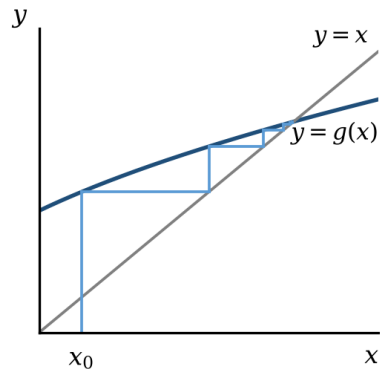


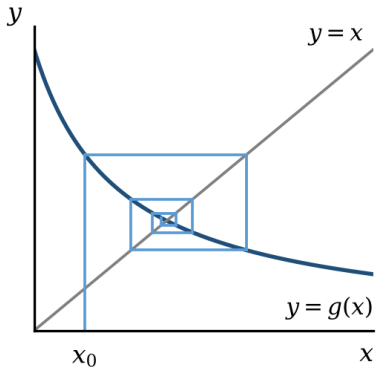
Numerical Solution of Equations

- 1 Show that the equation $x^3 + 2x - 7 = 0$ has a root between $x = 1$ and $x = 2$. [2]
- 2 The sequence defined by $x_{n+1} = \sqrt{3x_n + 4}$ with $x_0 = 3$ is used to approximate a root of an equation.
Find x_1 , x_2 and x_3 , giving each to 4 decimal places. [2]
- 3 Show that the equation $x^3 - 6x + 2 = 0$ can be rearranged into the form $x = \frac{x^3 + 2}{6}$. [2]
- 4 Show that the equation $\cos x = x$, where x is in radians, has a root in the interval $[0.7, 0.8]$. [2]
- 5 The equation $x^3 - 3x - 5 = 0$ has a root α in the interval $[2, 3]$.
Use one step of interval bisection to find a smaller interval, of width 0.5, that contains α . [2]
- 6 The iteration $x_{n+1} = \frac{12}{x_n + 1}$ is used to find a root of an equation.
(a) State the value of x_0 for which x_1 cannot be calculated. [1]
(b) Starting with $x_0 = 2$, find x_1 , x_2 and x_3 to 3 decimal places. [2]
- 7 $f(x) = e^x - 3x$.
(a) Show that the equation $f(x) = 0$ has a root in the interval $[0, 1]$. [2]
(b) Show that $f(x) = 0$ has a second root in the interval $[1, 2]$. [2]
- 8 The equation $2^x + 3x = 9$ has a root α .
(a) Show that α lies between 1.5 and 2. [2]
(b) The iteration $x_{n+1} = \frac{\ln(9 - 3x_n)}{\ln 2}$ is started with $x_0 = 2.9$. Find x_1 and x_2 to 3 decimal places, and explain why x_3 cannot be found. [3]
- 9 The equation $x^3 - x - 3 = 0$ has a root α with $1 < \alpha < 2$.
Use three steps of interval bisection, starting with the interval $[1, 2]$, to find an interval of width 0.125 that contains α . [4]
- 10 Let $f(x) = 3^x + x - 8$. The equation $f(x) = 0$ has a root α between 1 and 2.
Use linear interpolation with $f(1)$ and $f(2)$ to obtain an estimate x_1 of α . Then use linear interpolation again, on a smaller interval that contains α , to obtain a better estimate x_2 . Give x_1 and x_2 to 3 decimal places. [5]
- 11 The equation $x^3 - 5x + 2 = 0$ can be rearranged in different ways.
(a) Show that the equation can be written as $x = \frac{x^3 + 2}{5}$. [2]
(b) Show that the equation can also be written as $x = \sqrt[3]{5x - 2}$. [2]

- 12** The diagram shows the curve $y = g(x)$, where $g(x) = \sqrt{2x + 3}$, and the line $y = x$. The staircase shows the iteration $x_{n+1} = g(x_n)$ from the starting value x_0 .



- (a) State whether the sequence converges, and name the type of path (staircase or cobweb). [1]
 (b) Find the exact limit of the sequence. [2]
-
- 13** $f(x) = x^3 + 2x - 6$. Show that the root α of the equation $f(x) = 0$ is 1.5 correct to 1 decimal place. [3]
-
- 14** The curve C has equation $y = x^3 - 4 \ln x$ for $x > 0$. The point P on C is where the gradient of C is 6.
 (a) Show that the x -coordinate of P satisfies $3x^3 - 6x - 4 = 0$. [2]
 (b) Show that the x -coordinate of P lies between 1.6 and 1.8. [2]
 (c) Show that the equation in (a) can be written $x = \sqrt[3]{2x + \frac{4}{3}}$. Use the iteration $x_{n+1} = \sqrt[3]{2x_n + \frac{4}{3}}$ with $x_0 = 1.7$ to find the coordinates of P , to 2 decimal places. [4]
-
- 15** $f(x) = 5x - 2 \cos x - 1$, where x is in radians.
 (a) Show that the equation $f(x) = 0$ has a root α between 0.5 and 0.6. [2]
 (b) An iteration of the form $x_{n+1} = A + B \cos x_n$ is used to find α . Find the constants A and B . [2]
 (c) Starting with $x_1 = 0.55$, find x_2 , x_3 and x_4 to 5 decimal places. [2]
 (d) By considering the sign of f in a suitable interval, show that $\alpha = 0.543$ correct to 3 decimal places. [3]
-
- 16** The equation $x^3 - 7x + 3 = 0$ has a root α close to 0.5. Two iterations are proposed:
 A: $x_{n+1} = \frac{x_n^3 + 3}{7}$ B: $x_{n+1} = \sqrt[3]{7x_n - 3}$
 Starting with $x_0 = 0.5$ in each case, calculate x_1 , x_2 and x_3 for each iteration, to 4 decimal places, and state which iteration is suitable for finding α . [4]
-
- 17** The equation $\sin x = \frac{x}{3}$, where x is in radians, is to be solved.
 (a) By sketching $y = \sin x$ and $y = \frac{x}{3}$ on the same axes, state the number of real roots of the equation. [2]
 (b) Show that the positive root α lies between 2 and 2.5. [2]
 (c) Use the iteration $x_{n+1} = 3 \sin x_n$, with $x_0 = 2.5$, to find x_4 to 3 decimal places. [2]
-
- 18** A root of an equation lies in the interval $[1, 2]$. Interval bisection is used repeatedly, and after n steps the midpoint of the current interval is taken as the estimate of the root.
 (a) Show that the interval has width $\frac{1}{2^n}$ after n steps. [2]
 (b) Find the smallest n which guarantees that the estimate is within 0.001 of the root. [3]

- 19** The equation $x \ln x = 2$ has a single root α .
- (a) Show that $2 < \alpha < 3$. [2]
- (b) Show that the equation can be written $x = e^{2/x}$. [2]
- (c) Use $x_{n+1} = e^{2/x_n}$ with $x_0 = 2.3$ to find x_1, x_2, x_3, x_4 to 3 decimal places. Describe how the values behave relative to α . [3]
-
- 20** The equation $x^3 + 3x - 5 = 0$ has a root $\alpha \approx 1.154$. It is rearranged into the form $x = g(x)$ in two ways:
- $$g_1(x) = \frac{5 - x^3}{3} \quad g_2(x) = \frac{5}{x^2 + 3}$$
- An iteration $x_{n+1} = g(x_n)$ converges to α when $|g'(\alpha)| < 1$. Find $g_1'(\alpha)$ and $g_2'(\alpha)$, and hence decide which rearrangement gives an iteration that converges to α . [5]
-
- 21** Consider the equation $e^{-x} = x^2$.
- (a) By sketching $y = e^{-x}$ and $y = x^2$ on the same axes, show that the equation has exactly one positive root. [2]
- (b) Show that this root lies between 0.7 and 0.8. [2]
- (c) Show that the equation can be written $x = e^{-x/2}$, and use $x_{n+1} = e^{-x_n/2}$ with $x_0 = 0.7$ to find the root correct to 2 decimal places. [4]
-
- 22** The diagram shows the graphs of $y = g(x)$, where $g(x) = \frac{4}{x+1}$, and $y = x$, together with the path of the iteration $x_{n+1} = g(x_n)$ starting from $x_0 = 0.6$.
- 
- (a) The path is a cobweb rather than a staircase. Explain why, in terms of g . [1]
- (b) Show that the limit L of the sequence satisfies $L^2 + L - 4 = 0$ and find its exact value. [3]
- (c) Find $g'(L)$ and explain why the value confirms that the sequence converges. [2]

- 23** The iteration $x_{n+1} = e^{2/x_n}$ is used to find the root α of $x \ln x = 2$, where $\alpha = 2.345751$ correct to 6 decimal places. The table shows x_n for $n = 0$ to 4.

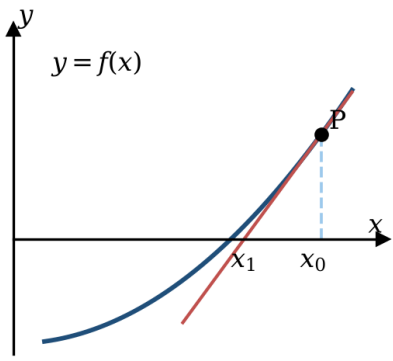
n	x_n
0	2.200000
1	2.482065
2	2.238443
3	2.443614
4	2.267006

- (a) Calculate the errors $e_n = x_n - \alpha$ for $n = 0, 1, 2, 3$, and the ratios $\frac{e_{n+1}}{e_n}$ for $n = 0, 1, 2$. [3]
- (b) Hence estimate $g'(\alpha)$, where $g(x) = e^{2/x}$, and state what this indicates about the convergence. [2]
- (c) Using the ratio in (b), estimate how many further iterations after x_4 are needed for the error to fall below 10^{-6} . [2]
-
- 24** Zara wants to show that the equation $\tan x = -1$ has a root between $x = 1$ and $x = 2$ (x in radians). She lets $f(x) = \tan x + 1$, finds $f(1) = 2.557$ and $f(2) = -1.185$, and states that the change of sign means that there is a root in $[1, 2]$.
- (a) Explain why Zara's conclusion is not valid. [2]
- (b) Show that the equation does have a root in the interval $[2, 2.5]$. [2]
- (c) Use two steps of interval bisection on $[2, 2.5]$ to obtain a smaller interval containing the root. [2]
-
- 25** The iteration $x_{n+1} = \frac{x_n^2 + 2}{3}$ is used with different starting values.
- (a) Find the two values to which the sequence could converge. [2]
- (b) Starting with $x_0 = 1.5$, calculate x_1, x_2 and x_3 to 4 decimal places, and state the value to which the sequence appears to converge. [3]
- (c) By finding $g'(x)$ at each fixed point, where $g(x) = \frac{x^2 + 2}{3}$, explain why a sequence with $x_0 = 2.5$ does not converge to 2. [3]
-
- 26** A hollow sphere of internal radius 10 cm is held with its centre vertically above its lowest point. Water is poured in. When the depth of water is h cm ($0 < h < 10$), the volume of water is $V = \frac{\pi h^2(30 - h)}{3} \text{ cm}^3$. The volume of water is 800 cm^3 .
- (a) Show that h satisfies $h^3 - 30h^2 + \frac{2400}{\pi} = 0$. [2]
- (b) Show that h lies between 5 and 6. [2]
- (c) Show that the equation can be written $h = \sqrt{\frac{h^3 + 2400}{\frac{\pi}{30}}}$, and use the iteration with $h_0 = 5$ to find the depth of water to 2 decimal places. [4]
-
- 27** The root α of $f(x) = x^3 + x - 3 = 0$ is found using the iteration $x_{n+1} = x_n - a f(x_n)$, where a is a positive constant.
- (a) Show that if the sequence converges, it converges to a root of $f(x) = 0$. [2]
- (b) Let $g(x) = x - a f(x)$. Show that the iteration converges to α when $0 < a < \frac{2}{f'(\alpha)}$, given that $\alpha \approx 1.213$. [3]
- (c) Taking $a = 0.2$ and $x_0 = 1$, find x_1, x_2 and x_3 to 5 decimal places. [2]
- (d) Find the value of a for which $g'(\alpha) = 0$ and explain why convergence is then fastest. [3]

- 28** A sequence converging to α satisfies $x_{n+1} - \alpha \approx \lambda(x_n - \alpha)$ for large n , where λ is a constant with $|\lambda| < 1$. The iteration $x_{n+1} = e^{2/x_n}$ with $x_0 = 2.3$ generates $x_1 = 2.385873$, $x_2 = 2.312357$, $x_3 = 2.374812$.
- Show that $\alpha \approx x_3 - \frac{(x_3 - x_2)^2}{(x_3 - x_2) - (x_2 - x_1)}$. [4]
 - Use this formula to estimate α , and compare your answer with $\alpha = 2.345751$ (6 d.p.). [3]
 - State, with a reason, whether the iteration converges monotonically or by oscillation. [1]
-
- 29** The equation $e^x = 4x$ has two real roots, $\alpha < \beta$.
- Show that α lies in $[0, 1]$ and β lies in $[2, 3]$. [3]
 - Show that the iteration $x_{n+1} = \ln(4x_n)$ with $x_0 = 2$ converges, and find β to 2 decimal places. [3]
 - Let $g(x) = \ln(4x)$. Show that $|g'(\beta)| < 1$ but $|g'(\alpha)| > 1$, and hence explain why this iteration can find β but not α . [2]
 - Use the rearrangement $x_{n+1} = \frac{e^{x_n}}{4}$ with $x_0 = 0.5$ to find α to 3 decimal places, and show that this rearrangement cannot find β . [4]
-
- 30** The iteration $x_{n+1} = g(x_n)$ converges to a root $\alpha = g(\alpha)$, and $e_n = x_n - \alpha$.
- Use a Taylor expansion of g about α to show that, for small e_n , $e_{n+1} \approx g'(\alpha) e_n$. [3]
 - For $g(x) = \frac{5}{x^2 + 3}$ the root of $x^3 + 3x - 5 = 0$ is $\alpha \approx 1.1542$. Find $g'(\alpha)$. [2]
 - Starting with $x_0 = 1.5$, $e_0 = 0.3458$. Use (a) to estimate e_4 , and compare with the actual $e_4 = 0.047$. [3]
 - Explain why $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ would be expected to converge faster, by finding $g'(\alpha)$ for this g . [3]

Newton-Raphson Method

- Write down the Newton-Raphson iteration for solving $f(x) = 0$, where $f(x) = x^3 - x - 1$. [2]
- Use one step of the Newton-Raphson method, with $x_0 = 3$, to find an approximation to the positive root of $x^2 - 10 = 0$. Give your answer to 4 decimal places. [2]
- $f(x) = x^3 + 2x - 4$. Taking $x_0 = 1.3$, use the Newton-Raphson method once to find x_1 to 4 decimal places. [2]
- $f(x) = e^x - 3x$. Find $f'(x)$ and use the Newton-Raphson method with $x_0 = 0.5$ to find x_1 to 4 decimal places. [3]
- Use the Newton-Raphson method with $x_0 = 0.7$ to find a better approximation to the root of $\cos x = x$. Work in radians and give your answer to 4 decimal places. [3]
- Use the Newton-Raphson method on $f(x) = x^2 - 2$ with $x_0 = 1.5$ to find x_1 and x_2 to 5 decimal places. [3]
- Show that applying the Newton-Raphson method to $f(x) = x^2 - N$ gives $x_{n+1} = \frac{1}{2}(x_n + \frac{N}{x_n})$. Use this with $N = 11$ and $x_0 = 3$ to find $\sqrt{11}$ to 4 decimal places. [4]
- Use the Newton-Raphson method with $f(x) = x^3 - 5$ and $x_0 = 2$ to find x_1 , x_2 and x_3 , and hence give the value of $\sqrt[3]{5}$ to 5 decimal places. [4]

- 9** The equation $x^3 - 2x - 5 = 0$ has a root close to $x = 2$.
- (a) Show that the Newton-Raphson iteration for this equation can be written $x_{n+1} = \frac{2x_n^3 + 5}{3x_n^2 - 2}$. [2]
- (b) Starting with $x_0 = 2$, find x_1 , x_2 and x_3 , giving x_3 to 6 decimal places. [2]
-
- 10** $f(x) = \ln x + x - 3$.
- (a) Show that the equation $f(x) = 0$ has a root in the interval $[2, 3]$. [2]
- (b) Use the Newton-Raphson method with $x_0 = 2$ to find the root to 4 decimal places. [3]
-
- 11** The curve with equation $x^3 + y^3 = 9$ meets the line $y = 2x - 1$ at the point A, where the x-coordinate of A is close to 1.4.
- (a) Show that the x-coordinate of A satisfies $9x^3 - 12x^2 + 6x - 10 = 0$. [2]
- (b) Use the Newton-Raphson method once, with $x_0 = 1.4$, to find a better approximation to the x-coordinate of A. Give your answer to 3 decimal places. [3]
-
- 12** The diagram shows part of the curve $y = f(x)$. The point P on the curve has x-coordinate x_0 , and the tangent to the curve at P meets the x-axis at the point with x-coordinate x_1 .
- Show that $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$.
- 
- [3]
-
- 13** The equation $x^2 = 3^x$ is to be solved.
- (a) Sketch the graphs of $y = x^2$ and $y = 3^x$ on the same axes, and hence explain why the equation has exactly one real root, which is negative. [2]
- (b) Show that the root lies between -1 and 0 . [1]
- (c) Use the Newton-Raphson method with $x_0 = -0.5$ to find the root to 4 decimal places. [3]
-
- 14** $f(x) = x^3 - 3x + 1$. The equation $f(x) = 0$ has a root near $x = 1.5$.
- (a) Explain why the Newton-Raphson method cannot be used with $x_0 = 1$. [2]
- (b) Starting with $x_0 = 1.1$, find x_1 , x_2 and x_3 to 4 decimal places. [3]
- (c) Explain why x_1 is much further from the root than x_0 . [1]
-
- 15** The equation $3 \sin x = x$, where x is in radians, has a positive root α close to 2.
- (a) Use the Newton-Raphson method with $x_0 = 2$ to find x_1 , x_2 and x_3 to 6 decimal places. [3]
- (b) By considering the sign of f in a suitable interval, show that $\alpha = 2.2789$ correct to 4 decimal places. [3]
-
- 16** The curves $y = x^3$ and $y = 2^x$ intersect at a point with x-coordinate close to 1.4. Use the Newton-Raphson method, with $x_0 = 1.4$, to find this x-coordinate to 5 decimal places. Show your values of x_1 , x_2 and x_3 . [5]

- 17** The curve C has equation $y = e^x - x^3$ for $x > 0$. The curve has a stationary point near $x = 1$. Show that the x-coordinate of the stationary point satisfies $e^x = 3x^2$, and use the Newton-Raphson method with $x_0 = 1$ to find it to 4 decimal places. [6]
- 18** A curve has equation $x = \ln(y^3 - 2y)$ for $y > \sqrt{2}$. The point P on the curve is where the gradient $\frac{dy}{dx} = 5$, and the y-coordinate of P is close to 15.
- (a) Show that the y-coordinate of P satisfies $y^3 - 15y^2 - 2y + 10 = 0$. [3]
- (b) Use the Newton-Raphson method twice, starting with $y_0 = 15$, to find the y-coordinate of P to 3 decimal places. [2]
- (c) Find the coordinates of P, to 2 decimal places. [2]
- 19** $f(x) = x^2 - 3$, with roots $\pm\sqrt{3}$.
- (a) Use the Newton-Raphson method with $x_0 = -1$ to find x_1 , x_2 and x_3 to 4 decimal places. [3]
- (b) State which root the sequence is approaching in (a). [1]
- (c) Starting with $x_0 = 0.5$, find x_1 and say which root the method is heading towards. [2]
- 20** The position of a planet in its orbit is found by solving Kepler's equation $E - 0.3 \sin E = M$, where E is in radians. For a particular time, $M = 1$. Use the Newton-Raphson method, with $E_0 = 1$, to find E to 5 decimal places, showing E_1 and E_2 . [6]
- 21** $f(x) = x^3 - 4x + 1$. The equation $f(x) = 0$ has three real roots.
- (a) Write down the Newton-Raphson formula for this equation in the form $x_{n+1} = \frac{2x_n^3 - 1}{3x_n^2 - 4}$. [2]
- (b) Using $x_0 = 2$, find the largest root to 4 decimal places. [2]
- (c) Using $x_0 = 0$, find the root close to 0 to 4 decimal places. [2]
- (d) Starting from $x_0 = -2$, find the third root to 3 decimal places, and show that starting at $x_0 = 1$ leads to $x_1 = -1$ and so does not give the nearby root. [2]
- 22** An arithmetic series has first term 3 and common difference d, and a geometric series has first term 3 and common ratio d, where $d > 1$. The sum of the 8th term of the arithmetic series and the 8th term of the geometric series is 500.
- (a) Show that $3d^7 + 7d = 497$. [2]
- (b) Show that $2 < d < 2.1$. [2]
- (c) Use the Newton-Raphson method twice, with $d_0 = 2.1$, to find d to 3 decimal places. [3]
- 23** A calculator needs to find reciprocals using only multiplication and subtraction.
- (a) Show that applying the Newton-Raphson method to $f(x) = \frac{1}{x} - a$ gives $x_{n+1} = x_n(2 - ax_n)$. [3]
- (b) Use this with $a = 7$ and $x_0 = 0.1$ to estimate $\frac{1}{7}$, giving x_1 to x_4 to 6 decimal places. [2]
- (c) Find the range of starting values x_0 for which $x_1 > 0$, and explain why $x_0 = 0.5$ is a poor choice. [2]
- 24** The curve C has equation $y = e^{2x} + 2$. The tangent to C at the point P, where $x = p$, passes through the origin.
- (a) Show that $(2p - 1)e^{2p} = 2$. [3]
- (b) Show that p lies between 0.7 and 0.8. [2]
- (c) Use the Newton-Raphson method twice, starting with $p_0 = 0.8$, to find p to 4 decimal places. [3]

- 25** A loan of £10 000 is repaid by 24 equal monthly payments of £450, the first paid one month after the loan is taken. The monthly interest rate r (as a decimal) satisfies $450(1 - (1 + r)^{-24}) = 10\,000r$.
- Show that r is a root of $f(r) = 450(1 - (1 + r)^{-24}) - 10\,000r$, and find $f'(r)$. [2]
 - Use the Newton-Raphson method with $r_0 = 0.01$ to find r_1, r_2, r_3 and r_4 , and hence give r to 5 decimal places. [4]
 - Find the equivalent nominal annual rate, as a percentage to 2 decimal places, taken as $12r$. [2]
-
- 26** The point $P(x, y)$ lies on the curve $y = \sqrt{x}$. The point A has coordinates $(2, 1)$.
- Show that the squared distance AP^2 is minimised when $2x - 3 - \frac{1}{\sqrt{x}} = 0$. [3]
 - Use the Newton-Raphson method with $x_0 = 2$ to find the value of x that minimises AP^2 , to 4 decimal places. [4]
 - Hence find the minimum distance AP to 3 significant figures. [2]
-
- 27** The Newton-Raphson iteration $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ converges to a simple root α of $f(x) = 0$, with f' and f'' continuous and $f'(\alpha) \neq 0$. Let $e_n = x_n - \alpha$.
- By expanding $f(\alpha) = 0$ as a Taylor series about x_n , show that $e_{n+1} \approx \frac{f''(\alpha)}{2f'(\alpha)} e_n^2$ for small e_n . [5]
 - For $f(x) = x^3 - 5$ and $\alpha = \sqrt[3]{5}$, find the value of $\frac{f''(\alpha)}{2f'(\alpha)}$ to 3 decimal places. [2]
 - Taking $x_0 = 2$, $e_0 = 0.2900$. Use (a) and (b) to estimate e_1 and e_2 , and compare with the actual values 0.0400 and 0.000908. [3]
-
- 28** The Newton-Raphson method is applied to $f(x) = x^{1/3}$ for $x \neq 0$, a function with a root at $x = 0$, starting with $x_0 = a$ where $a \neq 0$.
- Show that $x_{n+1} = -2x_n$. [3]
 - Write down x_n in terms of a and n , and explain what happens to the sequence. [2]
 - Explain, with reference to $f'(x)$ near $x = 0$, why the method fails for this function even though the root is simple in appearance. [2]
 - Taking $a = 0.1$, find the smallest n for which $|x_n| > 1000$. [2]
-
- 29** $f(x) = (x - 1)^2(x + 2)$, which has a repeated root at $x = 1$.
- Show that $f'(x) = 3(x - 1)(x + 1)$ and that the Newton-Raphson iteration simplifies to
$$x_{n+1} = x_n - \frac{(x_n - 1)(x_n + 2)}{3(x_n + 1)}.$$
 [3]
 - With $x_0 = 2$, find x_1, x_2, x_3 to 5 decimal places, and calculate the ratios $\frac{e_{n+1}}{e_n}$ where $e_n = x_n - 1$. [3]
 - Explain why convergence here is only linear (not quadratic). [1]
 - The modified iteration $x_{n+1} = x_n - 2\frac{f(x_n)}{f'(x_n)}$ is used with $x_0 = 2$. Find x_1, x_2, x_3 to 6 decimal places and comment. [3]
-
- 30** The Newton-Raphson method is applied to $f(x) = x^3 - 2x + k$, where k is a non-zero constant, with $x_0 = 0$.
- Show that $x_1 = \frac{k}{2}$. [2]
 - Show that $x_2 = 0$ if and only if $2x_1^3 = k$. [3]
 - Hence find the two values of k for which the iteration returns to 0 after two steps, so that it cycles forever. [3]
 - For $k = 2$, state the values of x_n for all n , and explain why the iteration never finds the real root of $x^3 - 2x + 2 = 0$. [2]

Numerical Integration

- 1 The table gives values of $y = f(x)$. Use the trapezium rule with all the values to estimate $\int_0^4 f(x) dx$.
- | | | | | | |
|----------|----------|----------|----------|----------|----------|
| x | 0 | 1 | 2 | 3 | 4 |
| y | 3.2 | 4.1 | 5.5 | 5 | 4.4 |
- [2]
-
- 2 Use Simpson's rule with 2 strips to estimate $\int_1^2 \frac{1}{x} dx$, and hence show that $\ln 2 \approx \frac{25}{36}$.
- [3]
-
- 3 Use the mid-ordinate rule with 2 strips to estimate $\int_0^2 e^x dx$. Give your answer to 3 decimal places.
- [2]
-
- 4 Use Simpson's rule with 2 strips to estimate $\int_0^4 x^2 dx$, and comment on your answer given that the exact value is $\frac{64}{3}$.
- [3]
-
- 5 The trapezium rule is to be used to estimate $\int_1^4 f(x) dx$ with 6 strips. Write down the strip width h and the x -values at which f must be evaluated.
- [2]
-
- 6 Copy and complete the table of values of $y = \sqrt{x+1}$, giving values to 3 decimal places.
 $x = 0, 1, 2, 3, 4$
- [2]
-
- 7 Use the trapezium rule with 4 strips to estimate $\int_0^1 \frac{1}{1+x^2} dx$, giving your answer to 4 decimal places.
State the exact value of the integral and the percentage error in your estimate.
- [4]
-
- 8 Use Simpson's rule with 4 strips to estimate $\int_1^3 \ln x dx$, giving your answer to 4 decimal places.
- [4]
-
- 9 Let $I = \int_0^{\pi/2} \cos^2\left(\frac{x}{2}\right) dx$.
- (a) Use Simpson's rule with 4 strips to estimate I , to 4 decimal places. [3]
- (b) Hence estimate $\int_0^{\pi/2} \sin^2\left(\frac{x}{2}\right) dx$. [2]
- (c) Hence estimate $\int_0^{\pi/2} \cos x dx$. [2]
-
- 10 Use the mid-ordinate rule with 4 strips to estimate $\int_0^2 \sqrt{x^3+1} dx$. Give your answer to 3 decimal places.
- [4]

- 11** A car accelerates from rest. Its speed $v \text{ m s}^{-1}$ is recorded every 5 seconds, as shown in the table. Use the trapezium rule to estimate the distance travelled in the first 30 seconds.

t	0	5	10	15	20	25	30
v	0	4	9	13	16	18	19

[3]

- 12** The table shows some values of a function $y = f(x)$, where a is a constant. Simpson's rule, using all of the values in the table, gives an estimate of 14 for the mean value of $f(x)$ over the interval $-3 \leq x \leq 5$. Find the value of a .

x	-3	-1	1	3	5
y	4	10	a	19	16

[4]

- 13** Use Simpson's rule with 4 strips to estimate $\int_0^{\pi/2} \frac{\sin x}{x} dx$, where the integrand takes the value 1 at $x = 0$. Give your answer to 4 decimal places.

[4]

- 14** The integral $I = \int_1^3 \frac{1}{x} dx$ is estimated using the trapezium rule. T_2 and T_4 denote the estimates with 2 and 4 strips.

- (a) Calculate T_2 and T_4 to 5 decimal places. [3]
 (b) Show that $\frac{4T_4 - T_2}{3}$ gives the value of Simpson's rule with 4 strips, and evaluate it. [2]
 (c) Compare the three values with the exact value $\ln 3$ and say which is the best estimate. [1]

- 15** The trapezium rule with 3 strips is used to estimate $\int_1^4 \sqrt{x} dx$.

- (a) Find the estimate to 4 decimal places. [2]
 (b) Explain, by considering the second derivative, whether the estimate is an overestimate or underestimate. [2]
 (c) Find the exact value of the integral, and the percentage error in the estimate. [2]

- 16** For a function f with $|f''(x)| \leq M$ on $[a, b]$, the error in the trapezium rule with n strips satisfies $|E| \leq \frac{(b-a)h^2}{12} M$, where $h = \frac{b-a}{n}$.

The trapezium rule is used to estimate $\int_0^1 e^x dx$. Find the smallest number of strips that guarantees an error of less than 0.0005.

[5]

- 17** The width w metres of a small lake is measured at 10 m intervals along its length, as shown in the table (x metres from one end). Estimate the area of the lake using

x	0	10	20	30	40	50	60
w	0	18	26	31	29	20	0

- (a) the trapezium rule, [3]
 (b) Simpson's rule. [3]

- 18** The region bounded by the curve $y = e^{-x^2}$, the x-axis and the lines $x = 0$ and $x = 1$ is rotated through 2π radians about the x-axis to form a solid.
Show that the volume is $V = \pi \int_0^1 e^{-2x^2} dx$, and use Simpson's rule with 4 strips to estimate V to 3 significant figures. [6]
-
- 19** Use Simpson's rule with 4 strips to estimate $\int_1^5 x \ln x dx$. Then use integration by parts to find the exact value, and calculate the error in the estimate. [7]
-
- 20** The curve C has equation $y = \frac{k}{x+2}$, where k is a positive integer. The trapezium rule with 4 equal strips is used to estimate the area bounded by C , the coordinate axes and the line $x = 2$. The estimate is 6.97, correct to 3 significant figures.
(a) Find the value of k . [3]
(b) State one way in which the accuracy of the estimate could be improved. [1]
(c) By considering a suitable stretch of the curve, use 6.97 to estimate $\int_0^1 \frac{20}{x+1} dx$. [2]
-
- 21** The integral $J = \int_0^1 \frac{e^{-x}}{\sqrt{x}} dx$ cannot be estimated directly by the trapezium or Simpson's rule, because the integrand is undefined at $x = 0$.
(a) Explain why the integrand causes a problem at $x = 0$. [1]
(b) Use the substitution $u = \sqrt{x}$ to show that $J = 2 \int_0^1 e^{-u^2} du$. [3]
(c) Use Simpson's rule with 4 strips on the new integral to estimate J to 4 decimal places. [4]
-
- 22** The integral $I = \int_0^1 x^4 dx = 0.2$ is estimated by the trapezium rule with n strips. T_n denotes the estimate and $E_n = T_n - 0.2$ the error.
(a) Calculate T_2 , T_4 and T_8 to 6 decimal places. [3]
(b) Calculate E_2 , E_4 , E_8 and the ratios $\frac{E_2}{E_4}$ and $\frac{E_4}{E_8}$ to 2 decimal places. [3]
(c) Explain what the ratios show about how E_n depends on h , and use E_8 to predict E_{16} to 2 significant figures. [2]
-
- 23** Ella estimates $\int_0^2 \sqrt{1+x^2} dx$ using $h = 0.5$, with ordinates $y_0, \dots, y_4 = 1.0000, 1.1180, 1.4142, 1.8028, 2.2361$.
(a) For the trapezium rule Ella writes: $\text{Area} \approx 0.5 \times [y_0 + y_4 + 2(y_1 + y_2 + y_3)] = 11.906$. She forgot a factor in the formula. Identify the error and give the correct estimate. [3]
(b) Ella then says: "Simpson's rule with 5 strips needs $h = 0.4$ and y -values at 0, 0.4, ..., 2." Explain why Simpson's rule with 5 strips cannot be used in the standard form. [1]
(c) Find Simpson's estimate using 4 strips and then using 6 strips, to 4 decimal places, and say which is likely to be more accurate. [2]
-

- 24** A surveyor measures the depth d metres of a river at distances x metres from the bank.
- (a) The measurements below are at uneven intervals. Use the trapezium rule on each interval separately to estimate the area of the cross-section from $x = 0$ to $x = 10$. [3]

x	0	1	3	6	10
d	2	2.8	4.1	3.5	1.2

- (b) A second cross-section is measured at equal intervals, as shown. Estimate its area using the trapezium rule and Simpson's rule. [4]

x	0	2	4	6	8
d	3	4.4	5.1	4.6	2.9

- 25** Let $f(x) = x \ln x$ and $I = \int_1^3 f(x) dx$.
- (a) Show that $f''(x) > 0$ for $x > 0$, and deduce whether the trapezium rule and the mid-ordinate rule over- or underestimate I . [3]
- (b) Calculate the trapezium estimate T_4 and the mid-ordinate estimate M_4 , each with 4 strips, to 4 decimal places. [3]
- (c) Find the exact value of I and verify that it lies between M_4 and T_4 . [3]

- 26** The arc length of a curve $y = g(x)$ between $x = a$ and $x = b$ is $L = \int_a^b \sqrt{1 + (g'(x))^2} dx$. Find an expression for the length of one arch of $y = \sin x$, from $x = 0$ to $x = \pi$, as an integral, and use Simpson's rule with 6 strips to estimate it to 3 significant figures. [8]

- 27** Simpson's rule with 2 strips of width h on $[-h, h]$ gives $S = \frac{h}{3}[f(-h) + 4f(0) + f(h)]$.
- (a) Let $f(x) = ax^3 + bx^2 + cx + d$. Show that $\int_{-h}^h f(x) dx = \frac{2bh^3}{3} + 2dh$. [2]
- (b) Show that $S = \frac{2bh^3}{3} + 2dh$ for the same f , so that Simpson's rule is exact for any cubic. [4]
- (c) Explain why the result holds for a cubic on any interval $[p, q]$, and verify it for $\int_0^2 (x^3 - 2x + 1) dx$ using 2 strips. [3]

- 28** The integral $I = \int_0^1 x^2 dx = \frac{1}{3}$ is estimated by the trapezium rule with n strips, giving T_n .
- (a) Show that $T_n = \frac{1}{n^3}[\frac{1}{2}n^2 + \sum_{r=1}^{n-1} r^2]$. [2]
- (b) Use $\sum_{r=1}^{n-1} r^2 = \frac{(n-1)n(2n-1)}{6}$ to show that $T_n = \frac{1}{3} + \frac{1}{6n^2}$. [3]
- (c) Hence find the smallest n for which T_n is within 10^{-4} of the exact value. [2]
- (d) Explain how the form of the error in (b) relates to the fact that the error of the trapezium rule is proportional to h^2 . [2]

- 29** The function $f(x) = ax^2 + bx^3$, where a and b are constants, is integrated over $[0, 2]$. The trapezium rule with 2 strips gives an estimate of 19, and Simpson's rule with 2 strips gives an estimate of 16.
- (a) Show that the trapezium estimate is $3a + 5b$, and find an expression for the Simpson estimate. [3]
- (b) Find a and b . [3]
- (c) Find the exact value of $\int_0^2 f(x) dx$, and explain why it equals the Simpson estimate. [3]

- 30** The exact value of $\int_0^1 \frac{4}{1+x^2} dx$ is π . Simpson's rule with $2n$ strips is denoted S_{2n} . It is known that the error in S_{2n} is approximately proportional to h^4 .
- (a) Calculate S_2 and S_4 to 7 decimal places. [4]
- (b) Explain why, since halving h reduces the error by a factor of about 16, the quantity $R = \frac{16S_4 - S_2}{15}$ should be a better estimate of π . [3]
- (c) Calculate R and the error in R , and compare with the errors in S_2 and S_4 . [3]

Numerical Differentiation

- 1** Use the forward difference approximation $f'(x) \approx \frac{f(x+h) - f(x)}{h}$ with $h = 0.1$ to estimate $f'(2)$ for $f(x) = x^2$. [2]
- 2** Use the central difference approximation $f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$ with $h = 0.1$ to estimate the gradient of $y = e^x$ at $x = 1$. Give your answer to 4 decimal places. [2]
- 3** Use the backward difference approximation $f'(x) \approx \frac{f(x) - f(x-h)}{h}$ with $h = 0.1$ to estimate the gradient of $y = \ln x$ at $x = 2$, to 4 decimal places. [2]
- 4** The table gives values of a function $f(x)$. Use the central difference approximation with $h = 1$ to estimate $f'(11)$.
- | | | | |
|-----|-----------|-----------|-----------|
| x | 10 | 11 | 12 |
| f | 3.162 | 3.317 | 3.464 |
- [2]
- 5** Use the approximation $f''(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$ with $h = 0.1$ to estimate $f''(1)$ for $f(x) = x^3$. [2]
- 6** The displacement s metres of a particle at time t seconds is recorded in the table. Use a central difference to estimate the velocity at $t = 3$.
- | | | | |
|-----|----------|----------|----------|
| t | 2 | 3 | 4 |
| s | 11 | 18.5 | 28 |
- [2]
- 7** For $f(x) = \sin x$, estimate $f'(\frac{\pi}{3})$ using, with $h = 0.1$,
- (a) the forward difference, [2]
- (b) the central difference. [2]
- (c) Calculate the error in each estimate by comparing with the exact value $\cos(\frac{\pi}{3}) = 0.5$. [2]
- 8** $f(x) = x e^x$. The exact value is $f'(1) = 2e = 5.43656$.
- (a) Use the forward difference with $h = 0.1$ and with $h = 0.01$ to estimate $f'(1)$, and find the error in each. [2]
- (b) Repeat with the central difference. [2]
- (c) Which method is more accurate for the same h ? Give a reason based on your results. [1]

- 9** The temperature T °C of a liquid is recorded at time t minutes, as shown in the table.
Estimate the rate of change of temperature dT/dt at

t	0	1	2	3	4
T	20	31.5	41.2	49	55.1

- (a) $t = 2$, using a central difference, [2]
 (b) $t = 0$, using a forward difference, [1]
 (c) $t = 4$, using a backward difference. [1]

- 10** Use the forward difference approximation to estimate the gradient of $y = \sqrt{x}$ at $x = 4$, using $h = 0.1$ and then $h = 0.01$. Give each answer to 5 decimal places and compare with the exact value. [4]

- 11** Use $f''(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$ with $h = 0.1$ to estimate $f''(2)$ for $f(x) = \ln x$. Compare with the exact value. [3]

- 12** The gradient of the curve $y = f(x)$ at $x = a$ is the limit of the gradient of the chord joining $(a, f(a))$ and $(a+h, f(a+h))$ as $h \rightarrow 0$.
Show that the forward difference is the gradient of this chord, and explain why it gives a better estimate of $f'(a)$ when h is small. [3]

- 13** $f(x) = x^3$, and the exact value of $f'(2)$ is 12. Take $h = 0.5$.
 (a) Use the forward difference to estimate $f'(2)$. [2]
 (b) Use the central difference to estimate $f'(2)$. [2]
 (c) State which estimate is more accurate, and give a reason. [1]

- 14** Use Taylor expansions of $f(x+h)$ and $f(x-h)$ about x to show that $\frac{f(x+h) - f(x-h)}{2h} = f'(x) + \frac{h^2}{6}f'''(x) + \dots$
Hence state the order of the error in the central difference approximation and compare it with that of the forward difference. [5]

- 15** Use Taylor expansions to show that $\frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = f''(x) + \frac{h^2}{12}f^{(4)}(x) + \dots$
State the order of the error. [5]

- 16** Let $D(h) = \frac{f(x+h) - f(x-h)}{2h}$. For small h , $D(h) \approx f'(x) + Ch^2$, where C is a constant.
Use $D(0.2)$ and $D(0.1)$ for $f(x) = \ln x$ at $x = 3$ to find an improved estimate of $f'(3)$, and compare it with the exact value. [6]

- 17** The table gives values of $f(x) = e^{-x^2}$.

x	1	1.1	1.2	1.3	1.4
f	0.3679	0.2982	0.2369	0.1845	0.1409

- (a) Use central differences with $h = 0.1$ and with $h = 0.2$ to estimate $f'(1.2)$. [3]
 (b) Estimate $f''(1.2)$ using $h = 0.1$. [2]
 (c) The exact derivative is $f'(x) = -2x e^{-x^2}$. State which estimate in (a) is closer to $f'(1.2) = -0.5686$. [1]

- 18** Values of e^x are recorded to 4 decimal places, with $e^1 = 2.7183$, $e^{1.1} = 3.0042$, $e^{1.01} = 2.7456$ and $e^{1.0001} = 2.7186$.
Use the forward difference with $h = 0.1$, 0.01 and 0.0001 to estimate the gradient of $y = e^x$ at $x = 1$, and explain why the estimate for $h = 0.0001$ is poor even though h is smallest. The exact gradient is $e = 2.71828...$ [6]

- 19** The table shows values of $y = x e^{-x}$, rounded to 4 decimal places. The curve has a maximum point near $x = 1$.

x	0.6	0.8	1	1.2	1.4
y	0.3293	0.3595	0.3679	0.3614	0.3452

- (a) Use central differences with $h = 0.2$ to estimate $\frac{dy}{dx}$ at $x = 0.8$, 1.0 and 1.2 . [3]
(b) Use linear interpolation on the gradient estimates to find an estimate of the x -coordinate of the maximum. [2]

- 20** $f(x) = x^3 - 2x - 5$ and the root of $f(x) = 0$ is to be found starting from $x_0 = 2$.
(a) Find x_1 using the Newton-Raphson method with the exact derivative. [2]
(b) Repeat, but replace $f'(2)$ by the forward difference approximation with $h = 0.01$. Give x_1 to 5 decimal places. [3]
(c) Comment on the difference between the two values of x_1 . [1]

- 21** A function is only known at x , $x + h$ and $x + 2h$, so a central difference cannot be used at x .
(a) Write down the Taylor series for $f(x + h)$ and $f(x + 2h)$ up to the term in h^3 . [2]
(b) By combining $4f(x + h) - f(x + 2h) - 3f(x)$, show that $f'(x) = \frac{-3f(x) + 4f(x + h) - f(x + 2h)}{2h} + \frac{h^2}{3}f'''(x) + \dots$ [4]
(c) Use the formula with $h = 0.1$ to estimate $f'(0)$ for $f(x) = e^x$, and compare the error with that of the forward difference with the same h . [2]

- 22** The central difference $D(h) = \frac{f(x + h) - f(x - h)}{2h}$ is used to estimate $f'(1)$ for $f(x) = e^x$, where each value of f is rounded to 4 decimal places, so each has an error of at most $\varepsilon = 5 \times 10^{-5}$.
(a) Show that the rounding errors cause an error in $D(h)$ of at most $\frac{\varepsilon}{h}$. [2]
(b) The truncation error is approximately $\frac{h^2}{6}f'''(1) = \frac{h^2 e}{6}$. Write down the total maximum error $E(h)$ and find the value of h that minimises it. [4]
(c) Calculate the minimum value of $E(h)$, and explain why a smaller h (for example $h = 0.001$) is worse. [2]

- 23** Sam estimates $f'(2)$ and $f''(2)$ for $f(x) = x^3$ using $h = 0.1$. He writes:
 $f'(2) \approx \frac{f(2.1) - f(1.9)}{0.1} = \frac{9.261 - 6.859}{0.1} = 24.02$
 $f''(2) \approx \frac{f(2.1) - 2f(2) + f(1.9)}{0.1} = \frac{9.261 - 16 + 6.859}{0.1} = 1.2$
(a) Identify the mistake in each calculation. [2]
(b) Calculate the correct estimates. [2]
(c) Differentiate to find the exact values of $f'(2)$ and $f''(2)$, and explain how this could have shown Sam that his answers were wrong. [2]

- 24** The displacement s metres of a particle from a fixed point at time t seconds is recorded every 0.5 s, as shown. Take $v = \frac{ds}{dt}$ and $a = \frac{d^2s}{dt^2}$.

t	0	0.5	1	1.5	2
s	0.000	1.500	3.000	6.000	12.000

- (a) Estimate v at $t = 0.5, 1$ and 1.5 using central differences. [2]
 (b) Estimate v at $t = 0$ using a forward difference and at $t = 2$ using a backward difference. [2]
 (c) Estimate the acceleration at $t = 1$. [2]
 (d) Given that $s = 2t^3 - 3t^2 + 4t$, find the exact values of $v(1)$ and $a(1)$, and comment on the accuracy of the estimates. [1]

- 25** The forward difference with step h is used to estimate $f'(2)$ for $f(x) = \ln x$, for which $f'(2) = 0.5$.

- (a) Calculate the estimates for $h = 0.1$ and $h = 0.05$ to 6 decimal places, and the errors. [3]
 (b) Show that the ratio of the errors is close to 2, and explain this using the leading term of the Taylor series for the error. [2]
 (c) Use the leading term $\frac{h}{2}f''(x)$ to predict the error for $h = 0.1$ and compare with the actual error. [2]

- 26** A curve is defined parametrically by $x = t^2 + 1$, $y = t^3 - 2t$.

- (a) Show that $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ and use it to find the exact value of $\frac{dy}{dx}$ when $t = 2$. [3]
 (b) Use central differences in t with $h = 0.1$ to estimate $\frac{dy}{dt}$ and $\frac{dx}{dt}$ at $t = 2$, and hence estimate $\frac{dy}{dx}$. [3]
 (c) Explain why the h in the central differences cancels in the ratio. [1]

- 27** A function f is known at x , $x + h$ and $x + 2h$ only.

- (a) Use Taylor series to show that $f(x) - 2f(x + h) + f(x + 2h) = h^2f''(x) + h^3f'''(x) + \frac{7h^4}{12}f^{(4)}(x) + \dots$ [4]
 (b) Hence write down a one-sided approximation to $f''(x)$ with its leading error term. [2]
 (c) For $f(x) = \ln x$, estimate $f''(1)$ using the formula in (b) with $h = 0.1$, and then use the error term (with $f'''(1) = 2$) to obtain a corrected estimate. Compare both with the exact value $f''(1) = -1$. [3]

- 28** Let $D(h) = \frac{f(x + h) - f(x - h)}{2h}$. Taylor series give $D(h) = f'(x) + \frac{h^2}{6}f'''(x) + \frac{h^4}{120}f^{(5)}(x) + \dots$

- (a) Show that $\frac{4D(h) - D(2h)}{3} = \frac{f(x - 2h) - 8f(x - h) + 8f(x + h) - f(x + 2h)}{12h}$. [3]
 (b) Show that the h^2 terms cancel in $\frac{4D(h) - D(2h)}{3}$, and state the order of the error that remains. [3]
 (c) For $f(x) = \sin x$ at $x = 1$ with $h = 0.1$, calculate $D(0.1)$, $D(0.2)$ and the five-point estimate to 8 decimal places, and compare their errors with $\cos 1 = 0.54030231$. [3]

- 29** Let $D_2(h) = \frac{f(x + h) - 2f(x) + f(x - h))}{h^2}$. It can be shown that $D_2(h) = f''(x) + \frac{h^2}{12}f^{(4)}(x) + \dots$

- (a) Show that the combination $\frac{4D_2(\frac{h}{2}) - D_2(h)}{3}$ has an error of order h^4 . [3]
 (b) For $f(x) = \ln x$ at $x = 2$, calculate $D_2(0.2)$ and $D_2(0.1)$ to 6 decimal places. [3]
 (c) Use (a) to obtain an improved estimate of $f''(2)$ and compare all three estimates with the exact value $f''(2) = -0.25$. [3]

30

The function $y(x)$ satisfies the boundary value problem $\frac{d^2y}{dx^2} = y$, with $y(0) = 0$ and $y(1) = 1$.

The interval $[0, 1]$ is divided into 4 equal strips of width $h = 0.25$ with $x_i = ih$ and $y_i = y(x_i)$, so that $y_0 = 0$ and $y_4 = 1$.

- (a) Replace the second derivative by $\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$ and show that $y_{i-1} - \frac{33}{16}y_i + y_{i+1} = 0$ for $i = 1, 2, 3$. [3]
- (b) Write the three equations for y_1, y_2, y_3 in matrix form and solve them to 5 decimal places. [5]
- (c) The exact solution is $y = \frac{\sinh x}{\sinh 1}$. Compare your values with the exact values at $x = 0.25, 0.5, 0.75$. [2]

Euler's Method

- 1 $\frac{dy}{dx} = x + y$, with $y = 1$ when $x = 0$. Use Euler's method with a step length of 0.1 to estimate y when $x = 0.1$. [2]
- 2 $\frac{dy}{dx} = 2x - y$, with $y = 3$ when $x = 1$. Use Euler's method with $h = 0.2$ to estimate y when $x = 1.2$. [2]
- 3 $\frac{dy}{dx} = y^2$, $y(0) = 1$. Use Euler's method with step 0.1 to find an estimate of $y(0.2)$. [3]
- 4 $\frac{dy}{dx} = \sin x$, $y(0) = 0$. Use two steps of Euler's method with $h = 0.5$ to estimate $y(1)$. Give your answer to 4 decimal places. [3]
- 5 The differential equation $\frac{dy}{dx} = x^2 + y$ has $y = y_0$ when $x = x_0$. Write down the Euler step formula giving y_{n+1} in terms of x_n, y_n and the step length h , and use it to find y_1 when $x_0 = 2, y_0 = 1$ and $h = 0.1$. [2]
- 6 $\frac{dy}{dx} = 3 - \frac{y}{2}$, $y(0) = 0$. Use Euler's method with $h = 0.5$ to estimate $y(0.5)$ and $y(1)$. [3]
- 7 $\frac{dy}{dx} = e^{-x}y$, $y(0) = 2$. Use Euler's method with a step length of 0.1 to estimate $y(0.3)$, giving your answer to 4 decimal places. [4]
- 8 $\frac{dy}{dx} = \frac{1}{1+x^2}$, $y(0) = 0$. Use Euler's method with $h = 0.25$ to estimate $y(1)$. Compare with the exact value $y(1) = \frac{\pi}{4}$. [4]

- 9** Euler's method with step $h = 0.1$ is used to solve $\frac{dy}{dx} = xy - 1$, $y(1) = 2$. The table shows the calculation, with some values replaced by ?. Find the values of the missing entries, and hence y_3 .

n	x_n	y_n	$f(x_n, y_n)$
0	1.0	2.0000	1.0000
1	1.1	2.1000	1.3100
2	1.2	?	?

[4]

- 10** $\frac{dy}{dx} = y$, $y(0) = 1$.
- (a) Show that Euler's method with step h gives $y_{n+1} = (1 + h)y_n$, and hence $y_n = (1 + h)^n$. [2]
- (b) Use this with $h = 0.1$ to estimate $y(1)$, and compare with the exact value e . [2]

- 11** $\frac{dy}{dx} = 2y$, $y(0) = 1$.
- (a) Use Euler's method with $h = 0.1$ to estimate $y(0.3)$. [3]
- (b) Solve the differential equation exactly, and find the error in your estimate of $y(0.3)$. [2]

- 12** A cup of tea cools according to $\frac{dT}{dt} = -0.1(T - 20)$, where T °C is the temperature after t minutes. Initially $T = 90$. Use Euler's method with step 2 minutes to estimate the temperature after 4 minutes. [4]

- 13** $\frac{dy}{dx} = xy$, $y(1) = 2$. Use Euler's method with $h = 0.2$ to find $y(1.4)$. [4]

- 14** $\frac{dy}{dx} = y$, $y(0) = 1$.
- (a) Use Euler's method with $h = 0.5$ to estimate $y(1)$. [2]
- (b) Solve the differential equation and find the percentage error in your estimate. [2]
- (c) Explain, by considering the gradient of the solution curve between the points, why Euler's method gives an underestimate here. [1]

- 15** $\frac{dy}{dx} = x^2 - y$. Euler's method with $h = 0.1$ is used from the point $(1, y_0)$ and gives the estimate $y(1.1) \approx 2.2$. Find the value of y_0 , giving your answer to 4 decimal places. [4]

- 16** $\frac{dy}{dx} = ky - x$, where k is a constant, and $y(0) = 1$. Using Euler's method with step length 0.2, the estimate of $y(0.2)$ is 1.1. Find the value of k , and hence estimate $y(0.4)$. [5]

- 17** Salt solution flows into a tank at 4 g per minute, and solution leaves so that the mass S grams of salt in the tank satisfies $\frac{dS}{dt} = 4 - 0.02S$, where t is in minutes. Initially $S = 0$. Use Euler's method with step 5 minutes to estimate S at $t = 15$. Find the value that S approaches in the long term, and comment on your estimate. [6]

- 18** $\frac{d^2y}{dx^2} = -y$, with $y = 1$ and $\frac{dy}{dx} = 0$ when $x = 0$. Let $v = \frac{dy}{dx}$. Use Euler's method with $h = 0.1$ for the pair of equations $\frac{dy}{dx} = v$, $\frac{dv}{dx} = -y$ to estimate $y(0.3)$ and $v(0.3)$. [6]

- 19** $\frac{dy}{dx} = x + y$, $y(0) = 1$.
- (a) Use Euler's method with $h = 0.2$ to estimate $y(0.4)$. [2]
- (b) Use Euler's method with $h = 0.1$ to estimate $y(0.4)$. [2]
- (c) The error is roughly proportional to h . Use this to find an improved estimate of the form $2y_{(h=0.1)} - y_{(h=0.2)}$, and compare it with the exact value $y(0.4) = 2e^{0.4} - 1.4 = 1.58365$. [3]
-
- 20** $\frac{dy}{dx} = x + y$, $y(0) = 1$.
- (a) Use one step of Euler's method with $h = 0.1$ to estimate $y(0.1)$. [1]
- (b) The exact solution is $y = 2e^x - x - 1$. Find the error in y_1 . [2]
- (c) Use the Taylor series $y(h) \approx y(0) + hy'(0) + \frac{h^2}{2}y''(0)$ to explain this error, where $y'' = 1 + y'$. [3]
-
- 21** The population P (in thousands) of a colony satisfies $\frac{dP}{dt} = 0.2P(1 - \frac{P}{50})$, where t is in years. Initially $P = 10$.
- (a) Use Euler's method with step 1 year to estimate P after 4 years, showing each step. [5]
- (b) Find the values of P for which the population is stationary, and explain why the population can never exceed 50 thousand in the model. [3]
-
- 22** The functions $x(t)$ and $y(t)$ satisfy the pair of differential equations
- $$\frac{dx}{dt} = -2x + y, \quad \frac{dy}{dt} = x - 2y,$$
- with $x(0) = 1$ and $y(0) = 0$. Use Euler's method with $h = 0.1$ to estimate $x(0.3)$ and $y(0.3)$. [7]
-
- 23** Kai uses Euler's method to estimate $y(0.3)$ for $\frac{dy}{dx} = x + y$, $y(0) = 1$, with $h = 0.1$. His working is:
- $$y_1 = 1 + 0.1(0.1 + 1) = 1.110, \quad y_2 = 1.110 + 0.1(0.2 + 1.110) = 1.2410, \quad y_3 = 1.2410 + 0.1(0.3 + 1.2410) = 1.39510$$
- (a) Identify Kai's error. [2]
- (b) Carry out the calculation correctly to find y_1 , y_2 and y_3 . [4]
-
- 24** The differential equation $\frac{dy}{dx} = -10y$ has $y(0) = 1$.
- (a) Solve the differential equation and describe the behaviour of the solution for large x . [2]
- (b) Use Euler's method with $h = 0.3$ to find y_1, y_2, y_3, y_4 , and describe how this compares with the exact solution. [3]
- (c) Show that for $y_{n+1} = (1 - 10h)y_n$ the estimates tend to 0 only if $0 < h < 0.2$. [2]
-
- 25** A parachutist falls from rest and her speed v m s⁻¹ after t seconds is modelled by $\frac{dv}{dt} = 9.8 - 0.2v$.
- (a) Use Euler's method with step 0.5 s to estimate her speed after 2 seconds. [4]
- (b) Find the terminal speed according to the model. [1]
- (c) The exact solution is $v = 49(1 - e^{-0.2t})$. Find the exact speed at $t = 2$ and the percentage error in your estimate. [3]
-
- 26** $\frac{dy}{dx} = \frac{y}{x}$, with $y = 2$ when $x = 1$.
- (a) Use Euler's method with $h = 0.25$ to estimate $y(2)$. Show your values of y_1 to y_4 . [4]
- (b) Solve the differential equation by separating variables, and find the exact $y(2)$. [3]
- (c) Comment on your answers to (a) and (b), and explain why this happens. [1]

- 27** Euler's method with step $h = \frac{1}{N}$ is used to estimate $y(1)$ for $\frac{dy}{dx} = y$, $y(0) = 1$, giving E_N .
- (a) Show that $E_N = (1 + \frac{1}{N})^N$. [2]
- (b) Calculate E_N for $N = 1, 2, 4, 10$ and 100 to 4 decimal places. [2]
- (c) Use $\ln E_N = N \ln(1 + \frac{1}{N})$ and the series $\ln(1 + u) = u - \frac{u^2}{2} + \frac{u^3}{3} - \dots$ to show that $E_N \approx e(1 - \frac{1}{2N})$ for large N . [4]
- (d) Hence state the order of the error and check it for $N = 100$. [2]
-
- 28** Euler's method with step $h = \frac{1}{N}$ is applied to $\frac{dy}{dx} = x$ with $y(0) = 0$, to estimate $y(1)$.
- (a) Show that $y_n = \frac{n(n-1)h^2}{2}$. [3]
- (b) Hence show that the estimate of $y(1)$ is $\frac{1}{2} - \frac{1}{2N}$. [2]
- (c) Find the exact value of $y(1)$, and the error of the estimate in terms of h . Explain why the error is exactly proportional to h here. [3]
- (d) Find the number of steps N needed so that the error is less than 0.0005 . [1]
-
- 29** A block on a spring has displacement x metres from equilibrium at time t seconds, satisfying $\frac{d^2x}{dt^2} + 0.5\frac{dx}{dt} + 4x = 0$. At $t = 0$, $x = 1$ and $\frac{dx}{dt} = 0$.
- (a) Let $v = \frac{dx}{dt}$. Show that the equation can be written as the pair $\frac{dx}{dt} = v$, $\frac{dv}{dt} = -0.5v - 4x$. [2]
- (b) Use Euler's method with $h = 0.1$ to find estimates of x and v at $t = 0.1, 0.2, 0.3$ and 0.4 , giving values to 4 decimal places. [6]
- (c) The block should pass through equilibrium ($x = 0$) first at about $t = 0.85$ in the exact model. Explain, using your results, whether the Euler estimates are plausible and why accuracy would deteriorate over a longer time. [2]
-
- 30** $\frac{dy}{dx} = y^2$, $y(0) = 1$.
- (a) Solve the differential equation, and state what happens to y as $x \rightarrow 1$. [3]
- (b) Use Euler's method with $h = 0.2$ to estimate $y(0.2)$, $y(0.4)$, ..., $y(1.0)$, to 3 decimal places. [4]
- (c) Compare with the exact values at $x = 0.2$ and $x = 0.8$, and explain why Euler's method is a poor tool close to $x = 1$. [2]

Improved Euler Methods

- 1** The improved Euler method uses $y_{n+1}^* = y_n + hf(x_n, y_n)$ followed by $y_{n+1} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_{n+1}, y_{n+1}^*)]$.
For $\frac{dy}{dx} = x + y$ with $y(0) = 1$, use $h = 0.1$ to find y_1 . [3]
-
- 2** The modified Euler (mid-point) method uses $k_1 = f(x_n, y_n)$, $k_2 = f(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_1)$ and $y_{n+1} = y_n + hk_2$.
For $\frac{dy}{dx} = 2x - y$ with $y(1) = 3$, use $h = 0.2$ to find y_1 . [3]

- 3** $\frac{dy}{dx} = y^2$ with $y(0) = 1$. Using $y^*_1 = 1.1$ from the first stage of the improved Euler method with $h = 0.1$, find y_1 . [2]
-
- 4** A numerical method for $\frac{dy}{dx} = f(x, y)$ uses $y_{r+1} = y_{r-1} + 2hf(x_r, y_r)$. For $\frac{dy}{dx} = x - y$, with $h = 0.1$, $y_0 = 1$ and $y_1 = 0.91$, find y_2 . [2]
-
- 5** $\frac{dy}{dx} = -y$, $y(0) = 1$. Use one step of the improved Euler method with $h = 0.5$ to estimate $y(0.5)$. [3]
-
- 6** For $\frac{dy}{dx} = y$, $y(0) = 1$, the exact solution is $y = e^x$. With $h = 0.1$, Euler's method gives $y(0.1) \approx 1.1$ and the improved Euler method gives $y(0.1) \approx 1.105$. Which is closer to the exact value of $y(0.1)$? Give $y(0.1)$ to 5 decimal places. [2]
-
- 7** $\frac{dy}{dx} = xy$, $y(0) = 1$. Use the improved Euler method with $h = 0.2$ to estimate $y(0.4)$. [5]
-
- 8** $\frac{dy}{dx} = x^2 + y$, $y(0) = 1$. Use the modified Euler (mid-point) method with $h = 0.2$ to estimate $y(0.4)$. [4]
-
- 9** $\frac{dy}{dx} = x + y^2$, $y(0) = 1$. Use Euler's method with $h = 0.1$ to find y_1 . Then use $y_{r+1} = y_{r-1} + 2hf(x_r, y_r)$ to estimate $y(0.2)$ and $y(0.3)$. [5]
-
- 10** $\frac{dy}{dx} = 2y$, $y(0) = 1$.
 (a) Use Euler's method with $h = 0.1$ to estimate $y(0.2)$. [1]
 (b) Use the improved Euler method with $h = 0.1$ to estimate $y(0.2)$. [3]
 (c) Compare both with the exact value $e^{0.4} = 1.4918$. [1]
-
- 11** The fourth-order Runge-Kutta method is $k_1 = f(x_n, y_n)$, $k_2 = f(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_1)$, $k_3 = f(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_2)$, $k_4 = f(x_n + h, y_n + hk_3)$, $y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$.
 For $\frac{dy}{dx} = x + y$, $y(0) = 1$, use $h = 0.2$ to find y_1 to 5 decimal places. [4]
-
- 12** Explain how the improved Euler formula $y_{n+1} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_{n+1}, y^*_{n+1})]$ is based on the trapezium rule, and why it is more accurate than Euler's method. [3]
-
- 13** A cup of tea cools according to $\frac{dT}{dt} = -0.1(T - 20)$, where T °C is its temperature after t minutes, and $T = 90$ when $t = 0$. Use the improved Euler method with step 2 minutes to estimate T when $t = 4$, and compare with the exact value $20 + 70e^{-0.1t}$. [5]
-
- 14** $\frac{dy}{dx} = \frac{1}{1 + x^2 + y^2}$, $y(0) = 1$. The corrector formula $y_1 = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]$ is applied with $h = 0.5$, with the predictor $y^*_1 = y_0 + hf(x_0, y_0)$.
 Use the predictor once, then apply the corrector twice (replacing y_1 on the right-hand side by the latest value each time), to find two successive improved values of y_1 , to 5 decimal places. [6]
-

- 15** $\frac{dy}{dx} = y$, $y(0) = 1$.
- (a) Show that one step of the improved Euler method with step h gives $y_{n+1} = (1 + h + \frac{h^2}{2})y_n$. [3]
- (b) Use $h = 0.1$ to estimate $y(1)$, and compare with e . [3]
-
- 16** $\frac{dy}{dx} = y$, $y(0) = 1$.
- (a) Show that the modified Euler (mid-point) method also gives $y_{n+1} = (1 + h + \frac{h^2}{2})y_n$. [3]
- (b) The exact solution gives $y(h) = e^h = 1 + h + \frac{h^2}{2} + \frac{h^3}{6} + \dots$. Write down the local error of the method and evaluate it for $h = 0.1$. [3]
-
- 17** $\frac{dy}{dx} = y$, $y(0) = 1$. The fourth-order Runge-Kutta method is used with step h .
- (a) Show that $k_1 = y_n$, $k_2 = (1 + \frac{h}{2})y_n$, $k_3 = (1 + \frac{h}{2} + \frac{h^2}{4})y_n$ and $k_4 = (1 + h + \frac{h^2}{2} + \frac{h^3}{4})y_n$. [3]
- (b) Use $h = 0.5$ and two steps to estimate $y(1)$, and compare with e . [3]
-
- 18** The functions $x(t)$ and $y(t)$ satisfy $\frac{dx}{dt} = -y$ and $\frac{dy}{dt} = x$, with $x(0) = 1$ and $y(0) = 0$.
Use one step of the improved Euler method for the pair of equations, with $h = 0.2$, to estimate $x(0.2)$ and $y(0.2)$. Compare with the exact values $x = \cos t$, $y = \sin t$. [7]
-
- 19** $\frac{dy}{dx} = ay$ with $y(0) = 1$, where a is a positive constant. One step of the improved Euler method with $h = 0.2$ gives $y_1 = 1.22$.
Find the value of a . [6]
-
- 20** Euler's method and the improved Euler method are each used to estimate $y(1)$ for $\frac{dy}{dx} = y$, $y(0) = 1$, with N steps of size $h = \frac{1}{N}$. Let y_N be the estimate. For $N = 10$ and $N = 20$, calculate the error $e - y_N$ for each method (to 5 significant figures) and the ratio of the errors for $N = 10$ to $N = 20$. Hence deduce the order of each method. [7]
-
- 21** A fish population N (in hundreds) is modelled by $\frac{dN}{dt} = 0.4N(1 - \frac{N}{80}) - 3$, where t is in years; the final term represents fishing. Initially $N = 30$.
- (a) Use the improved Euler method with step 1 year to estimate N after 3 years, showing the working for each step. [6]
- (b) Find the equilibrium populations of the model, by solving $\frac{dN}{dt} = 0$. [3]
-
- 22** $\frac{dy}{dx} = x^2 + y^2$, $y(0) = 1$. Use the fourth-order Runge-Kutta method with $h = 0.1$ to estimate $y(0.2)$ to 5 decimal places. [8]
-
- 23** Mia is using the improved Euler method with $h = 0.1$ for $\frac{dy}{dx} = x + y$, $y(0) = 1$. In the first step she writes:
 $y^*_1 = 1 + 0.1(0 + 1) = 1.1$, $y_1 = 1 + 0.05 f(0.1, 1.1) = 1 + 0.05(1.2) = 1.06$
- (a) Explain Mia's error. [2]
- (b) Carry out the improved Euler method correctly to find y_1 , y_2 and y_3 , to 5 decimal places. [4]

- 24** The method $y_{r+1} = y_{r-1} + 2hf(x_r, y_r)$ is applied to $\frac{dy}{dx} = -2y$, with $y(0) = 1$ and step $h = 0.5$. The exact solution is $y = e^{-2x}$. The first step is taken from the exact value, $y_1 = 0.368$.
- Show that $y_{r+1} = y_{r-1} - 2y_r$. [1]
 - Calculate y_2, y_3, y_4, y_5 and y_6 . [2]
 - Compare with the exact solution at $x = 3$, and describe the behaviour of the numerical solution. [2]
 - Show that solutions of $y_{r+1} + 2y_r - y_{r-1} = 0$ contain a term proportional to λ^r with $\lambda = -1 - \sqrt{2}$, and explain why this causes the instability. [2]
-
- 25** $\frac{dy}{dx} = x + y$, $y(0) = 1$. The exact solution is $y = 2e^x - x - 1$. Use $h = 0.2$ and two steps to estimate $y(0.4)$ by
- Euler's method, [1]
 - the improved Euler method, [2]
 - the mid-point method, [2]
 - the fourth-order Runge-Kutta method. [1]
 - Compare the errors with the exact value 1.58365. [1]
-
- 26** $\frac{dy}{dx} = \cos x$, $y(0) = 0$, which has exact solution $y = \sin x$. Use $h = \frac{\pi}{4}$ and 2 steps to estimate $y(\frac{\pi}{2})$ by
- Euler's method, [2]
 - the improved Euler method, [2]
 - the mid-point method. [2]
 - State which of the three estimates is closest to the exact value 1, and give the sign of each error. [1]
-
- 27** Consider $\frac{dy}{dx} = f(x, y)$. Let $f, f_x = \frac{\partial f}{\partial x}$ and $f_y = \frac{\partial f}{\partial y}$ be evaluated at (x, y) . The Taylor series gives $y(x + h) = y + hf + \frac{h^2}{2}(f_x + ff_y) + O(h^3)$.
- Show that $f(x + h, y + hf) = f + h(f_x + ff_y) + O(h^2)$. [2]
 - Hence show that the improved Euler method agrees with the Taylor series up to and including the h^2 term, so its local error is $O(h^3)$. [4]
 - Show that Euler's method agrees with the Taylor series only up to the term in h , so its local error is $O(h^2)$, and state the leading error term for Euler's method. [2]
 - Explain why the global error over a fixed interval $[a, b]$ is of one order lower than the local error. [2]
-
- 28** The two-step method $y_{r+1} = y_{r-1} + 2hf(x_r, y_r)$ is applied to $\frac{dy}{dx} = -y$, $y(0) = 1$, with starting values $y_0 = 1$ and $y_1 = e^{-h}$.
- Show that $y_{r+1} + 2hy_r - y_{r-1} = 0$; $y_r = A\lambda_1^r + B\lambda_2^r$, where λ_1, λ_2 are the roots of $\lambda^2 + 2h\lambda - 1 = 0$. [3]
 - Show that $\lambda_1 = -h + \sqrt{h^2 + 1}$ and $\lambda_2 = -h - \sqrt{h^2 + 1}$, and that $\lambda_1\lambda_2 = -1$. [2]
 - Show that $0 < \lambda_1 < 1$ and $|\lambda_2| > 1$ for every $h > 0$, and explain why λ_1 gives a good approximation to e^{-h} for small h . [3]
 - Take $h = 0.1$. Find A and B , and hence show that the numerical solution at $x = 10$ ($r = 100$) is dominated by the term $B\lambda_2^r$, and say what this means for the accuracy. [3]

- 29** The fourth-order Runge-Kutta method is used for $\frac{dy}{dx} = y$, $y(0) = 1$, so that $y_{n+1} = r(h)y_n$, where $r(h) = 1 + h + \frac{h^2}{2} + \frac{h^3}{6} + \frac{h^4}{24}$.
- Find the RK4 estimates of $y(1)$ with $h = 0.5, 0.25$ and 0.125 , to 7 decimal places. [3]
 - Calculate the errors $e = y(1) - y_n$ for the three step sizes and the ratios of successive errors. [3]
 - Deduce the order of the global error of RK4 and explain how the ratios show it. [2]
 - Show that $r(h) = e^h - \frac{h^5}{120} - \dots$, and use it to explain the error found at $h = 0.125$. [2]
-
- 30** For the test equation $\frac{dy}{dx} = -10y$, $y(0) = 1$, the exact solution is e^{-10x} , which decays to 0. Two methods with step h are compared.
- Show that Euler's method gives $y_{n+1} = (1 - 10h)y_n$, and that the estimates decay to 0 only if $h < 0.2$. [2]
 - The trapezium (corrector) method is $y_{n+1} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_{n+1}, y_{n+1})]$, used with y_{n+1} on both sides. Show that for this equation $y_{n+1} = \frac{1 - 5h}{1 + 5h}y_n$. [3]
 - Show that $|\frac{1 - 5h}{1 + 5h}| < 1$ for every $h > 0$, so that the trapezium method decays for any step size. [2]
 - Calculate y_4 for $h = 0.5$ by each method, and compare with the exact value of $y(2)$. [2]

Numerical Solutions of Differential Equations

- 1** $\frac{dy}{dx} = x^2 + y$, $y(0) = 1$. Using the approximation $y_{r+1} \approx y_r + hy'_r$ with $h = 0.1$, estimate $y(0.1)$. [2]
-
- 2** $\frac{dy}{dx} = x + y$. Given that $y(0) = 1$ and $y(0.1) = 1.11$, use the approximation $y'_r \approx \frac{y_{r+1} - y_{r-1}}{2h}$ with $h = 0.1$ to estimate $y(0.2)$. [2]
-
- 3** $\frac{d^2y}{dx^2} = -y$, with $y(0) = 1$ and $y(0.1) = 0.995$. Use the approximation $y''_r \approx \frac{y_{r+1} - 2y_r + y_{r-1}}{h^2}$ with $h = 0.1$ to estimate $y(0.2)$ to 5 decimal places. [3]
-
- 4** $\frac{dy}{dx} = x + y$, $y(0) = 1$. Find $y''(0)$ and $y'''(0)$, and use them in the Taylor series $y(x) \approx y(0) + xy'(0) + \frac{x^2}{2}y''(0) + \frac{x^3}{6}y'''(0)$ to estimate $y(0.1)$ to 5 decimal places. [3]
-
- 5** Write the second-order differential equation $\frac{d^2y}{dx^2} = xy - 2\frac{dy}{dx}$ as a pair of first-order differential equations, using $v = \frac{dy}{dx}$. [2]
-
- 6** $\frac{dy}{dx} = \frac{2x + y^2}{x + y}$, with $y = 2$ when $x = 1$. Using the approximation $y_{r+1} \approx y_r + hy'_r$, the value of y when $x = 1 + h$ is estimated as 2.5. Find h . [3]
-
- 7** The curve $y = y(x)$ passes through $(2, 1)$ and satisfies $\frac{dy}{dx} = \ln(2x + y)$. Use the approximation $y'_r \approx \frac{y_{r+1} - y_r}{h}$ with $h = 0.1$ to find $y(2.2)$, correct to 3 decimal places. [4]

- 8** $\frac{dy}{dx} = 3x^2 - 2y^2$, $y(1) = 0.5$. Use the approximation $y'_r \approx \frac{y_{r+1} - y_r}{h}$ with $h = 0.05$ to estimate $y(1.1)$ to 4 decimal places. (No credit will be given for solving the differential equation analytically.) [4]
- 9** $\frac{dy}{dx} = \cos(x^2 + y^2)$, $y(0.5) = 1$. Use $y_{r+1} \approx y_r + hy'_r$ with $h = 0.02$ to estimate $y(0.56)$ to 4 decimal places. [4]
- 10** $\frac{d^2y}{dx^2} = \frac{x+1}{y^2}$, $y(1) = 2$, $y(1.1) = 2.15$. Use $y''_r \approx \frac{y_{r+1} - 2y_r + y_{r-1}}{h^2}$ with $h = 0.1$ to estimate $y(1.2)$ and $y(1.3)$ to 4 decimal places. [5]
- 11** $\frac{dy}{dx} = x + y^2$, $y(0) = 1$. By differentiating the equation, find the Taylor series for y in ascending powers of x up to and including the term in x^4 , and use it to estimate $y(0.2)$ to 4 decimal places. [5]
- 12** $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$, with $y(0) = 1$ and $y'(0) = 0$. Let $v = \frac{dy}{dx}$. Use $y_{r+1} = y_r + hv_r$ and $v_{r+1} = v_r + hv'_r$ with $h = 0.1$ to estimate $y(0.2)$ and $y'(0.2)$. [4]
- 13** $\frac{dy}{dx} = \frac{x+y}{k+y}$, where k is a positive constant, and $y(0) = 1$. Using $y_{r+1} \approx y_r + hy'_r$ with $h = 0.1$, the estimate of $y(0.1)$ is 1.025. Find the value of k , and hence estimate $y(0.2)$ to 4 decimal places. [5]
- 14** The curve $y = y(x)$ satisfies $\frac{dy}{dx} = \frac{5x^2 - y^2}{2xy}$, with $y(k) = 1$ where $k > 0$.
 (a) Use $y_{r+1} \approx y_r + hy'_r$ with $h = 0.1$ to find k , given that $y(k + 0.1) \approx 1.2$. [3]
 (b) Use your answer and the approximation $y'_r \approx \frac{y_{r+1} - y_{r-1}}{2h}$, with $h = 0.1$, to estimate $y(k + 0.2)$, to 3 decimal places. [3]
- 15** $\frac{d^2y}{dx^2} = y + x$, with $y(0) = 0$ and $y(1) = 1$. Using step $h = 0.25$ and $y''_i \approx \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$, find $y(0.25)$, $y(0.5)$ and $y(0.75)$ to 4 decimal places, by forming and solving three simultaneous equations. [7]
- 16** $\frac{d^2y}{dx^2} = x - y$, with $y(0) = 1$ and $y'(0) = 0$. Use $h = 0.1$.
 Use the Taylor series $y(h) \approx y(0) + hy'(0) + \frac{h^2}{2}y''(0)$ to estimate $y(0.1)$. Then use $y_{r+1} = 2y_r - y_{r-1} + h^2y''_r$ to estimate $y(0.2)$ and $y(0.3)$ to 5 decimal places. [6]
- 17** $\frac{dy}{dx} = x^2 + y^2$, $y(0) = 0$. Assume that $y = ax^3 + bx^7 + \dots$ for small x .
 (a) Substitute this series into the differential equation and equate coefficients of x^2 and x^6 to find a and b . [4]
 (b) Hence estimate $y(0.5)$ to 5 decimal places, and compare with the value 0.04179 from a more accurate numerical method. [2]
- 18** The method $y_{r+1} = y_r + \frac{h}{2}[3f(x_r, y_r) - f(x_{r-1}, y_{r-1})]$ is used for $\frac{dy}{dx} = x - y$ with $h = 0.1$, $y(0) = 1$ and $y(0.1) = 0.9097$. Calculate estimates of $y(0.2)$ and $y(0.3)$ to 4 decimal places, and compare $y(0.3)$ with the exact value $x - 1 + 2e^{-x}$ at $x = 0.3$. [6]

- 19** $\frac{dy}{dx} + 2y = 4x$, with $y(0) = 1$.
- (a) Solve the differential equation exactly using an integrating factor. [4]
- (b) Use $y_{r+1} \approx y_r + hy'_r$ with $h = 0.1$ to estimate $y(0.3)$, and find the error compared with the exact value. [3]
-
- 20** $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = x + y$, with $y(0) = 1$ and $y(0.1) = 1.1$. Use the approximations
- $$y''_{r+1} \approx \frac{y_{r+2} - 2y_{r+1} + y_r}{h^2}, y'_{r+1} \approx \frac{y_{r+2} - y_r}{2h},$$
- (a) Show that $(1 + h)y_{r+2} = 2y_{r+1} - (1 - h)y_r + h^2(x_{r+1} + y_{r+1})$. [3]
- (b) Use $h = 0.1$ to estimate $y(0.2)$ and $y(0.3)$ to 4 decimal places. [3]
-
- 21** $\frac{d^2y}{dx^2} = 2 + x \cos y$, with $y = 1$ and $\frac{dy}{dx} = 0.5$ at $x = 2$. The solution is to be estimated at $x = 2.1, 2.2$ and 2.3 , using $h = 0.1$.
- (a) Explain why the recurrence $y_{r+1} = 2y_r - y_{r-1} + h^2y''_r$ cannot be used to find $y(2.1)$ directly. [1]
- (b) Use $y(2 + h) \approx y(2) + hy'(2) + \frac{h^2}{2}y''(2)$ to estimate $y(2.1)$ to 5 decimal places. [3]
- (c) Use the recurrence to estimate $y(2.2)$ and $y(2.3)$ to 5 decimal places. [4]
-
- 22** $\frac{dy}{dx} = x + y$, $y(0) = 1$.
- (a) By repeated differentiation, show that $y^{(n)}(0) = 2$ for $n \geq 2$, and write down the Taylor series of y up to the term in x^5 . [4]
- (b) Use the series to estimate $y(0.5)$ and $y(1)$, to 4 decimal places. [2]
- (c) The exact solution is $y = 2e^x - x - 1$. Find the error in each estimate and explain why the series is much less accurate at $x = 1$. [2]
-
- 23** Ravi solves $\frac{d^2y}{dx^2} = -4y$, with $y(0) = 0$ and $y'(0) = 2$, using $h = 0.1$. He estimates $y(0.1) = 0.2$ from $y(0) + hy'(0)$, and then writes:
- $$y_{r+1} = 2y_r - y_{r-1} + h(-4y_r), \text{ so } y(0.2) = 2(0.2) - 0 - 0.4(0.2) = 0.32$$
- (a) Identify the error in Ravi's recurrence. [1]
- (b) Calculate $y(0.2)$ and $y(0.3)$ correctly, to 5 decimal places. [3]
- (c) The exact solution is $y = \sin 2x$. Compare Ravi's $y(0.2)$ and the corrected $y(0.2)$ with the exact value. [2]
-
- 24** $\frac{d^2y}{dx^2} = y$, with $y(0) = 1$ and $y(1) = 2$. The exact solution is $y = \cosh x + B \sinh x$, where $B = \frac{2 - \cosh 1}{\sinh 1}$.
- (a) Use $y''_i \approx \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$ with $h = 0.5$ to estimate $y(0.5)$, to 4 decimal places. [3]
- (b) Repeat with $h = 0.25$ to estimate $y(0.5)$, by solving the three equations for $y(0.25)$, $y(0.5)$, $y(0.75)$. (Equations: $y_{i-1} - \frac{33}{16}y_i + y_{i+1} = 0$.) [3]
- (c) The errors are proportional to h^2 . Use $y_{\text{extrapolated}} = \frac{4y(h = 0.25) - y(h = 0.5)}{3}$ for $y(0.5)$ and compare all three with the exact value 1.3302. [2]
-
- 25** During an epidemic in a school, S is the number of susceptible pupils, I the number infected, and t the time in days. The model is $\frac{dS}{dt} = -0.002SI$, $\frac{dI}{dt} = 0.002SI - 0.4I$, with $S = 400$ and $I = 10$ at $t = 0$.
- Use Euler's method with step 1 day to estimate S and I after 3 days, giving S and I to 1 decimal place at each stage. [8]

- 26** A predictor-corrector method for $\frac{dy}{dx} = f(x, y)$ uses the predictor $y_{r+1}^* = y_{r-1} + 2hf(x_r, y_r)$ and the corrector $y_{r+1} = y_{r-1} + \frac{h}{3}[f(x_{r-1}, y_{r-1}) + 4f(x_r, y_r) + f(x_{r+1}, y_{r+1}^*)]$. For $\frac{dy}{dx} = x + y$ with $h = 0.1$, $y(0) = 1$ and $y(0.1) = 1.1103$, find $y(0.2)$ and $y(0.3)$ to 5 decimal places, and compare with the exact values $2e^x - x - 1$. [8]
-
- 27** The equation $\frac{d^2y}{dx^2} = y$, with $y(0) = 1$ and $y'(0) = 0$, has exact solution $y = \cosh x$. The central difference approximation $y''_r \approx \frac{y_{r+1} - 2y_r + y_{r-1}}{h^2}$ gives a recurrence for the estimates y_r , with $y_0 = 1$ and $y_1 = 1 + \frac{h^2}{2}$.
- (a) Show that $y_{r+1} = (2 + h^2)y_r - y_{r-1}$. [1]
- (b) Let $\cosh \theta = 1 + \frac{h^2}{2}$. Show that $y_r = \cosh(r\theta)$ satisfies the recurrence and both starting values. [4]
- (c) For $h = 0.25$ find θ and the estimate of $y(1)$, and compare with $\cosh 1 = 1.54308$. [3]
- (d) Use $\cosh \theta = 1 + \frac{h^2}{2}$ with $\theta = h - ch^3 + \dots$ to show that $c = \frac{1}{24}$, and so explain why the error is of order h^2 . [2]
-
- 28** $\frac{d^2y}{dx^2} + \frac{dy}{dx} = x$, with $y(0) = 0$ and $y'(0) = 1$.
- (a) Use Taylor series for $y(x + h)$ and $y(x - h)$ to justify the approximations $y''_r \approx \frac{y_{r+1} - 2y_r + y_{r-1}}{h^2}$ and $y'_r \approx \frac{y_{r+1} - y_{r-1}}{2h}$. [3]
- (b) Hence show that $(2 + h)y_{r+1} = 2h^2x_r + 4y_r - (2 - h)y_{r-1}$. [3]
- (c) With $h = 0.1$, use $y(h) \approx y(0) + hy'(0) + \frac{h^2}{2}y''(0)$ to estimate $y(0.1)$, and then the recurrence to estimate $y(0.2)$ and $y(0.3)$, to 5 decimal places. [3]
- (d) The exact solution is $y = \frac{x^2}{2} - x + 2 - 2e^{-x}$. Find the errors in your estimates at $x = 0.1, 0.2$ and 0.3 . [1]
-
- 29** The Taylor series method of order p for $\frac{dy}{dx} = y$ uses $y_{n+1} = (1 + h + \frac{h^2}{2!} + \dots + \frac{h^p}{p!})y_n$.
- (a) Show how this formula follows from the Taylor series $y(x + h) = y + hy' + \frac{h^2}{2}y'' + \dots + \frac{h^p}{p!}y^{(p)}$. [2]
- (b) For $h = 0.5$ calculate the one-step error $e^{0.5} - T_p(0.5)$ for $p = 1, 2, 3, 4$ to 3 significant figures. [3]
- (c) Show that the leading error term is $\frac{h^{p+1}}{(p+1)!}$, and find the smallest p for which the one-step error is below 10^{-6} when $h = 0.5$. [3]
- (d) Use two steps of the order 4 method with $h = 0.5$ to estimate $y(1)$, and find the error compared with e . [2]

30

A pendulum has angle θ radians from the vertical at time t seconds, where $\frac{d^2\theta}{dt^2} = -\sin \theta$. Initially $\theta = 1$ and $\frac{d\theta}{dt} = 0$.

- (a) Let $\omega = \frac{d\theta}{dt}$. Write the equation as a pair of first-order equations. [1]
- (b) Use the improved Euler method with $h = 0.1$ for three steps to estimate θ and ω at $t = 0.3$, to 5 decimal places. Show the predictor values at each step. [6]
- (c) For small oscillations $\sin \theta \approx \theta$, giving $\theta = \cos t$. Calculate $\cos 0.3$ and say whether it is a good model for these oscillations. [2]
- (d) A highly accurate numerical solution gives $\theta(0.3) = 0.962289$. Find the error in your improved Euler estimate. [1]

Error Analysis

- 1 The number $\pi = 3.14159\dots$ is rounded to 3 decimal places, giving 3.142. Find the absolute error and the relative error in using 3.142 for 3.14159, giving each to 2 significant figures. [2]
- 2 A length is measured as $x = 5.6$ cm, correct to 1 decimal place. Write down the range of possible values of x as an inequality. [2]
- 3 The fraction $\frac{22}{7}$ is used as an approximation to $\pi = 3.14159265\dots$ Find the absolute error and the percentage error, giving the percentage error to 2 significant figures. [3]
- 4 The number $e = 2.718281828\dots$ is approximated to 3 decimal places, once by rounding and once by truncating (cutting off the digits after the third decimal place).
 - (a) Write down the two approximations, and find the absolute error in each. [2]
 - (b) Give an example of a number for which truncating to 3 decimal places gives a larger error than rounding to 3 decimal places. [1]
- 5 $a = 3.4$ and $b = 2.1$ are each correct to 1 decimal place. Find the least and greatest possible values of $a + b$. [2]
- 6 $\sqrt{10}$ is approximated by 3.16. Calculate the absolute error and the percentage error to 2 significant figures. [2]
- 7 $a = 4.6$ and $b = 2.3$, each rounded to 2 significant figures.
 - (a) Find the greatest and least possible values of ab . [2]
 - (b) Find the greatest and least possible values of $\frac{a}{b}$. [2]
- 8 The sides of a rectangle are measured as 8.5 cm and 4.2 cm, each correct to 1 decimal place. Find the upper and lower bounds for the area, and explain how many significant figures you can safely quote for the area. [4]
- 9 The series $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$ is truncated after the term in x^3 , and used to estimate $e^{0.5}$.
 - (a) Calculate the estimate and the true error, given $e^{0.5} = 1.648721$. [2]
 - (b) Show that the error is approximately equal to the next term in the series, and give this as a percentage of the estimate. [2]
- 10 The radius of a circular disc is measured as 5.0 cm with a possible percentage error of 2%. Use $A = \pi r^2$ to estimate the greatest possible percentage error in the area, and the greatest possible error in A . [4]

- 11** The radius of a circle is $r = 10.0 \pm 0.1$ cm. Use the approximation $\delta A \approx \frac{dA}{dr} \delta r$ to estimate the maximum error in the area $A = \pi r^2$, and the relative error in A . [4]
-
- 12** $x = 1.234$ and $y = 1.230$ are each correct to 4 significant figures. Find the greatest and least possible values of $x - y$, and explain why the relative error in $x - y$ is much larger than the relative error in x or y . [4]
-
- 13** A calculator works to 6 significant figures. To find $\sqrt{10001} - \sqrt{10000}$ a student uses $\sqrt{10001} = 100.005$ (6 s.f.) and $\sqrt{10000} = 100.000$.
 (a) Show that this gives 0.005 and say why the answer is unreliable. [2]
 (b) Show that $\sqrt{x+1} - \sqrt{x} = \frac{1}{\sqrt{x+1} + \sqrt{x}}$ and use it to find the value correct to 3 significant figures. [3]
-
- 14** Values of a function are given to 4 decimal places, so each has an error of at most 0.00005. The forward difference $\frac{f(x+h) - f(x)}{h}$ is used with $h = 0.01$. Find the maximum possible error in the estimate of $f'(x)$ caused by rounding, and explain what happens as h is made smaller. [4]
-
- 15** $a = 12.4$ and $b = 3.7$ are correct to 3 significant figures. Find the greatest and least possible values of $\frac{a-b}{a+b}$, giving each to 4 decimal places. [5]
-
- 16** A cylinder has radius $r = 4.0$ cm and height $h = 10.0$ cm, with r correct to ± 0.05 cm and h to ± 0.1 cm. Use $\delta V \approx \frac{\partial V}{\partial r} \delta r + \frac{\partial V}{\partial h} \delta h$ for $V = \pi r^2 h$ to estimate the greatest error in V , and the percentage error. [5]
-
- 17** The sum $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is approximated by the first 10 terms.
 (a) Calculate the approximation to 5 decimal places. [2]
 (b) Show that the truncation error is less than $\int_{10}^{\infty} \frac{1}{x^2} dx$ and find this bound. Compare with the actual error $\frac{\pi^2}{6} - 1.54977$. [3]
-
- 18** Show that if x and y have small relative errors ε_x and ε_y , then the relative error in the product xy is approximately $\varepsilon_x + \varepsilon_y$, and in the quotient $\frac{x}{y}$ is approximately $\varepsilon_x - \varepsilon_y$ (so at most $|\varepsilon_x| + |\varepsilon_y|$ in size).
 Hence find the greatest percentage error in $\frac{xy}{z}$ when x, y, z each have a relative error of at most 1%. [5]
-
- 19** $\sin x$ is approximated by $x - \frac{x^3}{6}$ for small x .
 (a) Find the approximation to $\sin 0.1$ and its absolute error, given that $\sin 0.1 = 0.099833417$. [2]
 (b) The next term is $\frac{x^5}{120}$. Show that the error is less than 10^{-7} for $|x| \leq 0.1$ and explain why. [2]
-

- 20** The iteration $x_{n+1} = e^{-x_n}$ with $x_0 = 0.5$ converges to α , where $\alpha = e^{-\alpha}$. Near α the error satisfies $e_{n+1} \approx -\alpha e_n$, using $g'(\alpha) = -e^{-\alpha} = -\alpha$.
- (a) Given $\alpha = 0.567143$, find $e_0 = x_0 - \alpha$ and the predicted value of e_3 to 2 significant figures. [3]
- (b) Calculate x_3 and its actual error. [2]
-
- 21** The trapezium rule with n strips estimates $I = \int_0^{\pi/2} \sin x \, dx = 1$.
- (a) Find the estimates T_2 and T_4 to 5 decimal places and the errors. [3]
- (b) Show that the ratio of the errors is about 4, and use it to deduce that $E \approx Ch^2$. [2]
- (c) Use the error bound $|E| \leq \frac{(b-a)h^2}{12}M$ with $M = 1$ to find the bound for T_4 and check that it exceeds the actual error. [2]
-
- 22** The quadratic equation $x^2 - 5000x + 1 = 0$ has a large root and a small root. A student uses the formula $x = \frac{5000 \pm \sqrt{5000^2 - 4}}{2}$ on a calculator that keeps only 6 significant figures at every stage.
- (a) Show that the student obtains 0 for the smaller root. [3]
- (b) Use the product of the roots to find the smaller root correctly, to 3 significant figures, and explain why the formula method failed. [3]
-
- 23** T_4 and T_8 are the trapezium estimates of $\int_0^1 \frac{1}{1+x^2} \, dx = \frac{\pi}{4}$.
- (a) Calculate T_4 and T_8 to 6 decimal places. [3]
- (b) Given that the error is proportional to h^2 , show that $\frac{4T_8 - T_4}{3}$ is a better estimate and evaluate it. [3]
- (c) Compare the errors in T_8 and the improved value with $\frac{\pi}{4} = 0.785398$. [1]
-
- 24** In an iteration $x_{n+1} = g(x_n)$ with $|g'| \leq k < 1$ near the fixed point α , each calculation has a rounding error of at most ε . Show that the total error satisfies $|e_n| \leq k^n|e_0| + \frac{\varepsilon(1-k^n)}{1-k}$, and find the limiting bound as $n \rightarrow \infty$ when $k = 0.5$ and $\varepsilon = 5 \times 10^{-6}$. [6]
-
- 25** Consider the system $x + y = 2$, $x + 1.0001y = 2.0001$.
- (a) Solve the system exactly. [2]
- (b) The right-hand side 2.0001 is replaced by 2.0002 (a relative change of about 0.005%). Solve again and comment. [3]
-
- 26** Euler's method is applied to $\frac{dy}{dx} = y$, $y(0) = 1$ to estimate $y(1) = e$, with N steps of size $h = \frac{1}{N}$, giving $(1 + \frac{1}{N})^N$. Find the least N for which the relative error is below 1%, and explain why the error is approximately $\frac{1}{2N}$ relative. [6]
-
- 27** For small h , the central difference $D(h) = \frac{f(x+h) - f(x-h)}{2h}$ has truncation error $\approx \frac{h^2}{6}M_3$ where M_3 bounds $|f'''|$, and each function value has rounding error at most ε .
- (a) Show that the total error is bounded by $E(h) = \frac{M_3h^2}{6} + \frac{\varepsilon}{h}$. [2]
- (b) Show that $E(h)$ is least when $h = (3\varepsilon/M_3)^{1/3}$, and find the least value of E . [4]
- (c) For $\varepsilon = 5 \times 10^{-9}$ and $M_3 = 1$, find the optimal h and the least error to 2 significant figures, and explain why using $h = 10^{-10}$ would be a mistake. [3]

- 28** The sequence $x_{n+1} = \cos x_n$, $x_0 = 1$, converges slowly to $\alpha = 0.739085$. Aitken's formula $x'_n = x_{n+2} - \frac{(x_{n+2} - x_{n+1})^2}{(x_{n+2} - x_{n+1}) - (x_{n+1} - x_n)}$ accelerates convergence.
- (a) Calculate $x_1, \dots, x_4$ to 6 decimal places. [2]
- (b) Calculate x'_0, x'_1, x'_2 to 6 decimal places. [4]
- (c) Compare the errors in x_4 and x'_2 (both use x_0 to x_4), and explain why the method works when $e_{n+1} \approx \lambda e_n$. [3]
-
- 29** The polynomial $p(x) = (x - 1)(x - 2)(x - 3)$ has a root at $x = 3$. A small perturbation ϵx^2 is added, so that $p(x) + \epsilon x^2 = 0$ has a root $3 + \delta$.
- (a) Show that $\delta \approx -\frac{9\epsilon}{p'(3)}$ to first order in ϵ , and find $p'(3)$. [3]
- (b) Hence estimate δ when $\epsilon = 0.001$, and compare with the exact root of the perturbed equation, 2.99548 (to 5 d.p.). [2]
- (c) Now consider $p(x) = (x - 1)(x - 1.01)(x - 1.02)$, whose roots are close together. Estimate the change in the root near 1.01 when ϵx^2 is added with $\epsilon = 10^{-8}$, and compare it with the answer in (b) by finding the factor by which each change in the root exceeds ϵ . Explain why root-finding is ill-conditioned when roots are close. [3]
-
- 30** For Euler's method applied to $y' = f(x, y)$ with $|f_y| \leq L$ and $|y''| \leq M$, the global error at x_n satisfies $|e_n| \leq \frac{Mh}{2L}(e^{L(x_n - x_0)} - 1)$. Apply this to $y' = y$, $y(0) = 1$ ($L = 1$) on $0 \leq x \leq 1$ where $M = e$.
- (a) Find the bound on $|e_N|$ for $h = 0.1$ and compare with the actual error $e - 1.1^{10}$. [3]
- (b) Find the largest h of the form $\frac{1}{N}$ for which the bound guarantees an error below 0.01, and state N . [3]
- (c) Explain why the bound is pessimistic by comparing with the actual smallest N giving error below 0.01. [2]

Approximation Methods

- 1** Use the linear approximation $f(x) \approx f(a) + (x - a)f'(a)$ with $f(x) = \sqrt{x}$ and $a = 25$ to estimate $\sqrt{26}$ to 4 decimal places. [3]
- 2** Use the small angle approximations to estimate $\sin 0.1$ and $\cos 0.1$ (angle in radians). Give $\sin 0.1$ using $\sin x \approx x$ and $\cos 0.1$ using $\cos x \approx 1 - \frac{x^2}{2}$. [2]
- 3** Write down the first three non-zero terms of the Maclaurin series for e^x , and use them to estimate $e^{0.2}$ to 4 decimal places. [3]
- 4** Use the binomial approximation $(1 + x)^n \approx 1 + nx$ to estimate $(1.02)^5$ to 3 decimal places. [2]
- 5** Use linear interpolation between (1, 3) and (2, 7) to estimate $f(1.25)$. [2]
- 6** Find the quadratic Taylor polynomial for $f(x) = \ln(1 + x)$ about $x = 0$ and use it to estimate $\ln 1.1$ to 4 decimal places. [3]
- 7** Use the cubic Maclaurin polynomial for e^x to estimate $e^{-0.3}$, and find the absolute error given that $e^{-0.3} = 0.740818$. [3]
- 8** Write down the Maclaurin series for $\sin x$ up to and including x^5 , and use it to estimate $\sin 0.5$ to 5 decimal places. [3]

- 9** Use the first three terms of the binomial expansion of $(1 + x)^{1/2}$ to estimate $\sqrt{1.08}$ to 5 decimal places, and compare with the calculator value 1.03923. [4]
-
- 10** The points (0, 1), (1, 3), (2, 11) lie on the curve $y = f(x)$. Use linear interpolation between suitable points to estimate $f(0.5)$ and $f(1.5)$, and explain which estimate is likely to be less reliable. [4]
-
- 11** Use the approximation $(1 - x)^{-1/2} \approx 1 + \frac{x}{2} + \frac{3x^2}{8}$ to estimate $\frac{1}{\sqrt{0.99}}$ to 6 decimal places and the error. [3]
-
- 12** Show that for small x , $\frac{1 - \cos x}{x^2} \approx \frac{1}{2}$, using the Maclaurin series of $\cos x$. Evaluate $\frac{1 - \cos x}{x^2}$ for $x = 0.1$ to 5 decimal places. [4]
-
- 13** The values of $f(x) = x^3$ are known at $x = 1, 2, 3$. Use quadratic Lagrange interpolation through these points to estimate $f(2.5) = 15.625$ and find the error. [4]
-
- 14** Find the Maclaurin series for $e^x \cos x$ up to and including the term in x^4 , and use it to estimate $e^{0.2} \cos 0.2$ to 5 decimal places. [6]
-
- 15** Find the first three non-zero terms of the Maclaurin series for $\ln(1 + \sin x)$, and use them to estimate $\ln(1 + \sin 0.1)$ to 6 decimal places. [6]
-
- 16** Use the series $e^u = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \dots$ with $u = -x^2$ to find a series for e^{-x^2} up to x^6 , and integrate it to estimate $\int_0^{0.5} e^{-x^2} dx$ to 5 decimal places. [6]
-
- 17** Use the Taylor polynomial of degree 2 for $f(x) = x^{1/3}$ about $x = 8$ to estimate $\sqrt[3]{8.5}$ to 5 decimal places. [5]
-
- 18** The Lagrange form of the linear interpolating polynomial through (x_0, y_0) and (x_1, y_1) is $p(x) = y_0 \frac{x - x_1}{x_0 - x_1} + y_1 \frac{x - x_0}{x_1 - x_0}$. Use quadratic interpolation through (0, 1), (1, e), (2, e^2) to estimate $e^{1.5}$ and compare with the exact value 4.4817. [5]
-
- 19** A quantity y varies with x as shown: $y(0) = 2.00$, $y(1) = 2.72$, $y(2) = 4.39$, $y(3) = 7.39$. Use Newton's forward difference formula, with the first and second differences taken from the first three values, to estimate $y(0.5)$ to 3 decimal places. [5]
-
- 20** Explain why the linear approximation $\sqrt{x+h} \approx \sqrt{x} + \frac{h}{2\sqrt{x}}$ always overestimates, by considering the second derivative of $\sqrt{x}$, and show that the error is less than $\frac{h^2}{8x^{3/2}}$ for $h > 0$. [5]
-
- 21** The degree-5 Maclaurin polynomial for e^x is used to estimate $e^{0.5}$. The Lagrange form of the remainder is $R_5 = \frac{e^\xi x^6}{6!}$ for some ξ between 0 and x .
- (a) Calculate the estimate to 7 decimal places. [2]
- (b) Using $e^\xi < e^{0.5} < 2$, show that the error is less than $\frac{2 \times 0.5^6}{720}$ and evaluate this bound. [3]
- (c) Compare with the actual error $e^{0.5} - \text{your estimate}$, and find the least degree n for which the bound $\frac{2 \times 0.5^{n+1}}{(n+1)!}$ is below 10^{-8} . [3]

22

The rational function $R(x) = \frac{1 + \frac{x}{2} + \frac{x^2}{12}}{1 - \frac{x}{2} + \frac{x^2}{12}}$ is a Padé approximation to e^x .

- (a) Show that $(1 - \frac{x}{2} + \frac{x^2}{12})(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots) = 1 + \frac{x}{2} + \frac{x^2}{12} + O(x^5)$, by expanding up to x^4 . [4]
- (b) Evaluate $R(1)$ and the cubic Maclaurin polynomial at $x = 1$, and compare both with $e = 2.718282$. [3]
- (c) Show that $R(-x) = \frac{1}{R(x)}$, and hence explain why R is a sensible approximation to e^x , which satisfies $e^{-x} = \frac{1}{e^x}$. [2]

23

The data points $(0, 1.1)$, $(1, 2.9)$, $(2, 5.2)$, $(3, 6.8)$ are to be approximated by the line $y = mx + c$ that minimises $S = \sum(y_i - mx_i - c)^2$.

- (a) Show that the conditions $\partial S / \partial m = 0$ and $\partial S / \partial c = 0$ give $m \sum x^2 + c \sum x = \sum xy$ and $m \sum x + 4c = \sum y$. [3]
- (b) Calculate m and c , and the sum of squares S . [4]

24

Show that the quadratic $p(x)$ passing through $(-h, y_{-1})$, $(0, y_0)$, (h, y_1) has $\int_{-h}^h p(x) dx = \frac{h}{3}(y_{-1} + 4y_0 + y_1)$. Hence use it to estimate $\int_0^2 \frac{1}{1+x} dx$ with $h = 1$, and compare with $\ln 3$. [7]

25

Stirling's approximation states $n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$.

- (a) Use it to estimate $5!$ and $10!$, and find the percentage error in each. [4]
- (b) The percentage error is about $\frac{100}{12n}$. Check this for $n = 10$, and use it to find the smallest n for which the error is below 0.1% . [3]

26

The function f takes the values $f(1) = 0$, $f(2) = 3$, $f(4) = 15$, $f(5) = 24$.

- (a) Construct the divided difference table and write the interpolating polynomial in Newton's form. [4]
- (b) Simplify $p(x)$, and estimate $f(3)$. [2]
- (c) Explain why the third divided difference being zero is to be expected from the data. [1]

27

A function f has $|f''(x)| \leq M$ on $[a, b]$, and p is the linear interpolant through $(a, f(a))$ and $(b, f(b))$.

- (a) The error is $f(x) - p(x) = \frac{f''(\xi)}{2}(x-a)(x-b)$. Show that the greatest value of $|(x-a)(x-b)|$ on $[a, b]$ is $\frac{(b-a)^2}{4}$. [2]
- (b) Deduce that $|f(x) - p(x)| \leq \frac{M(b-a)^2}{8}$. [1]
- (c) A table of $\ln x$ is to be used with linear interpolation. Find the largest spacing h (between $x = 1$ and $x = 2$) that guarantees an error below 5×10^{-5} . [3]
- (d) Check with $x = 1.5$, using the interpolant through $\ln 1.48$ and $\ln 1.52$: find the actual error to 2 significant figures. [3]

28

For small x , $f(x) = \frac{x}{e^x - 1}$.

- (a) Show that $f(x) = 1 - \frac{x}{2} + \frac{x^2}{12} + O(x^4)$ by expanding $e^x - 1$ and dividing. [4]
- (b) Evaluate the series at $x = 0.5$ and compare with the exact $f(0.5) = 0.770747$. [2]
- (c) Use the series to show that $\lim_{x \rightarrow 0} \frac{1}{x} \left(\frac{x}{e^x - 1} - 1 + \frac{x}{2} \right) = 0$ and evaluate $\lim_{x \rightarrow 0} \frac{1}{x^2} \left(\frac{x}{e^x - 1} - 1 + \frac{x}{2} \right)$. [3]

29

The series $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$ is used to estimate π .

- (a) Use $x = 1$ and the first 4 terms to estimate π to 4 decimal places, and explain why the convergence is slow. [3]
- (b) Machin's formula is $\pi = 16 \arctan(\frac{1}{5}) - 4 \arctan(\frac{1}{239})$. Use two terms of the series for the first part and one term for the second, giving an estimate to 6 decimal places. [3]
- (c) Explain why Machin's formula converges much faster, and find the error in (b) to 2 significant figures. [2]

30

The function e^x on $[0, 1]$ is to be approximated by a straight line $L(x) = mx + c$ so that the greatest value of $|e^x - L(x)|$ is as small as possible (a minimax approximation). For the best line, $e^x - L(x)$ has the same value at both end points, and has the same size but opposite sign at the interior point where $e^x - L(x)$ is stationary.

- (a) Show that the line is parallel to the chord from $(0, 1)$ to $(1, e)$, so $m = e - 1$. [2]
- (b) Find the x -coordinate x_1 of the interior stationary point of $e^x - L(x)$. [2]
- (c) Use the condition that the error at x_1 is the negative of the error at 0 to find c , and the maximum error. [4]
- (d) Compare with the Maclaurin linear approximation $1 + x$, whose greatest error on $[0, 1]$ is $e - 2 = 0.7183$. [2]