

Definition and Standard MGFs

1 (2 marks)

$$M_X(t) = E(e^{tX}) = \sum e^{tx}P(X = x)$$

M1

$$M_X(t) = 0.3e^t + 0.7e^{2t}$$

A1

$$\text{Answer: } M_X(t) = 0.3e^t + 0.7e^{2t}$$

2 marks

2 (2 marks)

$$M_X(0) = E(e^{0 \times X})$$

M1

$$= E(1) = 1$$

A1

Answer: Shown

2 marks

3 (2 marks)

$$M_X(t) = \sum e^{tx}P(X = x)$$

M1

$$M_X(t) = \frac{1}{4}(e^t + e^{2t} + e^{3t} + e^{4t})$$

A1

$$\text{Answer: } \frac{1}{4}(e^t + e^{2t} + e^{3t} + e^{4t})$$

2 marks

4 (2 marks)

$$(a) M_X(t) = 0.6e^0 + 0.4e^t = 0.6 + 0.4e^t$$

B1

$$\text{Answer: } M_X(t) = 0.6 + 0.4e^t$$

1 mark

$$(b) e^{\ln 3} = 3, \text{ so } M_X(\ln 3) = 0.6 + 1.2 = 1.8$$

B1

Answer: 1.8

1 mark

5 (2 marks)

$$(a) M_X(t) = (0.8 + 0.2e^t)^5$$

B1

$$\text{Answer: } (0.8 + 0.2e^t)^5$$

1 mark

$$(b) e^{\ln 4} = 4, \text{ so } M_X(\ln 4) = (0.8 + 0.8)^5 = 1.6^5 = 10.48576$$

B1

Answer: 10.48576

1 mark

6 (2 marks)

$$(a) M_X(t) = e^{3(e^t - 1)}$$

B1

$$\text{Answer: } e^{3(e^t - 1)}$$

1 mark

$$(b) e^{\ln 2} = 2, \text{ so } M_X(\ln 2) = e^{3(2 - 1)} = e^3$$

B1

Answer: e^3

1 mark

7 (3 marks)

$$M_X(t) = \sum e^{tx}P(X = x)$$

M1

$$= 0.2e^{-t} + 0.5e^0 + 0.3e^{2t}$$

M1

$$M_X(t) = 0.2e^{-t} + 0.5 + 0.3e^{2t}$$

A1

$$\text{Answer: } 0.2e^{-t} + 0.5 + 0.3e^{2t}$$

3 marks

8 (5 marks)

$$M_X(t) = \sum_{x=1}^{\infty} e^{tx} q^{x-1} p \quad \text{M1}$$

$$= pe^t \sum_{x=1}^{\infty} (qe^t)^{x-1} \quad \text{M1}$$

This is a geometric series with first term 1 and common ratio qe^t M1

Sum to infinity gives $M_X(t) = \frac{pe^t}{1 - qe^t}$ A1

The series converges when $qe^t < 1$, that is $t < -\ln q$ B1

Answer: Shown; $t < -\ln q$ 5 marks

9 (4 marks)

$$M_X(t) = \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda} \lambda^x}{x!} \quad \text{M1}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!} \quad \text{M1}$$

$$= e^{-\lambda} e^{\lambda e^t} \text{ (exponential series)} \quad \text{A1}$$

$$= e^{\lambda(e^t - 1)}, \text{ valid for all real } t \quad \text{B1}$$

Answer: Shown; all real t 4 marks

10 (4 marks)

$$M_X(t) = \int_0^{\infty} e^{tx} \lambda e^{-\lambda x} dx = \lambda \int_0^{\infty} e^{-(\lambda - t)x} dx \quad \text{M1}$$

$$= \lambda \left[-\frac{e^{-(\lambda - t)x}}{\lambda - t} \right]_0^{\infty} \quad \text{M1}$$

$$= \frac{\lambda}{\lambda - t} \quad \text{A1}$$

The integral converges only when $\lambda - t > 0$, that is $t < \lambda$ B1

Answer: Shown; $t < \lambda$ 4 marks

11 (3 marks)

$$M_X(t) = \int_0^2 e^{tx} \times \frac{1}{2} dx \quad \text{M1}$$

$$= \frac{1}{2} \left[\frac{e^{tx}}{t} \right]_0^2 \quad \text{M1}$$

$$= \frac{e^{2t} - 1}{2t} \quad \text{A1}$$

Answer: Shown 3 marks

12 (3 marks)

$$M_X(t) = \sum P(X = x)e^{tx}, \text{ so the coefficient of } e^{tx} \text{ is } P(X = x) \quad \text{M1}$$

$$P(X = 0) = 0.1, P(X = 1) = 0.3, P(X = 2) = 0.4, P(X = 3) = 0.2 \quad \text{A1}$$

$$P(X \geq 2) = 0.4 + 0.2 = 0.6 \quad \text{A1}$$

Answer: $P(X \geq 2) = 0.6$ 3 marks

13 (3 marks)

$$\text{(a) } M_X(t) = e^{5t + \frac{9}{2}t^2} \quad \text{B1}$$

Answer: $e^{5t + \frac{9}{2}t^2}$ 1 mark

$$\text{(b) } Z \sim N(0, 1), \text{ so } M_Z(t) = e^{\frac{1}{2}t^2} \quad \text{B1}$$

Answer: $e^{\frac{1}{2}t^2}$ 1 mark

$$\text{(c) } M_X\left(\frac{1}{2}\right) = e^{\frac{5}{2} + \frac{9}{2} \times \frac{1}{4}} = e^{\frac{29}{8}} \quad \text{B1}$$

Answer: $e^{\frac{29}{8}}$ 1 mark

14 (4 marks)

$$M_X(t) = \sum_{x=0}^n e^{tx} {}^nC_x p^x q^{n-x} \quad \text{M1}$$

$$= \sum_{x=0}^n {}^nC_x (pe^t)^x q^{n-x} \quad \text{M1}$$

By the binomial theorem $\sum {}^nC_x a^x b^{n-x} = (a + b)^n$ with $a = pe^t$, $b = q$ A1

$M_X(t) = (q + pe^t)^n$, valid for all real t A1

Answer: Shown 4 marks

15 (5 marks)

$$M_X(t) = \int_0^1 3x^2 e^{tx} dx \quad \text{M1}$$

$$= 3[e^{tx}(\frac{x^2}{t} - \frac{2x}{t^2} + \frac{2}{t^3})]_0^1 \quad \text{M1 A1}$$

$$= 3[e^t(\frac{1}{t} - \frac{2}{t^2} + \frac{2}{t^3}) - \frac{2}{t^3}] \quad \text{M1}$$

$$= \frac{3e^t(t^2 - 2t + 2) - 6}{t^3} \quad \text{A1}$$

Answer: Shown 5 marks

16 (5 marks)

$$M_X(t) = \frac{1}{2} \int_{-\infty}^0 e^{tx} e^x dx + \frac{1}{2} \int_0^{\infty} e^{tx} e^{-x} dx \quad \text{M1}$$

$$\text{First integral: } \frac{1}{2} \left[\frac{e^{(1+t)x}}{1+t} \right]_{-\infty}^0 = \frac{1}{2(1+t)}, \text{ needs } t > -1 \quad \text{M1 A1}$$

$$\text{Second integral: } \frac{1}{2} \left[-\frac{e^{-(1-t)x}}{1-t} \right]_0^{\infty} = \frac{1}{2(1-t)}, \text{ needs } t < 1 \quad \text{A1}$$

$$M_X(t) = \frac{1}{2} \left(\frac{1}{1+t} + \frac{1}{1-t} \right) = \frac{1}{1-t^2} \text{ for } -1 < t < 1 \quad \text{B1}$$

$$\text{Answer: } M_X(t) = \frac{1}{1-t^2}, -1 < t < 1 \quad \text{5 marks}$$

17 (4 marks)

$$M_X(t) = \sum_{x=0}^{\infty} e^{tx} \times \frac{1}{4} \left(\frac{3}{4} \right)^x = \frac{1}{4} \sum_{x=0}^{\infty} \left(\frac{3e^t}{4} \right)^x \quad \text{M1}$$

$$\text{Geometric series with first term 1 and ratio } \frac{3e^t}{4} \quad \text{M1}$$

$$M_X(t) = \frac{1}{4} \times \frac{1}{1 - \frac{3e^t}{4}} = \frac{1}{4 - 3e^t} \quad \text{A1}$$

$$\text{Converges when } \frac{3e^t}{4} < 1, \text{ that is } t < \ln \frac{4}{3} \quad \text{B1}$$

$$\text{Answer: } M_X(t) = \frac{1}{4 - 3e^t}, t < \ln \frac{4}{3} \quad \text{4 marks}$$

18 (4 marks)

$$M_X(t) = \int_0^{\infty} e^{tx} \times 4xe^{-2x} dx = 4 \int_0^{\infty} xe^{-(2-t)x} dx \quad \text{M1 A1}$$

$$\text{Using the result with } k = 2 - t \text{ (which must be positive): } M_X(t) = \frac{4}{(2-t)^2} \quad \text{M1}$$

The result requires $2 - t > 0$, so $t < 2$ B1

$$\text{Answer: } M_X(t) = \frac{4}{(2-t)^2}, t < 2 \quad \text{4 marks}$$

19 (4 marks)

$$\sum P(X = x) = k(1 + 2 + 3 + 4) = 10k = 1, \text{ so } k = \frac{1}{10}$$

M1 A1

$$M_X(t) = \sum e^{tx} kx$$

M1

$$M_X(t) = \frac{1}{10}(e^t + 2e^{2t} + 3e^{3t} + 4e^{4t})$$

A1

$$\text{Answer: } k = \frac{1}{10}; M_X(t) = \frac{1}{10}(e^t + 2e^{2t} + 3e^{3t} + 4e^{4t})$$

4 marks

20 (4 marks)

(a) $e^{tx} = e^t \times e^{t(x-1)}$, so a factor e^t should have been taken outside the sum, not dropped

E1

The series should be $pe^t \sum (qe^t)^{x-1}$, with first term pe^t

E1

Answer: A factor e^t was lost when rewriting e^{tx}

2 marks

$$(b) M_X(t) = pe^t \sum_{x=1}^{\infty} (qe^t)^{x-1}$$

M1

$$= \frac{pe^t}{1 - qe^t}$$

A1

$$\text{Answer: } \frac{pe^t}{1 - qe^t}$$

2 marks

21 (7 marks)

$$(a) M_Z(t) = \int_{-\infty}^{\infty} e^{tz} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$$

M1

$$tz - \frac{1}{2}z^2 = -\frac{1}{2}(z - t)^2 + \frac{1}{2}t^2$$

M1 A1

$$M_Z(t) = e^{\frac{1}{2}t^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z - t)^2} dz$$

M1

The integrand is the density of $N(t, 1)$, so the integral is 1 and $M_Z(t) = e^{\frac{1}{2}t^2}$

A1

Answer: Shown

5 marks

$$(b) M_X(t) = e^{\mu t} M_Z(\sigma t)$$

M1

$$= e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

A1

$$\text{Answer: } e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

2 marks

22 (7 marks)

$$M_X(t) = \int_0^1 x e^{tx} dx + \int_1^2 (2 - x) e^{tx} dx$$

M1

$$\int_0^1 x e^{tx} dx = [e^{tx} (\frac{x}{t} - \frac{1}{t^2})]_0^1 = e^t (\frac{1}{t} - \frac{1}{t^2}) + \frac{1}{t^2}$$

M1 A1

$$\int_1^2 (2 - x) e^{tx} dx = [\frac{(2 - x)e^{tx}}{t}]_1^2 + \frac{1}{t} \int_1^2 e^{tx} dx = -\frac{e^t}{t} + \frac{e^{2t} - e^t}{t^2}$$

M1 A1

$$M_X(t) = \frac{e^t}{t} - \frac{e^t}{t^2} + \frac{1}{t^2} - \frac{e^t}{t} + \frac{e^{2t}}{t^2} - \frac{e^t}{t^2}$$

M1

$$= \frac{e^{2t} - 2e^t + 1}{t^2} = \frac{(e^t - 1)^2}{t^2}$$

A1

Answer: Shown

7 marks

23 (6 marks)

$$M_X(t) = \sum_{x=1}^{\infty} e^{tx} \frac{e^{-\lambda} \lambda^x}{x!(1 - e^{-\lambda})} = \frac{e^{-\lambda}}{1 - e^{-\lambda}} \sum_{x=1}^{\infty} \frac{(\lambda e^t)^x}{x!}$$
M1 A1

$$\sum_{x=1}^{\infty} \frac{(\lambda e^t)^x}{x!} = e^{\lambda e^t} - 1$$
M1 A1

$$M_X(t) = \frac{e^{-\lambda}(e^{\lambda e^t} - 1)}{1 - e^{-\lambda}}$$
M1

$$= \frac{e^{\lambda(e^t - 1)} - e^{-\lambda}}{1 - e^{-\lambda}}, \text{ valid for all real } t$$
A1

Answer: Shown; all real t

6 marks

24 (6 marks)

Condition on the two cases: $M_X(t) = \frac{1}{3}E(e^{tX} | X = 0) + \frac{2}{3}E(e^{tX} | X \text{ is exponential})$

M1

First case: $e^0 = 1$, so it contributes $\frac{1}{3}$

B1

Second case: $\frac{2}{3} \times \frac{2}{2-t} = \frac{4}{3(2-t)}$

M1 A1

$$M_X(t) = \frac{1}{3} + \frac{4}{3(2-t)} = \frac{(2-t) + 4}{3(2-t)} = \frac{6-t}{3(2-t)}$$
A1

Exists for $t < 2$

B1

Answer: $M_X(t) = \frac{6-t}{3(2-t)}, t < 2$

6 marks

25 (6 marks)

(a) For $t \leq 0$ and $x \geq 1$, $0 < e^{tx} \leq 1$

M1

So $E(e^{tX}) \leq \int_1^{\infty} \frac{2}{x^3} dx = 1$, which is finite

A1

Answer: Shown

2 marks

(b) For $t > 0$, $e^{tx} > \frac{t^3 x^3}{6}$

M1

So $\frac{2e^{tx}}{x^3} > \frac{t^3}{3}$, a positive constant

A1

$\int_1^{\infty} \frac{t^3}{3} dx$ diverges, so by comparison $E(e^{tX})$ diverges

A1

Answer: Shown

3 marks

(c) $M_X(t)$ is not defined on any open interval containing $t = 0$, so $M_X'(0)$ cannot be found

E1

Answer: $M_X(t)$ does not exist on any interval containing $t = 0$, so $M_X'(0)$ cannot be found

1 mark

26 (6 marks)

$$M_X(t) = \int_0^{\infty} e^{tx} \frac{\lambda^{\alpha} x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} dx = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} \int_0^{\infty} x^{\alpha-1} e^{-(\lambda-t)x} dx$$
M1 A1

For convergence we need $\lambda - t > 0$, so $t < \lambda$

B1

Let $u = (\lambda - t)x$, so $x = \frac{u}{\lambda - t}$ and $dx = \frac{1}{\lambda - t} du$

M1

The integral becomes $\frac{1}{(\lambda - t)^{\alpha}} \int_0^{\infty} u^{\alpha-1} e^{-u} du = \frac{\Gamma(\alpha)}{(\lambda - t)^{\alpha}}$

M1

$$M_X(t) = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} \times \frac{\Gamma(\alpha)}{(\lambda - t)^{\alpha}} = \left(\frac{\lambda}{\lambda - t}\right)^{\alpha}$$
A1

Answer: Shown

6 marks

27 (8 marks)

$$(a) M_X(t) = \sum_{x=r}^{\infty} e^{tx} x^{-1} C_{r-1} p^r q^{x-r} \quad \text{M1}$$

Let $k = x - r$, so $k = 0, 1, 2, \dots$ and $e^{tx} = e^{tr} e^{tk}$ M1 A1

$$M_X(t) = (pe^t)^r \sum_{k=0}^{\infty} k+r-1 C_{r-1} (qe^t)^k \quad \text{M1}$$

$$= (pe^t)^r (1 - qe^t)^{-r} = \left(\frac{pe^t}{1 - qe^t} \right)^r \quad \text{A1}$$

The series requires $qe^t < 1$, so $t < -\ln q$ B1

Answer: Shown; $t < -\ln q$ 6 marks

$$(b) M_Y(t) = e^{-rt} M_X(t) \quad \text{M1}$$

$$= \left(\frac{p}{1 - qe^t} \right)^r \quad \text{A1}$$

$$\text{Answer: } \left(\frac{p}{1 - qe^t} \right)^r \quad \text{2 marks}$$

28 (8 marks)

$$(a) M_Y(t) = E(e^{tZ^2}) = \int_{-\infty}^{\infty} e^{tz^2} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \quad \text{M1}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(1-2t)z^2/2} dz \quad \text{M1 A1}$$

For the integral to converge we need $1 - 2t > 0$, that is $t < \frac{1}{2}$ B1

Write $\frac{1}{s} = 1 - 2t$, so $s = \frac{1}{1-2t}$; the integral is $\sqrt{2\pi s}$ M1

$$M_Y(t) = \frac{1}{\sqrt{2\pi}} \times \sqrt{2\pi/1-2t} = (1-2t)^{-1/2} \quad \text{A1}$$

Answer: Shown 6 marks

$$(b) \text{ Put } t = -\frac{1}{2}: M_Y(-\frac{1}{2}) = 2^{-1/2} = \frac{1}{\sqrt{2}} \quad \text{B1}$$

$$\text{Answer: } \frac{1}{\sqrt{2}} \quad \text{1 mark}$$

(c) For $t \geq \frac{1}{2}$ the exponent $-(1-2t)z^2/2$ is ≥ 0 , so the integrand does not tend to 0 and the integral diverges E1

Answer: The integral diverges 1 mark

29 (8 marks)

$$(a) M_X(t) = \frac{1}{2} \int_0^{\pi} e^{tx} \sin x dx \quad \text{M1}$$

$$= \frac{1}{2} \left[\frac{e^{tx}(t \sin x - \cos x)}{1+t^2} \right]_0^{\pi} \quad \text{M1 A1}$$

$$= \frac{1}{2} \times \frac{e^{\pi t}(0+1) - (0-1)}{1+t^2} = \frac{1+e^{\pi t}}{2(1+t^2)} \quad \text{A1}$$

Answer: Shown 4 marks

$$(b) M_X(0) = \frac{1+1}{2(1+0)} = 1 \quad \text{B1}$$

Answer: Verified 1 mark

$$(c) M_{X-\pi/2}(t) = e^{-\pi t/2} M_X(t) = \frac{e^{-\frac{\pi t}{2}} + e^{\frac{\pi t}{2}}}{2(1+t^2)} \quad \text{M1}$$

$$= \frac{\cosh(\frac{\pi t}{2})}{1+t^2} \quad \text{A1}$$

$\cosh$ and t^2 are both even functions, so the quotient is even E1

Answer: Shown 3 marks

30 (8 marks)

$$(a) P(Y \leq y) = P(-\ln U \leq y) = P(U \geq e^{-y})$$

$$= 1 - e^{-y}$$

M1

A1

This is the distribution function of an exponential distribution with rate 1

A1

Answer: Y is exponential with rate 1

3 marks

$$(b) e^{tY} = e^{-t \ln U} = U^{-t}, \text{ so } M_Y(t) = \int_0^1 u^{-t} du$$

M1

$$= \left[\frac{u^{1-t}}{1-t} \right]_0^1 = \frac{1}{1-t} \text{ (the integral converges for } t < 1)$$

A1

Answer: Shown

2 marks

$$(c) \text{ Let } S = -\ln(U_1 U_2 U_3) = Y_1 + Y_2 + Y_3; \text{ then } M_S(t) = (1 - t)^{-3} \text{ for independent } Y_i$$

M1 A1

$$E((U_1 U_2 U_3)^{-1/2}) = E(e^{S/2}) = M_S\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^{-3} = 8$$

A1

Answer: 8

3 marks

Moments from MGFs**1 (2 marks)**

$$M_X'(t) = 0.7e^t$$

M1

$$E(X) = M_X'(0) = 0.7$$

A1

Answer: 0.7

2 marks

2 (2 marks)

$$M_X'(t) = (1 + 2t)e^{t + t^2}$$

M1

$$E(X) = M_X'(0) = 1$$

A1

Answer: 1

2 marks

3 (2 marks)

$$\text{Coefficient of } t \text{ is } E(X), \text{ so } E(X) = 4$$

B1

$$\text{Coefficient of } t^2 \text{ is } \frac{E(X^2)}{2!}, \text{ so } E(X^2) = 2 \times 9 = 18$$

B1

Answer: E(X) = 4, E(X^2) = 18

2 marks

4 (2 marks)

$$M_X'(t) = 4(0.5 + 0.5e^t)^3 \times 0.5e^t$$

M1

$$E(X) = M_X'(0) = 4 \times 1 \times 0.5 = 2$$

A1

Answer: 2

2 marks

5 (2 marks)

$$M_X'(t) = \frac{5}{(5 - t)^2}$$

M1

$$E(X) = M_X'(0) = \frac{5}{25} = \frac{1}{5}$$

A1

Answer: $\frac{1}{5}$

2 marks

6 (2 marks)

$$E(X) = 3 \text{ and } E(X^2) = 13$$

B1

$$\text{Var}(X) = 13 - 3^2 = 4$$

B1

Answer: 4

2 marks

7 (4 marks)

$$M_X'(t) = \frac{4}{(4-t)^2}, M_X''(t) = \frac{8}{(4-t)^3} \quad \text{M1 A1}$$

$$E(X) = M_X'(0) = \frac{1}{4} \text{ and } E(X^2) = M_X''(0) = \frac{1}{8} \quad \text{A1}$$

$$\text{Var}(X) = \frac{1}{8} - \left(\frac{1}{4}\right)^2 = \frac{1}{16} \quad \text{A1}$$

$$\text{Answer: } E(X) = \frac{1}{4}, \text{ Var}(X) = \frac{1}{16} \quad \text{4 marks}$$

8 (5 marks)

$$M_X'(t) = \lambda e^t e^{\lambda(e^t-1)} \quad \text{M1}$$

$$E(X) = M_X'(0) = \lambda \quad \text{A1}$$

$$M_X''(t) = \lambda e^t e^{\lambda(e^t-1)} + \lambda^2 e^{2t} e^{\lambda(e^t-1)} \quad \text{M1 A1}$$

$$E(X^2) = \lambda + \lambda^2, \text{ so } \text{Var}(X) = \lambda + \lambda^2 - \lambda^2 = \lambda \quad \text{A1}$$

$$\text{Answer: Shown} \quad \text{5 marks}$$

9 (4 marks)

$$(1-t)^{-2} = 1 + 2t + 3t^2 + 4t^3 + \dots \quad \text{M1 A1}$$

$$E(X^n) = n! \times (\text{coefficient of } t^n): E(X) = 2, E(X^2) = 2! \times 3 = 6, E(X^3) = 3! \times 4 = 24 \quad \text{A1}$$

$$\text{Var}(X) = 6 - 2^2 = 2 \quad \text{B1}$$

$$\text{Answer: } E(X) = 2, E(X^2) = 6, E(X^3) = 24; \text{ Var}(X) = 2 \quad \text{4 marks}$$

10 (5 marks)

$$M_X'(t) = n(q + pe^t)^{n-1} pe^t, \text{ so } E(X) = np(q + p)^{n-1} = np \quad \text{M1 A1}$$

$$M_X''(t) = n(n-1)(q + pe^t)^{n-2} p^2 e^{2t} + n(q + pe^t)^{n-1} pe^t \quad \text{M1 A1}$$

$$E(X^2) = n(n-1)p^2 + np, \text{ so } \text{Var}(X) = n^2 p^2 - np^2 + np - n^2 p^2 = np(1-p) = npq \quad \text{A1}$$

$$\text{Answer: Shown} \quad \text{5 marks}$$

11 (4 marks)

$$M_X'(t) = -0.2e^{-t} + 0.6e^{2t}, M_X''(t) = 0.2e^{-t} + 1.2e^{2t} \quad \text{M1 A1}$$

$$E(X) = -0.2 + 0.6 = 0.4 \text{ and } E(X^2) = 0.2 + 1.2 = 1.4 \quad \text{A1}$$

$$\text{Var}(X) = 1.4 - 0.4^2 = 1.24 \quad \text{A1}$$

$$\text{Answer: } E(X) = 0.4, \text{ Var}(X) = 1.24 \quad \text{4 marks}$$

12 (4 marks)

$$e^u = 1 + u + \frac{u^2}{2} + \frac{u^3}{6} + \dots \text{ with } u = t + \frac{1}{2}t^2 \quad \text{M1}$$

$$\text{Terms in } t^3: \text{ from } \frac{u^2}{2}, \frac{1}{2} \times 2 \times t \times \frac{1}{2}t^2 = \frac{1}{2}t^3; \text{ from } \frac{u^3}{6}, \frac{1}{6}t^3 \quad \text{M1 A1}$$

$$\text{Coefficient of } t^3 \text{ is } \frac{2}{3}, \text{ so } E(X^3) = 3! \times \frac{2}{3} = 4 \quad \text{A1}$$

$$\text{Answer: 4} \quad \text{4 marks}$$

13 (4 marks)

$$M_X'(t) = \frac{1}{3}(e^t + 2e^{2t} + 3e^{3t}), M_X''(t) = \frac{1}{3}(e^t + 4e^{2t} + 9e^{3t}) \quad \text{M1 A1}$$

$$E(X) = M_X'(0) = \frac{6}{3} = 2 \text{ and } E(X^2) = M_X''(0) = \frac{14}{3} \quad \text{A1}$$

$$\text{Var}(X) = \frac{14}{3} - 2^2 = \frac{2}{3}; X \text{ takes the values 1, 2, 3 each with probability } \frac{1}{3} \quad \text{A1}$$

$$\text{Answer: } E(X) = 2, \text{ Var}(X) = \frac{2}{3}; \text{ equally likely values 1, 2, 3} \quad \text{4 marks}$$

14 (6 marks)

$$M_X''(t) = \frac{pe^t(1 - qe^t)^2 + pe^t \times 2qe^t(1 - qe^t)}{(1 - qe^t)^4}$$

M1 A1

$$= \frac{pe^t(1 + qe^t)}{(1 - qe^t)^3}$$

A1

$$E(X^2) = M_X''(0) = \frac{p(1+q)}{p^3} = \frac{1+q}{p^2}$$

M1 A1

$$\text{Var}(X) = \frac{1+q}{p^2} - \frac{1}{p^2} = \frac{q}{p^2}$$

A1

Answer: Shown

6 marks

15 (5 marks)

$$e^t - 1 = t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} + \dots$$

M1

$$M_X(t) = 1 + \frac{t}{2} + \frac{t^2}{6} + \frac{t^3}{24} + \dots$$

A1

$$E(X) = \frac{1}{2}, E(X^2) = 2! \times \frac{1}{6} = \frac{1}{3}, E(X^3) = 3! \times \frac{1}{24} = \frac{1}{4}$$

M1 A1

$$\text{Var}(X) = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{12}$$

A1

$$\text{Answer: } E(X) = \frac{1}{2}, E(X^2) = \frac{1}{3}, E(X^3) = \frac{1}{4}; \text{Var}(X) = \frac{1}{12}$$

5 marks

16 (6 marks)

$$\text{Put } u = -2t: (1 - 2t)^{-3} = 1 - 3(-2t) + 6(-2t)^2 - 10(-2t)^3 + \dots$$

M1

$$= 1 + 6t + 24t^2 + 80t^3 + \dots$$

A1

$$E(X) = 6, E(X^2) = 2! \times 24 = 48, E(X^3) = 3! \times 80 = 480$$

M1 A1 A1

$$\text{Var}(X) = 48 - 36 = 12$$

A1

$$\text{Answer: } E(X) = 6, \text{Var}(X) = 12, E(X^3) = 480$$

6 marks

17 (6 marks)

$$(a) K'(t) = \frac{M'(t)}{M(t)}, \text{ and } M(0) = 1, \text{ so } K'(0) = M'(0) = E(X)$$

M1 A1

$$K''(t) = \frac{M''(t)M(t) - (M'(t))^2}{(M(t))^2}, \text{ so } K''(0) = M''(0) - (M'(0))^2 = E(X^2) - (E(X))^2 = \text{Var}(X)$$

A1

Answer: Shown

3 marks

$$(b) K(t) = 10 \ln(0.6 + 0.4e^t), \text{ so } K'(t) = \frac{4e^t}{0.6 + 0.4e^t}$$

M1

$$E(Y) = K'(0) = 4$$

A1

$$K''(t) = \frac{2.4e^t}{(0.6 + 0.4e^t)^2}, \text{ so } \text{Var}(Y) = K''(0) = 2.4$$

A1

$$\text{Answer: } E(Y) = 4, \text{Var}(Y) = 2.4$$

3 marks

18 (5 marks)

$$M_X(0) = 1 \text{ gives } \frac{k}{4} = 1, \text{ so } k = 4$$

M1 A1

$$M_X'(t) = \frac{8}{(2-t)^3}, M_X''(t) = \frac{24}{(2-t)^4}$$

M1

$$E(X) = \frac{8}{8} = 1, E(X^2) = \frac{24}{16} = \frac{3}{2}$$

A1

$$\text{Var}(X) = \frac{3}{2} - 1 = \frac{1}{2}$$

A1

$$\text{Answer: } k = 4, E(X) = 1, \text{Var}(X) = \frac{1}{2}$$

5 marks

19 (5 marks)

(a) $M_X''(0) = E(X^2)$, not the variance

E1

$E(X) = M_X'(0) = 2$, so $\text{Var}(X) = 6 - 2^2$

M1

$\text{Var}(X) = 2$

A1

Answer: $\text{Var}(X) = 2$

3 marks

(b) $M_X'''(t) = (2e^t + 12e^{2t} + 8e^{3t})e^{2(e^t-1)}$

M1

$E(X^3) = 2 + 12 + 8 = 22$

A1

Answer: 22

2 marks

20 (5 marks)

$\frac{1}{1-t^2} = 1 + t^2 + t^4 + \dots$

M1 A1

No odd powers of t , so $E(X) = E(X^3) = 0$

A1

$E(X^2) = 2! \times 1 = 2$, $E(X^4) = 4! \times 1 = 24$

A1

Coefficient of t^{2k} is 1, so $E(X^{2k}) = (2k)!$

B1

Answer: 0, 2, 0, 24; $E(X^{2k}) = (2k)!$

5 marks

21 (7 marks)

(a) $M_X'(t) = (\mu + \sigma^2 t)e^{\mu t + \sigma^2 t^2/2}$, so $E(X) = \mu$

M1 A1

$M_X''(t) = [\sigma^2 + (\mu + \sigma^2 t)^2]e^{\mu t + \sigma^2 t^2/2}$, so $E(X^2) = \sigma^2 + \mu^2$

M1

$\text{Var}(X) = \sigma^2 + \mu^2 - \mu^2 = \sigma^2$

A1

Answer: Shown

4 marks

(b) $M_X'''(t) = [\sigma^2(\mu + \sigma^2 t) + 2\sigma^2(\mu + \sigma^2 t) + (\mu + \sigma^2 t)^3]e^{\mu t + \sigma^2 t^2/2}$

M1 A1

$E(X^3) = 3\mu\sigma^2 + \mu^3$

A1

Answer: $\mu^3 + 3\mu\sigma^2$

3 marks

22 (7 marks)

(a) $M_X(t) = \frac{1}{1 - \frac{t}{\lambda}}$

M1

$= 1 + \frac{t}{\lambda} + \left(\frac{t}{\lambda}\right)^2 + \dots + \left(\frac{t}{\lambda}\right)^n + \dots$ for $|t| < \lambda$

M1 A1

Coefficient of t^n is $\frac{1}{\lambda^n}$ and $E(X^n) = n! \times \frac{1}{\lambda^n}$

A1

Answer: Shown

4 marks

(b) $E[(X - \mu)^3] = E(X^3) - 3\mu E(X^2) + 3\mu^2 E(X) - \mu^3$

M1

$= \frac{6}{\lambda^3} - 3 \times \frac{1}{\lambda} \times \frac{2}{\lambda^2} + 3 \times \frac{1}{\lambda^2} \times \frac{1}{\lambda} - \frac{1}{\lambda^3}$

A1

$= \frac{6 - 6 + 3 - 1}{\lambda^3} = \frac{2}{\lambda^3}$

A1

Answer: Shown

3 marks

23 (7 marks)

(a) $X(X - 1) = X^2 - X$

M1

$E[X(X - 1)] = E(X^2) - E(X) = M_X''(0) - M_X'(0)$

A1

Answer: Shown

2 marks

(b) $M_X'(t) = np e^t (1 - p + pe^t)^{n-1}$

B1

$M_X''(t) = npe^t(1 - p + pe^t)^{n-1} + n(n-1)p^2 e^{2t}(1 - p + pe^t)^{n-2}$

M1

$M_X''(0) - M_X'(0) = np + n(n-1)p^2 - np = n(n-1)p^2$

A1

Answer: Shown

3 marks

(c) $E(X^2) = E[X(X - 1)] + E(X) = n(n-1)p^2 + np$

M1

$\text{Var}(X) = n^2 p^2 - np^2 + np - n^2 p^2 = np(1 - p)$

A1

Answer: Shown

2 marks

24 (8 marks)

$$(a) e^u = 1 + u + \frac{u^2}{2} + \frac{u^3}{6} + \dots \text{ with } u = \frac{1}{2}t^2$$

M1

$$M_Z(t) = 1 + \frac{1}{2}t^2 + \frac{1}{8}t^4 + \frac{1}{48}t^6 + \dots$$

A1

$$\text{Answer: } 1 + \frac{1}{2}t^2 + \frac{1}{8}t^4 + \frac{1}{48}t^6$$

2 marks

$$(b) E(Z^4) = 4! \times \frac{1}{8} = 3$$

B1

$$E(Z^6) = 6! \times \frac{1}{48} = 15$$

B1

$$\text{Answer: } E(Z^4) = 3, E(Z^6) = 15$$

2 marks

$$(c) \text{Var}(Z^2) = E(Z^4) - (E(Z^2))^2 \\ = 3 - 1 = 2$$

M1

A1

$$\text{Answer: } 2$$

2 marks

$$(d) (Z^2 - 1)^3 = Z^6 - 3Z^4 + 3Z^2 - 1$$

M1

$$E = 15 - 9 + 3 - 1 = 8$$

A1

$$\text{Answer: } 8$$

2 marks

25 (7 marks)

$$M_X(0) = 1: a + b + c = 1$$

B1

$$M_X'(0) = b + 2c = 1.2$$

M1 A1

$$M_X''(0) = b + 4c = 2$$

M1 A1

$$\text{Subtracting: } 2c = 0.8, \text{ so } c = 0.4, b = 0.4, a = 0.2$$

A1

$$\text{Var}(X) = 2 - 1.2^2 = 0.56$$

A1

$$\text{Answer: } a = 0.2, b = 0.4, c = 0.4; \text{Var}(X) = 0.56$$

7 marks

26 (6 marks)

$$E(Y^n) = E(e^{nY}) = M_X(n)$$

M1

$$E(Y) = M_X(1) = e^{1/2}$$

M1 A1

$$E(Y^2) = M_X(2) = e^2$$

A1

$$\text{Var}(Y) = e^2 - (e^{1/2})^2$$

M1

$$= e^2 - e$$

A1

$$\text{Answer: } E(Y) = e^{1/2}, \text{Var}(Y) = e^2 - e$$

6 marks

27 (8 marks)

$$K(t) = r[\ln p + t - \ln(1 - qe^t)]$$

M1 A1

$$K'(t) = r\left[1 + \frac{qe^t}{1 - qe^t}\right]$$

M1 A1

$$= \frac{r}{1 - qe^t}$$

A1

$$E(X) = K'(0) = \frac{r}{1 - q} = \frac{r}{p}$$

A1

$$K''(t) = \frac{rqe^t}{(1 - qe^t)^2}$$

M1

$$\text{Var}(X) = K''(0) = \frac{rq}{p^2}$$

A1

$$\text{Answer: Shown}$$

8 marks

28 (8 marks)

$$(a) (1 + u)^{-1/2} = 1 - \frac{1}{2}u + \frac{3}{8}u^2 - \frac{5}{16}u^3 + \dots \quad \text{M1}$$

$$\text{Put } u = -2t: 1 + t + \frac{3}{8} \times 4t^2 + \frac{5}{16} \times 8t^3 \quad \text{M1}$$

$$M_Y(t) = 1 + t + \frac{3}{2}t^2 + \frac{5}{2}t^3 + \dots \quad \text{A1}$$

$$\text{Answer: } 1 + t + \frac{3}{2}t^2 + \frac{5}{2}t^3 \quad \text{3 marks}$$

$$(b) E(Y) = 1, E(Y^2) = 2! \times \frac{3}{2} = 3, E(Y^3) = 3! \times \frac{5}{2} = 15 \quad \text{M1 A1}$$

$$\text{Var}(Y) = 3 - 1 = 2 \quad \text{A1}$$

$$\text{Answer: } E(Y) = 1, \text{Var}(Y) = 2, E(Y^3) = 15 \quad \text{3 marks}$$

$$(c) M_Y'(t) = (1 - 2t)^{-3/2}, M_Y''(t) = 3(1 - 2t)^{-5/2} \quad \text{M1}$$

$$M_Y'(0) = 1 \text{ and } M_Y''(0) = 3, \text{ in agreement} \quad \text{A1}$$

$$\text{Answer: Verified} \quad \text{2 marks}$$

29 (8 marks)

$$(a) M_X(t) = 1 + \frac{t^2}{3!} + \frac{t^4}{5!} + \dots + \frac{t^{2k}}{(2k+1)!} + \dots \quad \text{M1 A1}$$

$$\text{No odd powers of } t, \text{ so all odd moments are } 0 \quad \text{B1}$$

$$E(X^{2k}) = (2k)! \times \frac{1}{(2k+1)!} = \frac{1}{2k+1} \quad \text{A1}$$

$$\text{Answer: Shown} \quad \text{4 marks}$$

$$(b) \text{Var}(X) = E(X^2) = \frac{1}{3} \quad \text{B1}$$

$$E(X^4) = \frac{1}{5} \quad \text{B1}$$

$$E[(X^2 - \frac{1}{3})^2] = E(X^4) - \frac{2}{3}E(X^2) + \frac{1}{9} \quad \text{M1}$$

$$= \frac{1}{5} - \frac{2}{9} + \frac{1}{9} = \frac{4}{45} \quad \text{A1}$$

$$\text{Answer: } \text{Var}(X) = \frac{1}{3}, E(X^4) = \frac{1}{5} \quad \text{4 marks}$$

30 (8 marks)

$$(a) M_{X-\lambda}(t) = e^{-\lambda t} M_X(t) \quad \text{M1}$$

$$= \exp[\lambda(e^t - 1) - \lambda t] = \exp[\lambda(e^t - 1 - t)] \quad \text{A1}$$

$$\text{Answer: Shown} \quad \text{2 marks}$$

$$(b) \lambda(e^t - 1 - t) = \lambda(\frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24}) + \dots \quad \text{M1}$$

$$\exp(w) = 1 + w + \frac{w^2}{2} + \dots; \text{ the term in } t^3 \text{ comes only from } w, \text{ with coefficient } \frac{\lambda}{6} \quad \text{M1}$$

$$E[(X - \lambda)^3] = 3! \times \frac{\lambda}{6} = \lambda \quad \text{A1}$$

$$\text{Answer: Shown} \quad \text{3 marks}$$

$$(c) \text{Term in } t^4: \text{ from } w, \frac{\lambda}{24}; \text{ from } \frac{w^2}{2}, \frac{1}{2}(\frac{\lambda}{2})^2 = \frac{\lambda^2}{8} \quad \text{M1 A1}$$

$$E[(X - \lambda)^4] = 4!(\lambda/24 + \lambda^2/8) = \lambda + 3\lambda^2 \quad \text{A1}$$

$$\text{Answer: Shown} \quad \text{3 marks}$$

Sums, Transformations and Uniqueness**1 (2 marks)**

$$M_{2X}(t) = M_X(2t) \quad \text{M1}$$

$$= 0.5 + 0.5e^{2t} \quad \text{A1}$$

$$\text{Answer: } 0.5 + 0.5e^{2t} \quad \text{2 marks}$$

2 (2 marks)

$$M_{X+3}(t) = e^{3t}M_X(t) \quad \text{M1}$$

$$= \frac{e^{3t}}{1-t} \quad \text{A1}$$

Answer: $\frac{e^{3t}}{1-t}$ 2 marks

3 (2 marks)

$$M_{X+Y}(t) = M_X(t)M_Y(t) = (0.5 + 0.5e^t)^5 \quad \text{M1}$$

$$X + Y \sim B(5, 0.5) \quad \text{A1}$$

Answer: **B(5, 0.5)** 2 marks

4 (2 marks)

This is the moment generating function of Po(3) B1

$$P(X = 2) = \frac{e^{-3}3^2}{2!} = 0.224 \quad \text{B1}$$

Answer: **Po(3); P(X = 2) = 0.224** 2 marks

5 (2 marks)

This is the moment generating function of B(6, 0.3) B1

$$P(X = 2) = {}^6C_2(0.3)^2(0.7)^4 = 0.324135 \quad \text{B1}$$

Answer: **B(6, 0.3); P(X = 2) = 0.324135** 2 marks

6 (2 marks)

Compare with $e^{\mu t + \sigma^2 t^2/2}$; $\mu = 5$ B1

$$\frac{1}{2}\sigma^2 = 1, \text{ so } \sigma^2 = 2; X \sim N(5, 2) \quad \text{B1}$$

Answer: **N(5, 2)** 2 marks

7 (3 marks)

$$M_{aX+b}(t) = E(e^{t(aX+b)}) \quad \text{M1}$$

$$= E(e^{bt}e^{atX}) = e^{bt}E(e^{(at)X}) \quad \text{M1}$$

$$= e^{bt}M_X(at) \quad \text{A1}$$

Answer: **Shown** 3 marks

8 (3 marks)

$$M_Y(t) = e^t M_X(2t) \quad \text{M1}$$

$$= e^t \times \frac{3}{3-2t} \quad \text{A1}$$

Exists when $3 - 2t > 0$, that is $t < \frac{3}{2}$ B1

Answer: $M_Y(t) = \frac{3e^t}{3-2t}, t < \frac{3}{2}$ 3 marks

9 (3 marks)

$$M_{X+Y}(t) = M_X(t)M_Y(t) = e^{2(e^t-1)}e^{5(e^t-1)} \quad \text{M1}$$

$$= e^{7(e^t-1)} \quad \text{A1}$$

By uniqueness $X + Y \sim \text{Po}(7)$ A1

Answer: **Po(7)** 3 marks

10 (4 marks)

$$M_X(t) = e^{t+2t^2}, M_Y(t) = e^{2t+4.5t^2} \quad \text{B1}$$

$$M_{X+Y}(t) = M_X(t)M_Y(t) = e^{3t+6.5t^2} \quad \text{M1 A1}$$

This is the moment generating function of N(3, 13), so $X + Y \sim N(3, 13)$ A1

Answer: **N(3, 13)** 4 marks

11 (3 marks)

Compare with the geometric moment generating function $\frac{pe^t}{1 - qe^t}$ ($x = 1, 2, \dots$): $p = 0.4$, $q = 0.6$

M1 A1

$$P(X = 3) = q^2p = 0.36 \times 0.4 = 0.144$$

A1

Answer: Geometric with $p = 0.4$; $P(X = 3) = 0.144$

3 marks

12 (3 marks)

By uniqueness, the coefficient of e^{tx} gives $P(X = x)$: $P(X = -1) = 0.1$, $P(X = 0) = 0.4$, $P(X = 2) = 0.5$

M1 A1

$$E(X) = -0.1 + 0 + 1.0 = 0.9$$

A1

Answer: $P(-1) = 0.1$, $P(0) = 0.4$, $P(2) = 0.5$; $E(X) = 0.9$

3 marks

13 (4 marks)

$$Z = \frac{1}{\sigma}X - \frac{\mu}{\sigma}, \text{ so } M_Z(t) = e^{-\mu/\sigma}M_X\left(\frac{t}{\sigma}\right)$$

M1 A1

$$= e^{-\mu/\sigma}e^{\mu/\sigma + \sigma^2 t^2/2\sigma^2}$$

M1

$$= e^{t^2/2}, \text{ so } Z \sim N(0, 1)$$

A1

Answer: Shown

4 marks

14 (4 marks)

(a) $M_{X+Y}(t) = M_X(t)M_Y(t)$ by independence

M1

$$= (q + pe^t)^n(q + pe^t)^m = (q + pe^t)^{n+m}$$

A1

This is the moment generating function of $B(n + m, p)$; by uniqueness $X + Y \sim B(n + m, p)$

A1

Answer: Shown

3 marks

(b) The product $(q + pe^t)^n(q' + p'e^t)^m$ is not of the form $(q'' + p''e^t)^{n+m}$

E1

Answer: The product is not a binomial moment generating function

1 mark

15 (5 marks)

$$M_W(t) = e^{5t}M_X(3t)M_Y(-2t)$$

M1 A1

$$M_X(3t) = e^{6t + 4.5t^2}, M_Y(-2t) = e^{2t + 8t^2}$$

M1 A1

$$M_W(t) = e^{13t + 12.5t^2}, \text{ so } W \sim N(13, 25)$$

A1

Answer: $N(13, 25)$

5 marks

16 (5 marks)

$$M_{X/4}(t) = M_{X_1}\left(\frac{t}{4}\right) = e^{\mu/4 + \sigma^2 t^2/32}$$

M1 A1

$$\text{By independence } M_{\bar{X}}(t) = [e^{\mu/4 + \sigma^2 t^2/32}]^4$$

M1

$$= e^{\mu t + \sigma^2 t^2/8}$$

A1

This is the moment generating function of $N(\mu, \frac{\sigma^2}{4})$; by uniqueness $\bar{X} \sim N(\mu, \frac{\sigma^2}{4})$

A1

Answer: Shown

5 marks

17 (6 marks)

$$(a) M_{Z+2}(t) = e^{2t}M_Z(t)$$

M1

$$= (0.3 + 0.7e^t)^4, \text{ so } Y \sim B(4, 0.7)$$

A1

Answer: $B(4, 0.7)$

2 marks

$$(b) E(Y) = 4 \times 0.7 = 2.8$$

M1

$$E(Z) = 2.8 - 2 = 0.8$$

A1

Answer: 0.8

2 marks

$$(c) P(Z = 0) = P(Y = 2)$$

M1

$$= {}^4C_2(0.7)^2(0.3)^2 = 0.2646$$

A1

Answer: 0.2646

2 marks

18 (6 marks)

$$M_X(t) = \frac{1}{3}M_{Po(2)}(t) + \frac{2}{3}M_{Po(5)}(t) \text{ (conditioning on the type)} \quad \text{M1 A1}$$

$$M_X'(t) = \frac{1}{3} \times 2e^t e^{2(e^t-1)} + \frac{2}{3} \times 5e^t e^{5(e^t-1)}, \text{ so } E(X) = \frac{2}{3} + \frac{10}{3} = 4 \quad \text{M1 A1}$$

$$E(X^2) = \frac{1}{3}(2 + 4) + \frac{2}{3}(5 + 25) = 22 \quad \text{M1}$$

$$\text{Var}(X) = 22 - 16 = 6 \quad \text{A1}$$

Answer: $E(X) = 4$, $\text{Var}(X) = 6$ 6 marks

19 (5 marks)

$$M_{-Y}(t) = M_Y(-t), \text{ so } M_D(t) = e^{3(e^t-1)}e^{2(e^{-t}-1)} \quad \text{M1 A1}$$

$$K(t) = \ln M_D(t) = 3(e^t - 1) + 2(e^{-t} - 1) \quad \text{M1}$$

$$E(D) = K'(0) = 3 - 2 = 1, \text{ Var}(D) = K''(0) = 3 + 2 = 5 \quad \text{A1}$$

A Poisson variable cannot take negative values, but D can (for example $X = 0$, $Y = 1$) E1

Answer: $E(D) = 1$, $\text{Var}(D) = 5$ 5 marks

20 (6 marks)

$$(a) M_{X+Y}(t) = (0.5 + 0.5e^t)^2(0.8 + 0.2e^t) = (0.25 + 0.5e^t + 0.25e^{2t})(0.8 + 0.2e^t) \quad \text{M1 A1}$$

$$P(X + Y = 3) \text{ is the coefficient of } e^{3t} \quad \text{M1}$$

$$= 0.25 \times 0.2 = 0.05 \quad \text{A1}$$

Answer: $P(X + Y = 3) = 0.05$ 4 marks

$$(b) E(X + Y) = 1 + 0.2 = 1.2, \text{ so a } B(3, p) \text{ distribution would need } p = 0.4 \quad \text{M1}$$

$$\text{Then } P(X + Y = 0) \text{ would be } 0.6^3 = 0.216, \text{ but from the expansion it is } 0.25 \times 0.8 = 0.2 \quad \text{E1}$$

Answer: Shown 2 marks

21 (6 marks)

$$\text{Jar: } M(t) = e^{200t + 12.5t^2}; \text{ bottle: } M(t) = e^{50t + 4.5t^2} \quad \text{B1}$$

$$M_T(t) = (e^{200t + 12.5t^2})^3 \times e^{50t + 4.5t^2} = e^{650t + 42t^2} \quad \text{M1 A1}$$

$$\text{So } T \sim N(650, 84) \quad \text{A1}$$

$$z = \frac{660 - 650}{\sqrt{84}} = 1.091 \quad \text{M1}$$

$$P(T > 660) = 1 - \Phi(1.091) = 0.138 \quad \text{A1}$$

Answer: $T \sim N(650, 84)$; $P(T > 660) = 0.138$ 6 marks

22 (7 marks)

$$(a) \text{ By independence } M_S(t) = M_{X_1}(t)M_{X_2}(t) \quad \text{M1}$$

$$= \left(\frac{0.4e^t}{1 - 0.6e^t} \right)^2 \quad \text{A1}$$

$$\text{Answer: } \left(\frac{0.4e^t}{1 - 0.6e^t} \right)^2 \quad \text{2 marks}$$

$$(b) M_S(t) = 0.16e^{2t}(1 - 0.6e^t)^{-2} \quad \text{M1}$$

$$= 0.16e^{2t}(1 + 2(0.6e^t) + 3(0.6e^t)^2 + \dots) \quad \text{A1}$$

$$\text{Coefficient of } e^{4t} \text{ is } 0.16 \times 3 \times 0.36 = 0.1728, \text{ so } P(S = 4) = 0.1728 \quad \text{A1}$$

Answer: 0.1728 3 marks

$$(c) P(S = 4) = P(1,3) + P(2,2) + P(3,1) \text{ with } P(x, y) = (0.4)^2(0.6)^{x+y-2} \quad \text{M1}$$

$$= 3 \times 0.16 \times 0.36 = 0.1728 \quad \text{A1}$$

Answer: 0.1728 2 marks

23 (6 marks)

$$(a) \text{ Comparing with } \frac{e^{bt} - e^{at}}{(b-a)t}: b = 4, a = -2, b - a = 6 \quad \text{M1 A1}$$

$$X \sim U[-2, 4], \text{ so } P(X > 1) = \frac{4 - 1}{6} = \frac{1}{2} \quad \text{A1}$$

Answer: $U[-2, 4]$; $P(X > 1) = \frac{1}{2}$ 3 marks

$$(b) Y = \frac{1}{6}X + \frac{1}{3}, \text{ so } M_Y(t) = e^{t/3}M_X\left(\frac{t}{6}\right)$$

M1

$$= e^{t/3} \times \frac{e^{\frac{2t}{3}} - e^{-\frac{t}{3}}}{t} = \frac{e^t - 1}{t}$$

A1

This is the moment generating function of $U[0, 1]$, so $Y \sim U[0, 1]$

A1

Answer: $Y \sim U[0, 1]$

3 marks

24 (6 marks)

$$(a) M_{\bar{X}}(t) = [M_X(\frac{t}{4})]^4 \text{ by independence}$$

M1

$$= \left[\frac{2}{2 - \frac{t}{4}} \right]^4$$

A1

$$= \left(\frac{8}{8 - t} \right)^4$$

A1

This is a gamma distribution with $\alpha = 4$ and $\lambda = 8$

A1

Answer: Gamma, $\alpha = 4$, $\lambda = 8$

4 marks

$$(b) E(\bar{X}) = \frac{4}{8} = \frac{1}{2}, \text{ Var}(\bar{X}) = \frac{4}{64} = \frac{1}{16}$$

B1

$$\text{Check: } E(X_i) = \frac{1}{2}, \text{ Var}(X_i) = \frac{1}{4}, \text{ so } \text{Var}(\bar{X}) = \frac{1}{4} \div 4 = \frac{1}{16}$$

B1

$$\text{Answer: } E = \frac{1}{2}, \text{ Var} = \frac{1}{16}$$

2 marks

25 (7 marks)

$$(a) M_{-Y}(t) = M_Y(-t) = \frac{1}{1+t}$$

M1 A1

$$M_D(t) = \frac{1}{1-t} \times \frac{1}{1+t} = \frac{1}{1-t^2}, \text{ valid for } -1 < t < 1$$

A1

Answer: Shown

3 marks

$$(b) \text{ Same moment generating function as } W, \text{ so by uniqueness } D \text{ has pdf } \frac{1}{2}e^{-|d|}$$

M1 A1

$$P(D > 2) = \int_2^\infty \frac{1}{2}e^{-d} dd$$

M1

$$= \frac{1}{2}e^{-2}$$

A1

$$\text{Answer: } \frac{1}{2}e^{-|d|}; P(D > 2) = \frac{1}{2}e^{-2}$$

4 marks

26 (8 marks)

$$(a) M_X(t) = \frac{1}{2}E(e^{tX} | N(0, 1)) + \frac{1}{2}E(e^{tX} | N(0, 4))$$

M1

$$= \frac{1}{2}e^{t^2/2} + \frac{1}{2}e^{2t^2}$$

A1

Answer: Shown

2 marks

$$(b) e^{t^2/2} = 1 + \frac{t^2}{2} + \frac{t^4}{8} + \dots, e^{2t^2} = 1 + 2t^2 + 2t^4 + \dots$$

M1 A1

$$M_X(t) = 1 + \frac{5}{4}t^2 + \frac{17}{16}t^4 + \dots$$

A1

$$E(X^2) = 2! \times \frac{5}{4} = \frac{5}{2} \text{ so } \text{Var}(X) = \frac{5}{2} \text{ (as } E(X) = 0); E(X^4) = 4! \times \frac{17}{16} = \frac{51}{2}$$

A1

$$\text{Answer: } \text{Var}(X) = \frac{5}{2}, E(X^4) = \frac{51}{2}$$

4 marks

$$(c) \text{ If } X \text{ were normal with } \text{Var}(X) = \frac{5}{2}, E(X^4) \text{ would be } 3 \times \left(\frac{5}{2}\right)^2 = \frac{75}{4}$$

M1

$$\text{But } E(X^4) = \frac{51}{2} = \frac{102}{4} \neq \frac{75}{4}$$

E1

Answer: Shown

2 marks

27 (9 marks)

$$(a) M_D(t) = M_X(t)M_{-Y}(t) = M_X(t)M_Y(-t) \quad \text{M1}$$

$$= e^{a(e^t-1)}e^{b(e^{-t}-1)} \quad \text{A1}$$

Answer: Shown 2 marks

$$(b) K(t) = a(e^t - 1) + b(e^{-t} - 1) \quad \text{B1}$$

$$K'(t) = ae^t - be^{-t}, \text{ so } E(D) = a - b \quad \text{M1 A1}$$

Answer: Shown (Var(D) = K''(0) = a + b as K''(t) = ae^t + be^{-t}) 3 marks

$$(c) K'''(t) = ae^t - be^{-t}, \text{ so third central moment} = K'''(0) = a - b \quad \text{M1}$$

If $a > b$ this is positive, so D is positively skewed A1

Answer: a - b; positively skewed 2 marks

$$(d) M_D(t) = \exp[a(e^t + e^{-t} - 2)] = \exp[2a(\cosh t - 1)] \quad \text{M1}$$

This is an even function of t, so $M_{-D}(t) = M_D(t)$ and D and -D have the same distribution A1

Answer: Shown 2 marks

28 (8 marks)

$$(a) M_X(t) = \int_0^1 e^{tx} dx \quad \text{M1}$$

$$= \left[\frac{e^{tx}}{t} \right]_0^1 = \frac{e^t - 1}{t} \quad \text{A1}$$

Answer: Shown 2 marks

$$(b) \text{ By independence } M_{X+Y}(t) = (M_X(t))^2 \quad \text{M1}$$

$$= \frac{(e^t - 1)^2}{t^2} \quad \text{A1}$$

$$\text{Answer: } \frac{(e^t - 1)^2}{t^2} \quad \text{2 marks}$$

(c) The moment generating function equals that of W, so by uniqueness X + Y has the pdf of W M1

$f(s) = s$ for $0 \leq s \leq 1$, $2 - s$ for $1 < s \leq 2$, 0 otherwise A1

Answer: $f(s) = s$ for $0 \leq s \leq 1$, $f(s) = 2 - s$ for $1 < s \leq 2$, $f(s) = 0$ otherwise 2 marks

$$(d) P(X + Y > 1.5) = \int_{1.5}^2 (2 - s) ds \quad \text{M1}$$

$$= \left[2s - \frac{s^2}{2} \right]_{1.5}^2 = 2 - 1.875 = \frac{1}{8} \quad \text{A1}$$

$$\text{Answer: } \frac{1}{8} \quad \text{2 marks}$$

29 (9 marks)

$$(a) \text{ By independence } M_{Y_3}(t) = [(1 - 2t)^{-1/2}]^3 \quad \text{M1}$$

$$= (1 - 2t)^{-3/2} \quad \text{A1}$$

Answer: $(1 - 2t)^{-3/2}$ 2 marks

$$(b) M'(t) = 3(1 - 2t)^{-5/2}, M''(t) = 15(1 - 2t)^{-7/2} \quad \text{M1 A1}$$

$$E(Y_3) = 3, E(Y_3^2) = 15, \text{ so } \text{Var}(Y_3) = 15 - 9 = 6 \quad \text{A1}$$

Answer: E = 3, Var = 6 3 marks

$$(c) M_{Y_3+W}(t) = (1 - 2t)^{-3/2}(1 - 2t)^{-5/2} \quad \text{M1}$$

$$= (1 - 2t)^{-4} = M_{Y_8}(t); \text{ by uniqueness the distributions are the same} \quad \text{A1}$$

Answer: Shown 2 marks

$$(d) M_{Y_2}(t) = (1 - 2t)^{-1} = \frac{1}{\frac{1}{2} - t}, \text{ the exponential moment generating function with rate } \frac{1}{2} \quad \text{M1}$$

$$P(Y_2 > 4) = e^{-4/2} = e^{-2} \quad \text{A1}$$

$$\text{Answer: } e^{-2} \quad \text{2 marks}$$

30 (10 marks)

$$(a) M_X(t) = \frac{1}{2} \times \frac{1}{1 - \frac{e^t}{2}} \quad \text{M1}$$

$$= \frac{1}{2} \sum_{x=0}^{\infty} \left(\frac{e^t}{2}\right)^x = \sum_{x=0}^{\infty} \left(\frac{1}{2}\right)^{x+1} e^{tx} \quad \text{M1}$$

$$\text{Coefficient of } e^{tx} \text{ gives } P(X = x) = \left(\frac{1}{2}\right)^{x+1} \quad \text{A1}$$

Answer: Shown 3 marks

$$(b) M_Y(t) = (M_X(t))^3 \text{ by independence} \quad \text{M1}$$

$$= \left(\frac{1}{2}\right)^3 \left(1 - \frac{e^t}{2}\right)^{-3} = \frac{1}{8} \left(1 - \frac{e^t}{2}\right)^{-3} \quad \text{A1}$$

Answer: Shown 2 marks

$$(c) M_Y(t) = \frac{1}{8} \sum_{k=2}^{\infty} C_2 \left(\frac{e^t}{2}\right)^k, \text{ so } P(Y = k) = \frac{1}{8} \times C_2 \times \left(\frac{1}{2}\right)^k \quad \text{M1}$$

$$P(Y = 2) = \frac{1}{8} \times 6 \times \frac{1}{4} = \frac{3}{16} \quad \text{A1}$$

$$\text{Answer: } P(Y = k) = C_2 \left(\frac{1}{2}\right)^{k+3}; P(Y = 2) = \frac{3}{16} \quad \text{2 marks}$$

$$(d) M_X'(t) = \frac{e^t}{(2 - e^t)^2}, \text{ so } E(X) = 1 \quad \text{M1 A1}$$

$$M_X''(0) = 3, \text{ so } \text{Var}(X) = 3 - 1 = 2; \text{ by independence } E(Y) = 3, \text{Var}(Y) = 6 \quad \text{A1}$$

Answer: E(Y) = 3, Var(Y) = 6 3 marks

Applications and Limits

1 (2 marks)

$$M_{X_n}(t) = \left(1 - \frac{3}{n} + \frac{3e^t}{n}\right)^n = \left(1 + \frac{3(e^t - 1)}{n}\right)^n, \text{ so } c = 3(e^t - 1) \quad \text{B1}$$

$$\text{Limit} = e^{3(e^t - 1)} \quad \text{B1}$$

Answer: c = 3(e^t - 1); limit e^{3(e^t - 1)} 2 marks

2 (2 marks)

$$P(X \geq 6) \leq e^{-6} e^{2(e - 1)} \quad \text{M1}$$

$$= e^{2e - 8} = 0.077 \quad \text{A1}$$

Answer: P(X ≥ 6) ≤ e^{2e - 8} = 0.077 2 marks

3 (2 marks)

$$M_{X+Y}(t) = M_X(t)M_Y(t) \quad \text{M1}$$

$$= \left(\frac{2}{2 - t}\right)^2 = \frac{4}{(2 - t)^2} \quad \text{A1}$$

Answer: $\frac{4}{(2 - t)^2}$ 2 marks

4 (2 marks)

$$E(S) = 300 \text{ and } \text{Var}(S) = 400 \quad \text{B1}$$

$$S \text{ is approximately } N(300, 400) \quad \text{B1}$$

Answer: S ≈ N(300, 400) 2 marks

5 (2 marks)

$$G_N(s) = 0.3 + 0.7s \quad \text{B1}$$

$$M_S(t) = 0.3 + 0.7 \times \frac{2}{2 - t} \quad \text{B1}$$

Answer: $0.3 + \frac{1.4}{2 - t}$ 2 marks

6 (2 marks)

$X \geq a$ and $t > 0$ give $tX \geq ta$, and e^u is an increasing function, so $e^{tX} \geq e^{ta}$

E1

If $t < 0$, multiplying by t reverses the inequality, so $tX \leq ta$ and $e^{tX} \leq e^{ta}$

E1

Answer: For $t > 0$, $tX \geq ta$ so $e^{tX} \geq e^{ta}$; for $t < 0$ the inequality reverses, $e^{tX} \leq e^{ta}$

2 marks

7 (4 marks)

$$M_{X_n}(t) = (1 - \frac{\lambda}{n} + \frac{\lambda e^t}{n})^n$$

M1

$$= (1 + \frac{\lambda(e^t - 1)}{n})^n$$

A1

As $n \rightarrow \infty$ this tends to $e^{\lambda(e^t - 1)}$

M1

This is the moment generating function of $Po(\lambda)$, so X_n tends in distribution to $Po(\lambda)$

A1

Answer: Shown

4 marks

8 (5 marks)

Bound $g(t) = \frac{e^{-5t}}{1-t}$; minimise $\ln g = -5t - \ln(1-t)$

M1

Derivative: $-5 + \frac{1}{1-t} = 0$, so $t = \frac{4}{5}$

M1 A1

$$g(\frac{4}{5}) = \frac{e^{-4}}{\frac{1}{5}} = 5e^{-4} \approx 0.0916$$

A1

Exact: $P(X \geq 5) = e^{-5} \approx 0.0067$, so the bound is valid but not tight

B1

Answer: $t = \frac{4}{5}$; bound $5e^{-4} \approx 0.0916$, exact $e^{-5} \approx 0.0067$

5 marks

9 (5 marks)

$$M_X(t) = \frac{1}{2}e^t + \frac{1}{2}e^{2t}, \text{ so } K(t) = 4(\frac{1}{2}e^t + \frac{1}{2}e^{2t} - 1)$$

M1 A1

$$K'(t) = 2e^t + 4e^{2t}, K''(t) = 2e^t + 8e^{2t}$$

M1

$$E(S) = K'(0) = 6$$

A1

$$\text{Var}(S) = K''(0) = 10$$

A1

Answer: $E(S) = 6$, $\text{Var}(S) = 10$

5 marks

10 (4 marks)

$$M_S(t) = (\frac{\lambda}{\lambda - t})^n \text{ by independence}$$

B1

$$K(t) = n[\ln \lambda - \ln(\lambda - t)], K'(t) = \frac{n}{\lambda - t}, K''(t) = \frac{n}{(\lambda - t)^2}$$

M1 A1

$$E(S) = K'(0) = \frac{n}{\lambda} \text{ and } \text{Var}(S) = K''(0) = \frac{n}{\lambda^2}$$

A1

Answer: Shown

4 marks

11 (4 marks)

$$E(Y) = \frac{E(X) - \mu}{\sigma} = 0$$

M1 A1

$$E(Y^2) = \frac{\text{Var}(X)}{\sigma^2} = 1 \text{ (since } E(Y) = 0)$$

M1

$$M_Y(\frac{t}{\sqrt{n}}) = 1 + \frac{t^2}{2n} + (\text{terms in } t^3 n^{-3/2} \text{ and higher powers})$$

A1

Answer: $1 + \frac{t^2}{2n} + \dots$

4 marks

12 (4 marks)

$$P(Z \geq a) \leq e^{-ta + t^2/2}$$

M1

The exponent $-ta + \frac{t^2}{2}$ is minimised when $t = a$

M1 A1

With $a = 2$ the bound is $e^{-2} = 0.1353$, which is much larger than 0.02275 (valid but not tight)

A1

Answer: $e^{-a^2/2}$; $a = 2$ gives 0.1353

4 marks

13 (4 marks)

Let $Y = e^{tX}$, which is non-negative

B1

For $t > 0$, $X \geq a$ if and only if $Y \geq e^{ta}$

M1

$$E(Y) \geq e^{ta}P(Y \geq e^{ta}) = e^{ta}P(X \geq a)$$

M1

$$\text{So } P(X \geq a) \leq e^{-ta}E(e^{tX}) = e^{-ta}M_X(t)$$

A1

Answer: Shown

4 marks

14 (5 marks)

$$M_S(t) = \left(\frac{\lambda}{\lambda - t}\right)^2$$

M1

This is the moment generating function of the gamma distribution with $\alpha = 2$ ($\Gamma(2) = 1$), so by uniqueness S has pdf $\lambda^2 se^{-\lambda s}$

A1

$$P(S > 1) = \int_1^\infty 2.25se^{-1.5s} ds$$

M1

$$= 2.25\left[-\frac{s}{1.5}e^{-1.5s} - \frac{1}{2.25}e^{-1.5s}\right]_1^\infty = (1.5 + 1)e^{-1.5}$$

A1

$$= 0.558$$

A1

Answer: S has pdf $\lambda^2 se^{-\lambda s}$; $P(S > 1) = 0.558$

5 marks

15 (6 marks)

$$\ln \text{ of bound: } h(t) = 20 \ln(0.5 + 0.5e^t) - 15t$$

M1

$$h'(t) = \frac{10e^t}{0.5 + 0.5e^t} - 15 = \frac{20e^t}{1 + e^t} - 15 = 0$$

M1 A1

$$20e^t = 15 + 15e^t, \text{ so } e^t = 3$$

A1

$$\text{Bound} = (0.5 + 1.5)^{20} \times 3^{-15} = \frac{2^{20}}{3^{15}} = 0.0731$$

M1

Exact $P(X \geq 15) = 0.0207$, so the bound holds

A1

Answer: $e^t = 3$; bound 0.0731; exact 0.0207

6 marks

16 (6 marks)

Given $N = n$, $Y \sim B(n, p)$, so $E(e^{tY} | N = n) = (q + pe^t)^n$

M1 A1

$$M_Y(t) = \sum P(N = n)(q + pe^t)^n = E((q + pe^t)^N) = G_N(q + pe^t)$$

M1

$$= \exp[\lambda(q + pe^t - 1)]$$

A1

Since $q - 1 = -p$ this is $\exp[\lambda p(e^t - 1)]$

A1

This is the moment generating function of $Po(\lambda p)$, so $Y \sim Po(\lambda p)$

A1

Answer: Shown

6 marks

17 (5 marks)

$$\text{Bound} = \exp[3(e^t - 1) - ta]$$

M1

$$\text{Derivative of exponent: } 3e^t - a = 0, \text{ giving } e^t = \frac{a}{3}$$

M1

$$\text{Bound} = \exp[a - 3 - a \ln(\frac{a}{3})] = e^{a-3}(\frac{3}{a})^a$$

A1

$$a = 9: e^6 \times (\frac{1}{3})^9 = 0.0205$$

A1

$$\text{Exact: } P(X \geq 9) = 0.0038$$

B1

Answer: $e^{a-3}(\frac{3}{a})^a$; $a = 9$ gives 0.0205; exact 0.0038

5 marks

18 (5 marks)

(a) $X \leq a$ if and only if $e^{-tX} \geq e^{-ta}$

M1

Applying the inequality $P(Y \geq k) \leq E(Y)/k$ with $Y = e^{-tX}$, $k = e^{-ta}$: $P(X \leq a) \leq e^{ta} E(e^{-tX}) = e^{ta} M_X(-t)$

A1

Answer: Shown

2 marks

(b) Bound = $\exp[ta + 10(e^{-t} - 1)]$; derivative $a - 10e^{-t} = 0$ gives $e^{-t} = \frac{a}{10}$

M1 A1

$a = 4$: bound = $(2.5)^4 e^{-6} = 0.0968$, while the exact $P(X \leq 4) = 0.0293$

A1

Answer: 0.0968 (exact 0.0293)

3 marks

19 (5 marks)

(a) The bracket tends to 1 but the power n tends to infinity at the same time, so the limit is of the form 1^∞ , which is not necessarily 1

E1

The correct result is $(1 + \frac{x}{n})^n \rightarrow e^x$

E1

Answer: Indeterminate form 1^∞

2 marks

(b) Use $(1 + \frac{x}{n})^n \rightarrow e^x$ with $x = \lambda(e^t - 1)$

M1

Limit = $e^{\lambda(e^t - 1)}$

A1

This is the moment generating function of $Po(\lambda)$

A1

Answer: $e^{\lambda(e^t - 1)}$, so $Po(\lambda)$

3 marks

20 (6 marks)

(a) By independence $M_Y(t) = [(1 - 2t)^{-1/2}]^4$

M1

$= (1 - 2t)^{-2}$

A1

Answer: Shown

2 marks

(b) With $\lambda = \frac{1}{2}$: $(\frac{1}{2}/\frac{1}{2} - t)^2 = (1 - 2t)^{-2}$

M1

By uniqueness the pdf is $(\frac{1}{2})^2 y e^{-y/2} = \frac{y}{4} e^{-y/2}$

A1

Answer: Shown

2 marks

(c) $P(Y > 6) = \int_6^\infty \frac{y}{4} e^{-y/2} dy = [-\frac{y}{2} e^{-y/2} - e^{-y/2}]_6^\infty$

M1

$= 3e^{-3} + e^{-3} = 4e^{-3} \approx 0.1991$

A1

Answer: $4e^{-3} \approx 0.1991$

2 marks

21 (7 marks)

(a) By independence $M_{Z_n}(t) = \prod M_{Y_i/\sqrt{n}}(t)$

M1

$= [M_Y(\frac{t}{\sqrt{n}})]^n$

A1

Answer: Shown

2 marks

(b) $\ln M_{Z_n}(t) = n \ln(1 + \frac{t^2}{2n} + o(\frac{1}{n}))$

M1

$= n(\frac{t^2}{2n} + o(\frac{1}{n}) + O(\frac{1}{n^2})) \rightarrow \frac{1}{2}t^2$

M1 A1

Answer: Shown

3 marks

(c) $M_{Z_n}(t) \rightarrow e^{t^2/2}$, the moment generating function of $N(0, 1)$

B1

By the continuity theorem for moment generating functions, Z_n tends in distribution to $N(0, 1)$:

E1

this is the central limit theorem

Answer: $N(0, 1)$

2 marks

22 (7 marks)

$$(a) M_{S_n}(t) = (1 - t)^{-n} \quad \text{M1}$$

$$Z_n = \frac{1}{\sqrt{n}}S_n - \sqrt{n}, \text{ so } M_{Z_n}(t) = e^{-\sqrt{n}t}M_{S_n}\left(\frac{t}{\sqrt{n}}\right) \quad \text{M1}$$

$$= e^{-t\sqrt{n}}\left(1 - \frac{t}{\sqrt{n}}\right)^{-n} \quad \text{A1}$$

Answer: Shown 3 marks

$$(b) \ln M_{Z_n}(t) = -t\sqrt{n} - n \ln\left(1 - \frac{t}{\sqrt{n}}\right) \quad \text{M1}$$

$$= -t\sqrt{n} + n\left(\frac{t}{\sqrt{n}} + \frac{t^2}{2n} + \frac{t^3}{3n\sqrt{n}} + \dots\right) \quad \text{M1}$$

$$= \frac{1}{2}t^2 + \frac{t^3}{3\sqrt{n}} + \dots \quad \text{A1}$$

Answer: Shown 3 marks

$$(c) M_{Z_n}(t) \rightarrow e^{t^2/2}, \text{ so } Z_n \text{ tends in distribution to } N(0, 1) \quad \text{E1}$$

Answer: $e^{t^2/2}$; $N(0, 1)$ 1 mark

23 (6 marks)

$$q_r = \frac{\mu}{r + \mu}, \text{ so } \frac{p_r}{1 - q_r e^t} = \frac{r}{r + \mu - \mu e^t} \quad \text{M1 A1}$$

$$= \frac{1}{1 + \frac{\mu(1 - e^t)}{r}} = \left(1 - \frac{\mu(e^t - 1)}{r}\right)^{-1} \quad \text{A1}$$

$$\text{So } M_{Y_r}(t) = \left(1 - \frac{\mu(e^t - 1)}{r}\right)^{-r} \quad \text{A1}$$

$$\left(1 - \frac{\mu(e^t - 1)}{r}\right)^r \rightarrow e^{-\mu(e^t - 1)} \quad \text{M1}$$

Taking the reciprocal, $M_{Y_r}(t) \rightarrow e^{\mu(e^t - 1)}$, the moment generating function of $\text{Po}(\mu)$ A1

Answer: Shown 6 marks

24 (7 marks)

$$(a) M_S(t) = \frac{p M_X}{1 - q M_X} \text{ with } M_X = \frac{\lambda}{\lambda - t} \quad \text{M1}$$

$$\text{Multiplying numerator and denominator by } \lambda - t: M_S(t) = \frac{p\lambda}{(\lambda - t) - q\lambda} \quad \text{M1 A1}$$

$$= \frac{p\lambda}{p\lambda - t} \quad \text{A1}$$

This is the moment generating function of an exponential distribution with rate $p\lambda$ A1

Answer: $S \sim \text{Exp}(p\lambda)$ 5 marks

$$(b) S \text{ has rate } 0.6, \text{ so } P(S > 5) = e^{-0.6 \times 5} \quad \text{M1}$$

$$= e^{-3} \approx 0.0498 \quad \text{A1}$$

Answer: $e^{-3} \approx 0.0498$ 2 marks

25 (6 marks)

$$M_{Z_\lambda}(t) = e^{-t\sqrt{\lambda}}M_{X_\lambda}\left(\frac{t}{\sqrt{\lambda}}\right) \quad \text{M1 A1}$$

$$= \exp[\lambda(e^{t/\sqrt{\lambda}} - 1) - t\sqrt{\lambda}] \quad \text{A1}$$

$$e^s - 1 = s + \frac{s^2}{2} + \frac{s^3}{6} + \dots \text{ with } s = \frac{t}{\sqrt{\lambda}} \quad \text{M1}$$

$$\lambda(e^s - 1) = t\sqrt{\lambda} + \frac{t^2}{2} + \frac{t^3}{6\sqrt{\lambda}} + \dots \quad \text{A1}$$

$$\text{So the exponent is } \frac{t^2}{2} + \frac{t^3}{6\sqrt{\lambda}} + \dots \rightarrow \frac{t^2}{2}, \text{ and } M_{Z_\lambda}(t) \rightarrow e^{t^2/2} \quad \text{A1}$$

Answer: Shown 6 marks

26 (6 marks)

$$M_{\bar{X}_n}(t) = [M(\frac{t}{n})]^n \quad \text{M1 A1}$$

$$\ln M_{\bar{X}_n}(t) = n \ln(1 + \frac{\mu t}{n} + o(\frac{1}{n})) \quad \text{M1}$$

$$= n(\frac{\mu t}{n} + o(\frac{1}{n})) \rightarrow \mu t \quad \text{A1}$$

$$\text{So } M_{\bar{X}_n}(t) \rightarrow e^{\mu t} \quad \text{A1}$$

$e^{\mu t}$ is the moment generating function of the constant μ , so $\bar{X}_n$ tends in distribution to μ (weak law of large numbers) E1

Answer: Shown

6 marks

27 (8 marks)

$$(a) M_{X_1}(t) = \frac{1}{2}e^t + \frac{1}{2}e^{-t} = \cosh t \quad \text{B1}$$

Answer: Shown

1 mark

$$(b) \cosh t = \sum \frac{t^{2k}}{(2k)!} \text{ and } e^{t^2/2} = \sum \frac{t^{2k}}{2^k k!} \quad \text{M1 A1}$$

$(2k)! = 2k(2k-1)\dots(k+1) \times k! \geq 2^k k!$ because each of the k factors $2k, \dots, k+1$ is at least 2; so each term of the cosh series is at most the matching term of the exponential series E1

Answer: Shown

3 marks

$$(c) \text{ For } t > 0, P(S_n \geq a) \leq e^{-ta}(\cosh t)^n \leq e^{-ta + nt^2/2} \quad \text{M1 A1}$$

$$\text{Choose } t = \frac{a}{n} \text{ to minimise the exponent: } -\frac{a^2}{n} + \frac{a^2}{2n} = -\frac{a^2}{2n} \quad \text{A1}$$

Answer: Shown

3 marks

$$(d) e^{-900/200} = e^{-4.5} = 0.0111 \quad \text{B1}$$

Answer: 0.0111

1 mark

28 (9 marks)

$$(a) M_{I_k}(t) = 1 + p_k(e^t - 1) \leq e^{p_k(e^t - 1)} \quad \text{M1 A1}$$

$$M_X(t) = \prod M_{I_k}(t) \leq e^{\sum p_k(e^t - 1)} = e^{\mu(e^t - 1)} \quad \text{A1}$$

Answer: Shown

3 marks

$$(b) \text{ For } t > 0, P(X \geq (1 + \delta)\mu) \leq e^{-t(1+\delta)\mu} M_X(t) \quad \text{M1}$$

$$\leq \exp[\mu(e^t - 1) - t(1 + \delta)\mu] \quad \text{A1}$$

$$\text{Choose } t = \ln(1 + \delta) > 0: \text{exponent} = \mu(\delta - (1 + \delta) \ln(1 + \delta)) \quad \text{M1}$$

$$\text{So the bound is } (\frac{e^\delta}{(1 + \delta)^{1+\delta}})^\mu \quad \text{A1}$$

Answer: Shown

4 marks

$$(c) \delta = 1: \text{bound} = (\frac{e}{4})^{10} \quad \text{M1}$$

$$= 0.0210 \quad \text{A1}$$

$$\text{Answer: } (\frac{e}{4})^{10} = 0.0210 \quad \text{2 marks}$$

29 (9 marks)

$$(a) K_S(t) = \ln M_S(t) = \ln M_N(K_X(t)) \quad \text{M1}$$

$$= K_N(K_X(t)) \quad \text{A1}$$

Answer: Shown

2 marks

$$(b) K_S'(t) = K_N'(K_X(t))K_X'(t) \quad \text{M1}$$

$$\text{At } t = 0, K_X(0) = 0, \text{ so } E(S) = K_N'(0)K_X'(0) = E(N)E(X) \quad \text{A1}$$

Answer: Shown

2 marks

$$(c) K_S''(t) = K_N''(K_X(t))(K_X'(t))^2 + K_N'(K_X(t))K_X''(t) \quad \text{M1 A1}$$

$$\text{At } t = 0: \text{Var}(S) = \text{Var}(N)(E(X))^2 + E(N)\text{Var}(X) \quad \text{M1 A1}$$

Answer: Shown

4 marks

(d) $\text{Var}(N) = E(N) = \lambda$, so $\text{Var}(S) = \lambda(\text{Var}(X) + (E(X))^2) = \lambda E(X^2)$

Answer: Shown

B1
1 mark

30 (9 marks)

(a) $M_X(t) = \frac{1}{1-t}$, so $M_S(t) = \exp[3(\frac{1}{1-t} - 1)]$

M1

$= \exp[3 \times \frac{t}{1-t}] = \exp[\frac{3t}{1-t}]$

A1

Answer: Shown

2 marks

(b) $K(t) = \frac{3t}{1-t} = 3(-1 + \frac{1}{1-t})$, so $K'(t) = \frac{3}{(1-t)^2}$

M1 A1

$K''(t) = \frac{6}{(1-t)^3}$, so $E(S) = 3$ and $\text{Var}(S) = 6$

A1

Answer: $E(S) = 3$, $\text{Var}(S) = 6$

3 marks

(c) In bound $g(t) = \frac{3t}{1-t} - 12t$, so $g'(t) = \frac{3}{(1-t)^2} - 12$

M1

$g'(t) = 0$ gives $(1-t)^2 = \frac{1}{4}$, $t = \frac{1}{2}$ ($t = \frac{3}{2}$ is outside the range)

A1

$g''(t) = \frac{6}{(1-t)^3} > 0$, so this is a minimum

E1

Bound $= \exp[3 - 6] = e^{-3} \approx 0.0498$

A1

Answer: $t = \frac{1}{2}$; $P(S \geq 12) \leq e^{-3} \approx 0.0498$

4 marks