

Functions of Two Variables

1 (3 marks)

$$f(2, -1) = 3(4) - (2)(-1) + 2(-1) = 12$$

M1 A1

$$f(-3, 2) = 3(9) - (-3)(2) + 2(2) = 37$$

A1

Answer: $f(2, -1) = 12$, $f(-3, 2) = 37$

3 marks

2 (3 marks)

$$g(3, 4) = \sqrt{9 + 16} = 5$$

M1 A1

$$g(-5, 12) = \sqrt{25 + 144} = 13$$

A1

Answer: 5 and 13

3 marks

3 (2 marks)

$$h(1, 2, 3) = 2 + 6 + 3$$

M1

$$= 11$$

A1

Answer: 11

2 marks

4 (2 marks)

The argument of a logarithm must be positive

B1

Domain: all (x, y) with $x + y > 0$

B1

Answer: $x + y > 0$

2 marks

5 (3 marks)

$$f(a, 2a) = a^2 - a(2a) + (2a)^2$$

M1

$$= a^2 - 2a^2 + 4a^2$$

A1

$$= 3a^2$$

A1

Answer: $3a^2$

3 marks

6 (3 marks)

$$4 + k^2 = 13$$

M1

$$k^2 = 9$$

A1

$$k = 3 \text{ or } k = -3$$

A1

Answer: $k = \pm 3$

3 marks

7 (4 marks)

(a) Need $25 - x^2 - y^2 \geq 0$, so $x^2 + y^2 \leq 25$

M1

The disc (including its boundary circle) with centre the origin and radius 5

A1

Answer: $x^2 + y^2 \leq 25$; closed disc, radius 5

2 marks

(b) $0 \leq 25 - x^2 - y^2 \leq 25$ over the domain

M1

$$0 \leq f \leq 5$$

A1

Answer: $0 \leq f \leq 5$

2 marks

8 (4 marks)

$$\frac{x^2}{16} + \frac{y^2}{4} = 1$$

M1 A1

It is an ellipse

B1

It crosses the x-axis at $(\pm 4, 0)$ and the y-axis at $(0, \pm 2)$

A1

Answer: $\frac{x^2}{16} + \frac{y^2}{4} = 1$; ellipse; $(\pm 4, 0)$, $(0, \pm 2)$

4 marks

9 (3 marks)

The denominator must not be zero

B1

$$x^2 - y \neq 0, \text{ so } y \neq x^2$$

M1

All points of the plane except those on the parabola $y = x^2$

A1

Answer: All (x, y) with $y \neq x^2$

3 marks

10 (4 marks)**(a)** Two circular ends and a curved surface

M1

$$S(r, h) = 2\pi r^2 + 2\pi rh$$

A1

$$\text{Answer: } S = 2\pi r^2 + 2\pi rh$$

2 marks

$$\text{(b)} S(3, 10) = 2\pi(9) + 2\pi(3)(10) = 18\pi + 60\pi = 78\pi$$

M1

A1

$$\text{Answer: } 78\pi \text{ cm}^2$$

2 marks

11 (4 marks)

$$\text{(a)} 12\sqrt{25 \times 16} = 12 \times 20 = 240$$

M1

A1

$$\text{Answer: } 240$$

2 marks

$$\text{(b)} 12 \times 5 \sqrt{K} = 300$$

M1

$$\sqrt{K} = 5, \text{ so } K = 25$$

A1

$$\text{Answer: } K = 25$$

2 marks

12 (3 marks)

$$f(x + h, y) = (x + h)^2 y = (x^2 + 2xh + h^2)y$$

M1

$$f(x + h, y) - f(x, y) = 2xhy + h^2y$$

M1 A1

$$\text{Answer: } 2xhy + h^2y = hy(2x + h)$$

3 marks

13 (3 marks)

$$f(2, 1) = 1 - 4 = -3$$

M1

$$y - x^2 = -3$$

M1

$$y = x^2 - 3$$

A1

$$\text{Answer: } y = x^2 - 3$$

3 marks

14 (6 marks)

$$\text{(a)} f(t, 3t) = 2t^2 - t(3t) + (3t)^2 = 8t^2$$

M1

A1

$$\text{Answer: } 8t^2$$

2 marks

$$\text{(b)} 8t^2 = 56, \text{ so } t^2 = 7$$

M1

$$t = \pm\sqrt{7}$$

A1

$$\text{Answer: } t = \pm\sqrt{7}$$

2 marks

$$\text{(c)} \text{ Along } y = 3x, f = 8t^2$$

M1

$$8t^2 \geq 0 \text{ for all real } t, \text{ and every value } \geq 0 \text{ occurs, so the range is } f \geq 0$$

A1

$$\text{Answer: } f \geq 0$$

2 marks

15 (5 marks)

$$\text{(a)} f(tx, ty) = t^3x^3 - 2(tx)(t^2y^2) + t^3y^3 = t^3(x^3 - 2xy^2 + y^3) = t^3f(x, y)$$

M1

A1

$$\text{Answer: Shown: } f(tx, ty) = t^3f(x, y) \text{ for any real number } t$$

2 marks

$$\text{(b)} f(2, 3) = 8 - 36 + 27 = -1$$

M1

$$f(6, 9) = f(3 \times 2, 3 \times 3) = 27 \times (-1) = -27$$

M1 A1

$$\text{Answer: } f(2, 3) = -1, f(6, 9) = -27$$

3 marks

16 (5 marks)

$$c > 0: x^2 - y^2 = c \text{ is a hyperbola crossing the } x\text{-axis at } (\pm\sqrt{c}, 0), \text{ opening left and right}$$

M1 A1

$$c < 0: \text{ a hyperbola crossing the } y\text{-axis at } (0, \pm\sqrt{-c}), \text{ opening up and down}$$

M1 A1

$$c = 0: (x - y)(x + y) = 0, \text{ the pair of lines } y = x \text{ and } y = -x, \text{ crossing at the origin}$$

A1

$$\text{Answer: } c > 0: x\text{-axis } (\pm\sqrt{c}, 0); c < 0: y\text{-axis } (0, \pm\sqrt{-c}); c = 0: y = \pm x$$

5 marks

17 (7 marks)

(a) $4(x + y + z) = 60$, so $z = 15 - x - y$

M1 A1

$V = xyz = xy(15 - x - y)$

A1

Answer: Shown: the volume of the box is $V(x, y) = xy(15 - x - y)$

3 marks

(b) Lengths are positive: $x > 0$, $y > 0$ and $z > 0$

M1

$x > 0$, $y > 0$, $x + y < 15$

A1

Answer: $x > 0$, $y > 0$, $x + y < 15$

2 marks

(c) $V(3, 4) = 12 \times 8$

M1

$= 96 \text{ cm}^3$

A1

Answer: 96 cm^3

2 marks

18 (5 marks)

Need $y - x^2 \geq 0$ and $2 - y \geq 0$

M1 A1

So $x^2 \leq y \leq 2$, which is the region on or above the parabola and on or below the line

A1

They meet where $x^2 = 2$

M1

$x = \pm\sqrt{2}$

A1

Answer: $x = \pm\sqrt{2}$

5 marks

19 (5 marks)

(a) $f(1, 2) = e^{-(1+4)}$

M1

$= e^{-5}$

A1

Answer: e^{-5}

2 marks

(b) $f = c$ gives $x^2 + y^2 = -\ln c$: circles centred at the origin

M1 A1

Need $-\ln c \geq 0$, so $0 < c \leq 1$: range $0 < f \leq 1$

A1

Answer: Circles; $0 < f \leq 1$

3 marks

20 (4 marks) $g = 1$ is a hyperboloid of one sheet (a single connected surface around the z-axis)

B1 B1

 $g = -1$ is a hyperboloid of two sheets (two separate bowls, one above and one below)

B1

 $g = 0$ is a cone with the z-axis as its axis

B1

Answer: One sheet hyperboloid; two sheet hyperboloid; cone

4 marks

21 (8 marks)

(a) On $y = 0$: $f = 0$

B1

On $y = x$: $f = \frac{x^2}{2x^2} = \frac{1}{2}$

M1 A1

Answer: 0 and $\frac{1}{2}$

3 marks

(b) $f = \frac{x(mx)}{x^2 + m^2x^2} = \frac{mx^2}{x^2(1 + m^2)}$

M1

$= \frac{m}{1 + m^2}$

A1

Answer: Shown: along the line $y = mx$, f takes the constant value $\frac{m}{1 + m^2}$

2 marks

(c) Points arbitrarily close to the origin give $f = 0$ (on $y = 0$) and $f = \frac{1}{2}$ (on $y = x$)

M1

So f has no single limit at $(0, 0)$, and no choice of $f(0, 0)$ can make it continuous

A1 E1

Answer: Different limits along different lines

3 marks

22 (7 marks)

(a) Take $(-2, -3)$: $xy = 6 > 0$

M1

$f(-2, -3) = \ln 6$ is defined

A1

Answer: For example $f(-2, -3) = \ln 6$

2 marks

(b) Need $xy > 0$

M1

 x and y both positive or both negative: the first and third quadrants, excluding the axes

A1 A1

Answer: $xy > 0$

3 marks

(c) $f = \ln(e^2)$

$= 2$

Answer: 2

M1

A1

2 marks

23 (8 marks)

(a) $x^2 - 4x = (x - 2)^2 - 4$ and $y^2 + 2y = (y + 1)^2 - 1$

M1 A1

$f = (x - 2)^2 + (y + 1)^2 - 5 + 7 = (x - 2)^2 + (y + 1)^2 + 2$

A1

Answer: Shown: $f(x, y) = (x - 2)^2 + (y + 1)^2 + 2$

3 marks

(b) Squares are non-negative, so $f \geq 2$, and $f = 2$ at $(2, -1)$

M1 A1

Answer: $f \geq 2$

2 marks

(c) $f = 2$: the single point $(2, -1)$

B1

$f = 6$: $(x - 2)^2 + (y + 1)^2 = 4$, a circle with centre $(2, -1)$ and radius 2

B1 B1

Answer: A point; a circle

3 marks

24 (9 marks)

(a) $T(2, 1) = 80e^{-7/20}$

M1

$= 56.4^\circ\text{C}$

A1

Answer: 56.4°C

2 marks

(b) Same temperature, so the exponent equals that at $(2, 1)$: $x^2 + 3y^2 = 4 + 3 = 7$

M1 A1

Crosses the x-axis at $(\pm\sqrt{7}, 0)$

A1

Crosses the y-axis at $(0, \pm\sqrt{\frac{7}{3}}) = (0, \pm 1.53)$

A1

Answer: $x^2 + 3y^2 = 7$; $(\pm\sqrt{7}, 0)$, $(0, \pm 1.53)$

4 marks

(c) $x^2 + 3y^2 \geq 0$, so the exponent is ≤ 0 and the exponential is ≤ 1

M1 E1

Equality needs $x^2 + 3y^2 = 0$, which means $x = y = 0$

A1

Answer: Shown: $T(x, y) \leq 80$ for all points, with equality only at the origin

3 marks

25 (8 marks)

(a) $V = \text{cylinder} + \text{hemisphere} = \pi r^2 h + \frac{1}{2} \times \frac{4}{3} \pi r^3$

M1 A1

$S = \text{base } \pi r^2 + \text{curved wall } 2\pi r h + \text{hemisphere } 2\pi r^2$

M1 A1

$= 2\pi r h + 3\pi r^2$

A1

Answer: Shown: $V = \pi r^2 h + \frac{2}{3} \pi r^3$, $S = 2\pi r h + 3\pi r^2$

5 marks

(b) $25\pi h + \frac{250}{3}\pi = \frac{850}{3}\pi$

M1

$25h = 200$, so $h = 8$

A1

$S = 2\pi(5)(8) + 3\pi(25) = 155\pi \text{ m}^2$

A1

Answer: $h = 8 \text{ m}$, $S = 155\pi \text{ m}^2$

3 marks

26 (8 marks)

(a) $x^2 = c(x^2 + y^2)$, so $cy^2 = (1 - c)x^2$

M1 A1

$y^2 = \frac{1 - c}{c}x^2$, so $y = \pm\sqrt{\frac{1 - c}{c}}x$

M1 A1

Two lines through the origin with gradients $\pm\sqrt{\frac{1 - c}{c}}$

A1

Answer: $y = \pm\sqrt{\frac{1 - c}{c}}x$

5 marks

(b) $c = 0$: $x^2 = 0$, so the y-axis (origin removed)

B1

$c = 1$: $y^2 = 0$, so the x-axis (origin removed)

B1

Since $0 \leq x^2 \leq x^2 + y^2$, $0 \leq f \leq 1$, and these are the extreme values attained

E1

Answer: y-axis; x-axis; $0 \leq f \leq 1$

3 marks

27 (11 marks)

$$(a) f(x, mx) = \frac{mx^3}{x^4 + m^2x^2} = \frac{mx}{x^2 + m^2}$$

M1 A1

As $x \rightarrow 0$ this tends to 0 (for $m \neq 0$; for $m = 0$ and on $x = 0$, $f = 0$)

A1

Answer: Shown: $f \rightarrow 0$ along every line $y = mx$

3 marks

$$(b) f(x, x^2) = \frac{x^4}{x^4 + x^4}$$

M1

$$= \frac{1}{2}$$

A1

Answer: $\frac{1}{2}$

2 marks

(c) Along lines the limit is 0, along $y = x^2$ it is $\frac{1}{2}$

M1 E1

A limit must be independent of the path taken, so none exists

E1

Answer: Path $y = x^2$ gives $\frac{1}{2}$

3 marks

$$(d) x^4 + y^2 \geq 2x^2|y| \text{ (since } (x^2 - |y|)^2 \geq 0)$$

M1 A1

$$\text{So } |f| \leq \frac{x^2|y|}{2x^2|y|} = \frac{1}{2}$$

A1

Answer: Shown: $f(x, y) \leq \frac{1}{2}$ for all (x, y) in the domain

3 marks

28 (11 marks)

$$(a) (x - y)^2 \geq 0 \text{ gives } x^2 + y^2 \geq 2xy, \text{ and } (x + y)^2 \geq 0 \text{ gives } x^2 + y^2 \geq -2xy$$

M1 A1

$$\text{So } x^2 + y^2 + 1 > 2|xy|$$

M1

$$\text{Dividing, } |f| < \frac{1}{2}$$

A1

Answer: Shown: $-\frac{1}{2} < f(x, y) < \frac{1}{2}$ for all real x and y

4 marks

$$(b) f(t, t) = \frac{t^2}{2t^2 + 1} = \frac{1}{2} - \frac{1}{2(2t^2 + 1)}$$

M1 A1

This tends to $\frac{1}{2}$ as $t \rightarrow \infty$, and by symmetry $f(t, -t)$ tends to $-\frac{1}{2}$. Continuity gives every value between

M1 E1

Answer: Range $-\frac{1}{2} < f < \frac{1}{2}$

4 marks

$$(c) 4xy = x^2 + y^2 + 1, \text{ so } x^2 - 4xy + y^2 = -1$$

M1

$$x^2 + y^2 = \frac{u^2 + v^2}{2} \text{ and } xy = \frac{u^2 - v^2}{4}$$

M1

$$\frac{u^2 + v^2}{2} - (u^2 - v^2) = -1 \text{ gives } u^2 - 3v^2 = 2$$

A1

Answer: Shown: $u^2 - 3v^2 = 2$

3 marks

29 (10 marks)

$$(a) P(1, 1) = c = 5$$

B1

$$5 \times 4^a = 10 \text{ gives } 4^a = 2, \text{ so } a = \frac{1}{2}$$

M1 A1

$$5 \times 16^b = 20 \text{ gives } 16^b = 4, \text{ so } b = \frac{1}{2}$$

M1 A1

Answer: $c = 5$, $a = b = \frac{1}{2}$

5 marks

$$(b) P(2L, 2K) = 5\sqrt{2L \times 2K} = 5 \times 2\sqrt{LK}$$

M1

$$= 2P(L, K)$$

A1

Answer: Shown: doubling both L and K doubles the output

2 marks

$$(c) 5\sqrt{LK} = 40, \text{ so } LK = 64$$

M1 A1

$$K = \frac{64}{L}; \text{ at } L = 64, K = 1$$

A1

Answer: $K = \frac{64}{L}; K = 1$

3 marks

30 (10 marks)

(a) $f(tx, ty) = \frac{t(y-x)}{t(y+x)}$

M1

$= f(x, y)$

A1

Answer: Shown: $f(tx, ty) = f(x, y)$ for every $t \neq 0$

2 marks

(b) $y - x = c(y + x)$

M1

$y(1 - c) = x(1 + c)$

M1

$y = \frac{1+c}{1-c}x$, valid since $c \neq 1$

A1

Answer: Shown: $y = \frac{1+c}{1-c}x$

3 marks

(c) $c = 1$: $y - x = y + x$, so $x = 0$, the y-axis with the origin removed

M1 A1

Every real c is attained, so the range is all real numbers

A1

Answer: $x = 0$ ($y \neq 0$); all real numbers

3 marks

(d) Near the origin f takes every real value (on lines of different gradients)

M1

So there is no single limit

E1

Answer: No single limit at the origin

2 marks

Surfaces and Contour Diagrams

1 (2 marks)

Sphere

B1

Centre the origin, radius 4

B1

Answer: Sphere, centre O, radius 4

2 marks

2 (3 marks)

It is a plane

B1

x-axis: (3, 0, 0), y-axis: (0, 2, 0), z-axis: (0, 0, -6)

B1 B1

Answer: Plane; (3, 0, 0), (0, 2, 0), (0, 0, -6)

3 marks

3 (2 marks)

$x^2 + y^2 = 9$

M1

A circle of radius 3, centred on the z-axis, in the plane $z = 9$

A1

Answer: Circle of radius 3 centred on the z-axis

2 marks

4 (2 marks)

Put $y = 0$: $z = x^2$

M1

A parabola

A1

Answer: $z = x^2$, a parabola

2 marks

5 (2 marks)

Independent of z , so it is a vertical circular cylinder

B1

Radius 2, with the z-axis as its axis

B1

Answer: Circular cylinder, radius 2, axis the z-axis

2 marks

6 (3 marks)

$3x - y = 6$, so $y = 3x - 6$

M1 A1

Gradient 3

A1

Answer: $y = 3x - 6$, gradient 3

3 marks

7 (4 marks)**(a)** Radii 1, 2, 3, 4 (read from the axes) **B1**Contour $z = c$ has radius $\sqrt{c}$, so $x^2 + y^2 = z$ **M1**

$$f(x, y) = x^2 + y^2$$
 A1

Answer: $f(x, y) = x^2 + y^2$ **3 marks****(b)** A paraboloid (bowl) with vertex at the origin, opening upwards **A1****Answer: Paraboloid** **1 mark****8 (4 marks)** $x^2 + y^2 \geq 0$ so $z \leq 9$, with equality at the origin: highest point (0, 0, 9) **M1 A1**

$$z = 0: x^2 + y^2 = 9$$
 M1

A circle of radius 3 centred on the z-axis, in the plane $z = 0$ **A1****Answer: (0, 0, 9); circle $x^2 + y^2 = 9, z = 0$** **4 marks****9 (6 marks)****(a)** $x = 0: z = -y^2$, a parabola opening downwards **M1 A1****Answer: $z = -y^2$, a parabola opening downwards** **2 marks****(b)** $y = 0: z = x^2$, a parabola opening upwards **M1 A1****Answer: $z = x^2$, a parabola opening upwards** **2 marks****(c)** $z = 0: x^2 = y^2$, so $(x - y)(x + y) = 0$, the pair of lines $y = x$ and $y = -x$ **M1 A1****Answer: The pair of lines $y = x$ and $y = -x$** **2 marks****10 (4 marks)**

$$xy = 4, \text{ so } y = \frac{4}{x}$$
 M1

$$\text{At } (2, 2): 2 \times 2 = 4$$
 A1

$$f(-1, 3) = -3, \text{ so the contour is } xy = -3$$
 M1 A1

Answer: $xy = 4; xy = -3$ **4 marks****11 (3 marks)**

$$\text{x-axis: } y = z = 0, x = 6$$
 B1

$$\text{y-axis: } x = z = 0, y = 3$$
 B1

$$\text{z-axis: } x = y = 0, z = 2$$
 B1

Answer: (6, 0, 0), (0, 3, 0), (0, 0, 2) **3 marks****12 (4 marks)**Centre (1, -2, 0), radius 5 **B1 B1**

$$z = 3: (x - 1)^2 + (y + 2)^2 = 25 - 9 = 16$$
 M1

Radius 4 **A1****Answer: Centre (1, -2, 0), radius 5; circle radius 4** **4 marks****13 (5 marks)****(a)** A is on the 60 m contour and B on the 0 m contour **B1**Horizontal distance $AB = 2$ km, so the average gradient is $\frac{60}{2} = 30$ m per km downhill **M1 A1****Answer: 30 m per km downhill** **3 marks****(b)** The contour curves between A and C (it bulges outwards), so the straight chord does not follow it **E1**The chord lies inside the 60 m contour, where the ground is higher than 60 m **E1****Answer: The chord lies on higher ground** **2 marks****14 (7 marks)****(a)** Near the origin, the contours with positive values lie to the left and right and those with negative values above and below **M1 A1**Moving along the x-axis f increases, and along the y-axis f decreases **M1**So the origin is a saddle point **E1****Answer: A saddle point** **4 marks**

(b) $z = x^2 - y^2$

B1

$x^2 - y^2 = 0$ gives $y = \pm x$, two lines through the origin

M1 A1

Answer: $z = x^2 - y^2$; $y = \pm x$

3 marks

15 (5 marks)

$x^2 + 4y^2 = c$ gives $\frac{x^2}{c} + \frac{y^2}{\frac{c}{4}} = 1$

M1 A1

It is an ellipse with semi-axes $\sqrt{c}$ (along x) and $\frac{\sqrt{c}}{2}$ (along y)

A1

Area = $\pi \times \sqrt{c} \times \frac{\sqrt{c}}{2}$

M1

= $\frac{\pi c}{2}$

A1

Answer: Ellipse, semi-axes $\sqrt{c}$ and $\frac{\sqrt{c}}{2}$; area $\frac{\pi c}{2}$

5 marks

16 (5 marks)

$x^2 + y^2 = 4x$

M1

$x^2 - 4x + y^2 = 0$

A1

$(x - 2)^2 + y^2 = 4$

M1 A1

A circle with centre (2, 0) and radius 2

A1

Answer: Circle, centre (2, 0), radius 2

5 marks

17 (6 marks)

(a) $x^2 + y^2 \geq 0$ so $z \leq e^0 = 1$

M1

Highest point (0, 0, 1)

A1

Answer: (0, 0, 1)

2 marks

(b) $x^2 + y^2 = -\ln c = \ln \frac{1}{c}$

M1 A1

Circle, centre the origin, radius $\sqrt{\ln \frac{1}{c}}$

A1

Answer: Radius $\sqrt{-\ln c}$

3 marks

(c) $z \rightarrow 0$ but is never 0, so the surface approaches the (x, y) plane without meeting it

B1

Answer: z tends to 0

1 mark

18 (6 marks)

(a) $z = 3$: $x^2 + y^2 = 9$, a circle of radius 3

M1 A1

Answer: $x^2 + y^2 = 9$, a circle of radius 3

2 marks

(b) $y = 0$: $z^2 = x^2$, so $z = \pm x$, a pair of lines

M1 A1

Answer: $z = \pm x$, a pair of lines

2 marks

(c) $x = 1$: $z^2 - y^2 = 1$, a hyperbola

M1 A1

Answer: $z^2 - y^2 = 1$, a hyperbola

2 marks

19 (5 marks)

$f(1, 2) = 2 + 4 = 6$, so the contour is $2x^2 + y^2 = 6$

M1 A1

$f(2, -1) = 8 + 1 = 9 > 6$, so (2, -1) lies outside

M1 A1

$x = 0$: $y^2 = 6$, so $(0, \pm\sqrt{6})$

A1

Answer: $2x^2 + y^2 = 6$; outside; $(0, \pm\sqrt{6})$

5 marks

20 (5 marks)

Contour $z = c$ has vertex (0, c) and opens upwards, so $y = x^2 + c$

M1 A1

$f(x, y) = y - x^2$

A1

Every contour is a parabola, so f takes every real value and for any point there are points nearby with higher and lower values (for example change y)

M1 E1

Answer: $f(x, y) = y - x^2$

5 marks

21 (8 marks)

(a) Put $x = 0$: $z = -y^2$

M1 A1

Answer: $z = -y^2$

2 marks

(b) Along $x = 0$ the surface has a maximum at the origin, so values both above and below 0 occur close to it

M1 E1

Answer: The origin is a maximum along $x = 0$

2 marks

(c) $z(0.1, 0.2) = 0.01 - 0.04 = -0.03$ and $z(0.2, 0.1) = 0.04 - 0.01 = 0.03$

M1 A1

Values are both positive and negative close to the origin: a saddle point

A1 E1

Answer: -0.03 and 0.03 ; saddle point

4 marks

22 (9 marks)

(a) Largest at $x = y = 0$, $h = 500$ m

B1

Answer: 500 m

1 mark

(b) $e^{-(x^2 + y^2)/8} = \frac{1}{5}$

M1

$x^2 + y^2 = 8 \ln 5$

A1

Circle with centre at the summit and radius $\sqrt{8 \ln 5} = 3.59$ km

A1

Answer: Radius 3.59 km

3 marks

(c) $h = 500e^{-4/8} = 500e^{-0.5} = 303$ m

M1 A1

$2 < 3.59$, so the walker is inside the 100 m contour, so above 100 m

E1

Answer: 303 m; above

3 marks

(d) $h \rightarrow 0$ as $x^2 + y^2 \rightarrow \infty$

B1

But an exponential is positive for all x, y so $h = 0$ is never attained

E1

Answer: Tends to 0, never reaches it

2 marks

23 (6 marks)

$z = 0$ meets the x -axis at $x = \pm 6$: $a - 36p = 0$

M1 A1

$z = 20$ meets the x -axis at $x = \pm 4$: $a - 16p = 20$

M1

Subtracting: $20p = 20$, so $p = 1$ and $a = 36$

A1

$z = 0$ meets the y -axis at $y = \pm 3$: $36 - 9q = 0$

M1

$q = 4$

A1

Answer: $a = 36$, $p = 1$, $q = 4$

6 marks

24 (7 marks)

(a) $y = x$: $z = x^2$

B1

$y = -x$: $z = -x^2$

B1

Answer: $z = x^2$; $z = -x^2$

2 marks

(b) $z = 6$, so $xy = 6$

M1

$x^2 = 6$, so $x = \pm\sqrt{6}$

M1

$(\sqrt{6}, \sqrt{6})$ and $(-\sqrt{6}, -\sqrt{6})$

A1

Answer: $xy = 6$; $(\pm\sqrt{6}, \pm\sqrt{6})$

3 marks

(c) Along $y = x$ the origin is a minimum and along $y = -x$ it is a maximum

M1 E1

Answer: Min in one direction, max in another

2 marks

25 (8 marks)

(a) Denominator is smallest, 1, at the origin

M1

$T = 100$ °C at $(0, 0)$

A1

Answer: 100 °C at the origin

2 marks

(b) $1 + x^2 + y^2 = \frac{100}{c}$

M1

$x^2 + y^2 = \frac{100}{c} - 1 = \frac{100 - c}{c}$

A1

Circles, centre the origin, radius $\sqrt{\frac{100 - c}{c}}$

A1

Answer: Radius $\sqrt{\frac{100 - c}{c}}$

3 marks

(c) $T = 20$: radius $\sqrt{\frac{80}{20}} = 2$ m

M1 A1

$T = \frac{100}{1 + 9} = 10$ °C

A1

Answer: 2 m; 10 °C

3 marks

26 (7 marks)

(a) Let $\rho = \sqrt{x^2 + y^2}$ then $\rho = \rho^2$, so $\rho = 0$ or $\rho = 1$

M1 A1

Origin, and the circle $x^2 + y^2 = 1$ at height $z = 1$, radius 1

A1 A1

Answer: Origin and circle $x^2 + y^2 = 1$, $z = 1$

4 marks

(b) Cone: $z = 0.5$; paraboloid: $z = 0.25$

M1 A1

The cone is higher for $0 < \rho < 1$

A1

Answer: Cone (0.5) higher than paraboloid (0.25)

3 marks

27 (10 marks)

(a) $\frac{x^2}{4} + \frac{y^2}{9} = 1 - k^2$

M1

$\frac{x^2}{4(1 - k^2)} + \frac{y^2}{9(1 - k^2)} = 1$

M1 A1

Semi-axes $2\sqrt{1 - k^2}$ and $3\sqrt{1 - k^2}$

A1

Answer: Shown: semi-axes $2\sqrt{1 - k^2}$ and $3\sqrt{1 - k^2}$

4 marks

(b) Area = $\pi \times 2\sqrt{1 - k^2} \times 3\sqrt{1 - k^2}$

M1

= $6\pi(1 - k^2)$

A1

Answer: Shown: the area of this ellipse is $6\pi(1 - k^2)$

2 marks

(c) $V = \int_{-1}^1 6\pi(1 - k^2) dk$

M1

= $6\pi[k - \frac{k^3}{3}]$ between -1 and $1 = 6\pi \times \frac{4}{3}$

M1 A1

= 8π

A1

Answer: 8π

4 marks

28 (9 marks)

(a) $x^2 + y^2 = 2x + 4y - 5$

M1

$(x - 1)^2 + (y - 2)^2 = 0$

M1 A1

Only $x = 1$, $y = 2$, so $z = 5$: the point $(1, 2, 5)$

A1

Answer: $(1, 2, 5)$

4 marks

(b) $x^2 + y^2 - 2px - 2qy + p^2 + q^2 = 0$

M1

$(x - p)^2 + (y - q)^2 = 0$

A1

The point $(p, q, p^2 + q^2)$

A1

Answer: $(p, q, p^2 + q^2)$

3 marks

(c) $p = 3$, $q = -1$: $z = 6x - 2y - 9 - 1$

M1

$z = 6x - 2y - 10$

A1

Answer: $z = 6x - 2y - 10$

2 marks

29 (10 marks)

(a) $x^2 + y^2 = cx$

M1

$x^2 - cx + \frac{c^2}{4} + y^2 = \frac{c^2}{4}$

M1 A1

$(x - \frac{c}{2})^2 + y^2 = \frac{c^2}{4}$; $x = 0$ is excluded, which removes only the origin

A1 E1

Answer: Shown: $(x - \frac{c}{2})^2 + y^2 = \frac{c^2}{4}$

5 marks

(b) Centre $(\frac{c}{2}, 0)$, radius $\frac{|c|}{2}$

B1 B1

Reflections of each other in the y-axis

B1

Answer: Centre $(\frac{c}{2}, 0)$, radius $\frac{|c|}{2}$

3 marks

(c) f is a function, so each point (with $x \neq 0$) has one value

E1

All the circles pass through the origin (excluded) and touch the y-axis there

E1

Answer: Each point has one value

2 marks

30 (10 marks)

(a) $1 + t^2 - t^2 = 1$

M1 A1

Similarly $(1, t, -t)$ lies on H

A1

Answer: $(1, t, -t)$

3 marks

(b) $x^2 + y^2 = (\cos \alpha - t \sin \alpha)^2 + (\sin \alpha + t \cos \alpha)^2$

M1

$= \cos^2 \alpha + \sin^2 \alpha + t^2(\sin^2 \alpha + \cos^2 \alpha) = 1 + t^2$

M1 A1

So $x^2 + y^2 - z^2 = 1$

A1

Answer: Shown: the line with points $(\cos \alpha - t \sin \alpha, \sin \alpha + t \cos \alpha, t)$ lies wholly on H

4 marks

(c) Direction $(-\sin \alpha, \cos \alpha, 1)$ from the line in (b)

B1

Reflecting in the plane $z = 0$ gives the line with points $(\cos \alpha - t \sin \alpha, \sin \alpha + t \cos \alpha, -t)$, direction $(-\sin \alpha, \cos \alpha, -1)$

B1

Answer: $(-\sin \alpha, \cos \alpha, 1)$ and $(-\sin \alpha, \cos \alpha, -1)$

2 marks

(d) Through every point of the circle two straight lines lie wholly on H, so H is made up of straight lines

E1

Answer: Made of straight lines

1 mark

Partial Differentiation

1 (3 marks)

$$\frac{\partial f}{\partial x} = 6xy - 2y^3$$

M1 A1

$$\frac{\partial f}{\partial y} = 3x^2 - 6xy^2 + 1$$

A1

Answer: $\frac{\partial f}{\partial x} = 6xy - 2y^3$, $\frac{\partial f}{\partial y} = 3x^2 - 6xy^2 + 1$

3 marks

2 (2 marks)

$$\frac{\partial f}{\partial x} = 2e^{2x+3y}$$

B1

$$\frac{\partial f}{\partial y} = 3e^{2x+3y}$$

B1

Answer: $2e^{2x+3y}$, $3e^{2x+3y}$

2 marks

3 (3 marks)

$$\frac{\partial f}{\partial x} = \sin y \text{ and } \frac{\partial f}{\partial y} = x \cos y$$

M1 A1

$$\text{At } (2, \frac{\pi}{2}): \frac{\partial f}{\partial x} = 1 \text{ and } \frac{\partial f}{\partial y} = 0$$

A1

Answer: $\sin y$, $x \cos y$; 1 and 0

3 marks

4 (2 marks)

Chain rule: derivative of $\ln u$ is $\frac{1}{u} \times \frac{\partial u}{\partial x}$

M1

$$\frac{\partial f}{\partial x} = \frac{2x}{x^2 + y^2}$$

A1

Answer: $\frac{2x}{x^2 + y^2}$

2 marks

5 (2 marks)

$$\frac{\partial f}{\partial y} = x^2 + 3y^2$$

M1

$$\text{At } (1, 2): 1 + 12 = 13$$

A1

Answer: 13

2 marks

6 (2 marks)

$$\frac{\partial V}{\partial r} = 2\pi rh$$

B1

$$\frac{\partial V}{\partial h} = \pi r^2$$

B1

Answer: $2\pi rh$, πr^2

2 marks

7 (4 marks)

$$\text{Product rule: } \frac{\partial f}{\partial x} = 2xe^{xy} + x^2 \times ye^{xy}$$

M1 A1

$$= xe^{xy}(2 + xy)$$

A1

$$\frac{\partial f}{\partial y} = x^2 \times xe^{xy} = x^3 e^{xy}$$

A1

Answer: $xe^{xy}(2 + xy)$, $x^3 e^{xy}$

4 marks

8 (4 marks)

$$\text{Quotient rule: } \frac{\partial f}{\partial x} = \frac{(x + y) \times 1 - x \times 1}{(x + y)^2}$$

M1 A1

$$= \frac{y}{(x + y)^2}$$

A1

$$\frac{\partial f}{\partial y} = \frac{-x}{(x + y)^2}$$

A1

Answer: $\frac{\partial f}{\partial y} = \frac{-x}{(x + y)^2}$

4 marks

9 (4 marks)

$$\text{Use } d(\tan^{-1}u) = \frac{1}{1 + u^2} du \text{ with } u = \frac{y}{x}$$

M1

$$\frac{\partial f}{\partial x} = \frac{1}{1 + y^2/x^2} \times \left(-\frac{y}{x^2}\right) = -\frac{y}{x^2 + y^2}$$

M1 A1

$$\frac{\partial f}{\partial y} = \frac{1}{1 + y^2/x^2} \times \frac{1}{x} = \frac{x}{x^2 + y^2}$$

A1

Answer: $\frac{\partial f}{\partial y} = \frac{x}{x^2 + y^2}$

4 marks

10 (4 marks)

$$\delta V \approx \frac{\partial V}{\partial r} \delta r + \frac{\partial V}{\partial h} \delta h \text{ with } V = \pi r^2 h$$

M1

$$= 2\pi rh \delta r + \pi r^2 \delta h = 2\pi(4)(12)(0.1) + \pi(16)(-0.3)$$

M1 A1

$$= \frac{24\pi}{5} = 15.1 \text{ cm}^3$$

A1

Answer: $\frac{24\pi}{5} \approx 15.1 \text{ cm}^3$

4 marks

11 (4 marks)

$$\frac{\partial f}{\partial x} = 2xy = 4 \text{ and } \frac{\partial f}{\partial y} = x^2 = 4 \text{ at } (2, 1)$$

M1 A1

$$z - 4 = 4(x - 2) + 4(y - 1)$$

M1

$$z = 4x + 4y - 8$$

A1

Answer: $z = 4x + 4y - 8$

4 marks

12 (4 marks)

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$$

M1

$$= (2x + 3y)(2) + 3x(2t)$$

M1 A1

$$\text{Substituting } x = 2t, y = t^2: = (4t + 3t^2)(2) + 12t^2 = 18t^2 + 8t$$

A1

$$\text{Answer: } 18t^2 + 8t$$

4 marks

13 (4 marks)

$$\text{Let } F = x^2 + xy + y^2 - 7. \text{ Then } \frac{dy}{dx} = -\frac{F_x}{F_y}$$

M1

$$F_x = 2x + y \text{ and } F_y = x + 2y$$

A1

$$\frac{dy}{dx} = -\frac{2x + y}{x + 2y}$$

A1

$$\text{At } (1, 2): -\frac{4}{5}$$

A1

$$\text{Answer: } \frac{dy}{dx} = -\frac{2x + y}{x + 2y}; -\frac{4}{5}$$

4 marks

14 (6 marks)

$$\frac{\partial g}{\partial L} = \frac{4\pi^2}{T^2} \text{ and } \frac{\partial g}{\partial T} = -\frac{8\pi^2 L}{T^3}$$

M1 A1

$$\delta g \approx \frac{4\pi^2}{T^2} \delta L - \frac{8\pi^2 L}{T^3} \delta T$$

M1

$$\frac{\delta g}{g} \approx \frac{\delta L}{L} - 2 \frac{\delta T}{T}$$

M1 A1

$$\text{Maximum when the terms add in size: } 1\% + 2 \times 0.5\% = 2\%$$

A1

$$\text{Answer: } 2\%$$

6 marks

15 (6 marks)

$$(a) \frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} \text{ and } \frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}$$

M1 A1

$$\text{At } P: \frac{3}{5} \text{ and } \frac{4}{5}$$

A1

$$\text{Answer: } \frac{3}{5} \text{ and } \frac{4}{5}$$

3 marks

$$(b) z - 5 = \frac{3}{5}(x - 3) + \frac{4}{5}(y - 4)$$

M1

$$5z - 25 = 3x - 9 + 4y - 16, \text{ so } 3x + 4y - 5z = 0$$

A1

$$\text{Normal vector } 3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$$

A1

$$\text{Answer: } 3x + 4y - 5z = 0; 3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$$

3 marks

16 (6 marks)

$$(a) \frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial x} = 2uv \times 1 + u^2 \times y$$

M1 A1

$$= 2(x + 2y)(xy) + (x + 2y)^2 y = y(x + 2y)(2x + x + 2y)$$

M1 A1

$$\text{Answer: Shown: } \frac{\partial f}{\partial x} = y(x + 2y)(3x + 2y)$$

4 marks

$$(b) \frac{\partial f}{\partial y} = 2uv \times 2 + u^2 \times x = 4(x + 2y)(xy) + (x + 2y)^2 x$$

M1

$$= x(x + 2y)(4y + x + 2y) = x(x + 2y)(x + 6y)$$

A1

$$\text{Answer: } x(x + 2y)(x + 6y)$$

2 marks

17 (5 marks)

$$(a) \frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \cos \theta + \frac{\partial z}{\partial y} \sin \theta = 2xy \cos \theta + x^2 \sin \theta$$

M1 A1

$$= 2r^2 \sin \theta \cos^2 \theta + r^2 \cos^2 \theta \sin \theta = 3r^2 \sin \theta \cos^2 \theta$$

A1

$$\text{Answer: } 3r^2 \sin \theta \cos^2 \theta$$

3 marks

$$(b) z = r^3 \cos^2 \theta \sin \theta$$

M1

$$\frac{\partial z}{\partial r} = 3r^2 \cos^2 \theta \sin \theta, \text{ which agrees}$$

A1

Answer: Agrees

2 marks

18 (5 marks)

$$V = \frac{1}{3} \pi r^2 h, \text{ so } \frac{\partial V}{\partial r} = \frac{2}{3} \pi r h \text{ and } \frac{\partial V}{\partial h} = \frac{1}{3} \pi r^2$$

M1 A1

$$\text{At } r = 6, h = 9: \frac{\partial V}{\partial r} = 36\pi \text{ and } \frac{\partial V}{\partial h} = 12\pi$$

A1

$$\frac{dV}{dt} = 36\pi(0.5) + 12\pi(-0.4)$$

M1

$$= \frac{66\pi}{5} = 41.5 \text{ cm}^3 \text{ s}^{-1} \text{ (increasing)}$$

A1

Answer: 41.5 cm³ s⁻¹ increasing

5 marks

19 (5 marks)

$$\text{(a) Let } F = x^3 + y^3 - 6xy: F_x = 3x^2 - 6y \text{ and } F_y = 3y^2 - 6x$$

M1 A1

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = \frac{6y - 3x^2}{3y^2 - 6x} = \frac{2y - x^2}{y^2 - 2x}$$

M1 A1

$$\text{Answer: Shown: } \frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x}$$

4 marks

$$\text{(b) } \frac{6-9}{9-6} = -1$$

A1

Answer: -1

1 mark

20 (8 marks)

$$\text{(a) } f(2, b) = 2b^2 - 8b = -6$$

M1

$$b^2 - 4b + 3 = 0$$

A1

$$b = 1 \text{ or } b = 3$$

A1

Answer: b = 1 or b = 3

3 marks

$$\text{(b) } f_x = y^2 - 4xy \text{ and } f_y = 2xy - 2x^2$$

M1 A1

$$\text{At } (2, 1): \nabla f = -7\mathbf{i} - 4\mathbf{j}, \text{ so } -7(x-2) - 4(y-1) = 0, \text{ i.e. } 7x + 4y = 18$$

M1 A1

$$\text{At } (2, 3): \nabla f = -15\mathbf{i} + 4\mathbf{j}, \text{ so } -15(x-2) + 4(y-3) = 0, \text{ i.e. } 15x - 4y = 18$$

A1

Answer: (2, 1): -7i - 4j, 7x + 4y = 18; (2, 3): -15i + 4j, 15x - 4y = 18

5 marks

21 (7 marks)

$$\text{(a) } \frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt}$$

M1

$$= 2xy \times 2t + (x^2 + z) \times 2 + y \times \left(-\frac{1}{t^2}\right)$$

M1 A1

$$= 2(t^2)(2t)(2t) + 2\left(t^4 + \frac{1}{t}\right) - 2t \times \frac{1}{t^2}$$

M1

$$= 8t^4 + 2t^4 + \frac{2}{t} - \frac{2}{t} = 10t^4$$

A1

Answer: 10t⁴

5 marks

$$\text{(b) } w = t^4(2t) + 2t \times \frac{1}{t} = 2t^5 + 2$$

M1

$$\frac{dw}{dt} = 10t^4$$

A1

Answer: 10t⁴

2 marks

22 (8 marks)

$$(a) \frac{\partial f}{\partial x} = g_u \frac{\partial u}{\partial x} + g_v \frac{\partial v}{\partial x} \quad \text{M1}$$

$$\frac{\partial u}{\partial x} = 1 \text{ and } \frac{\partial v}{\partial x} = 1, \text{ so } \frac{\partial f}{\partial x} = g_u + g_v \quad \text{A1}$$

$$\frac{\partial u}{\partial y} = 1 \text{ and } \frac{\partial v}{\partial y} = -1, \text{ so } \frac{\partial f}{\partial y} = g_u - g_v \quad \text{A1}$$

$$\text{Answer: Shown: } \frac{\partial f}{\partial x} = g_u + g_v \text{ and } \frac{\partial f}{\partial y} = g_u - g_v \quad \text{3 marks}$$

$$(b) (g_u + g_v)^2 - (g_u - g_v)^2 \quad \text{M1}$$

$$= 4g_u g_v \quad \text{A1}$$

$$\text{Answer: Shown: } \left(\frac{\partial f}{\partial x}\right)^2 - \left(\frac{\partial f}{\partial y}\right)^2 = 4g_u g_v \quad \text{2 marks}$$

$$(c) f = (x + y)^2(x - y): \frac{\partial f}{\partial x} = 2(x + y)(x - y) + (x + y)^2 = 6 + 9 = 15 \quad \text{M1}$$

$$\frac{\partial f}{\partial y} = 2(x + y)(x - y) - (x + y)^2 = 6 - 9 = -3, \text{ so } \left(\frac{\partial f}{\partial x}\right)^2 - \left(\frac{\partial f}{\partial y}\right)^2 = 225 - 9 = 216 \quad \text{M1}$$

$$\text{At } u = 3, v = 1: g_u = 2uv = 6, g_v = u^2 = 9, \text{ so } 4g_u g_v = 216 \quad \text{A1}$$

$$\text{Answer: Both sides equal 216} \quad \text{3 marks}$$

23 (9 marks)

$$(a) \frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} \text{ and } \frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}} \quad \text{M1 A1}$$

$$\text{Answer: } \frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \quad \text{2 marks}$$

$$(b) f(3, 4) = 5, \frac{\partial f}{\partial x} = \frac{3}{5}, \frac{\partial f}{\partial y} = \frac{4}{5} \quad \text{B1}$$

$$\delta f \approx \frac{3}{5}(0.02) + \frac{4}{5}(-0.02) \quad \text{M1 A1}$$

$$= -0.004, \text{ so the estimate is } 4.996 \quad \text{A1}$$

$$\text{Answer: 4.996} \quad \text{4 marks}$$

$$(c) 4.99608 \quad \text{B1}$$

$$\text{Error} = 4.99608 - 4.996 = 0.000078; \text{ percentage } 0.0016\% \quad \text{M1 A1}$$

$$\text{Answer: 4.99608; 0.0016\%} \quad \text{3 marks}$$

24 (8 marks)

$$(a) \text{ The student has not applied the chain rule: the derivative of } \sin(xy) \text{ with respect to } x \text{ is } y \cos(xy) \quad \text{M1 E1}$$

$$\frac{\partial z}{\partial x} = 2xy + y \cos(xy) \quad \text{A1}$$

$$\text{Answer: } 2xy + y \cos(xy) \quad \text{3 marks}$$

$$(b) \frac{\partial z}{\partial y} = x^2 + x \cos(xy) \quad \text{M1 A1}$$

$$\text{Answer: } x^2 + x \cos(xy) \quad \text{2 marks}$$

$$(c) z(1, 0) = 0, \frac{\partial z}{\partial x} = 2(1)(0) + 0 \times \cos 0 = 0 \text{ and } \frac{\partial z}{\partial y} = 1 + 1 \times \cos 0 = 2 \quad \text{M1 A1}$$

$$z = 2y \quad \text{A1}$$

$$\text{Answer: } z = 2y \quad \text{3 marks}$$

25 (7 marks)

$$(a) \frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial x} = 2u(2x) + 2v(2y) \quad \text{M1 A1}$$

$$= 4x(x^2 - y^2) + 4y(2xy) \quad \text{M1}$$

$$= 4x^3 + 4xy^2 = 4x(x^2 + y^2) \quad \text{A1}$$

$$\text{Answer: } 4x(x^2 + y^2) \quad \text{4 marks}$$

$$(b) u^2 + v^2 = x^4 - 2x^2y^2 + y^4 + 4x^2y^2 = (x^2 + y^2)^2 \quad \text{M1 A1}$$

$$\frac{\partial f}{\partial x} = 2(x^2 + y^2)(2x) = 4x(x^2 + y^2) \quad \text{A1}$$

Answer: Shown: $u^2 + v^2 = (x^2 + y^2)^2$

3 marks

26 (6 marks)

(a) $P = \frac{nRT}{V}$, $V = \frac{nRT}{P}$, $T = \frac{PV}{nR}$

B1

Answer: $P = \frac{nRT}{V}$, $V = \frac{nRT}{P}$, $T = \frac{PV}{nR}$

1 mark

(b) $\frac{\partial P}{\partial V} = -\frac{nRT}{V^2}$, $\frac{\partial V}{\partial T} = \frac{nR}{P}$, $\frac{\partial T}{\partial P} = \frac{V}{nR}$

M1 A1 A1

Answer: $-\frac{nRT}{V^2}$, $\frac{nR}{P}$, $\frac{V}{nR}$

3 marks

(c) Product = $-\frac{nRT}{V^2} \times \frac{nR}{P} \times \frac{V}{nR} = -\frac{nRT}{PV}$

M1

Since $PV = nRT$ this is -1

A1

Answer: Shown: $\frac{\partial P}{\partial V} \frac{\partial V}{\partial T} \frac{\partial T}{\partial P} = -1$

2 marks

27 (11 marks)

(a) $\frac{\partial z}{\partial r} = z_x \frac{\partial x}{\partial r} + z_y \frac{\partial y}{\partial r}$ with $\frac{\partial x}{\partial r} = \cos \theta$ and $\frac{\partial y}{\partial r} = \sin \theta$

M1 A1

$\frac{\partial z}{\partial \theta} = z_x \frac{\partial x}{\partial \theta} + z_y \frac{\partial y}{\partial \theta}$ with $\frac{\partial x}{\partial \theta} = -r \sin \theta$ and $\frac{\partial y}{\partial \theta} = r \cos \theta$

M1 A1

Answer: Shown: $\frac{\partial z}{\partial r} = z_x \cos \theta + z_y \sin \theta$ and $\frac{\partial z}{\partial \theta} = -rz_x \sin \theta + rz_y \cos \theta$

4 marks

(b) Square and add: $(\frac{\partial z}{\partial r})^2 + \frac{1}{r^2}(\frac{\partial z}{\partial \theta})^2$

$= (z_x \cos \theta + z_y \sin \theta)^2 + (-z_x \sin \theta + z_y \cos \theta)^2$

M1

$= z_x^2 (\cos^2 \theta + \sin^2 \theta) + z_y^2 (\sin^2 \theta + \cos^2 \theta) + 2z_x z_y (\sin \theta \cos \theta - \sin \theta \cos \theta)$

M1 A1

$= z_x^2 + z_y^2$

A1

Answer: Shown: $(z_x)^2 + (z_y)^2 = (\frac{\partial z}{\partial r})^2 + \frac{1}{r^2}(\frac{\partial z}{\partial \theta})^2$

4 marks

(c) At $x = 0$, $y = 2$: $z_x = 2x = 0$ and $z_y = 2$, so LHS = 4

M1

$z = r^2 \cos^2 \theta + 2r \sin \theta$: $\frac{\partial z}{\partial r} = 2r \cos^2 \theta + 2 \sin \theta = 2$ and $\frac{\partial z}{\partial \theta} = -2r^2 \cos \theta \sin \theta + 2r \cos \theta = 0$ at r

M1 A1

$= 2$, $\theta = \frac{\pi}{2}$

RHS = $4 + 0 = 4$, which agrees

Answer: Both equal 4

3 marks

28 (11 marks)

(a) $f(tx, ty) = t^3 x^3 + 3t^3 xy^2 = t^3 f(x, y)$

M1 A1

$f_x = 3x^2 + 3y^2$ and $f_y = 6xy$

M1

$xf_x + yf_y = 3x^3 + 3xy^2 + 6xy^2 = 3(x^3 + 3xy^2) = 3f$

A1

Answer: Shown: $f(x, y) = x^3 + 3xy^2$ is homogeneous of degree 3 and that $xf_x + yf_y = 3f$

4 marks

(b) Left side: by the chain rule, $\frac{d}{dt} f(tx, ty) = xf_x(tx, ty) + yf_y(tx, ty)$

M1 A1

Right side: $\frac{d}{dt} [t^n f(x, y)] = nt^{n-1} f(x, y)$

A1

Put $t = 1$: $xf_x + yf_y = nf$

M1 A1

Answer: Proved: for any differentiable homogeneous function of degree n , $xf_x + yf_y = nf$

5 marks

(c) $2g_x + g_y = 3g$ at $(2, 1)$

M1

$2g_x + 4 = 15$, so $g_x = \frac{11}{2}$

A1

Answer: $\frac{11}{2}$

2 marks

29 (11 marks)

(a) Quotient rule: $\frac{\partial R}{\partial R_1} = \frac{R_2(R_1 + R_2) - R_1 R_2}{(R_1 + R_2)^2}$ M1 A1

$= \frac{R_2^2}{(R_1 + R_2)^2}$; by symmetry $\frac{\partial R}{\partial R_2} = \frac{R_1^2}{(R_1 + R_2)^2}$ A1

Answer: $\frac{\partial R}{\partial R_2} = \frac{R_1^2}{(R_1 + R_2)^2}$ 3 marks

(b) $R = \frac{1800}{90} = 20$

$\frac{\partial R}{\partial R_1} = \frac{3600}{8100} = \frac{4}{9}$ and $\frac{\partial R}{\partial R_2} = \frac{900}{8100} = \frac{1}{9}$ M1 A1

Maximum $\delta R \approx \frac{4}{9}(0.5) + \frac{1}{9}(1) = \frac{1}{3}$ M1 A1

Percentage error $= \frac{1}{3 \times 20} \times 100 = 1.7\%$ A1

Answer: $R = 20$, error $\approx \frac{1}{3}$ (1.7%) 5 marks

(c) $R = \frac{k^2}{2k} = \frac{k}{2}$ B1

$\frac{\partial R}{\partial R_1} = \frac{k^2}{4k^2} = \frac{1}{4}$, so $\delta R \approx \frac{1}{4}\delta R_1$ M1 E1

Answer: $R = \frac{k}{2}$; the error in R is a quarter of the error in one resistor 3 marks

30 (11 marks)

(a) S_1 : $z = xy$ has $\frac{\partial z}{\partial x} = y = 1$ and $\frac{\partial z}{\partial y} = x = 1$, so $z - 1 = (x - 1) + (y - 1)$, $x + y - z = 1$ M1 A1

S_2 : $F = x^2 + y^2 + z^2 - 3$ has gradient $(2x, 2y, 2z) = (2, 2, 2)$ at P , so $x + y + z = 3$ M1 A1

Answer: $x + y - z = 1$; $x + y + z = 3$ 4 marks

(b) Normals $(1, 1, -1)$ and $(1, 1, 1)$ M1

$\cos \theta = \frac{|1 + 1 - 1|}{\sqrt{3}\sqrt{3}} = \frac{1}{3}$ M1 A1

$\theta = 70.5^\circ$ A1

Answer: 70.5° 4 marks

(c) Adding the equations: $2(x + y) = 4$, so $x + y = 2$ and $z = 1$ M1 A1

$\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} - \mathbf{j})$ A1

Answer: $\mathbf{r} = (1, 1, 1) + \lambda(1, -1, 0)$ 3 marks

Second Partial Derivatives**1 (3 marks)**

$f_x = 3x^2y^2$ and $f_y = 2x^3y$ M1

$f_{xx} = 6xy^2$ and $f_{yy} = 2x^3$ A1

$f_{xy} = 6x^2y$ A1

Answer: $f_{xx} = 6xy^2$, $f_{yy} = 2x^3$, $f_{xy} = 6x^2y$ 3 marks

2 (2 marks)

$\frac{\partial^2 z}{\partial x^2} = 9e^{3x} \sin y$ B1

$\frac{\partial^2 z}{\partial y^2} = -e^{3x} \sin y$ B1

Answer: $9e^{3x} \sin y$, $-e^{3x} \sin y$ 2 marks

3 (3 marks)

$$f_x = 2x + 3y \text{ so } f_{xy} = 3$$

M1 A1

$$f_y = 3x + 2y \text{ so } f_{yx} = 3; \text{ they are equal}$$

A1

$$\text{Answer: } f_{xy} = f_{yx} = 3$$

3 marks

4 (3 marks)

$$g_x = \frac{1}{x} \text{ and } g_y = \frac{1}{y}$$

M1

$$g_{xx} = -\frac{1}{x^2}, g_{yy} = -\frac{1}{y^2}$$

A1

$$g_{xy} = 0$$

A1

$$\text{Answer: } -\frac{1}{x^2}, -\frac{1}{y^2}, 0$$

3 marks

5 (2 marks)

$$f_{xx} = 12x^2y$$

M1

$$\text{At } (1, 2): 24$$

A1

$$\text{Answer: } 24$$

2 marks

6 (3 marks)

$$\frac{\partial^2 z}{\partial x^2} = -4 \sin(2x + 3y)$$

B1

$$\frac{\partial^2 z}{\partial x \partial y} = -6 \sin(2x + 3y)$$

B1

$$\frac{\partial^2 z}{\partial y^2} = -9 \sin(2x + 3y)$$

B1

$$\text{Answer: } -4 \sin(2x + 3y), -6 \sin(2x + 3y), -9 \sin(2x + 3y)$$

3 marks

7 (4 marks)

$$f_x = 2xe^y \text{ and } f_y = x^2e^y + 3y^2$$

M1 A1

$$f_{xx} = 2e^y, f_{xy} = 2xe^y$$

A1

$$f_{yy} = x^2e^y + 6y$$

A1

$$\text{Answer: } 2e^y, 2xe^y, x^2e^y + 6y$$

4 marks

8 (4 marks)

$$f_x = ye^{xy} \text{ and } f_y = xe^{xy}$$

M1

$$f_{xx} = y^2e^{xy} \text{ and } f_{yy} = x^2e^{xy}$$

A1

$$f_{xy} = e^{xy} + xye^{xy} = (1 + xy)e^{xy}$$

M1 A1

$$\text{Answer: } y^2e^{xy}, (1 + xy)e^{xy}, x^2e^{xy}$$

4 marks

9 (4 marks)

$$f_x = \frac{1}{y} \text{ and } f_y = -\frac{x}{y^2}$$

M1

$$f_{xx} = 0$$

A1

$$f_{xy} = -\frac{1}{y^2}$$

A1

$$f_{yy} = \frac{2x}{y^3}$$

A1

$$\text{Answer: } 0, -\frac{1}{y^2}, \frac{2x}{y^3}$$

4 marks

10 (4 marks)

$$f_x = 3x^2 - 3y^2, \text{ so } f_{xx} = 6x$$

M1 A1

$$f_y = -6xy, \text{ so } f_{yy} = -6x$$

A1

$$f_{xx} + f_{yy} = 6x - 6x = 0$$

A1

Answer: Shown: $f(x, y) = x^3 - 3xy^2$ satisfies Laplace's equation $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$

4 marks

11 (5 marks)

$$\frac{\partial z}{\partial x} = \frac{2x}{x^2 + y^2}$$

B1

$$\frac{\partial^2 z}{\partial x^2} = \frac{2(x^2 + y^2) - 4x^2}{(x^2 + y^2)^2} = \frac{2(y^2 - x^2)}{(x^2 + y^2)^2}$$

M1 A1

$$\text{By symmetry } \frac{\partial^2 z}{\partial y^2} = \frac{2(x^2 - y^2)}{(x^2 + y^2)^2}$$

A1

The sum is 0

A1

Answer: Shown: $z_{xx} + z_{yy} = 0$

5 marks

12 (4 marks)

$$f_x = 2xy^3, f_{xx} = 2y^3, \text{ so } f_{xxy} = 6y^2$$

M1 A1

$$f_{xy} = 6xy^2, \text{ so } f_{xyx} = 6y^2$$

M1

They are equal

A1

Answer: $6y^2$ in both cases

4 marks

13 (3 marks)

$$f_{xx} = 2 \text{ and } f_{yy} = 2k$$

M1 A1

$$2 + 2k = 0, \text{ so } k = -1$$

A1

Answer: $k = -1$

3 marks

14 (5 marks)

$$u_x = 2 \cos 2x \cos 6t, u_{xx} = -4 \sin 2x \cos 6t$$

M1 A1

$$u_t = -6 \sin 2x \sin 6t, u_{tt} = -36 \sin 2x \cos 6t$$

M1 A1

$$u_{tt} = 9 \times (-4 \sin 2x \cos 6t) = 9u_{xx}$$

A1

Answer: Shown: $u = \sin 2x \cos 6t$ satisfies the wave equation $\frac{\partial^2 u}{\partial t^2} = 9 \frac{\partial^2 u}{\partial x^2}$

5 marks

15 (5 marks)

$$u_t = \lambda e^{\lambda t} \cos 3x$$

B1

$$u_x = -3e^{\lambda t} \sin 3x, u_{xx} = -9e^{\lambda t} \cos 3x$$

M1 A1

$$\lambda = 2(-9)$$

M1

$$\lambda = -18$$

A1

Answer: $\lambda = -18$

5 marks

16 (5 marks)

$$f_x = 3x^2 - 3y \text{ and } f_y = 3y^2 - 3x$$

M1

$$f_{xx} = 6x, f_{yy} = 6y, f_{xy} = -3$$

A1 A1

$$\text{At } (1, 1): 36 - 9 = 27$$

M1

$$\text{At } (0, 0): 0 - 9 = -9$$

A1

Answer: $f_{xx} = 6x, f_{yy} = 6y, f_{xy} = -3; 27$ and -9

5 marks

17 (5 marks)

$$f_x = y \cos(xy) \text{ and } f_y = x \cos(xy) \quad \text{M1}$$

$$f_{xx} = -y^2 \sin(xy) \text{ and } f_{yy} = -x^2 \sin(xy) \quad \text{A1}$$

$$f_{xy} = \cos(xy) - xy \sin(xy) \quad \text{M1 A1}$$

$$f_{yx} = \cos(xy) - xy \sin(xy), \text{ equal to } f_{xy} \quad \text{A1}$$

Answer: $-y^2 \sin(xy), -x^2 \sin(xy), \cos(xy) - xy \sin(xy)$ (both)

5 marks

18 (7 marks)

$$\text{Let } \rho^2 = x^2 + y^2 + z^2. \text{ Then } u_x = -x\rho^{-3} \quad \text{M1 A1}$$

$$u_{xx} = -\rho^{-3} + 3x^2\rho^{-5} \quad \text{M1 A1}$$

$$\text{Similarly } u_{yy} = -\rho^{-3} + 3y^2\rho^{-5} \text{ and } u_{zz} = -\rho^{-3} + 3z^2\rho^{-5} \quad \text{A1}$$

$$\text{Sum} = -3\rho^{-3} + 3\rho^2\rho^{-5} = 0 \quad \text{M1 A1}$$

Answer: Shown: u satisfies the three-dimensional Laplace equation $u_{xx} + u_{yy} + u_{zz} = 0$

7 marks

19 (5 marks)

$$z_x = 2x \cos(x^2 + y^2) \quad \text{M1}$$

$$\frac{\partial^2 z}{\partial x^2} = 2\cos(x^2 + y^2) - 4x^2 \sin(x^2 + y^2) \quad \text{M1 A1}$$

$$\text{By symmetry } \frac{\partial^2 z}{\partial y^2} = 2\cos(x^2 + y^2) - 4y^2 \sin(x^2 + y^2) \quad \text{A1}$$

Adding gives the result

A1

Answer: $2\cos(x^2 + y^2) - 4x^2 \sin(x^2 + y^2); 2\cos(x^2 + y^2) - 4y^2 \sin(x^2 + y^2)$

5 marks

20 (5 marks)

$$\text{Integrate with respect to } x: f = x^2y + 3xy^2 + g(y) \quad \text{M1 A1}$$

$$f(0, y) = g(y) = y^2 \quad \text{M1}$$

$$f = x^2y + 3xy^2 + y^2 \quad \text{A1}$$

$$f_x = 2xy + 3y^2 \text{ so } f_{xy} = 2x + 6y \quad \text{A1}$$

Answer: $f = x^2y + 3xy^2 + y^2$

5 marks

21 (8 marks)

$$\text{(a) } u_x = F'(x + ct) + G'(x - ct) \text{ by the chain rule} \quad \text{M1}$$

$$\frac{\partial^2 u}{\partial x^2} = F''(x + ct) + G''(x - ct) \quad \text{A1}$$

$$u_t = cF'(x + ct) - cG'(x - ct), \text{ so } \frac{\partial^2 u}{\partial t^2} = c^2F''(x + ct) + c^2G''(x - ct) \quad \text{M1 A1}$$

$$\text{Answer: } \frac{\partial^2 u}{\partial t^2} = c^2[F''(x + ct) + G''(x - ct)]$$

4 marks

$$\text{(b) Comparing: } \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

A1

$$\text{Answer: Shown: u satisfies the wave equation } \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

1 mark

$$\text{(c) } u = (x + 2t)^2 + e^{x - 2t}$$

B1

$$u_{xx} = 2 + e^{x - 2t} = 3 \text{ at } (0, 0) \quad \text{M1}$$

$$u_{tt} = 8 + 4e^{x - 2t} = 12 \text{ at } (0, 0) = 4 \times 3 = c^2 u_{xx} \quad \text{A1}$$

Answer: $u_{tt} = 12 = 4u_{xx}$

3 marks

22 (9 marks)

$$(a) 2r \frac{\partial r}{\partial x} = 2x, \text{ so } \frac{\partial r}{\partial x} = \frac{x}{r} \quad \text{M1 A1}$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{r - x(x/r)}{r^2} = \frac{r^2 - x^2}{r^3} = \frac{y^2}{r^3} \quad \text{A1}$$

$$\text{Similarly } \frac{\partial r}{\partial y} = \frac{y}{r}$$

$$\text{Answer: } \frac{\partial^2 r}{\partial x^2} = \frac{y^2}{r^3}, \frac{\partial r}{\partial y} = \frac{y}{r} \quad \text{3 marks}$$

$$(b) u_x = f'(r) \frac{x}{r} \quad \text{M1}$$

$$\frac{\partial^2 u}{\partial x^2} = f''(r) \frac{x^2}{r^2} + f'(r) \times \frac{y^2}{r^3} \quad \text{A1}$$

$$\text{Answer: Shown: } \frac{\partial^2 u}{\partial x^2} = \frac{x^2}{r^2} f''(r) + \frac{y^2}{r^3} f'(r) \quad \text{2 marks}$$

$$(c) \frac{\partial^2 u}{\partial y^2} = \frac{y^2}{r^2} f'' + \frac{x^2}{r^3} f' \text{ by symmetry; adding gives } f'' + \frac{1}{r} f' \quad \text{A1}$$

$$\text{Laplace: } f'' + \frac{f'}{r} = 0, \text{ so } (rf')' = 0, rf' = A \quad \text{M1 A1}$$

$$f = A \ln r + B \quad \text{A1}$$

$$\text{Answer: } f(r) = A \ln r + B \quad \text{4 marks}$$

23 (8 marks)

$$(a) f_x = 4x^3 - 6xy^2, f_y = -6x^2y + 4y^3 \quad \text{M1}$$

$$f_{xx} = 12x^2 - 6y^2, f_{xy} = -12xy, f_{yy} = -6x^2 + 12y^2 \quad \text{A1 A1}$$

$$\text{Answer: } 12x^2 - 6y^2, -12xy, 12y^2 - 6x^2 \quad \text{3 marks}$$

$$(b) x^2 f_{xx} = 12x^4 - 6x^2y^2 \quad \text{M1}$$

$$2xy f_{xy} = -24x^2y^2, y^2 f_{yy} = -6x^2y^2 + 12y^4 \quad \text{A1}$$

$$\text{Total: } 12x^4 - 36x^2y^2 + 12y^4 = 12f \quad \text{A1}$$

$$\text{Answer: Shown: } x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy} = 12f \quad \text{3 marks}$$

$$(c) \text{Homogeneous of degree 4: } xf_x + yf_y = 4(x^4 - 3x^2y^2 + y^4) = 4f \quad \text{M1 A1}$$

$$\text{Answer: } 4f \quad \text{2 marks}$$

24 (7 marks)

$$(a) \text{The student differentiated } e^{xy} \text{ but not the factor } (1 + xy), \text{ which also depends on } y \quad \text{M1 E1}$$

$$f_{xy} = xe^{xy} + (1 + xy) \times xe^{xy} \quad \text{M1}$$

$$= (2x + x^2y)e^{xy} \quad \text{A1}$$

$$\text{Answer: } f_{xy} = (2x + x^2y)e^{xy} \quad \text{4 marks}$$

$$(b) f_y = x^2e^{xy} \quad \text{B1}$$

$$f_{yx} = 2xe^{xy} + x^2 \times ye^{xy} = (2x + x^2y)e^{xy}, \text{ equal to } f_{xy} \quad \text{M1 A1}$$

$$\text{Answer: Equal} \quad \text{3 marks}$$

25 (8 marks)

$$(a) F = x^2 + y^2 - 25: \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2x}{2y} \quad \text{M1 A1}$$

$$\text{Answer: Shown: } \frac{dy}{dx} = -\frac{x}{y} \quad \text{2 marks}$$

$$(b) \frac{d^2y}{dx^2} = -\frac{y - x \frac{dy}{dx}}{y^2} \quad \text{M1 A1}$$

$$= -\frac{y + x^2}{y/y^2} = -\frac{y^2 + x^2}{y^3} \quad \text{M1}$$

$$= -\frac{25}{y^3} \quad \text{A1}$$

Answer: Shown: $\frac{d^2y}{dx^2} = -\frac{25}{y^3}$

4 marks

(c) $-\frac{25}{64}$

A1

Negative: the curve is concave downwards there

B1

Answer: $-\frac{25}{64}$; concave down

2 marks

26 (7 marks)

(a) $f_x = e^x \cos y$, $f_y = -e^x \sin y$

M1

At (0, 0): $1, 0$; $f_{xx} = 1$, $f_{xy} = 0$, $f_{yy} = -1$

A1

$f \approx 1 + h + \frac{1}{2}(h^2 - k^2)$

A1

Answer: Shown: $f(h, k) \approx 1 + h + \frac{1}{2}(h^2 - k^2)$

3 marks

(b) Estimate: $1 + 0.1 + \frac{1}{2}(0.01 - 0.04) = 1.085$

M1 A1

Exact: $e^{0.1} \cos 0.2 = 1.08314$

A1

The estimate differs by 0.0019

A1

Answer: Estimate 1.085; exact 1.08314

4 marks

27 (12 marks)

(a) $f_x(0, y) = \lim_{h \rightarrow 0} \frac{f(h, y) - f(0, y)}{h}$

M1

$f(h, y) = \frac{hy(h^2 - y^2)}{h^2 + y^2}$ and $f(0, y) = 0$

A1

$\frac{f(h, y)}{h} = \frac{y(h^2 - y^2)}{h^2 + y^2} \rightarrow \frac{-y^3}{y^2} = -y$

M1 A1

Answer: Shown: $f_x(0, y) = -y$ for $y \neq 0$

4 marks

(b) $\frac{f(x, k)}{k} = \frac{x(x^2 - k^2)}{x^2 + k^2} \rightarrow x$ as $k \rightarrow 0$

M1 A1

Answer: Shown: $f_y(x, 0) = x$ for $x \neq 0$

2 marks

(c) $f_{xy}(0, 0) = \frac{d}{dy}[f_x(0, y)] = \frac{d}{dy}(-y) = -1$

M1 A1

$f_{yx}(0, 0) = \frac{d}{dx}[f_y(x, 0)] = 1$

M1 A1

Answer: -1 and 1

4 marks

(d) The mixed derivatives are not equal, so continuity of the second derivatives at (0, 0) is needed for $f_{xy} = f_{yx}$ to hold

E1 E1

Answer: Mixed derivatives differ

2 marks

28 (11 marks)

(a) $u_x = u_\xi \frac{\partial \xi}{\partial x} + u_\eta \frac{\partial \eta}{\partial x}$ with $\frac{\partial \xi}{\partial x} = \frac{\partial \eta}{\partial x} = 1$

M1 A1

$u_t = u_\xi(-c) + u_\eta(c)$

A1

Answer: Shown: $u_x = u_\xi + u_\eta$ and $u_t = c(u_\eta - u_\xi)$

3 marks

(b) Apply the same operator again: $\frac{\partial^2 u}{\partial x^2} = (\partial_\xi + \partial_\eta)(u_\xi + u_\eta) = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$

M1 A1

$\frac{\partial^2 u}{\partial t^2} = c(\partial_\eta - \partial_\xi)$ of $c(u_\eta - u_\xi) = c^2(u_{\eta\eta} - 2u_{\xi\eta} + u_{\xi\xi})$

M1 A1

Answer: Shown: $\frac{\partial^2 u}{\partial x^2} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$ and $\frac{\partial^2 u}{\partial t^2} = c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta})$

4 marks

(c) $\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = -4c^2 u_{\xi\eta} = 0$, so $u_{\xi\eta} = 0$

M1 A1

Integrate with respect to η : $u_\xi = p(\xi)$; integrate again: $u = P(\xi) + G(\eta)$

M1

$u = F(x + ct) + G(x - ct)$ for arbitrary twice differentiable F and G

A1

Answer: $u = F(x + ct) + G(x - ct)$

4 marks

29 (11 marks)

$$(a) u_x = \cos \theta u_r - \frac{\sin \theta}{r} u_\theta$$

$$\frac{\partial}{\partial r}(u_x) = \cos \theta u_{rr} - \frac{\sin \theta}{r} u_{r\theta} + \frac{\sin \theta}{r^2} u_\theta$$

M1 A1

$$\frac{\partial}{\partial \theta}(u_x) = -\sin \theta u_r + \cos \theta u_{r\theta} - \frac{\cos \theta}{r} u_\theta - \frac{\sin \theta}{r} u_{\theta\theta}$$

M1

$$\frac{\partial^2 u}{\partial x^2} = \cos \theta \frac{\partial}{\partial r}(u_x) - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta}(u_x); \text{collecting terms gives the result}$$

M1 A1

Answer: Shown: u_{xx} as given, in terms of u_{rr} , $u_{r\theta}$, u_θ , u_r , $u_{\theta\theta}$

5 marks

(b) Add the two expressions

M1

$u_{rr}(\cos^2 \theta + \sin^2 \theta) = u_{rr}$; the $u_{r\theta}$ and u_θ terms cancel

M1

$$\frac{u_r}{r}(\sin^2 \theta + \cos^2 \theta) = \frac{u_r}{r} \text{ and } \frac{u_{\theta\theta}}{r^2}(\sin^2 \theta + \cos^2 \theta) = \frac{u_{\theta\theta}}{r^2}$$

A1

Answer: Shown: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}$

3 marks

$$(c) u_{rr} = n(n-1)r^{n-2}\cos n\theta, u_r/r = nr^{n-2}\cos n\theta, u_{\theta\theta}/r^2 = -n^2r^{n-2}\cos n\theta$$

M1 A1

$$\text{Sum} = (n^2 - n + n - n^2)r^{n-2}\cos n\theta = 0$$

A1

Answer: $u = r^n \cos n\theta$ works

3 marks

30 (12 marks)

$$(a) \text{Chain rule with } w = \frac{y}{x}: \frac{\partial w}{\partial x} = -\frac{y}{x^2} \text{ and } \frac{\partial w}{\partial y} = \frac{1}{x}$$

M1 A1

Hence the results

A1

Answer: Shown: $z_x = -\frac{y}{x^2} f'(\frac{y}{x})$ and $z_y = \frac{1}{x} f'(\frac{y}{x})$

3 marks

$$(b) z_{xx} = \frac{2y}{x^3} f' + \frac{y^2}{x^4} f''$$

M1 A1

$$z_{xy} = -\frac{1}{x^2} f' - \frac{y}{x^3} f''$$

M1 A1

$$z_{yy} = \frac{1}{x^2} f''$$

A1

Answer: $z_{xx} = \frac{2y}{x^3} f' + \frac{y^2}{x^4} f''$; $z_{xy} = -\frac{1}{x^2} f' - \frac{y}{x^3} f''$; $z_{yy} = \frac{1}{x^2} f''$

5 marks

$$(c) x^2 z_{xx} = \frac{2y}{x} f' + \frac{y^2}{x^2} f''$$

M1

$$2xyz_{xy} = -\frac{2y}{x} f' - \frac{2y^2}{x^2} f''$$

A1

$$y^2 z_{yy} = \frac{y^2}{x^2} f''$$

A1

Adding: 0

A1

Answer: Shown: $x^2 z_{xx} + 2xyz_{xy} + y^2 z_{yy} = 0$

4 marks

Stationary Points of Functions of Two Variables**1 (2 marks)**

Each square is at least 0, and both are 0 when $x = 2$, $y = -1$

M1

Minimum at $(2, -1, 3)$

A1

Answer: Minimum at $(2, -1, 3)$

2 marks

2 (3 marks)

$$f_x = 2x - 4 = 0 \text{ and } f_y = 2y + 6 = 0, \text{ so } (2, -3)$$

M1 A1

$$f_{xx} = 2, f_{yy} = 2, f_{xy} = 0, \text{ so } D = 4 > 0 \text{ and } f_{xx} > 0: \text{minimum}$$

A1

Answer: Minimum at $(2, -3)$, $f = -13$

3 marks

3 (2 marks)

$$f_x = -2x \text{ and } f_y = -4y, \text{ both zero at the origin}$$

M1

$$f_{xx} = -2, f_{yy} = -4, f_{xy} = 0, \text{ so } D = 8 > 0 \text{ and } f_{xx} < 0: \text{ maximum, } z = 9$$

A1

Answer: Shown: the surface $z = 9 - x^2 - 2y^2$ has a maximum at (0, 0, 9)

2 marks

4 (2 marks)

$$f_x = y \text{ and } f_y = x \text{ are both zero at } (0, 0)$$

M1

$$f_{xx} = f_{yy} = 0, f_{xy} = 1, \text{ so } D = -1 < 0: \text{ a saddle point}$$

A1

Answer: Saddle point at (0, 0, 0)

2 marks

5 (2 marks)

$$D = 2 \times 3 - 1 = 5 > 0$$

M1

$$f_{xx} > 0, \text{ so a minimum}$$

A1

Answer: Minimum

2 marks

6 (2 marks)

$$D = 4 \times 1 - 2^2 = 0$$

M1

When $D = 0$ the test gives no information about the nature of the point

E1

Answer: $D = 0$

2 marks

7 (5 marks)

$$f_x = 2x + y - 6 = 0 \text{ and } f_y = x + 2y = 0$$

M1 A1

$$x = -2y, \text{ so } -4y + y = 6, y = -2, x = 4$$

A1

$$f_{xx} = 2, f_{yy} = 2, f_{xy} = 1: D = 3 > 0, f_{xx} > 0, \text{ so a minimum}$$

M1

$$f(4, -2) = 16 - 8 + 4 - 24 = -12$$

A1

Answer: Minimum at (4, -2), $f = -12$

5 marks

8 (5 marks)

$$f_x = 3x^2 - 12 = 0, f_y = 2y = 0, \text{ so } (2, 0) \text{ and } (-2, 0)$$

M1 A1

$$f_{xx} = 6x, f_{yy} = 2, f_{xy} = 0, \text{ so } D = 12x$$

M1

$$(2, 0): D = 24 > 0 \text{ and } f_{xx} = 12 > 0, \text{ a minimum } (f = -16)$$

A1

$$(-2, 0): D = -24 < 0, \text{ a saddle point } (f = 16)$$

A1

Answer: (2, 0, -16) minimum; (-2, 0, 16) saddle

5 marks

9 (6 marks)

$$z_x = 4y - 6x + 2 = 0 \text{ and } z_y = 4x - 2y + 4 = 0$$

M1 A1

$$\text{From the second, } y = 2x + 2; \text{ then } 8x + 8 - 6x + 2 = 0, \text{ so } x = -5, y = -8$$

M1 A1

$$z_{xx} = -6, z_{yy} = -2, z_{xy} = 4, \text{ so } z_{xx}z_{yy} - (z_{xy})^2 = 12 - 16 = -4 < 0$$

M1

A saddle point, with $z = -21$

A1

Answer: Saddle point at (-5, -8), $z = -21$

6 marks

10 (4 marks)

$$f_x = 2x - 2 = 0 \text{ and } f_y = -2y + 4 = 0, \text{ so } (1, 2)$$

M1 A1

$$f_{xx} = 2, f_{yy} = -2, f_{xy} = 0, \text{ so } D = -4 < 0$$

M1

A saddle point at (1, 2, 3)

A1

Answer: Saddle point at (1, 2, 3)

4 marks

11 (4 marks)

$$f_x = 4y - 2x = 0 \text{ and } f_y = 4x - 2y = 0$$

M1

$$x = 2y \text{ and } y = 2x, \text{ so } x = y = 0$$

A1

$$f_{xx} = -2, f_{yy} = -2, f_{xy} = 4: D = 4 - 16 = -12 < 0$$

M1

A saddle point at the origin

A1

Answer: Saddle point at (0, 0, 0)

4 marks

12 (4 marks)

 $f_x = 2x + ky$ and $f_y = kx + 2y$ both vanish at the origin

B1

 $f_{xx} = 2, f_{yy} = 2, f_{xy} = k$, so $D = 4 - k^2$

M1

Minimum needs $D > 0$: $k^2 < 4$

M1

 $-2 < k < 2$

A1

Answer: $-2 < k < 2$

4 marks

13 (4 marks)

(a) $P_x = 18 - 2x - y = 0$ and $P_y = 12 - 2y - x = 0$

M1

 $x = 8, y = 2$

A1

Answer: $x = 8, y = 2$

2 marks

(b) $P_{xx} = -2, P_{yy} = -2, P_{xy} = -1, D = 3 > 0, P_{xx} < 0$: maximum

M1

 $P = 144 + 24 - 64 - 4 - 16 = 84$, ie £84 000

A1

Answer: £84 000

2 marks

14 (7 marks)

 $f_x = 3x^2 - 3 = 0$ gives $x = \pm 1$; $f_y = 3y^2 - 12 = 0$ gives $y = \pm 2$

M1 A1

Four stationary points: $(\pm 1, \pm 2)$

A1

 $f_{xx} = 6x, f_{yy} = 6y, f_{xy} = 0$, so $D = 36xy$

M1

 $(1, 2)$: $D > 0, f_{xx} > 0$, minimum, $z = -18$

A1

 $(-1, -2)$: $D > 0, f_{xx} < 0$, maximum, $z = 18$

A1

 $(1, -2)$ and $(-1, 2)$: $D < 0$, saddle points, $z = 14$ and $z = -14$

A1

Answer: $(1, 2, -18)$ min; $(-1, -2, 18)$ max; $(1, -2, 14), (-1, 2, -14)$ saddles

7 marks

15 (6 marks)

 $f_x = y - \frac{8}{x^2} = 0$ and $f_y = x - \frac{8}{y^2} = 0$

M1 A1

 $y = \frac{8}{x^2}$, so $x - \frac{x^4}{8} = 0, x^3 = 8, x = 2, y = 2$

M1 A1

 $f_{xx} = \frac{16}{x^3} = 2, f_{yy} = 2, f_{xy} = 1$

M1

 $D = 3 > 0$ and $f_{xx} > 0$: a minimum, with $f = 12$

A1

Answer: Minimum at $(2, 2), f = 12$

6 marks

16 (5 marks)

 $f_x = \cos x \sin y = 0$ and $f_y = \sin x \cos y = 0$; $\sin x, \sin y \neq 0$ in the range

M1

 $\cos x = \cos y = 0$, so $x = y = \frac{\pi}{2}$

A1

 $f_{xx} = -\sin x \sin y = -1, f_{yy} = -1, f_{xy} = \cos x \cos y = 0$

M1 A1

 $D = 1 > 0$ and $f_{xx} < 0$: maximum, $f = 1$

A1

Answer: Maximum at $(\frac{\pi}{2}, \frac{\pi}{2}), f = 1$

5 marks

17 (6 marks)

 $f_x = 2x - \frac{1}{x} = 0$, so $x^2 = \frac{1}{2}, x = \frac{1}{\sqrt{2}}$; similarly $y = \frac{1}{\sqrt{2}}$

M1 A1

 $f_{xx} = 2 + \frac{1}{x^2} = 4, f_{yy} = 4, f_{xy} = 0$

M1 A1

 $D = 16 > 0$ and $f_{xx} > 0$: minimum

A1

 $f = \frac{1}{2} + \frac{1}{2} - \ln \frac{1}{2} = 1 + \ln 2$

A1

Answer: Minimum at $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), f = 1 + \ln 2$

6 marks

18 (7 marks)

$$f_x = 3x^2 - 3ay = 0 \text{ and } f_y = 3y^2 - 3ax = 0$$

M1

$$y = \frac{x^2}{a} \text{ gives } x^4 = a^3x, \text{ so } x = 0 \text{ or } x = a; \text{ points } (0, 0) \text{ and } (a, a)$$

M1 A1

$$f_{xx} = 6x, f_{yy} = 6y, f_{xy} = -3a$$

M1

$$(0, 0): D = -9a^2 < 0, \text{ saddle}$$

A1

$$(a, a): D = 36a^2 - 9a^2 = 27a^2 > 0, f_{xx} = 6a > 0; \text{ minimum, } f = -a^3$$

A1 A1

Answer: (0, 0) saddle; (a, a) minimum with $f = -a^3$

7 marks

19 (5 marks)

$$(a) f_x = 4x^3 \text{ and } f_y = 4y^3 \text{ are zero at the origin}$$

B1

$$f_{xx} = 12x^2, f_{yy} = 12y^2, f_{xy} = 0, \text{ all zero at the origin, so } D = 0$$

M1 A1

Answer: $D = 0$

3 marks

$$(b) x^4 + y^4 \geq 0 \text{ for all } x, y, \text{ with equality only at } (0, 0)$$

M1 A1

Answer: $f \geq 0 = f(0, 0)$

2 marks

20 (5 marks)

$$f = (x - y)^2 + 2y^2 - 4y + 5$$

M1 A1

$$= (x - y)^2 + 2(y - 1)^2 + 3$$

M1 A1

Both squares are at least 0, so $f \geq 3$, with equality when $x = y = 1$

A1

Answer: Minimum value 3 at (1, 1)

5 marks

21 (9 marks)

$$f_x = 3x^2 + 3y^2 - 15 = 0 \text{ and } f_y = 6xy - 12 = 0$$

M1 A1

$$x^2 + y^2 = 5 \text{ and } xy = 2, \text{ so } (x + y)^2 = 9 \text{ and } (x - y)^2 = 1$$

M1

Points (2, 1), (1, 2), (-1, -2), (-2, -1)

A1 A1

$$f_{xx} = 6x, f_{yy} = 6x, f_{xy} = 6y, \text{ so } D = 36(x^2 - y^2)$$

M1

$$(2, 1): D > 0, f_{xx} > 0, \text{ minimum, } z = -28; (-2, -1): D > 0, f_{xx} < 0, \text{ maximum, } z = 28$$

A1 A1

$$(1, 2), (-1, -2): D < 0, \text{ saddle points, } z = -26 \text{ and } z = 26$$

A1

Answer: (2, 1, -28) min; (-2, -1, 28) max; (1, 2, -26), (-1, -2, 26) saddles

9 marks

22 (8 marks)

$$(a) f_x = (1 - 2x^2)e^{-(x^2 + y^2)} \text{ and } f_y = -2xye^{-(x^2 + y^2)}$$

M1 A1

$$f_y = 0 \text{ gives } x = 0 \text{ or } y = 0; x = 0 \text{ gives } f_x = e^{-y^2} \neq 0, \text{ so } y = 0 \text{ and } x = \pm \frac{1}{\sqrt{2}}$$

M1 A1

Answer: Shown: the only stationary points are at $(\pm \frac{1}{\sqrt{2}}, 0)$

4 marks

$$(b) f_{xx} = (4x^3 - 6x)e^{-(x^2 + y^2)}, f_{yy} = 2x(2y^2 - 1)e^{-(x^2 + y^2)}, f_{xy} = 0 \text{ at } y = 0$$

M1

$$\text{At } x = \frac{1}{\sqrt{2}}: f_{xx} = -2\sqrt{2}e^{-1/2} < 0, f_{yy} = -\sqrt{2}e^{-1/2} \text{ and } D = 4e^{-1} > 0: \text{ a maximum, } f = \frac{1}{\sqrt{2}}e^{-1/2}$$

A1 A1

$$\text{At } x = -\frac{1}{\sqrt{2}}: f_{xx} > 0 \text{ and } D = 4e^{-1} > 0: \text{ a minimum, } f = -\frac{1}{\sqrt{2}}e^{-1/2}$$

A1

Answer: Max at $(\frac{1}{\sqrt{2}}, 0)$, min at $(-\frac{1}{\sqrt{2}}, 0)$

4 marks

23 (6 marks)

$$(a) f_x = 2x = 0 \text{ and } f_y = 3y^2 = 0, \text{ so } (0, 0)$$

M1 A1

$$f_{xx} = 2, f_{yy} = 6y = 0, f_{xy} = 0, \text{ so } D = 0$$

A1

Answer: Shown: the origin is the only stationary point and that $D = 0$ there

3 marks

$$(b) \text{ On } x = 0, f = y^3, \text{ which is positive for } y > 0 \text{ and negative for } y < 0$$

M1 A1

So f takes values above and below $f(0, 0) = 0$ near the origin: neither

A1

Answer: Neither

3 marks

24 (8 marks)

(a) The nature cannot be decided from f_{yy} alone: $D = f_{xx}f_{yy} - (f_{xy})^2$ and the sign of f_{xx} are needed

M1 E1

Answer: Need the full second derivative test

2 marks

(b) $f_{xx} = 6x$, $f_{yy} = 2$, $f_{xy} = 0$, so $D = 12x$

M1

(1, 0): $D = 12 > 0$, $f_{xx} = 6 > 0$: minimum

A1

(-1, 0): $D = -12 < 0$: saddle point

A1

Answer: (1, 0) min; (-1, 0) saddle

3 marks

(c) On $y = 0$, $f = x^3 - 3x$ with $f'' = 6x = -6 < 0$ at $x = -1$: a local maximum

M1 A1

But along $x = -1$, $f = 2 + y^2$ has a minimum at $y = 0$, so the point is a saddle

E1

Answer: Saddle

3 marks

25 (7 marks)

(a) $f_x = 3x^2 - 3 = 0$ and $f_y = -2y = 0$, so (1, 0) and (-1, 0)

M1 A1

$f_{xx} = 6x$, $f_{yy} = -2$, $f_{xy} = 0$, $D = -12x$

M1

(-1, 0): $D = 12 > 0$, $f_{xx} < 0$: maximum, $z = 2$

A1

(1, 0): $D = -12 < 0$: saddle, $z = -2$

A1

Answer: (-1, 0, 2) maximum; (1, 0, -2) saddle

5 marks

(b) Near a maximum: small closed loops around the point

E1

Near a saddle: contours of lower and higher values forming a cross shape (X) through the point

E1

Answer: Closed loops; crossing pattern

2 marks

26 (7 marks)

(a) $f_x = 2x + ay - 5 = 0$ at (2, 1): $4 + a - 5 = 0$, $a = 1$

M1 A1

$f_y = ax + 2by - 4 = 0$ at (2, 1): $2 + 2b - 4 = 0$, $b = 1$

M1 A1

Answer: a = 1, b = 1

4 marks

(b) $f_{xx} = 2$, $f_{yy} = 2$, $f_{xy} = 1$, $D = 3 > 0$, $f_{xx} > 0$

M1

$f(2, 1) = 4 + 2 + 1 - 10 - 4 = -7$

A1 A1

Answer: Minimum, f = -7

3 marks

27 (12 marks)

(a) $f_x = 2ax + 2hy$ and $f_y = 2hx + 2by$, both zero at the origin

B1

$f_{xx} = 2a$, $f_{yy} = 2b$, $f_{xy} = 2h$

M1

$D = 4ab - 4h^2 = 4(ab - h^2)$

A1

Answer: Shown: the origin is a stationary point and that $f_{xx}f_{yy} - (f_{xy})^2 = 4(ab - h^2)$

3 marks

(b) $f = a(x^2 + \frac{2h}{a}xy) + by^2$

M1

$= a[(x + \frac{h}{a}y)^2 - \frac{h^2}{a^2}y^2] + by^2$

M1 A1

$= a(x + \frac{h}{a}y)^2 + \frac{ab - h^2}{a}y^2$

A1

Answer: Shown: $f(x, y) = a(x + \frac{h}{a}y)^2 + \frac{ab - h^2}{a}y^2$

4 marks

(c) If $a > 0$ and $ab > h^2$, both coefficients are positive, so $f \geq 0$ with equality only when $y = 0$ and $x = 0$: a minimum

M1 A1

Conversely, if the origin is a minimum then $f \geq 0$ everywhere; taking $y = 0$ gives $a \geq 0$ so $a > 0$

M1

Taking $x = -\frac{h}{a}y$ gives $\frac{ab - h^2}{a}y^2 \geq 0$, so $ab \geq h^2$; if $ab = h^2$, $f = 0$ along that line so it is not a

E1 E1

strict minimum; hence $ab > h^2$

Answer: Proved: the origin is a minimum if and only if $a > 0$ and $ab > h^2$

5 marks

28 (12 marks)

(a) $f_x = \cos x + \cos(x + y) = 0$ and $f_y = \cos y + \cos(x + y) = 0$

M1 A1

So $\cos x = \cos y$; cosine is one-to-one on $(0, \pi)$, so $x = y$

M1 A1

Answer: Shown: $\cos x = \cos y = -\cos(x + y)$, so $x = y$

4 marks

(b) $\cos x + \cos 2x = 0$, so $2\cos^2 x + \cos x - 1 = 0$

M1

$$(2\cos x - 1)(\cos x + 1) = 0; \cos x \neq -1 \text{ in the range, so } \cos x = \frac{1}{2}, x = \frac{\pi}{3}$$

M1 A1

Answer: $(\frac{\pi}{3}, \frac{\pi}{3})$

3 marks

$$(c) f_{xx} = -\sin x - \sin(x+y) = -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = -\sqrt{3}, f_{yy} = -\sqrt{3}, f_{xy} = -\sin(x+y) = -\frac{\sqrt{3}}{2}$$

M1 A1

$$D = 3 - \frac{3}{4} = \frac{9}{4} > 0 \text{ and } f_{xx} < 0: \text{ a maximum}$$

M1 A1

$$f = \frac{3\sqrt{3}}{2}$$

A1

Answer: Maximum $\frac{3\sqrt{3}}{2}$

5 marks

29 (11 marks)

$$(a) f_x = 4x(x^2 + y^2 - 1) \text{ and } f_y = 4y(x^2 + y^2 + 1)$$

M1 A1

$$f_y = 0 \text{ gives } y = 0 \text{ (the bracket is positive); then } 4x(x^2 - 1) = 0 \text{ gives } x = 0, \pm 1$$

M1 A1

$$f_{xx} = 4(3x^2 + y^2 - 1), f_{yy} = 4(x^2 + 3y^2 + 1), f_{xy} = 8xy$$

$$(0, 0): D = (-4)(4) = -16 < 0, \text{ saddle}$$

A1

$$(\pm 1, 0): f_{xx} = 8, f_{yy} = 8, D = 64 > 0: \text{ minima, } f = -1$$

A1 A1

Answer: (0, 0, 0) saddle; (\pm 1, 0, -1) minima

7 marks

$$(b) \text{ Expand: } x^4 + 2x^2y^2 + y^4 - 2x^2 - 2y^2 + 1 + 4y^2$$

M1

$$(x^2 + y^2 - 1)^2 + 4y^2 = (x^2 + y^2)^2 - 2x^2 - 2y^2 + 1 + 4y^2 = f + 1$$

A1

Answer: Shown: $f(x, y) + 1 = (x^2 + y^2 - 1)^2 + 4y^2$

2 marks

$$(c) f + 1 \text{ is a sum of squares, so } f \geq -1$$

M1 A1

$$\text{Equality needs } y = 0 \text{ and } x^2 = 1: \text{ at } (\pm 1, 0)$$

Answer: $f \geq -1$, attained at $(\pm 1, 0)$

2 marks

30 (10 marks)

$$(a) f_x = 2x + y - 3 = 0, f_y = 2y + x + z = 0, f_z = 4z + y = 0$$

M1 A1

$$y = -4z; \text{ then } x = 7z \text{ from the second equation, and } 14z - 4z = 3$$

M1

$$z = \frac{3}{10}, y = -\frac{6}{5}, x = \frac{21}{10}$$

A1

Answer: $(\frac{21}{10}, -\frac{6}{5}, \frac{3}{10})$

4 marks

$$(b) \text{ Minors: } 2, \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3 \text{ and } \begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 4 \end{vmatrix} = 2(8 - 1) - 1(4) = 10$$

M1 A1 A1

$$\text{All positive, so } H \text{ is positive definite and the point is a minimum}$$

E1

Answer: Minimum

4 marks

$$(c) f = \frac{441}{100} + \frac{36}{25} + \frac{9}{50} - \frac{63}{50} - \frac{9}{25} - \frac{63}{10}$$

M1

$$= -\frac{63}{20}$$

A1

Answer: $-\frac{63}{20}$

2 marks

Multivariable Optimisation

1 (3 marks)

$$P = x(10 - x) = 10x - x^2, \frac{dP}{dx} = 10 - 2x = 0, \text{ so } x = 5$$

M1 A1

$$\text{Maximum product} = 25$$

A1

Answer: 25 (when $x = y = 5$)

3 marks

2 (3 marks)

$$L = xy - \lambda(x + y - 10): y = \lambda \text{ and } x = \lambda$$

M1 A1

$$\text{So } x = y \text{ and } x + y = 10 \text{ gives } x = y = 5$$

A1

Answer: (5, 5)

3 marks

3 (3 marks)

$$x^2h = 500, \text{ so } h = \frac{500}{x^2}$$

M1

$$A = 2x^2 + 4xh = 2x^2 + \frac{2000}{x}$$

M1 A1

Answer: Shown: its surface area is $A = 2x^2 + \frac{2000}{x}$

3 marks

4 (3 marks)

$$\frac{dS}{da} = 2(a - 1) + 2(a - 2) + 2(a - 4) = 6a - 14$$

M1 A1

$$a = \frac{7}{3}, \text{ and } S'' = 6 > 0 \text{ so this is a minimum}$$

A1

Answer: $a = \frac{7}{3}$

3 marks

5 (3 marks)

$$D = x^2 + (2x + 5)^2 = 5x^2 + 20x + 25$$

M1

$$\frac{dD}{dx} = 10x + 20 = 0, \text{ so } x = -2, y = 1$$

M1 A1

Answer: (-2, 1)

3 marks

6 (2 marks)

f has no stationary point, so extremes are on the boundary, at the corners

M1

Greatest 2 at (1, 1); least 0 at (0, 0)

A1

Answer: Greatest 2 at (1, 1); least 0 at (0, 0)

2 marks

7 (5 marks)

$$h = \frac{32}{x^2}, A = x^2 + 4xh = x^2 + \frac{128}{x}$$

M1 A1

$$\frac{dA}{dx} = 2x - \frac{128}{x^2} = 0, \text{ so } x^3 = 64, x = 4, h = 2$$

M1 A1

$$\frac{d^2A}{dx^2} = 2 + \frac{256}{x^3} > 0, \text{ a minimum, } A = 16 + 32 = 48 \text{ m}^2$$

A1

Answer: 4 m by 4 m by 2 m; 48 m²

5 marks

8 (6 marks)

$$\text{Minimise } f = x^2 + y^2 \text{ subject to } g = xy - 8 = 0$$

B1

$$2x = \lambda y \text{ and } 2y = \lambda x$$

M1 A1

$$\text{So } 2x^2 = \lambda xy = 2y^2, \text{ giving } x = y$$

M1

$$x^2 = 8, \text{ so the point is } (2\sqrt{2}, 2\sqrt{2})$$

A1

$$\text{Distance} = \sqrt{8 + 8} = 4$$

A1

Answer: $(2\sqrt{2}, 2\sqrt{2})$; distance 4

6 marks

9 (4 marks)

$$L = x^2 + y^2 - \lambda(x + 2y - 10): 2x = \lambda, 2y = 2\lambda$$

M1 A1

$$y = 2x, \text{ so } 5x = 10, x = 2, y = 4$$

M1

Minimum value $4 + 16 = 20$ (the squared distance from the origin to the line)

A1

Answer: 20 at (2, 4)

4 marks

10 (4 marks)

$$S = (2 - m)^2 + (3 - 2m)^2 + (7 - 3m)^2 \quad \text{M1}$$

$$\frac{dS}{dm} = -2(2 - m) - 4(3 - 2m) - 6(7 - 3m) = 28m - 58 \quad \text{M1 A1}$$

$$m = \frac{29}{14} \quad \text{A1}$$

$$\text{Answer: } m = \frac{29}{14} \quad \text{4 marks}$$

11 (4 marks)

$$f = (x - 1)^2 + (y - 2)^2 - 5, \text{ least when } (x, y) = (1, 2), \text{ which lies inside: } -5 \quad \text{M1 A1}$$

$$\text{The greatest value is at the corner furthest from } (1, 2): (3, 0) \quad \text{M1}$$

$$f(3, 0) = 3 \text{ (the other corners give } 0, -3, 0) \quad \text{A1}$$

$$\text{Answer: Least } -5 \text{ at } (1, 2); \text{ greatest } 3 \text{ at } (3, 0) \quad \text{4 marks}$$

12 (4 marks)

$$L = xy^2 - \lambda(x + y - 9): y^2 = \lambda \text{ and } 2xy = \lambda \quad \text{M1 A1}$$

$$y^2 = 2xy, \text{ so } y = 2x \text{ (} y > 0) \quad \text{M1}$$

$$3x = 9, x = 3, y = 6, f = 108 \quad \text{A1}$$

$$\text{Answer: } 108 \text{ at } (3, 6) \quad \text{4 marks}$$

13 (4 marks)

$$\text{Minimise } x^2 + y^2 + z^2: 2x = \lambda, 2y = 2\lambda, 2z = 2\lambda \quad \text{M1 A1}$$

$$y = z = 2x, \text{ so } x + 4x + 4x = 9, x = 1 \quad \text{M1}$$

$$\text{Point } (1, 2, 2), \text{ distance } \sqrt{1 + 4 + 4} = 3 \quad \text{A1}$$

$$\text{Answer: } (1, 2, 2); \text{ distance } 3 \quad \text{4 marks}$$

14 (7 marks)

$$(a) h = \frac{1000}{xy} \quad \text{M1}$$

$$S = 2xy + 2(x + y)h = 2xy + \frac{2000}{y} + \frac{2000}{x} \quad \text{M1 A1}$$

$$\text{Answer: Shown: the total surface area is } S = 2xy + \frac{2000}{x} + \frac{2000}{y} \quad \text{3 marks}$$

$$(b) S_x = 2y - \frac{2000}{x^2} = 0 \text{ and } S_y = 2x - \frac{2000}{y^2} = 0 \quad \text{M1}$$

$$x^2y = 1000 = xy^2, \text{ so } x = y \text{ and } x^3 = 1000: x = y = 10$$

$$S_{xx} = \frac{4000}{x^3} = 4, S_{yy} = 4, S_{xy} = 2, D = 12 > 0, S_{xx} > 0: \text{ minimum} \quad \text{M1 A1}$$

$$S_{min} = 200 + 200 + 200 = 600 \text{ cm}^2; \text{ the box is a cube} \quad \text{A1}$$

$$\text{Answer: } 10 \text{ cm cube; } 600 \text{ cm}^2 \quad \text{4 marks}$$

15 (8 marks)

$$(a) S_a = -2\sum x_i^2(y_i - ax_i^2 - b) = 0 \text{ and } S_b = -2\sum(y_i - ax_i^2 - b) = 0 \quad \text{M1}$$

$$\sum x_i^2 = 6, \sum x_i^4 = 18, \sum x_i^2 y_i = 60, \sum y_i = 22 \quad \text{M1 A1}$$

$$\text{So } 18a + 6b = 60 \text{ and } 6a + 4b = 22 \quad \text{A1}$$

$$\text{Answer: Shown: } 18a + 6b = 60 \text{ and } 6a + 4b = 22 \quad \text{4 marks}$$

$$(b) 3a + b = 10 \text{ and } 3a + 2b = 11, \text{ so } b = 1 \text{ and } a = 3 \quad \text{M1 A1}$$

$$S_{aa} = 36, S_{bb} = 8, S_{ab} = 12 \quad \text{M1}$$

$$S_{aa}S_{bb} - S_{ab}^2 = 144 > 0 \text{ and } S_{aa} > 0, \text{ so a minimum} \quad \text{A1}$$

$$\text{Answer: } a = 3, b = 1 \quad \text{4 marks}$$

16 (6 marks)

$$L = xyz - \lambda(x + y + z - 12); yz = xz = xy = \lambda$$

M1 A1

Positive values: $x = y = z$

M1 A1

$x = y = z = 4$, product 64

A1

The product tends to 0 as any variable tends to 0, so the only stationary value is the maximum

E1

Answer: 64 at (4, 4, 4)

6 marks

17 (7 marks)

$$(a) S_m = -2\sum x_i(y_i - mx_i - c) = 0, \text{ so } m\sum x^2 + c\sum x = \sum xy$$

M1

$$\sum x^2 = 14, \sum x = 6, \sum xy = 35, \sum y = 16, n = 4$$

A1

$$14m + 6c = 35 \text{ and, from } S_c = 0, 6m + 4c = 16$$

A1

Answer: Shown: the conditions $S_m = 0$ and $S_c = 0$ give $14m + 6c = 35$ and $6m + 4c = 16$

3 marks

$$(b) m = \frac{11}{5}, c = \frac{7}{10}$$

M1 A1

$$S_{mm} = 28, S_{cc} = 8, S_{mc} = 12 \text{ (all halved), } D = 224 - 144 = 80 > 0 \text{ and } S_{mm} > 0$$

M1 A1

Answer: $m = 2.2, c = 0.7$

4 marks

18 (7 marks)

$$f_x = 2x - y = 0 \text{ and } f_y = 2y - x - 3 = 0 \text{ give } (1, 2), \text{ inside the rectangle, } f = -3$$

M1 A1

$$D = 3 > 0 \text{ and } f_{xx} > 0: \text{ a minimum, so least value } -3$$

A1

$$\text{Corners: } f(0, 0) = 0, f(2, 0) = 4, f(0, 3) = 0, f(2, 3) = -2$$

M1 A1

On the edges the extremes are at the corners or at -2.25 (for example $x = 0, y = \frac{3}{2}$); none exceeds 4

M1

Greatest value 4 at (2, 0)

A1

Answer: Greatest 4 at (2, 0); least -3 at (1, 2)

7 marks

19 (6 marks)

$$\text{Minimise } S = 2\pi r^2 + 2\pi rh \text{ subject to } \pi r^2 h = 500$$

B1

$$L = S - \lambda(\pi r^2 h - 500); 4\pi r + 2\pi h = 2\pi r h \lambda \text{ and } 2\pi r = \pi r^2 \lambda$$

M1 A1

$$\lambda = \frac{2}{r}, \text{ so } 4\pi r + 2\pi h = 4\pi h, \text{ giving } h = 2r$$

M1 A1

$$r^3 = \frac{250}{\pi}, r = 4.30, S = 6\pi r^2 = 349 \text{ cm}^2$$

A1

Answer: $h = 2r; 349 \text{ cm}^2$

6 marks

20 (5 marks)

$$L = x + y - \lambda(x^2 + y^2 - 2); 1 = 2\lambda x \text{ and } 1 = 2\lambda y$$

M1 A1

$$x = y, \text{ so } 2x^2 = 2, x = \pm 1$$

M1 A1

Greatest value 2 at (1, 1); least value -2 at $(-1, -1)$

A1

Answer: Greatest 2, least -2

5 marks

21 (7 marks)

$$(a) \frac{dS}{da} = -2\sum \frac{1}{x_i}(y_i - \frac{a}{x_i}) = 0$$

M1 A1

$$\sum \frac{y_i}{x_i} = a\sum \frac{1}{x_i^2}$$

M1

$$\frac{d^2S}{da^2} = 2\sum \frac{1}{x_i^2} > 0, \text{ a minimum}$$

A1

Answer: Shown: S is minimised when $a = \frac{\sum(y_i/x_i)}{\sum(1/x_i^2)}$

4 marks

$$(b) \sum \frac{y}{x} = 6 + 1.25 + 0.25 = 7.5$$

M1

$$\sum \frac{1}{x^2} = 1 + \frac{1}{4} + \frac{1}{16} = \frac{21}{16}$$

A1

$$a = 7.5 \times \frac{16}{21} = \frac{40}{7} \approx 5.71$$

A1

Answer: $a = \frac{40}{7} \approx 5.71$

3 marks

22 (8 marks)

$L = x^2 + y^2 - \lambda(x^2 + 4y^2 - 16)$: $2x = 2\lambda x$ and $2y = 8\lambda y$

M1 A1

Case $x \neq 0$: $\lambda = 1$, so $2y = 8y$, $y = 0$ and $x = \pm 4$

M1 A1

Case $y \neq 0$: $\lambda = \frac{1}{4}$, so $2x = \frac{1}{2}x$, $x = 0$ and $y = \pm 2$

M1 A1

$f(\pm 4, 0) = 16$ and $f(0, \pm 2) = 4$

A1

Closest $(0, \pm 2)$ at distance 2; furthest $(\pm 4, 0)$ at distance 4

A1

Answer: Closest $(0, \pm 2)$, furthest $(\pm 4, 0)$

8 marks

23 (8 marks)

(a) $Lg = 20\sqrt{LK} - \lambda(4L + 9K - 360)$: $10\sqrt{\frac{K}{L}} = 4\lambda$ and $10\sqrt{\frac{L}{K}} = 9\lambda$

M1 A1

Dividing the equations: $\frac{K}{L} = \frac{4}{9}$, so $K = \frac{4L}{9}$

M1 A1

$4L + 4L = 360$, so $L = 45$ and $K = 20$

M1

$P = 20\sqrt{900} = 600$

A1

Answer: $L = 45$, $K = 20$, $P = 600$

6 marks

(b) $\lambda = \frac{10}{4}\sqrt{\frac{20}{45}} = \frac{10}{4} \times \frac{2}{3} = \frac{5}{3}$

M1

An extra £1 of budget increases the maximum output by about $\frac{5}{3}$ units

E1

Answer: $\lambda = \frac{5}{3}$

2 marks

24 (7 marks)

(a) $f_x = 2x$ and $f_y = 4y$ vanish only at the origin, which is inside the disc, and $f = 0$

M1 A1

$f \geq 0$ everywhere, so the least value is 0

A1

Answer: 0 at $(0, 0)$

3 marks

(b) $f = x^2 + 2(4 - x^2) = 8 - x^2$ for $-2 \leq x \leq 2$

M1 A1

Greatest when $x = 0$: $f = 8$ at $(0, \pm 2)$

A1

Least on the boundary is 4 at $(\pm 2, 0)$, which exceeds the interior minimum

A1

Answer: Greatest 8 at $(0, \pm 2)$

4 marks

25 (7 marks)

(a) The student has ignored the constraint: the conditions $f_x = f_y = 0$ find a stationary point of xy on the whole plane, not along the line $x + y = 6$

M1 E1

$(0, 0)$ does not satisfy $x + y = 6$

A1

Answer: Constraint ignored

3 marks

(b) $f = x(6 - x)$, $\frac{df}{dx} = 6 - 2x = 0$, $x = 3$, $f = 9$

M1 A1

Lagrange: $y = \lambda = x$, so $x = y = 3$

M1 A1

Answer: 9 at $(3, 3)$

4 marks

26 (8 marks)

$L = x^2 + y^2 + z^2 - \lambda(x + y + z - 3) - \mu(x - 2y + z - 1)$

M1

$L_x = 0$: $2x = \lambda + \mu$, $L_y = 0$: $2y = \lambda - 2\mu$, $L_z = 0$: $2z = \lambda + \mu$

M1 A1

So $x = z$

A1

The constraints become $2x + y = 3$ and $2x - 2y = 1$; subtracting gives $3y = 2$

M1

$y = \frac{2}{3}$, $x = z = \frac{7}{6}$

A1

$f = 2 \times \frac{49}{36} + \frac{4}{9} = \frac{19}{6}$

A1

f has no maximum on the line of intersection, so this stationary value is the minimum

E1

Answer: Minimum value $\frac{19}{6}$ at $(\frac{7}{6}, \frac{2}{3}, \frac{7}{6})$

8 marks

27 (12 marks)

(a) $S_m = -2\sum x_i(y_i - mx_i - c) = 0$, so $B - mA - cX = 0$

M1 A1

$S_c = -2\sum(y_i - mx_i - c) = 0$, so $Y - mX - nc = 0$

M1 A1

Answer: Shown: $S_m = 0$ and $S_c = 0$ give $mA + cX = B$ and $mX + nc = Y$

4 marks

(b) Multiply the second equation by X and the first by n : $mnA + cnX = nB$ and $mX^2 + ncX = XY$

M1 A1

Subtract: $m(nA - X^2) = nB - XY$

M1

$$m = \frac{nB - XY}{nA - X^2}$$

A1

Answer: Shown: $m = \frac{nB - XY}{nA - X^2}$

4 marks

(c) $nA - X^2 = n\sum(x_i - \bar{x})^2$ where $\bar{x} = \frac{X}{n}$, which is positive unless every $x_i = \bar{x}$

M1 A1

$S_{mm} = 2A$, $S_{cc} = 2n$, $S_{mc} = 2X$, so $D = 4(nA - X^2) > 0$

M1 A1

$S_{cc} > 0$, so the stationary point is a minimum

Answer: Shown: $nA - X^2 > 0$, so a minimum

4 marks

28 (12 marks)

(a) $L = \frac{1}{3}\pi r^2 h - \lambda(r^2 + h^2 - 100)$: $\frac{2}{3}\pi r h = 2\lambda r$ and $\frac{1}{3}\pi r^2 = 2\lambda h$

M1 A1

$\lambda = \frac{1}{3}\pi h$ from the first ($r \neq 0$)

A1

$\frac{1}{3}\pi r^2 = \frac{2}{3}\pi h^2$, so $r^2 = 2h^2$

M1 A1

Answer: Shown: $r^2 = 2h^2$ at a stationary point

5 marks

(b) $3h^2 = 100$, so $h = \frac{10}{\sqrt{3}} = 5.77$ and $r^2 = \frac{200}{3}$

M1 A1

$V = \frac{1}{3}\pi \times \frac{200}{3} \times \frac{10}{\sqrt{3}} = 403 \text{ cm}^3$

A1

$V = 0$ when $r = 0$ or $h = 0$ (the ends of the constraint) and $V > 0$ between, so the only stationary value is the maximum

E1 E1

Answer: 403 cm^3

5 marks

(c) $\tan \alpha = \frac{r}{h} = \sqrt{\frac{r^2}{h^2}}$

M1

$= \sqrt{2}$

A1

Answer: $\tan \alpha = \sqrt{2}$

2 marks

29 (12 marks)

(a) $P = q_1 p_1 + q_2 p_2 - C$

M1

$= 50q_1 - 3q_1^2 + 30q_2 - q_2^2 - 4q_1 - 8q_2 - q_1^2 - 2q_1 q_2 - q_2^2$

M1

$= 46q_1 + 22q_2 - 4q_1^2 - 2q_2^2 - 2q_1 q_2$

A1

Answer: Shown: the profit is $P = 46q_1 + 22q_2 - 4q_1^2 - 2q_2^2 - 2q_1 q_2$

3 marks

(b) $P_1 = 46 - 8q_1 - 2q_2 = 0$ and $P_2 = 22 - 4q_2 - 2q_1 = 0$

M1 A1

$q_1 = 5$, $q_2 = 3$

$P_{11} = -8$, $P_{22} = -4$, $P_{12} = -2$: $D = 32 - 4 = 28 > 0$ and $P_{11} < 0$, a maximum

M1 A1

$P = 230 + 66 - 100 - 18 - 30 = 148$

A1

Answer: $q_1 = 5$, $q_2 = 3$, $P = 148$

5 marks

(c) $P_1 = P_2 = \lambda$: $46 - 8q_1 - 2q_2 = 22 - 4q_2 - 2q_1$, so $q_2 = 3q_1 - 12$

M1 A1

With $q_1 + q_2 = 6$: $q_1 = \frac{9}{2}$, $q_2 = \frac{3}{2}$

A1

$P = 141$; $\lambda = 46 - 36 - 3 = 7$

One extra unit of capacity would raise the maximum profit by about £7

E1

Answer: $q_1 = 4.5$, $q_2 = 1.5$, $P = 141$, $\lambda = 7$

4 marks

30 (11 marks)**(a)** $yz = xz = xy = \lambda$ and $x, y, z > 0$, so $x = y = z$ **M1 A1** $x = y = z = 1$, $xyz = 1$ **A1**

As any variable tends to 0 the product tends to 0, so the stationary value 1 is the maximum

E1**Answer: Maximum 1 at (1, 1, 1)****4 marks****(b)** From part (a), $xyz \leq 1$, and equality is attained only at the stationary point**M1 A1****Answer: Shown: $xyz \leq 1$ for all such numbers, with equality only if $x = y = z = 1$** **2 marks****(c)** $x + y + z = 3$, so $xyz \leq 1$ **M1** $\frac{27abc}{(a + b + c)^3} \leq 1$, so $(a + b + c)^3 \geq 27abc$ **M1 A1**Taking cube roots: $\frac{a + b + c}{3} \geq (abc)^{1/3}$ **A1****Answer: Proved: $\frac{a + b + c}{3} \geq (abc)^{1/3}$** **4 marks****(d)** Equality holds when $x = y = z$, that is when $a = b = c$ **B1****Answer: $a = b = c$** **1 mark**