

Numerical Solution of Equations

1 (2 marks)

Let $f(x) = x^3 + 2x - 7$. Then $f(1) = -4$ and $f(2) = 5$

M1

f is continuous and changes sign, so there is a root in the interval $(1, 2)$

A1

Answer: Sign change from -4 to 5

2 marks

2 (2 marks)

$$x_1 = \sqrt{13} = 3.6056$$

M1

$$x_2 = 3.8492, x_3 = 3.9431$$

A1

Answer: $x_1 = 3.6056$, $x_2 = 3.8492$, $x_3 = 3.9431$

2 marks

3 (2 marks)

$$x^3 - 6x + 2 = 0 \implies 6x = x^3 + 2$$

M1

$$\text{Dividing by } 6 \text{ gives } x = \frac{x^3 + 2}{6}$$

A1

$$\text{Answer: } x = \frac{x^3 + 2}{6}$$

2 marks

4 (2 marks)

Let $f(x) = \cos x - x$. $f(0.7) = 0.0648 > 0$ and $f(0.8) = -0.1033 < 0$

M1

Sign change and f continuous, so a root lies in $[0.7, 0.8]$

A1

Answer: $f(0.7) > 0 > f(0.8)$

2 marks

5 (2 marks)

Midpoint 2.5: $f(2.5) = 3.125 > 0$

M1

$f(2) < 0$ and $f(2.5) > 0$, so α lies in $[2, 2.5]$

A1

Answer: $[2, 2.5]$

2 marks

6 (3 marks)

(a) $x_0 + 1 = 0$ would make the denominator zero

B1

Answer: $x_0 = -1$

1 mark

$$(b) x_1 = \frac{12}{3} = 4.000, x_2 = \frac{12}{5} = 2.400$$

M1

$$x_3 = \frac{12}{3.4} = 3.529$$

A1

Answer: $x_1 = 4.000$, $x_2 = 2.400$, $x_3 = 3.529$

2 marks

7 (4 marks)

(a) $f(0) = 1 > 0$ and $f(1) = e - 3 = -0.282 < 0$

M1

Sign change and f continuous, so a root lies in $[0, 1]$

A1

Answer: Root in $[0, 1]$

2 marks

(b) $f(1) = -0.282 < 0$ and $f(2) = e^2 - 6 = 1.389 > 0$

M1

Sign change and f continuous, so there is a root in $[1, 2]$

A1

Answer: Root in $[1, 2]$

2 marks

8 (5 marks)

(a) Let $f(x) = 2^x + 3x - 9$: $f(1.5) = -1.672 < 0$ and $f(2) = 1 > 0$

M1

Sign change and f continuous, so $1.5 < \alpha < 2$

A1

Answer: $f(1.5) < 0 < f(2)$

2 marks

$$(b) x_1 = \frac{\ln 0.3}{\ln 2} = -1.737$$

M1

$$x_2 = \frac{\ln(9 - 3(-1.737))}{\ln 2} = 3.829$$

A1

$9 - 3x_2 = -2.49 < 0$, and the logarithm of a negative number is undefined

E1

Answer: $x_1 = -1.737$, $x_2 = 3.829$; ln of a negative number

3 marks

9 (4 marks)Midpoint 1.5: $f(1.5) = -1.125 < 0$, so α is in $[1.5, 2]$

M1 A1

Midpoint 1.75: $f(1.75) = 0.609 > 0$, so α is in $[1.5, 1.75]$

A1

Midpoint 1.625: $f(1.625) = -0.334 < 0$, so α is in $[1.625, 1.75]$

A1

Answer: [1.625, 1.75]

4 marks

10 (5 marks) $f(1) = -4$ and $f(2) = 3$, so $x_1 = 1 + \frac{4}{4+3} \times 1 = 1.571$

M1 A1

 $f(1.571) = -0.808 < 0$, so α lies in $[1.571, 2]$

M1

$$x_2 = 1.571 + \frac{0.808}{0.808+3} \times (2 - 1.571)$$

M1

$$= 1.662$$

A1

Answer: $x_1 = 1.571$, $x_2 = 1.662$

5 marks

11 (4 marks)

(a) $5x = x^3 + 2$

M1

$$x = \frac{x^3 + 2}{5}$$

A1

Answer: $x = \frac{x^3 + 2}{5}$

2 marks

(b) $x^3 = 5x - 2$

M1

Taking cube roots, $x = \sqrt[3]{5x - 2}$

A1

Answer: $x = \sqrt[3]{5x - 2}$

2 marks

12 (3 marks)

(a) The path moves steadily towards an intersection, a staircase, so the sequence converges

E1

Answer: Staircase, converges

1 mark

(b) At the limit $L = \sqrt{2L + 3}$, so $L^2 - 2L - 3 = 0$

M1

 $(L - 3)(L + 1) = 0$ and $L > 0$, so $L = 3$

A1

Answer: 3

2 marks

13 (3 marks)

$$f(1.45) = -0.0514 < 0$$

M1

$$f(1.55) = 0.8239 > 0$$

A1

Sign change in $[1.45, 1.55]$ and f continuous, so $\alpha = 1.5$ to 1 d.p.

A1

Answer: $\alpha = 1.5$ to 1 d.p.

3 marks

14 (8 marks)

(a) $\frac{dy}{dx} = 3x^2 - \frac{4}{x}$

M1

$$3x^2 - \frac{4}{x} = 6, \text{ so } 3x^3 - 4 = 6x \text{ and } 3x^3 - 6x - 4 = 0$$

A1

Answer: $3x^3 - 6x - 4 = 0$

2 marks

(b) Let $f(x) = 3x^3 - 6x - 4$: $f(1.6) = -1.312 < 0$ and $f(1.8) = 2.696 > 0$

M1

Sign change and f continuous, so $1.6 < x < 1.8$

A1

Answer: $f(1.6) < 0 < f(1.8)$

2 marks

(c) $3x^3 = 6x + 4$, so $x^3 = 2x + \frac{4}{3}$ and $x = \sqrt[3]{2x + \frac{4}{3}}$

B1

$$x_1 = 1.67902, x_2 = 1.67404, x_3 = 1.67286, x_4 = 1.67258$$

M1 A1

$$x = 1.67 \text{ and } y = 4.678 - 4 \ln 1.672 = 2.62$$

A1

Answer: P(1.67, 2.62)

4 marks

15 (9 marks)(a) $f(0.5) = -0.2552 < 0$ and $f(0.6) = 0.3493 > 0$

M1

Sign change and f continuous, so $0.5 < \alpha < 0.6$

A1

Answer: $f(0.5) < 0 < f(0.6)$

2 marks

(b) $5x = 1 + 2 \cos x$	M1
$x = 0.2 + 0.4 \cos x$, so $A = 0.2$ and $B = 0.4$	A1
Answer: $A = 0.2$, $B = 0.4$	2 marks
(c) $x_2 = 0.2 + 0.4 \cos 0.55 = 0.54101$	M1
$x_3 = 0.54288$, $x_4 = 0.54249$	A1
Answer: $x_2 = 0.54101$, $x_3 = 0.54288$, $x_4 = 0.54249$	2 marks
(d) $f(0.5425) = -0.00034 < 0$	M1
$f(0.5435) = 0.0057 > 0$	A1
Sign change in $[0.5425, 0.5435]$, so $\alpha = 0.543$ to 3 d.p.	A1
Answer: $\alpha = 0.543$	3 marks
16 (4 marks)	
A: $x_1 = 0.4464$, $x_2 = 0.4413$, $x_3 = 0.4408$	M1 A1
B: $x_1 = 0.7937$, $x_2 = 1.3673$, $x_3 = 1.8730$	A1
A settles down near 0.44, whereas B moves away from it, so A is suitable	E1
Answer: A converges to α; B does not	4 marks
17 (6 marks)	
(a) Sketch $y = \sin x$ and the line $y = \frac{x}{3}$ through the origin with gradient $\frac{1}{3}$	M1
They meet at the origin and at one positive and one negative point, so 3 roots	A1
Answer: 3	2 marks
(b) $f(x) = \sin x - \frac{x}{3}$: $f(2) = 0.2426 > 0$ and $f(2.5) = -0.2349 < 0$	M1
Sign change and f continuous, so $2 < \alpha < 2.5$	A1
Answer: $2 < \alpha < 2.5$	2 marks
(c) $x_1 = 1.7954$, $x_2 = 2.9246$	M1
$x_3 = 0.6458$, $x_4 = 1.805$	A1
Answer: 1.805	2 marks
18 (5 marks)	
(a) The width halves at each step, starting from 1	M1
After n steps the width is $1 \times (1/2)^n = \frac{1}{2^n}$	A1
Answer: $\frac{1}{2^n}$	2 marks
(b) The midpoint is within half the width of the root: $\frac{1}{2^{n+1}} \leq 0.001$	M1
$2^{n+1} \geq 1000$, and $2^9 = 512 < 1000 \leq 1024 = 2^{10}$	M1
$n + 1 \geq 10$, so $n = 9$	A1
Answer: $n = 9$	3 marks
19 (7 marks)	
(a) $f(x) = x \ln x - 2$: $f(2) = -0.6137 < 0$ and $f(3) = 1.2958 > 0$	M1
Sign change and f continuous	A1
Answer: $2 < \alpha < 3$	2 marks
(b) $\ln x = \frac{2}{x}$	M1
$x = e^{2/x}$	A1
Answer: $x = e^{2/x}$	2 marks
(c) $x_1 = 2.386$, $x_2 = 2.312$, $x_3 = 2.375$, $x_4 = 2.321$	M1 A1
The values alternate either side of $\alpha = 2.346$, so the sequence converges slowly by oscillation	E1
Answer: 2.386, 2.312, 2.375, 2.321; oscillating convergence	3 marks

20 (5 marks)

$$g_1'(x) = -x^2, \text{ so } g_1'(\alpha) \approx -1.33$$

M1 A1

$|g_1'(\alpha)| > 1$, so the first iteration does not converge

E1

$$g_2'(x) = -\frac{10x}{(x^2 + 3)^2}, \text{ so } g_2'(\alpha) \approx -0.61$$

M1

$|g_2'(\alpha)| < 1$, so only the second rearrangement converges

A1

Answer: Only g_2 converges

5 marks

21 (8 marks)

(a) $y = e^{-x}$ is decreasing from 1 and $y = x^2$ is increasing from 0 for $x > 0$

M1

So the graphs cross exactly once for $x > 0$

A1

Answer: One positive root

2 marks

(b) $f(x) = e^{-x} - x^2$: $f(0.7) = 0.0066 > 0$ and $f(0.8) = -0.1907 < 0$

M1

Sign change and f continuous

A1

Answer: $0.7 < \alpha < 0.8$

2 marks

(c) $x^2 = e^{-x}$ and $x > 0$, so $x = e^{-x/2}$

M1

$$x_1 = 0.7047, x_2 = 0.7030, x_3 = 0.7036$$

M1 A1

The values settle at 0.7035, so $\alpha = 0.70$ to 2 d.p.

A1

Answer: 0.70

4 marks

22 (6 marks)

(a) g is a decreasing function, so successive values alternate on either side of the limit

E1

Answer: g is decreasing

1 mark

$$(b) L = \frac{4}{L+1}, \text{ so } L(L+1) = 4$$

M1

$$L^2 + L - 4 = 0$$

A1

$$L = \frac{-1 + \sqrt{17}}{2} \text{ since } L > 0$$

A1

$$\text{Answer: } L = \frac{-1 + \sqrt{17}}{2}$$

3 marks

$$(c) g'(x) = -\frac{4}{(x+1)^2}, \text{ so } g'(L) \approx -0.610$$

M1

$|g'(L)| < 1$, so each error is smaller than the last and the sequence converges

E1

Answer: $g'(L) \approx -0.610$

2 marks

23 (7 marks)

$$(a) e_0 = -0.14575, e_1 = 0.13631, e_2 = -0.10731, e_3 = 0.09786$$

M1 A1

Ratios: $-0.935, -0.787, -0.912$

A1

Answer: Ratios $\approx -0.935, -0.787, -0.912$

3 marks

$$(b) \frac{e_{n+1}}{e_n} \approx g'(\alpha), \text{ so } g'(\alpha) \approx -0.91$$

M1

Negative, so the errors alternate in sign; $|g'(\alpha)| < 1$ but not small, so convergence is slow

E1

Answer: $g'(\alpha) \approx -0.91$

2 marks

(c) $|e_4| = 0.079$, and each step multiplies the error by about 0.80

M1

$$\text{Need } |e_4| \times (0.80)^k < 10^{-6}, \text{ so } k = 52$$

A1

Answer: $k = 52$

2 marks

24 (6 marks)

(a) $\tan x$ has an asymptote at $x = \frac{\pi}{2} \approx 1.571$, which lies inside $[1, 2]$

E1

So f is not continuous on $[1, 2]$ and the sign change is caused by the asymptote, not a root

E1

Answer: Asymptote at $x = \pi/2$

2 marks

(b) $f(2) = -1.185 < 0$ and $f(2.5) = 0.253 > 0$

M1

f is continuous on $[2, 2.5]$ since $\frac{\pi}{2} < 2$ and $2.5 < \frac{3\pi}{2}$, so a root exists

A1

Answer: Root in $[2, 2.5]$

2 marks

(c) $f(2.25) = -0.239 < 0$, so the root is in $[2.25, 2.5]$

M1

$f(2.375) = 0.037 > 0$, so the root is in $[2.25, 2.375]$

A1

Answer: $[2.25, 2.375]$

2 marks

25 (8 marks)

(a) $L = \frac{L^2 + 2}{3} \implies L^2 - 3L + 2 = 0$

M1

$(L - 1)(L - 2) = 0$, so $L = 1$ or $L = 2$

A1

Answer: $L = 1$ or 2

2 marks

(b) $x_1 = 1.4167, x_2 = 1.3356, x_3 = 1.2613$

M1 A1

The values approach 1

A1

Answer: Converges to 1

3 marks

(c) $g'(x) = \frac{2x}{3}$

B1

$g'(1) = \frac{2}{3}$, so 1 is attracting, but $g'(2) = \frac{4}{3} > 1$, so 2 is repelling

E1

Starting at $x_0 = 2.5$ gives 2.750, 3.188, ..., moving away from 2

E1

Answer: $g'(2) = 4/3 > 1$

3 marks

26 (8 marks)

(a) $\frac{\pi h^2(30 - h)}{3} = 800 \implies \pi h^2(30 - h) = 2400$

M1

$30h^2 - h^3 = \frac{2400}{\pi}$, so $h^3 - 30h^2 + \frac{2400}{\pi} = 0$

A1

Answer: h satisfies $h^3 - 30h^2 + \frac{2400}{\pi} = 0$

2 marks

(b) Let $f(h) = h^3 - 30h^2 + \frac{2400}{\pi}$: $f(5) = 138.94$ and $f(6) = -100.06$

M1

Sign change and f continuous

A1

Answer: $5 < h < 6$

2 marks

(c) $30h^2 = h^3 + \frac{2400}{\pi}$, so $h = \sqrt{\frac{h^3 + 2400}{\frac{\pi}{30}}}$

M1

$h_1 = 5.4435, h_2 = 5.5535, h_3 = 5.5834$

M1 A1

The values converge to 5.5949, so the depth is 5.59 cm

A1

Answer: 5.59 cm

4 marks

27 (10 marks)

(a) At the limit L , $L = L - a f(L)$

M1

So $a f(L) = 0$ and since $a \neq 0$, $f(L) = 0$

A1

Answer: The limit L satisfies $f(L) = 0$

2 marks

(b) $g'(x) = 1 - a f'(x)$ with $f'(x) = 3x^2 + 1$, so $f'(\alpha) \approx 5.42$

M1

Convergence needs $|1 - a f'(\alpha)| < 1$, i.e. $0 < a f'(\alpha) < 2$

M1

So $0 < a < \frac{2}{f'(\alpha)} \approx 0.37$

A1

Answer: $0 < a < 0.37$

3 marks

(c) $x_1 = 1 - 0.2 \times f(1) = 1.20000$

M1

$x_2 = 1.21440, x_3 = 1.21333$

A1

Answer: 1.20000, 1.21440, 1.21333

2 marks

(d) $g'(\alpha) = 1 - a f'(\alpha) = 0$ gives $a = \frac{1}{f'(\alpha)}$

M1

$a \approx 0.185$

A1

Errors shrink faster than linearly ($g'(\alpha) = 0$), so each step gains accuracy rapidly

E1

Answer: $a \approx 0.185$

3 marks

28 (8 marks)

(a) $x_3 - \alpha \approx \lambda(x_2 - \alpha)$ and $x_2 - \alpha \approx \lambda(x_1 - \alpha)$; subtract: $x_3 - x_2 \approx \lambda(x_2 - x_1)$

M1 A1

Also $x_3 - \alpha = (x_3 - x_2) + (x_2 - \alpha) \approx (x_3 - x_2) + \frac{x_3 - \alpha}{\lambda}$

M1

Hence $(x_3 - \alpha)(1 - \frac{1}{\lambda}) = x_3 - x_2$, so $x_3 - \alpha = \frac{\lambda(x_3 - x_2)}{\lambda - 1}$, and with $\lambda = \frac{x_3 - x_2}{x_2 - x_1}$ the result

A1

follows

Answer: $\alpha \approx x_3 - \frac{(x_3 - x_2)^2}{(x_3 - x_2) - (x_2 - x_1)}$

4 marks

(b) $x_3 - x_2 = 0.062455$, $x_2 - x_1 = -0.073516$

M1

$\alpha \approx 2.34612$

A1

This is much closer to α than x_3 (error 0.0004 compared with 0.029)

E1

Answer: 2.34612

3 marks

(c) $\lambda = \frac{x_3 - x_2}{x_2 - x_1} \approx -0.85$ is negative, so the terms alternate either side of α

E1

Answer: Oscillation

1 mark

29 (12 marks)

(a) Let $f(x) = e^x - 4x$: $f(0) = 1$, $f(1) = -1.282$, $f(2) = -0.611$, $f(3) = 8.086$

M1 A1

Sign changes in $[0, 1]$ and $[2, 3]$ and f is continuous

A1

Answer: Sign changes in $[0, 1]$ and $[2, 3]$, so $\alpha \in [0, 1]$ and $\beta \in [2, 3]$

3 marks

(b) $x_1 = 2.0794$, $x_2 = 2.1184$, $x_3 = 2.1370 \dots$

M1 A1

The values settle at 2.1516, so $\beta = 2.15$ to 2 d.p.

A1

Answer: $\beta = 2.15$

3 marks

(c) $g'(x) = \frac{1}{x}$: $g'(\beta) \approx 0.46 < 1$ but $g'(\alpha) \approx 2.80 > 1$

M1

Near α the errors grow at each step, so the iteration moves away from α

E1

Answer: $g'(\beta) < 1 < g'(\alpha)$

2 marks

(d) $x_1 = 0.4122$, $x_2 = 0.3775 \dots$

M1 A1

The values settle at 0.3574, so $\alpha = 0.357$

A1

$h(x) = \frac{e^x}{4}$ has $h'(\beta) = \frac{e^\beta}{4} \approx 2.15 > 1$, so β is repelling

E1

Answer: $\alpha = 0.357$

4 marks

30 (11 marks)

(a) $x_{n+1} = g(\alpha + e_n) = g(\alpha) + e_n g'(\alpha) + \frac{e_n^2}{2} g''(\alpha) + \dots$

M1 A1

Since $g(\alpha) = \alpha$, $e_{n+1} = x_{n+1} - \alpha \approx g'(\alpha) e_n$ when e_n^2 is negligible

A1

Answer: $e_{n+1} \approx g'(\alpha) e_n$

3 marks

(b) $g'(x) = -\frac{10x}{(x^2 + 3)^2}$

M1

$g'(\alpha) \approx -0.615$

A1

Answer: -0.615

2 marks

(c) $e_4 \approx (g'(\alpha))^4 e_0$

M1

$\approx (-0.615)^4 \times 0.3458 \approx 0.049$

A1

This agrees well with the actual error of 0.047

E1

Answer: 0.049

3 marks

(d) Here $g(x) = x - \frac{f(x)}{f'(x)}$, so $g'(x) = \frac{f(x) f''(x)}{(f'(x))^2}$

M1 A1

At a root $f(\alpha) = 0$, so $g'(\alpha) = 0$ and the error decreases faster than linearly

E1

Answer: $g'(\alpha) = 0$

3 marks

Newton-Raphson Method

1 (2 marks)

$$f'(x) = 3x^2 - 1 \quad \text{B1}$$

$$x_{n+1} = x_n - \frac{x_n^3 - x_n - 1}{3x_n^2 - 1} \quad \text{B1}$$

$$\text{Answer: } x_{n+1} = x_n - \frac{x_n^3 - x_n - 1}{3x_n^2 - 1} \quad \text{2 marks}$$

2 (2 marks)

$$f'(x) = 2x, \text{ so } x_1 = 3 - \frac{9 - 10}{6} \quad \text{M1}$$

$$x_1 = 3.1667 \quad \text{A1}$$

$$\text{Answer: } 3.1667 \quad \text{2 marks}$$

3 (2 marks)

$$f'(x) = 3x^2 + 2, f(1.3) = 0.797, f'(1.3) = 7.07 \quad \text{M1}$$

$$x_1 = 1.3 - \frac{0.797}{7.07} = 1.1873 \quad \text{A1}$$

$$\text{Answer: } 1.1873 \quad \text{2 marks}$$

4 (3 marks)

$$f'(x) = e^x - 3 \quad \text{B1}$$

$$f(0.5) = 0.1487, f'(0.5) = -1.3513 \quad \text{M1}$$

$$x_1 = 0.5 - \frac{0.1487}{-1.3513} = 0.6101 \quad \text{A1}$$

$$\text{Answer: } 0.6101 \quad \text{3 marks}$$

5 (3 marks)

$$f(x) = \cos x - x, \text{ so } f'(x) = -\sin x - 1 \quad \text{B1}$$

$$f(0.7) = 0.0648, f'(0.7) = -1.6442 \quad \text{M1}$$

$$x_1 = 0.7 - \frac{0.0648}{-1.6442} = 0.7394 \quad \text{A1}$$

$$\text{Answer: } 0.7394 \quad \text{3 marks}$$

6 (3 marks)

$$x_{n+1} = x_n - \frac{x_n^2 - 2}{2x_n} \quad \text{B1}$$

$$x_1 = 1.5 - \frac{0.25}{3} = 1.41667 \quad \text{M1}$$

$$x_2 = 1.41422 \quad \text{A1}$$

$$\text{Answer: } x_1 = 1.41667, x_2 = 1.41422 \quad \text{3 marks}$$

7 (4 marks)

$$f'(x) = 2x, \text{ so } x_{n+1} = x_n - \frac{x_n^2 - N}{2x_n} \quad \text{M1}$$

$$= \frac{2x_n^2 - x_n^2 + N}{2x_n} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right) \quad \text{A1}$$

$$x_1 = \frac{1}{2}\left(3 + \frac{11}{3}\right) = 3.33333, x_2 = 3.31667 \quad \text{M1}$$

$$x_3 = 3.316625, \text{ so } \sqrt{11} = 3.3166 \quad \text{A1}$$

$$\text{Answer: } 3.3166 \quad \text{4 marks}$$

8 (4 marks)

$$f'(x) = 3x^2$$

B1

$$x_1 = 2 - \frac{3}{12} = 1.75000$$

M1

$$x_2 = 1.710884, x_3 = 1.709976$$

A1

$$\sqrt[3]{5} = 1.70998 \text{ to 5 d.p.}$$

A1

Answer: 1.70998

4 marks

9 (4 marks)

$$(a) x_{n+1} = x_n - \frac{x_n^3 - 2x_n - 5}{3x_n^2 - 2}$$

M1

$$= \frac{3x_n^3 - 2x_n - x_n^3 + 2x_n + 5}{3x_n^2 - 2} = \frac{2x_n^3 + 5}{3x_n^2 - 2}$$

A1

$$\text{Answer: } x_{n+1} = \frac{2x_n^3 + 5}{3x_n^2 - 2}$$

2 marks

$$(b) x_1 = 2.10000, x_2 = 2.09457$$

M1

$$x_3 = 2.094551$$

A1

Answer: 2.094551

2 marks

10 (5 marks)

$$(a) f(2) = -0.3069 < 0 \text{ and } f(3) = 1.0986 > 0$$

M1

Sign change and f continuous

A1

Answer: $f(2) < 0 < f(3)$, so a root lies in $[2, 3]$

2 marks

$$(b) f'(x) = \frac{1}{x} + 1$$

B1

$$x_1 = 2.20457, x_2 = 2.20794$$

M1

$$x_3 = 2.207940, \text{ so the root is } 2.2079$$

A1

Answer: 2.2079

3 marks

11 (5 marks)

$$(a) x^3 + (2x - 1)^3 = 9, \text{ and } (2x - 1)^3 = 8x^3 - 12x^2 + 6x - 1$$

M1

$$9x^3 - 12x^2 + 6x - 1 = 9, \text{ so } 9x^3 - 12x^2 + 6x - 10 = 0$$

A1

$$\text{Answer: } 9x^3 - 12x^2 + 6x - 10 = 0$$

2 marks

$$(b) f'(x) = 27x^2 - 24x + 6$$

B1

$$f(1.4) = -0.424, f'(1.4) = 25.32$$

M1

$$x_1 = 1.4 - \frac{-0.424}{25.32} = 1.417$$

A1

Answer: 1.417

3 marks

12 (3 marks)

The tangent passes through $(x_0, f(x_0))$ and $(x_1, 0)$, so its gradient is $\frac{f(x_0) - 0}{x_0 - x_1}$

M1

This gradient equals $f'(x_0)$, so $f'(x_0)(x_0 - x_1) = f(x_0)$

A1

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

A1

$$\text{Answer: } x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

3 marks

13 (6 marks)

(a) Sketch: $y = 3^x$ is positive, passes through $(0, 1)$ and rises faster than $y = x^2$ for $x > 0$

M1

The graphs cross only once, for $x < 0$, where 3^x decreases to 0 while x^2 increases

E1

Answer: One real root, negative

2 marks

(b) Let $f(x) = 3^x - x^2$: $f(-1) = -0.667 < 0$ and $f(0) = 1 > 0$

B1

Answer: $f(-1) < 0 < f(0)$

1 mark

$$(c) f'(x) = 3^x \ln 3 - 2x$$

B1

$$x_1 = -0.70030, x_2 = -0.68610$$

M1

$$x_3 = -0.686027, \text{ so the root is } -0.6860$$

A1

Answer: -0.6860

3 marks

14 (6 marks)

(a) $f'(x) = 3x^2 - 3$, so $f'(1) = 0$

M1

The tangent at $x = 1$ is horizontal, so it never meets the x-axis (division by zero)

E1

Answer: $f'(1) = 0$

2 marks

(b) $f(1.1) = -0.969$, $f(1.1) = 0.63$

M1

$$x_1 = 2.6381$$

A1

$$x_2 = 1.9979, x_3 = 1.6657$$

A1

Answer: 2.6381, 1.9979, 1.6657

3 marks

(c) x_0 is close to a stationary point, where f' is small, so the tangent is nearly horizontal and meets the axis far away

E1

Answer: Gradient close to 0

1 mark

15 (6 marks)

(a) $f(x) = 3 \sin x - x$, $f'(x) = 3 \cos x - 1$

B1

$$x_1 = 2.323732$$

M1

$$x_2 = 2.279595, x_3 = 2.278863$$

A1

Answer: 2.279595, 2.278863

3 marks

(b) $f(2.27885) = 0.000037 > 0$

M1

$$f(2.27895) = -0.00026 < 0$$

A1

Sign change in this interval, so $\alpha = 2.2789$ to 4 d.p.

A1

Answer: $\alpha = 2.2789$

3 marks

16 (5 marks)

Let $f(x) = x^3 - 2^x$. Then $f'(x) = 3x^2 - 2^x \ln 2$

M1 A1

$$x_1 = 1.4 - \frac{f(1.4)}{f'(1.4)} = 1.374083$$

M1

$$x_2 = 1.373467, x_3 = 1.373467$$

A1

The x-coordinate is 1.37347 to 5 d.p.

A1

Answer: 1.37347

5 marks

17 (6 marks)

$\frac{dy}{dx} = e^x - 3x^2 = 0$ at a stationary point, so $e^x = 3x^2$

B1

Let $g(x) = e^x - 3x^2$, $g'(x) = e^x - 6x$

M1 A1

$$x_1 = 1 - \frac{g(1)}{g'(1)} = 0.91416, x_2 = 0.91002$$

M1

$$x_3 = 0.910008, x_4 = 0.910008$$

A1

The x-coordinate is 0.9100

A1

Answer: 0.9100

6 marks

18 (7 marks)

(a) $\frac{dx}{dy} = \frac{3y^2 - 2}{y^3 - 2y}$

M1 A1

$\frac{dy}{dx} = 5$ gives $\frac{dx}{dy} = \frac{1}{5}$, so $5(3y^2 - 2) = y^3 - 2y$, i.e. $y^3 - 15y^2 - 2y + 10 = 0$

A1

Answer: $y^3 - 15y^2 - 2y + 10 = 0$

3 marks

(b) $f(y) = y^3 - 15y^2 - 2y + 10$, $f'(y) = 3y^2 - 30y - 2$, $y_1 = 15 - \frac{-20}{223} = 15.08969$

M1

$$y_2 = 15.08863$$

A1

Answer: 15.089

2 marks

(c) $y = 15.089$

B1

$$x = \ln(15.089^3 - 2(15.089)) = 8.13$$

A1

Answer: P(8.13, 15.09)**2 marks****19 (6 marks)**

$$(a) x_{n+1} = x_n - \frac{x_n^2 - 3}{2x_n}$$

B1

$$x_1 = -2.0000, x_2 = -1.7500$$

M1

$$x_3 = -1.7321$$

A1**Answer: -1.7321****3 marks**(b) The values approach $-\sqrt{3} \approx -1.7321$ **B1****Answer: $-\sqrt{3}$** **1 mark**

$$(c) x_1 = 0.5 - \frac{0.25 - 3}{1} = 3.25$$

M1 x_1 is positive and larger, and the sequence then approaches $+\sqrt{3}$ **A1****Answer: $x_1 = 3.25$; towards $+\sqrt{3}$** **2 marks****20 (6 marks)**Let $f(E) = E - 0.3 \sin E - 1$, so $f'(E) = 1 - 0.3 \cos E$ **M1 A1**

$$f(1) = -0.25244, f'(1) = 0.83791$$

M1

$$E_1 = 1 - \frac{-0.25244}{0.83791} = 1.301275$$

A1

$$E_2 = 1.288119$$

A1 $E = 1.28809$ to 5 d.p.**A1****Answer: 1.28809****6 marks****21 (8 marks)**

$$(a) x_{n+1} = x_n - \frac{x_n^3 - 4x_n + 1}{3x_n^2 - 4}$$

M1

$$= \frac{2x_n^3 - 1}{3x_n^2 - 4}$$

A1

$$\text{Answer: } x_{n+1} = \frac{2x_n^3 - 1}{3x_n^2 - 4}$$

2 marks

$$(b) x_1 = 1.87500, x_2 = 1.86098$$

M1

$$x_4 = 1.860806, \text{ so the root is } 1.8608$$

A1**Answer: 1.8608****2 marks**

$$(c) x_1 = 0.25000, x_2 = 0.25410$$

M1

The root is 0.2541

A1**Answer: 0.2541****2 marks**(d) From $x_0 = -2$ the iteration converges to -2.115 **M1**

$$\text{From } x_0 = 1, f(1) = -2 \text{ and } f'(1) = -1, \text{ so } x_1 = 1 - \frac{-2}{-1} = -1$$

A1**Answer: -2.115****2 marks****22 (7 marks)**(a) 8th term of the arithmetic series $= 3 + 7d$, 8th term of the geometric series $= 3d^7$ **M1**

$$3 + 7d + 3d^7 = 500, \text{ so } 3d^7 + 7d = 497$$

A1

$$\text{Answer: } 3d^7 + 7d = 497$$

2 marks(b) $f(d) = 3d^7 + 7d - 497$: $f(2) = -99 < 0$ and $f(2.1) = 58.03 > 0$ **M1**Sign change and f continuous, so $2 < d < 2.1$ **A1****Answer: $f(2) < 0 < f(2.1)$** **2 marks**

$$(c) f'(d) = 21d^6 + 7$$

B1

$$d_1 = 2.1 - \frac{58.03}{1808.1} = 2.06791$$

M1

$$d_2 = 2.06634, \text{ so } d = 2.066$$

A1**Answer: $d = 2.066$** **3 marks**

23 (7 marks)

$$(a) f'(x) = -\frac{1}{x^2} \quad \text{B1}$$

$$x_{n+1} = x_n - \frac{\frac{1}{x_n} - a}{-\frac{1}{x_n^2}} = x_n + x_n^2 \left(\frac{1}{x_n} - a \right) \quad \text{M1}$$

$$= x_n + x_n - a x_n^2 = x_n(2 - a x_n) \quad \text{A1}$$

$$\text{Answer: } x_{n+1} = x_n(2 - a x_n) \quad \text{3 marks}$$

$$(b) x_1 = 0.1 \times (2 - 0.7) = 0.130000 \quad \text{M1}$$

$$x_2 = 0.141700, x_3 = 0.142848, x_4 = 0.142857 \quad \text{A1}$$

$$\text{Answer: } 0.142857 \quad \text{2 marks}$$

$$(c) x_1 > 0 \text{ requires } 2 - 7x_0 > 0 \text{ for } x_0 > 0, \text{ so } 0 < x_0 < \frac{2}{7} \quad \text{M1}$$

$$x_0 = 0.5 \text{ gives } x_1 = 0.5(2 - 3.5) = -0.75 < 0 \text{ and the later values tend to } -\infty \quad \text{A1}$$

$$\text{Answer: } 0 < x_0 < 2/7 \quad \text{2 marks}$$

24 (8 marks)

$$(a) \frac{dy}{dx} = 2e^{2x}, \text{ so the tangent at P is } y - (e^{2p} + 2) = 2e^{2p}(x - p) \quad \text{M1 A1}$$

$$\text{Through } (0, 0): -e^{2p} - 2 = -2pe^{2p}, \text{ so } (2p - 1)e^{2p} = 2 \quad \text{A1}$$

$$\text{Answer: } (2p - 1)e^{2p} = 2 \quad \text{3 marks}$$

$$(b) \text{ Let } g(p) = (2p - 1)e^{2p} - 2: g(0.7) = -0.378 < 0 \text{ and } g(0.8) = 0.972 > 0 \quad \text{M1}$$

$$\text{Sign change and } g \text{ continuous} \quad \text{A1}$$

$$\text{Answer: } g(0.7) < 0 < g(0.8) \quad \text{2 marks}$$

$$(c) g'(p) = 2e^{2p} + 2(2p - 1)e^{2p} = 4pe^{2p} \quad \text{M1}$$

$$p_1 = 0.8 - \frac{0.9718}{15.850} = 0.73869 \quad \text{A1}$$

$$p_2 = 0.73161, \text{ so } p = 0.7316 \quad \text{A1}$$

$$\text{Answer: } p = 0.7316 \quad \text{3 marks}$$

25 (8 marks)

$$(a) \text{ Rearranging the given equation gives } f(r) = 0 \quad \text{B1}$$

$$f(r) = 450 \times 24(1 + r)^{-25} - 10\,000 \quad \text{B1}$$

$$\text{Answer: } f'(r) = 10\,800(1 + r)^{-25} - 10\,000 \quad \text{2 marks}$$

$$(b) f(0.01) = -4.4048, f'(0.01) = -1578.501 \quad \text{M1 A1}$$

$$r_1 = 0.007210, r_2 = 0.006357, r_3 = 0.006252 \quad \text{M1}$$

$$r_4 = 0.0062507, \text{ so } r = 0.00625 \text{ to 5 d.p.} \quad \text{A1}$$

$$\text{Answer: } 0.00625 \quad \text{4 marks}$$

$$(c) \text{ Annual rate} = 12 \times 0.006251 \quad \text{M1}$$

$$= 7.50\% \quad \text{A1}$$

$$\text{Answer: } 7.50\% \quad \text{2 marks}$$

26 (9 marks)

$$(a) AP^2 = (x - 2)^2 + (\sqrt{x} - 1)^2 = x^2 - 3x + 5 - 2\sqrt{x} \quad \text{M1}$$

$$\frac{d(AP^2)}{dx} = 2x - 3 - \frac{1}{\sqrt{x}} \quad \text{M1 A1}$$

$$\text{Answer: } AP^2 \text{ is minimised when } 2x - 3 - \frac{1}{\sqrt{x}} = 0 \quad \text{3 marks}$$

$$(b) f(x) = 2x - 3 - x^{-1/2}, f'(x) = 2 + \frac{1}{2}x^{-3/2} \quad \text{M1 A1}$$

$$x_1 = 1.86545, x_2 = 1.86603 \quad \text{M1}$$

$$x = 1.8660 \text{ to 4 d.p.} \quad \text{A1}$$

$$\text{Answer: } 1.8660 \quad \text{4 marks}$$

$$(c) AP^2 = (1.8660 - 2)^2 + (\sqrt{1.8660} - 1)^2 = 0.1519 \quad \text{M1}$$

$$AP = 0.390 \quad \text{A1}$$

Answer: 0.390**2 marks****27 (10 marks)**

$$(a) 0 = f(\alpha) = f(x_n) - e_n f'(x_n) + \frac{e_n^2}{2} f''(\xi) \text{ for some } \xi \text{ between } \alpha \text{ and } x_n$$

M1 A1

$$\text{Divide by } f'(x_n): \frac{f(x_n)}{f'(x_n)} = e_n - \frac{e_n^2 f''(\xi)}{2f'(x_n)}$$

M1

$$e_{n+1} = x_{n+1} - \alpha = e_n - \frac{f(x_n)}{f'(x_n)}$$

M1

$$\text{So } e_{n+1} = \frac{e_n^2 f''(\xi)}{2f'(x_n)} \approx \frac{f''(\alpha)}{2f'(\alpha)} e_n^2$$

A1

$$\text{Answer: } e_{n+1} \approx \frac{f''(\alpha)}{2f'(\alpha)} e_n^2$$

5 marks

$$(b) f'(x) = 3x^2, f''(x) = 6x, \text{ so } \frac{f''(\alpha)}{2f'(\alpha)} = \frac{6\alpha}{6\alpha^2} = \frac{1}{\alpha}$$

M1

$$= 0.585$$

A1**Answer: 0.585****2 marks**

$$(c) e_1 \approx 0.585 \times (0.2900)^2 = 0.0492$$

M1 A1

$$e_2 \approx 0.585 \times (0.0400)^2 = 0.00094, \text{ close to the actual values}$$

A1**Answer: 0.0492, 0.00094****3 marks****28 (9 marks)**

$$(a) f'(x) = \frac{1}{3}x^{-2/3}$$

B1

$$x_{n+1} = x_n - \frac{x_n^{1/3}}{\frac{1}{3}x_n^{-2/3}} = x_n - 3x_n$$

M1

$$= -2x_n$$

A1

$$\text{Answer: } x_{n+1} = -2x_n$$

3 marks

$$(b) x_n = (-2)^n a$$

B1

$|x_n|$ grows without limit and the sign alternates, so the iteration diverges from the root

E1**Answer: $x_n = (-2)^n a$; diverges****2 marks**

$$(c) f'(x) = \frac{1}{3}x^{-2/3} \rightarrow \infty \text{ as } x \rightarrow 0, \text{ so } f' \text{ is not defined (not continuous) at the root}$$

E1

The conditions for fast convergence (f' finite and non-zero at α , f'' bounded) do not hold

E1**Answer: f' unbounded at the root****2 marks**

$$(d) |x_n| = 2^n \times 0.1 > 1000, \text{ so } 2^n > 10\,000$$

M1

$$2^{13} = 8192 < 10\,000 < 16\,384 = 2^{14}, \text{ so } n = 14$$

A1**Answer: $n = 14$** **2 marks****29 (10 marks)**

$$(a) f'(x) = 2(x-1)(x+2) + (x-1)^2 = (x-1)(3x+3) = 3(x-1)(x+1)$$

M1 A1

Dividing f by f' cancels a factor $(x-1)$, giving the stated iteration

A1

$$\text{Answer: } f'(x) = 3(x-1)(x+1); x_{n+1} = x_n - \frac{(x_n-1)(x_n+2)}{3(x_n+1)}$$

3 marks

$$(b) x_1 = 1.55556, x_2 = 1.29791, x_3 = 1.15539$$

M1 A1

$$\text{Ratios } \frac{e_1}{e_0} = 0.556, \frac{e_2}{e_1} = 0.536, \frac{e_3}{e_2} = 0.522, \text{ tending to } \frac{1}{2}$$

A1**Answer: Ratios tend to $1/2$** **3 marks**

$$(c) f'(1) = 0, \text{ so the condition } f'(\alpha) \neq 0 \text{ fails and the error is only halved at each step}$$

E1**Answer: $f'(\alpha) = 0$** **1 mark**

$$(d) x_1 = 1.111111$$

M1

$$x_2 = 1.001949, x_3 = 1.000001$$

A1

The sequence reaches 1 much faster, so the factor 2 restores quadratic convergence for a double root

E1

Answer: $x_1 = 1.111111$, $x_2 = 1.001949$, $x_3 = 1.000001$ **3 marks****30 (10 marks)****(a)** $f(0) = k$ and $f'(0) = -2$ **B1**

$$x_1 = 0 - \frac{k}{-2} = \frac{k}{2}$$

A1**Answer: $x_1 = \frac{k}{2}$** **2 marks****(b)** $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0 \iff x_1 f'(x_1) = f(x_1)$ **M1**

$$x_1(3x_1^2 - 2) = x_1^3 - 2x_1 + k$$

M1

$$3x_1^3 - 2x_1 = x_1^3 - 2x_1 + k, \text{ so } 2x_1^3 = k$$

A1**Answer: $x_2 = 0$ if and only if $2x_1^3 = k$** **3 marks****(c)** Substituting $x_1 = \frac{k}{2}$: $2 \frac{k^3}{8} = k$, so $k^3 = 4k$ **M1**

$$k \neq 0, \text{ so } k^2 = 4$$

M1

$$k = 2 \text{ or } k = -2$$

A1**Answer: $k = \pm 2$** **3 marks****(d)** $x_n = 0$ for even n and $x_n = 1$ for odd n **B1**

The iterates cycle between 0 and 1 and never settle, because the tangents at these points cross the axis at each other; the real root (≈ -1.769) is never approached

E1**Answer: Cycle 0, 1, 0, 1, ...****2 marks**

Numerical Integration

1 (2 marks)

$$h = 1, \text{ so area } \approx \frac{1}{2} \times 1 \times [3.2 + 4.4 + 2(4.1 + 5.5 + 5.0)]$$

M1

$$= 18.40$$

A1**Answer: 18.40****2 marks****2 (3 marks)**

$$h = 0.5, y_0 = 1, y_1 = \frac{1}{1.5} = \frac{2}{3}, y_2 = \frac{1}{2}$$

B1

$$\text{Area} \approx \frac{1}{3} \times 0.5 \times [1 + \frac{1}{2} + 4 \times \frac{2}{3}]$$

M1

$$= \frac{1}{6} \times \frac{25}{6} = \frac{25}{36}$$

A1**Answer: $\ln 2 \approx \frac{25}{36}$** **3 marks****3 (2 marks)**

$$h = 1, \text{ mid-ordinates at } x = 0.5 \text{ and } x = 1.5: e^{0.5} = 1.6487 \text{ and } e^{1.5} = 4.4817$$

M1

$$\text{Area} \approx 1 \times (1.6487 + 4.4817) = 6.130$$

A1**Answer: 6.130****2 marks****4 (3 marks)**

$$h = 2, y_0 = 0, y_1 = 4, y_2 = 16$$

B1

$$\text{Area} \approx \frac{1}{3} \times 2 \times [0 + 16 + 4(4)]$$

M1

$$= \frac{64}{3}, \text{ which is exact}$$

A1**Answer: $\frac{64}{3}$ (exact)****3 marks**

5 (2 marks)

$$h = \frac{4-1}{6} = 0.5$$

B1

$$x = 1, 1.5, 2, 2.5, 3, 3.5, 4$$

B1

Answer: $h = 0.5$; $x = 1, 1.5, 2, 2.5, 3, 3.5, 4$

2 marks

6 (2 marks)

$$y = 1, 1.414, 1.732, 2.000, 2.236$$

B1 B1

Answer: 1, 1.414, 1.732, 2.000, 2.236

2 marks

7 (4 marks)

$$h = 0.25; y \text{ values: } 1.00000, 0.94118, 0.80000, 0.64000, 0.50000$$

B1

$$\text{Area} \approx \frac{1}{2} \times 0.25 \times [1.00000 + 0.50000 + 2(0.94118 + 0.80000 + 0.64000)]$$

M1

$$= 0.7828$$

A1

$$\text{Exact value} = \arctan 1 = \frac{\pi}{4} = 0.7854; \text{error} \approx 0.33\%$$

A1

Answer: 0.7828

4 marks

8 (4 marks)

$$h = 0.5; y \text{ values: } 0.00000, 0.40547, 0.69315, 0.91629, 1.09861$$

B1

$$\text{Area} \approx \frac{1}{3} \times 0.5 \times [y_0 + y_4 + 4(y_1 + y_3) + 2y_2]$$

M1

$$= \frac{1}{3} \times 0.5 \times [1.09861 + 4(1.32176) + 2(0.69315)]$$

A1

$$= 1.2953$$

A1

Answer: 1.2953

4 marks

9 (7 marks)

$$(a) h = \frac{\pi}{8}; y \text{ values: } 1.00000, 0.96194, 0.85355, 0.69134, 0.50000$$

B1

$$I \approx \frac{1}{3} \times \frac{\pi}{8} \times [1.00000 + 0.50000 + 4(1.65328) + 2(0.85355)]$$

M1

$$= 1.2855$$

A1

Answer: 1.2855

3 marks

$$(b) \cos^2\left(\frac{x}{2}\right) + \sin^2\left(\frac{x}{2}\right) = 1, \text{ so } \int \sin^2\left(\frac{x}{2}\right) dx = \int_0^{\pi/2} 1 dx - I$$

M1

$$\approx \frac{\pi}{2} - 1.2855 = 0.2853$$

A1

Answer: 0.2853

2 marks

$$(c) \cos x = \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right) = 2\cos^2\left(\frac{x}{2}\right) - 1$$

M1

$$\int \cos x dx \approx 2(1.2855) - \frac{\pi}{2} = 1.0001$$

A1

Answer: 1.0001

2 marks

10 (4 marks)

$$h = 0.5; \text{mid-ordinates } x = 0.25, 0.75, 1.25, 1.75$$

B1

$$y \text{ values: } 1.00778, 1.19242, 1.71847, 2.52178$$

M1

$$\text{Area} \approx 0.5 \times (6.44045)$$

A1

$$= 3.220$$

A1

Answer: 3.220

4 marks

11 (3 marks)

$$\text{Distance} = \text{area under the speed-time graph, with } h = 5$$

B1

$$\approx \frac{1}{2} \times 5 \times [0 + 19 + 2(4 + 9 + 13 + 16 + 18)]$$

M1

$$= 347.5 \text{ m}$$

A1

Answer: 347.5 m

3 marks

12 (4 marks)

$h = 2$ and the interval has width 8

B1

$$\int f(x) dx \approx \frac{1}{3} \times 2 \times [4 + 16 + 4(10 + 19) + 2a] = \frac{2}{3}(136 + 2a)$$

M1

Mean value = $\frac{\text{integral}}{8}$, so $\frac{2}{3}(136 + 2a) = 14 \times 8 = 112$

M1

$136 + 2a = 168$, so $a = 16$

A1

Answer: $a = 16$

4 marks

13 (4 marks)

$h = \frac{\pi}{8}$

B1

y values: 1, 0.97450, 0.90032, 0.78421, 0.63662

M1

Area $\approx \frac{1}{3} \times \frac{\pi}{8} \times [1 + 0.63662 + 4(0.97450 + 0.78421) + 2(0.90032)]$

A1

$= 1.3708$

A1

Answer: 1.3708

4 marks

14 (6 marks)

(a) $T_2: h = 1, \frac{1}{2}[1 + \frac{1}{3} + 2(0.5)] = 1.16667$

M1 A1

$T_4: h = 0.5, \frac{1}{2} \times 0.5 \times [1 + \frac{1}{3} + 2(0.66667 + 0.5 + 0.40000)] = 1.11667$

A1

Answer: $T_2 = 1.16667$, $T_4 = 1.11667$

3 marks

(b) $\frac{4T_4 - T_2}{3} = \frac{4(1.11667) - 1.16667}{3}$

M1

$= 1.10000$, which equals $S_4 = \frac{1}{3} \times 0.5 \times [1 + \frac{1}{3} + 4(0.66667 + 0.40000) + 2(0.5)]$

A1

Answer: 1.10000

2 marks

(c) $\ln 3 = 1.09861$; the errors are 0.068 (T_2), 0.018 (T_4) and 0.0014 (S_4), so Simpson's rule is best

E1

Answer: Simpson's rule

1 mark

15 (6 marks)

(a) $h = 1: \frac{1}{2}[1 + 2 + 2(\sqrt{2} + \sqrt{3})]$

M1

$= 4.6463$

A1

Answer: 4.6463

2 marks

(b) $y'' = -\frac{1}{4}x^{-3/2} < 0$ so the curve is concave

M1

The chords lie below the curve: underestimate

E1

Answer: Underestimate

2 marks

(c) $\int x^{1/2} dx = \frac{2}{3}x^{3/2}$, so $I = \frac{2}{3}(8 - 1) = \frac{14}{3} = 4.6667$

M1

Percentage error = 0.44%

A1

Answer: $\frac{14}{3}$; 0.44%

2 marks

16 (5 marks)

$f''(x) = e^x$, so $M = e$ on $[0, 1]$

B1

$|E| \leq \frac{1 \times h^2}{12} \times e = \frac{e}{12n^2}$

M1 A1

Need $\frac{e}{12n^2} < 0.0005$, so $n^2 > 453.0$

M1

$n > 21.28$, so $n = 22$ strips

A1

Answer: $n = 22$

5 marks

17 (6 marks)**(a)** $h = 10$, and the end widths are both 0

B1

$$\text{Area} \approx \frac{1}{2} \times 10 \times [0 + 0 + 2(18 + 26 + 31 + 29 + 20)]$$

M1

$$= 1240 \text{ m}^2$$

A1

Answer: 1240 m²

3 marks

$$\text{(b) Area} \approx \frac{1}{3} \times 10 \times [0 + 0 + 4(18 + 31 + 20) + 2(26 + 29)]$$

M1 A1

$$= 1286.7 \text{ m}^2$$

A1

Answer: 1287 m²

3 marks

18 (6 marks)

$$V = \pi \int y^2 dx = \pi \int_0^1 (e^{-x^2})^2 dx = \pi \int_0^1 e^{-2x^2} dx$$

M1 A1

$$h = 0.25; y^2 \text{ values: } 1.00000, 0.88250, 0.60653, 0.32465, 0.13534$$

B1

$$\int \approx \frac{1}{3} \times 0.25 \times [1.00000 + 0.13534 + 4(1.20715) + 2(0.60653)] = 0.59808$$

M1 A1

$$V \approx \pi \times 0.59808 = 1.88$$

A1

Answer: 1.88

6 marks

19 (7 marks)

$$h = 1; y \text{ values: } 0, 1.38629, 3.29584, 5.54518, 8.04719$$

B1

$$\text{Area} \approx \frac{1}{3} \times 1 \times [0 + 8.04719 + 4(1.38629 + 5.54518) + 2(3.29584)]$$

M1 A1

$$= 14.1216$$

A1

$$\int x \ln x dx = \frac{x^2}{2} \ln x - \int \frac{x}{2} dx = \frac{x^2}{2} \ln x - \frac{x^2}{4}$$

M1 A1

$$I = (12.5 \ln 5 - 6.25) - (0 - 0.25) = 14.1180$$

A1

Answer: 14.1216; exact 14.1180; error 0.0036

7 marks

20 (6 marks)

$$\text{(a) } h = 0.5; \text{ ordinates } \frac{k}{2}, \frac{k}{2.5}, \frac{k}{3}, \frac{k}{3.5}, \frac{k}{4}$$

B1

$$\frac{1}{2} \times 0.5 \times k[0.5000 + 0.2500 + 2(1.01905)] = 0.69702 k$$

M1

$$0.69702 k = 6.97, \text{ so } k = 10.000 \approx 10$$

A1

Answer: k = 10

3 marks

(b) Use more strips (a smaller strip width h)

E1

Answer: Use more strips

1 mark

$$\text{(c) Replacing } x \text{ by } 2t: \int_0^2 \frac{10}{x+2} dx = \int_0^1 \frac{20}{2t+2} dt = \int_0^1 \frac{10}{t+1} dt$$

M1

$$\text{So } \int_0^1 \frac{20}{x+1} dx = 2 \times 6.97 = 13.94$$

A1

Answer: 13.94

2 marks

21 (8 marks)**(a)** As $x \rightarrow 0$ the integrand $\frac{e^{-x}}{\sqrt{x}} \rightarrow \infty$, so there is no value y_0 to use

E1

Answer: Integrand is unbounded at 0

1 mark

(b) $x = u^2$, so $dx = 2u du$

B1

$$\frac{e^{-x}}{\sqrt{x}} dx = \frac{e^{-u^2}}{u} \times 2u du = 2e^{-u^2} du$$

M1

$$\text{Limits } u = 0 \text{ and } u = 1, \text{ so } J = 2 \int_0^1 e^{-u^2} du$$

A1

Answer: J = 2 \int_0^1 e^{-u^2} du

3 marks

$$\text{(c) } h = 0.25; y \text{ values: } 1.00000, 0.93941, 0.77880, 0.56978, 0.36788$$

B1

$$\int_0^1 e^{-u^2} du \approx \frac{1}{3} \times 0.25 \times [1.00000 + 0.36788 + 4(1.50920) + 2(0.77880)]$$

M1 A1

$$J \approx 2 \times 0.74686 = 1.4937$$

A1

Answer: 1.4937

4 marks

22 (8 marks)

$$(a) T_2 = \frac{1}{2} \times 0.5 \times [0 + 1 + 2(0.0625)] = 0.281250$$

M1 A1

$$T_4 = 0.220703 \text{ and } T_8 = 0.205200$$

A1

Answer: 0.281250, 0.220703, 0.205200

3 marks

$$(b) E_2 = 0.081250, E_4 = 0.020703, E_8 = 0.005200$$

M1 A1

Ratios: 3.92 and 3.98

A1

Answer: 3.92, 3.98

3 marks

(c) Halving h divides the error by about 4, so $E \propto h^2$

E1

$$E_{16} \approx \frac{E_8}{4} \approx 0.0013$$

A1

Answer: 0.0013

2 marks

23 (6 marks)(a) The formula is $\frac{h}{2}[\dots]$, so the factor is $\frac{1}{2} \times 0.5 = 0.25$, not 0.5

E1

$$\text{Correct estimate} = 0.25 \times [1.0000 + 2.2361 + 2(4.3350)]$$

M1

$$= 2.977$$

A1

Answer: 2.977

3 marks

(b) Simpson's rule requires an even number of strips, so that the ordinates can be paired by parabolas

E1

Answer: Needs an even number of strips

1 mark

$$(c) S_4 (h = 0.5) = 2.9580$$

M1

$$S_6 = 2.9579 (h = \frac{1}{3}); \text{ more strips, so } S_6 \text{ is likely to be more accurate}$$

A1

Answer: $S_4 = 2.9580$, $S_6 = 2.9579$

2 marks

24 (7 marks)(a) Area = sum of trapezia $\frac{1}{2}(\text{width})(\text{sum of parallel sides})$

M1

$$\frac{1}{2}[1(2.0 + 2.8) + 2(2.8 + 4.1) + 3(4.1 + 3.5) + 4(3.5 + 1.2)]$$

M1

$$= 30.10 \text{ m}^2$$

A1

Answer: 30.10 m²

3 marks

$$(b) \text{ Trapezium: } \frac{1}{2} \times 2 \times [3.0 + 2.9 + 2(4.4 + 5.1 + 4.6)] = 34.10$$

M1 A1

$$\text{Simpson: } \frac{1}{3} \times 2 \times [3.0 + 2.9 + 4(4.4 + 4.6) + 2(5.1)] = 34.73$$

M1 A1

Answer: Trapezium 34.1 m²; Simpson 34.7 m²

4 marks

25 (9 marks)

$$(a) f'(x) = \ln x + 1, f''(x) = \frac{1}{x} > 0$$

M1 A1

Convex: trapezium rule overestimates and mid-ordinate rule underestimates

E1

Answer: T over, M under

3 marks

$$(b) h = 0.5; T_4 = \frac{1}{2} \times 0.5 \times [f(1) + f(3) + 2(f(1.5) + f(2) + f(2.5))]$$

M1

$$T_4 = 2.9666$$

A1

$$M_4 = 0.5 \times [f(1.25) + f(1.75) + f(2.25) + f(2.75)] = 2.9324$$

A1

Answer: T = 2.9666, M = 2.9324

3 marks

$$(c) \int x \ln x \, dx = \frac{x^2}{2} \ln x - \frac{x^2}{4}$$

M1 A1

$$I = (4.5 \ln 3 - 2.25) - (-0.25) = 2.9438, \text{ and } 2.9324 < 2.9438 < 2.9666$$

A1

Answer: 2.9438**3 marks****26 (8 marks)**

$$g'(x) = \cos x, \text{ so } L = \int_0^{\pi} \sqrt{1 + \cos^2 x} \, dx$$

M1 A1

$$h = \frac{\pi}{6}; \text{ y values: } 1.41421, 1.32288, 1.11803, 1.00000, 1.11803, 1.32288, 1.41421$$

B1 M1

$$L \approx \frac{1}{3} \times \frac{\pi}{6} \times [1.41421 + 1.41421 + 4(3.64575) + 2(2.23607)]$$

M1 A1

$$= 3.8194$$

A1

$$L \approx 3.82$$

A1**Answer: 3.82****8 marks****27 (9 marks)**

$$(a) \int_{-h}^h x^3 \, dx = \int_{-h}^h x \, dx = 0 \text{ by symmetry (odd functions)}$$

M1

$$\int (bx^2 + d) \, dx = \left[\frac{bx^3}{3} + dx \right]_{-h}^h = \frac{2bh^3}{3} + 2dh$$

A1

$$\text{Answer: } \int_{-h}^h f(x) \, dx = \frac{2bh^3}{3} + 2dh$$

2 marks

$$(b) f(h) + f(-h) = 2bh^2 + 2d$$

M1 A1

$$f(0) = d, \text{ so } f(-h) + 4f(0) + f(h) = 2bh^2 + 6d$$

M1

$$S = \frac{h}{3}(2bh^2 + 6d) = \frac{2bh^3}{3} + 2dh, \text{ equal to the integral}$$

A1

$$\text{Answer: } S = \frac{2bh^3}{3} + 2dh \text{ for the same } f, \text{ so that Simpson's rule is exact for any cubic}$$

4 marks

(c) Translating the x-axis ($x \rightarrow x - m$) turns a cubic into another cubic, so the result holds on any interval

E1

$$S = \frac{1}{3} \times 1 \times [f(0) + 4f(1) + f(2)] = \frac{1}{3}[1 + 0 + 5] = 2$$

M1

$$\text{Exact: } \left[\frac{x^4}{4} - x^2 + x \right]_0^2 = 4 - 4 + 2 = 2$$

A1**Answer: S = 2 = exact****3 marks****28 (9 marks)**

$$(a) h = \frac{1}{n}, y_r = \left(\frac{r}{n}\right)^2, y_0 = 0, y_n = 1$$

M1

$$T_n = \frac{1}{2n} \left[0 + 1 + 2 \sum_{r=1}^{n-1} \frac{r^2}{n^2} \right] = \frac{1}{n^3} \left[\frac{n^2}{2} + \sum r^2 \right]$$

A1

$$\text{Answer: } T_n = \frac{1}{n^3} \left[\frac{1}{2}n^2 + \sum_{r=1}^{n-1} r^2 \right]$$

2 marks

$$(b) \frac{1}{n^3} \left[\frac{n^2}{2} + \frac{(n-1)n(2n-1)}{6} \right]$$

M1

$$= \frac{1}{n^3} \times \frac{3n^2 + 2n^3 - 3n^2 + n}{6} = \frac{2n^3 + n}{6n^3}$$

M1

$$= \frac{1}{3} + \frac{1}{6n^2}$$

A1

$$\text{Answer: } T_n = \frac{1}{3} + \frac{1}{6n^2}$$

3 marks

$$(c) \text{Error} = \frac{1}{6n^2} < 10^{-4}, \text{ so } n^2 > \frac{10^4}{6} = 1666.7$$

M1

$$n > 40.82, \text{ so } n = 41$$

A1**Answer: n = 41****2 marks**

$$(d) h = \frac{1}{n}, \text{ so the error } \frac{1}{6n^2} = \frac{h^2}{6}$$

M1

Halving h (doubling n) divides the error by 4

A1**Answer: Error = $h^2/6$** **2 marks**

29 (9 marks)

(a) $f(1) = a + b$ and $f(2) = 4a + 8b$

B1

$$T = \frac{1}{2} \times 1 \times [0 + f(2) + 2f(1)] = 2a + 4b + a + b = 3a + 5b$$

M1

$$S = \frac{1}{3}[0 + f(2) + 4f(1)] = \frac{8a + 12b}{3}$$

A1

Answer: $S = (8a + 12b)/3$

3 marks

(b) $3a + 5b = 19$ and $8a + 12b = 48$, i.e. $2a + 3b = 12$

M1

Eliminating a: $2(3a + 5b) - 3(2a + 3b) = 38 - 36$, so $b = 2$

M1

$a = 3$

A1

Answer: $a = 3, b = 2$

3 marks

(c) $\int_0^2 (3x^2 + 2x^3) dx = [x^3 + \frac{x^4}{2}]_0^2$

M1

$= 8 + 8 = 16$

A1

f is a cubic and Simpson's rule is exact for cubics

E1

Answer: 16

3 marks

30 (10 marks)

(a) S_2 : $h = 0.5$, ordinates 4.0, 3.200000, 2.0

B1

$$S_2 = \frac{1}{3} \times 0.5 \times [4 + 2 + 4(3.200000)] = 3.133333$$

M1 A1

S_4 : $h = 0.25$, ordinates 4.000000, 3.764706, 3.200000, 2.560000, 2.000000, so $S_4 = 3.1415686$

M1

Answer: $S_2 = 3.1333333$, $S_4 = 3.1415686$

4 marks

(b) Let $S_2 = \pi + e$ and $S_4 \approx \pi + \frac{e}{16}$

M1

$16S_4 - S_2 \approx 16\pi + e - \pi - e = 15\pi$

M1 A1

Answer: $R \approx \pi$

3 marks

(c) $R = \frac{16(3.1415686) - 3.1333333}{15} = 3.1421176$

M1

Errors: $S_2 - 0.0083$, $S_4 - 0.000024$, $R - 0.00052$

A1

R is far more accurate than either Simpson estimate

E1

Answer: $R = 3.1421176$

3 marks

Numerical Differentiation**1 (2 marks)**

$f(2.1) = 4.41$ and $f(2) = 4$

B1

$$f'(2) \approx \frac{4.41 - 4}{0.1} = 4.1$$

A1

Answer: 4.1

2 marks

2 (2 marks)

$f(1.1) = 3.00417$ and $f(0.9) = 2.45960$

M1

$$f'(1) \approx \frac{3.00417 - 2.45960}{0.2} = 2.7228$$

A1

Answer: 2.7228

2 marks

3 (2 marks)

$f(2) = 0.69315$ and $f(1.9) = 0.64185$

M1

$$f'(2) \approx \frac{0.69315 - 0.64185}{0.1} = 0.5129$$

A1

Answer: 0.5129

2 marks

4 (2 marks)

$$f'(11) \approx \frac{f(12) - f(10)}{2 \times 1}$$

M1

$$= \frac{3.464 - 3.162}{2} = 0.151$$

A1

Answer: 0.151

2 marks

5 (2 marks)

$$f(1.1) = 1.331, f(1) = 1, f(0.9) = 0.729$$

B1

$$f''(1) \approx \frac{1.331 - 2 + 0.729}{0.01} = 6.0$$

A1

Answer: 6.0

2 marks

6 (2 marks)

$$v \approx \frac{s(4) - s(2)}{2}$$

M1

$$= \frac{28.0 - 11.0}{2} = 8.5 \text{ m s}^{-1}$$

A1

Answer: 8.5 m s⁻¹

2 marks

7 (6 marks)

$$(a) f(x + h) = \sin(1.1472) = 0.91162, f\left(\frac{\pi}{3}\right) = 0.86603$$

M1

$$f' \approx 0.4559$$

A1

Answer: 0.4559

2 marks

$$(b) f(x - h) = \sin(0.9472) = 0.81178$$

M1

$$f' \approx \frac{0.91162 - 0.81178}{0.2} = 0.4992$$

A1

Answer: 0.4992

2 marks

$$(c) \text{Forward error} = -0.0441$$

B1

$$\text{Central error} = -0.0008, \text{ much smaller}$$

B1

Answer: Errors -0.0441 and -0.0008

2 marks

8 (5 marks)

$$(a) h = 0.1: 5.8630, \text{ error } 0.4264$$

M1

$$h = 0.01: 5.4775, \text{ error } 0.0410$$

A1

Answer: 5.8630, 5.4775

2 marks

$$(b) h = 0.1: 5.4547, \text{ error } 0.0181$$

M1

$$h = 0.01: 5.4367, \text{ error } 0.0002$$

A1

Answer: 5.4547, 5.4367

2 marks

(c) Central difference: its errors are about 100 times smaller and fall by a factor of about 100 when h is divided by 10

E1

Answer: Central difference

1 mark

9 (4 marks)

$$(a) \frac{dT}{dt} \approx \frac{T(3) - T(1)}{2}$$

M1

$$= \frac{49.0 - 31.5}{2} = 8.75 \text{ }^{\circ}\text{C per minute}$$

A1

Answer: 8.75 }^{\circ}\text{C per minute}

2 marks

$$(b) \frac{dT}{dt} \approx \frac{31.5 - 20.0}{1} = 11.5 \text{ }^{\circ}\text{C per minute}$$

B1

Answer: 11.5 }^{\circ}\text{C per minute}

1 mark

$$(c) \frac{dT}{dt} \approx \frac{55.1 - 49.0}{1} = 6.1 \text{ }^{\circ}\text{C per minute}$$

B1

Answer: 6.1 }^{\circ}\text{C per minute}

1 mark

10 (4 marks)

$$h = 0.1: \frac{\sqrt{4.1} - 2}{0.1} = 0.24846$$

M1 A1

$$h = 0.01: \frac{\sqrt{4.01} - 2}{0.01} = 0.24984$$

A1

Exact: $y' = \frac{1}{2\sqrt{x}} = 0.25$, and the smaller h gives the smaller error

A1

Answer: 0.24846 and 0.24984

4 marks

11 (3 marks)

$$f(2.1) = 0.741937, f(2) = 0.693147, f(1.9) = 0.641854$$

M1

$$f'(2) \approx \frac{0.741937 - 2(0.693147) + 0.641854}{0.01} = -0.2503$$

A1

Exact: $f'(x) = -\frac{1}{x^2} = -0.25$, so the error is -0.0003

A1

Answer: -0.2503

3 marks

12 (3 marks)

$$\text{Gradient of the chord} = \frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h}$$

M1 A1

As $h \rightarrow 0$ the chord approaches the tangent at $x = a$, so its gradient approaches $f'(a)$

E1

Answer: Chord gradient $\rightarrow$ tangent gradient

3 marks

13 (5 marks)

(a) $f(2.5) = 15.625$ and $f(2) = 8$

M1

$$f'(2) \approx \frac{15.625 - 8}{0.5} = 15.25$$

A1

Answer: 15.25

2 marks

(b) $f(2.5) = 15.625$ and $f(1.5) = 3.375$

M1

$$f'(2) \approx \frac{15.625 - 3.375}{1} = 12.25$$

A1

Answer: 12.25

2 marks

(c) The central estimate is more accurate: its error is 0.25 compared with 3.25 for the forward difference

E1

Answer: Central difference

1 mark

14 (5 marks)

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{6}f'''(x) + \dots$$

B1

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{6}f'''(x) + \dots$$

B1

Subtracting: $f(x+h) - f(x-h) = 2hf'(x) + \frac{h^3}{3}f'''(x) + \dots$

M1

Dividing by $2h$ gives $f'(x) + \frac{h^2}{6}f'''(x) + \dots$

A1

The error is of order h^2 (central), whereas the forward difference has error of order h

E1

Answer: Central error $O(h^2)$; forward $O(h)$

5 marks

15 (5 marks)

$$f(x \pm h) = f(x) \pm hf'(x) + \frac{h^2}{2}f''(x) \pm \frac{h^3}{6}f'''(x) + \frac{h^4}{24}f^{(4)}(x) \pm \dots$$

B1

Adding the two expansions, the odd powers cancel

M1

$$f(x+h) + f(x-h) = 2f(x) + h^2f''(x) + \frac{h^4}{12}f^{(4)}(x) + \dots$$

A1

Rearranging and dividing by h^2 gives the result

A1

The error is of order h^2

E1

Answer: Error $O(h^2)$

5 marks

16 (6 marks)

$$D(0.2) = \frac{\ln 3.2 - \ln 2.8}{0.4} = 0.333828$$

M1 A1

$$D(0.1) = \frac{\ln 3.1 - \ln 2.9}{0.2} = 0.333457$$

A1

$$D(0.2) - D(0.1) \approx C(0.04 - 0.01), \text{ so eliminate } C: f'(3) \approx \frac{4D(0.1) - D(0.2)}{3}$$

M1 A1

$$= 0.3333330, \text{ compared with } \frac{1}{3} = 0.3333333$$

A1

Answer: 0.333333

6 marks

17 (6 marks)

$$(a) h = 0.1: \frac{f(1.3) - f(1.1)}{0.2} = \frac{0.1845 - 0.2982}{0.2} = -0.5685$$

M1 A1

$$h = 0.2: \frac{f(1.4) - f(1.0)}{0.4} = -0.5675$$

A1

Answer: -0.5685, -0.5675

3 marks

$$(b) f''(1.2) \approx \frac{f(1.3) - 2f(1.2) + f(1.1)}{0.01}$$

M1

$$= \frac{0.1845 - 2(0.2369) + 0.2982}{0.01} = 0.89$$

A1

Answer: 0.89

2 marks

(c) The $h = 0.1$ estimate (error 0.00013) is closer than $h = 0.2$ (error 0.0011)

E1

Answer: $h = 0.1$

1 mark

18 (6 marks)

$$h = 0.1: \frac{3.0042 - 2.7183}{0.1} = 2.859$$

M1 A1

$$h = 0.01: 2.73$$

A1

$$h = 0.0001: 3.0$$

A1

The values are rounded to 4 d.p., so each has an error up to 0.00005

E1

Dividing the difference by a very small h magnifies this rounding error (up to about $\frac{0.0001}{h}$),

E1

which outweighs the gain from reducing the truncation error

Answer: 2.859, 2.73, 3.0

6 marks

19 (5 marks)

$$(a) \text{ At } 0.8: \frac{0.3679 - 0.3293}{0.4} = 0.0965$$

M1 A1

$$\text{At } 1.0: 0.0048, \text{ at } 1.2: -0.0567$$

A1

Answer: 0.097, 0.005, -0.057

3 marks

(b) The gradient changes from 0.0048 to -0.0567 between $x = 1.0$ and 1.2

M1

$$x \approx 1.0 + 0.2 \times \frac{0.0048}{0.0048 - (-0.0567)} = 1.015$$

A1

Answer: 1.02

2 marks

20 (6 marks)

$$(a) f(2) = -1 \text{ and } f'(2) = 10$$

B1

$$x_1 = 2 - \frac{-1}{10} = 2.1000$$

A1

Answer: 2.1000

2 marks

$$(b) f(2.01) = -0.899399$$

B1

$$f'(2) \approx \frac{-0.899399 - (-1)}{0.01} = 10.0601$$

M1

$$x_1 = 2 + \frac{1}{10.0601} = 2.09940$$

A1

Answer: 2.09940

3 marks

(c) The numerical derivative is slightly different (10.0601 against 10), but the effect on x_1 is tiny (about 0.0006), so the method still converges to the root

E1

Answer: Difference of about 0.0006

1 mark

21 (8 marks)

$$(a) f(x+h) = f(x) + hf' + \frac{h^2}{2}f'' + \frac{h^3}{6}f''' + \dots \quad \text{B1}$$

$$f(x+2h) = f(x) + 2hf' + 2h^2f'' + \frac{4h^3}{3}f''' + \dots \quad \text{B1}$$

$$\text{Answer: } f(x+h) = f + hf' + \frac{h^2}{2}f'' + \frac{h^3}{6}f''' + \dots \text{ and } f(x+2h) = f + 2hf' + 2h^2f'' + \frac{4h^3}{3}f''' + \dots \quad 2 \text{ marks}$$

$$(b) 4f(x+h) - f(x+2h) - 3f(x) = (4-1-3)f + (4h-2h)f' \quad \text{M1 A1}$$

$$+ (2h^2 - 2h^2)f'' + \left(\frac{2h^3}{3} - \frac{4h^3}{3}\right)f''' \quad \text{M1}$$

$$= 2hf' - \frac{2h^3}{3}f''' + \dots \text{ so dividing by } 2h \text{ and rearranging gives the result} \quad \text{A1}$$

$$\text{Answer: } f'(x) = \frac{-3f(x) + 4f(x+h) - f(x+2h)}{2h} + \frac{h^2}{3}f'''(x) + \dots \quad 4 \text{ marks}$$

$$(c) \text{ Estimate} = \frac{-3 + 4e^{0.1} - e^{0.2}}{0.2} = 0.99640, \text{ error } -0.0036 \quad \text{M1}$$

Forward difference 1.05171 has error 0.052: the three-point formula is much more accurate A1

Answer: 0.99640 2 marks

22 (8 marks)

$$(a) \text{ Error in } f(x+h) - f(x-h) \text{ is at most } 2\varepsilon \quad \text{M1}$$

$$\text{Dividing by } 2h \text{ gives a maximum error of } \frac{2\varepsilon}{2h} = \frac{\varepsilon}{h} \quad \text{A1}$$

$$\text{Answer: Rounding error in } D(h) \text{ is at most } \frac{\varepsilon}{h} \quad 2 \text{ marks}$$

$$(b) E(h) = \frac{h^2 e}{6} + \frac{\varepsilon}{h} \quad \text{B1}$$

$$E'(h) = \frac{he}{3} - \frac{\varepsilon}{h^2} = 0 \quad \text{M1 A1}$$

$$h^3 = \frac{3\varepsilon}{e}, \text{ so } h = 0.038 \quad \text{A1}$$

$$\text{Answer: 0.038} \quad 4 \text{ marks}$$

$$(c) E_{\min} = 0.0020 \quad \text{B1}$$

$$\text{For } h = 0.001, E \approx \frac{\varepsilon}{h} = 0.05, \text{ far larger: rounding error dominates} \quad \text{E1}$$

$$\text{Answer: 0.0020} \quad 2 \text{ marks}$$

23 (6 marks)

$$(a) \text{ The central difference divides by } 2h = 0.2, \text{ not } h \quad \text{E1}$$

$$\text{The second difference divides by } h^2 = 0.01, \text{ not } h \quad \text{E1}$$

$$\text{Answer: Denominators should be } 2h \text{ and } h^2 \quad 2 \text{ marks}$$

$$(b) f'(2) \approx \frac{2.402}{0.2} = 12.01 \quad \text{B1}$$

$$f''(2) \approx \frac{0.12}{0.01} = 12.0 \quad \text{B1}$$

$$\text{Answer: 12.01 and 12.0} \quad 2 \text{ marks}$$

$$(c) f'(x) = 3x^2 = 12 \text{ and } f''(x) = 6x = 12 \quad \text{B1}$$

Sam's answers 24.02 and 1.2 are not close to 12, but the corrected estimates (12.01 and 12) are E1

Answer: Both exact values are 12 2 marks

24 (7 marks)

$$(a) v(0.5) \approx \frac{s(1) - s(0)}{1} = 3.000 \quad \text{M1}$$

$$v(1) \approx 4.500 \text{ and } v(1.5) \approx 9.000 \quad \text{A1}$$

$$\text{Answer: 3.000, 4.500, 9.000} \quad 2 \text{ marks}$$

$$(b) v(0) \approx \frac{s(0.5) - s(0)}{0.5} = 3.000 \quad \text{B1}$$

$$v(2) \approx \frac{s(2) - s(1.5)}{0.5} = 12.000$$

B1

Answer: 3.000 and 12.000

2 marks

$$(c) a(1) \approx \frac{s(1.5) - 2s(1) + s(0.5)}{0.5^2}$$

M1

$$= \frac{6.000 - 2(3.000) + 1.500}{0.25} = 6.00$$

A1

Answer: 6.00

2 marks

(d) $v = 6t^2 - 6t + 4$ so $v(1) = 4$, and $a = 12t - 6$ so $a(1) = 6$; the central estimates are close but not exact because of the h^2 errors

E1

Answer: $v(1) = 4$, $a(1) = 6$

1 mark

25 (7 marks)

$$(a) h = 0.1: \frac{\ln 2.1 - \ln 2}{0.1} = 0.487902, \text{ error } -0.012098$$

M1 A1

$$h = 0.05: 0.493852, \text{ error } -0.006148$$

A1

Answer: 0.487902, 0.493852

3 marks

$$(b) \text{Ratio} = \frac{-0.012098}{-0.006148} = 1.97$$

M1

Error $\approx \frac{h}{2} f''(2)$, proportional to h , so halving h halves the error

E1

Answer: 1.97

2 marks

$$(c) f''(x) = -\frac{1}{x^2}, \text{ so } f''(2) = -0.25$$

B1

$$\text{Predicted error} = \frac{0.1}{2} \times (-0.25) = -0.0125, \text{ actual } -0.0121: \text{ agreement to 2 s.f.}$$

E1

Answer: -0.0125

2 marks

26 (7 marks)

$$(a) \frac{dy}{dt} = 3t^2 - 2, \frac{dx}{dt} = 2t$$

B1

$$\frac{dy}{dx} = \frac{3t^2 - 2}{2t}$$

M1

$$\text{At } t = 2: \frac{10}{4} = 2.5$$

A1

Answer: 2.5

3 marks

$$(b) y(2.1) - y(1.9) = 2.002, \text{ so } \frac{dy}{dt} \approx 10.010$$

M1

$$x(2.1) - x(1.9) = 0.80, \text{ so } \frac{dx}{dt} \approx 4.00$$

A1

$$\frac{dy}{dx} \approx \frac{2.002}{0.80} = 2.5025$$

A1

Answer: 2.5025

3 marks

$$(c) \text{ Each central difference has the same factor } 2h \text{ in its denominator, so } \frac{dy}{dx} \approx \frac{y(t+h) - y(t-h)}{x(t+h) - x(t-h)}$$

E1

Answer: Both denominators contain 2h

1 mark

27 (9 marks)

$$(a) f(x+h) = f + hf' + \frac{h^2}{2}f'' + \frac{h^3}{6}f''' + \frac{h^4}{24}f^{(4)} + \dots$$

B1

$$f(x+2h) = f + 2hf' + 2h^2f'' + \frac{4h^3}{3}f''' + \frac{2h^4}{3}f^{(4)} + \dots$$

B1

Combination $f - 2f(x+h) + f(x+2h)$: f terms $1 - 2 + 1 = 0$, f' terms $-2h + 2h = 0$

M1

$$h^2: -1 + 2 = 1; h^3: -\frac{1}{3} + \frac{4}{3} = 1; h^4: -\frac{1}{12} + \frac{2}{3} = \frac{7}{12}$$

A1

$$\text{Answer: } f(x) - 2f(x+h) + f(x+2h) = h^2f''(x) + h^3f'''(x) + \frac{7h^4}{12}f^{(4)}(x) + \dots$$

4 marks

$$(b) f''(x) = \frac{f(x) - 2f(x+h) + f(x+2h)}{h^2} - hf'''(x) - \dots$$

M1

The error is of order h

A1

Answer: $f''(x) \approx \frac{f(x) - 2f(x+h) + f(x+2h)}{h^2}$, **error** $\approx -hf'''(x)$

2 marks

(c) Formula: $\frac{\ln 1 - 2 \ln 1.1 + \ln 1.2}{0.01} = -0.8299$

M1

Correction: subtract $hf''' = 0.1 \times 2$, giving $-0.8299 - 0.2 = -1.0299$

A1

The corrected value is much closer to -1 (error 0.03) than the raw estimate (error 0.17)

E1

Answer: -0.8299 ; corrected -1.0299

3 marks

28 (9 marks)

(a) $4D(h) = \frac{2(f(x+h) - f(x-h))}{h}$ and $D(2h) = \frac{f(x+2h) - f(x-2h)}{4h}$

M1

$4D(h) - D(2h) = \frac{8(f(x+h) - f(x-h)) - (f(x+2h) - f(x-2h))}{4h}$

M1

Divide by 3: $\frac{-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)}{12h}$

A1

Answer: $\frac{4D(h) - D(2h)}{3} = \frac{f(x-2h) - 8f(x-h) + 8f(x+h) - f(x+2h)}{12h}$

3 marks

(b) $4D(h) - D(2h) = 3f' + (4 \times \frac{h^2}{6} - \frac{4h^2}{6})f''' + (4 \times \frac{h^4}{120} - \frac{16h^4}{120})f^{(5)} + \dots$

M1

The f''' terms cancel, leaving $3f' - \frac{h^4}{10}f^{(5)} + \dots$

M1

So the result is $f' - \frac{h^4}{30}f^{(5)}$; the error is of order h^4

A1

Answer: Error $O(h^4)$

3 marks

(c) $D(0.1) = 0.53940225$, $D(0.2) = 0.53670749$

M1

Five-point estimate = 0.54030051

A1

Errors: -0.00090 , -0.0036 and -0.0000018 ; the five-point formula is far more accurate

E1

Answer: 0.54030051

3 marks

29 (9 marks)

(a) $D_2(h/2) = f'' + \frac{h^2}{48}f^{(4)} + \dots$

B1

$4D_2(h/2) - D_2(h) = 3f'' + (\frac{h^2}{12} - \frac{h^2}{12})f^{(4)} + \dots$

M1

The h^2 terms cancel, so the error is $O(h^4)$

A1

Answer: $\frac{4D_2(\frac{h}{2}) - D_2(h)}{3}$ has an error of order h^4

3 marks

(b) $D_2(0.2) = \frac{\ln 2.2 - 2 \ln 2 + \ln 1.8}{0.04} = -0.251258$

M1 A1

$D_2(0.1) = -0.250313$

A1

Answer: -0.251258 , -0.250313

3 marks

(c) Improved = $\frac{4(-0.250313) - (-0.251258)}{3} = -0.2499979$

M1

Errors: $D_2(0.2)$: -0.0013 , $D_2(0.1)$: -0.00031 , improved: 0.0000021

A1

The extrapolated value is the most accurate

E1

Answer: -0.249998

3 marks

30 (10 marks)

(a) $y_{i+1} - 2y_i + y_{i-1} = h^2y_i$

M1

$y_{i-1} - (2 + h^2)y_i + y_{i+1} = 0$

M1

With $h^2 = \frac{1}{16}$, $2 + h^2 = \frac{33}{16}$

A1

Answer: $y_{i-1} - \frac{33}{16}y_i + y_{i+1} = 0$ for $i = 1, 2, 3$

3 marks

(b) $i = 1$: $-\frac{33}{16}y_1 + y_2 = 0$ (since $y_0 = 0$)

B1

$i = 2$: $y_1 - \frac{33}{16}y_2 + y_3 = 0$

B1

$$i = 3: y_2 - \frac{33}{16}y_3 = -1 \text{ (since } y_4 = 1)$$

B1

$$\text{Matrix form: } \begin{pmatrix} -\frac{33}{16} & 1 & 0 \\ 1 & -\frac{33}{16} & 1 \\ 0 & 1 & -\frac{33}{16} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

M1

$$y_1 = 0.21511, y_2 = 0.44367, y_3 = 0.69996$$

A1

Answer: $y_1 = 0.21511$, $y_2 = 0.44367$, $y_3 = 0.69996$

5 marks

(c) Exact values: 0.21495, 0.44341, 0.69972

B1

The numerical values agree to within 0.0003

E1

Answer: Agree to about 0.0003

2 marks

Euler's Method

1 (2 marks)

$$y_1 = y_0 + h f(x_0, y_0) = 1 + 0.1(0 + 1)$$

M1

$$= 1.1$$

A1

Answer: 1.1

2 marks

2 (2 marks)

$$y_1 = 3 + 0.2(2 \times 1 - 3)$$

M1

$$= 2.8$$

A1

Answer: 2.8

2 marks

3 (3 marks)

$$y_1 = 1 + 0.1(1^2) = 1.1$$

M1 A1

$$y_2 = 1.1 + 0.1(1.1^2) = 1.221$$

A1

Answer: 1.221

3 marks

4 (3 marks)

$$y_1 = 0 + 0.5 \sin 0 = 0$$

B1

$$y_2 = 0 + 0.5 \sin 0.5$$

M1

$$= 0.2397$$

A1

Answer: 0.2397

3 marks

5 (2 marks)

$$y_{n+1} = y_n + h(x_n^2 + y_n)$$

B1

$$y_1 = 1 + 0.1(4 + 1) = 1.5$$

B1

Answer: $y_1 = 1.5$

2 marks

6 (3 marks)

$$y_1 = 0 + 0.5(3 - 0) = 1.50$$

M1 A1

$$y_2 = 1.50 + 0.5(3 - \frac{1.50}{2}) = 2.625$$

A1

Answer: 1.50, 2.625

3 marks

7 (4 marks)

$$y_1 = 2 + 0.1 \times e^0 \times 2 = 2.2000$$

M1 A1

$$y_2 = 2.2000 + 0.1 \times e^{-0.1} \times 2.2000 = 2.3991$$

A1

$$y_3 = 2.5955$$

A1

Answer: 2.5955

4 marks

8 (4 marks)

$$y_1 = 0 + 0.25 \times 1 = 0.25$$

M1

$$y_2 = 0.25 + 0.25 \times \frac{1}{1.0625} = 0.4853$$

A1

$$y_3 = 0.6853, y_4 = 0.8453$$

A1

Exact $y(1) = \arctan 1 = 0.7854$, so the estimate is too large by 0.0599

A1

Answer: 0.8453

4 marks

9 (4 marks)

$$f(1, 2) = 1 \times 2 - 1 = 1, \text{ so } y_1 = 2 + 0.1 \times 1 = 2.1$$

M1

$$f(1.1, 2.1) = 1.1 \times 2.1 - 1 = 1.3100$$

M1 A1

$$y_2 = 2.2310, \text{ and then } f(1.2, 2.2310) = 1.6772 \text{ so } y_3 = 2.3987$$

A1

Answer: $y_3 = 2.3987$

4 marks

10 (4 marks)

$$(a) y_{n+1} = y_n + h \times y_n = (1 + h)y_n$$

M1

This is a geometric sequence with first term 1 and ratio $(1 + h)$, so $y_n = (1 + h)^n$

A1

Answer: Euler's method with step h gives $y_{n+1} = (1 + h)y_n$, and hence $y_n = (1 + h)^n$

2 marks

$$(b) y(1) \approx y_{10} = 1.1^{10} = 2.5937$$

M1

Exact $e = 2.7183$; the estimate is too small by 0.1245

A1

Answer: 2.5937

2 marks

11 (5 marks)

$$(a) y_1 = 1 + 0.1 \times 2 = 1.2$$

M1

$$y_2 = 1.2 + 0.1 \times 2.4 = 1.44$$

A1

$$y_3 = 1.728$$

A1

Answer: 1.728

3 marks

$$(b) y = e^{2x}, \text{ so } y(0.3) = e^{0.6} = 1.8221$$

M1

$$\text{Error} = 1.8221 - 1.7280 = 0.0941 \text{ (an underestimate)}$$

A1

Answer: $y = e^{2x}$; error 0.0941

2 marks

12 (4 marks)

$$T_1 = 90 + 2 \times (-0.1)(90 - 20)$$

M1

$$= 76.0$$

A1

$$T_2 = 76.0 + 2 \times (-0.1)(76.0 - 20)$$

M1

$$= 64.8^\circ\text{C}$$

A1

Answer: 64.8°C

4 marks

13 (4 marks)

$$y_1 = 2 + 0.2(1 \times 2)$$

M1

$$= 2.4$$

A1

$$y_2 = 2.4 + 0.2(1.2 \times 2.4)$$

M1

$$= 2.976$$

A1

Answer: 2.976

4 marks

14 (5 marks)

$$(a) y_1 = 1 + 0.5 \times 1 = 1.5$$

M1

$$y_2 = 1.5 + 0.5 \times 1.5 = 2.25$$

A1

Answer: 2.25

2 marks

$$(b) y = e^x, \text{ so } y(1) = e = 2.7183$$

M1

$$\text{Percentage error} = \frac{2.7183 - 2.25}{2.7183} \times 100 = 17.2\%$$

A1

Answer: 17.2%

2 marks

(c) The solution curve is convex ($y'' = y > 0$), so its gradient increases, and the tangent at the start of each step lies below the curve

E1

Answer: Convex curve, tangents lie below**1 mark****15 (4 marks)**

$$y_1 = y_0 + 0.1(1^2 - y_0)$$

M1

$$= 0.9y_0 + 0.1$$

A1

$$0.9y_0 + 0.1 = 2.2$$

M1

$$y_0 = \frac{2.1}{0.9} = 2.3333$$

A1**Answer: 2.3333****4 marks****16 (5 marks)**

$$y_1 = 1 + 0.2(k \times 1 - 0)$$

M1

$$1 + 0.2k = 1.1, \text{ so } k = 0.5$$

A1

$$f(x, y) = 0.5y - x, \text{ so } y_2 = 1.1 + 0.2(0.5 \times 1.1 - 0.2)$$

M1 A1

$$y(0.4) \approx 1.170$$

A1**Answer: $k = 0.5$; $y(0.4) \approx 1.170$** **5 marks****17 (6 marks)**

$$S_1 = 0 + 5(4 - 0) = 20.0$$

M1 A1

$$S_2 = 20.0 + 5(4 - 0.02 \times 20.0) = 38.0$$

A1

$$S_3 = 54.20$$

A1

$$\text{Long term } \frac{dS}{dt} = 0, \text{ so } S = \frac{4}{0.02} = 200$$

M1

The estimate is well below 200 and the approach is gradual, so after 15 minutes S is still far from its limit

A1**Answer: 54.20****6 marks****18 (6 marks)**

$$y_{n+1} = y_n + 0.1v_n, v_{n+1} = v_n - 0.1y_n$$

M1 A1

$$n = 0: y_1 = 1 + 0 = 1.000, v_1 = 0 - 0.1 = -0.100$$

A1

$$n = 1: y_2 = 0.990, v_2 = -0.200$$

A1

$$n = 2: y_3 = 0.970, v_3 = -0.299$$

A1 A1**Answer: $y(0.3) \approx 0.970$, $v(0.3) \approx -0.299$** **6 marks****19 (7 marks)**

$$(a) y_1 = 1 + 0.2(0 + 1) = 1.2$$

M1

$$y_2 = 1.2 + 0.2(0.2 + 1.2) = 1.48$$

A1**Answer: 1.48****2 marks**

$$(b) y_1 = 1.1, y_2 = 1.220, y_3 = 1.3620$$

M1

$$y_4 = 1.52820$$

A1**Answer: 1.52820****2 marks**

$$(c) \text{ Improved} = 2(1.52820) - 1.48$$

M1

$$= 1.57640$$

A1

$$\text{Errors: } h = 0.2: 0.1036; h = 0.1: 0.0554; \text{ improved: } 0.0072$$

A1**Answer: 1.57640****3 marks****20 (6 marks)**

$$(a) y_1 = 1 + 0.1(0 + 1) = 1.1$$

B1**Answer: 1.1****1 mark**

$$(b) y(0.1) = 2e^{0.1} - 1.1 = 1.110342$$

M1

$$\text{Error} = 0.010342$$

A1**Answer: 0.010342****2 marks**

(c) Euler gives $y(0) + hy'(0)$ only, so the error is approximately $\frac{h^2}{2}y''(0)$

E1

$$y'(0) = 1, \text{ so } y''(0) = 1 + 1 = 2$$

M1

$$\frac{0.01}{2} \times 2 = 0.01, \text{ close to the actual error } 0.0103$$

A1

Answer: ≈ 0.01

3 marks

21 (8 marks)

$$(a) P_1 = 10 + 1 \times 0.2 \times 10 \times \left(1 - \frac{10}{50}\right) = 11.600$$

M1 A1

$$P_2 = 13.3818$$

A1

$$P_3 = 15.3418$$

A1

$$P_4 = 17.469$$

A1

Answer: 17.469

5 marks

$$(b) \frac{dP}{dt} = 0 \text{ when } P = 0 \text{ or } P = 50$$

B1

For $0 < P < 50$ the rate is positive, so P increases towards 50 but cannot pass the stationary solution

M1

For $P > 50$ the rate is negative, so P falls back to 50

E1

Answer: $P = 0$ or 50

3 marks

22 (7 marks)

$$x_{n+1} = x_n + 0.1(-2x_n + y_n), y_{n+1} = y_n + 0.1(x_n - 2y_n)$$

M1 A1

$$n = 0: x_1 = 1 + 0.1(-2) = 0.80, y_1 = 0 + 0.1(1) = 0.10$$

M1 A1

$$n = 1: x_2 = 0.650, y_2 = 0.160$$

A1

$$n = 2: x_3 = 0.5360, y_3 = 0.1930$$

A1 A1

Answer: $x(0.3) \approx 0.5360, y(0.3) \approx 0.1930$

7 marks

23 (6 marks)

(a) Kai has used x_{n+1} in place of x_n in $f(x, y)$

E1

Euler's method uses the gradient at the start of each step: $f(x_n, y_n)$ with $x_0 = 0$

E1

Answer: Uses x_{n+1} instead of x_n

2 marks

$$(b) y_1 = 1 + 0.1(0 + 1) = 1.10$$

M1 A1

$$y_2 = 1.10 + 0.1(0.1 + 1.10) = 1.220$$

A1

$$y_3 = 1.3620$$

A1

Answer: 1.10, 1.220, 1.3620

4 marks

24 (7 marks)

$$(a) y = e^{-10x}$$

M1

y decays rapidly towards 0 and is always positive

A1

Answer: $y = e^{-10x}$

2 marks

$$(b) y_{n+1} = y_n + 0.3(-10y_n) = -2y_n$$

M1

$$y_1 = -2, y_2 = 4, y_3 = -8, y_4 = 16$$

A1

The estimates alternate in sign and grow, whereas the exact solution decays to 0

E1

Answer: -2, 4, -8, 16

3 marks

$$(c) y_n = (1 - 10h)^n y_0 \rightarrow 0 \text{ requires } |1 - 10h| < 1$$

M1

$$-1 < 1 - 10h < 1, \text{ so } 0 < h < 0.2$$

A1

Answer: $0 < h < 0.2$

2 marks

25 (8 marks)

$$(a) v_1 = 0 + 0.5(9.8 - 0) = 4.90$$

M1 A1

$$v_2 = 4.90 + 0.5(9.8 - 0.2 \times 4.90) = 9.310$$

A1

$$v_3 = 13.279, v_4 = 16.851 \text{ m s}^{-1}$$

A1

Answer: 16.85 m s^{-1}

4 marks

$$(b) \text{Terminal speed when } \frac{dv}{dt} = 0: v = \frac{9.8}{0.2} = 49 \text{ m s}^{-1}$$

B1

Answer: 49 m s^{-1}

1 mark

$$(c) v(2) = 49(1 - e^{-0.4}) = 16.154$$

M1 A1

Percentage error = 4.3%

Answer: 4.3%

A1

3 marks

26 (8 marks)

(a) $y_{n+1} = y_n + 0.25 \frac{y_n}{x_n} = y_n \left(1 + \frac{0.25}{x_n}\right)$

M1

$y_1 = 2(1.25) = 2.50, y_2 = 3.0000$

A1

$y_3 = 3.5000$

A1

$y_4 = 4.0000$

A1

Answer: 4.0000

4 marks

(b) $\int \frac{1}{y} dy = \int \frac{1}{x} dx$

M1

$\ln y = \ln x + c$, and $y(1) = 2$ gives $y = 2x$

A1

$y(2) = 4$

A1

Answer: $y = 2x$; $y(2) = 4$

3 marks

(c) Euler's estimate is exactly 4 because the exact solution $y = 2x$ is a straight line, so each tangent line coincides with the solution curve

E1

Answer: Euler is exact here

1 mark

27 (10 marks)

(a) $y_{n+1} = y_n(1 + h)$, so $y_n = (1 + h)^n$

M1

N steps are needed to reach $x = 1$, so $E_N = \left(1 + \frac{1}{N}\right)^N$

A1

Answer: $E_N = \left(1 + \frac{1}{N}\right)^N$

2 marks

(b) $E_1 = 2, E_2 = 2.2500, E_4 = 2.4414$

M1

$E_{10} = 2.5937, E_{100} = 2.7048$

A1

Answer: 2, 2.2500, 2.4414, 2.5937, 2.7048

2 marks

(c) $\ln E_N = N\left(\frac{1}{N} - \frac{1}{2N^2} + \frac{1}{3N^3} - \dots\right)$

M1

$= 1 - \frac{1}{2N} + \frac{1}{3N^2} - \dots$

A1

$E_N = e^{1 - 1/2N + \dots} = e \times e^{-1/2N + \dots}$

M1

$\approx e\left(1 - \frac{1}{2N}\right)$ for large N

A1

Answer: $E_N \approx e\left(1 - \frac{1}{2N}\right)$ for large N

4 marks

(d) Error $e - E_N \approx \frac{e}{2N} = \frac{e h}{2}$, so it is proportional to h (first order)

B1

$N = 100: \frac{e}{200} = 0.01359$, and actual $e - E_{100} = 0.01347$

E1

Answer: $\approx e/2N$

2 marks

28 (9 marks)

(a) $y_{n+1} = y_n + hx_n = y_n + nh^2$ (since $x_n = nh$)

M1

$y_n = h^2(0 + 1 + 2 + \dots + (n - 1))$

M1

$= h^2 \times \frac{n(n-1)}{2}$

A1

Answer: $y_n = \frac{n(n-1)h^2}{2}$

3 marks

(b) With $n = N$ and $h = \frac{1}{N}$: $y_N = \frac{N(N-1)}{2N^2}$

M1

$= \frac{1}{2} - \frac{1}{2N}$

A1

Answer: $y(1) \approx \frac{1}{2} - \frac{1}{2N}$

2 marks

(c) Exact $y = \frac{x^2}{2}$, so $y(1) = \frac{1}{2}$

B1

$$\text{Error} = \frac{1}{2} - \left(\frac{1}{2} - \frac{1}{2N}\right) = \frac{1}{2N} = \frac{h}{2}$$

M1

Euler's method here is a left-hand rectangle rule for $\int_0^1 x \, dx$, which underestimates an increasing function by $\frac{h}{2}$

E1

Answer: Error $h/2$

3 marks

(d) $\frac{1}{2N} < 0.0005$, so $N > 1000$, $N = 1001$

B1

Answer: $N = 1001$

1 mark

29 (10 marks)

(a) $\frac{dx}{dt} = v$ by definition

B1

Then $\frac{d^2x}{dt^2} = \frac{dv}{dt} = -0.5v - 4x$

B1

Answer: $\frac{dx}{dt} = v$, $\frac{dv}{dt} = -0.5v - 4x$

2 marks

(b) $x_{n+1} = x_n + 0.1v_n$, $v_{n+1} = v_n + 0.1(-0.5v_n - 4x_n)$

M1 A1

$t = 0.1$: $x = 1.0000$, $v = -0.4000$

A1

$t = 0.2$: $x = 0.9600$, $v = -0.7800$

A1

$t = 0.3$: $x = 0.8820$, $v = -1.1250$

A1

$t = 0.4$: $x = 0.7695$, $v = -1.4215$

A1

Answer: $x(0.4) \approx 0.7695$, $v(0.4) \approx -1.4215$

6 marks

(c) x is decreasing, and at $t = 0.4$ it is $0.769 > 0$, consistent with equilibrium not yet reached

E1

Errors from each step accumulate, so over many steps (or for oscillating solutions) Euler's method can drift in amplitude

E1

Answer: Plausible; errors accumulate

2 marks

30 (9 marks)

(a) $\int y^{-2} \, dy = \int dx$, so $-\frac{1}{y} = x + c$

M1

$y(0) = 1$ gives $c = -1$, so $y = \frac{1}{1-x}$

A1

As $x \rightarrow 1^-$, $y \rightarrow \infty$ (the solution blows up)

A1

Answer: $y = 1/(1-x)$

3 marks

(b) $y_1 = 1 + 0.2(1) = 1.200$

M1

$y_2 = 1.488$, $y_3 = 1.931$

A1

$y_4 = 2.676$

A1

$y_5 = 4.109$

A1

Answer: $y(1.0) \approx 4.109$

4 marks

(c) Exact: $y(0.2) = 1.250$, $y(0.8) = 5.000$; the Euler values 1.200 and 2.676 are well below

M1

The solution is unbounded near $x = 1$ and the Euler estimate stays finite, so the underestimate grows without limit

E1

Answer: Estimates far below the true values

2 marks

Improved Euler Methods

1 (3 marks)

$y^*_1 = 1 + 0.1(0 + 1) = 1.1$

B1

$y_1 = 1 + 0.05[(0 + 1) + (0.1 + 1.1)]$

M1

$= 1.110$

A1

Answer: 1.110

3 marks

2 (3 marks)

$$k_1 = 2(1) - 3 = -1$$

B1

$$k_2 = f(1.1, 3 + 0.1(-1)) = f(1.1, 2.9) = 2.2 - 2.9 = -0.7$$

M1

$$y_1 = 3 + 0.2(-0.7) = 2.86$$

A1

Answer: 2.86

3 marks

3 (2 marks)

$$y_1 = 1 + 0.05[f(0, 1) + f(0.1, 1.1)] = 1 + 0.05[1 + 1.21]$$

M1

$$= 1.1105$$

A1

Answer: 1.1105

2 marks

4 (2 marks)

$$y_2 = y_0 + 2 \times 0.1 \times f(0.1, 0.91)$$

M1

$$= 1 + 0.2(0.1 - 0.91) = 0.838$$

A1

Answer: 0.838

2 marks

5 (3 marks)

$$y^*_1 = 1 + 0.5(-1) = 0.5$$

B1

$$y_1 = 1 + 0.25[-1 + (-0.5)]$$

M1

$$= 0.625$$

A1

Answer: 0.625

3 marks

6 (2 marks)

$$y(0.1) = e^{0.1} = 1.10517$$

B1

Errors: 0.00517 (Euler) and 0.00017 (improved Euler), so the improved method is closer

E1

Answer: Improved Euler; $e^{0.1} = 1.10517$

2 marks

7 (5 marks)

$$y^*_1 = 1 + 0.2(0) = 1$$

B1

$$y_1 = 1 + 0.1[0 + 0.2 \times 1] = 1.020$$

M1 A1

$$y^*_2 = 1.020 + 0.2(0.2 \times 1.020) = 1.06080$$

M1

$$y_2 = 1.020 + 0.1[0.2040 + 0.4 \times 1.06080] = 1.0828$$

A1

Answer: 1.0828

5 marks

8 (4 marks)

$$k_1 = 0 + 1 = 1, k_2 = f(0.1, 1.1) = 1.11$$

M1 A1

$$y_1 = 1 + 0.2 \times 1.11 = 1.222$$

A1

$$k_1 = 1.262, k_2 = f(0.3, 1.3482) = 1.4382, y_2 = 1.5096$$

A1

Answer: 1.5096

4 marks

9 (5 marks)

$$y_1 = 1 + 0.1(0 + 1) = 1.1$$

M1 A1

$$y_2 = y_0 + 0.2 f(0.1, 1.1) = 1 + 0.2(0.1 + 1.21)$$

M1

$$= 1.262$$

A1

$$y_3 = y_1 + 0.2 f(0.2, 1.262) = 1.4585$$

A1

Answer: $y(0.2) \approx 1.262, y(0.3) \approx 1.4585$

5 marks

10 (5 marks)

$$(a) y_1 = 1.2, y_2 = 1.2 + 0.1(2.4) = 1.44$$

B1

Answer: 1.44

1 mark

$$(b) y^*_1 = 1.2, y_1 = 1 + 0.05[2 + 2.4] = 1.22$$

M1 A1

$$y_2 = 1.4884$$

A1

Answer: 1.4884

3 marks

(c) Errors: 0.0518 (Euler), 0.0034 (improved Euler); improved Euler is far closer

E1

Answer: Improved Euler

1 mark

11 (4 marks)

$$k_1 = 1, k_2 = f(0.1, 1.1) = 1.20$$

B1 M1

$$k_3 = f(0.1, 1.12) = 1.22, k_4 = f(0.2, 1.244) = 1.444$$

A1

$$y_1 = 1 + \frac{0.2}{6}(1 + 2.4 + 2.44 + 1.444) = 1.24280$$

A1

Answer: 1.24280

4 marks

12 (3 marks)

$$y(x+h) = y(x) + \int_x^{x+h} f(t, y) dt$$

M1

The trapezium rule approximates the integral by $\frac{h}{2}[f(x, y) + f(x+h, y(x+h))]$, using the predicted value y^* for the unknown $y(x+h)$

A1

Euler's method uses only the left-hand gradient (a rectangle), whereas this uses the average of the gradients at both ends

E1

Answer: Trapezium rule on the gradient

3 marks

13 (5 marks)

$$T_1^* = 90 + 2(-7) = 76.0$$

B1

$$T_1 = 90 + 1 \times [-7 + (-0.1)(76 - 20)] = 77.40$$

M1 A1

$$T_2 = 67.068$$

A1

Exact: 66.922, so the error is -0.146

A1

Answer: 67.068

5 marks

14 (6 marks)

$$f(0, 1) = 0.500, y_1^* = 1 + 0.5 \times 0.500 = 1.25000$$

M1 A1

$$\text{First correction: } y_1^{(1)} = 1 + 0.25[0.500 + f(0.5, 1.25000)] = 1.21389$$

M1

A1

$$\text{Second correction: } y_1^{(2)} = 1 + 0.25[0.500 + f(0.5, 1.21389)] = 1.21679$$

M1 A1

Answer: 1.21389, 1.21679

6 marks

15 (6 marks)

$$(a) y_{n+1}^* = y_n + hy_n = (1+h)y_n$$

B1

$$y_{n+1} = y_n + \frac{h}{2}[y_n + (1+h)y_n]$$

M1

$$= y_n[1 + \frac{h}{2}(2+h)] = (1+h+\frac{h^2}{2})y_n$$

A1

$$\text{Answer: } y_{n+1} = (1+h+\frac{h^2}{2})y_n$$

3 marks

$$(b) y_n = (1.105)^n, \text{ so } y(1) \approx 1.105^{10}$$

M1

$$= 2.71408$$

A1

Error = $e - 2.71408 = 0.00420$, far smaller than for Euler's method (0.1245)

A1

Answer: 2.71408

3 marks

16 (6 marks)

$$(a) k_1 = y_n, k_2 = y_n + \frac{h}{2}y_n = (1+\frac{h}{2})y_n$$

M1 A1

$$y_{n+1} = y_n + h(1+\frac{h}{2})y_n = (1+h+\frac{h^2}{2})y_n$$

A1

$$\text{Answer: } y_{n+1} = (1+h+\frac{h^2}{2})y_n$$

3 marks

$$(b) \text{ Local error} = e^h - (1+h+\frac{h^2}{2}) \approx \frac{h^3}{6}$$

M1

$$\text{For } h = 0.1: \frac{0.001}{6} = 0.000167$$

A1

Actual $e^{0.1} - 1.105 = 0.000171$, in close agreement

A1

Answer: $h^3/6 \approx 0.000167$

3 marks

17 (6 marks)

$$(a) k_1 = y_n, k_2 = y_n + \frac{h}{2}k_1 = (1 + \frac{h}{2})y_n \quad \text{M1}$$

$$k_3 = y_n + \frac{h}{2}k_2 = (1 + \frac{h}{2} + \frac{h^2}{4})y_n \quad \text{A1}$$

$$k_4 = y_n + hk_3 = (1 + h + \frac{h^2}{2} + \frac{h^3}{4})y_n \quad \text{A1}$$

$$\text{Answer: } k_1 = y_n, k_2 = (1 + \frac{h}{2})y_n, k_3 = (1 + \frac{h}{2} + \frac{h^2}{4})y_n \text{ and } k_4 = (1 + h + \frac{h^2}{2} + \frac{h^3}{4})y_n \quad \text{3 marks}$$

$$(b) y_{n+1}/y_n = 1 + h + \frac{h^2}{2} + \frac{h^3}{6} + \frac{h^4}{24} = 1.648438 \quad \text{M1 A1}$$

$$y(1) \approx 2.71735, \text{ and } e = 2.71828, \text{ error } 0.00094 \quad \text{A1}$$

$$\text{Answer: } 2.71735 \quad \text{3 marks}$$

18 (7 marks)

$$\text{Predictor: } x^* = 1 + 0.2(-0) = 1.00, y^* = 0 + 0.2(1) = 0.20 \quad \text{M1 A1}$$

$$\text{Gradients at the start: } \frac{dx}{dt} = 0, \frac{dy}{dt} = 1 \quad \text{B1}$$

$$\text{Gradients at the predicted point: } \frac{dx}{dt} = -0.20, \frac{dy}{dt} = 1.00 \quad \text{M1 A1}$$

$$x(0.2) \approx 1 + 0.1[0 + (-0.2)] = 0.98, y(0.2) \approx 0 + 0.1[1 + 1] = 0.20 \quad \text{A1}$$

$$\text{Exact: } \cos 0.2 = 0.9801, \sin 0.2 = 0.1987: \text{ close agreement} \quad \text{A1}$$

$$\text{Answer: } x \approx 0.98, y \approx 0.20 \quad \text{7 marks}$$

19 (6 marks)

$$y^*_1 = 1 + 0.2a \quad \text{B1}$$

$$y_1 = 1 + 0.1[a + a(1 + 0.2a)] \quad \text{M1 A1}$$

$$= 1 + 0.2a + 0.02a^2 \quad \text{A1}$$

$$0.02a^2 + 0.2a - 0.22 = 0, \text{ so } a^2 + 10a - 11 = 0 \quad \text{M1}$$

$$(a + 11)(a - 1) = 0, \text{ and } a > 0 \text{ so } a = 1 \quad \text{A1}$$

$$\text{Answer: } a = 1 \quad \text{6 marks}$$

20 (7 marks)

$$\text{Euler: } y_{10} = 1.1^{10} = 2.59374, \text{ error } 0.12454 \quad \text{M1 A1}$$

$$\text{Euler: } y_{20} = 1.05^{20}, \text{ error } 0.064984, \text{ ratio } 1.92 \quad \text{A1}$$

$$\text{Improved Euler: } y_{10} = 1.105^{10}, \text{ error } 0.0042010 \quad \text{M1 A1}$$

$$\text{Improved Euler: } y_{20} = (1.0525)^{20}, \text{ error } 0.0010908, \text{ ratio } 3.85 \quad \text{A1}$$

$$\text{Ratio } \approx 2 \text{ means error } \propto h \text{ (first order); ratio } \approx 4 \text{ means error } \propto h^2 \text{ (second order)} \quad \text{E1}$$

$$\text{Answer: Euler first order; improved Euler second order} \quad \text{7 marks}$$

21 (9 marks)

$$(a) t = 0: f = 4.5000, N^* = 34.5000, f^* = 4.8487, N = 34.6744 \quad \text{M1 A1}$$

$$t = 1: f = 4.8582, N^* = 39.5326, f^* = 4.9989, N = 39.6029 \quad \text{M1 A1}$$

$$t = 2: f = 4.9992, N^* = 44.6021, f^* = 4.8941, N = 44.5496 \quad \text{A1 A1}$$

$$\text{Answer: } 44.550 \quad \text{6 marks}$$

$$(b) 0.4N - \frac{N^2}{200} - 3 = 0, \text{ so } N^2 - 80N + 600 = 0 \quad \text{M1}$$

$$N = \frac{80 \pm \sqrt{6400 - 2400}}{2} = \frac{80 \pm \sqrt{4000}}{2} \quad \text{M1}$$

$$N = 40 \pm 10\sqrt{10}, \text{ so } N \approx 8.38 \text{ or } 71.62 \quad \text{A1}$$

$$\text{Answer: } N \approx 8.4 \text{ or } 71.6 \quad \text{3 marks}$$

22 (8 marks)

Step 1: $k_1 = f(0, 1) = 1.0000$

B1

$k_2 = f(0.05, 1 + 0.05k_1) = f(0.05, 1.0500) = 1.10500$

M1 A1

$k_3 = f(0.05, 1.05525) = 1.11605, k_4 = f(0.1, 1.11161) = 1.24567$

A1

$y_1 = 1 + \frac{0.1}{6}(1.0000 + 2(1.10500) + 2(1.11605) + 1.24567) = 1.11146$

A1

Step 2: $k_1 = f(0.1, 1.11146) = 1.24535, k_2 = 1.40014$

M1

$k_3 = 1.41837, k_4 = 1.61076$

A1

$y_2 = 1.25302$

A1

Answer: 1.25302

8 marks

23 (6 marks)

(a) The corrector averages two gradients: $y_1 = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^*)]$

E1

Mia has omitted $f(x_0, y_0)$, and has used $y_0 + \frac{h}{2} f(x_1, y_1^*)$, which halves the step

E1

Answer: Left out $f(x_0, y_0)$

2 marks

(b) $y_1 = 1 + 0.05[1 + 1.2] = 1.110$

M1 A1

$y_2 = 1.24205$

A1

$y_3 = 1.39847$

A1

Answer: 1.110, 1.24205, 1.39847

4 marks

24 (8 marks)

(a) $y_{r+1} = y_{r-1} + 2(0.5)(-2y_r) = y_{r-1} - 2y_r$

B1

Answer: $y_{r+1} = y_{r-1} - 2y_r$

1 mark

(b) $y_2 = 1 - 2(0.368) = 0.264$

M1

$y_3 = 0.368 - 2(0.264) = -0.160, y_4 = 0.584$

A1

$y_5 = -1.328, y_6 = 3.240$

A1

Answer: 0.264, -0.160, 0.584, -1.328, 3.240

3 marks

(c) Exact $y(3) = e^{-6} = 0.00248$, but $y_6 = 3.240$

M1

The numerical values alternate in sign and grow in size, so the method is unstable for this problem

A1

Answer: Unstable, oscillating growth

2 marks

(d) Try $y_r = \lambda^r$: $\lambda^2 + 2\lambda - 1 = 0$, so $\lambda = -1 \pm \sqrt{2}$

M1

$|-1 - \sqrt{2}| \approx 2.41 > 1$, so this component grows and alternates in sign

E1

Answer: $\lambda = -1 - \sqrt{2}, |\lambda| > 1$

2 marks

25 (7 marks)

(a) $y_1 = 1.2, y_2 = 1.2 + 0.2(1.4) = 1.4800$

B1

Answer: 1.4800

1 mark

(b) $y_1 = 1.240$

M1

$y_2 = 1.57680$

A1

Answer: 1.57680

2 marks

(c) $y_1 = 1.240$

M1

$y_2 = 1.57680$

A1

Answer: 1.57680

2 marks

(d) $y_1 = 1.24280, y_2 = 1.58364$

B1

Answer: 1.58364

1 mark

(e) Errors: Euler 0.10, improved Euler 0.0068, mid-point 0.0068, RK4 0.000013; each method improves on Euler, with RK4 by far the best

E1

Answer: RK4 is the most accurate

1 mark

26 (7 marks)

(a) $y_1 = 0 + \frac{\pi}{4} \cos 0 = 0.7854$ M1

$y_2 = 0.7854 + \frac{\pi}{4} \cos(\frac{\pi}{4}) = 1.3408$ A1

Answer: 1.3408 2 marks

(b) Since f does not depend on y , $y_{n+1} = y_n + \frac{h}{2}[\cos x_n + \cos x_{n+1}]$, the trapezium rule M1

$y_2 = \frac{\pi}{8}[\cos 0 + 2\cos(\frac{\pi}{4}) + \cos(\frac{\pi}{2})] = 0.9481$ A1

Answer: 0.9481 2 marks

(c) $y_{n+1} = y_n + h \cos(x_n + \frac{h}{2})$, the mid-ordinate rule M1

$y_2 = \frac{\pi}{4}[\cos(\frac{\pi}{8}) + \cos(\frac{3\pi}{8})] = 1.0262$ A1

Answer: 1.0262 2 marks

(d) Mid-point (error 0.0262), then improved Euler (error -0.0519), then Euler (error 0.3408):
Euler overestimates, the others underestimate E1

Answer: Mid-point is closest 1 mark

27 (10 marks)

(a) Two-variable Taylor expansion of f about (x, y) with increments h and hf M1

$f(x + h, y + hf) = f + hf_x + hff_y + O(h^2)$ A1

Answer: $f(x + h, y + hf) = f + h(f_x + ff_y) + O(h^2)$ 2 marks

(b) $y^*_1 = y + hf$ B1

$y_1 = y + \frac{h}{2}[f + f(x + h, y + hf)]$ M1

$= y + \frac{h}{2}[2f + h(f_x + ff_y)] + O(h^3)$ M1

$= y + hf + \frac{h^2}{2}(f_x + ff_y) + O(h^3)$, which matches $y(x + h)$ to this order A1

Answer: Agrees with the Taylor series to the h^2 term, so the local error is $O(h^3)$ 4 marks

(c) Euler: $y_1 = y + hf$, missing the $\frac{h^2}{2}(f_x + ff_y)$ term M1

Leading error $= \frac{h^2}{2}y''(x) = \frac{h^2}{2}(f_x + ff_y)$ A1

Answer: $\frac{h^2}{2}(f_x + ff_y)$ 2 marks

(d) About $\frac{b-a}{h}$ steps are taken over $[a, b]$ M1

Local errors of $O(h^{p+1})$ accumulate to a global error of about $\frac{b-a}{h} \times O(h^{p+1}) = O(h^p)$ A1

Answer: Number of steps $\propto \frac{1}{h}$ 2 marks

28 (11 marks)

(a) $y_{r+1} = y_{r-1} + 2h(-y_r)$, which rearranges to the given equation B1

Trying $y_r = \lambda^r$ gives $\lambda^2 + 2h\lambda - 1 = 0$ M1

Two distinct roots give the general solution $A\lambda_1^r + B\lambda_2^r$ A1

Answer: $y_{r+1} + 2hy_r - y_{r-1} = 0$, and that $y_r = A\lambda_1^r + B\lambda_2^r$, where λ_1, λ_2 are the roots of $\lambda^2 + 2h\lambda - 1 = 0$ 3 marks

(b) Quadratic formula: $\lambda = \frac{-2h \pm \sqrt{4h^2 + 4}}{2} = -h \pm \sqrt{h^2 + 1}$ M1

$\lambda_1\lambda_2 = h^2 - (h^2 + 1) = -1$ A1

Answer: $\lambda_1 = -h + \sqrt{h^2 + 1}$ and $\lambda_2 = -h - \sqrt{h^2 + 1}$, with $\lambda_1\lambda_2 = -1$ 2 marks

(c) $\lambda_1 = \frac{1}{h + \sqrt{h^2 + 1}} < 1$ and > 0 M1

$ \lambda_2 = \frac{1}{\lambda_1} > 1$	A1
$\lambda_1 = 1 - h + \frac{h^2}{2} - \dots \approx e^{-h}$ for small h , so $A\lambda_1^r$ follows the true solution	E1
Answer: $0 < \lambda_1 < 1$ and $\lambda_2 > 1$	3 marks
(d) $\lambda_1 = 0.90499$, $\lambda_2 = -1.10499$; $A + B = 1$ and $A\lambda_1 + B\lambda_2 = e^{-0.1}$	M1
$B = 0.000075$ (tiny), $A \approx 1$	A1
$ B\lambda_2^{100} \approx 1.6$, whereas the true solution at $x = 10$ is $e^{-10} = 0.000045$: the parasitic term overwhelms it and the estimate is meaningless	E1
Answer: Parasitic term dominates	3 marks
29 (10 marks)	
(a) $h = 0.5$: $r = 1.648438$, $y(1) = r^2 = 2.7173462$	M1 A1
$h = 0.25$: $r^4 = 2.7182099$; $h = 0.125$: $r^8 = 2.7182768$	A1
Answer: 2.7173462, 2.7182099, 2.7182768	3 marks
(b) Errors: 0.000936, 0.0000719, 0.00000498	M1 A1
Ratios: 13.0 and 14.4	A1
Answer: Ratios about 14	3 marks
(c) Halving h divides the error by about 2^p ; the ratios are close to $16 = 2^4$, so $p = 4$	M1
The global error is $O(h^4)$, consistent with a local error $O(h^5)$	A1
Answer: Order 4	2 marks
(d) $e^h - r(h) = \frac{h^5}{120} + \dots$, so the error in one step is about $\frac{h^5}{120}$	M1
Over $\frac{1}{h}$ steps the global error is $\approx e \times \frac{h^4}{120} = 0.0000055$ for $h = 0.125$, compared with the actual 0.0000050	A1
Answer: $e h^4/120$	2 marks
30 (9 marks)	
(a) $y_{n+1} = y_n + h(-10y_n) = (1 - 10h)y_n$	B1
$ 1 - 10h < 1$ requires $0 < h < 0.2$	A1
Answer: $h < 0.2$	2 marks
(b) $y_{n+1} = y_n + \frac{h}{2}[-10y_n - 10y_{n+1}]$	M1
$y_{n+1}(1 + 5h) = y_n(1 - 5h)$	M1
$y_{n+1} = \frac{1 - 5h}{1 + 5h}y_n$	A1
Answer: $y_{n+1} = \frac{1 - 5h}{1 + 5h}y_n$	3 marks
(c) For $h > 0$, $-(1 + 5h) < 1 - 5h < 1 + 5h$	M1
So $ 1 - 5h < 1 + 5h$ and the ratio has modulus less than 1	A1
Answer: $\frac{1 - 5h}{1 + 5h} < 1$ for every $h > 0$, so that the trapezium method decays for any step size	2 marks
(d) Euler ($h = 0.5$): $(1 - 5)^4 = 256$ (growing, alternating)	B1
Trapezium: $(\frac{1 - 2.5}{1 + 2.5})^4 = (\frac{-1.5}{3.5})^4 = 0.0337$; exact $e^{-20} = 0.0000000021$	B1
Answer: 256 and 0.0337	2 marks

Numerical Solutions of Differential Equations

1 (2 marks)

$y'_0 = 0^2 + 1 = 1$	B1
$y_1 \approx 1 + 0.1 \times 1 = 1.1$	B1
Answer: 1.1	2 marks

2 (2 marks)

$$y_2 \approx y_0 + 2hy'_1 = 1 + 0.2(0.1 + 1.11) \\ = 1.242$$

M1

A1

Answer: 1.242

2 marks

3 (3 marks)

$$y_{r+1} = 2y_r - y_{r-1} + h^2 y''_r \\ y_2 = 2(0.995) - 1 + 0.01(-0.995) \\ = 0.98005$$

M1

M1

A1

Answer: 0.98005

3 marks

4 (3 marks)

$$y' = x + y, \text{ so } y'(0) = 1; y'' = 1 + y', \text{ so } y''(0) = 2; y''' = y'', \text{ so } y'''(0) = 2 \\ y(0.1) \approx 1 + 0.1 + \frac{0.01}{2}(2) + \frac{0.001}{6}(2) = 1.11033$$

M1 A1

A1

Answer: 1.11033

3 marks

5 (2 marks)

$$\frac{dy}{dx} = v$$

B1

$$\frac{dv}{dx} = xy - 2v$$

B1

$$\text{Answer: } \frac{dy}{dx} = v, \frac{dv}{dx} = xy - 2v$$

2 marks

6 (3 marks)

$$y'_0 = \frac{2(1) + 2^2}{1 + 2} = \frac{6}{3} = 2$$

M1 A1

$$y(1 + h) \approx 2 + 2h = 2.5, \text{ so } h = 0.25$$

A1

Answer: h = 0.25

3 marks

7 (4 marks)

$$y_1 = y_0 + hf(x_0, y_0) = 1 + 0.1 \ln 5 \\ = 1.16094$$

M1

A1

$$y_2 = 1.16094 + 0.1 \ln(2(2.1) + 1.16094) \\ = 1.329$$

M1

A1

Answer: 1.329

4 marks

8 (4 marks)

$$y_1 = 0.5 + 0.05(3 - 2(0.25)) = 0.625$$

M1 A1

$$y_2 = 0.625 + 0.05(3(1.05)^2 - 2(0.625)^2) \\ = 0.7513$$

M1

A1

Answer: 0.7513

4 marks

9 (4 marks)

$$y_1 = 1 + 0.02 \cos(0.25 + 1) = 1.00631$$

M1 A1

$$y_2 = 1.00631 + 0.02 \cos(0.52^2 + 1.00631^2) = 1.01198$$

A1

$$y_3 = 1.01703, \text{ so } y(0.56) \approx 1.0170$$

A1

Answer: 1.0170

4 marks

10 (5 marks)

$$y_{r+1} = 2y_r - y_{r-1} + h^2 \frac{x_r + 1}{y_r^2} \quad \text{M1}$$

$$y(1.2) \approx 2(2.15) - 2 + 0.01 \times \frac{2.1}{2.15^2} \quad \text{M1 A1}$$

$$= 2.3045 \quad \text{A1}$$

$$y(1.3) \approx 2(2.3045) - 2.15 + 0.01 \times \frac{2.2}{2.3045^2} = 2.4632 \quad \text{A1}$$

Answer: $y(1.2) \approx 2.3045$, $y(1.3) \approx 2.4632$ 5 marks

11 (5 marks)

$$y' = x + y^2, \text{ so } y'(0) = 1 \quad \text{B1}$$

$$y'' = 1 + 2yy', y''' = 2y'^2 + 2yy'', y'''' = 6y'y'' + 2yy''' \quad \text{M1}$$

$$\text{At } x = 0: y''(0) = 3, y'''(0) = 8, y''''(0) = 34 \quad \text{A1}$$

$$y \approx 1 + x + \frac{3x^2}{2} + \frac{4x^3}{3} + \frac{17x^4}{12} \quad \text{A1}$$

$$y(0.2) \approx 1.2729 \quad \text{A1}$$

Answer: 1.2729 5 marks

12 (4 marks)

$$v' = -v - y, \text{ so } y_{r+1} = y_r + 0.1v_r \text{ and } v_{r+1} = v_r + 0.1(-v_r - y_r) \quad \text{M1}$$

$$r = 0: y_1 = 1, v_1 = 0 + 0.1(-1) = -0.10 \quad \text{A1}$$

$$r = 1: y_2 = 1 + 0.1(-0.10) = 0.99 \quad \text{M1}$$

$$v_2 = -0.10 + 0.1(0.10 - 1) = -0.190 \quad \text{A1}$$

Answer: $y(0.2) \approx 0.99$, $y'(0.2) \approx -0.190$ 4 marks

13 (5 marks)

$$y_1 = 1 + 0.1 \times \frac{0 + 1}{k + 1} \quad \text{M1}$$

$$1 + \frac{0.1}{k + 1} = 1.025, \text{ so } k + 1 = 4 \text{ and } k = 3 \quad \text{A1}$$

$$y_2 = 1.025 + 0.1 \times \frac{0.1 + 1.025}{3 + 1.025} \quad \text{M1 A1}$$

$$= 1.0530 \quad \text{A1}$$

Answer: $k = 3$; $y(0.2) \approx 1.0530$ 5 marks

14 (6 marks)

$$\text{(a) } y(k + 0.1) \approx 1 + 0.1 \times \frac{5k^2 - 1}{2k} = 1.2 \quad \text{M1}$$

$$5k^2 - 1 = 4k, \text{ so } 5k^2 - 4k - 1 = 0 \quad \text{M1}$$

$$(5k + 1)(k - 1) = 0, \text{ and } k > 0, \text{ so } k = 1 \quad \text{A1}$$

Answer: $k = 1$ 3 marks

$$\text{(b) } y'(1.1) = \frac{5(1.21) - 1.44}{2(1.1)(1.2)} = 1.7462 \quad \text{M1 A1}$$

$$y(1.2) \approx y(1) + 0.2 \times 1.7462 = 1.349 \quad \text{A1}$$

Answer: 1.349 3 marks

15 (7 marks)

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} = y_i + x_i, \text{ so } y_{i-1} - \frac{33}{16}y_i + y_{i+1} = \frac{x_i}{16}$$

M1 A1

$$i = 1: -\frac{33}{16}y_1 + y_2 = \frac{1}{64}$$

A1

$$i = 2: y_1 - \frac{33}{16}y_2 + y_3 = \frac{1}{32}$$

A1

$$i = 3: y_2 - \frac{33}{16}y_3 = \frac{3}{64} - 1$$

A1

$$\text{Solving: } y_1 = 0.1802, y_2 = 0.3873$$

M1

$$y_3 = 0.6499$$

A1

Answer: 0.1802, 0.3873, 0.6499

7 marks

16 (6 marks)

$$y''(0) = 0 - 1 = -1$$

B1

$$y_1 = 1 + 0 + \frac{0.01}{2}(-1) = 0.995$$

M1 A1

$$y_2 = 2(0.995) - 1 + 0.01(0.1 - 0.995)$$

M1

$$= 0.98105$$

A1

$$y_3 = 2(0.98105) - 0.995 + 0.01(0.2 - 0.98105) = 0.95929$$

A1

Answer: $y(0.2) \approx 0.98105$, $y(0.3) \approx 0.95929$

6 marks

17 (6 marks)

$$(a) y' = 3ax^2 + 7bx^6 + \dots \text{ and } y^2 = a^2x^6 + \dots$$

M1 A1

$$x^2: 3a = 1, \text{ so } a = \frac{1}{3}$$

A1

$$x^6: 7b = a^2 = \frac{1}{9}, \text{ so } b = \frac{1}{63}$$

A1

Answer: $a = 1/3$, $b = 1/63$

4 marks

$$(b) y(0.5) \approx \frac{0.125}{3} + \frac{0.0078125}{63} = 0.04179$$

M1

Difference from 0.04179 is 0.0000005, so the two terms give good accuracy

A1

Answer: 0.04179

2 marks

18 (6 marks)

$$f(0, 1) = -1 \text{ and } f(0.1, 0.9097) = -0.8097$$

B1

$$y_2 = 0.9097 + 0.05[3(-0.8097) - (-1)]$$

M1 A1

$$= 0.8382$$

A1

$$y_3 = 0.8382 + 0.05[3(-0.6382) - (-0.8097)] = 0.7830$$

M1

Exact $y(0.3) = 0.7816$, so the error is -0.0014

A1

Answer: $y(0.2) \approx 0.8382$, $y(0.3) \approx 0.7830$

6 marks

19 (7 marks)

$$(a) \text{ Integrating factor } e^{2x}: \frac{d}{dx}(ye^{2x}) = 4xe^{2x}$$

M1

$$\int 4xe^{2x} dx = 2xe^{2x} - e^{2x} + c$$

M1 A1

$$y = 2x - 1 + ce^{-2x}, \text{ and } y(0) = 1 \text{ gives } c = 2, \text{ so } y = 2x - 1 + 2e^{-2x}$$

A1

Answer: $y = 2x - 1 + 2e^{-2x}$

4 marks

$$(b) y_1 = 1 + 0.1(0 - 2) = 0.80, y_2 = 0.680$$

M1 A1

$$y_3 = 0.6240; \text{ exact } y(0.3) = 0.6976, \text{ error } 0.0736$$

A1

Answer: 0.6240

3 marks

20 (6 marks)

$$(a) \frac{y_{r+2} - 2y_{r+1} + y_r}{h^2} + 2\frac{y_{r+2} - y_r}{2h} = x_{r+1} + y_{r+1} \quad \text{M1}$$

$$\text{Multiply by } h^2: y_{r+2} - 2y_{r+1} + y_r + h(y_{r+2} - y_r) = h^2(x_{r+1} + y_{r+1}) \quad \text{M1}$$

$$(1+h)y_{r+2} = 2y_{r+1} - (1-h)y_r + h^2(x_{r+1} + y_{r+1}) \quad \text{A1}$$

$$\text{Answer: } (1+h)y_{r+2} = 2y_{r+1} - (1-h)y_r + h^2(x_{r+1} + y_{r+1}) \quad \text{3 marks}$$

$$(b) 1.1y_2 = 2(1.1) - 0.9(1) + 0.01(0.1 + 1.1) = 1.3120 \quad \text{M1}$$

$$y(0.2) \approx 1.1927 \quad \text{A1}$$

$$1.1y_3 = 2(1.1927) - 0.9(1.1) + 0.01(0.2 + 1.1927), \text{ so } y(0.3) \approx 1.2813 \quad \text{A1}$$

$$\text{Answer: } y(0.2) \approx 1.1927, y(0.3) \approx 1.2813 \quad \text{3 marks}$$

21 (8 marks)

$$(a) \text{ The recurrence needs two earlier values, } y_r \text{ and } y_{r-1}, \text{ but only } y(2) \text{ is known} \quad \text{E1}$$

$$\text{Answer: Needs two starting values} \quad \text{1 mark}$$

$$(b) y''(2) = 2 + 2 \cos 1 = 3.08060 \quad \text{M1}$$

$$y(2.1) \approx 1 + 0.1(0.5) + \frac{0.01}{2}(3.0806) \quad \text{M1}$$

$$= 1.06540 \quad \text{A1}$$

$$\text{Answer: } 1.06540 \quad \text{3 marks}$$

$$(c) y_2 = 2(1.06540) - 1 + 0.01(2 + 2.1 \cos 1.06540) \quad \text{M1}$$

$$= 1.16097 \quad \text{A1}$$

$$y_3 = 2(1.16097) - 1.06540 + 0.01(2 + 2.2 \cos 1.16097) \quad \text{M1}$$

$$= 1.28531 \quad \text{A1}$$

$$\text{Answer: } y(2.2) \approx 1.16097, y(2.3) \approx 1.28531 \quad \text{4 marks}$$

22 (8 marks)

$$(a) y' = x + y, \text{ so } y'(0) = 1; y'' = 1 + y' \text{ so } y''(0) = 2 \quad \text{M1 A1}$$

$$\text{For } n \geq 3, y^{(n)} = y^{(n-1)}, \text{ so every higher derivative at 0 equals 2} \quad \text{M1}$$

$$y \approx 1 + x + \frac{2x^2}{2!} + \frac{2x^3}{3!} + \frac{2x^4}{4!} + \frac{2x^5}{5!} = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{60} \quad \text{A1}$$

$$\text{Answer: } 1 + x + x^2 + x^3/3 + x^4/12 + x^5/60 \quad \text{4 marks}$$

$$(b) y(0.5) \approx 1.7974 \quad \text{B1}$$

$$y(1) \approx 3.4333 \quad \text{B1}$$

$$\text{Answer: } 1.7974, 3.4333 \quad \text{2 marks}$$

$$(c) \text{ Errors: } x = 0.5: 0.000047; x = 1: 0.0032 \quad \text{M1}$$

$$\text{The first omitted term is } \frac{2x^6}{6!}, \text{ which is larger for larger } x, \text{ so more terms are needed as } x \text{ increases} \quad \text{E1}$$

$$\text{Answer: Errors } 0.000047, 0.0032 \quad \text{2 marks}$$

23 (6 marks)

$$(a) \text{ The term should be } h^2 y''_r \text{ (not } hy''_r), \text{ because the approximation to } y'' \text{ has } h^2 \text{ in the denominator} \quad \text{E1}$$

$$\text{Answer: } h \text{ should be } h^2 \quad \text{1 mark}$$

$$(b) y_2 = 2(0.2) - 0 + 0.01(-4)(0.2) \quad \text{M1}$$

$$= 0.392 \quad \text{A1}$$

$$y_3 = 2(0.392) - 0.2 - 0.04(0.392) = 0.56832 \quad \text{A1}$$

$$\text{Answer: } y(0.2) \approx 0.392, y(0.3) \approx 0.56832 \quad \text{3 marks}$$

$$(c) \text{ Exact } \sin 0.4 = 0.38942 \quad \text{B1}$$

$$\text{Ravi's value } 0.32 \text{ is too small by } 0.0694, \text{ the corrected } 0.392 \text{ is far closer (error } -0.0026) \quad \text{E1}$$

$$\text{Answer: Corrected value is closer} \quad \text{2 marks}$$

24 (8 marks)

$$(a) \frac{1 - 2y_1 + 2}{0.25} = y_1$$

M1

$$3 = 2.25y_1$$

A1

$$y(0.5) \approx 1.3333$$

A1

Answer: 1.3333

3 marks

$$(b) i = 1: 1 - \frac{33}{16}y_1 + y_2 = 0; i = 2: y_1 - \frac{33}{16}y_2 + y_3 = 0; i = 3: y_2 - \frac{33}{16}y_3 + 2 = 0$$

M1 A1

$$\text{Solving: } y_1 = 1.1302, y_2 = 1.3310, y_3 = 1.6150$$

A1

Answer: 1.3310

3 marks

$$(c) \text{Extrapolated} = \frac{4(1.3310) - 1.3333}{3} = 1.3303$$

M1

$$\text{Errors: } h = 0.5: -0.0031, h = 0.25: -0.00079, \text{ extrapolated: } -0.000024$$

A1

Answer: 1.3303

2 marks

25 (8 marks)

$$S_{r+1} = S_r - 0.002S_r I_r, I_{r+1} = I_r + 0.002S_r I_r - 0.4I_r$$

M1 A1

$$r = 0: 0.002SI = 8, S_1 = 392.0, I_1 = 14.0$$

M1 A1

$$r = 1: 0.002SI = 10.976, S_2 = 381.0, I_2 = 19.4$$

M1 A1

$$r = 2: S_3 = 366.3, I_3 = 26.4$$

A1 A1

Answer: S ≈ 366.3, I ≈ 26.4

8 marks

26 (8 marks)

$$\text{Predictor: } y^*_2 = 1 + 0.2(0.1 + 1.1103) = 1.24206$$

M1 A1

$$f(0.2, y^*_2) = 1.44206; f(0.1, 1.1103) = 1.2103, f(0, 1) = 1$$

B1

$$\text{Corrector: } y_2 = 1 + \frac{0.1}{3}[1 + 4(1.2103) + 1.44206] = 1.24278$$

M1 A1

$$y^*_3 = 1.1103 + 0.2 f(0.2, 1.24278) = 1.39886$$

A1

$$y_3 = 1.39964$$

A1

$$\text{Exact } y(0.2) = 1.24281 \text{ and } y(0.3) = 1.39972, \text{ so the errors are } 0.000030 \text{ and } 0.000076$$

E1

Answer: y(0.2) ≈ 1.24278, y(0.3) ≈ 1.39964

8 marks

27 (10 marks)

$$(a) y_{r+1} - 2y_r + y_{r-1} = h^2 y_r, \text{ which rearranges to the given form}$$

B1

$$\text{Answer: } y_{r+1} = (2 + h^2)y_r - y_{r-1}$$

1 mark

$$(b) \cosh((r+1)\theta) + \cosh((r-1)\theta) = 2\cosh(r\theta)\cosh\theta$$

M1 A1

$$= 2(1 + \frac{h^2}{2})\cosh(r\theta) = (2 + h^2)\cosh(r\theta)$$

A1

$$y_0 = \cosh 0 = 1 \text{ and } y_1 = \cosh \theta = 1 + \frac{h^2}{2}$$

B1

$$\text{Answer: } y_r = \cosh(r\theta) \text{ satisfies the recurrence, } y_0 = 1 \text{ and } y_1 = 1 + \frac{h^2}{2}$$

4 marks

$$(c) \cosh \theta = 1.03125, \text{ so } \theta = 0.249353$$

M1

$$y(1) \approx y_4 = \cosh(4\theta) = \cosh(0.99741) = 1.54005$$

A1

$$\text{Error} = 0.00303: \text{ the numerical solution is slightly small}$$

A1

Answer: 1.54005

3 marks

$$(d) \cosh \theta = 1 + \frac{\theta^2}{2} + \frac{\theta^4}{24} + \dots \text{ and } \theta^2 = h^2 - 2ch^4; \text{ matching the } h^4 \text{ terms: } -c + \frac{1}{24} = 0 \text{ so } c =$$

M1

$$\frac{1}{24}$$

$$\text{So } r\theta = rh - \frac{rh^3}{24} = x(1 - \frac{h^2}{24}), \text{ i.e. the error in the argument is } O(h^2)$$

A1

Answer: c = 1/24

2 marks

28 (10 marks)

(a) $y(x \pm h) = y \pm hy' + \frac{h^2}{2}y'' \pm \frac{h^3}{6}y''' + \dots$ B1

Subtracting gives $y(x + h) - y(x - h) = 2hy' + O(h^3)$, so $y' \approx \frac{y_{r+1} - y_{r-1}}{2h}$ M1

Adding gives $y(x + h) + y(x - h) = 2y + h^2y'' + O(h^4)$, so $y'' \approx \frac{y_{r+1} - 2y_r + y_{r-1}}{h^2}$ A1

Answer: $y''_r \approx \frac{y_{r+1} - 2y_r + y_{r-1}}{h^2}$ and $y'_r \approx \frac{y_{r+1} - y_{r-1}}{2h}$, each with error $O(h^2)$ 3 marks

(b) $\frac{y_{r+1} - 2y_r + y_{r-1}}{h^2} + \frac{y_{r+1} - y_{r-1}}{2h} = x_r$ M1

Multiply by $2h^2$: $2y_{r+1} - 4y_r + 2y_{r-1} + hy_{r+1} - hy_{r-1} = 2h^2x_r$ M1

$(2 + h)y_{r+1} = 2h^2x_r + 4y_r - (2 - h)y_{r-1}$ A1

Answer: $(2 + h)y_{r+1} = 2h^2x_r + 4y_r - (2 - h)y_{r-1}$ 3 marks

(c) $y''(0) = 0 - 1 = -1$, so $y(0.1) \approx 0.1 - 0.005 = 0.095$ M1

$2.1y_2 = 0.02(0.1) + 4(0.095) - 1.9(0)$, so $y_2 = 0.18190$ A1

$2.1y_3 = 0.02(0.2) + 4(0.18190) - 1.9(0.095)$, so $y_3 = 0.26244$ A1

Answer: $y(0.2) \approx 0.18190$, $y(0.3) \approx 0.26244$ 3 marks

(d) Exact: 0.09533, 0.18254, 0.26336; errors 0.00033, 0.00063, 0.00093 B1

Answer: Errors of order 10^{-4} 1 mark

29 (10 marks)

(a) For $y' = y$, every derivative equals y : $y^{(k)} = y$ M1

Substituting gives $y(x + h) \approx y(1 + h + \frac{h^2}{2!} + \dots + \frac{h^p}{p!})$ A1

Answer: $y(x + h) \approx y(1 + h + \frac{h^2}{2!} + \dots + \frac{h^p}{p!})$ 2 marks

(b) $e^{0.5} = 1.648721$ B1

$p = 1$: 0.149, $p = 2$: 0.0237 M1

$p = 3$: 0.00289, $p = 4$: 0.000284 A1

Answer: 0.149, 0.0237, 0.00289, 0.000284 3 marks

(c) The first omitted term is $\frac{h^{p+1}}{(p+1)!}$ and the error is about this size M1

$p = 6$: error 0.00000165, $p = 7$: error 0.000000103 M1

So the smallest p is 7 A1

Answer: $p = 7$ 3 marks

(d) $T_4(0.5) = 1.648438$, so $y(1) \approx T_4(0.5)^2 = 2.717346$ M1

Error = $e - 2.717346 = 0.000936$ A1

Answer: $y(1) \approx 2.717346$, error 0.000936 2 marks

30 (10 marks)

(a) $\frac{d\theta}{dt} = \omega$, $\frac{d\omega}{dt} = -\sin \theta$ B1

Answer: $\frac{d\theta}{dt} = \omega$, $\frac{d\omega}{dt} = -\sin \theta$ 1 mark

(b) Step 1: predictor $\theta^* = 1.00000$, $\omega^* = -0.08415$; then $\theta_1 = 0.99579$, $\omega_1 = -0.08415$ M1 A1

Step 2: predictor $\theta^* = 0.98738$, $\omega^* = -0.16807$; then $\theta_2 = 0.98318$, $\omega_2 = -0.16784$ M1 A1

Step 3: $\theta_3 = 0.96224$, $\omega_3 = -0.25059$ M1 A1

Answer: $\theta \approx 0.96224$, $\omega \approx -0.25059$ 6 marks

(c) $\cos 0.3 = 0.95534$, compared with the numerical $\theta(0.3) = 0.96224$ M1

Since $\theta = 1$ rad is not small, $\sin \theta < \theta$, so the true pendulum accelerates less: it stays higher; the small-angle model is not accurate here E1

Answer: $\cos 0.3 = 0.95534$ 2 marks

(d) Error = $0.962289 - 0.962237 = 0.000052$ B1

Answer: 0.000052 1 mark

Error Analysis

1 (2 marks)

$$\text{Absolute error} = |3.142 - 3.14159| = 0.00041$$

B1

$$\text{Relative error} = \frac{0.00041}{3.14159} = 0.00013$$

B1

Answer: 0.00041; 0.00013

2 marks

2 (2 marks)

The maximum error is half the unit of rounding, 0.05 cm

M1

$$5.55 \leq x < 5.65$$

A1

Answer: $5.55 \leq x < 5.65$

2 marks

3 (3 marks)

$$\frac{22}{7} = 3.142857...$$

B1

$$\text{Absolute error} = 3.142857 - 3.141593 = 0.00126$$

M1

$$\text{Percentage error} = \frac{0.00126}{\pi} \times 100 = 0.040\%$$

A1

Answer: 0.00126; 0.040%

3 marks

4 (3 marks)

(a) Rounding: the next digit is 2, so the result is 2.718, error = 0.000282

B1

Truncating: 2.718, error = 0.000282

B1

Answer: 2.718 in each case; error 0.000282

2 marks

(b) For 2.7189: truncating gives 2.718 (error 0.0009) but rounding gives 2.719 (error 0.0001)

E1

Answer: e.g. 2.7189

1 mark

5 (2 marks)

$$3.35 \leq a < 3.45 \text{ and } 2.05 \leq b < 2.15$$

B1

$$5.40 \leq a + b < 5.60$$

B1

Answer: $5.40 \leq a + b < 5.60$

2 marks

6 (2 marks)

$$\text{Absolute error} = |3.16 - 3.16228| = 0.0023$$

B1

$$\text{Percentage error} = \frac{0.0023}{3.16228} \times 100 = 0.072\%$$

B1

Answer: 0.0023; 0.072%

2 marks

7 (4 marks)

$$(a) 4.55 \leq a < 4.65, 2.25 \leq b < 2.35$$

B1

$$\text{Least} = 4.55 \times 2.25 = 10.2375, \text{ greatest} < 4.65 \times 2.35 = 10.9275$$

B1

Answer: $10.2375 \leq ab < 10.9275$

2 marks

$$(b) \text{Least} = \frac{4.55}{2.35} = 1.9362$$

M1

$$\text{Greatest} < \frac{4.65}{2.25} = 2.0667$$

A1

Answer: $1.9362 \leq a/b < 2.0667$

2 marks

8 (4 marks)

$$\text{Lower bound} = 8.45 \times 4.15 = 35.0675$$

M1 A1

$$\text{Upper bound} = 8.55 \times 4.25 = 36.3375$$

A1

Both bounds round to 35 (2 s.f.), but to 3 s.f. they would be 35.1 and 36.3, so only 2 s.f. can be quoted

E1

Answer: $35.0675 \leq A < 36.3375$; 2 s.f.

4 marks

9 (4 marks)

$$(a) \text{ Estimate} = 1 + 0.5 + 0.125 + \frac{0.125}{6} = 1.645833$$

M1

$$\text{Error} = 1.648721 - 1.645833 = 0.002888$$

A1

Answer: 0.002888

2 marks

$$(b) \text{ Next term} = \frac{0.5^4}{24} = 0.002604, \text{ close to the true error } 0.002888$$

M1

$$\text{Percentage} = \frac{0.002604}{1.6458} \times 100 = 0.16\%$$

A1

Answer: 0.16%

2 marks

10 (4 marks)

r ranges from 4.9 to 5.1

B1

$$\text{Greatest area} = \pi(5.1)^2 = 81.713, \text{ least} = \pi(4.9)^2 = 75.430, \text{ nominal} = 78.540$$

M1

$$\text{Greatest error} \approx 3.17 \text{ cm}^2$$

A1

Percentage error $\approx 4.04\%$, about double the percentage error in r

A1

Answer: $\approx 4\%$

4 marks

11 (4 marks)

$$\frac{dA}{dr} = 2\pi r$$

B1

$$\delta A \approx 2\pi(10)(0.1)$$

M1

$$= 6.28 \text{ cm}^2$$

A1

$$\text{Relative error} = \frac{\delta A}{A} = \frac{6.28}{314.2} = 0.020 = 2 \times (\text{relative error in } r)$$

A1

Answer: 6.28 cm^2

4 marks

12 (4 marks)

$$1.2335 \leq x < 1.2345 \text{ and } 1.2295 \leq y < 1.2305$$

B1

$$\text{Least } x - y = 1.2335 - 1.2305 = 0.0030$$

M1

$$\text{Greatest } x - y < 1.2345 - 1.2295 = 0.0050, \text{ so } x - y = 0.004 \pm 0.001$$

A1

The absolute errors add but the answer is tiny (0.004), so the relative error is up to 25% compared with about 0.04% for x and y: cancellation

E1

Answer: $0.003 \leq x - y < 0.005$

4 marks

13 (5 marks)

$$(a) 100.005 - 100.000 = 0.005, \text{ with at most 1 significant figure surviving}$$

M1

The rounding errors in the 6 s.f. values (up to 0.0005) are as large as the answer's size scale, so there is cancellation

E1

Answer: 0.005; unreliable

2 marks

$$(b) \text{ Multiply by } \frac{\sqrt{(x+1)} + \sqrt{x}}{\sqrt{(x+1)} + \sqrt{x}}: \text{ numerator } (x+1) - x = 1$$

M1

$$= \frac{1}{100.005 + 100} = \frac{1}{200.005}$$

A1

$$= 0.00500$$

A1

Answer: 0.00500

3 marks

14 (4 marks)

$$\text{Error in } f(x+h) - f(x) \text{ is at most } 2 \times 0.00005 = 0.0001$$

M1

$$\text{Divided by } h = 0.01: \text{ maximum error} = \frac{0.0001}{0.01} = 0.01$$

A1

In general the rounding error is up to $\frac{0.0001}{h}$, which grows as h decreases

M1

So very small h makes the estimate worse: rounding error eventually exceeds truncation error

E1

Answer: 0.01

4 marks

15 (5 marks)

$$12.35 \leq a < 12.45 \text{ and } 3.65 \leq b < 3.75$$

B1

The expression $\frac{a-b}{a+b}$ increases with a and decreases with b

M1

$$\text{Greatest} < \frac{12.45 - 3.65}{12.45 + 3.65} = 0.5466$$

M1 A1

$$\text{Least} = \frac{12.35 - 3.75}{12.35 + 3.75} = 0.5342$$

A1

Answer: $0.5342 \leq \text{value} < 0.5466$

5 marks

16 (5 marks)

$$\frac{\partial V}{\partial r} = 2\pi rh = 251.33, \quad \frac{\partial V}{\partial h} = \pi r^2 = 50.27$$

M1 A1

$$\delta V \approx 251.33 \times 0.05 + 50.27 \times 0.1$$

M1

$$= 17.6 \text{ cm}^3$$

A1

$$V = 502.7, \text{ so percentage error} \approx 3.50\%$$

A1

Answer: 17.6 cm^3 ; 3.5%

5 marks

17 (5 marks)

$$(a) \sum_{k=1}^{10} \frac{1}{k^2} = 1 + 0.25 + 0.11111 + \dots$$

M1

$$= 1.54977$$

A1

Answer: 1.54977

2 marks

$$(b) \frac{1}{k^2} < \frac{1}{x^2} \text{ for } k-1 < x < k, \text{ so the tail} < \int_{10}^{\infty} \frac{1}{x^2} dx$$

M1

$$= \left[-\frac{1}{x}\right]_{10}^{\infty} = 0.1$$

A1

$$\text{Actual error} = 0.09517, \text{ which is below the bound } 0.1$$

A1

Answer: Bound 0.1; actual 0.09517

3 marks

18 (5 marks)

$$(x(1 + \epsilon_x))(y(1 + \epsilon_y)) = xy(1 + \epsilon_x + \epsilon_y + \epsilon_x\epsilon_y) \approx xy(1 + \epsilon_x + \epsilon_y)$$

M1 A1

$$\frac{x(1 + \epsilon_x)}{y(1 + \epsilon_y)} \approx \frac{x}{y}(1 + \epsilon_x)(1 - \epsilon_y) \approx \frac{x}{y}(1 + \epsilon_x - \epsilon_y)$$

A1

$$\text{For } \frac{xy}{z} \text{ the relative errors combine as } \epsilon_x + \epsilon_y - \epsilon_z$$

M1

$$\text{Greatest} = 1\% + 1\% + 1\% = 3\%$$

A1

Answer: 3%

5 marks

19 (4 marks)

$$(a) 0.1 - \frac{0.001}{6} = 0.099833333$$

M1

$$\text{Error} = 0.0000000833$$

A1

Answer: 0.0000000833

2 marks

$$(b) \frac{0.1^5}{120} = 0.0000000833 < 10^{-7}$$

M1

The series alternates with decreasing terms, so the error is smaller than the first omitted term

E1

Answer: Error $< 8.4 \times 10^{-8}$

2 marks

20 (5 marks)

$$(a) e_0 = 0.5 - 0.567143 = -0.067143$$

B1

$$e_3 \approx (-\alpha)^3 e_0$$

M1

$$\approx 0.012$$

A1

Answer: 0.012

3 marks

$$(b) x_1 = 0.60653, x_2 = 0.54524, x_3 = 0.57970$$

M1

$$\text{Error} = 0.013, \text{ similar to the prediction}$$

A1

Answer: 0.57970

2 marks

21 (7 marks)

$$(a) T_2 = \frac{\pi}{8}[0 + 1 + 2 \sin(\frac{\pi}{4})] = 0.94806$$

M1 A1

$$T_4 = 0.98712; \text{ errors } 0.05194 \text{ and } 0.01288$$

A1

Answer: 0.94806, 0.98712

3 marks

$$(b) \frac{E_2}{E_4} = 4.03 \approx 4$$

M1

Halving h divides the error by about 4, so $E \propto h^2$

E1

Answer: 4.03

2 marks

$$(c) h = \frac{\pi}{8}, \text{ bound} = \frac{\frac{\pi}{2}(\frac{\pi}{8})^2}{12} = 0.0202$$

M1

Actual error 0.0129 is below the bound

A1

Answer: 0.0202

2 marks

22 (6 marks)

$$(a) 5000^2 - 4 = 24\,999\,996, \text{ which the calculator stores as } 25\,000\,000 \text{ (6 s.f.)}$$

M1

$$\sqrt{25\,000\,000} = 5000.00, \text{ and the larger root is } \frac{5000 + 5000.00}{2} = 5000.00$$

A1

$$\text{Smaller root} = \frac{5000 - 5000.00}{2} = 0$$

A1

Answer: 0

3 marks

$$(b) \text{ Product of roots} = 1, \text{ so smaller root} = \frac{1}{5000.00}$$

M1

$$= 0.000200$$

A1

The formula subtracts two nearly equal numbers, so almost all the significant figures are lost: cancellation error

E1

Answer: 0.000200

3 marks

23 (7 marks)

$$(a) T_4 = 0.782794$$

M1 A1

$$T_8 = 0.784747$$

A1

Answer: 0.782794, 0.784747

3 marks

$$(b) I - T_n \approx Ch^2: I - T_8 \approx \frac{1}{4}(I - T_4)$$

M1

$$\text{So } 4(I - T_8) = I - T_4, \text{ giving } I \approx \frac{4T_8 - T_4}{3}$$

M1

$$= 0.785398$$

A1

Answer: 0.785398

3 marks

$$(c) \text{ Errors: } 0.00065 \text{ and } 0.000000038; \text{ the improved value is far more accurate}$$

E1

Answer: Errors 0.00065 (T_8) and 0.000000038 (improved)

1 mark

24 (6 marks)

$$|e_{n+1}| \leq k|e_n| + \varepsilon$$

M1 A1

$$\text{Assume } |e_n| \leq k^n|e_0| + \varepsilon(1 + k + \dots + k^{n-1})$$

M1

$$\text{Then } |e_{n+1}| \leq k^{n+1}|e_0| + \varepsilon(k + \dots + k^n) + \varepsilon, \text{ which is the geometric sum } \frac{\varepsilon(1 - k^{n+1})}{1 - k}$$

A1

$$\text{As } n \rightarrow \infty, k^n \rightarrow 0 \text{ and the bound} \rightarrow \frac{\varepsilon}{1 - k}$$

M1

$$= \frac{5 \times 10^{-6}}{0.5} = 10^{-5}$$

A1

Answer: 10^{-5}

6 marks

25 (5 marks)

$$(a) \text{ Subtract: } 0.0001y = 0.0001, \text{ so } y = 1$$

M1

$$x = 1$$

A1

Answer: $x = 1, y = 1$

2 marks

$$(b) 0.0001y = 0.0002, \text{ so } y = 2$$

M1

$$x = 0$$

A1

A tiny change in the data changes the solution drastically, so the system is ill-conditioned (the lines are nearly parallel)

E1

Answer: $x = 0, y = 2$

3 marks

26 (6 marks)

$$\text{Relative error} = \frac{e - (1 + \frac{1}{N})^N}{e}$$

M1

$$\ln(1 + \frac{1}{N})^N \approx 1 - \frac{1}{2N}, \text{ so } (1 + \frac{1}{N})^N \approx e(1 - \frac{1}{2N}), \text{ relative error} \approx \frac{1}{2N}$$

M1 A1

$$\text{Need } \frac{1}{2N} < 0.01, \text{ so } N > 50$$

M1

$$\text{Check } N = 50: (1.02)^{50} = 2.6916, \text{ relative error } 0.982\%$$

A1

$$N = 51 \text{ gives } 0.963\%, \text{ so check which is smallest: first } N \text{ with error } < 1\% \text{ is } N = 50$$

A1

Answer: $N = 50$

6 marks

27 (9 marks)

(a) Rounding: the numerator has error $\leq 2\varepsilon$, divided by $2h$ gives $\frac{\varepsilon}{h}$

M1

Adding the truncation error gives $E(h)$

A1

Answer: $E(h) = \frac{M_3 h^2}{6} + \frac{\varepsilon}{h}$

2 marks

(b) $E'(h) = \frac{M_3 h}{3} - \frac{\varepsilon}{h^2} = 0$

M1 A1

$$h^3 = \frac{3\varepsilon}{M_3}, \text{ and } E''(h) > 0 \text{ so this is a minimum}$$

A1

$$\text{Then } \frac{M_3 h^2}{6} = \frac{\varepsilon}{2h}, \text{ so } E_{\min} = \frac{3\varepsilon}{2h}$$

A1

Answer: $h = (3\varepsilon/M_3)^{1/3}; E_{\min} = 3\varepsilon/2h$

4 marks

(c) $h = (1.5 \times 10^{-8})^{1/3} = 0.0025$

B1

$$E_{\min} = \frac{3\varepsilon}{2h} = 0.0000030$$

M1

$$\text{At } h = 10^{-10}, \frac{\varepsilon}{h} = 0.05, \text{ dominated by rounding error}$$

E1

Answer: 0.0000030

3 marks

28 (9 marks)

(a) $x_1 = 0.540302, x_2 = 0.857553$

M1

$$x_3 = 0.654290, x_4 = 0.793480$$

A1

Answer: 0.540302, 0.857553, 0.654290, 0.793480

2 marks

(b) $x'_0 = 0.728010$

M1 A1

$$x'_1 = 0.733665$$

A1

$$x'_2 = 0.736906$$

A1

Answer: 0.728010, 0.733665, 0.736906

4 marks

(c) Errors: $x_4: 0.054, x'_2: -0.0022$

M1 A1

$$\text{If } e_{n+1} = \lambda e_n \text{ exactly, the formula returns } \alpha \text{ exactly, since } \lambda = \frac{x_{n+2} - x_{n+1}}{x_{n+1} - x_n}$$

E1

Answer: x'_2 is far closer

3 marks

29 (8 marks)

(a) $p(3 + \delta) + \varepsilon(3 + \delta)^2 = 0$, and $p(3 + \delta) \approx p'(3)\delta$

M1

$$p'(3)\delta + 9\varepsilon \approx 0, \text{ so } \delta \approx -\frac{9\varepsilon}{p'(3)}$$

A1

$$p'(3) = (3 - 1)(3 - 2) = 2$$

A1

Answer: $p'(3) = 2$

3 marks

(b) $\delta \approx -\frac{9 \times 0.001}{2} = -0.0045$, so the root ≈ 2.9955

M1

This agrees with 2.99548 to 4 decimal places

Answer: -0.0045

A1

2 marks

(c) $p'(1.01) = (0.01)(-0.01) = -0.0001$, so $\delta \approx -\frac{1.0201 \times 10^{-8}}{-0.0001}$

M1

$\delta \approx 0.000102$, which is about 10201 times ϵ , whereas in (b) δ was only 4.5 times ϵ

A1

p' is tiny when roots are close, so a small change in the coefficients moves the root a great deal:
the problem is ill-conditioned

E1

Answer: $\delta \approx 1.02 \times 10^{-4}$, about 10^4 times ϵ

3 marks

30 (8 marks)

(a) Bound = $\frac{e(0.1)}{2}(e - 1) = 0.2335$

M1 A1

Actual error = $2.7183 - 2.5937 = 0.1245$, within the bound

A1

Answer: 0.2335

3 marks

(b) $\frac{e h(e - 1)}{2} < 0.01$, so $h < \frac{0.02}{e(e - 1)} = 0.00428$

M1

$N > 233.5$

M1

$N = 234, h = \frac{1}{234}$

A1

Answer: $N = 234$

3 marks

(c) The actual error $\approx \frac{e}{2N}$, so error < 0.01 needs $N \geq 135$

M1

The bound requires about 234, around 1.7 times as many steps, because it uses worst-case M and L everywhere

E1

Answer: $N = 135$

2 marks

Approximation Methods

1 (3 marks)

$f(x) = \frac{1}{2\sqrt{x}}$, so $f'(25) = \frac{1}{10}$

B1

$\sqrt{26} \approx 5 + 1 \times 0.1$
 $= 5.1000$

M1

A1

Answer: 5.1000

3 marks

2 (2 marks)

$\sin 0.1 \approx 0.1$

B1

$\cos 0.1 \approx 1 - 0.005 = 0.995$

B1

Answer: 0.1; 0.995

2 marks

3 (3 marks)

$e^x \approx 1 + x + \frac{x^2}{2}$

B1

$e^{0.2} \approx 1 + 0.2 + 0.02$
 $= 1.2200$

M1

A1

Answer: 1.2200

3 marks

4 (2 marks)

$(1.02)^5 \approx 1 + 5(0.02)$
 $= 1.100$

M1

A1

Answer: 1.100

2 marks

5 (2 marks)

Gradient = $\frac{7 - 3}{2 - 1} = 4$

B1

$f(1.25) \approx 3 + 4(0.25) = 4$

B1

Answer: 4

2 marks

6 (3 marks)

$$f(0) = 0, f'(0) = 1, f''(0) = -1$$

B1

$$\ln(1+x) \approx x - \frac{x^2}{2}$$

M1

$$\ln 1.1 \approx 0.1 - 0.005 = 0.0950$$

A1

Answer: 0.0950

3 marks

7 (3 marks)

$$e^{-0.3} \approx 1 - 0.3 + 0.045 - 0.0045$$

M1

$$= 0.740500$$

A1

$$\text{Error} = |0.740818 - 0.740500| = 0.00032$$

A1

Answer: 0.740500

3 marks

8 (3 marks)

$$\sin x \approx x - \frac{x^3}{6} + \frac{x^5}{120}$$

B1

$$\sin 0.5 \approx 0.5 - 0.0208333 + 0.0002604$$

M1

$$= 0.47943$$

A1

Answer: 0.47943

3 marks

9 (4 marks)

$$(1+x)^{1/2} \approx 1 + \frac{x}{2} - \frac{x^2}{8}$$

B1 M1

$$\sqrt{1.08} \approx 1 + 0.04 - 0.0008$$

M1

$$= 1.03920, \text{ error } 0.000030$$

A1

Answer: 1.03920

4 marks

10 (4 marks)

$$f(0.5) \approx \frac{1+3}{2} = 2$$

M1 A1

$$f(1.5) \approx \frac{3+11}{2} = 7$$

A1

The gradient is increasing fast (2 then 8), so the curve is strongly convex and the chord overestimates, especially over the second interval

E1

Answer: 2 and 7

4 marks

11 (3 marks)

$$\text{Put } x = 0.01: 1 + 0.005 + \frac{3 \times 0.0001}{8}$$

M1

$$= 1.005037$$

A1

$$\text{Exact} = 1.005038, \text{ so error} = 0.00000032$$

A1

Answer: 1.005037

3 marks

12 (4 marks)

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \dots$$

M1

$$1 - \cos x = \frac{x^2}{2} - \frac{x^4}{24} + \dots$$

A1

$$\frac{1 - \cos x}{x^2} = \frac{1}{2} - \frac{x^2}{24} + \dots \approx \frac{1}{2}$$

A1

$$x = 0.1: 0.49958$$

A1

Answer: 0.49958

4 marks

13 (4 marks)

$$L_0 = \frac{(2.5-2)(2.5-3)}{(1-2)(1-3)} = -\frac{1}{8}, L_1 = \frac{(2.5-1)(2.5-3)}{(2-1)(2-3)} = \frac{3}{4}, L_2 = \frac{(2.5-1)(2.5-2)}{(3-1)(3-2)} = \frac{3}{8}$$

M1 A1

$$f(2.5) \approx -\frac{1}{8}(1) + \frac{3}{4}(8) + \frac{3}{8}(27) = 16.000$$

A1

$$\text{Error} = 15.625 - 16.000 = -0.375$$

A1

Answer: 16.000

4 marks

14 (6 marks)

$$e^x \cos x = (1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24})(1 - \frac{x^2}{2} + \frac{x^4}{24})$$

M1

$$x^2: \frac{1}{2} - \frac{1}{2} = 0; x^3: \frac{1}{6} - \frac{1}{2} = -\frac{1}{3}$$

M1 A1

$$x^4: \frac{1}{24} - \frac{1}{4} + \frac{1}{24} = -\frac{1}{6}$$

A1

$$e^x \cos x \approx 1 + x - \frac{x^3}{3} - \frac{x^4}{6}$$

A1

$$\text{At } x = 0.2: 1 + 0.2 - 0.002667 - 0.000267 = 1.19707$$

A1

Answer: 1.19707

6 marks

15 (6 marks)

$$\sin x = x - \frac{x^3}{6} + \dots$$

B1

$$\ln(1+u) = u - \frac{u^2}{2} + \frac{u^3}{3} - \dots \text{ with } u = \sin x$$

M1

$$u^2 = x^2 - \frac{x^4}{3}, u^3 = x^3 + \dots$$

M1

$$\ln(1 + \sin x) = x - \frac{x^2}{2} + (-\frac{1}{6} + \frac{1}{3})x^3 + \dots$$

A1

$$= x - \frac{x^2}{2} + \frac{x^3}{6}$$

A1

$$\text{At } x = 0.1: 0.1 - 0.005 + 0.000167 = 0.095167$$

A1

Answer: 0.095167

6 marks

16 (6 marks)

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \dots$$

M1 A1

$$\int = [x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42}]_0^{0.5}$$

M1 A1

$$= 0.5 - 0.041667 + 0.003125 - 0.000186$$

M1

$$= 0.46127$$

A1

Answer: 0.46127

6 marks

17 (5 marks)

$$f'(x) = \frac{1}{3}x^{-2/3} \text{ so } f'(8) = \frac{1}{12}; f''(x) = -\frac{2}{9}x^{-5/3} \text{ so } f''(8) = -\frac{1}{144}$$

M1 A1

$$f(8.5) \approx 2 + 0.5 \times \frac{1}{12} - \frac{0.25}{2} \times \frac{1}{144}$$

M1 A1

$$= 2 + 0.041667 - 0.000868 = 2.04080$$

A1

Answer: 2.04080

5 marks

18 (5 marks)

$$L_0(1.5) = \frac{(1.5-1)(1.5-2)}{(0-1)(0-2)} = -\frac{1}{8}, L_1 = \frac{(1.5)(-0.5)}{(1)(-1)} = \frac{3}{4}, L_2 = \frac{(1.5)(0.5)}{(2)(1)} = \frac{3}{8}$$

M1 A1

$$e^{1.5} \approx -\frac{1}{8} + \frac{3}{4}e + \frac{3}{8}e^2$$

M1

$$= 4.6846$$

A1

$$\text{Error} = -0.2029$$

A1

Answer: 4.6846

5 marks

19 (5 marks)

First differences: 0.72, 1.67; second difference: 0.95

M1 A1

$$y(x) \approx y_0 + x\Delta y_0 + \frac{x(x-1)}{2}\Delta^2 y_0$$

M1

$$x = 0.5: 2 + 0.36 + \frac{0.5(-0.5)}{2}(0.95)$$

A1

$$= 2.241$$

A1

Answer: 2.241

5 marks

20 (5 marks)

$$f(x) = x^{1/2}, f''(x) = -\frac{1}{4}x^{-3/2} < 0, \text{ so } f \text{ is concave}$$

M1 A1

A concave curve lies below its tangent, so the tangent approximation overestimates

E1

$$\text{Taylor: } f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(\xi)$$

M1

$$|\text{error}| = \frac{h^2}{2} \times \frac{1}{4\xi^{3/2}} \leq \frac{h^2}{8x^{3/2}} \text{ as } \xi > x$$

A1

$$\text{Answer: Tangent approximation overestimates; } |\text{error}| \leq \frac{h^2}{8x^{3/2}}$$

5 marks

21 (8 marks)

$$(a) 1 + 0.5 + 0.125 + 0.0208333 + 0.0026042 + 0.0002604$$

M1

$$= 1.6486979$$

A1

Answer: 1.6486979

2 marks

$$(b) 0 < \xi < 0.5 \text{ so } e^\xi < e^{0.5} < 2$$

M1

$$R_5 < \frac{2(0.5)^6}{720}$$

A1

$$= 0.0000434$$

A1

Answer: 0.0000434

3 marks

$$(c) \text{ Actual error} = 1.6487213 - 1.6486979 = 0.0000234, \text{ inside the bound}$$

B1

$$n = 8: \text{bound } 0.000000011; n = 9: 0.00000000054$$

M1

$$\text{Least } n = 9$$

A1

Answer: n = 9

3 marks

22 (9 marks)

$$(a) x^1: 1 - \frac{1}{2} = \frac{1}{2}$$

B1

$$x^2: \frac{1}{2} - \frac{1}{2} + \frac{1}{12} = \frac{1}{12}$$

M1 A1

$$x^3: \frac{1}{6} - \frac{1}{4} + \frac{1}{12} = 0 \text{ and } x^4: \frac{1}{24} - \frac{1}{12} + \frac{1}{24} = 0, \text{ so the product equals } 1 + \frac{x}{2} + \frac{x^2}{12} \text{ up to } x^4$$

A1

$$\text{Answer: } (1 - \frac{x}{2} + \frac{x^2}{12})(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots) = 1 + \frac{x}{2} + \frac{x^2}{12} + O(x^5)$$

4 marks

$$(b) R(1) = \frac{1 + 0.5 + \frac{1}{12}}{1 - 0.5 + \frac{1}{12}} = \frac{\frac{19}{12}}{\frac{7}{12}} = \frac{19}{7} = 2.714286$$

M1 A1

$$\text{Cubic: } 1 + 1 + 0.5 + \frac{1}{6} = 2.666667; \text{ errors: R: } 0.0040, \text{ cubic: } 0.052, \text{ so R is more accurate}$$

A1

Answer: 2.714286

3 marks

$$(c) R(-x) = \frac{1 - \frac{x}{2} + \frac{x^2}{12}}{1 + \frac{x}{2} + \frac{x^2}{12}}, \text{ the reciprocal of } R(x)$$

M1

$$\text{So R shares the property } e^{-x}e^x = 1$$

E1

Answer: R(-x) = 1/R(x)

2 marks

23 (7 marks)

$$(a) \partial S / \partial m = -2 \sum x_i (y_i - mx_i - c) = 0 \quad \text{M1}$$

$$\partial S / \partial c = -2 \sum (y_i - mx_i - c) = 0 \quad \text{M1}$$

Expanding the sums gives the two normal equations A1

$$\text{Answer: } m \sum x^2 + c \sum x = \sum xy \text{ and } m \sum x + 4c = \sum y \quad \text{3 marks}$$

$$(b) \sum x = 6, \sum x^2 = 14, \sum y = 16, \sum xy = 2.9 + 10.4 + 20.4 = 33.7 \quad \text{M1 A1}$$

$$14m + 6c = 33.7 \text{ and } 6m + 4c = 16 \quad \text{A1}$$

$$m = 1.94, c = 1.09, S = 0.082 \quad \text{A1}$$

$$\text{Answer: } y = 1.94x + 1.09 \quad \text{4 marks}$$

24 (7 marks)

$$\text{Let } p(x) = a + bx + cx^2: y_0 = a, y_1 + y_{-1} = 2a + 2ch^2 \quad \text{M1 A1}$$

$$\int_{-h}^h p \, dx = 2ah + \frac{2ch^3}{3} \quad \text{M1}$$

$$\text{From above } ch^2 = \frac{y_1 + y_{-1} - 2y_0}{2}, \text{ so integral} = 2hy_0 + \frac{h}{3}(y_1 + y_{-1} - 2y_0) = \frac{h}{3}(y_{-1} + 4y_0 + y_1) \quad \text{A1}$$

$$\text{With } h = 1 \text{ and } y = 1, 0.5, \frac{1}{3} \text{ at } x = 0, 1, 2: \frac{1}{3}(1 + 2 + \frac{1}{3}) \quad \text{M1}$$

$$= \frac{10}{9} = 1.11111 \quad \text{A1}$$

$$\ln 3 = 1.09861, \text{ an overestimate of } 0.01250 \quad \text{B1}$$

$$\text{Answer: } 1.11111 \quad \text{7 marks}$$

25 (7 marks)

$$(a) 5! \approx \sqrt{10\pi} \left(\frac{5}{e}\right)^5 = 118.02, \text{ error } 1.65\% \quad \text{M1 A1}$$

$$10! \approx 3598696, \text{ error } 0.83\% \quad \text{M1 A1}$$

$$\text{Answer: } 1.65\% \text{ and } 0.83\% \quad \text{4 marks}$$

$$(b) \frac{100}{120} = 0.833\%, \text{ close to } 0.830\% \quad \text{M1}$$

$$\frac{100}{12n} < 0.1 \text{ gives } n > 83.3 \quad \text{M1}$$

$$n = 84 \quad \text{A1}$$

$$\text{Answer: } n = 84 \quad \text{3 marks}$$

26 (7 marks)

$$(a) \text{ First divided differences: } 3, 6, 9 \quad \text{M1 A1}$$

$$\text{Second: } 1, 1; \text{ third: } 0 \quad \text{A1}$$

$$p(x) = 0 + 3(x - 1) + (x - 1)(x - 2) + 0 \quad \text{A1}$$

$$\text{Answer: } p(x) = 3(x - 1) + (x - 1)(x - 2) \quad \text{4 marks}$$

$$(b) p(x) = (x - 1)(3 + x - 2) = (x - 1)(x + 1) = x^2 - 1 \quad \text{M1}$$

$$f(3) \approx 8 \quad \text{A1}$$

$$\text{Answer: } x^2 - 1; 8 \quad \text{2 marks}$$

$$(c) \text{ All four points satisfy } y = x^2 - 1, \text{ so a quadratic fits exactly and the cubic coefficient is } 0 \quad \text{E1}$$

$$\text{Answer: Data lie on } y = x^2 - 1 \quad \text{1 mark}$$

27 (9 marks)

$$(a) g(x) = (x - a)(x - b) \text{ is a quadratic with minimum at the midpoint } x = \frac{a + b}{2} \quad \text{M1}$$

$$|g| = \frac{(b - a)}{2} \times \frac{(b - a)}{2} = \frac{(b - a)^2}{4} \quad \text{A1}$$

$$\text{Answer: Greatest } |(x - a)(x - b)| = \frac{(b - a)^2}{4} \quad \text{2 marks}$$

$$(b) |f - p| \leq \frac{M}{2} \times \frac{(b - a)^2}{4} \quad \text{B1}$$

$$\text{Answer: } |f(x) - p(x)| \leq \frac{M(b - a)^2}{8} \quad \text{1 mark}$$

$$(c) |f''| = \frac{1}{x^2} \leq 1 \text{ on } [1, 2] \quad \text{B1}$$

$$\frac{h^2}{8} \leq 5 \times 10^{-5}, \text{ so } h^2 \leq 4 \times 10^{-4} \quad \text{M1}$$

$$h \leq 0.02 \quad \text{A1}$$

Answer: $h = 0.02$ 3 marks

$$(d) p(1.5) = \frac{\ln 1.48 + \ln 1.52}{2} = 0.40537621 \quad \text{M1 A1}$$

$$\text{Error} = \ln 1.5 - p = 0.000089, \text{ below the bound } 0.000091 \quad \text{A1}$$

Answer: 0.000089 3 marks

28 (9 marks)

$$(a) \frac{e^x - 1}{x} = 1 + \frac{x}{2} + \frac{x^2}{6} + \frac{x^3}{24} + \dots \quad \text{M1 A1}$$

$$\text{Its reciprocal: } 1 - \left(\frac{x}{2} + \frac{x^2}{6}\right) + \left(\frac{x}{2}\right)^2 + \dots \quad \text{M1}$$

$$= 1 - \frac{x}{2} + \left(\frac{1}{4} - \frac{1}{6}\right)x^2 = 1 - \frac{x}{2} + \frac{x^2}{12} \quad \text{A1}$$

$$\text{Answer: } f(x) = 1 - \frac{x}{2} + \frac{x^2}{12} + O(x^4) \quad \text{4 marks}$$

$$(b) 1 - 0.25 + \frac{0.25}{12} = 0.770833 \quad \text{M1}$$

$$\text{Error} = -0.000086: \text{ very accurate for small } x \quad \text{A1}$$

Answer: 0.770833 2 marks

$$(c) \frac{x}{e^x - 1} - 1 + \frac{x}{2} = \frac{x^2}{12} + O(x^4) \quad \text{M1}$$

$$\text{Dividing by } x \text{ gives } \frac{x}{12} \rightarrow 0 \quad \text{A1}$$

$$\text{Dividing by } x^2 \text{ gives } \frac{1}{12} + O(x^2) \rightarrow \frac{1}{12} \quad \text{A1}$$

Answer: 0 and 1/12 3 marks

29 (8 marks)

$$(a) \pi = 4 \arctan 1 \approx 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}\right) \quad \text{M1}$$

$$= 2.8952 \quad \text{A1}$$

$$\text{The terms } \frac{1}{2k+1} \text{ decrease slowly (alternating series error about the next term, 4/9 here)} \quad \text{E1}$$

Answer: 2.8952 3 marks

$$(b) \arctan\left(\frac{1}{5}\right) \approx 0.2 - \frac{0.008}{3} = 0.1973333 \quad \text{M1}$$

$$16 \times 0.1973333 - 4 \times \frac{1}{239} = 3.140597 \quad \text{M1}$$

$$\approx 3.140597 \quad \text{A1}$$

Answer: 3.140597 3 marks

$$(c) \text{ The arguments } \frac{1}{5} \text{ and } \frac{1}{239} \text{ are small, so the terms } x^{2k+1} \text{ shrink quickly} \quad \text{E1}$$

$$\text{Error} = \pi - 3.140597 = 0.00100 \quad \text{B1}$$

Answer: 0.00100 2 marks

30 (10 marks)

$$(a) \text{ Equal errors at } x = 0 \text{ and } x = 1: 1 - c = e - m - c, \text{ so } m = e - 1 \quad \text{M1 A1}$$

Answer: $m = e - 1$ 2 marks

$$(b) \frac{d}{dx}(e^x - mx - c) = e^x - (e - 1) = 0 \quad \text{M1}$$

$$x_1 = \ln(e - 1) = 0.54132 \quad \text{A1}$$

Answer: 0.54132 2 marks

$$(c) \text{ Error at } 0 \text{ is } 1 - c; \text{ error at } x_1 \text{ is } e^{x_1} - mx_1 - c = m - mx_1 - c \quad \text{M1}$$

$$1 - c = -(m - mx_1 - c), \text{ so } 2c = 1 + m - mx_1 \quad \text{M1}$$

$$c = 0.89407$$

A1

$$\text{Maximum error} = 1 - c = 0.10593$$

A1

Answer: $c = 0.8941$; max error 0.1059

4 marks

(d) The minimax line has greatest error 0.1059, about 6.8 times smaller

M1

Maclaurin is best near $x = 0$ but worst at $x = 1$; the minimax line spreads the error evenly

E1

Answer: Minimax error is smaller

2 marks