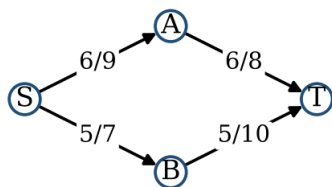


Maximum Flow Problems

- 1 In a network flow problem, state what is meant by
- (a) the source, [1]
 - (b) the sink, [1]
 - (c) the capacity of an arc. [1]

- 2 A flow passes through vertex P. The arcs into P carry flows of 6, 9 and 4 units. One arc out of P carries 7 units and the other arc out of P carries x units. Find x. [2]

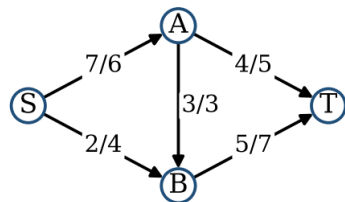
- 3 The diagram shows a feasible flow through a network from S to T. The numbers on each arc show flow/capacity. Write down the value of the flow.



[1]

- 4 State the two conditions that a flow must satisfy for it to be feasible. [2]

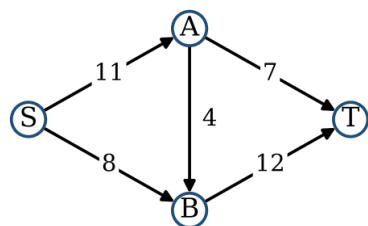
- 5 A student claims that the numbers on the arcs of this network, shown as flow/capacity, form a feasible flow. Identify the arc which shows that the student is wrong, and give a reason.



[2]

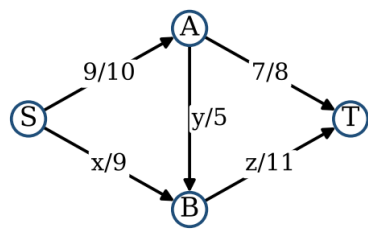
- 6 Two factories A and B can supply at most 15 and 10 tonnes per day respectively to a network which delivers to a single depot T. Explain how this can be modelled using a network with a single source, and state the capacities of any extra arcs you add. [2]

- 7 Find the maximum flow through this network from S to T.



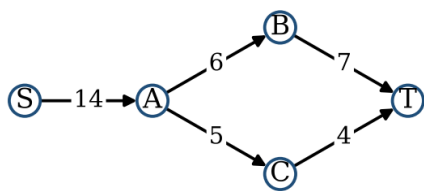
[4]

- 8 The diagram shows a feasible flow of value 15, with flow/capacity on each arc. The flows on three arcs are shown as x , y and z .
Use conservation of flow to find x , y and z .



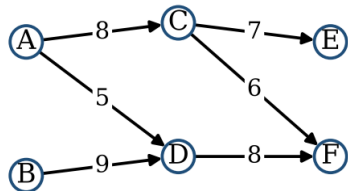
[4]

- 9 Water flows through a system of pipes from a reservoir S to a tank T. The numbers on the arcs show the capacities of the pipes in litres per second.
Find the maximum rate at which water can reach T.



[3]

- 10 Two warehouses A and B can supply at most 12 and 9 pallets per hour respectively. Two shops E and F can receive at most 8 and 10 pallets per hour respectively. The diagram shows the capacities of the delivery routes in pallets per hour.



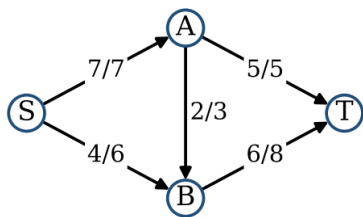
- (a) Add a supersource S and a supersink T, stating the capacity of each extra arc. [2]
(b) Find the maximum number of pallets per hour that can be delivered. [3]

- 11 Crude oil is pumped from a refinery R to a port P through three pumping stations 1, 2 and 3. Oil can only flow in the direction shown in the table, which gives the capacity of each pipeline in thousands of barrels per hour.
Draw a network to represent the system, and find the maximum rate of flow from R to P.

Pipeline	Capacity
R to 1	40
R to 2	30
1 to 2	10
1 to 3	25
2 to 3	20
2 to P	15
3 to P	45

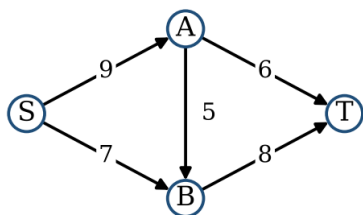
[5]

- 12** The diagram shows a feasible flow in a network, with flow/capacity on each arc.



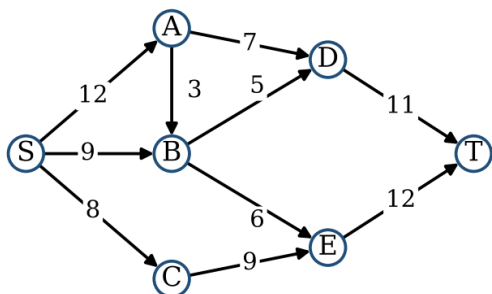
- (a) Write down the value of the flow. [1]
 (b) Show that the flow is not a maximum flow by finding a route from S to T, using only arcs that are not full, and stating how much extra flow it can carry. [2]

- 13** The network shows a system of one-way roads with capacities in hundreds of vehicles per hour. The direction of road AB is then reversed, so that it runs from B to A with the same capacity.



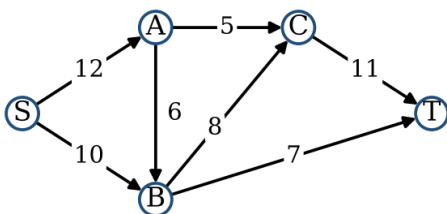
- (a) Find the maximum flow from S to T before the change. [2]
 (b) Find the maximum flow after the road is reversed. [2]

- 14** The diagram shows a network of conveyors carrying parcels from a depot S to a lorry bay T. The numbers show the capacities in parcels per minute.



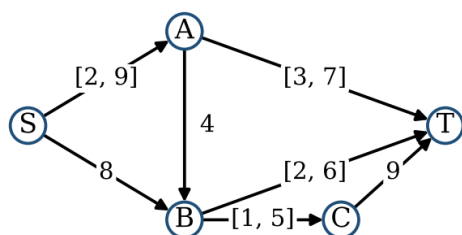
- (a) Explain why the maximum flow cannot exceed 23. [2]
 (b) Find the maximum flow and the flow along each arc. [5]

- 15** The network shows the capacities of routes in a distribution system, in tonnes per hour. Depot B can handle at most 9 tonnes per hour in total of goods passing through it.



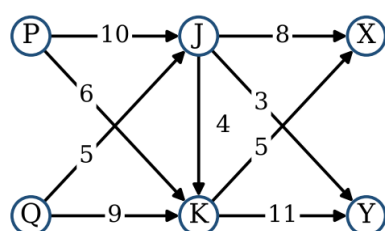
- (a) Explain how the limit at B can be included in the network by replacing B with two vertices, and state the capacity of the new arc. [2]
 (b) Find the maximum flow from S to T, and the capacity of a cut which proves it. [4]

- 16** In the network, each arc is labelled with its capacity. Arcs labelled $[m, c]$ must carry a flow of at least m and at most c ; other arcs have no minimum. Find the maximum flow from S to T, and list the flow along each arc used.



[6]

- 17** Reservoirs P and Q can supply at most 14 and 11 megalitres per day respectively. Towns X and Y require at most 12 and 9 megalitres per day respectively. The diagram shows the capacities of the pipes in megalitres per day, with water flowing from left to right.



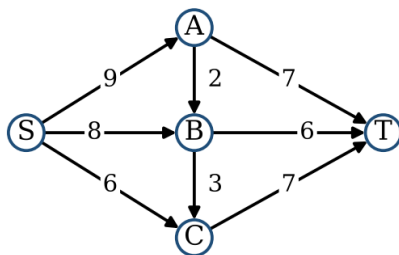
- (a) Draw a network with a single source S and a single sink T which can be used to find the maximum supply to the towns. [2]
 (b) Find the maximum number of megalitres per day that can be delivered to the towns, and show that your answer is the maximum. [4]

- 18** Cars enter a one-way road system at junction A and leave at junction F. The table shows the maximum number of cars per minute that each road can carry, in the direction given (for example AB means from A to B). Model the system as a network, and hence find the maximum number of cars per minute that can pass through the system.

Road	Cars/min
AB	30
AC	20
BC	10
BD	25
CD	15
CE	20
DF	30
EF	25
DE	5

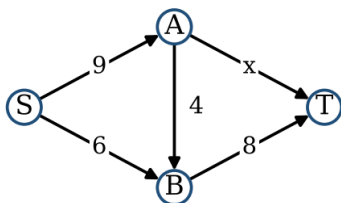
[5]

- 19** Show that the maximum flow through the network from S to T is 20.



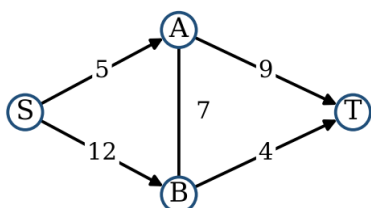
[5]

- 20** The diagram shows a network in which the capacity of arc AT is x , where x is a positive integer. The maximum flow from S to T is 13. Find the value of x .



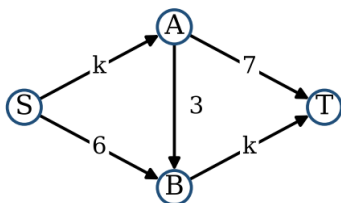
[5]

- 21** In the network shown, the road AB is two-way: up to 7 units of flow can pass along it in either direction (but the total flow in the two directions together is at most 7 at any time). All other arcs are one-way, with the capacities shown.



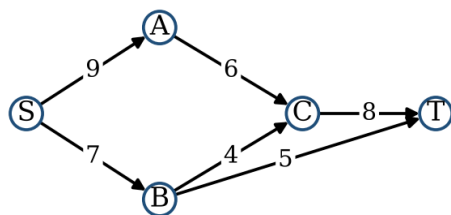
- (a) Explain how the two-way road can be modelled using directed arcs, and why it is never optimal to send flow along both directions of the road at once. [2]
- (b) Find the maximum flow from S to T. Show that your answer is a maximum, and state the value it would have if AB were one-way from A to B only. [5]

- 22** In the network, the arcs SA and BT each have capacity k , where k is a positive integer. The other capacities are as shown.



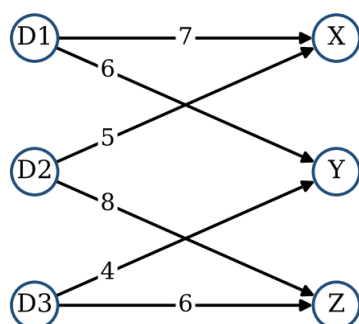
- (a) Find, in terms of k , the capacity of each of the four cuts which separate S from T. [3]
- (b) Hence find the maximum flow in terms of k , for all positive integers k . [5]

- 23** The diagram shows a pipe network with capacities in litres per second. A single pipe may be shut down for maintenance. Find the maximum flow with all pipes open. Then find which single pipe, when shut, reduces the maximum flow by the greatest amount, and state that reduction.



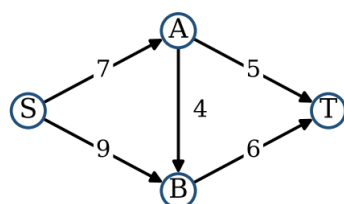
[7]

- 24** A company has three depots D1, D2 and D3 holding at most 14, 11 and 9 pallets respectively. Three shops X, Y and Z need 12, 10 and 13 pallets respectively. Each delivery route has a limit on the number of pallets that can be sent along it, as shown in the diagram.



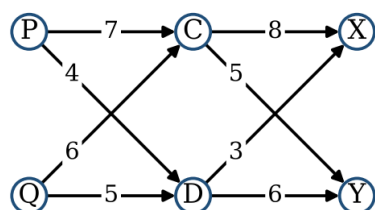
- Draw a network with a supersource and supersink that can be used to find the greatest number of pallets that can be delivered. [2]
- Find the maximum number of pallets that can be delivered, and prove that it is a maximum. [4]
- Hence find the total number of pallets by which the demand of the shops cannot be met, and explain which arcs cause this. [2]

- 25** A student finds the maximum flow from S to T in this network as follows. "Route SAT can carry 5, route SBT can carry 6, and route SABT can carry 4. These are three different routes, so I add them. The maximum flow is $5 + 6 + 4 = 15$."



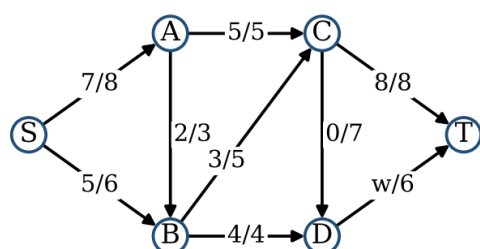
- Explain the error in the student's method. [2]
- Find the correct maximum flow, with a proof that it is a maximum. [4]

- 26** Two power stations P and Q can generate at most 10 and 8 units of power respectively. Two cities X and Y need at most 9 and 7 units respectively. The diagram shows the capacities of the transmission lines, in units of power. In addition, substation C can handle at most 6 units of power in total. Find the maximum amount of power that can be delivered to the cities, giving your reasoning.



[9]

- 27** A flow in a network has source S and sink T. For each arc (u, v) write $f(u, v)$ for the flow along it. Conservation of flow holds at every vertex other than S and T, no arc enters S and no arc leaves T.

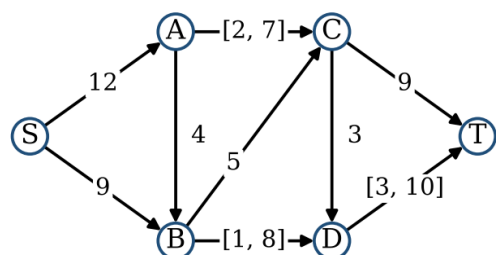


- (a) By summing the quantity (flow in) – (flow out) over every vertex of the network, prove that the flow out of S equals the flow into T. [5]
- (b) In the network shown (flow/capacity on each arc) the flow along DT is w . Use the result of part (a) to find w , and verify your answer by using conservation at D. [3]

- 28** A network has source S, sink T and n intermediate vertices $A_1, A_2, \dots, A_n$. The only arcs are SA_i of capacity a_i and A_iT of capacity b_i , for $i = 1, 2, \dots, n$, where all capacities are positive integers.

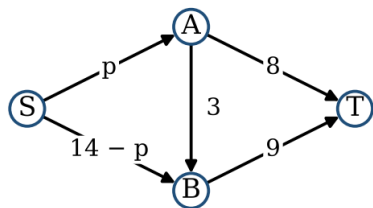
- (a) Prove that the maximum flow is $\sum_{i=1}^n \min(a_i, b_i)$. [4]
- (b) Find the maximum flow when $n = 4$, $(a_1, a_2, a_3, a_4) = (9, 6, 12, 5)$ and $(b_1, b_2, b_3, b_4) = (7, 8, 10, 9)$. [2]
- (c) An extra arc A_1A_2 of capacity 5 is added to the network in part (b). Find the new maximum flow, and show that it is a maximum. [3]

- 29** In the network, an arc labelled $[m, c]$ must carry a flow of at least m and at most c ; other arcs have capacity as shown and no minimum. In addition, vertex D can pass at most 9 units of flow in total.



- (a) Explain how the restriction at D can be included in the network. [2]
- (b) Find the smallest possible value of a flow that satisfies all of the restrictions, and give a flow of that value. Prove that no smaller flow is possible. [4]
- (c) Find the maximum flow, with a proof that it is a maximum. [6]

- 30** The capacities of the arcs SA and SB in the network are p and $14 - p$ respectively, where p is an integer with $0 \leq p \leq 14$. The other capacities are shown in the diagram.

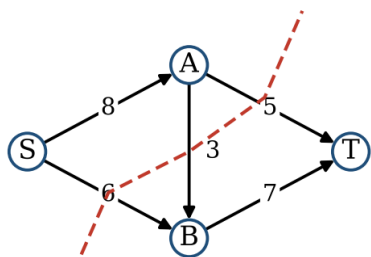


- (a) Find, in terms of p , the capacities of the four cuts separating S from T. [4]
- (b) Hence find the set of values of p for which the maximum flow is 14. [3]
- (c) Find the maximum flow when $p = 2$, and give a flow with this value. [3]

Minimum Cut Problems

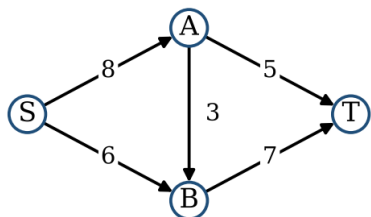
- 1** In a network with source S and sink T, state what is meant by
- (a) a cut, [1]
 - (b) the capacity of a cut. [1]

- 2** The dashed line in the network separates S and A from B and T. The numbers on the arcs are capacities.
List the arcs which the cut crosses from the source side to the sink side, and find the capacity of the cut.



[2]

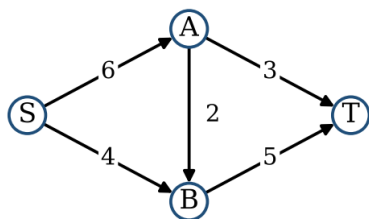
- 3** Find the capacity of the cut which has S and B on the source side and A and T on the sink side.



[2]

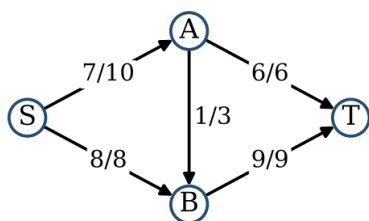
- 4** State the maximum flow minimum cut theorem. [2]
- 5** A feasible flow of value 12 is found in a network, and a cut of capacity 12 is also found. What can be concluded about the flow and the cut? Give a reason. [2]
- 6** When the capacity of a cut is calculated, arcs directed from the sink side to the source side are not included. Explain why. [2]

- 7 List all four cuts which separate S from T in the network, give the capacity of each, and state the minimum cut.



[4]

- 8 The diagram shows a flow in a network, with flow/capacity on each arc. Show that this is a maximum flow by finding a cut whose capacity equals the value of the flow.



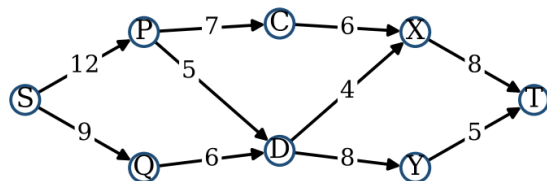
[3]

- 9 Four conveyors are joined in series to carry boxes from S to T, as shown, with capacities in boxes per minute. Find the minimum cut and hence the maximum flow.



[2]

- 10 The network shows two factories P and Q (with a supersource S) delivering to two warehouses X and Y (with a supersink T). The numbers are capacities in vans per day. Find a cut of minimum capacity and state the maximum flow.



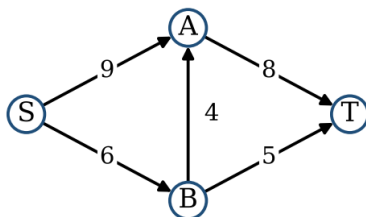
[4]

- 11** Water flows from a reservoir R to a farm F along canals, in the directions given in the table. The table shows the maximum flow along each canal in megalitres per day, and 1, 2 and 3 are junctions. Find the minimum cut for this network and state which canals form the bottleneck of the system.

Canal	Capacity
R to 1	14
R to 2	10
1 to 2	5
1 to 3	6
2 to 3	8
2 to F	7
3 to F	9

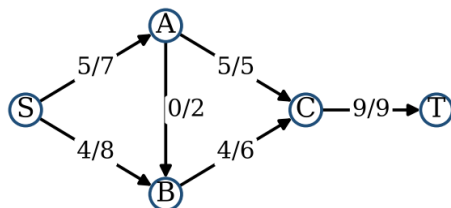
[4]

- 12** In the network shown, the arc BA is directed from B to A. Cut C_1 has S and A on its source side and B and T on its sink side. Cut C_2 has S and B on its source side and A and T on its sink side. Find the capacity of each cut, and state which of the two is smaller.



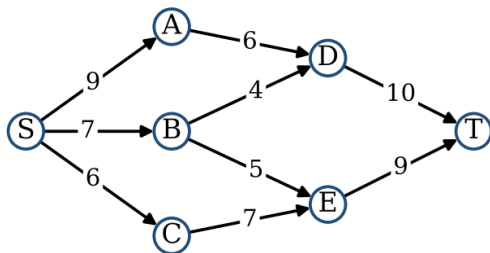
[4]

- 13** The diagram shows a maximum flow in a network, with flow/capacity on each arc.



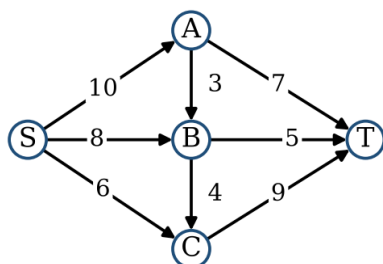
- (a) List the arcs which are saturated (flow equal to capacity). [2]
 (b) Find a cut whose arcs are all saturated, and show that its capacity equals the value of the flow. [2]

- 14** The network shows pipes carrying water from a source S to a sink T. The numbers are capacities in litres per second. Find a minimum cut for the network, listing the pipes it contains, and hence state the maximum flow.



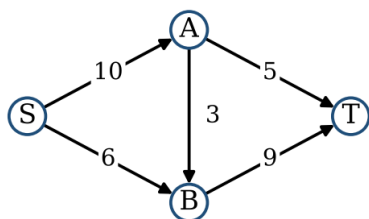
[6]

- 15** The network has three intermediate vertices A, B and C, so there are eight cuts separating S from T. By listing all eight cuts, show that the minimum cut has capacity 21.



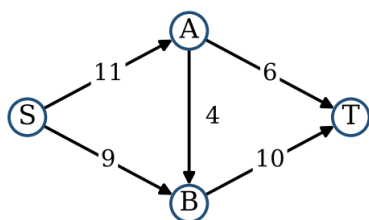
[6]

- 16** The network shows the capacities of the pipes in a system.



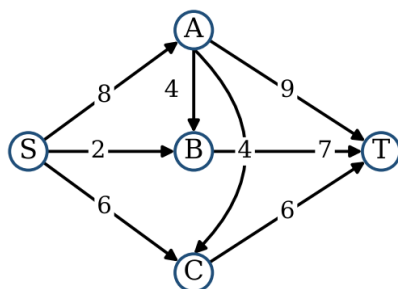
- (a) Find the maximum flow and a minimum cut. [3]
 (b) The capacity of AT is increased by 3. Find the new maximum flow. [2]
 (c) Instead, the capacity of BT is increased by 3. Find the new maximum flow, and explain why the increase is not worthwhile. [2]

- 17** The network shows the capacities of the routes between a depot S and a depot T. Vertex B is a junction that can handle at most 8 units in total.



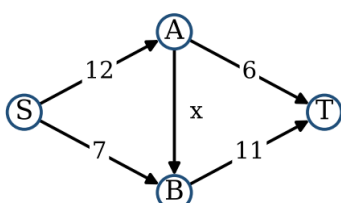
- (a) Show how the limit at B is modelled by splitting B into B and B' joined by an arc, stating the capacity of that arc. [2]
 (b) Find the minimum cut in the new network. Show that without the limit at B the maximum flow would be larger. [4]

- 18** The diagram shows a network with capacities on the arcs.



- (a) Find the minimum cut capacity, and write down every cut of that capacity. [5]
 (b) Which arcs would have to be increased in capacity for the maximum flow to increase? Give a reason. [2]

- 19** In the network, the capacity of AB is x , where $x > 0$. The other capacities are as shown.



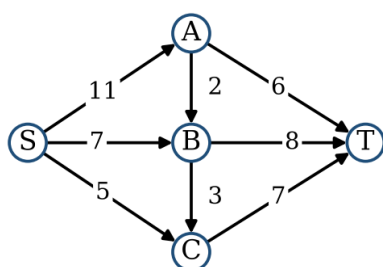
- (a) Find, in terms of x , the capacity of each of the four cuts separating S from T. [3]
 (b) Find the smallest value of x for which the maximum flow takes its greatest possible value, and state that value. [3]

- 20** The table gives the capacities of arcs in a network with source S and sink T; a dash means that there is no arc. Each row gives the start vertex of an arc and each column gives its end vertex, so, for example, there is an arc from S to A with capacity 8. Draw the network, find the minimum cut and state the maximum flow.

	A	B	C	T
S	8	6	5	-
A	-	3	4	5
B	-	-	2	7
C	-	-	-	9

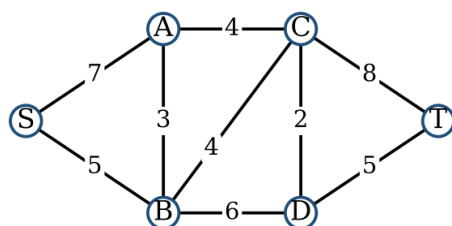
[6]

- 21** The network has source S, sink T and three intermediate vertices.



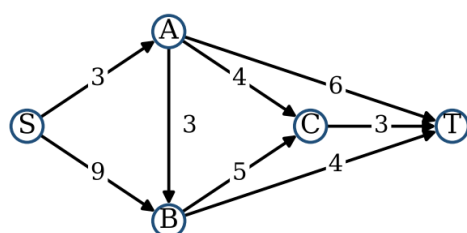
- (a) Find the maximum flow, and state a minimum cut. [3]
 (b) Show that this is the only minimum cut, by considering all of the cuts. [4]

- 22** The diagram shows a network of undirected pipes in which water can flow in either direction. The number on each pipe is its capacity in litres per second, and the pipe can carry this flow in either direction. Water enters at S and leaves at T.



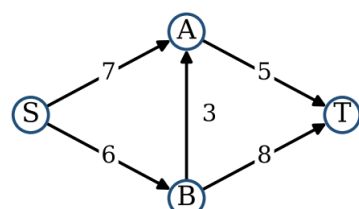
- (a) Explain how each pipe can be replaced by two arcs, so that the methods for directed networks can be used. [2]
 (b) Find the maximum flow from S to T, and a minimum cut, giving the pipes it cuts. [5]

- 23** The network shows the capacities of the links in a communications system, in gigabits per second. The capacity of any one link can be increased by 4.



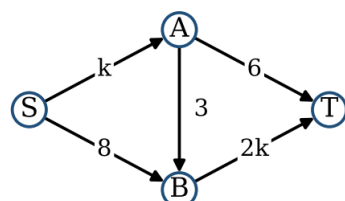
- (a) Find the maximum flow and a minimum cut for the original network. [3]
 (b) Find the new maximum flow when each link in turn has its capacity increased by 4, and hence state which single link should be upgraded. [5]

- 24** A student calculates the capacity of the cut with S and A on the source side and B and T on the sink side, as follows.
 “The arcs between the two sides are SB, BA and AT, so the capacity is $6 + 3 + 5 = 14$.”



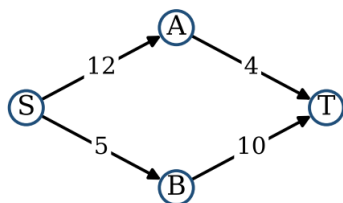
- (a) Explain the error in the student’s calculation. [2]
 (b) Find the correct capacity of this cut, and show that it is the minimum cut. [4]

- 25** In the network, the capacities of SA and BT are k and $2k$ respectively, where $k > 0$.



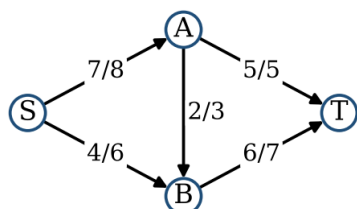
- (a) Find the capacity of each cut in terms of k . [4]
 (b) Find the maximum flow as a function of k . [4]

- 26** The diagram shows the capacities of the routes in a distribution network. A new route from A to B is to be built. It will have capacity c units, and it costs £2000 for each unit of capacity.



- (a) Find the maximum flow before the new route is built, and the cost of the cheapest new route which raises the maximum flow to at least 12. [4]
- (b) Find the greatest maximum flow that can be achieved, however large c is, and explain why. [3]

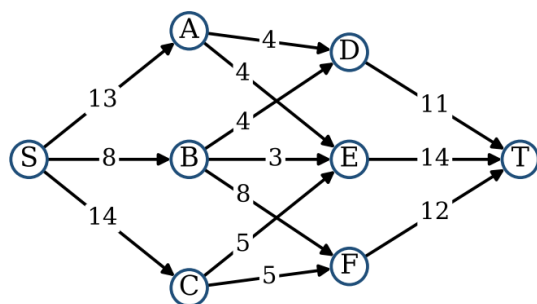
- 27** A network has source S and sink T , and a flow f . Let X be any set of vertices which contains S but not T , so that the arcs from X to the other vertices form a cut. Let F_{out} be the total flow along arcs from X to vertices outside X , and F_{in} the total flow along arcs from outside X into X . No arc enters S and every vertex other than S and T obeys conservation of flow.



- (a) By summing (flow out) – (flow in) over all vertices of X , prove that the value of the flow is $F_{\text{out}} - F_{\text{in}}$. [5]
- (b) Hence prove that the value of any flow is at most the capacity of any cut. [2]
- (c) The diagram shows a flow, with flow/capacity on each arc. For the cut with $X = \{S, B\}$, find F_{out} and F_{in} , and verify that $F_{\text{out}} - F_{\text{in}}$ is equal to the value of the flow. [2]

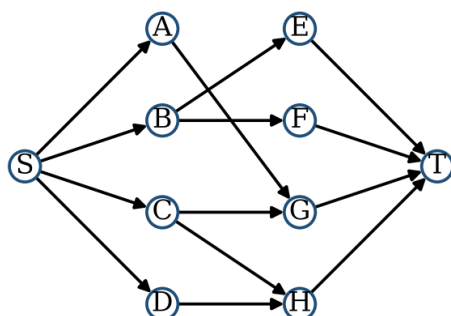
- 28** A network has source S and sink T , and the capacity of every arc is a positive integer.
- (a) Prove that if the capacity of every arc is even then the maximum flow is even. [3]
- (b) Generalise the result of part (a) to the case where every capacity is a multiple of m , for a positive integer m . [1]
- (c) Show by means of examples that if every capacity is odd the maximum flow may be even or odd. [2]
- (d) Explain why a network with integer capacities cannot have a maximum flow which is not an integer. [2]

- 29** The diagram shows a network of conveyor belts with capacities in tonnes per hour.



- (a) Find the maximum flow from S to T, and a minimum cut, listing the arcs of the cut. [6]
 (b) By considering which vertices can be reached from S, and which can reach T, using arcs that are not full (or backwards along arcs that carry flow), show that the minimum cut is unique. [3]
 (c) Find the new maximum flow if the capacity of CF is increased by 2. Explain why increasing the capacity of BF by 3 would make no difference. [2]

- 30** In the network shown every arc has capacity 1, so each unit of flow uses a different arc. A set of S to T paths is arc-disjoint if no two of the paths share an arc.



- (a) Find the greatest number of arc-disjoint paths from S to T, and show that no larger number is possible by finding a cut. [5]
 (b) Explain why the number of arc-disjoint paths from S to T cannot exceed the number of arcs in any cut. [2]
 (c) Explain why the maximum number of arc-disjoint paths equals the value of a maximum flow in the unit network. [3]

Flow-Augmenting Algorithms

- 1** In a network flow problem, state what is meant by a flow-augmenting route. [2]
2 The arc SA has capacity 9 and currently carries a flow of 4. Write down
 (a) the amount by which the flow along SA can be increased, [1]
 (b) the amount by which the flow along SA can be decreased. [1]
3 The diagram shows part of a network with flow/capacity on each arc. A flow-augmenting route S, A, T is to be used. Find the greatest amount by which the flow can be increased along this route.



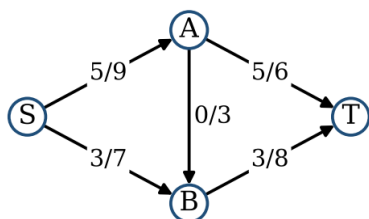
[2]

- 4 Arcs SA, AB and BT currently carry flows of 4, 1 and 5 respectively. The flow along the route S, A, B, T is increased by 3. Write down the new flow on each of the three arcs. [2]

- 5 A flow-augmenting route uses the arc AB backwards, and carries 2 extra units. State what happens to the flow along AB, and explain why this is allowed. [2]

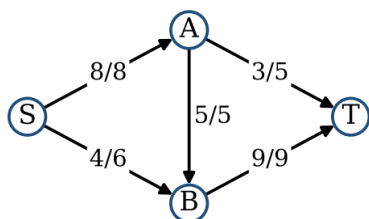
- 6 State how it is recognised that the labelling procedure for finding a maximum flow has finished, and what this tells you about the flow. [2]

- 7 The diagram shows a feasible flow in a network, with flow/capacity on each arc. Find the greatest amount by which the flow can be increased along the route S, B, T, and state the value of the flow after this increase.



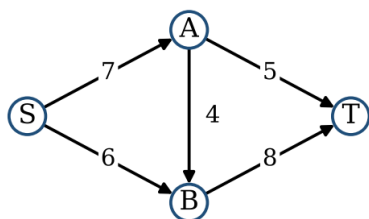
[3]

- 8 The diagram shows a flow with flow/capacity on each arc. For each of the five arcs, write down the amount by which the flow along it can be increased and the amount by which it can be decreased.



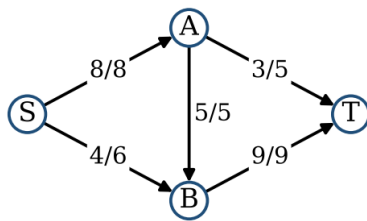
[4]

- 9 The network has capacities as shown, and the initial flow is zero. Flow-augmenting routes S, A, T then S, B, T then S, A, B, T are used in turn, each being increased by as much as possible. Find the amount added along each route and the final flow in each arc.



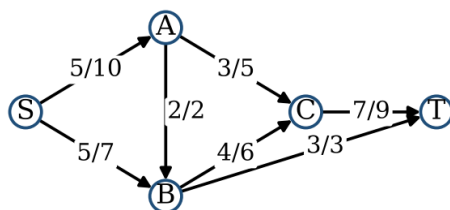
[4]

- 10** The diagram shows a flow of value 12, with flow/capacity on each arc. Show that the flow can be increased by a route that uses arc AB backwards, and find the new flow in each arc.



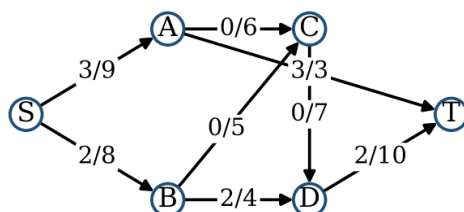
[4]

- 11** The diagram shows a flow in a network, with flow/capacity on each arc. Use the labelling procedure, labelling vertices in alphabetical order, to find a flow-augmenting route. State the label given to each vertex, the route, and the new value of the flow.



[4]

- 12** The diagram shows a flow in a network, with flow/capacity on each arc. Several flow-augmenting routes exist. Find a flow-augmenting route from S to T which uses as few arcs as possible, and find the amount by which it can increase the flow.



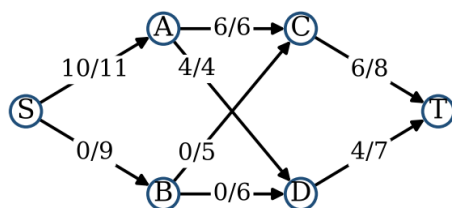
[3]

- 13** A distribution system carries goods from a depot R to a depot P through hubs A, B and C. The table shows the capacity of each route and the flow currently carried, both in tonnes per hour. Use flow-augmenting routes to find the maximum flow that the system can carry.

Pipe	Capacity	Flow
RA	12	9
RB	9	8
AB	5	2
AC	7	7
BC	8	4
BP	10	6
CP	11	11

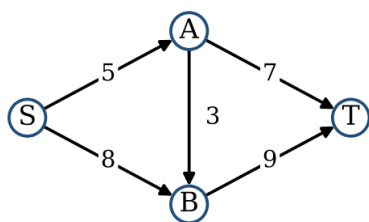
[5]

- 14** The diagram shows a network with a flow, with flow/capacity on each arc. Use the labelling procedure to increase the flow to a maximum, stating each flow-augmenting route and the amount added, and prove that your final flow is a maximum.



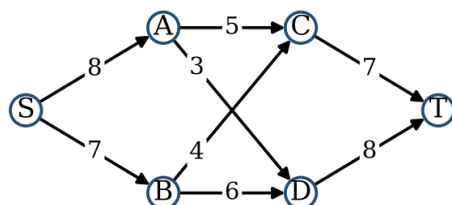
[5]

- 15** The diagram shows a network with the capacities on each arc. A student starts with zero flow and uses the routes S, A, B, T then S, A, T then S, B, T in turn, adding as much flow as possible each time.



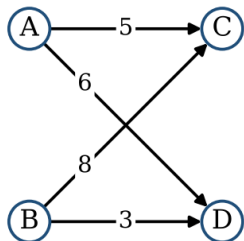
- (a) Find the amount added along each route, the final flow on each arc, and the value of the flow. [3]
 (b) Show that the flow can be increased further by a route which uses an arc backwards, and hence find the maximum flow, justifying that it is a maximum. [4]

- 16** Starting from zero flow, use the labelling procedure to find a maximum flow through the network, labelling vertices in alphabetical order at each stage. Give each flow-augmenting route used and the amount added, and justify that the final flow is a maximum.



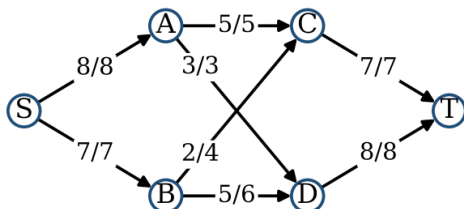
[7]

- 17** Two bakeries A and B can produce at most 12 and 11 trays of bread per hour respectively. The bread is sent to shops C and D, which can take at most 9 and 6 trays per hour respectively. The diagram shows the capacities of the delivery routes between bakeries and shops, in trays per hour. Model the problem as a network with a supersource S and a supersink T. Use the labelling procedure from zero flow to find the maximum number of trays per hour that can be delivered.



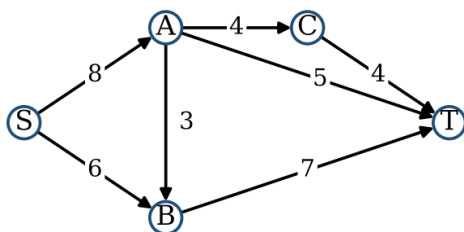
[7]

- 18** The diagram shows a flow in a network, with flow/capacity on each arc. Apply the labelling procedure to this flow. Show that the sink cannot be labelled, and use the vertices that are labelled to find a minimum cut. State the maximum flow.



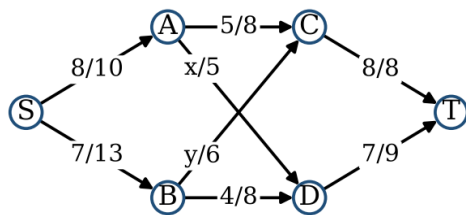
[5]

- 19** The network shows pipes carrying water from S to T, with capacities in litres per second. Starting with zero flow, use the labelling procedure to find a maximum flow, labelling vertices in alphabetical order. Give each route used and the amount added, and justify that the final flow is a maximum.



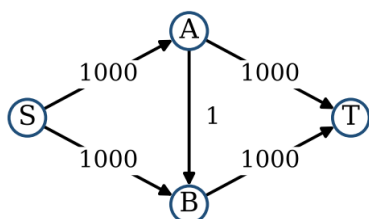
[7]

- 20** The diagram shows a feasible flow in a network, with flow/capacity on each arc. The flows on arcs AD and BC are shown as x and y . Use conservation of flow at A and at B to find x and y . Then use the labelling procedure to increase the flow to a maximum, stating each route and the amount added.



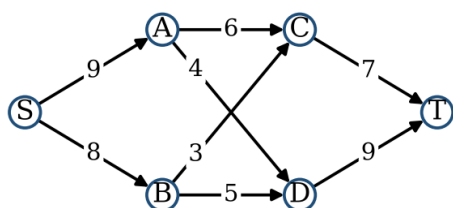
[7]

- 21** In the network shown, the labels are capacities. A student uses the labelling procedure but always chooses the route S, A, B, T when it is available, and otherwise the route S, B, A, T.



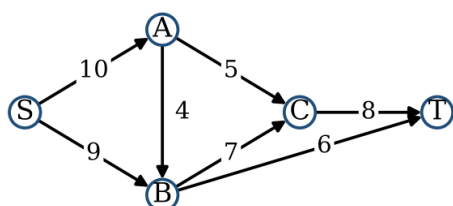
- (a) Starting from zero flow, show the flows after the first two augmentations, stating the amount added each time and the value of the flow. [3]
- (b) Explain why each further augmentation also adds only 1 to the flow, and how many augmentations are needed to reach a maximum flow by this method. [3]
- (c) Show that if the routes are chosen with the fewest arcs the maximum flow is found in only 2 augmentations. [1]

- 22** Starting from zero flow, use the labelling procedure, labelling vertices in alphabetical order, to find the maximum flow through the network. Give each flow-augmenting route and the amount added, state which route uses an arc backwards, and prove that your final flow is a maximum.



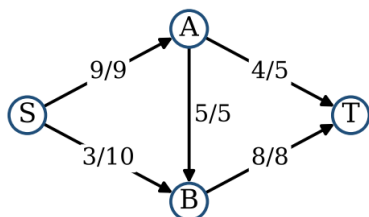
[8]

- 23** The network shows the capacities of the links in a system, in units per minute. Junction B can handle at most 6 units per minute in total. Show how the limit at B can be included in the network, and use the labelling procedure from zero flow to find the maximum flow, justifying that it is a maximum.



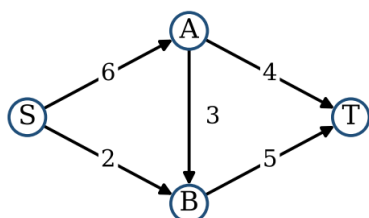
[8]

- 24** The diagram shows a flow in a network, with flow/capacity on each arc. A student finds the flow-augmenting route S, B, A, T and writes:
 “The route can carry 1 extra unit, so I add 1 to the flow on every arc of the route. New flows: SB 4, BA 6 (written as AB 6), AT 5.”



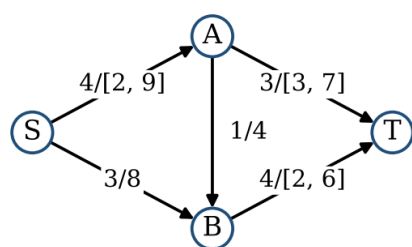
- (a) Explain the error in the student’s working. [2]
 (b) Write down the correct new flow on every arc and the value of the flow. [2]
 (c) Show that the new flow is a maximum flow. [2]

- 25** The network has capacities as shown.



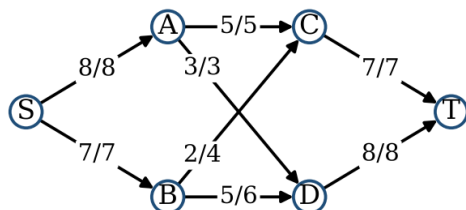
- (a) Find the maximum flow and a minimum cut. [3]
 (b) Find two different maximum flows, giving the flow along each arc. [3]
 (c) Show that the flow obtained by taking the average of your two flows on every arc is also a feasible maximum flow. [2]

- 26** In the network, an arc labelled flow/[m, c] currently carries that flow, must carry at least m and may carry at most c; other arcs show flow/capacity with no minimum. Check that the flow shown is feasible. Then use the labelling procedure (where a flow can only be reduced down to the minimum on an arc) to find the maximum flow.



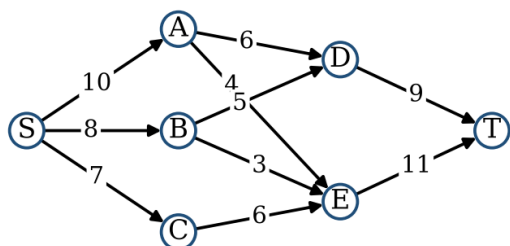
[7]

- 27** The labelling procedure is applied to a feasible flow in a network and finishes with the sink T unlabelled. Let X be the set of labelled vertices, so S is in X and T is not. You may use the result that for any set X containing S but not T, the value of a flow equals $F_{\text{out}} - F_{\text{in}}$, where F_{out} is the total flow along arcs from X to vertices outside X and F_{in} is the total flow along arcs from outside X into X.



- Prove that every arc from X to a vertex outside X is full, and every arc from a vertex outside X into X carries no flow. [3]
 - Hence prove that the value of the flow equals the capacity of the cut which separates X from the other vertices. [3]
 - Explain why this proves that the flow is a maximum flow, and the cut is a minimum cut. [2]
 - The diagram shows a flow with flow/capacity on each arc, for which the labelling procedure labels only S. Verify the results of part (a) for this flow. [2]
- 28** A network has source S and sink T, and every arc capacity is a positive integer. The labelling procedure is started from zero flow.
- Prove that after every augmentation, the flow on every arc is an integer, and that each augmentation increases the value of the flow by at least 1. [3]
 - Deduce that if the maximum flow has value v then the procedure ends after at most v augmentations. [2]
 - The capacity of every arc is a multiple of 3. Show that every augmentation adds a multiple of 3 to the value of the flow, and deduce that if the maximum flow has value v then the procedure ends after at most $\frac{v}{3}$ augmentations. [3]
 - Hence find the greatest number of augmentations that the procedure could use on a network whose capacities are all multiples of 3 and whose maximum flow is 60. [2]

- 29** The diagram shows a network with capacities on the arcs.



- Starting from zero flow, use the labelling procedure with vertices labelled in alphabetical order, to find a maximum flow. Give each flow-augmenting route and the amount added, and prove that your flow is a maximum. [8]
- Write down the final flow on every arc, and verify conservation of flow at vertices A and D. [3]

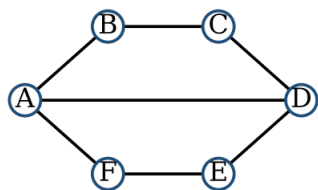
- 30** Let f_1 and f_2 be two different maximum flows in a network, each of value v . Define a new flow g by $g(e) = \frac{1}{2}(f_1(e) + f_2(e))$ on every arc e .

Arc	Capacity	Flow 1	Flow 2
SA	4	4	4
SB	4	4	4
AC	3	3	2
AD	3	1	2
BC	3	3	3
BD	3	1	1
CT	6	6	5
DT	6	2	3

- (a) Prove that g is a feasible flow of value v . [4]
 (b) Deduce that if a network has two different maximum flows then it has infinitely many. [2]
 (c) The table gives the capacities of the arcs of a network and two maximum flows of value 8. Find the flow g on each arc, and show that it is not made up of integers. [2]
 (d) Explain why, nevertheless, a network with integer capacities always has a maximum flow in which every arc carries an integer flow. [2]

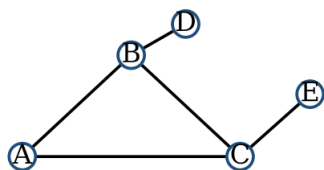
Bipartite Graphs

- 1** State what is meant by a bipartite graph. [2]
2 The complete bipartite graph $K_{3,5}$ has 3 vertices in one set and 5 in the other, with every vertex of the first set joined to every vertex of the second set.
 (a) Write down the number of vertices of $K_{3,5}$. [1]
 (b) Write down the number of edges of $K_{3,5}$. [1]
 (c) Write down the degree of each vertex in the set of 3 vertices. [1]
3 Show that the graph is bipartite by stating two sets of vertices that can be used.



[2]

- 4** Explain why the graph shown is not bipartite.



[2]

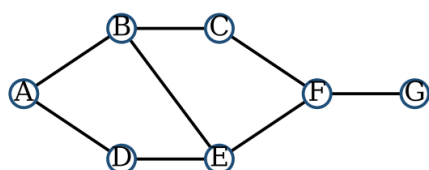
- 5 The cycle graph C_n consists of n vertices joined in a single cycle. State which of C_4 , C_5 , C_6 and C_7 are bipartite. [2]

- 6 Three students Amy, Ben and Cal each take some of the subjects Art, Music and Drama. The table shows with a cross which subjects each student takes. List the edges of the bipartite graph with students on one side and subjects on the other.

	Art	Music	Drama
Amy	X	-	X
Ben	-	X	X
Cal	X	X	-

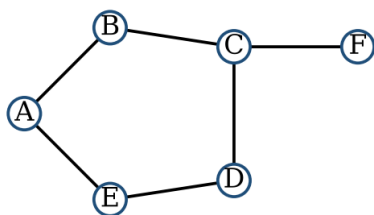
[2]

- 7 Show that the graph is bipartite by colouring its vertices with two colours, and state the two sets of vertices.



[3]

- 8 Show that the graph is not bipartite by finding a cycle of odd length.



[3]

- 9 A bipartite graph has vertices in two sets X and Y . The four vertices in X have degrees 3, 2, 4 and 1, and the five vertices in Y have degrees 2, x , 3, 1 and 2. Find the number of edges and the value of x . [3]

- 10 The complete bipartite graph $K_{m,n}$ has 9 vertices and 20 edges, with $m < n$. Find m and n . [3]

- 11 The complete bipartite graph $K_{2,3}$ has two vertices X_1, X_2 in one set and three vertices Y_1, Y_2, Y_3 in the other.

- (a) Draw $K_{2,3}$. [2]
 (b) State the degree of each vertex. [1]
 (c) Explain why $K_{2,3}$ does not have an Eulerian circuit. [1]

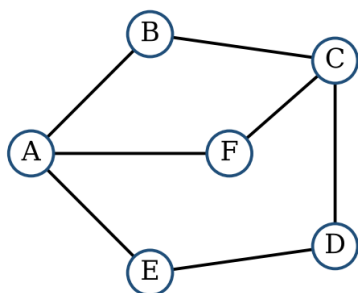
- 12 A graph has vertices A, B, C, D, E and adjacency matrix

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix}, \text{ with rows and columns in the order } A, B, C, D, E.$$

Show that the graph is bipartite, and state the two sets of vertices.

[4]

- 13** Five students join clubs as follows: Anna joins Chess and Drama; Ben joins Drama and Film; Cara joins Chess, Film and Hockey; Dev joins Hockey; Eli joins Drama and Hockey. A graph has a vertex for each student and for each club, and an edge for each membership. Explain why the graph is bipartite, and find the number of edges and the degree of the vertex Drama. [4]
- 14** Prove that every tree is bipartite. [5]
- 15** Prove that the cycle graph C_n is bipartite if and only if n is even. [5]
- 16** A bipartite graph has no loops and no repeated edges.
 (a) Show that a bipartite graph with 10 vertices has at most 25 edges. [3]
 (b) Find the largest possible number of edges of a bipartite graph with 11 vertices. [2]
- 17** Euler's formula states that a connected planar graph with v vertices, e edges and f faces satisfies $v - e + f = 2$. Every face is bounded by a closed walk of at least 3 edges, and every edge borders two faces (or one face twice). Prove that a connected bipartite planar graph with at least 3 vertices has $e \leq 2v - 4$, and hence show that $K_{3,3}$ is not planar. [6]
- 18** A bipartite graph has two sets of vertices X and Y , and every vertex has the same degree k , where $k \geq 1$.
 (a) Prove that X and Y contain the same number of vertices. [3]
 (b) Given that $k = 3$ and the graph has 12 edges, find the number of vertices. [2]
- 19** A bipartite graph has 4 vertices in one set and 5 in the other.
 (a) Prove that the graph does not contain a Hamiltonian cycle. [3]
 (b) Explain why $K_{4,5}$ does have a path which visits every vertex exactly once. [2]
- 20** The complete bipartite graph $K_{3,4}$ has 7 vertices. The complement of a graph on a set of vertices has an edge between two vertices exactly when the original graph does not. Find the number of edges in the complement of $K_{3,4}$, and show that the complement is not bipartite. [5]
- 21** A student looks at the graph shown and writes: "There are no triangles, so the graph is bipartite."



- (a) Explain why the student's reasoning is wrong. [2]
 (b) Find an odd cycle in the graph. [2]
 (c) Find every edge whose removal leaves a bipartite graph, and give a reason. [2]
- 22** G is a connected graph which is bipartite.
 (a) Prove that the two sets of the bipartition are uniquely determined, apart from swapping the names of the two sets. [5]
 (b) A bipartite graph (not necessarily connected) has 3 connected components. In how many ways can its vertices be split into labelled sets X and Y so that every edge joins X to Y ? Explain. [2]

23

A graph has vertices P, Q, R, S, T, U and adjacency matrix $A = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$, with rows and

columns in the order P, Q, R, S, T, U.

The entry in row i and column j of A^2 is the number of walks of length 2 from vertex i to vertex j , and the entry in A^3 is the number of walks of length 3.

- Show that the graph is bipartite, naming the two sets. [2]
- Calculate the first row of A^2 , and explain what it shows about vertices in the same set as P. [2]
- Calculate the diagonal entries of A^3 , and explain why they must all be zero for a bipartite graph. [3]

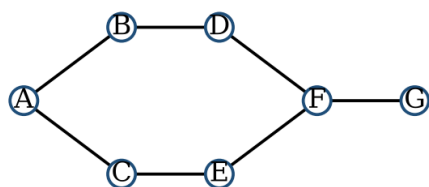
24 The degrees of the four vertices in one set X of a bipartite graph are 3, 3, 2, 2 and the degrees of the four vertices in the other set Y are 4, 3, 2, 1.

- Draw a bipartite graph (with no repeated edges) with these degrees. [4]
- Show that no bipartite graph without repeated edges has degrees 4, 4, 1, 1 in X and 3, 3, 3, 1 in Y . [4]

25 The complete bipartite graph $K_{m,n}$ has m vertices in one set and n in the other, where $m, n \geq 2$.

- Prove that $K_{m,n}$ has an Eulerian circuit if and only if m and n are both even. [3]
- Find the number of edges in $K_{4,6}$, and state whether it has an Eulerian circuit. [1]
- Determine whether each of $K_{2,3}$ and $K_{3,4}$ has an Eulerian trail (which may start and finish at different vertices). [3]

26 In a connected bipartite graph, prove that any path between two vertices in the same set has even length, and any path between vertices in different sets has odd length. Use this result, with a shortest path in each case, to decide whether (i) B and G, (ii) C and F are in the same set in the bipartite graph shown.



[7]

27 Prove that a connected graph is bipartite if and only if it contains no cycle of odd length.

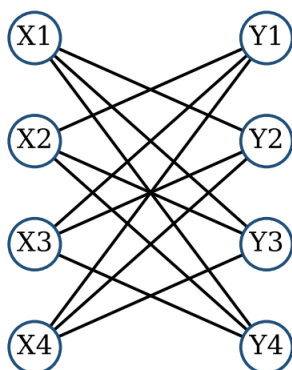
- Prove that a bipartite graph contains no cycle of odd length. [3]
- Prove that a connected graph with no cycle of odd length is bipartite. You may use the fact that a closed walk of odd length contains a cycle of odd length. [6]

28 A bipartite graph has two sets of three vertices, $X = x_1, x_2, x_3$ and $Y = y_1, y_2, y_3$, and no repeated edges. A 4-cycle is a cycle of length 4.

- Show that the graph with the six edges $x_1y_1, x_1y_2, x_2y_2, x_2y_3, x_3y_3, x_3y_1$ has no 4-cycle. [2]
- Prove that every such bipartite graph with 7 edges contains a 4-cycle. [7]

- 29** Hall's theorem states that a bipartite graph with sets X and Y has a matching which uses every vertex of X if and only if, for every subset S of X , the set $N(S)$ of vertices of Y adjacent to at least one vertex of S satisfies $|N(S)| \geq |S|$.

A bipartite graph is k -regular if every vertex has degree k , where $k \geq 1$.



- Prove that a k -regular bipartite graph has a perfect matching (a matching which uses every vertex). [6]
- The graph shown has vertices $X_1, \dots, X_4$ and $Y_1, \dots, Y_4$, with X_i joined to Y_j for every $i \neq j$. Show that it is 3-regular and find a perfect matching. [2]
- Show that the graph in (b) has exactly 9 perfect matchings. [2]

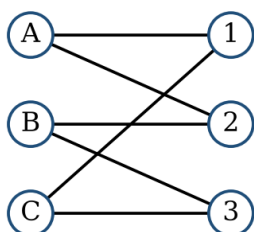
- 30** A bipartite graph has n vertices and no repeated edges.

- Prove that the graph has at most $\frac{n^2}{4}$ edges. [4]
- Show that equality holds if and only if n is even and the graph is $K_{n/2, n/2}$. [2]
- A graph has 13 vertices and 43 edges. Use part (a) to show that the graph cannot be bipartite. [2]
- Find the smallest possible number of edges in a connected bipartite graph with n vertices, and give an example. [2]

Matching Problems

- In a bipartite graph, state what is meant by
 - a matching, [1]
 - a complete matching. [1]

- Three people A, B, C can do the tasks 1, 2, 3 as shown in the bipartite graph. Decide whether each set of edges is a matching: (i) A1, B2, C3, (ii) A1, B2, C1.



[2]

- The complete bipartite graph $K_{3,5}$ has 3 vertices in one set and 5 in the other. Write down the number of edges in the largest possible matching in $K_{3,5}$. [2]
- State what is meant by an alternating path in a bipartite graph with respect to a matching M . [2]

- 5 Write down
- (a) the number of different complete matchings of the complete bipartite graph $K_{3,3}$, [1]
- (b) the number of different matchings of size 3 of the complete bipartite graph $K_{3,4}$. [2]

- 6 Ann, Ben and Cat each volunteer for the jobs they can do, as shown in the table (X means they can do the job).
Explain why Ben must be given the Hall, and hence find a complete matching.

	Kitchen	Hall	Garden
Ann	X	X	-
Ben	-	X	-
Cat	-	X	X

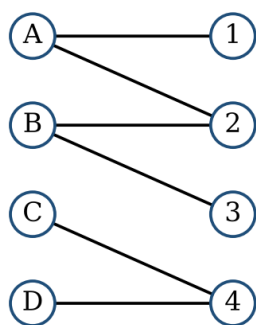
[2]

- 7 Five people A, B, C, D, E can do the jobs 1 to 5 as shown (X means that the person can do the job).
Find a complete matching.

	1	2	3	4	5
A	X	X	-	-	-
B	-	X	X	-	-
C	X	-	-	X	-
D	-	-	X	X	-
E	-	-	-	X	X

[3]

- 8 Four workers A, B, C, D can do the tasks 1 to 4 as shown in the bipartite graph.
Explain why it is impossible to give each worker a different task, and find a matching of size 3.



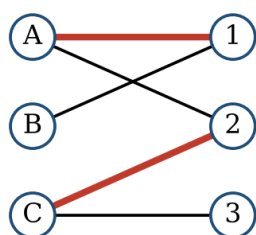
[3]

- 9** Four staff A, B, C, D can be given the tasks 1 to 4 according to the table (X means that the person can do the task).
Find two different complete matchings.

	1	2	3	4
A	X	X	-	-
B	-	X	X	-
C	-	-	X	X
D	X	-	-	X

[4]

- 10** The bold edges show a matching M in the bipartite graph.
Find an alternating path starting at the unmatched vertex B and ending at an unmatched vertex, and use it to find a complete matching.



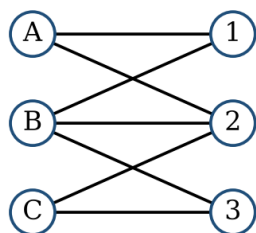
[4]

- 11** Five people A, B, C, D, E can do the tasks 1 to 4 as shown in the table (X means that the person can do the task). Each person is given at most one task and each task is given to at most one person.
Find a maximum matching, and show that it is not possible to find a matching of 5 people.

	1	2	3	4
A	X	X	-	-
B	X	X	-	-
C	X	X	-	-
D	-	-	X	-
E	-	-	X	X

[4]

- 12** Three people A, B, C and three jobs 1, 2, 3 are represented by the bipartite graph shown.
Find all of the complete matchings.



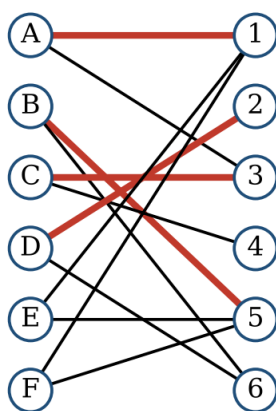
[4]

- 13** Four runners P, Q, R, S are to be assigned to the four legs 1 to 4 of a relay. The table shows which legs each runner can run (X means that the runner can run that leg). Draw the bipartite graph and find a complete matching, explaining your choices.

	1	2	3	4
P	X	-	X	-
Q	-	X	-	-
R	-	X	X	X
S	X	-	-	X

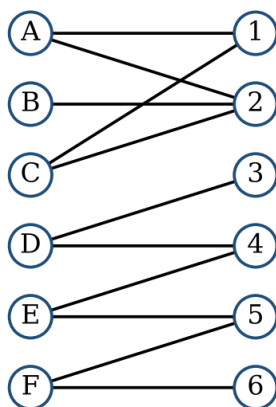
[4]

- 14** Six workers A to F can do the six tasks 1 to 6 as shown in the bipartite graph. The bold edges show an initial matching M. Use the maximum matching algorithm (alternating paths) to improve M to a complete matching, showing each alternating path.



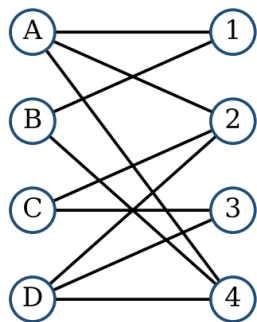
[6]

- 15** The bipartite graph shows which of the six tasks 1 to 6 each of six workers A to F can do.



- (a) Use Hall's condition to show that there is no complete matching. [3]
 (b) Find a matching of 5 workers, and explain why this is a maximum matching. [3]

- 16** Find, by systematic listing, the number of complete matchings of the bipartite graph shown, and write them all down.



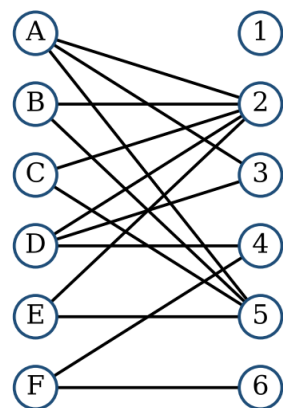
[6]

- 17** Six students A to F are to be put into three teams 1, 2 and 3 with exactly two students in each team. The table shows the teams that each student is willing to join (X means that the student is willing to join the team).

	1	2	3
A	X	X	-
B	X	-	-
C	-	X	X
D	-	-	X
E	X	X	-
F	-	X	X

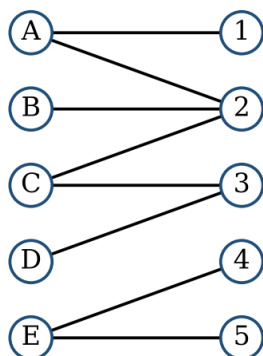
- (a) Explain how this problem can be modelled as a matching problem in a bipartite graph, by splitting each team into two vertices. State how many vertices there are in each set. [2]
- (b) Find an allocation of the students to the teams in which every student is willing to join the team given. [3]
- (c) Students C and F both now refuse to join team 3, so each of them is only willing to join team 2. Use Hall’s condition to show that no allocation is now possible. [2]

- 18** The bipartite graph shows which of six tasks 1 to 6 each of six workers A to F can do.



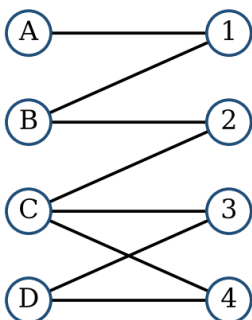
- (a) Find a matching of size 5. [2]
- (b) Show that the set of vertices A, D, F, 2, 5 includes at least one end of every edge, and use this to prove that no matching of size 6 exists. [4]

- 19** The bipartite graph shows which of five tasks 1 to 5 each of five workers A to E can do.



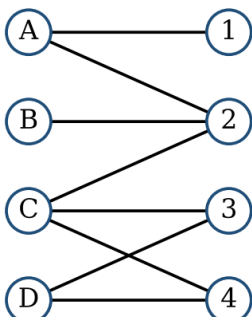
- (a) Show that there is no complete matching. [2]
 (b) One extra task is allowed for one worker. Show that if worker D is also able to do task 4 then there is a complete matching. [2]
 (c) Describe all of the single edges which could be added to give a complete matching. [2]

- 20** The bipartite graph shows the tasks 1 to 4 which workers A to D can do.



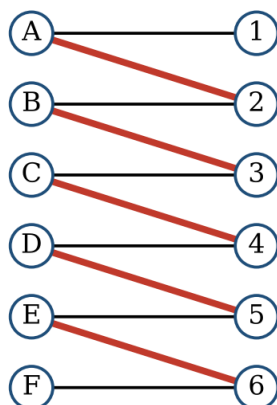
- (a) Find all of the complete matchings. [3]
 (b) State which edges belong to every complete matching, and explain how this is seen from the graph. [2]

- 21** Hall's theorem states that a bipartite graph with sets X and Y has a matching which uses every vertex of X if and only if $|N(S)| \geq |S|$ for every subset S of X , where $N(S)$ is the set of vertices adjacent to at least one vertex of S . Use this theorem, by checking every subset of $X = A, B, C, D$, to show that the graph shown has a complete matching, and then find one.



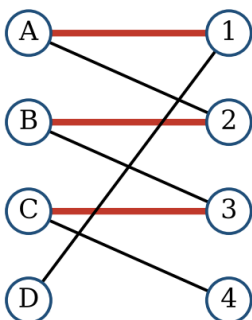
[7]

- 22** The bipartite graph shows which of six tasks 1 to 6 each of six workers A to F can do. The bold edges show a matching M of size 5.



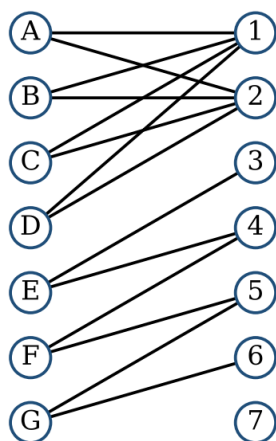
- (a) Find the alternating path which extends M to a complete matching, and write down the complete matching. [4]
 (b) Explain why every alternating path starting at F must pass through all six workers. [3]

- 23** The bold edges show a matching M in the bipartite graph. A student wants to improve M , and writes: "Start at the unmatched vertex D . The alternating path is $D, 1, A, 3, C, 4$, because $D-1$ is not in M , $1-A$ is in M , $A-3$ is not in M , $3-C$ is in M and $C-4$ is not in M . Swapping gives $D-1, A-3, C-4$ and $B-2$, which is a complete matching."



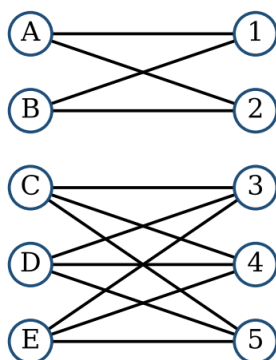
- (a) Identify the error in the student's path. [2]
 (b) Find the correct alternating path from D . [2]
 (c) Use your path to write down the complete matching. [2]

- 24** The bipartite graph shows which of seven tasks 1 to 7 each of seven workers A to G can do. Task 7 cannot be done by any of the workers shown.



- (a) Find a subset S of workers with $|N(S)| = |S| - 2$. [3]
 (b) Deduce that no matching has more than 5 edges, and find a matching with 5 edges. [3]
 (c) State the maximum size of a matching of this graph. [1]

- 25** The bipartite graph shows which of five tasks 1 to 5 each of five workers A to E can do. A complete matching gives each worker a different task.



- (a) Explain why the graph splits into two parts, and find the number of ways of matching the workers A and B with the tasks 1 and 2, and the number of ways of matching C, D, E with tasks 3, 4, 5. [3]
 (b) Hence find the number of complete matchings. [1]
 (c) Worker E becomes unable to do task 5. Find the new number of complete matchings. [3]

- 26** Six workers A to F can do the tasks 1 to 6 as shown in the table (X means that the worker can do the task).
Show how the problem of finding a maximum matching can be turned into a maximum flow problem, by describing a suitable network, and use a flow of maximum value to find a maximum matching.

	1	2	3	4	5	6
A	-	-	X	X	-	-
B	X	-	X	-	-	-
C	-	X	-	X	-	-
D	X	-	-	-	X	-
E	-	-	-	-	X	X
F	-	X	-	-	-	X

[8]

- 27** Let M be a matching in a bipartite graph. An augmenting path is an alternating path (edges alternately not in M and in M) which starts at an unmatched vertex and ends at an unmatched vertex. For two sets of edges M and M' , write $M \oplus M'$ for the set of edges which are in exactly one of M and M' .

- (a) Prove that if there is an augmenting path P for M then $M \oplus P$ is a matching with one more edge than M . [4]
(b) Prove that if M is not a maximum matching, then there is an augmenting path for M . [6]

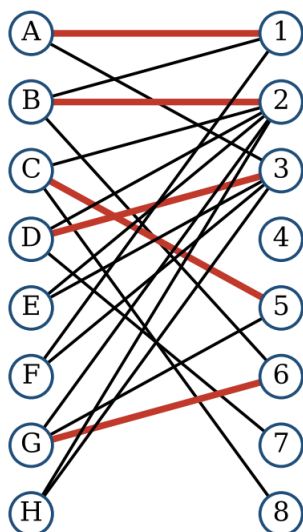
- 28** A bipartite graph has vertex sets X and Y . A vertex cover is a set of vertices such that every edge has at least one end in the set. Let M be a maximum matching, and let Z be the set of vertices which can be reached from unmatched vertices of X by alternating paths (paths whose edges alternately are not in M and in M).

- (a) Prove that every vertex cover has at least $|M|$ vertices. [2]
(b) Let $K = (X \setminus Z) \cup (Y \cap Z)$. Prove that K is a vertex cover. [3]
(c) Prove that K has exactly $|M|$ vertices, and deduce that the size of a maximum matching equals the size of a minimum vertex cover. [5]

- 29** A bipartite graph has sets X and Y . Every vertex of X has degree at least d , and every vertex of Y has degree at most d , where d is a positive integer. Hall's theorem states that there is a matching using every vertex of X if and only if $|N(S)| \geq |S|$ for every subset S of X .

- (a) Prove that there is a matching using every vertex of X . [6]
(b) Six workers each can do at least 3 of 6 tasks, and each task can be done by at most 3 of the workers. Explain why a complete matching must exist. [2]
(c) Show by an example with three workers that the conclusion can fail if the vertices of Y are allowed to have degree greater than d . [2]

- 30** Eight workers A to H are to be assigned to eight tasks 1 to 8. The bipartite graph shows which tasks each worker can do, and the bold edges show an initial matching M. No worker can do task 4.



- Use alternating paths to improve M as far as possible, showing each path, and state the size of the matching you reach. [6]
- Show that no complete matching is possible by finding a set of workers which violates Hall's condition. [3]
- Find a set of 7 vertices which includes at least one end of every edge, and explain why this proves that 7 is the maximum size of a matching. [3]

Allocation Algorithms

- 1** Describe the allocation problem that the Hungarian algorithm solves. [2]

- 2** Reduce the rows of the cost matrix $\begin{pmatrix} 9 & 7 & 13 \\ 8 & 20 & 19 \\ 20 & 17 & 11 \end{pmatrix}$ by subtracting the smallest entry in each row from every entry in that row. [2]

- 3** Reduce the columns of the matrix $\begin{pmatrix} 3 & 1 & 6 \\ 1 & 12 & 11 \\ 9 & 7 & 2 \end{pmatrix}$ by subtracting the smallest entry in each column from every entry in that column. [2]

- 4** Three workers A, B, C are to be given three jobs 1, 2, 3, with costs in pounds as shown in the table. After reducing rows and columns the matrix is $\begin{pmatrix} 2 & 0 & 6 \\ 0 & 12 & 11 \\ 9 & 6 & 0 \end{pmatrix}$.

Use the zeros to find an allocation, and find its total cost from the original table.

	1	2	3
A	9	7	13
B	8	20	19
C	20	17	11

[3]

5 State how a problem in which the total profit is to be maximised can be changed into a problem which can be solved by the Hungarian algorithm. [2]

6 Three workers are to be allocated to four jobs, with each worker given one job and each job done by at most one worker. State what must be added to the cost table before the Hungarian algorithm is used, and what values it contains. [2]

7 Four workers A, B, C, D are to be given four jobs 1, 2, 3, 4, with costs in pounds as shown in the table. Use the Hungarian algorithm to find the allocation with the minimum total cost, and state that cost.

	1	2	3	4
A	5	16	19	15
B	17	18	10	10
C	12	12	5	10
D	15	10	9	16

[4]

8 The table shows the cost in pounds of giving four jobs to four workers. Find the total cost of each of the allocations (i) A-4, B-1, C-3, D-2, (ii) A-1, B-2, C-3, D-4, and state which is cheaper.

	1	2	3	4
A	9	11	14	11
B	6	15	13	13
C	12	13	6	8
D	11	9	10	12

[3]

9 Three salespeople A, B, C are to be sent to three regions 1, 2, 3. The table shows the expected profit in thousands of pounds for each pairing. Use the Hungarian algorithm to find the allocation which maximises total profit, and state the profit.

	1	2	3
A	14	9	11
B	8	16	10
C	12	7	15

[5]

10 Three workers A, B, C are to be given three of four jobs 1, 2, 3, 4, one job each. The table shows the cost in pounds of each pairing. Use the Hungarian algorithm, after adding a dummy worker, to find the allocation of least cost, stating the job which is not done.

	1	2	3	4
A	8	12	9	14
B	11	7	13	10
C	10	9	8	12

[5]

- 11** Four mechanics A, B, C, D are to be given four vehicles 1, 2, 3, 4 to service, one each. The cost in pounds of each pairing is the mechanic's call-out charge plus the parts cost of the vehicle, as shown in the table.
- Reduce the rows and then the columns of the cost matrix, and explain what this shows about the total cost of the possible allocations.

	1	2	3	4
A	18	23	16	20
B	21	26	19	23
C	15	20	13	17
D	26	31	24	28

[4]

- 12** A manager gives jobs to workers A, B, C one at a time in that order, giving each worker the cheapest job which is still available. The cost in pounds of each pairing is shown in the table.

	1	2	3
A	14	17	22
B	6	20	13
C	7	11	9

- (a) Find the allocation which the manager makes, and its total cost. [3]
- (b) Use the Hungarian algorithm to find the allocation of least cost, and show that the manager's allocation is not the cheapest. [5]

- 13** After reducing the rows and columns of a cost matrix for allocating workers A, B, C to jobs 1, 2, 3, the matrix is $\begin{pmatrix} 0 & 0 & 7 \\ 12 & 2 & 0 \\ 5 & 1 & 0 \end{pmatrix}$.

Show that an allocation using zeros is not yet possible, carry out the next step of the algorithm, and hence find the allocation. The original costs were A: 12, 9, 16; B: 20, 7, 5; C: 20, 13, 12 for jobs 1, 2, 3, and the allocation has minimum cost.

[4]

- 14** A swimming coach assigns four swimmers, Ann, Bo, Cy and Di, to the four strokes of a relay team, one stroke each. The table shows each swimmer's time in seconds over one length for each stroke. Use the Hungarian algorithm, writing A, B, C, D for the swimmers and 1, 2, 3, 4 for the strokes in the order shown, to find the allocation with the least total time, and state that time.

	Back	Breast	Fly	Free
Ann	35	38	14	15
Bo	37	12	26	17
Cy	29	35	22	24
Di	27	16	17	33

[6]

- 15** A company assigns four sales staff A, B, C, D to four regions 1, 2, 3, 4, one each. The table shows the expected weekly sales, in thousands of pounds, of each member of staff in each region. Use the Hungarian algorithm to find the allocation which maximises the total expected sales, and state that total.

	1	2	3	4
A	47	23	56	21
B	57	36	39	46
C	32	31	27	57
D	53	56	23	40

[5]

- 16** Four workers A, B, C, D are to be given four jobs 1, 2, 3, 4, one each. The table shows the cost in pounds of each pairing; worker B is not able to do job 3, shown by a dash. Use the Hungarian algorithm, replacing the dash by a suitably large number, to find the allocation of least cost.

	1	2	3	4
A	18	24	27	16
B	24	19	-	12
C	13	27	26	30
D	15	27	27	17

[7]

- 17** Five workers A, B, C, D, E are available for four jobs 1, 2, 3, 4. Each job is done by exactly one worker and each worker does at most one job. The table shows the cost in pounds of each pairing. Use the Hungarian algorithm, after adding a dummy job, to find the allocation of least cost, stating which worker is not used.

	1	2	3	4
A	20	13	14	13
B	14	13	10	12
C	17	8	22	22
D	11	8	24	13
E	23	22	17	23

[7]

- 18** The table shows the cost in pounds of giving four jobs 1 to 4 to four workers A to D, one each. Use the Hungarian algorithm to find a least cost allocation, and show that there is a second allocation with the same total cost.

	1	2	3	4
A	20	10	28	29
B	17	28	10	15
C	29	18	26	13
D	18	28	11	11

[5]

- 19** The table shows the cost in pounds of four workers A to D doing four jobs 1 to 4, one each.

	1	2	3	4
A	16	18	16	13
B	28	25	21	21
C	28	19	11	25
D	29	11	13	10

- (a) Find the sum of the row minima, and then the sum of the column minima of the row reduced matrix. [3]
- (b) Explain why every allocation costs at least the total of the two sums in (a). [2]
- (c) Show that the zeros of the reduced matrix give an allocation costing exactly this total, and state the optimal allocation. [2]

- 20** The table shows the cost in pounds of four workers A to D doing four jobs 1 to 4, one each. Find the allocation of least cost, and explain why the zeros of the column reduced matrix are not enough to give the allocation.

	1	2	3	4
A	17	25	12	33
B	30	28	30	12
C	29	23	34	23
D	33	11	31	24

[7]

- 21** Five delivery drivers A, B, C, D, E are to be sent to five towns 1, 2, 3, 4, 5, one driver to each town. The table shows the cost in pounds of each possible journey. Use the Hungarian algorithm, showing the matrix after each stage, to find the allocation of least total cost and state that cost.

	1	2	3	4	5
A	35	23	33	21	16
B	20	30	36	17	25
C	11	18	20	19	20
D	28	22	25	30	30
E	36	20	40	16	33

[8]

- 22** A firm has four consultants A, B, C, D and five projects 1 to 5. Each consultant is assigned to one project, and each project has at most one consultant. The table shows the profit in thousands of pounds from each pairing. Consultant C is not able to work on project 2, shown by a dash.

Use the Hungarian algorithm, converting to a minimisation problem and adding a dummy consultant, to find the allocation with maximum total profit. State the project which is not staffed.

	1	2	3	4	5
A	39	16	24	12	20
B	13	11	26	13	33
C	15	-	10	23	11
D	39	11	27	21	17

[8]

- 23** The table shows the cost in pounds of giving four jobs 1 to 4 to four workers A to D, one each. A student

reduces the rows and columns and obtains $\begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 4 & 14 & 9 \\ 0 & 16 & 16 & 4 \\ 0 & 2 & 2 & 0 \end{pmatrix}$. The zeros need 3 lines, so the student writes:

“The smallest uncovered entry is 4. I subtract 4 from every uncovered entry and leave all the other entries

unchanged, giving $\begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 10 & 5 \\ 0 & 12 & 12 & 0 \\ 0 & 2 & 2 & 0 \end{pmatrix}$. The zeros give A-3, B-2, C-4, D-1, so the cost is 63.”

	1	2	3	4
A	11	13	12	19
B	15	21	30	30
C	11	29	28	21
D	9	13	12	15

- (a) Identify the error in the student’s adjustment. [2]
 (b) Carry out the adjustment correctly and find the allocation of least cost. [3]
 (c) Find the cost of the student’s allocation from the table, and the cost of the correct allocation, and comment. [3]

- 24** The table shows the cost in pounds of giving three jobs 1, 2, 3 to three workers A, B, C, one each. The allocation problem is to be written as a linear programming problem with variables x_{ij} , where $x_{ij} = 1$ if worker i is given job j and $x_{ij} = 0$ otherwise.

	1	2	3
A	16	5	12
B	5	17	16
C	7	18	14

- (a) Write down the objective function to be minimised. [2]
 (b) Write down the constraints which ensure that every worker gets exactly one job and every job is done by exactly one worker, and state how many constraints there are. [3]
 (c) Use the Hungarian algorithm to solve the problem and state the values of the variables that equal 1, and the minimum cost. [4]

- 25** The table shows the time in minutes that each of four workers A to D needs for each of four tasks 1 to 4, one task each.

	1	2	3	4
A	34	37	39	35
B	18	30	21	35
C	32	39	28	39
D	34	13	21	10

- (a) Use the Hungarian algorithm to find the allocation which minimises the total time, and state the longest individual time in it. [5]
- (b) The tasks must be done at the same time, so the aim is now to minimise the longest individual time. Find the allocation which does this, justifying that no smaller longest time is possible. [4]

- 26** The table shows the cost in pounds of giving three jobs to three workers, one each. The cost k of worker A doing job 1 depends on a delay and k is a positive constant.

The aim is to find the values of k for which the cheapest allocation gives job 1 to worker A.

	1	2	3
A	k	9	8
B	17	10	8
C	24	13	15

- (a) Find the least cost of an allocation in which A does not do job 1. [2]
- (b) Find, in terms of k , the least cost of an allocation in which A does job 1. [2]
- (c) Hence find the range of values of k for which the cheapest allocation gives job 1 to worker A, and state the value of k for which there are two cheapest allocations. [3]

- 27** A cost matrix has entries c_{ij} , for workers $i = 1, \dots, n$ and jobs $j = 1, \dots, n$. A new matrix has entries $d_{ij} = c_{ij} - r_i - s_j$, where r_i is subtracted from every entry of row i and s_j from every entry of column j .

	1	2	3	4
A	10	25	17	19
B	21	19	9	24
C	14	24	21	10
D	8	8	24	19

- (a) Show that, for every allocation, the total of the d values differs from the total of the c values by the same constant, and give that constant in terms of the r_i and s_j . [3]
- (b) Deduce that, if every $d_{ij} \geq 0$ and an allocation uses only entries with $d_{ij} = 0$, then it is a least cost allocation for the original matrix. [3]
- (c) Use these ideas on the cost matrix in the table, with r_i the row minima and s_j the minima of the row reduced columns, to find the least cost, and show that the allocation found is the only one. [3]

- 28** In the Hungarian algorithm for an n by n cost matrix, the zeros of a reduced matrix need k lines to cover them, where $k < n$. The smallest uncovered entry is m .

	1	2	3	4	5
A	40	34	28	22	17
B	39	29	12	26	25
C	14	23	26	17	10
D	22	21	33	27	39
E	31	37	40	29	26

- (a) Show that the adjustment of subtracting m from every uncovered entry and adding m to every entry covered twice is equivalent to subtracting m from each row which is not covered by a line and adding m to each column which is covered by a line. [3]
- (b) Hence show that the lower bound on the cost increases by $m(n - k)$ at each adjustment. [2]
- (c) The table gives the costs of five workers A to E doing five jobs 1 to 5. Find the lower bound after the row and column reductions, and use (b) to find the lower bound after each of the two adjustments needed. State the minimum cost. [4]

- 29** Six engineers A to F are to be assigned to six repair jobs 1 to 6, one job each. The table shows the cost in pounds of each possible assignment. Use the Hungarian algorithm, showing the matrix after each stage and the lines used, to find the assignment of least total cost and state that cost.

	1	2	3	4	5	6
A	44	37	12	34	28	39
B	10	16	37	29	13	28
C	10	20	29	45	48	32
D	40	41	12	11	13	42
E	10	38	42	44	29	27
F	16	24	10	26	50	32

[10]

- 30** A cost matrix has integer entries. After row and column reduction the lower bound for the cost of an allocation is L , and the cost of the least cost allocation is C .

	1	2	3	4
A	38	33	24	36
B	10	26	38	33
C	14	11	30	16
D	29	32	24	37

- (a) Explain why every adjustment in the Hungarian algorithm leaves all entries non-negative. [2]
- (b) Show that each adjustment increases the lower bound by at least 1, and deduce that the algorithm needs at most $C - L$ adjustments. [3]
- (c) For the cost matrix in the table, find L and C , and state the greatest number of adjustments that (b) allows. Verify that the algorithm needs fewer adjustments than this bound. [4]