

## Maximum Flow Problems

### 1 (3 marks)

(a) The vertex from which the flow starts, with no flow entering it

B1

**Answer: Where the flow starts**

1 mark

(b) The vertex at which the flow ends, with no flow leaving it

B1

**Answer: Where the flow ends**

1 mark

(c) The maximum amount of flow that the arc can carry

B1

**Answer: Maximum flow along the arc**

1 mark

### 2 (2 marks)

Conservation of flow: flow in = flow out, so  $6 + 9 + 4 = 7 + x$

M1

$x = 12$

A1

**Answer:  $x = 12$**

2 marks

### 3 (1 mark)

Flow out of S =  $6 + 5 = 11$

B1

**Answer: 11**

1 mark

### 4 (2 marks)

The flow along each arc cannot exceed the capacity of that arc (and cannot be negative)

B1

At every vertex other than the source and sink, the total flow in equals the total flow out

B1

**Answer: Capacity constraint and conservation**

2 marks

### 5 (2 marks)

Arc SA has flow 7 but capacity 6

M1

The flow exceeds the capacity of SA, so the flow is not feasible

A1

**Answer: SA: flow 7 exceeds capacity 6**

2 marks

### 6 (2 marks)

Add a supersource S joined to A and to B by directed arcs SA and SB

B1

Capacity of SA is 15 and capacity of SB is 10

B1

**Answer: Supersource S with SA = 15, SB = 10**

2 marks

### 7 (4 marks)

Augmenting routes (route: amount added) SAT: 7; SBT: 8; SABT: 4, total 19

M1 A1

Cut with {S} on the source side and {A, B, T} on the sink side: arcs SA, SB, capacity  $11 + 8 = 19$

B1

Max flow = min cut, so the maximum flow is 19

A1

**Answer: Maximum flow 19**

4 marks

### 8 (4 marks)

Vertex A: flow in 9 = flow out  $y + 7$ , so  $y = 2$

M1 A1

Value of the flow = flow out of S =  $9 + x = 15$ , so  $x = 6$

M1

Vertex B: flow in  $x + y$  = flow out  $z$ , so  $z = 6 + 2 = 8$

A1

**Answer:  $x = 6$ ,  $y = 2$ ,  $z = 8$**

4 marks

### 9 (3 marks)

Augmenting routes (route: amount added) SABT: 6; SACT: 4, total 10

M1

Cut with {S, A, C} on the source side and {B, T} on the sink side: arcs AB, CT, capacity  $6 + 4 = 10$

B1

Max flow = min cut, so the maximum flow is 10

A1

**Answer: 10 litres per second**

3 marks

### 10 (5 marks)

(a) Supersource S with arcs SA of capacity 12 and SB of capacity 9

B1

Supersink T with arcs ET of capacity 8 and FT of capacity 10

B1

**Answer: SA = 12, SB = 9, ET = 8, FT = 10**

2 marks

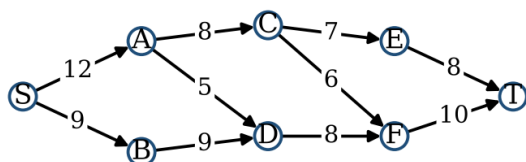
(b) Augmenting routes: SACET: 7; SACFT: 1; SADFT: 4; SBDFT: 4

M1 A1

Total = 16 pallets per hour (a cut of capacity 16 is AC, DF)

**Answer: 16 pallets per hour**

**B1**  
**3 marks**



### 11 (5 marks)

Network drawn with the seven directed arcs and their capacities shown in the model answer diagram

**B1 B1**

Augmenting routes:  $R \rightarrow 2 \rightarrow P$ : 15;  $R \rightarrow 1 \rightarrow 3 \rightarrow P$ : 25;  $R \rightarrow 2 \rightarrow 3 \rightarrow P$ : 15;  $R \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow P$ : 5

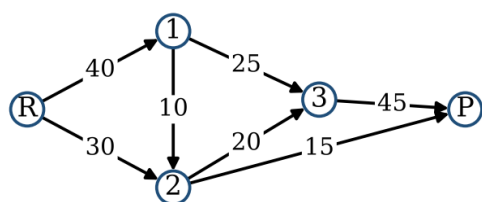
**M1 A1**

Cut  $\{1, 2, R\}$ : arcs 13, 23, 2P with capacity 60, so the maximum rate is 60 thousand barrels per hour

**B1**

**Answer: 60 thousand barrels per hour**

**5 marks**



### 12 (3 marks)

(a) Flow out of S =  $7 + 4 = 11$

**B1**

**Answer: 11**

**1 mark**

(b) Route SBT: SB has spare capacity  $6 - 4 = 2$  and BT has spare capacity  $8 - 6 = 2$

**M1**

Flow can be increased by 2 along SBT, so the flow is not maximal

**A1**

**Answer: SBT, increase by 2**

**2 marks**

### 13 (4 marks)

(a) Routes: SAT: 6; SBT: 7; SABT: 1

**M1**

Cut with  $\{S, A, B\}$  on the source side and  $\{T\}$  on the sink side: arcs AT, BT, capacity  $6 + 8 = 14$ , so the maximum flow is 14

**A1**

**Answer: 14**

**2 marks**

(b) After the change the only arcs into T are AT (capacity 6) and BT; B is now fed only by S, so BT carries at most  $SB = 7$

**M1**

Maximum flow =  $6 + 7 = 13$  (routes SAT: 6; SBT: 7)

**A1**

**Answer: 13**

**2 marks**

### 14 (7 marks)

(a) All flow into T passes along DT or ET

**M1**

Capacity of DT + capacity of ET =  $11 + 12 = 23$ , so the flow cannot exceed 23

**A1**

**Answer: Arcs into T have total capacity 23**

**2 marks**

(b) Augmenting routes: SADT: 7; SBDT: 4; SBET: 5; SCET: 7

**M1 A1**

Flows (arc: amount) AD 7, BD 4, BE 5, CE 7, DT 11, ET 12, SA 7, SB 9, SC 7

**A1**

Cut with  $\{S, A, B, C, D, E\}$  on the source side and  $\{T\}$  on the sink side: arcs DT, ET, capacity  $11 + 12 = 23$

**B1**

Maximum flow = 23 parcels per minute

**A1**

**Answer: Maximum flow 23**

**5 marks**

### 15 (6 marks)

(a) Replace B by B (receiving all arcs into B) and B' (sending all arcs out of B), joined by a new arc BB'

**M1**

BB' has capacity 9, so that no more than 9 tonnes per hour can pass through the depot

**A1**

**Answer: Arc BB' of capacity 9**

**2 marks**

(b) Routes in the new network: SACT: 5;  $S \rightarrow B \rightarrow B' \rightarrow T$ : 7;  $S \rightarrow B \rightarrow B' \rightarrow C \rightarrow T$ : 2

M1 A1

Cut with  $\{S, A, B\}$  on the source side and  $\{B', C, T\}$  on the sink side: arcs AC, BB', capacity  $5 + 9 = 14$

B1

Maximum flow = 14 tonnes per hour

A1

**Answer: 14 tonnes per hour**

4 marks

### 16 (6 marks)

The minimum flows on SA (2), AT (3), BC (1) and BT (2) mean that a feasible flow must be found first

M1

A feasible flow of value 17: AB 3, AT 6, BC 5, BT 6, CT 5, SA 9, SB 8

M1 A1

Check each arc lies between its minimum and maximum, and conservation holds at A, B and C

B1

Cut with S on the source side: capacity =  $(9 + 8 \text{ upper}) - (0 \text{ lower}) = 17$

M1

Maximum flow = 17

A1

**Answer: Maximum flow 17**

6 marks

### 17 (6 marks)

(a) Supersource S with arcs SP (14) and SQ (11); supersink T with arcs XT (12) and YT (9), as shown in the diagram

B1 B1

**Answer: SP = 14, SQ = 11, XT = 12, YT = 9**

2 marks

(b) Augmenting routes: SPJXT: 8; SPJYT: 2; SPKXT: 4; SQJYT: 1; SQKYT: 6

M1 A1

Cut with  $\{S, J, K, P, Q, X, Y\}$  on the source side and  $\{T\}$  on the sink side: arcs XT, YT, capacity  $12 + 9 = 21$

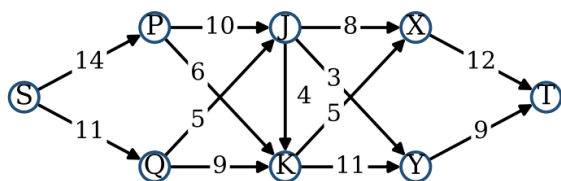
M1

Maximum delivery = 21 megalitres per day

A1

**Answer: 21 megalitres per day**

4 marks



### 18 (5 marks)

Network with a vertex for each junction and a directed arc for each road, as in the diagram

B1

Augmenting routes: ABDF: 25; ACDF: 5; ACEF: 15; ABCEF: 5

M1 A1

Cut with  $\{A\}$  on the source side and  $\{B, C, D, E, F\}$  on the sink side: arcs AB, AC, capacity  $30 + 20 = 50$

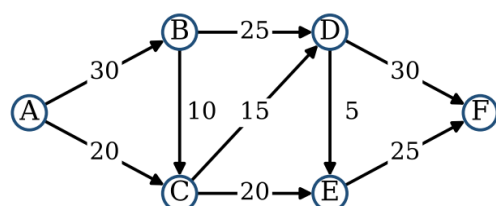
B1

Maximum flow = 50 cars per minute

A1

**Answer: 50 cars per minute**

5 marks



### 19 (5 marks)

Every unit of flow reaches T along AT, BT or CT, so flow  $\leq 7 + 6 + 7 = 20$

M1 A1

A flow of 20 exists: AT 7, BC 1, BT 6, CT 7, SA 7, SB 7, SC 6

M1 A1

Flow = 20 equals the upper bound, so the maximum flow is 20

A1

**Answer: Upper bound 20 is achieved**

5 marks

**20 (5 marks)**

Cut capacities with source side {S}: 15; {S, A}:  $6 + 4 + x = x + 10$ ; {S, B}:  $9 + 8 = 17$ ; {S, A, B}:  $x + 8$

M1 A1

Max flow = minimum cut capacity, and  $x + 8 < x + 10$ , so max flow =  $\min(15, 17, x + 8)$

M1

Max flow is  $13 < 15$ , so  $x + 8 = 13$

M1

$x = 5$

A1

**Answer:  $x = 5$**

5 marks

**21 (7 marks)**

(a) Replace AB by two directed arcs AB and BA, each of capacity 7

B1

If flow  $x$  went A to B and  $y$  went B to A, then replacing both by a single flow  $|x - y|$  in the larger direction gives the same flows at A and B (so the same through-flow) and uses less capacity

E1

**Answer: Two arcs AB and BA, each 7**

2 marks

(b) Routes in the modelled network: SAT: 5; SBT: 4; SBAT: 4

M1 A1

Cut with {S, A, B} on the source side and {T} on the sink side: arcs AT, BT, capacity  $9 + 4 = 13$

M1 A1

Maximum flow = 13. If AB were one-way A to B only, flow into T is at most  $SA + BT = 5 + 4 = 9$  since B could not pass flow to A, giving 9

A1

**Answer: 13 (and 9 if one-way)**

5 marks

**22 (8 marks)**

(a) Source side {S}: arcs SA, SB, capacity  $k + 6$

B1

{S, A}: arcs SB, AB, AT, capacity  $6 + 3 + 7 = 16$ ; {S, B}: arcs SA, BT, capacity  $2k$

B1

{S, A, B}: arcs AT, BT, capacity  $k + 7$

B1

**Answer:  $k + 6$ , 16,  $2k$ ,  $k + 7$**

3 marks

(b) Max flow =  $\min(k + 6, 16, 2k, k + 7)$  and  $k + 7 > k + 6$  so the fourth cut is never the minimum

M1

$2k < k + 6$  exactly when  $k < 6$ , so for  $k \leq 6$  the minimum is  $2k$

M1 A1

$k + 6 < 16$  exactly when  $k < 10$ , so for  $6 \leq k \leq 10$  the minimum is  $k + 6$

M1

For  $k \geq 10$  the minimum is 16

A1

**Answer:  $2k$  for  $k \leq 6$ ;  $k + 6$  for  $6 \leq k \leq 10$ ; 16 for  $k \geq 10$**

5 marks

**23 (7 marks)**

Routes with all pipes open: SBT: 5; SACT: 6; SBCT: 2

M1 A1

Cut with {S, A} on the source side and {B, C, T} on the sink side: arcs SB, AC, capacity  $7 + 6 = 13$

B1

Maximum flow with all open = 13

A1

Maximum flow with each pipe in turn removed (pipe: flow): SA: 7, SB: 6, AC: 7, BC: 11, BT: 8, CT: 5

M1 A1

Largest reduction is when CT is shut, a reduction of 8

A1

**Answer: Open: 13; shutting CT gives the greatest reduction, 8**

7 marks

**24 (8 marks)**

(a) Supersource S joined to D1, D2, D3 by arcs of capacity 14, 11, 9

B1

Supersink T joined from X, Y, Z by arcs of capacity 12, 10, 13, as in the diagram

B1

**Answer: SD1 = 14, SD2 = 11, SD3 = 9, XT = 12, YT = 10, ZT = 13**

2 marks

(b) Augmenting routes:  $S \rightarrow D1 \rightarrow X \rightarrow T$ : 7;  $S \rightarrow D1 \rightarrow Y \rightarrow T$ : 6;  $S \rightarrow D2 \rightarrow X \rightarrow T$ : 5;  $S \rightarrow D2 \rightarrow Z \rightarrow T$ : 6;

M1 A1

$S \rightarrow D3 \rightarrow Y \rightarrow T$ : 4;  $S \rightarrow D3 \rightarrow Z \rightarrow T$ : 5

Cut with {S, D1} on the source side and {D2, D3, T, X, Y, Z} on the sink side: arcs D1X, D1Y, SD2, SD3, capacity  $7 + 6 + 11 + 9 = 33$

M1

Maximum delivery = 33 pallets

A1

**Answer: 33 pallets**

4 marks

(c) Total demand =  $12 + 10 + 13 = 35$ , so shortfall =  $35 - 33 = 2$

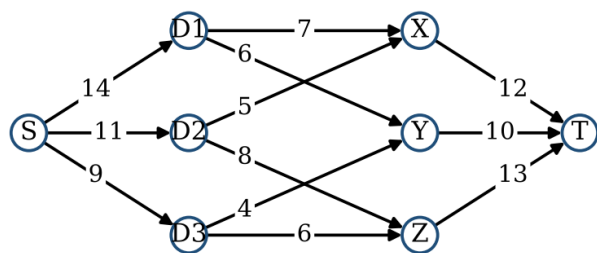
M1

The arcs of the minimum cut (D1X, D1Y, SD2, SD3) are the bottleneck

A1

**Answer: Shortfall of 2 pallets**

2 marks



### 25 (6 marks)

(a) The three routes share arcs (SA and AT are used by two routes, BT by two routes)

M1

SAT and SABT together send  $5 + 4 = 9$  units along SA (capacity 7), and SBT and SABT send 10 units along BT (capacity 6), so the flows exceed arc capacities

E1

**Answer: The routes share arcs, so their capacities cannot all be used**

2 marks

(b) Augmenting routes: SAT: 5; SBT: 6

M1 A1

Cut with  $\{S, A, B\}$  on the source side and  $\{T\}$  on the sink side: arcs AT, BT, capacity  $5 + 6 = 11$

M1

Maximum flow = 11

A1

**Answer: 11**

4 marks

### 26 (9 marks)

Add a supersource S (arcs SP = 10, SQ = 8) and a supersink T (arcs XT = 9, YT = 7)

B1

Without the limit at C the maximum flow would be 16, so the limit matters

B1

Split C into C and C' with an arc CC' of capacity 6; arcs into C go to C and arcs out of C come from C'

M1 A1

Augmenting routes: SPDXT: 3; SPDYT: 1; SQDYT: 5;  $S \rightarrow P \rightarrow C \rightarrow C' \rightarrow X \rightarrow T$ : 6

M1 A1

Cut with  $\{S, C, P, Q\}$  on the source side and  $\{C', D, T, X, Y\}$  on the sink side: arcs PD, QD, CC', capacity  $4 + 5 + 6 = 15$

M1 A1

Maximum power delivered = 15 units

A1

**Answer: 15 units**

9 marks

### 27 (8 marks)

(a) For every vertex  $v$  let  $\text{in}(v)$  be the total flow into  $v$  and  $\text{out}(v)$  the total flow out of  $v$ . Then conservation gives  $\text{in}(v) - \text{out}(v) = 0$  for every vertex other than S and T

B1

Sum over all vertices  $v$ :  $\sum (\text{in}(v) - \text{out}(v)) = \text{in}(S) - \text{out}(S) + \text{in}(T) - \text{out}(T) + 0$

M1

Each arc  $(u, v)$  with flow  $f$  contributes  $+f$  to  $\text{in}(v)$  and  $-f$  to  $\text{out}(u)$ , so each arc contributes  $+f - f = 0$  to the sum, so the sum is 0

M1 A1

No arc enters S and no arc leaves T, so  $\text{in}(S) = 0$  and  $\text{out}(T) = 0$ , giving  $-\text{out}(S) + \text{in}(T) = 0$ , that is  $\text{out}(S) = \text{in}(T)$

A1

**Answer: Proved:  $\text{out}(S) = \text{in}(T)$**

5 marks

(b) Flow out of S =  $7 + 5 = 12$

B1

Flow into T =  $8 + w = 12$ , so  $w = 4$

M1

Check at D:  $\text{in} = \text{BD} + \text{CD} = 4 + 0 = 4 = \text{DT}$ , so  $w = 4$

A1

**Answer:  $w = 4$**

3 marks

### 28 (9 marks)

(a) The only route through  $A_i$  is  $SA_iT$ , so the flow through  $A_i$  is at most  $\min(a_i, b_i)$  by the capacities of its two arcs

M1

The routes are independent (they share no arcs), so the total flow is at most  $\sum \min(a_i, b_i)$

A1

Sending  $\min(a_i, b_i)$  along each route  $SA_iT$  is feasible and has this value

M1

Hence the maximum flow is exactly  $\sum_{i=1}^n \min(a_i, b_i)$

A1

**Answer: Proved**

4 marks

(b) min values: 7, 6, 10, 5

M1

Maximum flow =  $7 + 6 + 10 + 5 = 28$

A1

**Answer: 28**

2 marks

(c)  $A_1$  receives 9 but can pass only 7 to T, so 2 units are spare;  $A_2$  can send 8 to T but receives

M1

only 6, so 2 units of spare capacity

Route  $SA_1A_2T$  carries 2 extra units (limited by the spare capacities), new total = 30

A1

Cut with  $\{S, A_3\}$  on the source side and  $\{A_1, A_2, A_4, T\}$  on the sink side: arcs  $SA_1, SA_2, A_3T, SA_4$ , capacity  $9 + 6 + 10 + 5 = 30$

B1

**Answer: 30**

3 marks

## 29 (12 marks)

(a) Split D into D (receiving the arcs BD and CD) and D' (sending the arc DT)

M1

Join them by an arc DD' with capacity 9

A1

**Answer: Arc DD' of capacity 9**

2 marks

(b) A flow of value 3: AB 1, AC 2, BD 1, CD 2, DD' 3, D'T 3, SA 3

M1 A1

Consider the cut with source side  $\{S, A, B\}$ : the arcs forward across the cut AC, BC, BD have minimum flows summing to 3 and the arcs back across the cut (none) carry at most 0

M1

So the flow is at least 3, hence the minimum is 3

A1

**Answer: Minimum 3**

4 marks

(c) Feasible flow of value 18: AB 4, AC 7, BC 5, BD 6, CD 3, CT 9, DD' 9, D'T 9, SA 11, SB 7

M1 A1 B1

Cut with source side  $\{S, A, B, C, D\}$ : forward arcs CT, DD' have capacities summing to 18; the arcs back across the cut (none) have minimum flow 0

M1 A1

Flow  $\leq 18 - 0 = 18$ , so the maximum flow is 18

A1

**Answer: Maximum 18**

6 marks

## 30 (10 marks)

(a)  $\{S\}$ : arcs SA, SB with capacity  $p + (14 - p) = 14$

B1

$\{S, A\}$ : arcs SB, AB, AT with capacity  $(14 - p) + 3 + 8 = 25 - p$

B1

$\{S, B\}$ : arcs SA, BT with capacity  $p + 9$

B1

$\{S, A, B\}$ : arcs AT, BT with capacity  $8 + 9 = 17$

B1

**Answer: 14,  $25 - p$ ,  $p + 9$ , 17**

4 marks

(b) Max flow =  $\min(14, 25 - p, p + 9, 17)$

M1

This equals 14 when  $25 - p \geq 14$  and  $p + 9 \geq 14$ , that is  $p \leq 11$  and  $p \geq 5$

M1

$5 \leq p \leq 11$  (integers)

A1

**Answer:  $5 \leq p \leq 11$**

3 marks

(c)  $p + 9 = 11$  is the smallest of the cut capacities (the others are 14, 23 and 17), so the maximum flow is 11

M1 A1

A flow of 11: SA 2, AT 2, SB 9, BT 9, AB 0 (SB cannot carry more than 9 usefully since B can only send flow on to T)

B1

**Answer: Max flow 11**

3 marks

## Minimum Cut Problems

### 1 (2 marks)

(a) A set of arcs whose removal separates S from T (equivalently, a partition of the vertices into two sets, one containing S and the other containing T)

B1

**Answer: Separates S from T**

1 mark

(b) The sum of the capacities of the arcs which are directed from the set containing S to the set containing T

B1

**Answer: Sum of capacities of arcs from the S side to the T side**

1 mark

### 2 (2 marks)

Arcs SB, AB and AT cross from  $\{S, A\}$  to  $\{B, T\}$

M1

Capacity =  $6 + 3 + 5 = 14$

A1

**Answer: SB, AB, AT; capacity 14**

2 marks

**3 (2 marks)**

Arcs from  $\{S, B\}$  to  $\{A, T\}$ : SA (8) and BT (7); arc AB runs from the sink side to the source side so it is not counted

M1

Capacity =  $8 + 7 = 15$

A1

**Answer: 15**

2 marks

**4 (2 marks)**

In any network the maximum value of a flow from S to T is equal to the minimum capacity of a cut separating S from T

B1

B1

**Answer: Max flow = min cut**

2 marks

**5 (2 marks)**

The flow is a maximum flow and the cut is a minimum cut

B1

Every flow has value at most the capacity of any cut, so a flow equal to a cut capacity cannot be improved

E1

**Answer: Flow is maximal, cut is minimal**

2 marks

**6 (2 marks)**

The flow that crosses the cut from the source side to the sink side is at most the sum of the capacities of the forward arcs

M1

Flow along a backward arc crosses the cut from the sink side to the source side, so it subtracts from the net flow across the cut and cannot add to the amount that the cut can carry

E1

**Answer: Backward arcs only reduce the net flow across the cut**

2 marks

**7 (4 marks)**

$\{S\}$ : SA (6) + SB (4) = 10

B1

$\{S, A\}$ : SB (4) + AB (2) + AT (3) = 9

B1

$\{S, B\}$ : SA (6) + BT (5) = 11

B1

$\{S, A, B\}$ : AT (3) + BT (5) = 8

B1

**Answer: Minimum cut capacity 8 (source side  $\{S, A, B\}$ )**

4 marks

**8 (3 marks)**

Value of the flow = flow out of S = 15

B1

Cut with  $\{S, A, B\}$  on the source side and  $\{T\}$  on the sink side: arcs AT, BT, capacity  $6 + 9 = 15$

M1 A1

**Answer: Cut capacity 15 = flow value**

3 marks

**9 (2 marks)**

Each cut contains exactly one of the four arcs, so the minimum cut is the arc of smallest capacity, AB

M1

Minimum cut capacity = 7, so the maximum flow is 7 boxes per minute

A1

**Answer: 7**

2 marks

**10 (4 marks)**

Augmenting routes: SPCXT: 6; SPDXT: 2; SPDYT: 3; SQDYT: 2

M1 A1

Cut with  $\{S, C, D, P, Q, X, Y\}$  on the source side and  $\{T\}$  on the sink side: arcs XT, YT, capacity  $8 + 5 = 13$

M1

Maximum flow = 13 vans per day

A1

**Answer: Min cut capacity 13**

4 marks

**11 (4 marks)**

Augmenting routes:  $R \rightarrow 2 \rightarrow F$ : 7;  $R \rightarrow 1 \rightarrow 3 \rightarrow F$ : 6;  $R \rightarrow 2 \rightarrow 3 \rightarrow F$ : 3

M1 A1

Cut with  $\{1, 2, 3, R\}$  on the source side and  $\{F\}$  on the sink side: arcs 2F, 3F, capacity  $7 + 9 = 16$

M1

The bottleneck canals are 2F, 3F, giving a maximum flow of 16 megalitres per day

A1

**Answer: Bottleneck: 2F, 3F, 16 megalitres per day**

4 marks

**12 (4 marks)**

$C_1$ : the arcs from  $\{S, A\}$  to  $\{B, T\}$  are SB (6) and AT (8); BA runs from the sink side to the source side so is not counted

M1 A1

$C_1$  capacity =  $6 + 8 = 14$

B1

$C_2$ : arcs from  $\{S, B\}$  to  $\{A, T\}$  are SA (9), BA (4) and BT (5), capacity 18; so  $C_1$  is smaller

A1

**Answer:  $C_1 = 14$ ,  $C_2 = 18$ ;  $C_1$  is smaller**

4 marks

**13 (4 marks)**

(a) Compare flow and capacity on each arc

M1

Saturated arcs: AC, CT

A1

**Answer: AC, CT**

2 marks

(b) Cut with  $\{S, A, B, C\}$  on the source side and  $\{T\}$  on the sink side: arcs CT, capacity  $9 = 9$ ; all of these arcs are saturated

M1

Capacity of the cut =  $9 =$  value of the flow (flow out of S is 9)

A1

**Answer: Cut of capacity 9**

2 marks

**14 (6 marks)**

Augmenting routes: SADT: 6; SBDT: 4; SBET: 3; SCET: 6

M1 A1

Cut with  $\{S, A\}$  on the source side and  $\{B, C, D, E, T\}$  on the sink side: arcs SB, SC, AD, capacity  $7 + 6 + 6 = 19$

M1 A1

Pipes in the minimum cut: SB, SC, AD

B1

Maximum flow = minimum cut = 19 litres per second

A1

**Answer: Minimum cut SB, SC, AD, capacity 19**

6 marks

**15 (6 marks)**

$\{S\}$ : SA (10) + SB (8) + SC (6) = 24

B1

$\{S, A\}$ : SB (8) + SC (6) + AB (3) + AT (7) = 24

B1

$\{S, B\}$ : SA (10) + SC (6) + BC (4) + BT (5) = 25

B1

$\{S, C\}$ : SA (10) + SB (8) + CT (9) = 27

B1

$\{S, A, B\}$ : SC (6) + BC (4) + AT (7) + BT (5) = 22 ;  $\{S, A, C\}$ : SB (8) + AB (3) + AT (7) + CT (9) = 27 ;  $\{S, B, C\}$ : SA (10) + BT (5) + CT (9) = 24 ;  $\{S, A, B, C\}$ : AT (7) + BT (5) + CT (9) = 21

B1

The smallest of the eight capacities is 21, so the minimum cut has capacity 21

A1

**Answer: Minimum cut capacity 21**

6 marks

**16 (7 marks)**

(a) Cuts:  $\{S\}$ : 16;  $\{S, A\}$ : SB + AB + AT =  $6 + 3 + 5 = 14$ ;  $\{S, B\}$ : SA + BT = 19;  $\{S, A, B\}$ : AT + BT = 14

M1 A1

Maximum flow = 14, minimum cuts  $\{S, A\}$  and  $\{S, A, B\}$

A1

**Answer: 14**

3 marks

(b) Cut capacities now:  $\{S\}$ : 16;  $\{S, A\}$ :  $6 + 3 + 8 = 17$ ;  $\{S, B\}$ : 19;  $\{S, A, B\}$ :  $8 + 9 = 17$

M1

Maximum flow = 16

A1

**Answer: 16**

2 marks

(c)  $\{S, A\}$  still has capacity 14 and does not contain BT, so the maximum flow stays 14

M1

BT is not in the minimum cut  $\{S, A\}$ , so increasing it cannot increase the flow

E1

**Answer: 14**

2 marks

**17 (6 marks)**

(a) Arcs into B end at B, arcs out of B start at B', and arc BB' joins them

M1

Capacity of BB' is 8

A1

**Answer: BB' has capacity 8**

2 marks

(b) Augmenting routes in the new network: SAT: 6;  $S \rightarrow B \rightarrow B' \rightarrow T$ : 8

M1

Cut with  $\{S, A, B\}$  on the source side and  $\{B', T\}$  on the sink side: arcs AT, BB', capacity  $6 + 8 = 14$

M1 A1

Without the limit: the cut  $\{S, A, B\}$  has capacity AT + BT =  $6 + 10 = 16$  and the maximum flow is  $16 > 14$

A1

**Answer: Minimum cut 14**

4 marks

**18 (7 marks)**

(a) {S}: 16; {S, A}: 25; {S, B}: 21; {S, C}: 16; {S, A, B}: 26; {S, A, C}: 21; {S, B, C}: 21; {S, A, B, C}: 22

M1

Minimum cut capacity is 16 (maximum flow 16)

A1

The cuts of capacity 16 are {S} (arcs SA, SB, SC) and {S, C} (arcs SA, SB, CT)

M1 A1 A1

**Answer: 16: source sides {S} and {S, C}**

5 marks

(b) To increase the flow every minimum cut must be raised, so the arc must lie in both minimum cuts: SA and SB

M1

Increasing SC or CT alone leaves the other minimum cut at 16

A1

**Answer: SA or SB**

2 marks

**19 (6 marks)**

(a) {S}:  $SA + SB = 12 + 7 = 19$ ; {S, A}:  $SB + AB + AT = 7 + x + 6 = x + 13$

B1

{S, B}:  $SA + BT = 12 + 11 = 23$

B1

{S, A, B}:  $AT + BT = 6 + 11 = 17$

B1

**Answer: 19,  $x + 13$ , 23, 17**

3 marks

(b) Max flow =  $\min(19, x + 13, 23, 17) = \min(x + 13, 17)$

M1

$x + 13 \geq 17$  gives  $x \geq 4$

M1

Smallest  $x$  is 4, with maximum flow 17

A1

**Answer:  $x = 4$ , maximum flow 17**

3 marks

**20 (6 marks)**

Network drawn with the nine arcs shown in the table (see diagram)

B1 B1

Augmenting routes: SAT: 5; SBT: 6; SCT: 5; SABT: 1; SACT: 2

M1

Cut with {S} on the source side and {A, B, C, T} on the sink side: arcs SA, SB, SC, capacity  $8 + 6 + 5 = 19$

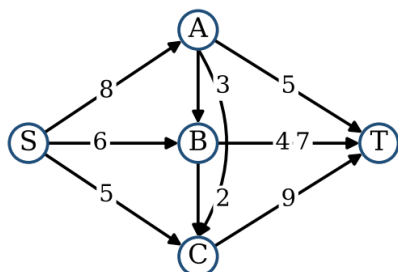
M1 A1

Maximum flow = minimum cut = 19

A1

**Answer: 19**

6 marks

**21 (7 marks)**

(a) Augmenting routes: SAT: 6; SBT: 7; SCT: 5; SABT: 1; SABCT: 1

M1 A1

Cut with {S, A} on the source side and {B, C, T} on the sink side: arcs SB, SC, AB, AT, capacity  $7 + 5 + 2 + 6 = 20$

A1

**Answer: Maximum flow 20**

3 marks

(b) There are eight cuts (one for each choice of intermediate vertices on the source side)

M1

Capacities: {S}:  $SA (11) + SB (7) + SC (5) = 23$ ; {S, A}:  $SB (7) + SC (5) + AB (2) + AT (6) =$

B1

$20$ ; {S, B}:  $SA (11) + SC (5) + BC (3) + BT (8) = 27$ ; {S, C}:  $SA (11) + SB (7) + CT (7) = 25$

{S, A, B}:  $SC (5) + BC (3) + AT (6) + BT (8) = 22$ ; {S, A, C}:  $SB (7) + AB (2) + AT (6) + CT$

B1

$(7) = 22$ ; {S, B, C}:  $SA (11) + BT (8) + CT (7) = 26$ ; {S, A, B, C}:  $AT (6) + BT (8) + CT (7) =$

21

Every other cut has capacity greater than 20, so the minimum cut is unique

A1

**Answer: Unique**

4 marks

## 22 (7 marks)

- (a) Replace each undirected pipe UV by two directed arcs UV and VU, each with the capacity of the pipe M1  
 Flow would never be sent both ways along a pair at once, so the maximum flow is unchanged E1  
**Answer: Two opposite arcs, each with the pipe capacity** 2 marks  
 (b) Augmenting routes: SACT: 4; SBCT: 4; SBDT: 1; SABDT: 3 M1 A1  
 Source side {S}: pipes cut are SA, SB M1 A1  
 Capacity =  $7 + 5 = 12$ , which equals the flow, so the maximum flow is 12 litres per second A1  
**Answer: 12 litres per second** 5 marks

## 23 (8 marks)

- (a) Augmenting routes: SAT: 3; SBT: 4; SBCT: 3 M1 A1  
 Cut with {S, B, C} on the source side and {A, T} on the sink side: arcs SA, BT, CT, capacity  $3 + 4 + 3 = 10$  A1  
**Answer: Maximum flow 10** 3 marks  
 (b) Only links in a minimum cut can raise the flow; the other links leave the maximum flow at 10 M1  
 New maximum flows (link: flow): SA: 13, SB: 10, AC: 10, BC: 10, AT: 10, BT: 12, CT: 12, AB: 10 M1 A1 A1  
 Upgrade SA, giving maximum flow 13 A1  
**Answer: Upgrade SA: flow 13** 5 marks

## 24 (6 marks)

- (a) Arc BA runs from B (sink side) to A (source side) M1  
 Only arcs from the source side to the sink side count towards the capacity of a cut, so BA should not be included E1  
**Answer: BA should not be counted** 2 marks  
 (b) Correct capacity =  $SB + AT = 6 + 5 = 11$  B1  
 Augmenting routes: SAT: 5; SBT: 6 M1 A1  
 A flow of 11 and a cut of capacity 11, so by the maximum flow minimum cut theorem 11 is the minimum cut capacity A1  
**Answer: 11** 4 marks

## 25 (8 marks)

- (a) {S}:  $SA + SB = k + 8$  B1  
 {S, A}:  $SB + AB + AT = 8 + 3 + 6 = 17$  B1  
 {S, B}:  $SA + BT = k + 2k = 3k$  B1  
 {S, A, B}:  $AT + BT = 6 + 2k$  B1  
**Answer:  $k + 8$ , 17,  $3k$ ,  $2k + 6$**  4 marks  
 (b)  $3k \leq k + 8$  exactly when  $k \leq 4$ , and  $3k \leq 2k + 6$  when  $k \leq 6$ , so for  $0 < k \leq 4$  the minimum is  $3k$  M1 A1  
 For  $k \geq 4$ :  $k + 8 \leq 2k + 6$  when  $k \geq 2$ , and  $k + 8 \leq 17$  when  $k \leq 9$ , so for  $4 \leq k \leq 9$  the minimum is  $k + 8$  M1  
 For  $k \geq 9$  the minimum is 17 A1  
**Answer:  $3k$  ( $k \leq 4$ );  $k + 8$  ( $4 \leq k \leq 9$ ); 17 ( $k \geq 9$ )** 4 marks

## 26 (7 marks)

- (a) Cuts without AB: {S}: 17; {S, A}:  $5 + 4 = 9$ ; {S, B}: 22; {S, A, B}: 14, so the maximum flow is 9 M1 A1  
 With AB: {S, A} becomes  $9 + c$ , others unchanged, so the maximum flow is  $\min(9 + c, 14)$  M1  
 $9 + c \geq 12$  gives  $c \geq 3$ , so the cheapest route has  $c = 3$  and costs £6000 A1  
**Answer: Before: 9;  $c = 3$  costing £6000** 4 marks  
 (b) For large  $c$ ,  $\min(9 + c, 14) = 14$  M1  
 The cut {S, A, B} (arcs AT and BT, capacity  $4 + 10 = 14$ ) does not contain AB, so it is unaffected by building the route M1 E1  
**Answer: 14** 3 marks

**27 (9 marks)**

(a) For S, (flow out) – (flow in) equals the value  $v$  of the flow; for every other vertex of X it equals 0 B1

So the sum over all vertices of X of (flow out) – (flow in) is  $v$  M1

An arc with both ends in X contributes  $+f$  at its tail and  $-f$  at its head, which cancel in the sum M1

An arc from X to outside contributes  $+f$  and an arc from outside into X contributes  $-f$  M1

So the sum is  $F_{\text{out}} - F_{\text{in}}$ , hence  $v = F_{\text{out}} - F_{\text{in}}$  A1

**Answer: Proved** 5 marks

(b)  $F_{\text{in}} \geq 0$  so  $v \leq F_{\text{out}}$  M1

Each arc leaving X carries at most its capacity, so  $F_{\text{out}} \leq \text{capacity of the cut}$  and  $v \leq \text{capacity of the cut}$  A1

**Answer: Proved** 2 marks

(c) Arcs out of X: SA (7) and BT (6), so  $F_{\text{out}} = 13$ ; arc into X: AB (2), so  $F_{\text{in}} = 2$  B1

$F_{\text{out}} - F_{\text{in}} = 11 = \text{flow out of S} = 7 + 4$  B1

**Answer:  $F_{\text{out}} = 13$ ,  $F_{\text{in}} = 2$ , value 11** 2 marks

**28 (8 marks)**

(a) By the maximum flow minimum cut theorem, the maximum flow equals the capacity of a minimum cut M1

The capacity of any cut is a sum of capacities of arcs, each of which is even M1

A sum of even integers is even, so the maximum flow is even A1

**Answer: Proved** 3 marks

(b) The same argument shows the maximum flow is a multiple of  $m$  B1

**Answer: Maximum flow is a multiple of  $m$**  1 mark

(c) Arcs SA = 3 and AT = 5: the maximum flow is 3 (odd) B1

Arcs SA = 3, SB = 3, AT = 5, BT = 5: maximum flow is  $3 + 3 = 6$  (even) B1

**Answer: Maximum flow 3 (odd) and 6 (even)** 2 marks

(d) Maximum flow = minimum cut capacity M1

A cut capacity is a sum of integers, so the maximum flow is an integer A1

**Answer: Min cut capacity is an integer** 2 marks

**29 (11 marks)**

(a) Augmenting routes: SADT: 4; SAET: 4; SBDT: 4; SBET: 3; SBFT: 1; SCET: 5; SCFT: 5 M1 A1

Cut with {S, A, C} on the source side and {B, D, E, F, T} on the sink side: arcs SB, AD, AE, CE, M1 A1

CF, capacity  $8 + 4 + 4 + 5 + 5 = 26$

Arcs in the minimum cut: SB, AD, AE, CE, CF B1

Maximum flow = 26 tonnes per hour A1

**Answer: 26 tonnes per hour** 6 marks

(b) After the maximum flow, the vertices which can be reached from S are {S, A, C} only: every arc leaving this set is full M1

Every other vertex (B, D, E, F, T) can reach T using spare capacity or by cancelling flow M1

A minimum cut must have all of {S, A, C} on the source side and all of the others on the sink E1

side, so it is unique

**Answer: Unique minimum cut** 3 marks

(c) CF is in the minimum cut and the new maximum flow is 28 B1

BF has both ends on the sink side of the unique minimum cut, so the cut is unchanged and the flow stays 26 E1

**Answer: New maximum flow 28** 2 marks

**30 (10 marks)**

(a) Arc-disjoint paths: SAGT, SBET, SCHAT M1 A1

Cut with {S, A, C, D, G, H} on the source side and {B, E, F, T} on the sink side: arcs SB, GT, M1 A1

HT, capacity  $1 + 1 + 1 = 3$

Any set of arc-disjoint paths must use a different arc of this cut for each path, so there are at most 3 A1

**Answer: 3 paths** 5 marks

- (b) Every path from S to T must contain at least one arc from the source side to the sink side, i.e. an arc of the cut M1  
 Paths are arc-disjoint, so different paths use different arcs of the cut A1  
**Answer: Each path uses its own arc of the cut** 2 marks  
 (c) A flow of integer value  $v$  on unit arcs uses  $v$  arcs out of S, each carrying 1 unit; following the flow gives  $v$  arc-disjoint paths M1  
 Conversely,  $v$  arc-disjoint paths give a feasible flow of value  $v$  by sending 1 unit along each M1  
 So the two maxima are equal A1  
**Answer: Equal** 3 marks

## Flow-Augmenting Algorithms

### 1 (2 marks)

- A route (path) from the source S to the sink T B1  
 along which the flow can be increased, using arcs which are not full (forward) or arcs which carry flow that can be reduced (backward) B1  
**Answer: A route from S to T along which the flow can be increased** 2 marks

### 2 (2 marks)

- (a)  $9 - 4 = 5$  B1  
**Answer: 5** 1 mark  
 (b) The flow can be reduced to 0, so by 4 B1  
**Answer: 4** 1 mark

### 3 (2 marks)

- Spare capacity on SA =  $9 - 5 = 4$  and on AT =  $8 - 3 = 5$  M1  
 The increase is limited by the smaller, so it is 4 A1  
**Answer: 4** 2 marks

### 4 (2 marks)

- Each arc on the route gains 3 units of flow M1  
 SA: 7, AB: 4, BT: 8 A1  
**Answer: SA 7, AB 4, BT 8** 2 marks

### 5 (2 marks)

- The flow along AB is reduced by 2 B1  
 Reducing flow on AB frees up capacity at A and B so that the flow can be rerouted; the net flow into and out of each vertex is still balanced E1  
**Answer: Flow on AB falls by 2** 2 marks

### 6 (2 marks)

- The procedure ends when the sink T cannot be labelled, so there is no flow-augmenting route B1  
 The current flow is then a maximum flow B1  
**Answer: T cannot be labelled; the flow is maximal** 2 marks

### 7 (3 marks)

- Value of the flow now =  $5 + 3 = 8$  B1  
 Spare capacity: SB  $7 - 3 = 4$  and BT  $8 - 3 = 5$ , so the flow increases by 4 M1  
 New value =  $8 + 4 = 12$  A1  
**Answer: Increase by 4; new value 12** 3 marks

### 8 (4 marks)

- SA: increase  $8 - 8 = 0$ , decrease 8 B1  
 SB: increase  $6 - 4 = 2$ , decrease 4 B1  
 AB: increase  $5 - 5 = 0$ , decrease 5 B1  
 AT: increase  $5 - 3 = 2$ , decrease 3; BT: increase  $9 - 9 = 0$ , decrease 9 B1  
**Answer: SA +0/-8, SB +2/-4, AB +0/-5, AT +2/-3, BT +0/-9** 4 marks

**9 (4 marks)**SAT:  $\min(7, 5) = 5$ ; flows SA 5, AT 5

M1

SBT:  $\min(6, 8) = 6$ ; flows SB 6, BT 6

M1

SABT: SA spare 2, AB spare 4, BT spare 2, so add 2

M1

Final flows: AB 2, AT 5, BT 8, SA 7, SB 6

A1

**Answer: Added 5, 6, 2; final: AB 2, AT 5, BT 8, SA 7, SB 6**

4 marks

**10 (4 marks)**Route S, B, A, T: SB has spare capacity  $6 - 4 = 2$ , AB can have flow reduced by up to 5, and AT has spare capacity  $5 - 3 = 2$ 

M1 A1

Amount added = 2: SB 6, AB  $5 - 2 = 3$ , AT 5 (the flow on AB falls)

M1

New flows: AB 3, AT 5, BT 9, SA 8, SB 6, so the value of the flow is 14

A1

**Answer: New flows: AB 3, AT 5, BT 9, SA 8, SB 6 (value 14)**

4 marks

**11 (4 marks)**

Labels (vertex: amount, labelled from): A: 5 (from S); B: 2 (from S); C: 2 (from A); T: 2 (from C)

M1 A1

Route S, A, C, T, which can carry 2 extra units

B1

New value of the flow =  $10 + 2 = 12$ 

B1

**Answer: Route SACT, increase 2**

4 marks

**12 (3 marks)**

The labelling procedure (breadth first) finds the route S, B, D, T, which has the fewest arcs and uses only arcs that are not full

M1 A1

Spare capacities: SB 6, BD 2, DT 8, so the flow can be increased by 2

A1

**Answer: Route SBDT, increase 2**

3 marks

**13 (5 marks)**

Route R, B, P: add 1, flow now 18

M1 A1

Route R, A, B, P: add 3, flow now 21

A1

Cut with {R} on the source side and {A, B, C, P} on the sink side: arcs RA, RB, capacity  $12 + 9 = 21$ 

B1

P cannot be labelled, so the flow of 21 is maximal

A1

**Answer: Maximum flow 21**

5 marks

**14 (5 marks)**

Route S, B, C, T: add 2, flow now 12

M1 A1

Route S, B, D, T: add 3, flow now 15

A1

Cut with {S, A, B, C, D} on the source side and {T} on the sink side: arcs CT, DT, capacity  $8 + 7 = 15$ 

B1

T cannot be labelled, so the flow of 15 is maximal

A1

**Answer: Maximum flow 15**

5 marks

**15 (7 marks)****(a)** Amounts added: SABT 3, SAT 2, SBT 6

M1 A1

Flows: AB 3, AT 2, BT 9, SA 5, SB 6, value 11

A1

**Answer: Added 3, 2, 6; value 11**

3 marks

**(b)** Route S, B, A, T (AB used backwards): SB spare 2, AB can be reduced by 3, AT spare 5, so add 2

M1 A1

New flows: AB 1, AT 4, BT 9, SA 5, SB 8, value 13

A1

Cut with {S} on the source side and {A, B, T} on the sink side: arcs SA, SB, capacity  $5 + 8 = 13$ 

B1

**Answer: 13**

4 marks

**16 (7 marks)**

Route S, A, C, T: add 5, flow now 5

M1 A1

Route S, A, D, T: add 3, flow now 8

A1

Route S, B, C, T: add 2, flow now 10

A1

Route S, B, D, T: add 5, flow now 15

A1

Cut with {S} on the source side and {A, B, C, D, T} on the sink side: arcs SA, SB, capacity  $8 + 7 = 15$ 

B1

T cannot be labelled, so the flow of 15 is maximal

A1

**Answer: Maximum flow 15**

7 marks

**17 (7 marks)**

Supersource S with arcs SA (12) and SB (11); supersink T with arcs CT (9) and DT (6), as shown in the diagram

B1

Route S, A, C, T: add 5, flow now 5

M1 A1

Route S, A, D, T: add 6, flow now 11

A1

Route S, B, C, T: add 4, flow now 15

A1

Cut with {S, A, B, C, D} on the source side and {T} on the sink side: arcs CT, DT, capacity  $9 + 6 = 15$ 

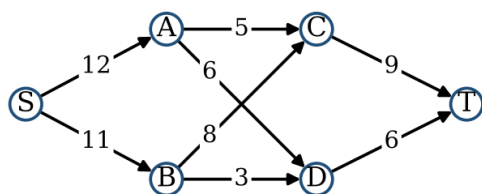
B1

T cannot be labelled, so the flow of 15 is maximal

A1

**Answer: Maximum 15 trays per hour**

7 marks

**18 (5 marks)**

Labelled vertices and labels:

M1 A1

T is not labelled, so there is no flow-augmenting route

B1

Cut with {S} on the source side and {A, B, C, D, T} on the sink side: arcs SA, SB, capacity  $8 + 7 = 15$ , which is a minimum cut

M1

Cut capacity 15 equals the value of the flow, so the maximum flow is 15

A1

**Answer: Maximum flow 15**

5 marks

**19 (7 marks)**

Route S, A, T: add 5, flow now 5

M1 A1

Route S, B, T: add 6, flow now 11

A1

Route S, A, B, T: add 1, flow now 12

A1

Route S, A, C, T: add 2, flow now 14

A1

Cut with {S} on the source side and {A, B, C, T} on the sink side: arcs SA, SB, capacity  $8 + 6 = 14$ 

B1

T cannot be labelled, so the flow of 14 is maximal

A1

**Answer: Maximum flow 14**

7 marks

**20 (7 marks)**Vertex A: flow in  $8 =$  flow out to C and D, so  $x = 8 - 5 = 3$ 

M1 A1

Vertex B: flow in  $7 =$  flow out, so  $y = 7 - 4 = 3$ 

A1

Route S, A, D, T: add 2, flow now 17

M1 A1

Cut with {S, A, B, C, D} on the source side and {T} on the sink side: arcs CT, DT, capacity  $8 + 9 = 17$ 

B1

T cannot be labelled, so the flow of 17 is maximal

A1

**Answer:  $x = 3$ ,  $y = 3$ ; maximum flow 17**

7 marks

**21 (7 marks)**

(a) Route SABT: AB has capacity 1, so add 1; flows SA 1, AB 1, BT 1, value 1

M1 A1

Route SBAT (AB used backwards, AB carries 1 which can be removed): add 1; flows SA 1, SB 1, AT 1, BT 1, AB 0, value 2

A1

**Answer: Value 1 then 2**

3 marks

(b) The flow on AB alternates between 1 and 0, so one of the routes is always limited by AB, which has capacity 1

M1 E1

A maximum flow of 2000 is reached (the cuts  $\{S\}$  and  $\{S, A, B\}$  both have capacity 2000), so 2000 augmentations are needed

A1

**Answer: 2000 augmentations**

3 marks

(c) Routes S, A, T (add 1000) and S, B, T (add 1000) have 2 arcs, and give a flow of 2000

B1

**Answer: 2 augmentations**

1 mark

**22 (8 marks)**

Route S, A, C, T: add 6, flow now 6

M1 A1

Route S, A, D, T: add 3, flow now 9

A1

Route S, B, C, T: add 1, flow now 10

A1

Route S, B, D, T: add 5, flow now 15

A1

Route S, B, C, A, D, T (using arc AC backwards): add 1, flow now 16

A1

Cut with  $\{S, A, B, C\}$  on the source side and  $\{D, T\}$  on the sink side: arcs AD, BD, CT, capacity  $4 + 5 + 7 = 16$

B1

T cannot be labelled, so the flow of 16 is maximal

A1

**Answer: Maximum flow 16**

8 marks

**23 (8 marks)**

Split B into B (receiving arcs SB and AB) and B' (sending arcs BC and BT), and join them by an arc BB' of capacity 6

M1 A1

Route S, A, C, T: add 5, flow now 5

M1 A1

Route S, B, B', T: add 6, flow now 11

A1

Cut with  $\{S, A, B\}$  on the source side and  $\{B', C, T\}$  on the sink side: arcs AC, BB', capacity  $5 + 6 = 11$

B1

T cannot be labelled, so the flow of 11 is maximal

A1

Without the limit at B the maximum flow would be 14, so the limit matters

B1

**Answer: Maximum flow 11**

8 marks

**24 (6 marks)**

(a) The route goes from B to A, against the direction of arc AB, so AB is used backwards

M1

The flow along AB must be reduced by 1 (to 4), not increased: increasing it to 6 would exceed the capacity of 5

E1

**Answer: AB is used backwards, so its flow decreases by 1**

2 marks

(b) Flows: AB 4, AT 5, BT 8, SA 9, SB 4

M1

Value of the flow = 13

A1

**Answer: Value 13**

2 marks

(c) Cut with  $\{S, A, B\}$  on the source side and  $\{T\}$  on the sink side: arcs AT, BT, capacity  $5 + 8 = 13$

M1

The cut capacity equals the value 13, so the flow is a maximum

A1

**Answer: Cut of capacity 13**

2 marks

**25 (8 marks)**

(a) Cut with source side  $\{S, A, B\}$ :  $AT + BT = 4 + 5 = 9$ ;  $\{S\}$ :  $SA + SB = 8$ ;  $\{S, A\}$ :  $SB + AB + AT = 2 + 3 + 4 = 9$ ;  $\{S, B\}$ :  $SA + BT = 11$

M1 A1

Minimum cut  $\{S\}$  has capacity 8, a flow of 8 saturating SA and SB exists, so the maximum flow is 8

A1

**Answer: 8**

3 marks

(b) Both must have  $SA = 6$  and  $SB = 2$

B1

Flow 1: AB 2, AT 4, BT 4

B1

Flow 2: AB 3, AT 3, BT 5

B1

**Answer: e.g. AB 2, AT 4, BT 4 and AB 3, AT 3, BT 5**

3 marks

(c) Average: SA 6, SB 2, AB  $\frac{5}{2}$ , AT  $\frac{7}{2}$ , BT  $\frac{9}{2}$ , all within capacity

M1

Conservation holds (it holds for both flows, so for their average) and the value is 8

A1

**Answer: Average is a maximum flow**

2 marks

## 26 (7 marks)

Each flow lies between its minimum and maximum, and conservation holds: at A,  $4 = 1 + 3$ ; at B,  $3 + 1 = 4$

M1 A1

Route S, A, T: add 4, value now 11

M1 A1

Route S, B, T: add 2, value now 13

A1

Cut with source side  $\{S, A, B\}$ : forward arcs AT, BT have capacity 13; the arcs returning across the cut (none) have total minimum flow 0; flow  $\leq 13 - 0 = 13$

B1

T cannot be labelled, so the maximum flow is 13

A1

**Answer: Maximum flow 13**

7 marks

## 27 (10 marks)

(a) Suppose an arc from  $u$  in  $X$  to  $v$  outside  $X$  had spare capacity: then  $v$  would be labelled from  $u$ , contradicting  $v$  not being in  $X$

M1

Suppose an arc from  $v$  outside  $X$  to  $u$  in  $X$  carried positive flow: then  $v$  would be labelled from  $u$  by a backward step, again a contradiction

M1

So every arc leaving  $X$  is full and every arc entering  $X$  carries zero flow

A1

**Answer: Proved**

3 marks

(b) By (a),  $F_{\text{out}}$  equals the sum of the capacities of the arcs leaving  $X$ , which is the capacity of the cut

M1

By (a),  $F_{\text{in}} = 0$

B1

So value  $= F_{\text{out}} - F_{\text{in}} = \text{capacity of the cut}$

A1

**Answer: Value = capacity of the cut**

3 marks

(c) Every flow has value at most the capacity of every cut

M1

This flow attains the capacity of this cut, so no flow is larger and no cut is smaller

A1

**Answer: Max flow = min cut**

2 marks

(d) Arcs leaving  $X$ : SA (8/8), SB (7/7), all full

B1

Arcs entering  $X$ : none, which carry no flow; the flow value is 15 = capacity of the cut

B1

**Answer: Verified**

2 marks

## 28 (10 marks)

(a) Initially all flows are integers. If all flows are integers then every spare capacity (capacity – flow) and every backward capacity (flow) is an integer

M1

The amount added is the minimum of these, which are positive integers, so it is an integer of at least 1, and the new flows are integers

M1

By induction every flow is an integer and every augmentation adds at least 1

A1

**Answer: Proved**

3 marks

(b) The value starts at 0, increases by at least 1 each time, and can never exceed  $v$

M1

So there are at most  $v$  augmentations

A1

**Answer: At most  $v$  augmentations**

2 marks

(c) Initially every flow is 0, a multiple of 3. If every flow is a multiple of 3 then every spare capacity (capacity – flow) and every backward capacity (flow) is a multiple of 3

M1

The amount added is the smallest of these positive multiples of 3, so it is a multiple of 3 of at least 3, and the new flows are again multiples of 3

M1

Each augmentation adds at least 3 and the value cannot exceed  $v$ , so there are at most  $\frac{v}{3}$  augmentations

A1

**Answer: Each augmentation adds a multiple of 3, so at most  $v/3$  augmentations**

3 marks

(d) By (c) the number of augmentations is at most  $\frac{60}{3}$

M1

= 20 augmentations

A1

**Answer: 20 augmentations**

2 marks

**29 (11 marks)**

|                                                                                                              |         |
|--------------------------------------------------------------------------------------------------------------|---------|
| (a) Route S, A, D, T: add 6, flow now 6                                                                      | M1 A1   |
| Route S, A, E, T: add 4, flow now 10                                                                         | A1      |
| Route S, B, D, T: add 3, flow now 13                                                                         | A1      |
| Route S, B, E, T: add 3, flow now 16                                                                         | A1      |
| Route S, C, E, T: add 4, flow now 20                                                                         | A1      |
| Cut with {S, A, B, C, D, E} on the source side and {T} on the sink side: arcs DT, ET, capacity $9 + 11 = 20$ | B1      |
| T cannot be labelled, so the flow of 20 is maximal                                                           | A1      |
| <b>Answer: Maximum flow 20</b>                                                                               | 8 marks |
| (b) Final flows: AD 6, AE 4, BD 3, BE 3, CE 4, DT 9, ET 11, SA 10, SB 6, SC 4                                | B1      |
| Vertex A: flow in 10 = flow out $6 + 4 = 10$                                                                 | M1      |
| Vertex D: flow in $6 + 3 = 9$ = flow out 9                                                                   | A1      |
| <b>Answer: Flows as listed</b>                                                                               | 3 marks |

**30 (10 marks)**

|                                                                                                                                                            |         |
|------------------------------------------------------------------------------------------------------------------------------------------------------------|---------|
| (a) Capacity: $0 \leq f_1(e), f_2(e) \leq c(e)$ , so $0 \leq g(e) \leq c(e)$                                                                               | M1      |
| Conservation: at every vertex other than S and T, (flow in) – (flow out) is 0 for $f_1$ and for $f_2$ , so it is $\frac{1}{2}(0 + 0) = 0$ for g            | M1 A1   |
| Value of $g = \frac{1}{2}(v + v) = v$                                                                                                                      | A1      |
| <b>Answer: Proved</b>                                                                                                                                      | 4 marks |
| (b) For every t with $0 \leq t \leq 1$ , the same argument shows that $t f_1 + (1 - t) f_2$ is a maximum flow                                              | M1      |
| These are different for different t, so there are infinitely many maximum flows                                                                            | A1      |
| <b>Answer: Infinitely many</b>                                                                                                                             | 2 marks |
| (c) g: SA 4, SB 4, AC $\frac{5}{2}$ , AD $\frac{3}{2}$ , BC 3, BD 1, CT $\frac{11}{2}$ , DT $\frac{5}{2}$                                                  | M1      |
| For example AC carries $\frac{5}{2}$ , which is not an integer                                                                                             | A1      |
| <b>Answer: SA 4, SB 4, AC <math>\frac{5}{2}</math>, AD <math>\frac{3}{2}</math>, BC 3, BD 1, CT <math>\frac{11}{2}</math>, DT <math>\frac{5}{2}</math></b> | 2 marks |
| (d) The labelling procedure from zero flow keeps all flows integer (every amount added is an integer)                                                      | M1      |
| It ends at a maximum flow, so a maximum flow with integer values exists                                                                                    | A1      |
| <b>Answer: Labelling procedure gives integer flows</b>                                                                                                     | 2 marks |

**Bipartite Graphs****1 (2 marks)**

|                                                                                                      |         |
|------------------------------------------------------------------------------------------------------|---------|
| The vertices can be divided into two sets (X and Y)                                                  | B1      |
| so that every edge joins a vertex in X to a vertex in Y (no edge joins two vertices of the same set) | B1      |
| <b>Answer: Vertices split into two sets with every edge between the sets</b>                         | 2 marks |

**2 (3 marks)**

|                                                                           |        |
|---------------------------------------------------------------------------|--------|
| (a) $3 + 5 = 8$                                                           | B1     |
| <b>Answer: 8</b>                                                          | 1 mark |
| (b) $3 \times 5 = 15$                                                     | B1     |
| <b>Answer: 15</b>                                                         | 1 mark |
| (c) Each is joined to all 5 vertices of the other set, so the degree is 5 | B1     |
| <b>Answer: 5</b>                                                          | 1 mark |

**3 (2 marks)**

|                                                               |         |
|---------------------------------------------------------------|---------|
| Sets {A, C, E} and {B, D, F}                                  | B1      |
| Every edge joins a vertex in one set to a vertex in the other | B1      |
| <b>Answer: {A, C, E} and {B, D, F}</b>                        | 2 marks |

**4 (2 marks)**

The edges AB, BC and CA form a cycle of length 3 (a triangle)

B1

A bipartite graph cannot contain a cycle of odd length, since vertices of a cycle must alternate between the two sets

E1

**Answer: It has an odd cycle (triangle ABC)**

2 marks

**5 (2 marks)**

A cycle is bipartite exactly when its length is even

M1

$C_4$  and  $C_6$

A1

**Answer:  $C_4$  and  $C_6$**

2 marks

**6 (2 marks)**

Amy: Art, Drama; Ben: Music, Drama; Cal: Art, Music

M1

Edges: Amy-Art, Amy-Drama, Ben-Music, Ben-Drama, Cal-Art, Cal-Music (6 edges)

A1

**Answer: 6 edges as listed**

2 marks

**7 (3 marks)**

Colour A red; its neighbours B and D blue

M1

Then C, E (neighbours of B and D) red, then F blue and G red, and check every edge joins different colours

M1

Sets  $\{A, C, E, G\}$  and  $\{B, D, F\}$

A1

**Answer:  $\{A, C, E, G\}$  and  $\{B, D, F\}$**

3 marks

**8 (3 marks)**

Look for a closed path A, B, C, D, E, A

M1

All edges AB, BC, CD, DE, EA are present

A1

This cycle has 5 edges, which is odd, so the graph is not bipartite

A1

**Answer: Cycle ABCDEA of length 5**

3 marks

**9 (3 marks)**

The number of edges equals the sum of the degrees in X, so  $3 + 2 + 4 + 1 = 10$  edges

M1 A1

The degrees in Y also sum to 10:  $2 + x + 3 + 1 + 2 = 10$ , so  $x = 2$

A1

**Answer: 10 edges,  $x = 2$**

3 marks

**10 (3 marks)**

$m + n = 9$  and  $mn = 20$ , so  $m(9 - m) = 20$

M1

$m^2 - 9m + 20 = 0$ , so  $(m - 4)(m - 5) = 0$

M1

$m = 4$  and  $n = 5$

A1

**Answer:  $m = 4$ ,  $n = 5$**

3 marks

**11 (4 marks)**

(a)  $X_1$  and  $X_2$  on one side,  $Y_1, Y_2, Y_3$  on the other, as in the diagram

M1

All six edges  $X_i Y_j$  drawn and no others

A1

**Answer: 6 edges**

2 marks

(b)  $X_1, X_2$  have degree 3 and  $Y_1, Y_2, Y_3$  have degree 2

B1

**Answer: 3, 3, 2, 2, 2**

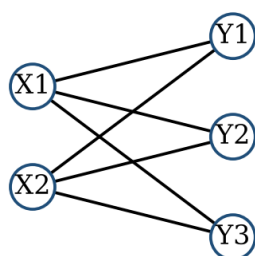
1 mark

(c) Two vertices ( $X_1$  and  $X_2$ ) have odd degree, but an Eulerian circuit needs every degree even

B1

**Answer: Vertices of odd degree**

1 mark



**12 (4 marks)**

Edges (from the 1s above the diagonal): AB, AE, CB, CE, DB

M1 A1

Colour A red; its neighbours B and E blue; neighbours of B and E (A, C, D) are red

M1

Sets {A, C, D} and {B, E} with all edges between them

A1

**Answer: {A, C, D} and {B, E}**

4 marks

**13 (4 marks)**

The students form one set and the clubs the other, and every edge joins a student to a club

M1 A1

Number of edges =  $2 + 2 + 3 + 1 + 2 = 10$

A1

Drama is joined to Anna, Ben and Eli, so its degree is 3

A1

**Answer: 10 edges; degree of Drama is 3**

4 marks

**14 (5 marks)**

Let  $T$  be a tree, choose any vertex  $r$  and let  $d(v)$  be the length of the unique path from  $r$  to  $v$

B1

Put  $v$  in set  $X$  if  $d(v)$  is even and in set  $Y$  if  $d(v)$  is odd

M1

Let  $uv$  be an edge with  $d(u) \leq d(v)$ . The path from  $r$  to  $v$  is unique, and the path from  $r$  to  $u$  followed by the edge  $uv$  is a path to  $v$ , so  $d(v) = d(u) + 1$

M1 A1

So  $d(u)$  and  $d(v)$  have different parity and every edge joins  $X$  to  $Y$ , so  $T$  is bipartite

A1

**Answer: Proved**

5 marks

**15 (5 marks)**

Label the vertices  $v_1, v_2, \dots, v_n$  in order round the cycle, so that the edges are  $v_i v_{i+1}$  and  $v_n v_1$

B1

If  $n$  is even, put  $v_i$  in  $X$  when  $i$  is odd and in  $Y$  when  $i$  is even. Edges  $v_i v_{i+1}$  join odd to even

M1

and  $v_n v_1$  joins  $v_n$  (even) to  $v_1$  (odd), so the graph is bipartite

A1

If  $n$  is odd and the graph were bipartite, the vertices  $v_1, v_2, v_3, \dots$  would have to alternate sets, so  $v_n$  would be in the same set as  $v_1$

M1

But  $v_n v_1$  is an edge, a contradiction, so  $C_n$  is not bipartite

A1

**Answer: Proved**

5 marks

**16 (5 marks)**

(a) If the two sets have  $m$  and  $10 - m$  vertices there are at most  $m(10 - m)$  edges

M1

$$m(10 - m) = 25 - (m - 5)^2 \leq 25$$

M1

with equality when  $m = 5$ , so there are at most 25 edges

A1

**Answer: At most 25**

3 marks

(b)  $m(11 - m)$  is largest at  $m = 5$  or  $6$

M1

$$5 \times 6 = 30$$

A1

**Answer: 30**

2 marks

**17 (6 marks)**

A bipartite graph has no odd cycles, so in particular no triangles; every face is bounded by at least 4 edges

M1

$$\text{Counting edge-sides: } 2e \geq 4f, \text{ so } f \leq \frac{e}{2}$$

M1

$$v - e + f = 2 \text{ gives } 2 \leq v - e + \frac{e}{2} = v - \frac{e}{2}, \text{ so } e \leq 2v - 4$$

M1 A1

$$K_{3,3} \text{ has } v = 6 \text{ and } e = 9, \text{ but } 2v - 4 = 8 < 9$$

M1

So  $K_{3,3}$  is not planar

A1

**Answer:  $e \leq 2v - 4$ ;  $K_{3,3}$  is not planar**

6 marks

**18 (5 marks)**

(a) The number of edges equals the sum of the degrees of the vertices in  $X$ , which is  $k|X|$

M1

It also equals the sum of the degrees in  $Y$ , which is  $k|Y|$

M1

So  $k|X| = k|Y|$  and, since  $k \geq 1$ ,  $|X| = |Y|$

A1

**Answer: Proved**

3 marks

(b)  $3|X| = 12$ , so  $|X| = 4$

M1

$|Y| = 4$  as well, so there are 8 vertices

A1

**Answer: 8**

**2 marks**

**19 (5 marks)**

(a) Consecutive vertices in a cycle lie in different sets, so a cycle alternates between the two sets

**M1**

So it uses equal numbers of vertices from each set

**M1**

A Hamiltonian cycle would use all  $4 + 5 = 9$  vertices, which is not an equal split, a contradiction

**A1**

**Answer: Proved**

**3 marks**

(b) Start at a vertex in the set of 5 and alternate

**M1**

Y, X, Y, X, Y, X, Y, X, Y uses all  $5 + 4 = 9$  vertices, and consecutive vertices are joined in  $K_{4,5}$

**A1**

**Answer: Alternate between the sets starting and ending in the larger set**

**2 marks**

**20 (5 marks)**

The complete graph on 7 vertices has  $\frac{7 \times 6}{2} = 21$  edges

**M1**

$K_{3,4}$  has  $3 \times 4 = 12$  edges, so the complement has  $21 - 12 = 9$  edges

**M1 A1**

In the complement, the 3 vertices of the first set are all joined to each other, forming a triangle

**B1**

A triangle is an odd cycle, so the complement is not bipartite

**E1**

**Answer: 9 edges**

**5 marks**

**21 (6 marks)**

(a) A graph is bipartite only if it has no cycle of odd length, not just no triangles

**M1**

Odd cycles of length 5 also prevent a graph being bipartite

**E1**

**Answer: Cycles of length 5 also matter**

**2 marks**

(b) Cycle A, B, C, D, E, A has the five edges AB, BC, CD, DE, EA

**M1**

Length 5 is odd, so the graph is not bipartite

**A1**

**Answer: ABCDEA**

**2 marks**

(c) The odd cycles are ABCDEA and AFCDEA; the edges common to both are CD, DE and EA

**M1**

Removing any one of these edges destroys both odd cycles and leaves a bipartite graph; removing any other edge leaves an odd cycle

**A1**

**Answer: CD, DE, EA**

**2 marks**

**22 (7 marks)**

(a) Pick a vertex  $v$  and suppose it is in set  $X$

**B1**

Any vertex  $u$  is joined to  $v$  by a path, and along a path the vertices alternate between  $X$  and  $Y$

**M1**

So  $u$  is in  $X$  if the path has even length and in  $Y$  if it has odd length

**M1**

Every path from  $v$  to  $u$  has the same parity of length for this reason, so the set of  $u$  is determined by  $v$ 's set

**M1**

Putting  $v$  in  $Y$  instead gives the swapped sets, so there are exactly two ways to name the sets

**A1**

**Answer: Proved**

**5 marks**

(b) Each component can be split in exactly 2 ways by (a)

**M1**

Independent choices for 3 components gives  $2^3 = 8$  ways

**A1**

**Answer: 8**

**2 marks**

**23 (7 marks)**

(a) Edges: PS, PT, QS, QU, RT, RU; colour P red then its neighbours S, T blue, then Q, R, U red

**M1**

Sets  $\{P, Q, R\}$  and  $\{S, T, U\}$

**A1**

**Answer:  $\{P, Q, R\}$  and  $\{S, T, U\}$**

**2 marks**

(b) First row of  $A^2$  is  $(2, 1, 1, 0, 0, 0)$

**M1**

It is non-zero only for P, Q, R, which are in the same set as P: walks of length 2 return to the same set

**A1**

**Answer:  $(2, 1, 1, 0, 0, 0)$**

**2 marks**

$$(c) A^3 = \begin{pmatrix} 0 & 0 & 0 & 3 & 3 & 2 \\ 0 & 0 & 0 & 3 & 2 & 3 \\ 0 & 0 & 0 & 2 & 3 & 3 \\ 3 & 3 & 2 & 0 & 0 & 0 \\ 3 & 2 & 3 & 0 & 0 & 0 \\ 2 & 3 & 3 & 0 & 0 & 0 \end{pmatrix}$$

**M1**

The diagonal entries are all 0

A1

A walk of length 3 from a vertex back to itself would be a closed walk of odd length, which cannot exist in a bipartite graph

E1

**Answer: All zero**

3 marks

## 24 (8 marks)

(a) The sums of the degrees are both 10, so 10 edges are needed

B1

$Y_1$  has degree 4 so it is joined to every vertex of X

M1

A valid graph:  $X_1Y_1, X_1Y_2, X_1Y_3, X_2Y_1, X_2Y_2, X_2Y_4, X_3Y_1, X_3Y_3, X_4Y_1, X_4Y_2$

M1

Degrees: X: 3, 3, 2, 2 and Y: 4, 3, 2, 1, as required

A1

**Answer: Edges:  $X_1Y_1, X_1Y_2, X_1Y_3, X_2Y_1, X_2Y_2, X_2Y_4, X_3Y_1, X_3Y_3, X_4Y_1, X_4Y_2$**

4 marks

(b) The degree sums are both 10

B1

The two vertices of degree 4 in X must each be joined to all 4 vertices of Y

M1

So every vertex of Y has degree at least 2

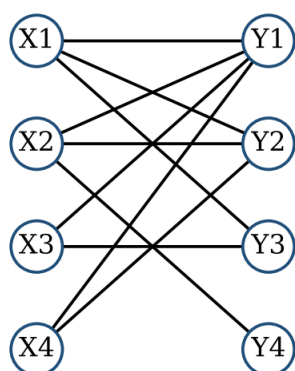
M1

But Y has a vertex of degree 1, a contradiction

A1

**Answer: Impossible**

4 marks



## 25 (7 marks)

(a) Each vertex in the set of  $m$  has degree  $n$  and each vertex in the other set has degree  $m$

M1

An Eulerian circuit exists exactly when the graph is connected and every degree is even

M1

$K_{m,n}$  is connected, so this happens exactly when  $n$  and  $m$  are both even

A1

**Answer: Proved**

3 marks

(b) 24 edges;  $m = 4$  and  $n = 6$  are both even, so yes

B1

**Answer: 24; yes**

1 mark

(c)  $K_{2,3}$ : the two vertices in the set of 2 have degree 3 (odd), the other three have degree 2: exactly two odd vertices, so an Eulerian trail exists

M1 A1

$K_{3,4}$ : the four vertices in the set of 4 have degree 3 (odd), so there are four odd vertices and no Eulerian trail

A1

**Answer:  $K_{2,3}$  yes;  $K_{3,4}$  no**

3 marks

## 26 (7 marks)

Every path alternates between the two sets, so after an even number of edges it is back in the starting set and after an odd number it is in the other set

M1 A1

Hence the length of any path between two vertices is even if they are in the same set and odd if they are in different sets

A1

(i) B to G: shortest path B, D, F, G has length 3, which is odd, so B and G are in different sets

M1 A1

(ii) C to F: shortest path C, E, F has length 2, which is even, so C and F are in the same set

M1 A1

**Answer: (i) different sets (ii) the same set**

7 marks

## 27 (9 marks)

(a) Suppose the vertices of a cycle are  $v_1, v_2, \dots, v_k, v_1$  and  $v_1$  is in X

B1

Consecutive vertices are in different sets, so  $v_1, v_3, v_5, \dots$  are in X and  $v_2, v_4, \dots$  are in Y

M1

The closing edge  $v_k v_1$  needs  $v_k$  in Y, so  $k$  is even; hence there is no odd cycle

A1

**Answer: Proved**

3 marks

(b) Choose a vertex  $v$  and let  $d(x)$  be the length of a shortest path from  $v$  to  $x$ . Let  $X$  be the vertices with  $d(x)$  even and  $Y$  those with  $d(x)$  odd B1

Suppose an edge  $xy$  has both ends in the same set, so  $d(x)$  and  $d(y)$  have the same parity M1

Join shortest paths from  $v$  to  $x$  and from  $v$  to  $y$  to the edge  $xy$  to form a closed walk  $v$  to  $x$  to  $y$  to  $v$  M1

Its length is  $d(x) + 1 + d(y)$ , which is odd, because  $d(x) + d(y)$  is even M1 A1

So the graph contains an odd cycle, a contradiction; every edge joins  $X$  to  $Y$  and the graph is bipartite A1

**Answer: Proved** 6 marks

**28 (9 marks)**

(a) This graph is a single cycle  $x_1, y_2, x_2, y_3, x_3, y_1, x_1$  of length 6, and every vertex has degree 2 M1

A 4-cycle would need two vertices of  $X$  with two common neighbours; here each pair of vertices of  $X$  shares exactly one neighbour A1

**Answer: No 4-cycle** 2 marks

(b) Let the degrees of  $x_1, x_2, x_3$  be  $d_1, d_2, d_3$ , with sum 7 and each at most 3 B1

A vertex of degree  $d$  in  $X$  is adjacent to  $C(d, 2)$  pairs of vertices in  $Y$ :  $\frac{d(d-1)}{2}$  M1

The sum is smallest when the degrees are as equal as possible:  $(3, 2, 2)$  gives  $3 + 1 + 1 = 5$  M1 A1

Check other splits:  $(3, 3, 1)$  gives  $3 + 3 + 0 = 6 > 5$ , so every split gives at least 5 pairs A1

There are only  $C(3, 2) = 3$  pairs of vertices in  $Y$ , and  $5 > 3$ , so by the pigeonhole principle two vertices of  $X$  are adjacent to the same pair of vertices of  $Y$  M1

These four vertices form a 4-cycle A1

**Answer: Proved** 7 marks

**29 (10 marks)**

(a) Counting edges,  $k|X| = k|Y|$ , so  $|X| = |Y|$  M1 A1

Let  $S$  be a subset of  $X$ . The  $k|S|$  edges leaving  $S$  all end in  $N(S)$  M1

The vertices of  $N(S)$  have degree  $k$ , so they are the end of at most  $k|N(S)|$  edges M1

So  $k|S| \leq k|N(S)|$ , that is  $|N(S)| \geq |S|$  A1

By Hall's theorem there is a matching using every vertex of  $X$ ; as  $|X| = |Y|$  it also uses every vertex of  $Y$ , so it is perfect A1

**Answer: Proved** 6 marks

(b) Each  $X_i$  is joined to the three  $Y_j$  with  $j \neq i$ , and similarly for each  $Y_j$ , so every degree is 3 M1

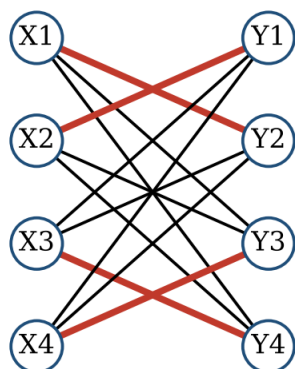
Perfect matching:  $X1Y2, X2Y1, X3Y4, X4Y3$  A1

**Answer:  $X1Y2, X2Y1, X3Y4, X4Y3$**  2 marks

(c) A perfect matching pairs each  $X_i$  with a different  $Y_j$ ,  $j \neq i$ , so it corresponds to a rearrangement of 1, 2, 3, 4 with no number in its own place M1

There are 9 such arrangements (derangements): 2143, 2341, 2413, 3142, 3412, 3421, 4123, 4312, 4321 A1

**Answer: 9** 2 marks



**30 (10 marks)**

(a) Suppose the two sets have  $p$  and  $q$  vertices,  $p + q = n$  B1

Each edge joins a vertex of the first set to one of the second, so the number of edges  $e \leq pq$  M1

$(p - q)^2 \geq 0$  gives  $(p + q)^2 \geq 4pq$ , so  $pq \leq \frac{n^2}{4}$  M1

Hence  $e \leq \frac{n^2}{4}$  A1

**Answer: Proved** 4 marks

(b) Equality needs  $p = q$ , so  $n = 2p$  is even, and  $e = pq$  so every possible edge is present M1

That is the complete bipartite graph  $K_{n/2, n/2}$  A1

**Answer:  $K_{n/2, n/2}$**  2 marks

(c) By (a) a bipartite graph with 13 vertices has at most  $\frac{13^2}{4} = 42.25$  edges, so at most 42 edges M1

since the number of edges is an integer

The graph has  $43 > 42$  edges, so it is not bipartite A1

**Answer: At most 42 edges are possible, but there are 43** 2 marks

(d) A connected graph on  $n$  vertices needs at least  $n - 1$  edges M1

A path on  $n$  vertices has  $n - 1$  edges and is bipartite A1

**Answer:  $n - 1$  (a path)** 2 marks

## Matching Problems

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**1 (2 marks)**

(a) A set of edges in which no two edges share a vertex B1

**Answer: Edges with no common vertex** 1 mark

(b) A matching in which every vertex of the (smaller or first) set is matched B1

**Answer: Every vertex of one set is matched** 1 mark

**2 (2 marks)**

(i) No two edges share a vertex, so it is a matching (and a complete matching) B1

(ii) A1 and C1 share vertex 1, so it is not a matching B1

**Answer: (i) yes (ii) no** 2 marks

**3 (2 marks)**

Each edge of a matching uses one vertex from each set M1

So the matching has at most 3 edges, and 3 is achievable A1

**Answer: 3** 2 marks

**4 (2 marks)**

A path whose edges are alternately not in  $M$  and in  $M$  B1

(An alternating path starting at an unmatched vertex and ending at an unmatched vertex in the other set is called an augmenting path) B1

**Answer: Edges alternately not in  $M$  and in  $M$**  2 marks

**5 (3 marks)**

(a) The first vertex can be matched in 3 ways, the next in 2 ways and the last in 1 way:  $3 \times 2 \times 1 = 6$  B1

**Answer: 6** 1 mark

(b) Match the three vertices of the smaller set in turn to different vertices of the larger set:  $4 \times 3 \times 2 = 24$  M1

**Answer: 24** A1

2 marks

**6 (2 marks)**

Ben can only do the Hall

B1

Then Cat must do the Garden and Ann the Kitchen: Ann-Kitchen, Ben-Hall, Cat-Garden

B1

**Answer: Ann-Kitchen, Ben-Hall, Cat-Garden**

2 marks

**7 (3 marks)**

Job 5 can only be done by E, so E-5

B1

Then D must do 3 or 4, C must do 1 or 4, A: 1 or 2, B: 2 or 3

M1

A complete matching: A-1, B-2, C-4, D-3, E-5

A1

**Answer: A-1, B-2, C-4, D-3, E-5**

3 marks

**8 (3 marks)**

C and D can only do task 4

B1

Two workers cannot both be given task 4, so there is no complete matching

E1

A matching of size 3: A-1, B-2, C-4

B1

**Answer: No complete matching (C and D both need task 4); A-1, B-2, C-4**

3 marks

**9 (4 marks)**

Try A-1: then D must do 4, C must do 3, B must do 2

M1

First matching: A-1, B-2, C-3, D-4

A1

Try A-2: then B must do 3, C must do 4, D must do 1

M1

Second matching: A-2, B-3, C-4, D-1

A1

**Answer: A-1, B-2, C-3, D-4; A-2, B-3, C-4, D-1**

4 marks

**10 (4 marks)**

Unmatched vertices: B and 3

B1

Alternating path: B – 1 (not in M), 1 – A (in M), A – 2 (not in M), 2 – C (in M), C – 3 (not in M)

M1 A1

Remove A1 and C2 from M and add B1, A2 and C3: complete matching A-2, B-1, C-3

A1

**Answer: A-2, B-1, C-3**

4 marks

**11 (4 marks)**

A, B, C can only do tasks 1 and 2, so at most 2 of them can be given a task

B1

D and E can be given tasks 3 and 4 (D-3, E-4)

M1

A maximum matching: A-1, B-2, D-3, E-4 (size 4)

A1

There are only 4 tasks, so at most 4 people can be matched, hence no matching of 5

B1

**Answer: 4 matched, e.g. A-1, B-2, D-3, E-4**

4 marks

**12 (4 marks)**

Consider A-1: then C must do 2 or 3, B the other

M1

A-1, B-3, C-2 and A-1, B-2, C-3

A1

Consider A-2: then C must do 3 and B must do 1

M1

A-2, B-1, C-3. There are exactly 3 complete matchings

A1

**Answer: A-1, B-2, C-3; A-1, B-3, C-2; A-2, B-1, C-3**

4 marks

**13 (4 marks)**

Bipartite graph with runners on one side and legs on the other, with edges P1, P3, Q2, R2, R3, R4, S1, S4

B1

Q can only run leg 2, so Q-2, and then R must run 3 or 4

M1

Try R-3: P must run 1 and S must run 4

M1

Complete matching: P-1, Q-2, R-3, S-4

A1

**Answer: P-1, Q-2, R-3, S-4**

4 marks

**14 (6 marks)**

Unmatched vertices: E, F and 4, 6

B1

Alternating path from E: E – 5 (not in M), 5 – B (in M), B – 6 (not in M). Remove B5 from M and add E5, B6; M is now A-1, B-6, C-3, D-2, E-5

M1 A1

Alternating path from F: F – 1 (not in M), 1 – A (in M), A – 3 (not in M), 3 – C (in M), C – 4 (not in M). Remove A1, C3 from M and add F1, A3, C4; M is now A-3, B-6, C-4, D-2, E-5, F-1

M1 A1

Complete matching: A-3, B-6, C-4, D-2, E-5, F-1

A1

**Answer: A-3, B-6, C-4, D-2, E-5, F-1**

6 marks

**15 (6 marks)**

(a) Take  $S = \{A, B, C\}$ , the workers A, B and C

B1

They can only do tasks 1 and 2, so the set  $N(S) = \{1, 2\}$  has 2 elements

M1

$|N(S)| = 2 < 3 = |S|$ , so Hall's condition fails and there is no complete matching

A1

**Answer: S = A, B, C, N(S) = 1, 2**

3 marks

(b) A matching of size 5: A-1, B-2, D-3, E-4, F-5

M1 A1

Only 5 of the 6 workers can be matched, since A, B, C can use only two tasks between them (so at most 5 workers in total can be matched)

B1

**Answer: A-1, B-2, D-3, E-4, F-5**

3 marks

**16 (6 marks)**

Consider the task given to A in turn, and then to B, C and D, checking that every task is used exactly once

M1

Matching 1: A-1, B-4, C-2, D-3

B1

Matching 2: A-1, B-4, C-3, D-2

B1

Matching 3: A-2, B-1, C-3, D-4

B1

Matching 4: A-4, B-1, C-2, D-3

B1

Matching 5: A-4, B-1, C-3, D-2

B1

**Answer: 5 complete matchings**

6 marks

**17 (7 marks)**

(a) Replace each team by two vertices, one for each place in the team (1a, 1b, 2a, 2b, 3a, 3b), each joined to every student willing to join that team

M1

There are 6 vertices (students) in one set and 6 vertices (places) in the other

A1

**Answer: Six students and six places**

2 marks

(b) B can only join team 1 and D can only join team 3, so B-1 and D-3

B1

Each team needs one more student: A takes the second place in team 1, F takes the second place in team 3, and C and E fill team 2

M1

Team 1: A, B; team 2: C, E; team 3: D, F. Each student is willing to join the team given

A1

**Answer: Team 1: A, B; team 2: C, E; team 3: D, F**

3 marks

(c) The two places in team 3 can now only be filled by D, so the set of these two places has only 1 neighbour

M1

$1 < 2$ , so Hall's condition fails for this set of places and no allocation exists

A1

**Answer: Only D is willing to join team 3, but it needs two students**

2 marks

**18 (6 marks)**

(a) A matching: A-3, B-5, C-2, D-4, F-6

M1 A1

**Answer: A-3, B-5, C-2, D-4, F-6**

2 marks

(b) Every edge from B, C, E goes to 2 or 5; every edge from A, D, F contains A, D or F

M1 A1

Each edge of a matching must contain a different vertex of this set

M1

The set has only 5 vertices, so a matching has at most 5 edges, so no matching of size 6 exists

A1

**Answer: No matching of size 6**

4 marks

**19 (6 marks)**

(a) Workers B, C and D can only do tasks 2 and 3

B1

Three workers share two tasks, so  $|N(S)| = 2 < 3 = |S|$  for  $S = \{B, C, D\}$  and Hall's condition fails

E1

**Answer: Hall's condition fails for B, C, D**

2 marks

(b) With D-4: complete matching A-1, B-2, C-3, D-4, E-5

M1 A1

**Answer: A-1, B-2, C-3, D-4, E-5****2 marks**

(c) Workers A, B, C, D can only do tasks 1, 2 and 3, so the extra edge must join one of them to task 4 or 5. If it starts at A, workers B, C, D still have only tasks 2 and 3, so it must start at B, C or D

**M1**

Each of the six edges B4, B5, C4, C5, D4, D5 works, by the same method as in (b)

**A1****Answer: Any of B, C, D joined to 4 or 5****2 marks****20 (5 marks)**

(a) A can only do task 1, so A-1; B must then do 2

**M1**

C and D share tasks 3 and 4: C-3, D-4 or C-4, D-3

**M1**

Complete matchings: A-1, B-2, C-3, D-4 and A-1, B-2, C-4, D-3

**A1****Answer: A-1, B-2, C-3, D-4; A-1, B-2, C-4, D-3****3 marks**

(b) A-1 and B-2 are in both matchings

**B1**

A has only one possible task and B is then forced onto task 2

**E1****Answer: A-1 and B-2****2 marks****21 (7 marks)**

There are 15 non-empty subsets of X, which are checked by size

**M1**

Subsets of size 1 (S: N(S)): {A}: {1, 2}; {B}: {2}; {C}: {2, 3, 4}; {D}: {3, 4}; in every case

**B1**

$|N(S)| \geq |S|$

Subsets of size 2 (S: N(S)): {A, B}: {1, 2}; {A, C}: {1, 2, 3, 4}; {A, D}: {1, 2, 3, 4}; {B, C}: {2, 3, 4}; {B, D}: {2, 3, 4}; {C, D}: {2, 3, 4}; in every case  $|N(S)| \geq |S|$

**B1**

Subsets of size 3 (S: N(S)): {A, B, C}: {1, 2, 3, 4}; {A, B, D}: {1, 2, 3, 4}; {A, C, D}: {1, 2, 3, 4}; {B, C, D}: {2, 3, 4}; in every case  $|N(S)| \geq |S|$

**B1**

Subsets of size 4 (S: N(S)): {A, B, C, D}: {1, 2, 3, 4}; in every case  $|N(S)| \geq |S|$

**B1**

By Hall's theorem a complete matching exists

**A1**

B can only do 2, so B-2; A must do 1, then C and D share 3 and 4: A-1, B-2, C-3, D-4

**A1****Answer: A-1, B-2, C-3, D-4****7 marks****22 (7 marks)**

(a) Unmatched vertices: F and 1

**B1**

Alternating path from F: F – 6 (not in M), 6 – E (in M), E – 5 (not in M), 5 – D (in M), D – 4 (not in M), 4 – C (in M), C – 3 (not in M), 3 – B (in M), B – 2 (not in M), 2 – A (in M), A – 1 (not in M). Remove E6, D5, C4, B3, A2 from M and add F6, E5, D4, C3, B2, A1; M is now A-1, B-2, C-3, D-4, E-5, F-6

**M1 A1**

Complete matching: A-1, B-2, C-3, D-4, E-5, F-6

**A1****Answer: A-1, B-2, C-3, D-4, E-5, F-6****4 marks**

(b) F can only do task 6, so the path begins F – 6

**B1**

Task 6 is matched to E, and E can then only continue to 5 (the only other task E can do), and so on: at each stage there is only one new vertex available

**M1**

So the path is forced through E, D, C, B, A, finishing at the unmatched task 1

**A1****Answer: The path is forced****3 marks****23 (6 marks)**

(a) A-3 is not an edge of the graph (A can only do tasks 1 and 2)

**M1**

So D, 1, A, 3, C, 4 is not a path in the graph and the swap is invalid

**E1****Answer: A3 is not an edge****2 marks**

(b) From A the only other edge is A-2 (not in M), and 2 is matched to B

**M1**

D, 1, A, 2, B, 3, C, 4

**A1****Answer: D, 1, A, 2, B, 3, C, 4****2 marks**

(c) Remove A1, B2, C3 from M and add D1, A2, B3, C4

**M1**

Complete matching: A-2, B-3, C-4, D-1

**A1****Answer: A-2, B-3, C-4, D-1****2 marks**

**24 (7 marks)****(a)** Take  $S = \{A, B, C, D\}$ 

M1

These four workers can only do tasks 1 and 2, so  $N(S) = \{1, 2\}$ 

A1

 $|N(S)| = 2 = 4 - 2$ 

A1

**Answer:  $S = A, B, C, D$** 

3 marks

**(b)** At most 2 of A, B, C, D can be matched, and at most 3 of E, F, G, so at most 5 workers can be matched

M1 A1

A matching of size 5: A-2, B-1, E-3, F-4, G-5

A1

**Answer: A-2, B-1, E-3, F-4, G-5**

3 marks

**(c)** From (b), 5

B1

**Answer: 5**

1 mark

**25 (7 marks)****(a)** A and B can only do tasks 1 and 2, so they use up those tasks and C, D, E must use 3, 4, 5

M1

A and B: 2 ways (A-1, B-2 or A-2, B-1)

A1

C, D, E:  $3! = 6$  ways

A1

**Answer: 2 and 6**

3 marks

**(b)** Independent choices, so  $2 \times 6 = 12$ 

B1

**Answer: 12**

1 mark

**(c)** Count the ways for C, D, E with E restricted to tasks 3 or 4: E has 2 choices

M1

Then C and D can take the other two tasks (including task 5) in 2 ways:  $2 \times 2 = 4$ 

M1

Total =  $2 \times 4 = 8$ 

A1

**Answer: 8**

3 marks

**26 (8 marks)**

Network: source S, a vertex for each worker, a vertex for each task and sink T

B1

Arcs S to each worker, each task to T, and a worker to a task whenever the worker can do the task; every arc has capacity 1

B1 B1

Augmenting routes (route: amount):  $S \rightarrow A \rightarrow 3 \rightarrow T: 1$ ;  $S \rightarrow B \rightarrow 1 \rightarrow T: 1$ ;  $S \rightarrow C \rightarrow 2 \rightarrow T: 1$ ;  $S \rightarrow D \rightarrow 5 \rightarrow T: 1$ ;  $S \rightarrow E \rightarrow 6 \rightarrow T: 1$ ;  $S \rightarrow F \rightarrow 2 \rightarrow C \rightarrow 4 \rightarrow T: 1$ 

M1 A1

Maximum flow is 6, so the maximum matching has 6 edges

B1

Flow along worker to task arcs: A-3, B-1, C-4, D-5, E-6, F-2

M1

Every worker and every task is used once, so this is a complete matching

A1

**Answer: Maximum matching of size 6: A-3, B-1, C-4, D-5, E-6, F-2**

8 marks

**27 (10 marks)****(a)** P has  $k$  edges not in M and  $k - 1$  edges in M (it begins and ends with an edge not in M)

M1

 $M \oplus P$  removes the  $k - 1$  edges of M on P and adds the  $k$  other edges, so it has  $|M| + 1$  edges

M1

Each internal vertex of P was matched to one neighbour on P and is now matched to the other; the two end vertices were unmatched and are now matched once

M1

Vertices off P are unaffected, so no vertex lies on two edges and  $M \oplus P$  is a matching

A1

**Answer: Proved**

4 marks

**(b)** Let  $M'$  be a larger matching, and consider the graph H formed by the edges of  $M \oplus M'$ 

B1

Every vertex has at most one edge from M and at most one from  $M'$  so has degree at most 2 in H

M1

So each component of H is a path or an even cycle, and along each component the edges alternate between M and  $M'$ 

M1

In a cycle there are equal numbers of M and  $M'$  edges

B1

Since  $|M'| > |M|$ , some component is a path with more edges from  $M'$  than from M, so it starts and ends with an edge of  $M'$ 

M1

The ends of this path are not matched in M, so it is an augmenting path for M

A1

**Answer: Proved**

6 marks

**28 (10 marks)****(a)** The edges of M share no vertices, so a vertex cover needs a different vertex for each edge of M

M1

So any cover has at least  $|M|$  vertices

A1

**Answer: Proved**

2 marks

|                                                                                                                                                                                                 |                |
|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------|
| <b>(b)</b> Consider an edge $xy$ with $x$ in $X$ and $y$ in $Y$ . If $x$ is not in $Z$ then $x$ is in $K$                                                                                       | <b>B1</b>      |
| If $x$ is in $Z$ and the edge $xy$ is not in $M$ , then $y$ is reached by one more step along the alternating path, so $y$ is in $Z$ and in $K$                                                 | <b>M1</b>      |
| If $xy$ is in $M$ , then $x$ in $Z$ must have been reached via $y$ , so $y$ is in $Z$ and in $K$ ; so every edge is covered                                                                     | <b>A1</b>      |
| <b>Answer: <math>K</math> is a cover</b>                                                                                                                                                        | <b>3 marks</b> |
| <b>(c)</b> Every unmatched vertex of $X$ is in $Z$ , so is not in $K$ ; every vertex of $K$ is therefore matched                                                                                | <b>M1</b>      |
| An unmatched vertex $y$ in $Y \cap Z$ would give an augmenting path, impossible as $M$ is maximum                                                                                               | <b>M1</b>      |
| For each edge $xy$ of $M$ , not both $x$ and $y$ are in $K$ : if $y$ is in $Z$ then $x$ is in $Z$ (reached through $y$ ), and if $y$ is not in $Z$ then $x$ is in $K$ only if $x$ is not in $Z$ | <b>M1</b>      |
| Each edge of $M$ therefore contributes exactly one vertex to $K$ , so $ K  =  M $                                                                                                               | <b>A1</b>      |
| With (a), the minimum vertex cover has size $ M $                                                                                                                                               | <b>A1</b>      |
| <b>Answer: Proved</b>                                                                                                                                                                           | <b>5 marks</b> |

**29 (10 marks)**

|                                                                                                                                                  |                |
|--------------------------------------------------------------------------------------------------------------------------------------------------|----------------|
| <b>(a)</b> Let $S$ be a subset of $X$ . Count the edges leaving $S$ : there are at least $d S $ of them                                          | <b>M1</b>      |
| All of these edges end at vertices in $N(S)$                                                                                                     | <b>B1</b>      |
| Each vertex of $N(S)$ has degree at most $d$ , so receives at most $d N(S) $ edges                                                               | <b>M1</b>      |
| So $d S  \leq d N(S) $                                                                                                                           | <b>M1</b>      |
| Since $d > 0$ , $ N(S)  \geq  S $                                                                                                                | <b>A1</b>      |
| By Hall's theorem there is a matching using every vertex of $X$                                                                                  | <b>A1</b>      |
| <b>Answer: Proved</b>                                                                                                                            | <b>6 marks</b> |
| <b>(b)</b> Apply (a) with $d = 3$ : every worker has degree at least 3 and every task has degree at most 3                                       | <b>M1</b>      |
| So there is a matching using every worker, which is complete since there are 6 tasks                                                             | <b>A1</b>      |
| <b>Answer: <math>d = 3</math> satisfies the conditions</b>                                                                                       | <b>2 marks</b> |
| <b>(c)</b> Let workers $A, B, C$ each be able to do only tasks 1 and 2, with a third task 3 which nobody can do: every worker has degree $2 = d$ | <b>M1</b>      |
| Tasks 1 and 2 have degree $3 > d$ , and the three workers cannot all be matched to two tasks                                                     | <b>A1</b>      |
| <b>Answer: <math>A, B, C</math> all adjacent to 1 and 2 only</b>                                                                                 | <b>2 marks</b> |

**30 (12 marks)**

|                                                                                                                                                                                                                                                                                                                                           |                |
|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------|
| <b>(a)</b> Unmatched vertices: $E, F, H$ and $4, 7, 8$                                                                                                                                                                                                                                                                                    | <b>B1</b>      |
| Alternating path from $E$ : $E - 3$ (not in $M$ ), $3 - D$ (in $M$ ), $D - 7$ (not in $M$ ). Remove $D3$ from $M$ and add $E3, D7$ ; $M$ is now $A-1, B-2, C-5, D-7, E-3, G-6$                                                                                                                                                            | <b>M1 A1</b>   |
| Alternating path from $F$ : $F - 3$ (not in $M$ ), $3 - E$ (in $M$ ), $E - 2$ (not in $M$ ), $2 - B$ (in $M$ ), $B - 6$ (not in $M$ ), $6 - G$ (in $M$ ), $G - 5$ (not in $M$ ), $5 - C$ (in $M$ ), $C - 8$ (not in $M$ ). Remove $E3, B2, G6, C5$ from $M$ and add $F3, E2, B6, G5, C8$ ; $M$ is now $A-1, B-6, C-8, D-7, E-2, F-3, G-5$ | <b>M1 A1</b>   |
| The matching $A-1, B-6, C-8, D-7, E-2, F-3, G-5$ has size 7                                                                                                                                                                                                                                                                               | <b>B1</b>      |
| <b>Answer: Matching of size 7: <math>A-1, B-6, C-8, D-7, E-2, F-3, G-5</math></b>                                                                                                                                                                                                                                                         | <b>6 marks</b> |
| <b>(b)</b> Take $S = \{A, E, F, H\}$                                                                                                                                                                                                                                                                                                      | <b>B1</b>      |
| These workers can only do tasks 1, 2 and 3, so $N(S) = \{1, 2, 3\}$                                                                                                                                                                                                                                                                       | <b>M1</b>      |
| $ N(S)  = 3 < 4 =  S $ , so no complete matching exists                                                                                                                                                                                                                                                                                   | <b>A1</b>      |
| <b>Answer: <math>S = A, E, F, H</math></b>                                                                                                                                                                                                                                                                                                | <b>3 marks</b> |
| <b>(c)</b> Workers $B, C, D, G$ and tasks 1, 2, 3: every edge from $A, E, F, H$ goes to task 1, 2 or 3, and every other edge starts at $B, C, D$ or $G$                                                                                                                                                                                   | <b>M1 A1</b>   |
| A matching needs a different vertex of this set for each of its edges, so has at most 7 edges                                                                                                                                                                                                                                             | <b>A1</b>      |
| <b>Answer: <math>B, C, D, G</math> and 1, 2, 3</b>                                                                                                                                                                                                                                                                                        | <b>3 marks</b> |

**Allocation Algorithms****1 (2 marks)**

|                                                                                                                               |                |
|-------------------------------------------------------------------------------------------------------------------------------|----------------|
| There are $n$ workers and $n$ jobs, and each worker is given exactly one job, with a cost (or time) for each possible pairing | <b>B1</b>      |
| The aim is to allocate the jobs so that the total cost is a minimum                                                           | <b>B1</b>      |
| <b>Answer: Match <math>n</math> workers to <math>n</math> jobs with least total cost</b>                                      | <b>2 marks</b> |

**2 (2 marks)**

Row minima are 7, 8 and 11

M1

Reduced matrix  $\begin{pmatrix} 2 & 0 & 6 \\ 0 & 12 & 11 \\ 9 & 6 & 0 \end{pmatrix}$ 

A1

**Answer:**  $\begin{pmatrix} 2 & 0 & 6 \\ 0 & 12 & 11 \\ 9 & 6 & 0 \end{pmatrix}$ 

2 marks

**3 (2 marks)**

Column minima are 1, 1, 2

M1

Reduced matrix  $\begin{pmatrix} 2 & 0 & 4 \\ 0 & 11 & 9 \\ 8 & 6 & 0 \end{pmatrix}$ 

A1

**Answer:**  $\begin{pmatrix} 2 & 0 & 4 \\ 0 & 11 & 9 \\ 8 & 6 & 0 \end{pmatrix}$ 

2 marks

**4 (3 marks)**

Zeros are at A2, B1, C3, so the allocation is A-2, B-1, C-3

M1 A1

Cost = 7 + 8 + 11 = 26

A1

**Answer: A-2, B-1, C-3, cost 26**

3 marks

**5 (2 marks)**

Subtract every entry of the table from the largest entry in the table

B1

Then minimise the total of the new values: this gives the allocation which maximises the original total

B1

**Answer: Subtract all entries from the largest and minimise**

2 marks

**6 (2 marks)**

Add a dummy worker (a fourth row) so that the table is square

B1

All entries in the dummy row are the same, e.g. 0

B1

**Answer: A dummy row of zeros**

2 marks

**7 (4 marks)**Subtract the row minimum from each row:  $\begin{pmatrix} 0 & 11 & 14 & 10 \\ 7 & 8 & 0 & 0 \\ 7 & 7 & 0 & 5 \\ 6 & 1 & 0 & 7 \end{pmatrix}$ 

M1

Subtract the column minimum from each column:  $\begin{pmatrix} 0 & 10 & 14 & 10 \\ 7 & 7 & 0 & 0 \\ 7 & 6 & 0 & 5 \\ 6 & 0 & 0 & 7 \end{pmatrix}$ 

A1

4 lines are now needed (rows A, B, C, D), so an allocation using zeros exists: A-1, B-4, C-3, D-2

M1

Cost from the original table = 5 + 10 + 5 + 10 = 30

A1

**Answer: A-1, B-4, C-3, D-2, cost 30**

4 marks

**8 (3 marks)**

(i) 11 + 6 + 6 + 9 = 32

M1 A1

(ii) 9 + 15 + 6 + 12 = 42, so (i) is cheaper

A1

**Answer: (i) 32, (ii) 42: (i) is cheaper**

3 marks

**9 (5 marks)**

Largest entry is 16: subtract every entry from 16:  $\begin{pmatrix} 2 & 7 & 5 \\ 8 & 0 & 6 \\ 4 & 9 & 1 \end{pmatrix}$  M1

Subtract the row minimum from each row:  $\begin{pmatrix} 0 & 5 & 3 \\ 8 & 0 & 6 \\ 3 & 8 & 0 \end{pmatrix}$  M1

Subtract the column minimum from each column:  $\begin{pmatrix} 0 & 5 & 3 \\ 8 & 0 & 6 \\ 3 & 8 & 0 \end{pmatrix}$  A1

3 lines are now needed (rows A, B, C), so an allocation using zeros exists: A-1, B-2, C-3 M1

Profit from the original table = 14 + 16 + 15 = 45 A1

**Answer: A-1, B-2, C-3, profit 45 (thousand pounds)** 5 marks

**10 (5 marks)**

Add a dummy worker with a cost of 0 for every job:  $\begin{pmatrix} 8 & 12 & 9 & 14 \\ 11 & 7 & 13 & 10 \\ 10 & 9 & 8 & 12 \\ 0 & 0 & 0 & 0 \end{pmatrix}$  B1

Subtract the row minimum from each row:  $\begin{pmatrix} 0 & 4 & 1 & 6 \\ 4 & 0 & 6 & 3 \\ 2 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$  M1

Subtract the column minimum from each column:  $\begin{pmatrix} 0 & 4 & 1 & 6 \\ 4 & 0 & 6 & 3 \\ 2 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$  A1

4 lines are now needed (rows A, B, C, Dummy), so an allocation using zeros exists: A-1, B-2, C-3, Dummy-4 M1

Cost from the original table (ignoring the dummy worker) = 8 + 7 + 8 = 23 A1

**Answer: A-1, B-2, C-3, cost 23, job 4 not done** 5 marks

**11 (4 marks)**

Row minima are 16, 19, 13, 24; the reduced matrix is  $\begin{pmatrix} 2 & 7 & 0 & 4 \\ 2 & 7 & 0 & 4 \\ 2 & 7 & 0 & 4 \\ 2 & 7 & 0 & 4 \end{pmatrix}$  M1

Column minima are 2, 7, 0, 4; the matrix is now  $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$  A1

Every entry is zero, so every allocation uses only zeros and every allocation has the same total cost E1

That cost is 16 + 19 + 13 + 24 + 2 + 7 + 0 + 4 = 85 pounds B1

**Answer: Every allocation costs 85 pounds** 4 marks

**12 (8 marks)**

**(a)** A takes job 1 (cost 14), the cheapest of A's costs M1

B then takes job 3 (cost 13), the cheaper of jobs 2 and 3, and C is left with job 2 (cost 11) A1

Allocation A-1, B-3, C-2 with total cost 14 + 13 + 11 = 38 pounds A1

**Answer: A-1, B-3, C-2, cost 38 pounds** 3 marks

**(b)** Subtract the row minimum from each row:  $\begin{pmatrix} 0 & 3 & 8 \\ 0 & 14 & 7 \\ 0 & 4 & 2 \end{pmatrix}$  M1

Subtract the column minimum from each column:  $\begin{pmatrix} 0 & 0 & 6 \\ 0 & 11 & 5 \\ 0 & 1 & 0 \end{pmatrix}$  A1

3 lines are now needed (rows A, B, C), so an allocation using zeros exists: A-2, B-1, C-3 M1

Cost from the original table =  $17 + 6 + 9 = 32$  pounds A1

The least cost is 32 pounds, which is less than the manager's 38 pounds B1

**Answer: A-2, B-1, C-3, cost 32 pounds (manager's allocation costs 6 pounds more)** 5 marks

### 13 (4 marks)

Zeros can be covered by 2 lines (row A and column 3), fewer than 3, so no allocation using zeros exists B1

Smallest uncovered entry is 1: subtract it from uncovered entries, add it to doubly covered entries: M1 A1

$$\begin{pmatrix} 0 & 0 & 8 \\ 11 & 1 & 0 \\ 4 & 0 & 0 \end{pmatrix}$$

Zeros give A-1, B-3, C-2 with cost 30 A1

**Answer: A-1, B-3, C-2, cost 30** 4 marks

### 14 (6 marks)

Subtract the row minimum from each row:  $\begin{pmatrix} 21 & 24 & 0 & 1 \\ 25 & 0 & 14 & 5 \\ 7 & 13 & 0 & 2 \\ 11 & 0 & 1 & 17 \end{pmatrix}$  M1

Subtract the column minimum from each column:  $\begin{pmatrix} 14 & 24 & 0 & 0 \\ 18 & 0 & 14 & 4 \\ 0 & 13 & 0 & 1 \\ 4 & 0 & 1 & 16 \end{pmatrix}$  A1

Zeros need 3 lines (rows A, C and column 2), fewer than 4. Smallest uncovered entry is 1: M1 A1

subtract it from uncovered entries and add it to entries covered twice:  $\begin{pmatrix} 14 & 25 & 0 & 0 \\ 17 & 0 & 13 & 3 \\ 0 & 14 & 0 & 1 \\ 3 & 0 & 0 & 15 \end{pmatrix}$

4 lines are now needed (rows A, B, C, D), so an allocation using zeros exists: A-4, B-2, C-1, D-3 M1

Cost from the original table =  $15 + 12 + 29 + 17 = 73$  seconds A1

**Answer: A-4, B-2, C-1, D-3, total 73 seconds** 6 marks

### 15 (5 marks)

Largest entry is 57: subtract every entry from 57:  $\begin{pmatrix} 10 & 34 & 1 & 36 \\ 0 & 21 & 18 & 11 \\ 25 & 26 & 30 & 0 \\ 4 & 1 & 34 & 17 \end{pmatrix}$  M1

Subtract the row minimum from each row:  $\begin{pmatrix} 9 & 33 & 0 & 35 \\ 0 & 21 & 18 & 11 \\ 25 & 26 & 30 & 0 \\ 3 & 0 & 33 & 16 \end{pmatrix}$  M1

Subtract the column minimum from each column:  $\begin{pmatrix} 9 & 33 & 0 & 35 \\ 0 & 21 & 18 & 11 \\ 25 & 26 & 30 & 0 \\ 3 & 0 & 33 & 16 \end{pmatrix}$  A1

4 lines are now needed (rows A, B, C, D), so an allocation using zeros exists: A-3, B-1, C-4, D-2 M1

Profit from the original table =  $56 + 57 + 57 + 56 = 226$  A1

**Answer: A-3, B-1, C-4, D-2, total 226 (thousand pounds)** 5 marks

**16 (7 marks)**

Replace the impossible entry for B and job 3 by 99, which is larger than every other cost:

**B1**

$$\begin{pmatrix} 18 & 24 & 27 & 16 \\ 24 & 19 & 99 & 12 \\ 13 & 27 & 26 & 30 \\ 15 & 27 & 27 & 17 \end{pmatrix}$$

Subtract the row minimum from each row:

$$\begin{pmatrix} 2 & 8 & 11 & 0 \\ 12 & 7 & 87 & 0 \\ 0 & 14 & 13 & 17 \\ 0 & 12 & 12 & 2 \end{pmatrix}$$

**M1**

Subtract the column minimum from each column:

$$\begin{pmatrix} 2 & 1 & 0 & 0 \\ 12 & 0 & 76 & 0 \\ 0 & 7 & 2 & 17 \\ 0 & 5 & 1 & 2 \end{pmatrix}$$

**A1**

Zeros need 3 lines (rows A, B and column 1), fewer than 4. Smallest uncovered entry is 1:

**M1 A1**

subtract it from uncovered entries and add it to entries covered twice:

$$\begin{pmatrix} 3 & 1 & 0 & 0 \\ 13 & 0 & 76 & 0 \\ 0 & 6 & 1 & 16 \\ 0 & 4 & 0 & 1 \end{pmatrix}$$

4 lines are now needed (rows A, B, C, D), so an allocation using zeros exists: A-4, B-2, C-1, D-3

**M1**

Cost from the original table =  $16 + 19 + 13 + 27 = 75$

**A1**

**Answer: A-4, B-2, C-1, D-3, cost 75**

**7 marks****17 (7 marks)**

Add a dummy job with a cost of 0 for every worker:

$$\begin{pmatrix} 20 & 13 & 14 & 13 & 0 \\ 14 & 13 & 10 & 12 & 0 \\ 17 & 8 & 22 & 22 & 0 \\ 11 & 8 & 24 & 13 & 0 \\ 23 & 22 & 17 & 23 & 0 \end{pmatrix}$$

**B1**

Subtract the row minimum from each row:

$$\begin{pmatrix} 20 & 13 & 14 & 13 & 0 \\ 14 & 13 & 10 & 12 & 0 \\ 17 & 8 & 22 & 22 & 0 \\ 11 & 8 & 24 & 13 & 0 \\ 23 & 22 & 17 & 23 & 0 \end{pmatrix}$$

**M1**

Subtract the column minimum from each column:

$$\begin{pmatrix} 9 & 5 & 4 & 1 & 0 \\ 3 & 5 & 0 & 0 & 0 \\ 6 & 0 & 12 & 10 & 0 \\ 0 & 0 & 14 & 1 & 0 \\ 12 & 14 & 7 & 11 & 0 \end{pmatrix}$$

**A1**

Zeros need 4 lines (rows B, C, D and column Dummy), fewer than 5. Smallest uncovered entry is

**M1 A1**

1: subtract it from uncovered entries and add it to entries covered twice:

$$\begin{pmatrix} 8 & 4 & 3 & 0 & 0 \\ 3 & 5 & 0 & 0 & 1 \\ 6 & 0 & 12 & 10 & 1 \\ 0 & 0 & 14 & 1 & 1 \\ 11 & 13 & 6 & 10 & 0 \end{pmatrix}$$

5 lines are now needed (rows A, B, C, D, E), so an allocation using zeros exists: A-4, B-3, C-2, D-1, E-Dummy

**M1**

Cost from the original table (ignoring the dummy job) =  $13 + 10 + 8 + 11 = 42$

**A1**

**Answer: A-4, B-3, C-2, D-1, cost 42, worker E not used**

**7 marks**

**18 (5 marks)**

Subtract the row minimum from each row:  $\begin{pmatrix} 10 & 0 & 18 & 19 \\ 7 & 18 & 0 & 5 \\ 16 & 5 & 13 & 0 \\ 7 & 17 & 0 & 0 \end{pmatrix}$  M1

Subtract the column minimum from each column:  $\begin{pmatrix} 3 & 0 & 18 & 19 \\ 0 & 18 & 0 & 5 \\ 9 & 5 & 13 & 0 \\ 0 & 17 & 0 & 0 \end{pmatrix}$  A1

4 lines are now needed (rows A, B, C, D), so an allocation using zeros exists: A-2, B-3, C-4, D-1 M1

Cost from the original table =  $10 + 10 + 13 + 18 = 51$  A1

The zeros also give a second allocation: A-2, B-1, C-4, D-3, with the same cost 51 B1

**Answer: A-2, B-1, C-4, D-3 or A-2, B-3, C-4, D-1, cost 51** 5 marks

**19 (7 marks)**

(a) Row minima are 13, 21, 11, 10, sum 55 M1 A1

Column minima of the row reduced matrix are 3, 1, 0, 0, sum 4 A1

**Answer: 55 and 4** 3 marks

(b) Every allocation uses exactly one entry from each row and each column, so its cost is at least the total of the row minima plus the column minima after reduction M1

That total is  $55 + 4 = 59$  E1

**Answer: At least 59** 2 marks

(c) The reduced matrix  $\begin{pmatrix} 0 & 4 & 3 & 0 \\ 4 & 3 & 0 & 0 \\ 14 & 7 & 0 & 14 \\ 16 & 0 & 3 & 0 \end{pmatrix}$  has an allocation on zeros: A-1, B-4, C-3, D-2 M1

Its cost is  $16 + 21 + 11 + 11 = 59$ , equal to the lower bound, so it is optimal A1

**Answer: A-1, B-4, C-3, D-2, cost 59** 2 marks

**20 (7 marks)**

Subtract the row minimum from each row:  $\begin{pmatrix} 5 & 13 & 0 & 21 \\ 18 & 16 & 18 & 0 \\ 6 & 0 & 11 & 0 \\ 22 & 0 & 20 & 13 \end{pmatrix}$  M1

Subtract the column minimum from each column:  $\begin{pmatrix} 0 & 13 & 0 & 21 \\ 13 & 16 & 18 & 0 \\ 1 & 0 & 11 & 0 \\ 17 & 0 & 20 & 13 \end{pmatrix}$  A1

Zeros need 3 lines (row A and columns 2, 4), fewer than 4. Smallest uncovered entry is 1: subtract M1 A1

it from uncovered entries and add it to entries covered twice:  $\begin{pmatrix} 0 & 14 & 0 & 22 \\ 12 & 16 & 17 & 0 \\ 0 & 0 & 10 & 0 \\ 16 & 0 & 19 & 13 \end{pmatrix}$

The zeros of the column reduced matrix can be covered by only 3 lines, fewer than 4, so no allocation uses zeros only E1

4 lines are now needed (rows A, B, C, D), so an allocation using zeros exists: A-3, B-4, C-1, D-2 M1

Cost from the original table =  $12 + 12 + 29 + 11 = 64$  A1

**Answer: A-3, B-4, C-1, D-2, cost 64** 7 marks

**21 (8 marks)**

Subtract the row minimum from each row:

$$\begin{pmatrix} 19 & 7 & 17 & 5 & 0 \\ 3 & 13 & 19 & 0 & 8 \\ 0 & 7 & 9 & 8 & 9 \\ 6 & 0 & 3 & 8 & 8 \\ 20 & 4 & 24 & 0 & 17 \end{pmatrix}$$

**M1**

Subtract the column minimum from each column:

$$\begin{pmatrix} 19 & 7 & 14 & 5 & 0 \\ 3 & 13 & 16 & 0 & 8 \\ 0 & 7 & 6 & 8 & 9 \\ 6 & 0 & 0 & 8 & 8 \\ 20 & 4 & 21 & 0 & 17 \end{pmatrix}$$

**A1**

Zeros need 4 lines (rows A, C, D and column 4), fewer than 5. Smallest uncovered entry is 3:

**M1 A1**

subtract it from uncovered entries and add it to entries covered twice:

$$\begin{pmatrix} 19 & 7 & 14 & 8 & 0 \\ 0 & 10 & 13 & 0 & 5 \\ 0 & 7 & 6 & 11 & 9 \\ 6 & 0 & 0 & 11 & 8 \\ 17 & 1 & 18 & 0 & 14 \end{pmatrix}$$

Zeros need 4 lines (rows A, D and columns 1, 4), fewer than 5. Smallest uncovered entry is 1:

**M1 A1**

subtract it from uncovered entries and add it to entries covered twice:

$$\begin{pmatrix} 20 & 7 & 14 & 9 & 0 \\ 0 & 9 & 12 & 0 & 4 \\ 0 & 6 & 5 & 11 & 8 \\ 7 & 0 & 0 & 12 & 8 \\ 17 & 0 & 17 & 0 & 13 \end{pmatrix}$$

5 lines are now needed (rows A, B, C, D, E), so an allocation using zeros exists: A-5, B-4, C-1, D-3, E-2

**M1**

Cost from the original table = 16 + 17 + 11 + 25 + 20 = 89 pounds

**A1****Answer: A-5, B-4, C-1, D-3, E-2, cost 89****8 marks**

**22 (8 marks)**

Subtract every profit from the largest value, 40, to form a cost matrix, and replace the impossible entry for C and project 2 by 99, which is larger than every other entry

**M1**

Add a dummy worker with entries 0:

$$\begin{pmatrix} 1 & 24 & 16 & 28 & 20 \\ 27 & 29 & 14 & 27 & 7 \\ 25 & 99 & 30 & 17 & 29 \\ 1 & 29 & 13 & 19 & 23 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

**B1**

Subtract the row minimum from each row:

$$\begin{pmatrix} 0 & 23 & 15 & 27 & 19 \\ 20 & 22 & 7 & 20 & 0 \\ 8 & 82 & 13 & 0 & 12 \\ 0 & 28 & 12 & 18 & 22 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

**M1**

Subtract the column minimum from each column:

$$\begin{pmatrix} 0 & 23 & 15 & 27 & 19 \\ 20 & 22 & 7 & 20 & 0 \\ 8 & 82 & 13 & 0 & 12 \\ 0 & 28 & 12 & 18 & 22 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

**A1**

Zeros need 4 lines (rows B, C, Dummy and column 1), fewer than 5. Smallest uncovered entry is 12: subtract it from uncovered entries and add it to entries covered twice:

**M1 A1**

$$\begin{pmatrix} 0 & 11 & 3 & 15 & 7 \\ 32 & 22 & 7 & 20 & 0 \\ 20 & 82 & 13 & 0 & 12 \\ 0 & 16 & 0 & 6 & 10 \\ 12 & 0 & 0 & 0 & 0 \end{pmatrix}$$

5 lines are now needed (rows A, B, C, D, Dummy), so an allocation using zeros exists: A-1, B-5, C-4, D-3, Dummy-2

**M1**

Profit from the original table (ignoring the dummy) = 39 + 33 + 23 + 27 = 122

**A1**

**Answer: A-1, B-5, C-4, D-3, profit 122 (thousand pounds), project 2 unstaffed**

**8 marks****23 (8 marks)**

(a) The smallest uncovered entry should also be added to every entry covered by two lines (a row line and a column line)

**M1**

The student left the doubly covered entries unchanged

**E1**

**Answer: Doubly covered entries were not increased**

**2 marks**

(b) Correct matrix 
$$\begin{pmatrix} 4 & 0 & 0 & 2 \\ 0 & 0 & 10 & 5 \\ 0 & 12 & 12 & 0 \\ 4 & 2 & 2 & 0 \end{pmatrix}$$

**M1 A1**

Zeros give A-3, B-2, C-1, D-4

**A1**

**Answer: A-3, B-2, C-1, D-4**

**3 marks**

(c) Student: 12 + 21 + 21 + 9 = 63

**M1**

Correct: 12 + 21 + 11 + 15 = 59

**A1**

The student's allocation is not optimal (costs 4 more)

**E1**

**Answer: 63 against 59: student not optimal**

**3 marks****24 (9 marks)**

(a) Total cost = sum of cost times x over all nine pairs

**M1**

$$C = 16 x_{A1} + 5 x_{A2} + 12 x_{A3} + 5 x_{B1} + 17 x_{B2} + 16 x_{B3} + 7 x_{C1} + 18 x_{C2} + 14 x_{C3}$$

**A1**

**Answer: C = 16 x<sub>A1</sub> + 5 x<sub>A2</sub> + 12 x<sub>A3</sub> + 5 x<sub>B1</sub> + 17 x<sub>B2</sub> + 16 x<sub>B3</sub> + 7 x<sub>C1</sub> + 18 x<sub>C2</sub> + 14 x<sub>C3</sub>**

**2 marks**

(b) Each worker:  $x_{i1} + x_{i2} + x_{i3} = 1$  for  $i = A, B, C$

**B1**

Each job:  $x_{A_j} + x_{B_j} + x_{C_j} = 1$  for  $j = 1, 2, 3$

**B1**

3 + 3 = 6 constraints

**B1**

**Answer: 6 equations**

**3 marks**

(c) Subtract the row minimum from each row:  $\begin{pmatrix} 11 & 0 & 7 \\ 0 & 12 & 11 \\ 0 & 11 & 7 \end{pmatrix}$  M1

Subtract the column minimum from each column:  $\begin{pmatrix} 11 & 0 & 0 \\ 0 & 12 & 4 \\ 0 & 11 & 0 \end{pmatrix}$  A1

3 lines are now needed (rows A, B, C), so an allocation using zeros exists: A-2, B-1, C-3 M1

Cost from the original table =  $5 + 5 + 14 = 24$  A1

**Answer:  $x_{A2} = 1, x_{B1} = 1, x_{C3} = 1$ , minimum cost 24** 4 marks

## 25 (9 marks)

(a) Subtract the row minimum from each row:  $\begin{pmatrix} 0 & 3 & 5 & 1 \\ 0 & 12 & 3 & 17 \\ 4 & 11 & 0 & 11 \\ 24 & 3 & 11 & 0 \end{pmatrix}$  M1

Subtract the column minimum from each column:  $\begin{pmatrix} 0 & 0 & 5 & 1 \\ 0 & 9 & 3 & 17 \\ 4 & 8 & 0 & 11 \\ 24 & 0 & 11 & 0 \end{pmatrix}$  A1

4 lines are now needed (rows A, B, C, D), so an allocation using zeros exists: A-2, B-1, C-3, D-4 M1

Cost from the original table =  $37 + 18 + 28 + 10 = 93$  A1

Longest individual time is 37 minutes B1

**Answer: A-2, B-1, C-3, D-4, total 93, longest 37** 5 marks

(b) Allow only times of at most 32: then at least one worker has no available task, so no complete allocation is possible M1

Allowing times of at most 34 gives edges A1, B1, B2, B3, C1, C3, D1, D2, D3, D4 M1

A complete matching exists: A-1, B-2, C-3, D-4 A1

So the least possible longest time is 34 minutes A1

**Answer: A-1, B-2, C-3, D-4, longest time 34** 4 marks

## 26 (7 marks)

(a) There are four such allocations, with costs: A-2, B-1, C-3: 41; A-2, B-3, C-1: 41; A-3, B-1, C-2: 38; A-3, B-2, C-1: 42 M1

Least cost is 38, with A-3, B-1, C-2 A1

**Answer: 38** 2 marks

(b) Remaining workers B and C take jobs 2 and 3 in two ways: B2, C3 costs 25 and B3, C2 costs 21 M1

Least cost is  $k + 21$  A1

**Answer:  $k + 21$**  2 marks

(c) A does job 1 in a cheapest allocation if  $k + 21 \leq 38$  M1

$k \leq 17$  A1

At  $k = 17$  the two allocations cost the same (38) B1

**Answer:  $0 < k \leq 17$ , equal costs at  $k = 17$**  3 marks

## 27 (9 marks)

(a) An allocation uses exactly one entry from each row and each column B1

Total of d = total of c -  $\sum_{i=1}^n r_i - \sum_{j=1}^n s_j$  M1

The subtracted amount does not depend on which allocation is chosen E1

**Answer: Constant =  $\sum_{i=1}^n r_i + \sum_{j=1}^n s_j$**  3 marks

(b) That allocation has total d equal to 0 B1

Every other allocation has total  $d \geq 0$ , since all  $d_{ij} \geq 0$  M1

So by (a) it has the least total c as well, equal to the constant E1

**Answer: Zero total of d gives minimum** 3 marks

(c) Row minima 10, 9, 10, 8, column minima 0, 0, 0, 0, constant 37

M1

Zeros of  $\begin{pmatrix} 0 & 15 & 7 & 9 \\ 12 & 10 & 0 & 15 \\ 4 & 14 & 11 & 0 \\ 0 & 0 & 16 & 11 \end{pmatrix}$  give A-1, B-3, C-4, D-2

A1

The zeros give only one complete allocation, so it is the only one with total  $d = 0$  and hence the only least cost allocation; least cost 37

A1

**Answer: Least cost 37, A-1, B-3, C-4, D-2 only**

3 marks

## 28 (9 marks)

(a) Let  $r$  rows and  $s$  columns be covered, with  $r + s = k$ . An entry in an uncovered row and uncovered column is reduced by  $m$

M1

An entry in a covered row and covered column gains  $m$  (from the column) and is unchanged by the row, giving  $+m$ ; an entry in a covered row and uncovered column is unchanged; an entry in an uncovered row and covered column loses  $m$  and gains  $m$ , so is unchanged

M1 A1

**Answer: Equivalent**

3 marks

(b) The total subtracted from rows and columns increases by  $m(n - r) - ms$   
 $= m(n - r - s) = m(n - k)$

M1

A1

**Answer:  $m(n - k)$**

2 marks

(c) Row minima 17, 12, 10, 21, 26, then column minima 1, 0, 0, 3, 0 of the row reduced matrix: lower bound 90

M1

First adjustment: 4 lines,  $m = 2$ , increase  $2 \times (5 - 4) = 2$ , bound 92

M1 A1

Second adjustment: 4 lines,  $m = 1$ , bound 93, equal to the cost of the allocation A-5, B-3, C-1, D-2, E-4

A1

**Answer: 90, 92, 93; minimum cost 93**

4 marks

**29 (10 marks)**

Subtract the row minimum from each row:

|    |    |    |    |    |    |
|----|----|----|----|----|----|
| 32 | 25 | 0  | 22 | 16 | 27 |
| 0  | 6  | 27 | 19 | 3  | 18 |
| 0  | 10 | 19 | 35 | 38 | 22 |
| 29 | 30 | 1  | 0  | 2  | 31 |
| 0  | 28 | 32 | 34 | 19 | 17 |
| 6  | 14 | 0  | 16 | 40 | 22 |

**M1**

Subtract the column minimum from each column:

|    |    |    |    |    |    |
|----|----|----|----|----|----|
| 32 | 19 | 0  | 22 | 14 | 10 |
| 0  | 0  | 27 | 19 | 1  | 1  |
| 0  | 4  | 19 | 35 | 36 | 5  |
| 29 | 24 | 1  | 0  | 0  | 14 |
| 0  | 22 | 32 | 34 | 17 | 0  |
| 6  | 8  | 0  | 16 | 38 | 5  |

**A1**

Zeros need 5 lines (rows B, C, D, E and column 3), fewer than 6. Smallest uncovered entry is 5:  
 subtract it from uncovered entries and add it to entries covered twice:

**M1 A1**

|    |    |    |    |    |    |
|----|----|----|----|----|----|
| 27 | 14 | 0  | 17 | 9  | 5  |
| 0  | 0  | 32 | 19 | 1  | 1  |
| 0  | 4  | 24 | 35 | 36 | 5  |
| 29 | 24 | 6  | 0  | 0  | 14 |
| 0  | 22 | 37 | 34 | 17 | 0  |
| 1  | 3  | 0  | 11 | 33 | 0  |

Zeros need 5 lines (rows B, D and columns 1, 3, 6), fewer than 6. Smallest uncovered entry is 3:  
 subtract it from uncovered entries and add it to entries covered twice:

**M1 A1**

|    |    |    |    |    |    |
|----|----|----|----|----|----|
| 27 | 11 | 0  | 14 | 6  | 5  |
| 3  | 0  | 35 | 19 | 1  | 4  |
| 0  | 1  | 24 | 32 | 33 | 5  |
| 32 | 24 | 9  | 0  | 0  | 17 |
| 0  | 19 | 37 | 31 | 14 | 0  |
| 1  | 0  | 0  | 8  | 30 | 0  |

Zeros need 5 lines (row D and columns 1, 2, 3, 6), fewer than 6. Smallest uncovered entry is 1:  
 subtract it from uncovered entries and add it to entries covered twice:

**M1 A1**

|    |    |    |    |    |    |
|----|----|----|----|----|----|
| 27 | 11 | 0  | 13 | 5  | 5  |
| 3  | 0  | 35 | 18 | 0  | 4  |
| 0  | 1  | 24 | 31 | 32 | 5  |
| 33 | 25 | 10 | 0  | 0  | 18 |
| 0  | 19 | 37 | 30 | 13 | 0  |
| 1  | 0  | 0  | 7  | 29 | 0  |

6 lines are now needed (rows A, B, C, D, E, F), so an allocation using zeros exists: A-3, B-5, C-1, D-4, E-6, F-2

**M1**

Cost from the original table =  $12 + 13 + 10 + 11 + 27 + 24 = 97$  pounds

**A1**

**Answer: A-3, B-5, C-1, D-4, E-6, F-2, cost 97**

**10 marks****30 (9 marks)**

(a) Only uncovered entries are reduced, and each is at least  $m$ , the smallest of them, so they stay at least 0

**M1**

Covered entries are not reduced (some are increased), so no entry becomes negative

**E1**

**Answer: Entries stay non-negative**

**2 marks**

(b) By the previous result the bound increases by  $m(n - k)$  at each adjustment, where  $m \geq 1$  as the costs are integers and there are no uncovered zeros, and  $n - k \geq 1$

**M1 A1**

The bound cannot exceed  $C$ , since it equals the cost of an allocation at the end, so there are at most  $C - L$  adjustments

**E1**

**Answer: At most  $C - L$**

**3 marks**

(c) Row minima 24, 10, 11, 24, column minima 0, 0, 0, 5, so  $L = 74$

**M1**

The algorithm gives A-4, B-1, C-2, D-3, so  $C = 81$

**M1 A1**

At most  $C - L = 7$  adjustments; the algorithm needs 2, which is fewer

**A1**

**Answer:  $L = 74$ ,  $C = 81$ , bound 7, actual 2**

**4 marks**