

Functions of Two Variables

- 1 The function f is given by $f(x, y) = 3x^2 - xy + 2y$. Find $f(2, -1)$ and $f(-3, 2)$. [3]
- 2 The function g is given by $g(x, y) = \sqrt{x^2 + y^2}$. Find $g(3, 4)$ and $g(-5, 12)$. [3]
- 3 The function h is given by $h(x, y, z) = xy + yz + zx$. Find $h(1, 2, 3)$. [2]
- 4 State the largest possible domain of the function $f(x, y) = \ln(x + y)$, where x and y are real. [2]
- 5 The function f is given by $f(x, y) = x^2 - xy + y^2$. Find and simplify $f(a, 2a)$. [3]
- 6 The function f is given by $f(x, y) = x^2 + y^2$. The point $(2, k)$ satisfies $f(2, k) = 13$. Find the possible values of k . [3]
- 7 The function f is given by $f(x, y) = \sqrt{25 - x^2 - y^2}$.
 - (a) Find the largest possible domain of f , describing it geometrically. [2]
 - (b) Find the range of f . [2]
- 8 The function f is given by $f(x, y) = x^2 + 4y^2$. The level curve $f(x, y) = 16$ is drawn in the (x, y) plane.
Write down the equation of the curve in the form $\frac{x^2}{p} + \frac{y^2}{q} = 1$, name its shape and state where it crosses the axes. [4]
- 9 State the largest possible domain of the function $f(x, y) = \frac{1}{x^2 - y}$, explaining your answer. [3]
- 10 A closed cylindrical can has radius r cm and height h cm.
 - (a) Write down the total surface area $S(r, h)$ of the can. [2]
 - (b) Find the exact value of $S(3, 10)$. [2]
- 11 A firm's output P units is modelled by $P(L, K) = 12\sqrt{LK}$, where L is the labour input in hours and K is the capital input in units.
 - (a) Find $P(25, 16)$. [2]
 - (b) Find the capital K needed to produce $P = 300$ when $L = 25$. [2]
- 12 The function f is given by $f(x, y) = x^2y$. Find and simplify $f(x + h, y) - f(x, y)$. [3]
- 13 The function f is given by $f(x, y) = y - x^2$. Find the equation of the level curve of f that passes through the point $(2, 1)$, giving y in terms of x . [3]
- 14 The function f is given by $f(x, y) = 2x^2 - xy + y^2$.
 - (a) Find and simplify $f(t, 3t)$. [2]
 - (b) Find the values of t for which $f(t, 3t) = 56$. [2]
 - (c) State the range of values taken by f along the line $y = 3x$, and explain your answer. [2]
- 15 The function f is given by $f(x, y) = x^3 - 2xy^2 + y^3$.
 - (a) Show that $f(tx, ty) = t^3f(x, y)$ for any real number t . [2]
 - (b) Find $f(2, 3)$ and hence write down the value of $f(6, 9)$. [3]
- 16 The function f is given by $f(x, y) = x^2 - y^2$. Describe the level curves $f(x, y) = c$ in the cases $c > 0$, $c < 0$ and $c = 0$, stating where each crosses an axis. [5]

- 17** A rectangular box has a base of dimensions x cm by y cm and height z cm. The total length of all its edges is 60 cm.
- Show that the volume of the box is $V(x, y) = xy(15 - x - y)$. [3]
 - State the domain of V in this context. [2]
 - Find $V(3, 4)$. [2]
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- 18** The function f is given by $f(x, y) = \sqrt{y - x^2} + \sqrt{2 - y}$.
Show that the domain of f is the region between the parabola $y = x^2$ and the line $y = 2$, and find the exact x -coordinates of the points where they meet. [5]
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- 19** The function f is given by $f(x, y) = e^{-(x^2 + y^2)}$.
- Find the exact value of $f(1, 2)$. [2]
 - Describe the level curves of f and find the range of f . [3]
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- 20** The function g is given by $g(x, y, z) = x^2 + y^2 - z^2$. Describe the level surfaces $g = 1$, $g = -1$ and $g = 0$. [4]
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- 21** The function f is given by $f(x, y) = \frac{xy}{x^2 + y^2}$, defined for $(x, y) \neq (0, 0)$.
- Find the value that f takes along the x -axis, and along the line $y = x$. [3]
 - Show that along the line $y = mx$, f takes the constant value $\frac{m}{1 + m^2}$. [2]
 - Hence explain why $f(x, y)$ cannot be given a value at the origin that makes f continuous there. [3]
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- 22** A student writes: " $f(x, y) = \ln(xy)$ is defined when $x > 0$ and $y > 0$, so its domain is the first quadrant."
- Show that the student's statement is wrong by finding a point outside the first quadrant at which f is defined. [2]
 - State the correct domain. [3]
 - Find the exact value of $f(-e, -e)$. [2]
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- 23** The function f is given by $f(x, y) = x^2 + y^2 - 4x + 2y + 7$.
- By completing the square, show that $f(x, y) = (x - 2)^2 + (y + 1)^2 + 2$. [3]
 - Find the range of f . [2]
 - Describe the level curves $f = 2$ and $f = 6$. [3]
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- 24** The temperature in degrees Celsius at the point (x, y) of a thin metal plate is modelled by $T(x, y) = 80e^{-(x^2 + 3y^2)/20}$, where x and y are in centimetres.
- Find $T(2, 1)$ to 3 significant figures. [2]
 - Show that the curve of constant temperature through $(2, 1)$ has equation $x^2 + 3y^2 = 7$, and find where it crosses the axes. [4]
 - Show that $T(x, y) \leq 80$ for all points, with equality only at the origin. [3]
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- 25** A storage tank is a vertical cylinder of radius r m and height h m, with a flat circular base and a hemispherical roof of radius r .
- Show that the volume is $V(r, h) = \pi r^2 h + \frac{2}{3}\pi r^3$, and that the total outer surface area is $S(r, h) = 2\pi r h + 3\pi r^2$. [5]
 - The tank has radius 5 m and volume $\frac{850}{3}\pi$ m³. Find its height and its surface area, in terms of π . [3]

- 26** The function f is given by $f(x, y) = \frac{x^2}{x^2 + y^2}$, defined for $(x, y) \neq (0, 0)$.
- (a) Show that the level curve $f = c$, where $0 < c < 1$, consists of two straight lines through the origin (with the origin removed), and find their gradients in terms of c . [5]
- (b) State, with a reason, what the level curves are when $c = 0$ and when $c = 1$, and the range of f . [3]
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- 27** The function f is given by $f(x, y) = \frac{x^2 y}{x^4 + y^2}$, defined for $(x, y) \neq (0, 0)$.
- (a) Show that along every straight line $y = mx$ the value of f approaches 0 as (x, y) approaches $(0, 0)$. [3]
- (b) Find the value of f along the parabola $y = x^2$. [2]
- (c) Hence explain why f has no limit at the origin, even though its limit along every straight line is the same. [3]
- (d) Show that $f(x, y) \leq \frac{1}{2}$ for all (x, y) in the domain. [3]
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- 28** The function f is given by $f(x, y) = \frac{xy}{x^2 + y^2 + 1}$.
- (a) Show that $-\frac{1}{2} < f(x, y) < \frac{1}{2}$ for all real x and y . [4]
- (b) Find $f(t, t)$ and hence show that the values $\frac{1}{2}$ and $-\frac{1}{2}$ are approached but never attained, so the range of f is $-\frac{1}{2} < f < \frac{1}{2}$. [4]
- (c) Show that the level curve $f = \frac{1}{4}$ is the hyperbola with equation $u^2 - 3v^2 = 2$ in terms of $u = x + y$ and $v = x - y$. [3]
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- 29** A firm's output is modelled by $P(L, K) = cL^a K^b$, where c , a and b are positive constants. Measurements give $P(1, 1) = 5$, $P(4, 1) = 10$ and $P(1, 16) = 20$.
- (a) Find c , a and b . [5]
- (b) Show that doubling both L and K doubles the output. [2]
- (c) The curve $P = 40$ is drawn in the (L, K) plane. Show that it has equation $K = \frac{64}{L}$, and find K when $L = 64$. [3]
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- 30** The function f is given by $f(x, y) = \frac{y - x}{y + x}$, defined when $y + x \neq 0$.
- (a) Show that $f(tx, ty) = f(x, y)$ for every $t \neq 0$. [2]
- (b) Show that for $f = c$, where $c \neq 1$, the level curve is the line $y = \frac{1+c}{1-c}x$ (excluding the origin). [3]
- (c) Find the level curve when $c = 1$, and write down the range of f . [3]
- (d) Explain why f cannot be defined at the origin so as to be continuous there. [2]

Surfaces and Contour Diagrams

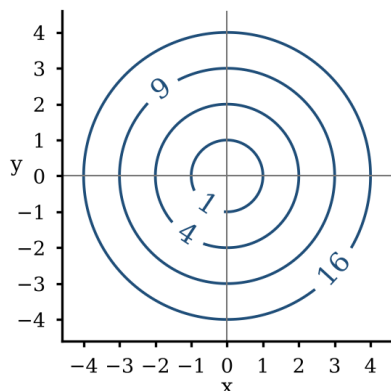
- 1** Name the surface with equation $x^2 + y^2 + z^2 = 16$, giving its centre and radius. [2]
- 2** The surface S has equation $z = 2x + 3y - 6$. Name the type of surface S and find the coordinates of the points where it meets the three coordinate axes. [3]
- 3** The surface $z = x^2 + y^2$ is cut by the plane $z = 9$. Describe the curve of intersection. [2]

4 The surface $z = x^2 - y^2$ is cut by the plane $y = 0$. Find the equation of the curve of intersection and name its shape. [2]

5 Describe the surface in three dimensions with equation $x^2 + y^2 = 4$. [2]

6 The function f is given by $f(x, y) = 3x - y$. Find the equation of the contour $f(x, y) = 6$, giving y in terms of x , and state its gradient. [3]

7 The diagram shows the contours $z = 1$, $z = 4$, $z = 9$ and $z = 16$ of a surface $z = f(x, y)$. Each contour is a circle centred at the origin.



(a) Use the diagram to find the radius of each circle, and hence suggest a formula for $f(x, y)$. [3]

(b) Describe the shape of the surface. [1]

8 The surface S has equation $z = 9 - x^2 - y^2$. Find the coordinates of the highest point of S and the equation of the curve in which S meets the plane $z = 0$. [4]

9 The surface S has equation $z = x^2 - y^2$. In each case find the equation of the curve in which S meets the given plane, and describe the curve.

(a) The plane $x = 0$. [2]

(b) The plane $y = 0$. [2]

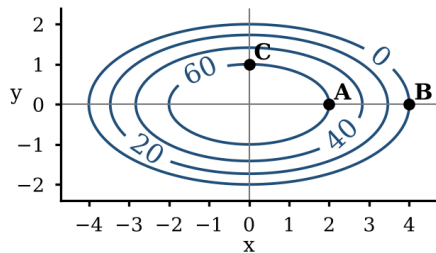
(c) The plane $z = 0$. [2]

10 The function f is given by $f(x, y) = xy$. Find the equation of the contour $f(x, y) = 4$, and show that the point $(2, 2)$ lies on it. State the equation of the contour through $(-1, 3)$. [4]

11 The plane Π has equation $x + 2y + 3z = 6$. Find the coordinates of the points where Π meets the x , y and z axes. [3]

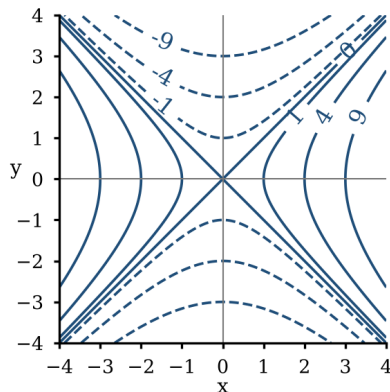
12 A sphere has equation $(x - 1)^2 + (y + 2)^2 + z^2 = 25$. Find the centre and radius of the sphere, and the radius of the circle in which the plane $z = 3$ cuts it. [4]

- 13** The diagram is a contour map of a hill. The contours are labelled with heights in metres above sea level, and the axes are marked in kilometres.



- (a) Read off the heights of A and B, and find the average gradient of the straight path from A to B, in metres per kilometre. [3]
- (b) A and C lie on the same contour. Explain why a walker going in a straight line from A to C is not walking on level ground. [2]

- 14** The diagram shows contours of a surface $z = f(x, y)$ at $z = -9, -4, -1, 0, 1, 4$ and 9 .



- (a) Explain what the contour diagram shows about the point at the origin, and why it is neither a maximum nor a minimum. [4]
- (b) Suggest an equation for the surface and find the equation of the contour $z = 0$. [3]

- 15** The surface S has equation $z = x^2 + 4y^2$. Show that the contour $z = c$, where $c > 0$, is an ellipse and find its semi-axes. Hence find, in terms of c , the area enclosed by the contour. [5]

- 16** The paraboloid $z = x^2 + y^2$ is cut by the plane $z = 4x$. Show that the projection of the curve of intersection onto the (x, y) plane is a circle, and find its centre and radius. [5]

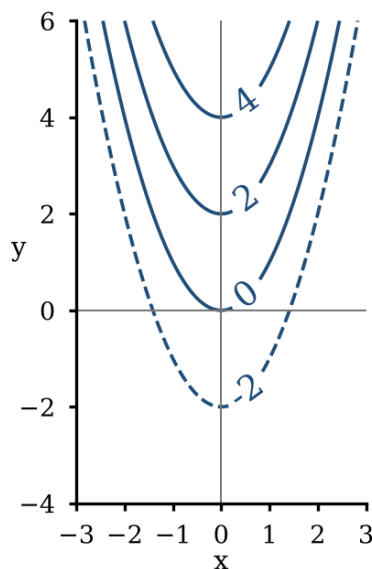
- 17** The surface S has equation $z = e^{-(x^2 + y^2)}$.
- (a) Find the coordinates of the highest point of S. [2]
- (b) Show that the contour $z = c$, where $0 < c < 1$, is a circle, and find its radius in terms of c . [3]
- (c) State what happens to z as the distance from the origin increases. [1]

- 18** The cone C has equation $z^2 = x^2 + y^2$. In each case find the equation of the curve in which C meets the given plane, and name the curve.
- (a) The plane $z = 3$. [2]
- (b) The plane $y = 0$. [2]
- (c) The plane $x = 1$. [2]

- 19** The function f is given by $f(x, y) = 2x^2 + y^2$. Find the equation of the contour of f through the point $(1, 2)$. Determine whether the point $(2, -1)$ lies inside or outside this contour, and state where the contour crosses the y -axis. [5]

- 20** The diagram shows the contours $z = -2, 0, 2$ and 4 of a surface $z = f(x, y)$, each of which is a parabola.

Use the diagram to find an equation for $f(x, y)$, and explain how you know that no contour has a turning point on the surface itself, ie that the surface has no highest or lowest point.



[5]

- 21** A student writes: "The trace of the surface $z = x^2 - y^2$ in the plane $y = 0$ is $z = x^2$, which has a minimum at the origin, so the surface has a minimum at the origin."

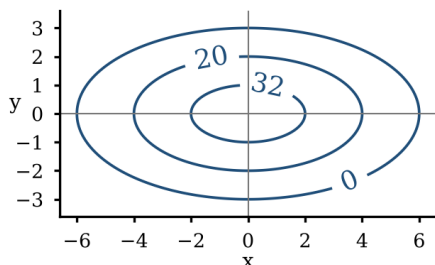
- Find the trace of the surface in the plane $x = 0$. [2]
- Explain why the student's conclusion is wrong, referring to your answer to part (a). [2]
- Find the value of z at $(0.1, 0.2)$ and at $(0.2, 0.1)$, and describe the type of point at the origin. [4]

- 22** A hill has height $h(x, y) = 500e^{-(x^2 + y^2)/8}$ metres above the level ground, where x and y are in kilometres from the summit.

- Write down the height of the summit. [1]
- Show that the contour of height 100 m is a circle, and find its radius to 3 significant figures. [3]
- A walker is 2 km from the summit. Find the height at that point, and say whether the ground is above or below the 100 m contour. [3]
- State the height of the ground far from the hill and explain why no contour has height 0. [2]

- 23** The diagram shows the contours $z = 0, z = 20$ and $z = 32$ of a surface with equation $z = a - px^2 - qy^2$, where a, p and q are positive constants.

Use readings from the diagram to find a, p and q .



[6]

- 24** The surface S has equation $z = xy$.
- Find the traces of S in the planes $y = x$ and $y = -x$. [2]
 - Find the equation of the contour of S through $(2, 3)$ and the coordinates of the points where it meets the line $y = x$. [3]
 - Hence explain why the origin is a saddle point. [2]
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- 25** The temperature in degrees Celsius at the point (x, y) on a thin plate is $T(x, y) = \frac{100}{1 + x^2 + y^2}$, where x and y are in metres.
- Find the maximum temperature and where it occurs. [2]
 - Show that the curves of constant temperature $T = c$, where $0 < c \leq 100$, are circles, and find the radius in terms of c . [3]
 - Find the radius of the circle on which $T = 20$, and the temperature at a distance of 3 m from the centre. [3]
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- 26** The cone K has equation $z = \sqrt{x^2 + y^2}$ and the paraboloid P has equation $z = x^2 + y^2$.
- Show that K and P intersect in the origin and in a circle, and find the equation and radius of this circle together with its height. [4]
 - Hence determine, for points above a circle of radius 0.5 centred at the origin, which of K and P is higher, and give the heights. [3]
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- 27** The ellipsoid E has equation $\frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1$.
- Show that the trace of E in the plane $z = k$, where $-1 < k < 1$, is an ellipse with semi-axes $2\sqrt{1 - k^2}$ and $3\sqrt{1 - k^2}$. [4]
 - Show that the area of this ellipse is $6\pi(1 - k^2)$. [2]
 - By integrating these areas, find the exact volume enclosed by E . [4]
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- 28** The paraboloid P has equation $z = x^2 + y^2$.
- Show that the plane $z = 2x + 4y - 5$ meets P at exactly one point, and find the coordinates of that point. [4]
 - Show that for any real numbers p and q the plane $z = 2px + 2qy - p^2 - q^2$ meets P at exactly one point, and state that point. [3]
 - Hence write down the equation of the plane that meets P only at $(3, -1, 10)$. [2]
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- 29** The function f is given by $f(x, y) = \frac{x^2 + y^2}{x}$, defined for $x \neq 0$.
- Show that the contour $f = c$ is the circle $(x - \frac{c}{2})^2 + y^2 = \frac{c^2}{4}$ with the origin removed, for $c \neq 0$. [5]
 - State the centre and radius of the contour $f = c$, and describe how the contours for c and $-c$ are related. [3]
 - Explain why the contours $f = c_1$ and $f = c_2$ do not meet when $c_1 \neq c_2$, and say what they have in common at the origin. [2]
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- 30** The hyperboloid H has equation $x^2 + y^2 - z^2 = 1$.
- Show that for every real t the point $(1, t, t)$ lies on H , and find a second line, in the plane $x = 1$, that lies wholly on H . [3]
 - For any angle α , show that the line with points $(\cos \alpha - t \sin \alpha, \sin \alpha + t \cos \alpha, t)$ lies wholly on H . [4]
 - The point $(\cos \alpha, \sin \alpha, 0)$ lies on the circle in which H meets the plane $z = 0$. Write down the direction vectors of two different lines on H through this point. [2]
 - Explain why H is called a ruled surface. [1]

Partial Differentiation

- 1 Given that $f(x, y) = 3x^2y - 2xy^3 + y$, find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. [3]
- 2 Given that $f(x, y) = e^{2x+3y}$, find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. [2]
- 3 Given that $f(x, y) = x \sin y$, find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, and evaluate both at the point $(2, \frac{\pi}{2})$. [3]
- 4 Given that $f(x, y) = \ln(x^2 + y^2)$, find $\frac{\partial f}{\partial x}$. [2]
- 5 Given that $f(x, y) = x^2y + y^3$, find the value of $\frac{\partial f}{\partial y}$ at the point $(1, 2)$. [2]
- 6 The volume of a cylinder is $V = \pi r^2 h$. Find $\frac{\partial V}{\partial r}$ and $\frac{\partial V}{\partial h}$. [2]
- 7 Given that $f(x, y) = x^2 e^{xy}$, find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, factorising your answers. [4]
- 8 Given that $f(x, y) = \frac{x}{x+y}$, show that $\frac{\partial f}{\partial x} = \frac{y}{(x+y)^2}$ and find a similar expression for $\frac{\partial f}{\partial y}$. [4]
- 9 Given that $f(x, y) = \tan^{-1}(\frac{y}{x})$, show that $\frac{\partial f}{\partial x} = -\frac{y}{x^2 + y^2}$ and find $\frac{\partial f}{\partial y}$. [4]
- 10 A cylinder has radius 4 cm and height 12 cm. The radius increases by 0.1 cm and the height decreases by 0.3 cm. Use a differential approximation to estimate the change in the volume, giving your answer in terms of π and to 3 significant figures. [4]
- 11 The surface S has equation $z = x^2y$. Find the equation of the tangent plane to S at the point $(2, 1, 4)$. [4]
- 12 The function z is given by $z = x^2 + 3xy$, where $x = 2t$ and $y = t^2$. Use the chain rule to find $\frac{dz}{dt}$ in terms of t . [4]
- 13 A curve has equation $x^2 + xy + y^2 = 7$. Use partial differentiation to find $\frac{dy}{dx}$ in terms of x and y , and find its value at the point $(1, 2)$. [4]
- 14 The acceleration due to gravity is found using $g = \frac{4\pi^2 L}{T^2}$, where L is the length of a pendulum and T its period.
The length L is measured with a maximum error of 1% and T with a maximum error of 0.5%.
Use partial differentiation to find the approximate maximum percentage error in g . [6]
- 15 The surface S has equation $z = \sqrt{x^2 + y^2}$.
(a) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at the point $P(3, 4, 5)$. [3]
(b) Find the equation of the tangent plane to S at P , and a vector normal to S at P . [3]

- 16** The function f depends on u and v , where $f = u^2v$, $u = x + 2y$ and $v = xy$.
- (a) Use the chain rule to show that $\frac{\partial f}{\partial x} = y(x + 2y)(3x + 2y)$. [4]
- (b) Find a similar factorised expression for $\frac{\partial f}{\partial y}$. [2]
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- 17** Plane polar coordinates are defined by $x = r \cos \theta$ and $y = r \sin \theta$. The function z is $z = x^2y$.
- (a) Use the chain rule to find $\frac{\partial z}{\partial r}$ in terms of r and θ . [3]
- (b) Verify your answer by first writing z in terms of r and θ and then differentiating. [2]
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- 18** The radius of a cone is increasing at 0.5 cm s^{-1} and its height is decreasing at 0.4 cm s^{-1} . Find the rate at which the volume is changing at the instant when the radius is 6 cm and the height is 9 cm, giving your answer to 3 significant figures. [5]
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- 19** A curve has equation $x^3 + y^3 = 6xy$.
- (a) Use partial differentiation to show that $\frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x}$. [4]
- (b) Find the gradient of the curve at (3, 3). [1]
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- 20** The point $P(2, b)$ lies on the contour $f(x, y) = -6$ of the function $f(x, y) = xy^2 - 2x^2y$.
- (a) Find the two possible values of b . [3]
- (b) For each possible point P , find a vector normal to the contour at P and the equation of the tangent line to the contour at P . [5]
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- 21** The function w is given by $w = x^2y + yz$, where $x = t^2$, $y = 2t$ and $z = \frac{1}{t}$, for $t \neq 0$.
- (a) Use the chain rule to find $\frac{dw}{dt}$ in terms of t . [5]
- (b) Verify your answer by writing w in terms of t only. [2]
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- 22** The function $f(x, y)$ is written as $f = g(u, v)$, where $u = x + y$ and $v = x - y$.
- (a) Show that $\frac{\partial f}{\partial x} = g_u + g_v$ and $\frac{\partial f}{\partial y} = g_u - g_v$. [3]
- (b) Hence show that $(\frac{\partial f}{\partial x})^2 - (\frac{\partial f}{\partial y})^2 = 4g_u g_v$. [2]
- (c) Verify this result for $g = u^2v$ at $(x, y) = (2, 1)$. [3]
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- 23** Let $f(x, y) = \sqrt{x^2 + y^2}$.
- (a) Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. [2]
- (b) Use a differential approximation at (3, 4) to estimate the value of $\sqrt{3.02^2 + 3.98^2}$. [4]
- (c) Find the exact value of $\sqrt{3.02^2 + 3.98^2}$ on a calculator, to 5 decimal places, and the percentage error in your estimate to 2 significant figures. [3]
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- 24** A student is asked to find the partial derivatives of $z = x^2y + \sin(xy)$. The student writes:
- (a) " $\frac{\partial z}{\partial x} = 2xy + \cos(xy)$ " Identify the error in the student's working and write down the correct expression for $\frac{\partial z}{\partial x}$. [3]
- (b) Find $\frac{\partial z}{\partial y}$. [2]
- (c) Find the equation of the tangent plane at the point where $x = 1$ and $y = 0$, giving z in terms of x and y . [3]

- 25** The functions u and v are defined by $u = x^2 - y^2$ and $v = 2xy$, and $f = u^2 + v^2$.
- (a) Use the chain rule to find $\frac{\partial f}{\partial x}$ in terms of x and y . [4]
- (b) Show that $u^2 + v^2 = (x^2 + y^2)^2$, and verify your answer to part (a) by differentiating this directly. [3]
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- 26** For a fixed mass of ideal gas, the pressure P , volume V and temperature T are related by $PV = nRT$, where n and R are constants.
- (a) Write P as a function of V and T , V as a function of P and T , and T as a function of P and V . [1]
- (b) Find $\frac{\partial P}{\partial V}$, $\frac{\partial V}{\partial T}$ and $\frac{\partial T}{\partial P}$. [3]
- (c) Show that $\frac{\partial P}{\partial V} \frac{\partial V}{\partial T} \frac{\partial T}{\partial P} = -1$. [2]
-
- 27** Plane polar coordinates are given by $x = r \cos \theta$ and $y = r \sin \theta$. The function $z = f(x, y)$ has partial derivatives z_x and z_y .
- (a) Show that $\frac{\partial z}{\partial r} = z_x \cos \theta + z_y \sin \theta$ and $\frac{\partial z}{\partial \theta} = -rz_x \sin \theta + rz_y \cos \theta$. [4]
- (b) Hence show that $(z_x)^2 + (z_y)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial z}{\partial \theta}\right)^2$. [4]
- (c) Verify the identity for $z = x^2 + 2y$ at the point where $r = 2$ and $\theta = \frac{\pi}{2}$. [3]
-
- 28** A function $f(x, y)$ is homogeneous of degree n if $f(tx, ty) = t^n f(x, y)$ for all $t > 0$.
- (a) Show that $f(x, y) = x^3 + 3xy^2$ is homogeneous of degree 3 and that $xf_x + yf_y = 3f$. [4]
- (b) Prove that for any differentiable homogeneous function of degree n , $xf_x + yf_y = nf$.
(Hint: differentiate $f(tx, ty) = t^n f(x, y)$ with respect to t .) [5]
- (c) A function g is homogeneous of degree 3. Given that $g(2, 1) = 5$ and $g_y(2, 1) = 4$, find $g_x(2, 1)$. [2]
-
- 29** The two resistors of resistances R_1 and R_2 ohms are connected in parallel, so that the combined resistance R satisfies $R = \frac{R_1 R_2}{R_1 + R_2}$.
- (a) Show that $\frac{\partial R}{\partial R_1} = \frac{R_2^2}{(R_1 + R_2)^2}$ and write down the corresponding expression for $\frac{\partial R}{\partial R_2}$. [3]
- (b) The resistors are measured as $R_1 = 30 \pm 0.5$ and $R_2 = 60 \pm 1$. Find R and use a differential approximation to find the approximate maximum error in R , and the percentage error to 2 significant figures. [5]
- (c) Show that when $R_1 = R_2 = k$ the combined resistance is $\frac{k}{2}$, and hence explain why a small error in one resistor then affects R by exactly a quarter of that error. [3]
-
- 30** The surface S_1 has equation $z = xy$ and the sphere S_2 has equation $x^2 + y^2 + z^2 = 3$. Both pass through the point $P(1, 1, 1)$.
- (a) Find the equation of the tangent plane to each surface at P . [4]
- (b) Find the acute angle at which the surfaces intersect at P . [4]
- (c) Find a vector equation of the line in which the two tangent planes meet. [3]

Second Partial Derivatives

- 1** Given that $f(x, y) = x^3 y^2$, find f_{xx} , f_{yy} and f_{xy} . [3]

- 2 Given that $z = e^{3x} \sin y$, find $\frac{\partial^2 z}{\partial x^2}$ and $\frac{\partial^2 z}{\partial y^2}$. [2]

- 3 Given that $f(x, y) = x^2 + 3xy + y^2$, find f_{xy} and f_{yx} and state what you notice. [3]

- 4 Given that $g(x, y) = \ln(xy)$, for $x > 0$ and $y > 0$, find g_{xx} , g_{yy} and g_{xy} . [3]

- 5 Given that $f(x, y) = x^4 y$, find the value of f_{xx} at the point $(1, 2)$. [2]

- 6 Given that $z = \sin(2x + 3y)$, find $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial x \partial y}$ and $\frac{\partial^2 z}{\partial y^2}$. [3]

- 7 Given that $f(x, y) = x^2 e^y + y^3$, find f_{xx} , f_{xy} and f_{yy} . [4]

- 8 Given that $f(x, y) = e^{xy}$, find f_{xx} , f_{xy} and f_{yy} . [4]

- 9 Given that $f(x, y) = \frac{x}{y}$, find f_{xx} , f_{xy} and f_{yy} . [4]

- 10 Show that $f(x, y) = x^3 - 3xy^2$ satisfies Laplace's equation $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$. [4]

- 11 Show that $z = \ln(x^2 + y^2)$ satisfies Laplace's equation $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$ for $(x, y) \neq (0, 0)$. [5]

- 12 Given that $f(x, y) = x^2 y^3$, find f_{xxy} and f_{xyx} and show that they are equal. [4]

- 13 The function $f(x, y) = x^2 + ky^2 + xy$ satisfies Laplace's equation $f_{xx} + f_{yy} = 0$. Find the value of k . [3]

- 14 Show that $u = \sin 2x \cos 6t$ satisfies the wave equation $\frac{\partial^2 u}{\partial t^2} = 9 \frac{\partial^2 u}{\partial x^2}$. [5]

- 15 The function $u(x, t) = e^{\lambda t} \cos 3x$ satisfies the heat equation $\frac{\partial u}{\partial t} = 2 \frac{\partial^2 u}{\partial x^2}$. Find the value of λ . [5]

- 16 Given that $f(x, y) = x^3 + y^3 - 3xy$, find f_{xx} , f_{yy} and f_{xy} , and evaluate $f_{xx}f_{yy} - (f_{xy})^2$ at $(1, 1)$ and at $(0, 0)$. [5]

- 17 Given that $f(x, y) = \sin(xy)$, find f_{xx} , f_{yy} , f_{xy} and f_{yx} , and confirm that $f_{xy} = f_{yx}$. [5]

- 18 Let $u = (x^2 + y^2 + z^2)^{-1/2}$, for $(x, y, z) \neq (0, 0, 0)$. Show that u satisfies the three-dimensional Laplace equation $u_{xx} + u_{yy} + u_{zz} = 0$. [7]

- 19 Given that $z = \sin(x^2 + y^2)$, find $\frac{\partial^2 z}{\partial x^2}$ and $\frac{\partial^2 z}{\partial y^2}$. Hence show that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 4\cos(x^2 + y^2) - 4(x^2 + y^2)\sin(x^2 + y^2)$. [5]

- 20 A function f satisfies $f_x = 2xy + 3y^2$ and $f(0, y) = y^2$ for all y . Find $f(x, y)$, and verify that $\frac{\partial^2 f}{\partial x \partial y} = 2x + 6y$. [5]

- 21** The function F is twice differentiable, and $u(x, t) = F(x + ct) + G(x - ct)$ where G is also twice differentiable and c is a non-zero constant.
- (a) Show that $\frac{\partial^2 u}{\partial x^2} = F''(x + ct) + G''(x - ct)$ and find $\frac{\partial^2 u}{\partial t^2}$. [4]
- (b) Hence show that u satisfies the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$. [1]
- (c) Verify with $F(s) = s^2$ and $G(s) = e^s$, $c = 2$, at the point $(x, t) = (0, 0)$. [3]
-
- 22** Let $u = f(r)$, where $r = \sqrt{x^2 + y^2}$ and f is twice differentiable.
- (a) Show that $\frac{\partial r}{\partial x} = \frac{x}{r}$ and find $\frac{\partial^2 r}{\partial x^2}$, and write $\frac{\partial r}{\partial y}$ in the same way. [3]
- (b) Show that $\frac{\partial^2 u}{\partial x^2} = \frac{x^2}{r^2} f''(r) + \frac{y^2}{r^3} f'(r)$. [2]
- (c) Hence show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r)$, and find f such that $u = f(r)$ satisfies Laplace's equation. [4]
-
- 23** The function f is given by $f(x, y) = x^4 - 3x^2y^2 + y^4$.
- (a) Find f_{xx} , f_{xy} and f_{yy} . [3]
- (b) Show that $x^2f_{xx} + 2xyf_{xy} + y^2f_{yy} = 12f$. [3]
- (c) Hence, or otherwise, find $xf_x + yf_y$ in terms of f . [2]
-
- 24** A student is asked to show that $f_{xy} = f_{yx}$ for $f(x, y) = xe^{xy}$. The student writes:
- (a) " $f_x = (1 + xy)e^{xy}$, so $f_{xy} = xe^{xy}$." Identify the mistake in this working and find the correct f_{xy} . [4]
- (b) Find f_y and f_{yx} and confirm that $f_{yx} = f_{xy}$. [3]
-
- 25** The curve C has equation $x^2 + y^2 = 25$.
- (a) Show that $\frac{dy}{dx} = -\frac{x}{y}$. [2]
- (b) By differentiating $\frac{dy}{dx} = -\frac{x}{y}$ with respect to x , show that $\frac{d^2y}{dx^2} = -\frac{25}{y^3}$. [4]
- (c) Hence find the value of $\frac{d^2y}{dx^2}$ at $(3, 4)$ and state what it shows about the curvature. [2]
-
- 26** The function f is given by $f(x, y) = e^x \cos y$. Near the origin, the second-order approximation $f(h, k) \approx f(0, 0) + hf_x + kf_y + \frac{1}{2}(h^2f_{xx} + 2hkf_{xy} + k^2f_{yy})$ is used, with all derivatives evaluated at $(0, 0)$.
- (a) Show that $f(h, k) \approx 1 + h + \frac{1}{2}(h^2 - k^2)$. [3]
- (b) Use this to estimate $f(0.1, 0.2)$ and compare with the exact value to 5 decimal places. [4]
-
- 27** The function f is defined by $f(x, y) = \frac{xy(x^2 - y^2)}{x^2 + y^2}$ for $(x, y) \neq (0, 0)$, and $f(0, 0) = 0$.
- (a) By using the definition of the partial derivative as a limit, show that $f_x(0, y) = -y$ for $y \neq 0$. [4]
- (b) Similarly show that $f_y(x, 0) = x$ for $x \neq 0$. [2]
- (c) Hence find $f_{xy}(0, 0)$ and $f_{yx}(0, 0)$, where f_{xy} means differentiate with respect to x first. [4]
- (d) Comment on your answer. [2]

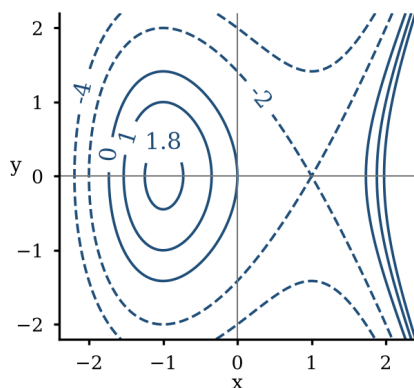
- 28** The function $u(x, t)$ satisfies the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$. New variables are defined by $\xi = x - ct$ and $\eta = x + ct$, and u is regarded as a function of ξ and η .
- (a) Show that $u_x = u_\xi + u_\eta$ and $u_t = c(u_\eta - u_\xi)$. [3]
- (b) Hence show that $\frac{\partial^2 u}{\partial x^2} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$ and $\frac{\partial^2 u}{\partial t^2} = c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta})$. [4]
- (c) Deduce that the wave equation becomes $u_{\xi\eta} = 0$ and find the general solution. [4]
-
- 29** Plane polar coordinates are $x = r \cos \theta$ and $y = r \sin \theta$, and u is a twice differentiable function of x and y . It is given that $\frac{\partial}{\partial x} = \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta}$ and $\frac{\partial}{\partial y} = \sin \theta \frac{\partial}{\partial r} + \frac{\cos \theta}{r} \frac{\partial}{\partial \theta}$ as operators.
- (a) Show that $\frac{\partial^2 u}{\partial x^2} = \cos^2 \theta u_{rr} - \frac{2 \sin \theta \cos \theta}{r} u_{r\theta} + \frac{2 \sin \theta \cos \theta}{r^2} u_\theta + \frac{\sin^2 \theta}{r} u_r + \frac{\sin^2 \theta}{r^2} u_{\theta\theta}$. [5]
- (b) Using the same method, the second derivative with respect to y is $\frac{\partial^2 u}{\partial y^2} = \sin^2 \theta u_{rr} + \frac{2 \sin \theta \cos \theta}{r} u_{r\theta} - \frac{2 \sin \theta \cos \theta}{r^2} u_\theta + \frac{\cos^2 \theta}{r} u_r + \frac{\cos^2 \theta}{r^2} u_{\theta\theta}$. Hence show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}$. [3]
- (c) Find a solution of Laplace's equation of the form $u = r^n \cos n\theta$ for integer $n \geq 1$, and check it using the result in (b). [3]
-
- 30** The function f is twice differentiable and $z = f(\frac{y}{x})$ for $x \neq 0$.
- (a) Show that $z_x = -\frac{y}{x^2} f'(\frac{y}{x})$ and $z_y = \frac{1}{x} f'(\frac{y}{x})$. [3]
- (b) Find z_{xx} , z_{xy} and z_{yy} . [5]
- (c) Hence show that $x^2 z_{xx} + 2xyz_{xy} + y^2 z_{yy} = 0$. [4]

Stationary Points of Functions of Two Variables

- 1** The function f is given by $f(x, y) = (x - 2)^2 + (y + 1)^2 + 3$. Write down the coordinates of the stationary point of the surface $z = f(x, y)$ and state its nature. [2]
- 2** Find the stationary point of $f(x, y) = x^2 + y^2 - 4x + 6y$, and show that it is a minimum. [3]
- 3** Show that the surface $z = 9 - x^2 - 2y^2$ has a maximum at $(0, 0, 9)$. [2]
- 4** Show that the surface $z = xy$ has a stationary point at the origin, and state its nature. [2]
- 5** At a stationary point of a function f , $f_{xx} = 2$, $f_{yy} = 3$ and $f_{xy} = 1$. Determine the nature of the stationary point. [2]
- 6** At a stationary point of a function f , $f_{xx} = 4$, $f_{yy} = 1$ and $f_{xy} = 2$. Explain why the second derivative test is inconclusive. [2]
- 7** Find the stationary point of $f(x, y) = x^2 + xy + y^2 - 6x$, and determine its nature, giving the value of f there. [5]
- 8** Find and classify the stationary points of $f(x, y) = x^3 - 12x + y^2$. [5]
- 9** Find the stationary point of $z = 4xy - 3x^2 - y^2 + 2x + 4y$ and determine its nature, giving the value of z there. [6]

- 10** Find the stationary point of $f(x, y) = x^2 - y^2 - 2x + 4y$ and determine its nature. [4]
- 11** Show that $f(x, y) = 4xy - x^2 - y^2$ has exactly one stationary point, and find its nature. [4]
- 12** The function f is given by $f(x, y) = x^2 + kxy + y^2$, where k is a constant. Find the set of values of k for which the origin is a minimum. [4]
- 13** A firm makes two products. When it sells x hundred units of the first and y hundred of the second, its weekly profit in thousands of pounds is $P(x, y) = 18x + 12y - x^2 - y^2 - xy$.
 (a) Find the values of x and y that give a stationary value of P . [2]
 (b) Show that this gives the maximum profit and find it. [2]
- 14** Find the coordinates of all the stationary points of the surface $z = x^3 + y^3 - 3x - 12y$ and determine their nature. [7]
- 15** Find the stationary point of $f(x, y) = xy + \frac{8}{x} + \frac{8}{y}$, for $x > 0$ and $y > 0$, and show that it is a minimum. [6]
- 16** Find the stationary point of $f(x, y) = \sin x \sin y$ for $0 < x < \pi$ and $0 < y < \pi$, and show that it is a maximum. [5]
- 17** Find the stationary point of $f(x, y) = x^2 + y^2 - \ln(xy)$, for $x > 0$ and $y > 0$, and show that it is a minimum. Give the minimum value exactly. [6]
- 18** The function f is given by $f(x, y) = x^3 + y^3 - 3axy$, where a is a positive constant. Find the stationary points in terms of a and determine their nature. [7]
- 19** The function f is given by $f(x, y) = x^4 + y^4$.
 (a) Show that the origin is a stationary point and that the second derivative test is inconclusive there. [3]
 (b) Show directly that the origin is a minimum. [2]
- 20** By completing the square, show that $f(x, y) = x^2 - 2xy + 3y^2 - 4y + 5$ has a minimum value, and find the point where it occurs. [5]
- 21** Find the coordinates of the stationary points of the surface $z = x^3 + 3xy^2 - 15x - 12y$ and determine their nature. [9]
- 22** The function f is given by $f(x, y) = xe^{-(x^2 + y^2)}$.
 (a) Show that the only stationary points are at $(\pm \frac{1}{\sqrt{2}}, 0)$. [4]
 (b) Determine the nature of each stationary point and the value of f there. [4]
- 23** The function f is given by $f(x, y) = x^2 + y^3$.
 (a) Show that the origin is the only stationary point and that $D = 0$ there. [3]
 (b) By considering the values of f along the y -axis, show that the origin is neither a maximum nor a minimum. [3]
- 24** A student is asked to find and classify the stationary points of $f(x, y) = x^3 - 3x + y^2$. The student writes: " $f_x = 3x^2 - 3 = 0$ and $f_y = 2y = 0$, so $(1, 0)$ and $(-1, 0)$. $f_{yy} = 2 > 0$ so both are minima."
 (a) Explain the error in the student's method. [2]
 (b) Find the correct nature of each point. [3]
 (c) Show that, along the x -axis, the function has a local maximum at $x = -1$, and explain why this does not make $(-1, 0)$ a maximum of the surface. [3]

- 25** The diagram shows contours of $z = f(x, y)$ where $f(x, y) = x^3 - 3x - y^2$. The surface has two stationary points.



- (a) Find the coordinates of the stationary points and classify them using derivatives. [5]
- (b) Explain what you would expect to see in the contour diagram close to each stationary point. [2]
-
- 26** The function f is given by $f(x, y) = x^2 + axy + by^2 - 5x - 4y$, where a and b are constants. It has a stationary point at $(2, 1)$.
- (a) Find a and b . [4]
- (b) Show that the stationary point is a minimum and find the minimum value. [3]
-
- 27** Let $f(x, y) = ax^2 + 2hxy + by^2$, where a, h and b are constants with $a \neq 0$.
- (a) Show that the origin is a stationary point and that $f_{xx}f_{yy} - (f_{xy})^2 = 4(ab - h^2)$. [3]
- (b) By completing the square, show that $f(x, y) = a(x + \frac{h}{a}y)^2 + \frac{ab - h^2}{a}y^2$. [4]
- (c) Hence prove that the origin is a minimum if and only if $a > 0$ and $ab > h^2$. [5]
-
- 28** The function f is given by $f(x, y) = \sin x + \sin y + \sin(x + y)$, for $0 < x < \pi$ and $0 < y < \pi$.
- (a) Show that any stationary point satisfies $\cos x = \cos y = -\cos(x + y)$, and deduce that $x = y$. [4]
- (b) Find the stationary point. [3]
- (c) Show that it is a maximum and find the maximum value. [5]
-
- 29** The function f is given by $f(x, y) = (x^2 + y^2)^2 - 2(x^2 - y^2)$.
- (a) Show that f has three stationary points and classify them. [7]
- (b) Show that $f(x, y) + 1 = (x^2 + y^2 - 1)^2 + 4y^2$. [2]
- (c) Hence show that -1 is the global minimum value of f , and state where it is attained. [2]
-
- 30** The function f of three variables is $f(x, y, z) = x^2 + y^2 + 2z^2 + xy + yz - 3x$.
- (a) Find the coordinates of the stationary point. [4]
- (b) The matrix of second derivatives is $H = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 4 \end{pmatrix}$. Show that the leading principal minors of H are all positive, and deduce that the stationary point is a minimum. [4]
- (c) Find the minimum value of f . [2]

Multivariable Optimisation

- 1** Two positive numbers x and y satisfy $x + y = 10$. By substituting $y = 10 - x$, find the maximum value of their product xy . [3]

- 2 Use the method of Lagrange multipliers to find the stationary point of $f(x, y) = xy$ subject to $x + y = 10$. [3]

- 3 A closed cuboid has a square base of side x cm and height h cm, and volume 500 cm^3 . Show that its surface area is $A = 2x^2 + \frac{2000}{x}$. [3]

- 4 Find the value of a that minimises $S(a) = (a - 1)^2 + (a - 2)^2 + (a - 4)^2$. [3]

- 5 Find the point on the line $y = 2x + 5$ that is closest to the origin, by minimising the squared distance $x^2 + y^2$. [3]

- 6 The function $f(x, y) = x + y$ is defined on the square $0 \leq x \leq 1, 0 \leq y \leq 1$. Write down its greatest and least values and where they occur. [2]

- 7 An open-topped box has a square base of side x m, height h m and volume 32 m^3 . Find the dimensions that minimise the area of material used, and find this minimum area. [5]

- 8 Use Lagrange multipliers to find the point on the curve $xy = 8$, where $x > 0$ and $y > 0$, that is closest to the origin, and find its distance from the origin. [6]

- 9 Use Lagrange multipliers to find the minimum value of $x^2 + y^2$ subject to $x + 2y = 10$. [4]

- 10 A line through the origin with equation $y = mx$ is fitted to the data points $(1, 2)$, $(2, 3)$ and $(3, 7)$ by minimising $S = \sum (y_i - mx_i)^2$. Find m . [4]

- 11 Find the greatest and least values of $f(x, y) = x^2 + y^2 - 2x - 4y$ on the rectangle $0 \leq x \leq 3, 0 \leq y \leq 3$. [4]

- 12 Use Lagrange multipliers to find the maximum value of $f(x, y) = xy^2$ subject to $x + y = 9$, with $x > 0$ and $y > 0$. [4]

- 13 Find the point on the plane $x + 2y + 2z = 9$ that is closest to the origin, and the distance from the origin to the plane. [4]

- 14 A closed rectangular box has a base of x cm by y cm and a volume of 1000 cm^3 .
 - (a) Show that the total surface area is $S = 2xy + \frac{2000}{x} + \frac{2000}{y}$. [3]
 - (b) Find the dimensions that minimise S , justifying that it is a minimum, and find the minimum value. [4]

- 15 A curve of the form $y = ax^2 + b$ is fitted to the data points $(-1, 5)$, $(0, 1)$, $(1, 3)$ and $(2, 13)$ by minimising $S(a, b) = \sum (y_i - ax_i^2 - b)^2$.
 - (a) Show that $S_a = 0$ and $S_b = 0$ give $18a + 6b = 60$ and $6a + 4b = 22$. [4]
 - (b) Find a and b , and show that this gives a minimum. [4]

- 16 Use Lagrange multipliers to find the maximum value of xyz subject to $x + y + z = 12$, where x, y and z are positive. Justify briefly that the stationary point gives a maximum. [6]

- 17 The line $y = mx + c$ is fitted to the points $(0, 1)$, $(1, 3)$, $(2, 4)$ and $(3, 8)$ by minimising $S(m, c) = \sum (y_i - mx_i - c)^2$.
 - (a) Show that the conditions $S_m = 0$ and $S_c = 0$ give $14m + 6c = 35$ and $6m + 4c = 16$. [3]
 - (b) Solve for m and c , and show that this gives a minimum. [4]

- 18** Find the greatest and least values of $f(x, y) = x^2 + y^2 - xy - 3y$ on the rectangle $0 \leq x \leq 2$, $0 \leq y \leq 3$. [7]
- 19** A closed cylindrical can has radius r cm, height h cm and volume 500 cm^3 . Use Lagrange multipliers to show that the surface area is least when $h = 2r$, and find the least surface area to 3 significant figures. [6]
- 20** Use Lagrange multipliers to find the greatest and least values of $x + y$ on the circle $x^2 + y^2 = 2$. [5]
- 21** A curve of the form $y = \frac{a}{x}$ is to be fitted to data points (x_i, y_i) by minimising $S(a) = \sum (y_i - \frac{a}{x_i})^2$.
 (a) Show that S is minimised when $a = \frac{\sum(y_i x_i)}{\sum(1/x_i^2)}$. [4]
 (b) Find a for the points $(1, 6)$, $(2, 2.5)$ and $(4, 1)$. [3]
- 22** Find the points on the ellipse $x^2 + 4y^2 = 16$ that are closest to and furthest from the origin, by using Lagrange multipliers on $f = x^2 + y^2$. [8]
- 23** A firm's output is $P = 20\sqrt{LK}$, where L and K are the labour and capital used. Labour costs £4 per unit and capital costs £9 per unit, and the total budget is £360, so $4L + 9K = 360$.
 (a) Use Lagrange multipliers to find L and K that maximise output, and the maximum output. [6]
 (b) Find λ and interpret it. [2]
- 24** The function $f(x, y) = x^2 + 2y^2$ is defined on the disc $x^2 + y^2 \leq 4$.
 (a) Find the least value of f on the disc. [3]
 (b) On the boundary, substitute $y^2 = 4 - x^2$ to find the greatest value of f , and state where it occurs. [4]
- 25** A student is asked to find the maximum of xy subject to $x + y = 6$. The student writes: "For a maximum, $f_x = y = 0$ and $f_y = x = 0$, so the maximum value of xy is 0."
 (a) Explain what is wrong with the student's method. [3]
 (b) Find the correct maximum value, by substitution, and verify it with Lagrange multipliers. [4]
- 26** Use Lagrange multipliers to find the minimum value of $f(x, y, z) = x^2 + y^2 + z^2$ subject to the two constraints $x + y + z = 3$ and $x - 2y + z = 1$. [8]
- 27** Data points (x_i, y_i) , for $i = 1$ to n , are to be fitted by a line $y = mx + c$ by minimising $S(m, c) = \sum (y_i - mx_i - c)^2$. Write $X = \sum x_i$, $Y = \sum y_i$, $A = \sum x_i^2$ and $B = \sum x_i y_i$.
 (a) Show that $S_m = 0$ and $S_c = 0$ give $mA + cX = B$ and $mX + nc = Y$. [4]
 (b) Solve to show that $m = \frac{nB - XY}{nA - X^2}$. [4]
 (c) Show that $nA - X^2 > 0$ unless all the x_i are equal, and hence that the stationary point is a minimum. [4]

- 28** A cone has base radius r and height h and its slant height is fixed at $l = 10$ cm, so $r^2 + h^2 = 100$. Its volume is $V = \frac{1}{3}\pi r^2 h$.
- Use Lagrange multipliers on V subject to $r^2 + h^2 = 100$ to show that $r^2 = 2h^2$ at a stationary point. [5]
 - Find the maximum volume, to 3 significant figures, and explain why the stationary value is a maximum. [5]
 - Show that the semi-vertical angle of this cone satisfies $\tan \alpha = \sqrt{2}$. [2]
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- 29** A firm sells in two markets. The prices are $p_1 = 50 - 3q_1$ and $p_2 = 30 - q_2$ pounds, where q_1 and q_2 are the quantities sold. The total cost of production is $C = 4q_1 + 8q_2 + (q_1 + q_2)^2$ pounds.
- Show that the profit is $P = 46q_1 + 22q_2 - 4q_1^2 - 2q_2^2 - 2q_1q_2$. [3]
 - Find the quantities that maximise the profit, show that this is a maximum, and find the maximum profit. [5]
 - Production is limited by capacity to $q_1 + q_2 = 6$. Use Lagrange multipliers to find the new best quantities and profit, and give the value and meaning of the multiplier. [4]
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- 30** The positive numbers x , y and z satisfy $x + y + z = 3$.
- Use Lagrange multipliers to find the stationary value of xyz , and explain why it is the maximum. [4]
 - Deduce that $xyz \leq 1$ for all such numbers, with equality only if $x = y = z = 1$. [2]
 - Let a , b and c be any positive numbers. By choosing $x = \frac{3a}{a+b+c}$ and corresponding expressions for y and z , prove that $\frac{a+b+c}{3} \geq (abc)^{1/3}$. [4]
 - State when equality holds in the inequality in part (c). [1]
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