

Polar Coordinate Systems

- 1 The point A has polar coordinates $(4, \frac{\pi}{3})$. Find the Cartesian coordinates of A. [2]

- 2 The point B has Cartesian coordinates (3, 3). Find the polar coordinates (r, θ) of B, where $r > 0$ and $0 \leq \theta < 2\pi$. [2]

- 3 The point C has polar coordinates $(5, \frac{7\pi}{4})$. Write down polar coordinates of C of the form (r, θ) with $r > 0$ and $-\pi < \theta \leq \pi$. [1]

- 4 Describe the set of points whose polar coordinates (r, θ) satisfy
 - (a) $r = 3, 0 \leq \theta < 2\pi$, [1]
 - (b) $\theta = \frac{\pi}{4}, r \geq 0$. [1]

- 5 The points A and B have polar coordinates $(3, 0)$ and $(4, \frac{\pi}{2})$. Find the distance AB. [2]

- 6 The point P has polar coordinates $(-3, \frac{\pi}{2})$. Find the Cartesian coordinates of P. [2]

- 7 The points A and B have polar coordinates $(6, \frac{\pi}{6})$ and $(6, \frac{5\pi}{6})$ respectively. Find the exact length of AB. [3]

- 8 The points A and B have polar coordinates $(5, \frac{\pi}{12})$ and $(8, \frac{5\pi}{12})$. Find the exact area of triangle OAB, where O is the pole. [3]

- 9 The point Q has polar coordinates $(-2, \frac{2\pi}{3})$. Find the exact Cartesian coordinates of Q. [2]

- 10 The point R has Cartesian coordinates $(-2, 2\sqrt{3})$. Find the polar coordinates (r, θ) of R, with $r > 0$ and $0 \leq \theta < 2\pi$, giving θ in terms of π . [3]

- 11 The points P and Q have polar coordinates $(4, \frac{\pi}{6})$ and $(2, \frac{\pi}{2})$. Find the exact distance PQ. [3]

- 12 Sam says that the points $(-3, \frac{\pi}{4})$ and $(3, \frac{5\pi}{4})$ are different points because one has a negative radius and the other a positive radius. Explain why Sam is wrong. [2]

- 13 The point A has polar coordinates $(3, \frac{2\pi}{3})$. Write down three other pairs of polar coordinates for A, one of which must have a negative value of r . Give all angles in the range $-2\pi \leq \theta \leq 2\pi$. [3]

- 14 The region R is defined by the inequalities $2 \leq r \leq 4$ and $0 \leq \theta \leq \frac{\pi}{2}$. Sketch R, and find its exact area. [4]

- 15 The points A and B have polar coordinates $(4, \frac{\pi}{12})$ and $(6, \theta)$, where $0 < \theta < \frac{\pi}{2}$. The area of triangle OAB is $6\sqrt{3}$. Find the exact value of θ . [4]

- 16** A radar station is at the pole O, with the initial line pointing East and angles measured anticlockwise. Radar contact A is 12 km from O on a bearing of 060° . Radar contact B is 9 km from O on a bearing of 150° .
Find the distance between A and B. [4]
- 17** The point P has polar coordinates $(4, \theta)$, where $0 \leq \theta < 2\pi$. The y-coordinate of P is $-2\sqrt{2}$.
Find the possible values of θ . [3]
- 18** O is the pole, and A and B are the points $(2, 0)$ and $(2, \frac{2\pi}{3})$. The point D is such that OADB is a rhombus. Find the polar coordinates of D, with $r > 0$. [4]
- 19** The points A and B have polar coordinates $(a, 0)$ and $(b, \frac{\pi}{2})$, where $0 < a < b$. The area of triangle OAB is 30 and the length of AB is 13. Find the values of a and b. [5]
- 20** The points A and B have polar coordinates $(3, \frac{\pi}{6})$ and $(4, \frac{2\pi}{3})$. The point C is such that OACB is a rectangle, where O is the pole.
Find the polar coordinates of C, giving θ to 3 significant figures. [5]
- 21** The points A, B and C have polar coordinates $(2, \frac{\pi}{6})$, $(2, \frac{5\pi}{6})$ and $(2, \frac{3\pi}{2})$ respectively.
Show that triangle ABC is equilateral, and find its exact area. [6]
- 22** ABCDEF is a regular hexagon with centre at the pole O. Vertex A has polar coordinates $(4, \frac{\pi}{12})$, and the vertices are labelled anticlockwise.
(a) Find the polar coordinates of C and E, with $r > 0$ and $0 \leq \theta < 2\pi$. [2]
(b) Find the exact area of the hexagon. [3]
(c) Find the exact length of AC. [2]
- 23** The points A, B and C have polar coordinates $(3, \frac{\pi}{6})$, $(5, \frac{\pi}{2})$ and $(4, \frac{5\pi}{6})$ respectively.
By considering triangles OAB, OBC and OAC, find the exact area of triangle ABC. [6]
- 24** The point A has polar coordinates $(6, 0)$ and the point B has polar coordinates (r, θ) , where $0 < \theta < \frac{\pi}{2}$. The triangle OAB is isosceles with $OB = AB$.
Show that $r = 3 \sec \theta$. [4]
- 25** The points A and B have polar coordinates (a, α) and (b, β) respectively.
Show that the distance AB is given by $AB = \sqrt{a^2 + b^2 - 2ab \cos(\alpha - \beta)}$. [5]
- 26** The points O, A and B have polar coordinates $(0, 0)$, $(6, 0)$ and $(8, \frac{\pi}{2})$ respectively. The circle C passes through O, A and B.
Find the polar coordinates of the centre of C, giving the angle to 3 significant figures, and the radius of C. [6]
- 27** The points A and B have polar coordinates $(5, 0)$ and $(3, \theta)$ respectively, where $0 \leq \theta < 2\pi$.
(a) Show that $AB^2 = 34 - 30 \cos \theta$. [2]
(b) Find the least and greatest possible values of AB. [3]
(c) Find the values of θ for which $AB = 7$, giving your answers to 3 significant figures. [4]

- 28** The points A, B and C have polar coordinates $(2, 0)$, $(2, \frac{2\pi}{3})$ and $(2, \frac{4\pi}{3})$ respectively. The point P has polar coordinates $(2, \theta)$ and lies on the same circle.
Show that $PA^2 + PB^2 + PC^2$ is independent of θ , and state its value. [8]
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- 29** The points A, B and C have polar coordinates (r_1, θ_1) , (r_2, θ_2) and (r_3, θ_3) , with $\theta_1 < \theta_2 < \theta_3 < \theta_1 + \pi$, and the pole O lies inside triangle ABC.
Show that the area of triangle ABC is $\frac{1}{2}[r_1 r_2 \sin(\theta_2 - \theta_1) + r_2 r_3 \sin(\theta_3 - \theta_2) + r_3 r_1 \sin(\theta_1 + 2\pi - \theta_3)]$. [8]
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- 30** The point A has polar coordinates $(6, 0)$. The point B has polar coordinates $(6, \varphi)$, where $0 < \varphi < \pi$, and M is the midpoint of AB.
(a) Show that M has polar coordinates $(6 \cos \frac{\varphi}{2}, \frac{\varphi}{2})$. [4]
(b) Find the maximum area of triangle OAM as φ varies, and the value of φ for which it occurs. [4]
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Converting Between Polar and Cartesian Coordinates

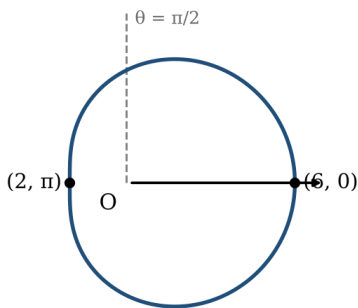
- 1** Write down a polar equation for the circle with Cartesian equation $x^2 + y^2 = 25$. [1]
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- 2** Find a Cartesian equation for the curve with polar equation $r = 4$. [1]
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- 3** Find a polar equation for the line $y = x$, giving your answer in the form $\theta = k$. [2]
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- 4** Find a polar equation for the line $x = 3$, giving your answer in the form $r = f(\theta)$. [2]
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- 5** Find the Cartesian equation of the curve with polar equation $r \sin \theta = 2$. [1]
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- 6** A curve has polar equation $r = 6 \cos \theta$. Show that its Cartesian equation is $x^2 + y^2 = 6x$. [2]
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- 7** A curve has Cartesian equation $xy = 8$. Show that its polar equation can be written as $r^2 \sin 2\theta = 16$. [3]
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- 8** A parabola has Cartesian equation $y^2 = 4x$. Find its polar equation in the form $r = f(\theta)$. [3]
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- 9** A line has Cartesian equation $y = 2x + 3$. Show that its polar equation is $r = \frac{3}{\sin \theta - 2 \cos \theta}$. [3]
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- 10** A circle has Cartesian equation $(x - 3)^2 + y^2 = 9$. Find its polar equation in the form $r = f(\theta)$. [3]
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- 11** A curve has Cartesian equation $x^2 - y^2 = 4$. Find its polar equation in the form $r^2 = f(\theta)$. [3]
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- 12** A curve has polar equation $r = \frac{2}{\cos \theta + \sin \theta}$. Find its Cartesian equation and describe the curve. [3]
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- 13** A curve has polar equation $r = 4 \cos \theta$.
Show that the point with polar coordinates $(2, \frac{\pi}{3})$ lies on the curve, and find the Cartesian equation of the curve. [4]
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- 14** A curve C has polar equation $r = 4 \cos \theta + 2 \sin \theta$, $0 \leq \theta < 2\pi$.
Show that C is a circle, and find the Cartesian coordinates of its centre and the exact length of its radius. [5]
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- 15** A curve has polar equation $r^2 = 9 \cos 2\theta$. Find a Cartesian equation of the curve. [4]
- 16** A line has polar equation $r = 4 \sec(\theta - \frac{\pi}{6})$. Find its Cartesian equation in the form $ax + by = c$, where a , b and c are exact. [4]
- 17** A curve has polar equation $r = \frac{2}{1 - \cos \theta}$. Show that its Cartesian equation is $y^2 = 4(x + 1)$. [4]
- 18** A circle has Cartesian equation $x^2 + y^2 - 6x + 8y = 0$. Find its polar equation in the form $r = a \cos \theta + b \sin \theta$, stating the values of a and b . [3]
- 19** A curve has polar equation $r = \sin 2\theta$. Show that its Cartesian equation is $(x^2 + y^2)^3 = 4x^2y^2$. [5]
- 20** A parabola has Cartesian equation $y = x^2$. Show that its polar equation can be written as $r = \tan \theta \sec \theta$, where $r \geq 0$. [3]
- 21** A curve has Cartesian equation $(x^2 + y^2)^2 = 8xy$. Show that a polar equation of the curve is $r^2 = 4 \sin 2\theta$. [4]
- 22** The points A and B have Cartesian coordinates $(-2, 0)$ and $(2, 0)$. The point P moves so that $AP \times BP = 4$. Show that a polar equation of the locus of P , with the origin as the pole, is $r^2 = 8 \cos 2\theta$. [5]
- 23** A curve has polar equation $r = \sec \theta + 2$. Find a Cartesian equation of the curve. [5]
- 24** A curve C has Cartesian equation $(x^2 + y^2)^3 = (x - y)^4$. Show that a polar equation for C is $r = 1 - \sin 2\theta$, $r \geq 0$. [5]
- 25** A curve has polar equation $r = \frac{4}{2 - \cos \theta}$. Show that its Cartesian equation can be written in the form $3x^2 - 8x + 4y^2 = 16$. [5]
- 26** A student is asked to find the centre and radius of the circle with polar equation $r = 2 \cos \theta + 2 \sin \theta$. The student writes:
 $"r^2 = 2r \cos \theta + 2r \sin \theta$, so $x^2 + y^2 = 2x + 2y$, so $(x - 1)^2 + (y - 1)^2 = 2$. The centre is $(1, 1)$ and the radius is 2."
 Identify the error in the student's work, give the correct radius, and confirm it by finding the greatest value of r on the curve. [4]
- 27** A curve C has Cartesian equation $(x^2 + y^2 - 2ax)^2 = 4a^2(x^2 + y^2)$, where a is a positive constant.
 (a) Show that C has polar equation $r = 2a(1 + \cos \theta)$, explaining why the other solution you obtain gives the same curve. [7]
 (b) For $a = 2$, find the Cartesian coordinates of the point on C where $\theta = \frac{\pi}{3}$. [2]
- 28** A curve has polar equation $r = \cos 3\theta$, $-\frac{\pi}{6} \leq \theta \leq \frac{\pi}{6}$. Show that its Cartesian equation is $(x^2 + y^2)^2 = x^3 - 3xy^2$. [7]
- 29** A curve C has polar equation $r = \frac{6}{1 + 2 \cos \theta}$.
 (a) Show that the Cartesian equation of C can be written as $\frac{(x - 4)^2}{4} - \frac{y^2}{12} = 1$. [6]
 (b) Hence state the Cartesian coordinates of the centre of C and the gradients of its asymptotes. [3]

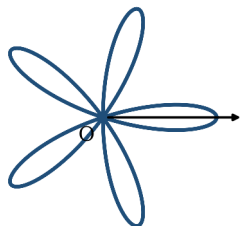
- 30** The circle C passes through the pole O, and its centre is the point with polar coordinates $(4, \frac{\pi}{3})$.
- (a) Show that a polar equation of C is $r = 8 \cos(\theta - \frac{\pi}{3})$. [3]
- (b) Find a Cartesian equation of C. [3]
- (c) Find the polar coordinates of the points where C meets the initial line and the line $\theta = \frac{\pi}{2}$, other than O. [3]

Sketching Polar Curves

- 1** Sketch the curve with polar equation $r = 3$, $0 \leq \theta < 2\pi$. [2]
- 2** Sketch the curve with polar equation $\theta = \frac{2\pi}{3}$, $r \geq 0$. [2]
- 3** The curve C has polar equation $r = 3 + 2 \cos \theta$, $0 \leq \theta < 2\pi$. Find the greatest and least values of r , and the value of θ at which each occurs. [2]
- 4** The curve C has polar equation $r = 2 + 2 \cos \theta$. Find the value of r when $\theta = \frac{\pi}{3}$ and when $\theta = \frac{2\pi}{3}$. [2]
- 5** The curve C has polar equation $r = 1 + \cos \theta$. State a line of symmetry of C, and give a reason. [2]
- 6** The curve S has polar equation $r = \theta$, $\theta \geq 0$. Find r when $\theta = \frac{\pi}{2}$, π and 2π , and describe how r changes as θ increases. [2]
- 7** Sketch the curve with polar equation $r = 4 \cos \theta$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, showing where it crosses the initial line. [3]
- 8** Sketch the curve with polar equation $r = 6 \sin \theta$, $0 \leq \theta \leq \pi$, indicating the greatest distance of the curve from O. [3]
- 9** Sketch the curve with polar equation $r = 2(1 + \cos \theta)$, $0 \leq \theta < 2\pi$, labelling the points where it crosses the initial line and the line $\theta = \frac{\pi}{2}$. [4]
- 10** Sketch the curve with polar equation $r = 3 + 2 \cos \theta$, $0 \leq \theta < 2\pi$, marking the points where it crosses the initial line and its extension. [4]
- 11** Sketch the curve with polar equation $r = \theta$ for $0 \leq \theta \leq 2\pi$. [3]
- 12** Sketch the curve with polar equation $r = 2 \sec \theta$, $-\frac{\pi}{3} \leq \theta \leq \frac{\pi}{3}$. [3]
- 13** Sketch the curve with polar equation $r = 4 \cos \theta + 3 \sin \theta$, for values of θ where $r \geq 0$, marking where it meets the initial line and the line $\theta = \frac{\pi}{2}$. State the diameter of the curve. [4]
- 14** Sketch the curve with polar equation $r = 4 \sin 2\theta$, $0 \leq \theta < 2\pi$. Give the greatest value of r and the values of θ at which the curve passes through O. [5]

- 15** Sketch the curve with polar equation $r^2 = 4 \cos 2\theta$, indicating the values of θ for which the curve exists in $0 \leq \theta < 2\pi$, and the greatest value of r . [5]
- 16** Sketch the curve with polar equation $r = 1 + 2 \cos \theta$, $0 \leq \theta < 2\pi$, giving the values of θ for which $r = 0$. [5]
- 17** Sketch the curve with polar equation $r = 3 \cos 3\theta$, $-\frac{\pi}{6} \leq \theta \leq \frac{\pi}{6}$, and state how many petals the complete curve has. [5]
- 18** Sketch the curve with polar equation $r = e^{\theta/2}$ for $0 \leq \theta \leq 2\pi$, giving the polar coordinates of its end points. [4]
- 19** Sketch the curve with polar equation $r = \frac{2}{1 + \cos \theta}$, $-\pi < \theta < \pi$, and state its Cartesian equation. [5]
- 20** The diagram shows the curve C with polar equation $r = a + b \cos \theta$, where a and b are positive constants and $a > b$. The curve crosses the initial line at the points $(6, 0)$ and $(2, \pi)$. Find the values of a and b .
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- [3]
- 21** The curve C_1 has polar equation $r = 2(1 + \cos \theta)$, $0 \leq \theta < 2\pi$. The curve C_2 is obtained by rotating C_1 anticlockwise through $\frac{\pi}{2}$ about O . Find a polar equation for C_2 , and sketch C_1 and C_2 on the same diagram, marking the greatest distance of C_2 from O . [5]
- 22** The curve C has polar equation $r = 1 + \frac{\theta}{\pi}$, $0 \leq \theta \leq 4\pi$. Sketch C . Find the polar coordinates of the points where C crosses the initial line, and show that the distance between consecutive crossings is constant. [6]
- 23** The curve C has polar equation $r = \frac{1}{\theta}$, $\theta > 0$. Show that as $\theta \rightarrow 0$ the curve approaches the line $y = 1$, and sketch C . [5]
- 24** The curve C has polar equation $r = 2 + \sec \theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Show that $x \rightarrow 1$ as $\theta \rightarrow \pm \frac{\pi}{2}$, and hence sketch C , stating the equation of the asymptote. [6]

- 25** The diagram shows the curve C with polar equation $r = a \cos n\theta$, where a and n are positive integers and $0 \leq \theta < 2\pi$. The curve has five petals, and the greatest value of r is 4. Find a and n , and write down the polar coordinates of the tips of the petals in the range $0 \leq \theta < 2\pi$, with $r > 0$.



[5]

- 26** A student sketches the curve with polar equation $r = 2 \sin 3\theta$, $0 \leq \theta < 2\pi$, and draws six petals, because r is positive for three ranges of θ and negative for three further ranges. Explain why the curve has only three petals, and state the equations of the lines of symmetry through the tips of the petals. [4]

- 27** The curve C has polar equation $r = 2 + 3 \cos \theta$, $0 \leq \theta < 2\pi$.
 (a) Find the values of θ , in the range $0 \leq \theta < 2\pi$, for which $r = 0$. Give your answers to 3 significant figures. [3]
 (b) Sketch C, showing the polar coordinates of the points where C crosses the initial line. [4]
 (c) Find the range of values of k for which the curve $r = 2 + k \cos \theta$ has an inner loop. [2]

- 28** The curve C has polar equation $r = \frac{2}{1 + 2 \cos \theta}$, $0 \leq \theta < 2\pi$.
 (a) Find the values of θ at which r is undefined, and describe what happens to the curve near these values. [3]
 (b) Show that the Cartesian equation of C can be written as $3(x - \frac{4}{3})^2 - y^2 = \frac{4}{3}$. [4]
 (c) Hence sketch C, and state the coordinates of the points at which it crosses the initial line. [3]

- 29** The curve C has polar equation $r = 2 \sin \theta \tan \theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.
 (a) Show that the Cartesian equation of C is $x(x^2 + y^2) = 2y^2$. [3]
 (b) Show that the line $x = 2$ is an asymptote of C. [3]
 (c) Sketch C, and describe its behaviour at O. [3]

- 30** The path of a comet around the Sun is modelled by the curve $r = \frac{12}{1 + 0.6 \cos \theta}$, where r is the distance in millions of km from the Sun, which is at the pole O. P is the point of the orbit nearest to the Sun and A is the point furthest from the Sun.
 (a) Find the distance of P from the Sun and the distance of A from the Sun, and state the value of θ at each point. [4]
 (b) Show that the Cartesian equation of the orbit is $\frac{(x + 11.25)^2}{18.75^2} + \frac{y^2}{15^2} = 1$, and hence find the distance PA. [6]

Intersections of Polar Curves

- 1** The curves C_1 and C_2 have polar equations $r = 2$ and $r = 1 + \cos \theta$ respectively, $0 \leq \theta < 2\pi$. Find the polar coordinates of their point of intersection. [2]

- 2 The curves with polar equations $r = 3 + \cos \theta$ and $r = 5 - \cos \theta$ meet at the point P, where $0 \leq \theta < 2\pi$. Find the polar coordinates of P. [2]

- 3 The circle C has polar equation $r = 4 \cos \theta$. The line L has equation $\theta = \frac{\pi}{3}$, $r \geq 0$. Find the polar coordinates of the point, other than O, where L meets C. [2]

- 4 The curve C has polar equation $r = 1 + \cos \theta$, $0 \leq \theta < 2\pi$. Show that C passes through the pole, and state the value of θ there. [2]

- 5 The curves $r = 2 \sin \theta$ and $r = 2 \cos \theta$ meet at a point P for $0 < \theta < \frac{\pi}{2}$. Find the polar coordinates of P. [2]

- 6 The circles $r = 2 \sin \theta$ and $r = 1$ meet at two points for $0 < \theta < \pi$. Find their polar coordinates. [3]

- 7 The curves $C_1: r = 4 \cos \theta$ and $C_2: r \cos \theta = 1$ meet at two points. Find their polar coordinates, with $r > 0$ and $-\frac{\pi}{2} < \theta \leq \frac{\pi}{2}$. [4]

- 8 The cardioid C has polar equation $r = 2(1 + \cos \theta)$ and the circle D has polar equation $r = 3$, for $-\pi < \theta \leq \pi$. Find the polar coordinates of the points where C and D intersect. [4]

- 9 The curves C_1 and C_2 have polar equations $r = 1 + \sin \theta$, $0 \leq \theta < 2\pi$, and $r = 3 \sin \theta$, $0 \leq \theta \leq \pi$. Find the polar coordinates of all the points where C_1 and C_2 intersect. [5]

- 10 The circle C has polar equation $r = 4 \sin \theta$, and the line L has polar equation $r \sin \theta = 2$, where $0 < \theta < \pi$. Find the polar coordinates of the points of intersection of C and L. [4]

- 11 The curves with polar equations $r = 6 \cos \theta$ and $r = 2$ meet at two points P and Q. Find the exact Cartesian coordinates of P and Q. [4]

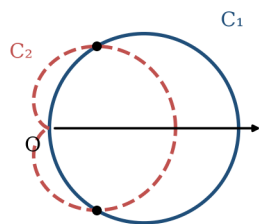
- 12 The circle $r = 4$ and the line $r \sin \theta = 2$ intersect at two points for $0 < \theta < \pi$. Find their polar coordinates. [3]

- 13 The cardioid C has polar equation $r = 4(1 + \cos \theta)$, $-\pi < \theta \leq \pi$, and the line L has polar equation $r \cos \theta = 3$. Find the polar coordinates of the points where L meets C. [4]

- 14 The curves C_1 and C_2 have polar equations $r = \theta$ and $r = 2 \sin \theta$ respectively, for $0 < \theta < \pi$. Show that the curves intersect at a point for which $1.8 < \theta < 2.0$, and find the value of θ at the point of intersection to 3 significant figures. [5]

- 15 The curve C_1 has polar equation $r = 2 \cos \theta$ and the curve C_2 has polar equation $r = 2 + \cos \theta$, $0 \leq \theta < 2\pi$. Show that C_1 and C_2 do not intersect. [4]

- 16** The diagram shows the circle $C_1: r = 3 \cos \theta$ and the cardioid $C_2: r = 1 + \cos \theta$. Find the polar coordinates of all the points where C_1 and C_2 intersect.



[5]

- 17** The lemniscate L has polar equation $r^2 = 2 \cos 2\theta$ and the circle C has polar equation $r = 1$. Find the polar coordinates, with $r > 0$ and $0 \leq \theta < 2\pi$, of all the points where L and C intersect.

[4]

- 18** The cardioids $C_1: r = 1 + \cos \theta$ and $C_2: r = 1 - \cos \theta$ are drawn for $0 \leq \theta < 2\pi$. Find the polar coordinates of all their points of intersection.

[5]

- 19** The curves $C_1: r = 2 \sin \theta$ and $C_2: r = \tan \theta$ are drawn for $0 \leq \theta < \frac{\pi}{2}$. Find the polar coordinates of all the points where they intersect.

[5]

- 20** A drone flies along the curve with polar equation $r = 100(1 + \cos \theta)$, where r is measured in metres from a control tower at the pole O . A circular safety fence has polar equation $r = 150$. Find the polar coordinates of the points where the drone's path crosses the fence, and the straight-line distance between these two points.

[5]

- 21** The curve C has polar equation $r = 2(1 + \sin \theta)$, $0 \leq \theta < 2\pi$. The line L has polar equation $r \sin \theta = 3$. Find the exact value of r , and the values of θ to 3 significant figures, at the points where L meets C .

[6]

- 22** The curve C_1 has polar equation $r = 2 \sin 2\theta$ and the circle C_2 has polar equation $r = 1$, $0 \leq \theta < 2\pi$. Find the polar coordinates of all the points of intersection of C_1 and C_2 , giving $r > 0$ and θ in terms of π .

[6]

- 23** A student is asked to find the points of intersection of the curves $r = 4 \cos \theta$ and $r = 2 \sec \theta$ for $0 \leq \theta < 2\pi$. The student writes:

" $4 \cos \theta = 2 \sec \theta$, so $\cos^2 \theta = \frac{1}{2}$, so $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$. There are four points of intersection."

Show that the student is wrong, and find the correct number of points of intersection.

[3]

- 24** The curves $C_1: r = 2(1 + \sin \theta)$ and $C_2: r = 2(1 + \cos \theta)$ are defined for $0 \leq \theta < 2\pi$. Find the polar coordinates of all the points of intersection of C_1 and C_2 , giving exact values.

[6]

- 25** The curve P has polar equation $r = \frac{4}{1 + \cos \theta}$, $-\pi < \theta < \pi$, and the circle C has polar equation $r = 4$, $0 \leq \theta < 2\pi$.

(a) Find the polar coordinates of the points of intersection of P and C .

[3]

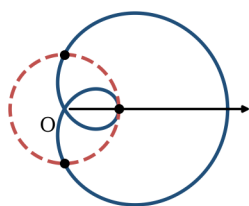
(b) Show that the Cartesian equation of P is $y^2 = 16 - 8x$, and verify your answer to (a) using Cartesian coordinates.

[4]

- 26** The curve C has polar equation $r = 3 + 2 \cos \theta$, $0 \leq \theta < 2\pi$. The circle D_k has polar equation $r = k$, where $k > 0$.
Find, in terms of k , the number of points of intersection of C and D_k , justifying your answer. [5]

- 27** The curves C_1 and C_2 have polar equations $r = 2 + 2 \cos \theta$ and $r = 3 \sin \theta$ respectively, $0 \leq \theta < 2\pi$.
Find the polar coordinates of all the points of intersection of C_1 and C_2 . Give θ to 3 significant figures where it is not an exact multiple of π . [8]

- 28** The diagram shows the limaçon C_1 : $r = 1 + 2 \cos \theta$, with its inner loop, and the circle C_2 : $r = 1$, for $0 \leq \theta < 2\pi$.
Find the polar coordinates, with $r > 0$, of all the points where C_2 meets C_1 , and explain why the circle C : $r = 2$ meets C_1 at only two points.



[8]

- 29** The circles C_1 : $r = 2p \cos \theta$ and C_2 : $r = 2q \sin \theta$, where p and q are positive constants, meet at the pole O and at a second point P .
(a) Show that the polar coordinates of P are $(r, \theta) = \left(\frac{2pq}{\sqrt{p^2 + q^2}}, \theta\right)$, where $\tan \theta = \frac{p}{q}$. [4]
(b) Find the Cartesian equations of C_1 and C_2 , and hence show that the circles meet at right angles at P . [6]

- 30** The ellipse E has polar equation $r = \frac{4}{2 - \cos \theta}$ and the hyperbola H has polar equation $r = \frac{3}{1 + 2 \cos \theta}$, for $0 \leq \theta < 2\pi$. For H , points with $r < 0$ are drawn using the usual convention.
Find the polar coordinates, with $r > 0$, of all the points where E and H intersect, giving angles to 3 significant figures. [10]

Tangents to Polar Curves

- 1** A curve has polar equation $r = 2 + \cos \theta$. By writing $y = r \sin \theta$, find $\frac{dy}{d\theta}$, giving your answer in terms of θ . [3]
- 2** The curve C has polar equation $r = 2 \sin \theta$, $0 \leq \theta < \pi$. Find the equation of the tangent to C at the pole. [2]
- 3** The curve C has polar equation $r = 4 \cos 2\theta$, $0 \leq \theta < \pi$. Find the equations of the tangents to C at the pole. [2]
- 4** A spiral has polar equation $r = \theta$. By writing $x = r \cos \theta$, find $\frac{dx}{d\theta}$. [2]
- 5** A curve has polar equation $r = f(\theta)$, and $x = r \cos \theta$, $y = r \sin \theta$. Show that $\frac{dy}{d\theta} = \frac{dr}{d\theta} \sin \theta + r \cos \theta$. [2]

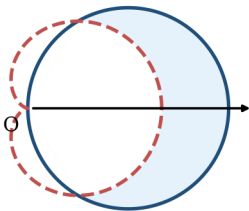
- 6** The circle C has polar equation $r = 5$, $0 \leq \theta < 2\pi$. Find a polar equation for the tangent to C at the point $(5, \frac{\pi}{2})$, in the form $r = f(\theta)$. [2]
-
- 7** The curve C has polar equation $r = 4 \cos \theta$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. Find the polar coordinates, with $r > 0$, of the points on C at which the tangent is parallel to the initial line. [4]
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- 8** The curve C has polar equation $r = 3 \sin \theta$, $0 \leq \theta \leq \pi$. Find the polar coordinates, with $r > 0$, of the points on C at which the tangent is perpendicular to the initial line. [4]
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- 9** The curve C has polar equation $r = 2 \cos 3\theta$, $0 \leq \theta < \pi$. Find the equations of all the tangents to C at the pole. [3]
-
- 10** The spiral S has polar equation $r = \theta$, $\theta \geq 0$. Find the gradient of the tangent to S at the point where $\theta = \frac{\pi}{2}$. [4]
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- 11** The cardioid C has polar equation $r = a(1 + \cos \theta)$, where $a > 0$. Show that the gradient of the tangent to C at the point where $\theta = \frac{\pi}{2}$ is 1. [3]
-
- 12** The curve C has polar equation $r = 3 \sin \theta + 4 \cos \theta$, $0 \leq \theta < \pi$. Find the Cartesian equation of the tangent to C at the pole. [3]
-
- 13** The circle C has polar equation $r = 6 \cos \theta$. Find the gradient of the tangent to C at the point where $\theta = \frac{\pi}{6}$, giving your answer in exact form. [4]
-
- 14** The cardioid C has polar equation $r = 1 + \cos \theta$, $0 \leq \theta < 2\pi$. Find the Cartesian equation of the normal to C at the point where $\theta = \frac{\pi}{2}$. [4]
-
- 15** The circle C has polar equation $r = 2 \cos \theta$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. The point P on C has polar coordinates $(1, \frac{\pi}{3})$. Find the equation of the tangent to C at P , in the form $y = mx + c$, giving exact values of m and c . [5]
-
- 16** The curve C has polar equation $r = 2 + \cos \theta$, $0 \leq \theta < 2\pi$. Find the greatest distance of a point of C above the initial line, giving your answer to 3 significant figures. [6]
-
- 17** The spiral S has polar equation $r = \theta$, $\theta \geq 0$. Find the Cartesian equation of the tangent to S at the point where $\theta = \pi$. [4]
-
- 18** The cardioid C has polar equation $r = 2 + 2 \cos \theta$, $0 \leq \theta < 2\pi$. Find the equations of the tangents to C that are perpendicular to the initial line. [5]
-
- 19** The limaçon C has polar equation $r = 1 + 2 \cos \theta$, $0 \leq \theta < 2\pi$. The curve passes through the pole twice. Find the equations of the tangents to C at the pole. [3]
-
- 20** The curve C has polar equation $r = e^\theta$, $0 \leq \theta \leq \pi$. Find the polar coordinates of the point on C at which the tangent is parallel to the initial line. [5]
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- 21** The equiangular spiral C has polar equation $r = a e^{k\theta}$, where a and k are positive constants. The tangent to C at the point P makes an angle φ with the initial line, and OP makes an angle θ with the initial line.
Show that $\tan \varphi = \frac{k \sin \theta + \cos \theta}{k \cos \theta - \sin \theta}$, and hence show that the angle $\varphi - \theta$ between the tangent and OP is the same at every point of C , stating its value. [6]
-
- 22** The lemniscate L has polar equation $r^2 = 4 \cos 2\theta$, $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$ (and its reflection in the line $\theta = \frac{\pi}{2}$).
Find the greatest value of y on L , and the polar coordinates of the point where it occurs in the range $0 < \theta < \frac{\pi}{4}$. [6]
-
- 23** The curve C has polar equation $r = \sin 2\theta$, $0 \leq \theta \leq \frac{\pi}{2}$. Show that the greatest value of y on C occurs when $\tan \theta = \sqrt{2}$, and find this greatest value in exact form. [6]
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- 24** A student states: "The tangent to a polar curve is parallel to the initial line where $\frac{dr}{d\theta} = 0$."
Using the curve $r = 3 + 2 \cos \theta$, show that the student is wrong, by finding the tangent at the points where $\frac{dr}{d\theta} = 0$. [4]
-
- 25** The curves $C_1: r = a(1 + \cos \theta)$ and $C_2: r = a(1 - \cos \theta)$, where $a > 0$, intersect at the point P where $\theta = \frac{\pi}{2}$.
Show that the tangents to C_1 and C_2 at P are perpendicular. [6]
-
- 26** The spiral S has polar equation $r = \theta$, $\theta > 0$. At the point P with polar angle θ , the tangent to S makes an angle ψ with the line OP , where $\tan \psi = \frac{r}{d r / d \theta}$.
Find $\tan \psi$ for S , and describe what happens to the angle ψ as $\theta \rightarrow \infty$ and as $\theta \rightarrow 0$. [5]
-
- 27** The circle C has polar equation $r = 2a \cos \theta$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, where $a > 0$. The point P on C has polar angle α , where $0 < \alpha < \frac{\pi}{2}$.
(a) Show that the Cartesian coordinates of P are $(2a \cos^2 \alpha, a \sin 2\alpha)$, and that the radius from the centre of C to P makes an angle 2α with the initial line. [3]
(b) Hence show that the tangent to C at P has polar equation $r \cos(\theta - 2\alpha) = 2a \cos^2 \alpha$. [5]
-
- 28** The circle C has polar equation $r = 2 \sin \theta$, $0 \leq \theta \leq \pi$. The point A has polar coordinates $(4, 0)$.
Find the equations of the two tangents to C that pass through A , giving the second in the form $r = \frac{k}{a \cos \theta + b \sin \theta}$, where k , a and b are integers. [9]
-
- 29** The cardioid C has polar equation $r = a(1 + \cos \theta)$, $a > 0$. The points P and Q on C have polar angles θ and $\theta + \pi$ respectively, so that PQ is a chord of C passing through the pole O .
Show that the tangents to C at P and Q are perpendicular. [8]
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- 30** The cardioid C has polar equation $r = a(1 + \cos \theta)$, $0 \leq \theta < 2\pi$, where $a > 0$. The smallest rectangle with sides parallel and perpendicular to the initial line that encloses C is drawn.
- (a) Show that the greatest value of y on C is $\frac{3\sqrt{3}}{4}a$. [5]
- (b) Find the greatest and least values of x on C . [4]
- (c) Hence find the area of the rectangle. [2]

Areas Using Polar Coordinates

- 1** Find the exact area of the sector bounded by the curve $r = 4$ and the half-lines $\theta = 0$ and $\theta = \frac{\pi}{3}$. [2]
- 2** The curve C has polar equation $r = 3 + \cos \theta$, $0 \leq \theta < 2\pi$. Write down an integral, in terms of θ , for the area enclosed by C . You do not need to evaluate it. [2]
- 3** The curve C has polar equation $r = e^\theta$. Find the exact area of the region bounded by C and the half-lines $\theta = 0$ and $\theta = 1$. [3]
- 4** Use integration to find the area enclosed by the curve $r = 2 \cos \theta$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. [3]
- 5** The line L has polar equation $r = 2 \sec \theta$. Use the formula for area in polar coordinates to find the exact area of the region bounded by L and the half-lines $\theta = 0$ and $\theta = \frac{\pi}{3}$. [3]
- 6** The curve C has polar equation $r = f(\theta)$ and is symmetrical about the initial line, with $f(\theta) = f(-\theta)$. The curve is closed and $0 \leq \theta \leq \pi$ gives its upper half. Explain why the area enclosed by C is $\int_0^\pi r^2 d\theta$. [2]
- 7** The spiral S has polar equation $r = \theta$, $0 \leq \theta \leq \pi$. Find the exact area of the region bounded by S and the initial line. [3]
- 8** The line L has polar equation $r = 4 \operatorname{cosec} \theta$. Find the area of the region bounded by L and the half-lines $\theta = \frac{\pi}{4}$ and $\theta = \frac{\pi}{2}$. [4]
- 9** The region R is bounded by the curves $r = 1$ and $r = 3$, and the half-lines $\theta = 0$ and $\theta = \frac{\pi}{2}$. Find the exact area of R . [3]
- 10** The curve C has polar equation $r^2 = 4 \cos 2\theta$, $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$. Find the area enclosed by this loop of C . [4]
- 11** The circle C has polar equation $r = \sin \theta + \cos \theta$, $-\frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}$. Find the exact area enclosed by C . [4]
- 12** The curve C has polar equation $r = 3 \sin 2\theta$, $0 \leq \theta \leq \frac{\pi}{2}$. Find the exact area of the petal enclosed by C . [4]
- 13** The limaçon C has polar equation $r = 2 + \cos \theta$, $0 \leq \theta < 2\pi$. Find the exact area enclosed by C . [4]

- 14** Show that the area enclosed by the cardioid $r = a(1 + \cos \theta)$, $0 \leq \theta < 2\pi$, is $\frac{3\pi a^2}{2}$. [5]
- 15** The lemniscate L has polar equation $r^2 = a^2 \cos 2\theta$, where $a > 0$ and the curve is made up of two loops. Show that the total area enclosed by L is a^2 . [5]
- 16** The cardioid C has polar equation $r = 1 + \cos \theta$ and the circle D has polar equation $r = 2$, $0 \leq \theta < 2\pi$. The cardioid lies inside the circle. Find the exact area of the region between C and D. [4]
- 17** The curve C has polar equation $r = 2 \cos 3\theta$, $-\pi \leq \theta \leq \pi$. One petal of C is bounded by the half-lines $\theta = -\frac{\pi}{6}$ and $\theta = \frac{\pi}{6}$. Find the exact area of this petal, and hence the area enclosed by the whole curve. [5]
- 18** The curve C has polar equation $r = 1 + \sin \theta$, $0 \leq \theta \leq 2\pi$. The region R is bounded by C and the half-lines $\theta = 0$ and $\theta = \frac{\pi}{2}$. Find the exact area of R. [5]
- 19** A garden is bounded by the curve with polar equation $r = 20 + 10 \cos \theta$, $0 \leq \theta < 2\pi$, where r is measured in metres and the origin is a fountain at the centre. Find the area of the garden, to 3 significant figures. [4]
- 20** The curve C has polar equation $r = a + 3 \sin \theta$, $0 \leq \theta < 2\pi$, where a is a constant and $a > 3$. The area enclosed by C is $\frac{59\pi}{2}$. Find the value of a . [5]
- 21** The diagram shows the circle C: $r = 3 \cos \theta$ and the cardioid D: $r = 1 + \cos \theta$. The circle and the cardioid meet at the points where $\theta = \pm \frac{\pi}{3}$. Find the exact area of the region that is inside C and outside D. [6]
- 
- 22** The limaçon C has polar equation $r = 1 + 2 \cos \theta$, $0 \leq \theta < 2\pi$. For $\frac{2\pi}{3} < \theta < \frac{4\pi}{3}$, r is negative and the curve forms an inner loop. Find the exact area enclosed by the inner loop. [6]
- 23** The circles C_1 : $r = 2 \cos \theta$ and C_2 : $r = 2 \sin \theta$ overlap, as shown. Find the polar coordinates of their point of intersection other than O, and find the exact area of the region common to both circles. [6]
- 24** The spiral S has polar equation $r = \theta$, $\theta \geq 0$. The region R lies between the first turn of S ($0 \leq \theta \leq 2\pi$) and the second turn of S ($2\pi \leq \theta \leq 4\pi$), as measured along each ray from the pole. Show that the area of R is $8\pi^3$. [6]
- 25** The circle C has polar equation $r = 4$ and the line L has polar equation $r = 2 \sec \theta$. The line cuts off a segment of the circle. Find the exact area of this segment (the region inside C and to the right of L). [6]

- 26** The curve C has polar equation $r = a(3 + 2 \cos \theta)$, $0 \leq \theta < 2\pi$, where a is a positive constant. The region R is bounded by C and the half-lines $\theta = 0$ and $\theta = \frac{\pi}{2}$. Show that the area of R is $a^2(\frac{11\pi}{4} + 6)$. [5]
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- 27** The cardioid C has polar equation $r = 2 + 2 \cos \theta$ and the circle D has polar equation $r = 3$, $0 \leq \theta < 2\pi$. Find the exact area of the region that is inside both C and D . [7]
-
- 28** The limaçon C has polar equation $r = 1 + 2 \cos \theta$, $0 \leq \theta < 2\pi$, with an outer loop and an inner loop. Find the exact area of the region between the outer loop and the inner loop. [9]
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- 29** The ellipse E has polar equation $r = \frac{4}{2 - \cos \theta}$, $0 \leq \theta < 2\pi$. Use the substitution $t = \tan \frac{\theta}{2}$ to show that the area enclosed by E is $\frac{32\sqrt{3}\pi}{9}$. [9]
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- 30** The curves C_1 : $r = 2\theta^2$ and C_2 : $r = 3 - \theta$, for $0 \leq \theta \leq 3$, meet at the point A .
(a) Show that the polar coordinates of A are $(2, 1)$. [3]
(b) The finite region R is bounded by C_1 , C_2 and the initial line, and lies in the sector $0 \leq \theta \leq 1$. Find the exact area of R . [5]
-