}}{2})$

B1

$$\frac{dr}{d\theta} = -2 \sin \theta = -\sqrt{3}; \frac{dy}{d\theta} = -\sqrt{3}(\frac{\sqrt{3}}{2}) + 1(\frac{1}{2}) = -1$$

M1

$$\frac{dx}{d\theta} = -\sqrt{3}(\frac{1}{2}) - 1(\frac{\sqrt{3}}{2}) = -\sqrt{3}, \text{ so } \frac{dy}{dx} = \frac{1}{\sqrt{3}}$$

A1

$$y - \frac{\sqrt{3}}{2} = \frac{1}{\sqrt{3}}(x - \frac{1}{2})$$

M1

$$y = \frac{\sqrt{3}}{3}x + \frac{\sqrt{3}}{3}$$

A1

Answer: $y = \frac{\sqrt{3}}{3}(x + 1)$

5 marks

16 (6 marks)

$$y = (2 + \cos \theta) \sin \theta$$

B1

$$\frac{dy}{d\theta} = 2 \cos \theta + \cos 2\theta = 2\cos^2\theta + 2 \cos \theta - 1$$

M1 A1

$$\frac{dy}{d\theta} = 0: \cos \theta = \frac{-1 + \sqrt{3}}{2} (\approx 0.366) \text{ (the other root is below } -1)$$

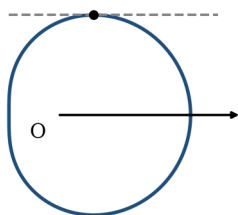
M1 A1

$$y = (2 + 0.366)(\sqrt{1 - 0.366^2}) = 2.20$$

A1

Answer: 2.20

6 marks

**17 (4 marks)**

At $\theta = \pi$, $r = \pi$, so the point is $(-\pi, 0)$

B1

$$\frac{dy}{d\theta} = \sin \theta + \theta \cos \theta = -\pi, \frac{dx}{d\theta} = \cos \theta - \theta \sin \theta = -1$$

M1

$$\frac{dy}{dx} = \pi$$

A1

$$y = \pi(x + \pi)$$

A1

Answer: $y = \pi(x + \pi)$

4 marks

18 (5 marks)

$$x = (2 + 2 \cos \theta) \cos \theta$$

B1

$$\frac{dx}{d\theta} = -2 \sin \theta - 4 \sin \theta \cos \theta = -2 \sin \theta(1 + 2 \cos \theta)$$

M1 A1

$$\frac{dx}{d\theta} = 0: \sin \theta = 0 \text{ (}\theta = 0 \text{ gives } x = 4, \theta = \pi \text{ is the pole) or } \cos \theta = -\frac{1}{2}, \text{ where } x = (1)(-\frac{1}{2}) =$$

M1

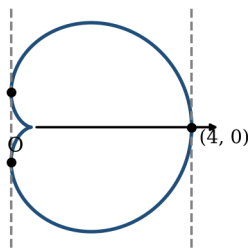
$$-\frac{1}{2}$$

$$\text{Tangents: } x = 4 \text{ and } x = -\frac{1}{2}$$

A1

Answer: $x = 4$ and $x = -\frac{1}{2}$

5 marks



19 (3 marks)

$$r = 0 \text{ when } \cos \theta = -\frac{1}{2}$$

M1

$$\theta = \frac{2\pi}{3} \text{ or } \frac{4\pi}{3}$$

A1

The tangents at O are the lines $\theta = \frac{2\pi}{3}$ and $\theta = \frac{4\pi}{3}$

A1

$$\text{Answer: } \theta = \frac{2\pi}{3}, \theta = \frac{4\pi}{3}$$

3 marks

20 (5 marks)

$$y = e^{\theta} \sin \theta$$

B1

$$\frac{dy}{d\theta} = e^{\theta}(\sin \theta + \cos \theta)$$

M1 A1

$$\frac{dy}{d\theta} = 0, \text{ so } \tan \theta = -1$$

M1

$$\theta = \frac{3\pi}{4}, r = e^{3\pi/4}: \text{ the point is } (e^{3\pi/4}, \frac{3\pi}{4})$$

A1

$$\text{Answer: } (e^{3\pi/4}, \frac{3\pi}{4})$$

5 marks

21 (6 marks)

$$\frac{dr}{d\theta} = k \text{ a } e^{k\theta} = kr$$

B1

$$\frac{dy}{d\theta} = kr \sin \theta + r \cos \theta \text{ and } \frac{dx}{d\theta} = kr \cos \theta - r \sin \theta$$

M1 A1

$$\tan \phi = \frac{dy}{dx} = \frac{k \sin \theta + \cos \theta}{k \cos \theta - \sin \theta}$$

A1

$$\text{Divide top and bottom by } \cos \theta: \tan \phi = \frac{k \tan \theta + 1}{k - \tan \theta} = \frac{\tan \theta + \frac{1}{k}}{1 - \frac{1}{k} \tan \theta} = \tan(\theta + \alpha), \text{ where } \tan \alpha = \frac{1}{k}$$

M1

$$\text{So } \phi - \theta = \alpha = \arctan \frac{1}{k}, \text{ which is constant}$$

A1

$$\text{Answer: } \phi - \theta = \arctan \frac{1}{k}$$

6 marks

22 (6 marks)

$$y^2 = r^2 \sin^2 \theta = 4 \cos 2\theta \sin^2 \theta$$

M1

$$= 4(1 - 2 \sin^2 \theta) \sin^2 \theta. \text{ Let } u = \sin^2 \theta: y^2 = 4u - 8u^2$$

M1 A1

$$\frac{d}{du}(4u - 8u^2) = 4 - 16u = 0, \text{ so } u = \frac{1}{4}$$

M1

$$y^2 = 1 - \frac{1}{2} = \frac{1}{2}, \text{ so the greatest } y \text{ is } \frac{1}{\sqrt{2}}$$

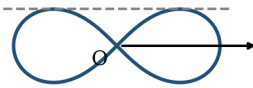
A1

$$\sin \theta = \frac{1}{2}, \theta = \frac{\pi}{6}, r^2 = 4 \cos \frac{\pi}{3} = 2: \text{ the point is } (\sqrt{2}, \frac{\pi}{6})$$

A1

$$\text{Answer: } y_{\max} = \frac{1}{\sqrt{2}} \text{ at } (\sqrt{2}, \frac{\pi}{6})$$

6 marks

**23 (6 marks)**

$$y = \sin 2\theta \sin \theta = 2 \sin^2 \theta \cos \theta$$

M1 A1

$$\frac{dy}{d\theta} = 4 \sin \theta \cos^2 \theta - 2 \sin^3 \theta = 2 \sin \theta (2 \cos^2 \theta - \sin^2 \theta)$$

M1 A1

$$\frac{dy}{d\theta} = 0 \text{ with } 0 < \theta < \frac{\pi}{2}: 2 \cos^2 \theta = \sin^2 \theta, \text{ so } \tan \theta = \sqrt{2}$$

A1

$$\sin \theta = \frac{\sqrt{2}}{\sqrt{3}}, \cos \theta = \frac{1}{\sqrt{3}}: y = 2\left(\frac{2}{3}\right)\left(\frac{1}{\sqrt{3}}\right) = \frac{4\sqrt{3}}{9}$$

A1

$$\text{Answer: } \frac{4\sqrt{3}}{9}$$

6 marks

24 (4 marks)

$$\frac{dr}{d\theta} = -2 \sin \theta = 0 \text{ at } \theta = 0 \text{ and } \theta = \pi$$

M1

At $\theta = 0$, the point is $(5, 0)$ and $\frac{dy}{d\theta} = \frac{dr}{d\theta} \sin \theta + r \cos \theta = 5 \neq 0$, while $\frac{dx}{d\theta} = \frac{dr}{d\theta} \cos \theta - r \sin \theta = 0$

M1 A1

The tangent at $(5, 0)$ is vertical: $x = 5$. Similarly at $\theta = \pi$, the point is $(1, \pi) = (-1, 0)$ and the

E1

tangent is $x = -1$. Neither is parallel to the initial line, so $\frac{dr}{d\theta} = 0$ does not give a tangent parallel to the initial line

Answer: Tangents $x = 5$ at $(5, 0)$ and $x = -1$ at $(1, \pi)$

4 marks

25 (6 marks)

P is $(a, \frac{\pi}{2})$, Cartesian $(0, a)$

B1

$$C_1: \frac{dy}{d\theta} = a(\cos \theta + \cos 2\theta) = -a, \frac{dx}{d\theta} = -a(\sin \theta + \sin 2\theta) = -a \text{ at } \theta = \frac{\pi}{2}, \text{ so gradient} = 1$$

M1 A1

$$C_2: r = a(1 - \cos \theta), \frac{dr}{d\theta} = a \sin \theta = a$$

B1

$$\frac{dy}{d\theta} = \frac{dr}{d\theta} \sin \theta + r \cos \theta = a \text{ and } \frac{dx}{d\theta} = \frac{dr}{d\theta} \cos \theta - r \sin \theta = -a, \text{ so gradient} = -1$$

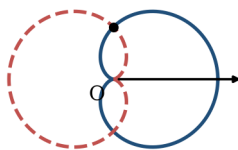
M1

Product of gradients = -1 , so the tangents are perpendicular

A1

Answer: Gradients 1 and -1 , so the tangents are perpendicular

6 marks

**26 (5 marks)**

$$\frac{dr}{d\theta} = 1$$

B1

$$\tan \psi = \frac{\theta}{1} = \theta$$

M1 A1

As $\theta \rightarrow \infty$, $\tan \psi \rightarrow \infty$, so $\psi \rightarrow \frac{\pi}{2}$: far from O the tangent is almost perpendicular to OP

A1

As $\theta \rightarrow 0$, $\tan \psi \rightarrow 0$, so $\psi \rightarrow 0$: near O the tangent is almost along OP

A1

Answer: $\tan \psi = \theta$; $\psi \rightarrow \frac{\pi}{2}$ as $\theta \rightarrow \infty$, $\psi \rightarrow 0$ as $\theta \rightarrow 0$

5 marks

27 (8 marks)

$$(a) x = 2a \cos \alpha \cos \alpha = 2a \cos^2 \alpha, y = 2a \cos \alpha \sin \alpha = a \sin 2\alpha$$

M1 A1

The centre is $(a, 0)$; P – centre $= (2a \cos^2 \alpha - a, a \sin 2\alpha) = a(\cos 2\alpha, \sin 2\alpha)$, which has direction 2α

A1

Answer: P = $(2a \cos^2 \alpha, a \sin 2\alpha)$

3 marks

(b) The tangent is perpendicular to the radius, so has normal direction $u = (\cos 2\alpha, \sin 2\alpha)$

M1

Tangent: $(x, y) \cdot u = P \cdot u$

M1

$$P \cdot u = 2a \cos^2 \alpha \cos 2\alpha + a \sin 2\alpha \sin 2\alpha = a(1 + \cos 2\alpha) \cos 2\alpha + a \sin^2 2\alpha$$

M1

$$= a \cos 2\alpha + a(\cos^2 2\alpha + \sin^2 2\alpha) = a(1 + \cos 2\alpha) = 2a \cos^2 \alpha$$

A1

$$(x, y) \cdot u = r \cos \theta \cos 2\alpha + r \sin \theta \sin 2\alpha = r \cos(\theta - 2\alpha), \text{ so } r \cos(\theta - 2\alpha) = 2a \cos^2 \alpha$$

A1

Answer: $r \cos(\theta - 2\alpha) = 2a \cos^2 \alpha$

5 marks

28 (9 marks)

Convert: $x^2 + y^2 = 2y$, so $x^2 + (y - 1)^2 = 1$: centre $(0, 1)$, radius 1

M1 A1

A line through $A(4, 0)$: $y = m(x - 4)$, i.e. $mx - y - 4m = 0$

B1

Distance from $(0, 1)$ to the line equals 1: $\frac{|-1 - 4m|}{\sqrt{m^2 + 1}} = 1$

M1

$$(1 + 4m)^2 = 1 + m^2, \text{ so } 15m^2 + 8m = 0$$

M1 A1

$m = 0$: the tangent is the initial line, $\theta = 0$ (and its extension)

A1

$$m = -\frac{8}{15}: y = -\frac{8}{15}(x - 4), \text{ so } 15y + 8x = 32$$

M1

$$r(15 \sin \theta + 8 \cos \theta) = 32, \text{ so } r = \frac{32}{8 \cos \theta + 15 \sin \theta}$$

A1

Answer: $\theta = 0$ ($y = 0$) and $r = \frac{32}{8 \cos \theta + 15 \sin \theta}$

9 marks

29 (8 marks)

$$\text{At P: } r = a(1 + \cos \theta), \frac{dr}{d\theta} = -a \sin \theta$$

B1

$$\frac{dy}{d\theta} = -a \sin^2 \theta + a(1 + \cos \theta) \cos \theta = a(\cos \theta + \cos 2\theta)$$

M1 A1

$$\frac{dx}{d\theta} = -a \sin \theta \cos \theta - a(1 + \cos \theta) \sin \theta = -a(\sin \theta + \sin 2\theta)$$

A1

$$\text{Gradient at P: } m_P = -\frac{\cos \theta + \cos 2\theta}{\sin \theta + \sin 2\theta} = -\frac{2 \cos \frac{3\theta}{2} \cos \frac{\theta}{2}}{2 \sin \frac{3\theta}{2} \cos \frac{\theta}{2}} = -\cot \frac{3\theta}{2}$$

M1 A1

$$\text{At Q replace } \theta \text{ by } \theta + \pi: m_Q = -\cot\left(\frac{3\theta}{2} + \frac{3\pi}{2}\right) = \tan \frac{3\theta}{2}$$

M1

$$m_P m_Q = -\cot \frac{3\theta}{2} \tan \frac{3\theta}{2} = -1, \text{ so the tangents are perpendicular}$$

A1

Answer: Product of gradients = -1 , so the tangents are perpendicular

8 marks

30 (11 marks)

$$(a) y = a(1 + \cos \theta) \sin \theta, \frac{dy}{d\theta} = a(\cos \theta + \cos 2\theta) = a(2\cos^2 \theta + \cos \theta - 1)$$

M1 A1

$$\frac{dy}{d\theta} = 0: (2 \cos \theta - 1)(\cos \theta + 1) = 0$$

M1

$$\cos \theta = \frac{1}{2} \left(\theta = \frac{\pi}{3} \right); \cos \theta = -1 \text{ is the pole}$$

A1

$$y = a\left(\frac{3}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{3\sqrt{3}}{4}a$$

A1

Answer: $y_{\max} = \frac{3\sqrt{3}}{4}a$

5 marks

$$(b) x = a(1 + \cos \theta) \cos \theta, \frac{dx}{d\theta} = -a \sin \theta(1 + 2 \cos \theta)$$

M1 A1

$$\theta = 0: x = 2a \text{ (greatest)}$$

A1

$$\cos \theta = -\frac{1}{2}: x = a\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) = -\frac{a}{4} \text{ (least)}$$

A1

Answer: Greatest $2a$, least $-\frac{a}{4}$ **4 marks****(c)** Width = $2a + \frac{a}{4} = \frac{9a}{4}$; height = $2 \times \frac{3\sqrt{3}}{4}a = \frac{3\sqrt{3}}{2}a$ **M1**

Area = $\frac{27\sqrt{3}}{8}a^2$

A1**Answer: $\frac{27\sqrt{3}}{8}a^2$** **2 marks**

Areas Using Polar Coordinates

1 (2 marks)

Area = $\frac{1}{2} \int_0^{\pi/3} 16 \, d\theta$

M1

= $8 \times \frac{\pi}{3} = \frac{8\pi}{3}$

A1**Answer: $\frac{8\pi}{3}$** **2 marks**

2 (2 marks)

Area = $\frac{1}{2} \int r^2 \, d\theta$

M1

= $\frac{1}{2} \int_0^{2\pi} (3 + \cos \theta)^2 \, d\theta$

A1**Answer: $\frac{1}{2} \int_0^{2\pi} (3 + \cos \theta)^2 \, d\theta$** **2 marks**

3 (3 marks)

Area = $\frac{1}{2} \int_0^1 e^{2\theta} \, d\theta$

M1

= $\frac{1}{2} \left[\frac{1}{2} e^{2\theta} \right]_0^1$

M1

= $\frac{1}{4} (e^2 - 1)$

A1**Answer: $\frac{1}{4} (e^2 - 1)$** **3 marks**

4 (3 marks)

Area = $\frac{1}{2} \int_{-\pi/2}^{\pi/2} 4 \cos^2 \theta \, d\theta = \int_{-\pi/2}^{\pi/2} (1 + \cos 2\theta) \, d\theta$

M1

= $\left[\theta + \frac{1}{2} \sin 2\theta \right]_{-\pi/2}^{\pi/2}$

M1

= π

A1**Answer: π** **3 marks**

5 (3 marks)

Area = $\frac{1}{2} \int_0^{\pi/3} 4 \sec^2 \theta \, d\theta$

M1

= $[2 \tan \theta]_0^{\pi/3}$

M1

= $2\sqrt{3}$

A1**Answer: $2\sqrt{3}$** **3 marks**

6 (2 marks)

The upper half has area $\frac{1}{2} \int_0^{\pi} r^2 \, d\theta$

E1

By symmetry the whole area is twice this, $\int_0^{\pi} r^2 \, d\theta$

E1**Answer: Twice the upper half****2 marks**

7 (3 marks)

$$\text{Area} = \frac{1}{2} \int_0^{\pi} \theta^2 \, d\theta \quad \text{M1}$$

$$= \frac{1}{2} \left[\frac{\theta^3}{3} \right]_0^{\pi} \quad \text{M1}$$

$$= \frac{\pi^3}{6} \quad \text{A1}$$

Answer: $\frac{\pi^3}{6}$ 3 marks

8 (4 marks)

$$\text{Area} = \frac{1}{2} \int_{\pi/4}^{\pi/2} 16 \operatorname{cosec}^2 \theta \, d\theta \quad \text{M1}$$

$$= [-8 \cot \theta]_{\pi/4}^{\pi/2} \quad \text{M1 A1}$$

$$= 0 - (-8) = 8 \quad \text{A1}$$

Answer: 8 4 marks

9 (3 marks)

$$\text{Area} = \frac{1}{2} \int_0^{\pi/2} (3^2 - 1^2) \, d\theta \quad \text{M1}$$

$$= 4 \times \frac{\pi}{2} \quad \text{M1}$$

$$= 2\pi \quad \text{A1}$$

Answer: 2π 3 marks

10 (4 marks)

$$\text{Area} = \frac{1}{2} \int_{-\pi/4}^{\pi/4} 4 \cos 2\theta \, d\theta \quad \text{M1}$$

$$= [\sin 2\theta]_{-\pi/4}^{\pi/4} \quad \text{M1 A1}$$

$$= 1 - (-1) = 2 \quad \text{A1}$$

Answer: 2 4 marks

11 (4 marks)

$$r^2 = \sin^2 \theta + \cos^2 \theta + 2 \sin \theta \cos \theta = 1 + \sin 2\theta \quad \text{M1}$$

$$\text{Area} = \frac{1}{2} \int_{-\pi/4}^{3\pi/4} (1 + \sin 2\theta) \, d\theta \quad \text{M1}$$

$$= \frac{1}{2} \left[\theta - \frac{1}{2} \cos 2\theta \right]_{-\pi/4}^{3\pi/4} \quad \text{M1}$$

$$= \frac{1}{2} (\pi + 0) = \frac{\pi}{2} \quad \text{A1}$$

Answer: $\frac{\pi}{2}$ 4 marks

12 (4 marks)

$$\text{Area} = \frac{1}{2} \int_0^{\pi/2} 9 \sin^2 2\theta \, d\theta \quad \text{M1}$$

$$\sin^2 2\theta = \frac{1}{2} (1 - \cos 4\theta) \quad \text{M1}$$

$$= \frac{9}{4} \left[\theta - \frac{1}{4} \sin 4\theta \right]_0^{\pi/2} \quad \text{M1}$$

$$= \frac{9\pi}{8} \quad \text{A1}$$

Answer: $\frac{9\pi}{8}$ 4 marks

13 (4 marks)

$$r^2 = 4 + 4 \cos \theta + \cos^2 \theta = \frac{9}{2} + 4 \cos \theta + \frac{1}{2} \cos 2\theta$$

M1 A1

$$\text{Area} = \frac{1}{2} \left[\frac{9}{2} \theta + 4 \sin \theta + \frac{1}{4} \sin 2\theta \right]_0^{2\pi}$$

M1

$$= \frac{1}{2} (9\pi) = \frac{9\pi}{2}$$

A1

$$\text{Answer: } \frac{9\pi}{2}$$

4 marks

14 (5 marks)

$$\text{Area} = \frac{1}{2} \int_0^{2\pi} a^2 (1 + \cos \theta)^2 d\theta$$

M1

$$(1 + \cos \theta)^2 = 1 + 2 \cos \theta + \cos^2 \theta = \frac{3}{2} + 2 \cos \theta + \frac{1}{2} \cos 2\theta$$

M1 A1

$$\text{Area} = \frac{a^2}{2} \left[\frac{3}{2} \theta + 2 \sin \theta + \frac{1}{4} \sin 2\theta \right]_0^{2\pi}$$

M1

$$= \frac{a^2}{2} (3\pi) = \frac{3\pi a^2}{2}$$

A1

$$\text{Answer: Area} = \frac{3\pi a^2}{2}$$

5 marks

15 (5 marks)

$$\text{The right-hand loop is for } -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$$

B1

$$\text{Area of one loop} = \frac{1}{2} \int_{-\pi/4}^{\pi/4} a^2 \cos 2\theta d\theta$$

M1

$$= \frac{a^2}{2} \left[\frac{1}{2} \sin 2\theta \right]_{-\pi/4}^{\pi/4}$$

M1 A1

$$= \frac{a^2}{2}; \text{ by symmetry the two loops have total area } a^2$$

A1

$$\text{Answer: Total area} = a^2$$

5 marks

16 (4 marks)

$$\text{Area of the circle} = \pi(2)^2 = 4\pi$$

B1

$$\text{Area of the cardioid} = \frac{1}{2} \int_0^{2\pi} (1 + \cos \theta)^2 d\theta = \frac{3\pi}{2}$$

M1 A1

$$\text{Area between} = 4\pi - \frac{3\pi}{2} = \frac{5\pi}{2}$$

A1

$$\text{Answer: } \frac{5\pi}{2}$$

4 marks

17 (5 marks)

$$\text{Area of one petal} = \frac{1}{2} \int_{-\pi/6}^{\pi/6} 4 \cos^2 3\theta d\theta$$

M1

$$= \int_{-\pi/6}^{\pi/6} (1 + \cos 6\theta) d\theta$$

M1 A1

$$= \left[\theta + \frac{1}{6} \sin 6\theta \right]_{-\pi/6}^{\pi/6} = \frac{\pi}{3}$$

A1

$$C \text{ has three identical petals, so the total area is } \pi$$

A1

$$\text{Answer: Petal } \frac{\pi}{3}; \text{ total } \pi$$

5 marks

18 (5 marks)

$$(1 + \sin \theta)^2 = 1 + 2 \sin \theta + \sin^2 \theta = \frac{3}{2} + 2 \sin \theta - \frac{1}{2} \cos 2\theta$$

M1 A1

$$\text{Area} = \frac{1}{2} \left[\frac{3}{2} \theta - 2 \cos \theta - \frac{1}{4} \sin 2\theta \right]_0^{\pi/2}$$

M1

$$= \frac{1}{2} \left[\left(\frac{3\pi}{4} - 0 - 0 \right) - (0 - 2 - 0) \right]$$

M1

$$= \frac{3\pi}{8} + 1$$

A1

$$\text{Answer: } \frac{3\pi}{8} + 1$$

5 marks

19 (4 marks)

$$\text{Area} = \frac{1}{2} \int_0^{2\pi} (20 + 10 \cos \theta)^2 d\theta$$

M1

$$(20 + 10 \cos \theta)^2 = 400 + 400 \cos \theta + 50 + 50 \cos 2\theta$$

M1

$$\text{Area} = \frac{1}{2} [450\theta + 400 \sin \theta + 25 \sin 2\theta]_0^{2\pi} = 450\pi$$

A1

$$\approx 1410 \text{ m}^2$$

A1

$$\text{Answer: } 1410 \text{ m}^2$$

4 marks

20 (5 marks)

$$\text{Area} = \frac{1}{2} \int_0^{2\pi} (a + 3 \sin \theta)^2 d\theta$$

M1

$$(a + 3 \sin \theta)^2 = a^2 + 6a \sin \theta + \frac{9}{2} - \frac{9}{2} \cos 2\theta$$

M1

$$\text{Area} = \frac{1}{2} \left[\left(a^2 + \frac{9}{2} \right) \theta - 6a \cos \theta - \frac{9}{4} \sin 2\theta \right]_0^{2\pi}$$

M1

$$= \pi \left(a^2 + \frac{9}{2} \right)$$

A1

$$\pi \left(a^2 + \frac{9}{2} \right) = \frac{59\pi}{2}, \text{ so } a^2 = 25 \text{ and } a = 5 \text{ (since } a > 3)$$

A1

$$\text{Answer: } a = 5$$

5 marks

21 (6 marks)

$$\text{For } -\frac{\pi}{3} \leq \theta \leq \frac{\pi}{3}, 3 \cos \theta \geq 1 + \cos \theta, \text{ so the region lies between the curves}$$

B1

$$\text{Area} = \frac{1}{2} \int_{-\pi/3}^{\pi/3} (9 \cos^2 \theta - (1 + \cos \theta)^2) d\theta = \int_0^{\pi/3} (8 \cos^2 \theta - 1 - 2 \cos \theta) d\theta$$

M1 A1

$$8 \cos^2 \theta = 4 + 4 \cos 2\theta, \text{ so integrand} = 3 + 4 \cos 2\theta - 2 \cos \theta$$

M1

$$= [3\theta + 2 \sin 2\theta - 2 \sin \theta]_0^{\pi/3}$$

A1

$$= \pi + \sqrt{3} - \sqrt{3} = \pi$$

A1

$$\text{Answer: } \pi$$

6 marks

22 (6 marks)

$$\text{Inner loop: } \frac{2\pi}{3} \leq \theta \leq \frac{4\pi}{3}$$

B1

$$r^2 = 1 + 4 \cos \theta + 4 \cos^2 \theta = 3 + 4 \cos \theta + 2 \cos 2\theta$$

M1 A1

$$\text{Area} = \frac{1}{2} [3\theta + 4 \sin \theta + \sin 2\theta]_{2\pi/3}^{4\pi/3}$$

M1

$$3\theta \text{ gives } 2\pi; 4 \sin \theta \text{ gives } 4 \left(-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} \right) = -4\sqrt{3}; \sin 2\theta \text{ gives } \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \sqrt{3}$$

A1

$$\text{Area} = \frac{1}{2} (2\pi - 3\sqrt{3}) = \pi - \frac{3\sqrt{3}}{2}$$

A1

$$\text{Answer: } \pi - \frac{3\sqrt{3}}{2}$$

6 marks

23 (6 marks)

$2 \cos \theta = 2 \sin \theta$ gives $\theta = \frac{\pi}{4}$, $r = \sqrt{2}$: the point is $(\sqrt{2}, \frac{\pi}{4})$

M1 A1

For $0 \leq \theta \leq \frac{\pi}{4}$ the boundary of the common region is C_2 ; for $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$ it is C_1 . By symmetry the two parts are equal

M1

$$\text{Area} = 2 \times \frac{1}{2} \int_0^{\pi/4} 4 \sin^2 \theta \, d\theta = 2 \int_0^{\pi/4} (1 - \cos 2\theta) \, d\theta$$

M1

$$= 2[\theta - \frac{1}{2} \sin 2\theta]_0^{\pi/4} = 2(\frac{\pi}{4} - \frac{1}{2})$$

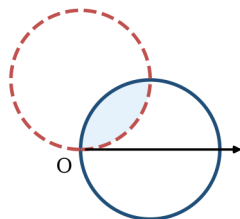
M1

$$= \frac{\pi}{2} - 1$$

A1

Answer: $(\sqrt{2}, \frac{\pi}{4}); \frac{\pi}{2} - 1$

6 marks

**24 (6 marks)**

Along the ray at angle θ ($0 \leq \theta \leq 2\pi$), the first turn has $r = \theta$ and the second turn has $r = \theta + 2\pi$

B1

$$\text{Area} = \frac{1}{2} \int_0^{2\pi} ((\theta + 2\pi)^2 - \theta^2) \, d\theta$$

M1 A1

$$= \frac{1}{2} \int_0^{2\pi} (4\pi\theta + 4\pi^2) \, d\theta$$

M1

$$= \frac{1}{2} [2\pi\theta^2 + 4\pi^2\theta]_0^{2\pi}$$

M1

$$= \frac{1}{2} (8\pi^3 + 8\pi^3) = 8\pi^3$$

A1

Answer: Area = $8\pi^3$

6 marks

25 (6 marks)

Intersection: $4 = 2 \sec \theta$, $\cos \theta = \frac{1}{2}$, $\theta = \pm \frac{\pi}{3}$

M1 A1

$$\text{Area} = \frac{1}{2} \int_{-\pi/3}^{\pi/3} (16 - 4 \sec^2 \theta) \, d\theta$$

M1 A1

$$= [8\theta - 2 \tan \theta]_{-\pi/3}^{\pi/3}$$

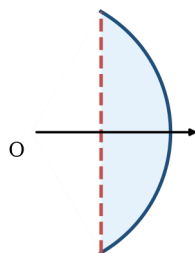
M1

$$= \frac{16\pi}{3} - 4\sqrt{3}$$

A1

Answer: $\frac{16\pi}{3} - 4\sqrt{3}$

6 marks



26 (5 marks)

$$r^2 = a^2(9 + 12 \cos \theta + 4\cos^2\theta) = a^2(11 + 12 \cos \theta + 2 \cos 2\theta)$$

M1 A1

$$\text{Area} = \frac{a^2}{2}[11\theta + 12 \sin \theta + \sin 2\theta]_0^{\pi/2}$$

M1 A1

$$= \frac{a^2}{2}\left(\frac{11\pi}{2} + 12\right) = a^2\left(\frac{11\pi}{4} + 6\right)$$

A1

$$\text{Answer: Area} = a^2\left(\frac{11\pi}{4} + 6\right)$$

5 marks

27 (7 marks)

$$\text{Intersections: } 2 + 2 \cos \theta = 3, \cos \theta = \frac{1}{2}, \theta = \pm \frac{\pi}{3}$$

M1 A1

For $|\theta| \leq \frac{\pi}{3}$ the cardioid is outside D, so the boundary of the common region is D; for $\frac{\pi}{3} \leq |\theta| \leq \pi$ the boundary is C

M1

$$\text{Area} = 2\left[\frac{1}{2}\int_0^{\pi/3} 9 \, d\theta + \frac{1}{2}\int_{\pi/3}^{\pi} (2 + 2 \cos \theta)^2 \, d\theta\right]$$

M1

$$\text{First part: } 2 \times \frac{1}{2} \times 9 \times \frac{\pi}{3} = 3\pi$$

A1

$$\text{Second part: } 4\int_{\pi/3}^{\pi} \left(\frac{3}{2} + 2 \cos \theta + \frac{1}{2}\cos 2\theta\right) \, d\theta = 4\left[\frac{3\theta}{2} + 2 \sin \theta + \frac{1}{4}\sin 2\theta\right]_{\pi/3}^{\pi} = 4\pi - \frac{9\sqrt{3}}{2}$$

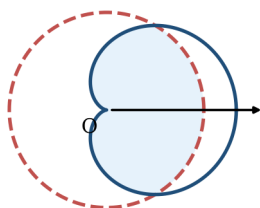
M1

$$\text{Area} = 7\pi - \frac{9\sqrt{3}}{2}$$

A1

$$\text{Answer: } 7\pi - \frac{9\sqrt{3}}{2}$$

7 marks

**28 (9 marks)**

$r = 0$ at $\theta = \frac{2\pi}{3}$ and $\frac{4\pi}{3}$; the outer loop is traced for $-\frac{2\pi}{3} \leq \theta \leq \frac{2\pi}{3}$ and the inner loop for $\frac{2\pi}{3} \leq \theta \leq \frac{4\pi}{3}$

B1

$$r^2 = 3 + 4 \cos \theta + 2 \cos 2\theta$$

M1

$$\text{Outer loop: } \frac{1}{2}[3\theta + 4 \sin \theta + \sin 2\theta]_{-2\pi/3}^{2\pi/3} = \frac{1}{2}(4\pi + 4\sqrt{3} - \sqrt{3})$$

M1 A1

$$= 2\pi + \frac{3\sqrt{3}}{2}$$

A1

$$\text{Inner loop: } \frac{1}{2}[3\theta + 4 \sin \theta + \sin 2\theta]_{2\pi/3}^{4\pi/3} = \pi - \frac{3\sqrt{3}}{2}$$

M1 A1

The inner loop lies inside the outer loop, so the required area is $(2\pi + \frac{3\sqrt{3}}{2}) - (\pi - \frac{3\sqrt{3}}{2})$

M1

$$= \pi + 3\sqrt{3}$$

A1

$$\text{Answer: } \pi + 3\sqrt{3}$$

9 marks

29 (9 marks)

$$\text{Area} = \frac{1}{2} \int_0^{2\pi} \frac{16}{(2 - \cos \theta)^2} d\theta = 8 \int_0^{2\pi} \frac{d\theta}{(2 - \cos \theta)^2} \quad \text{M1}$$

$$t = \tan \frac{\theta}{2}; \cos \theta = \frac{1 - t^2}{1 + t^2}, d\theta = \frac{2 dt}{1 + t^2} \quad \text{B1}$$

$$2 - \cos \theta = \frac{1 + 3t^2}{1 + t^2} \quad \text{M1 A1}$$

The range $0 \leq \theta < 2\pi$ becomes $-\infty < t < \infty$ (the integrand is even in t), so $\text{Area} = 16 \int_{-\infty}^{\infty} \frac{1 + t^2}{(1 + 3t^2)^2} dt$ M1 A1

$$\frac{1 + t^2}{(1 + 3t^2)^2} dt = 32 \int_0^{\infty} \frac{1 + t^2}{(1 + 3t^2)^2} dt$$

$$\text{Let } u = \sqrt{3}t: \int_0^{\infty} \frac{1 + t^2}{(1 + 3t^2)^2} dt = \frac{1}{3\sqrt{3}} \int_0^{\infty} \frac{3 + u^2}{(1 + u^2)^2} du \quad \text{M1}$$

$$\int_0^{\infty} \frac{du}{(1 + u^2)^2} = \frac{\pi}{4} \text{ and } \int_0^{\infty} \frac{u^2}{(1 + u^2)^2} du = \frac{\pi}{4}, \text{ so the integral} = \frac{1}{3\sqrt{3}} \left(3 \times \frac{\pi}{4} + \frac{\pi}{4} \right) = \frac{\pi}{3\sqrt{3}} \quad \text{A1}$$

$$\text{Area} = \frac{32\pi}{3\sqrt{3}} = \frac{32\sqrt{3}\pi}{9} \quad \text{A1}$$

Answer: Area = $\frac{32\sqrt{3}\pi}{9}$ 9 marks

30 (8 marks)

(a) $2\theta^2 = 3 - \theta$, so $2\theta^2 + \theta - 3 = 0$ M1

$(2\theta + 3)(\theta - 1) = 0$, so $\theta = 1$ ($\theta = -\frac{3}{2}$ is outside the range) M1

$r = 2$, so A is (2, 1) A1

Answer: A is (2, 1) 3 marks

(b) For $0 \leq \theta \leq 1$, C_2 is further from O than C_1 , so R lies between the curves B1

$$\text{Area} = \frac{1}{2} \int_0^1 ((3 - \theta)^2 - 4\theta^4) d\theta \quad \text{M1}$$

$$= \frac{1}{2} \left[-\frac{(3 - \theta)^3}{3} - \frac{4\theta^5}{5} \right]_0^1 \quad \text{M1 A1}$$

$$= \frac{1}{2} \left(-\frac{8}{3} - \frac{4}{5} + 9 \right) = \frac{83}{30} \quad \text{A1}$$

Answer: $\frac{83}{30}$ 5 marks

