

Definition and Standard MGFs

- 1 The discrete random variable X takes the value 1 with probability 0.3 and the value 2 with probability 0.7.
Write down the moment generating function $M_X(t)$ of X . [2]
- 2 Show that $M_X(0) = 1$ for any random variable X whose moment generating function exists. [2]
- 3 The random variable X takes each of the values 1, 2, 3, 4 with probability $\frac{1}{4}$.
Write down $M_X(t)$. [2]
- 4 The random variable X has $P(X = 0) = 0.6$ and $P(X = 1) = 0.4$.
(a) Find $M_X(t)$. [1]
(b) Find the exact value of $M_X(\ln 3)$. [1]
- 5 The moment generating function of the binomial distribution $B(n, p)$ is $(1 - p + pe^t)^n$. The random variable $X \sim B(5, 0.2)$.
(a) Write down $M_X(t)$. [1]
(b) Find $M_X(\ln 4)$. [1]
- 6 The moment generating function of the Poisson distribution $Po(\lambda)$ is $e^{\lambda(e^t - 1)}$. The random variable $X \sim Po(3)$.
(a) Write down $M_X(t)$. [1]
(b) Find the exact value of $M_X(\ln 2)$. [1]
- 7 The random variable X has the probability distribution shown.
 x : -1, 0, 2
 $P(X = x)$: 0.2, 0.5, 0.3
 Find $M_X(t)$. [3]
- 8 The random variable X has a geometric distribution with $P(X = x) = q^{x-1}p$ for $x = 1, 2, 3, \dots$, where $q = 1 - p$.
Show that $M_X(t) = \frac{pe^t}{1 - qe^t}$ and state the range of values of t for which this holds. [5]
- 9 The random variable $X \sim Po(\lambda)$, so $P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}$ for $x = 0, 1, 2, \dots$
Show from the definition that $M_X(t) = e^{\lambda(e^t - 1)}$, and state the values of t for which this holds. [4]
- 10 The random variable X is exponentially distributed with probability density function $f(x) = \lambda e^{-\lambda x}$ for $x \geq 0$, where $\lambda > 0$.
Show that $M_X(t) = \frac{\lambda}{\lambda - t}$ and state the range of values of t for which the integral converges. [4]
- 11 The continuous random variable X is uniformly distributed on the interval $[0, 2]$, so its probability density function is $f(x) = \frac{1}{2}$ for $0 \leq x \leq 2$.
Show that $M_X(t) = \frac{e^{2t} - 1}{2t}$ for $t \neq 0$. [3]
- 12 The random variable X has moment generating function $M_X(t) = 0.1 + 0.3e^t + 0.4e^{2t} + 0.2e^{3t}$.
Write down the probability distribution of X and find $P(X \geq 2)$. [3]

- 13** The moment generating function of the normal distribution $N(\mu, \sigma^2)$ is $e^{\mu t + \frac{1}{2}\sigma^2 t^2}$. The random variable $X \sim N(5, 9)$.
- (a) Write down $M_X(t)$. [1]
- (b) The random variable $Z = \frac{X - 5}{3}$ has the standard normal distribution. Write down $M_Z(t)$. [1]
- (c) Find the exact value of $M_X(\frac{1}{2})$. [1]
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- 14** The random variable $X \sim B(n, p)$, so $P(X = x) = {}^nC_x p^x q^{n-x}$ for $x = 0, 1, \dots, n$, where $q = 1 - p$.
Use the definition of the moment generating function and the binomial theorem to show that $M_X(t) = (q + pe^t)^n$. [4]
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- 15** The continuous random variable X has probability density function $f(x) = 3x^2$ for $0 \leq x \leq 1$ and $f(x) = 0$ otherwise.
You may use the result $\int x^2 e^{tx} dx = e^{tx}(\frac{x^2}{t} - \frac{2x}{t^2} + \frac{2}{t^3}) + c$.
Show that $M_X(t) = \frac{3e^t(t^2 - 2t + 2) - 6}{t^3}$ for $t \neq 0$. [5]
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- 16** The continuous random variable X has probability density function $f(x) = \frac{1}{2}e^{-|x|}$ for all real x .
Find $M_X(t)$, stating the range of values of t for which it exists. [5]
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- 17** The random variable X has probability distribution $P(X = x) = \frac{1}{4}(\frac{3}{4})^x$ for $x = 0, 1, 2, \dots$
Find $M_X(t)$, and state the range of values of t for which it exists. [4]
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- 18** The continuous random variable X has probability density function $f(x) = 4xe^{-2x}$ for $x \geq 0$ and $f(x) = 0$ otherwise.
You may use the result $\int_0^\infty xe^{-kx} dx = \frac{1}{k^2}$ for $k > 0$.
Find $M_X(t)$ and state the range of t for which it exists. [4]
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- 19** The discrete random variable X has probability distribution $P(X = x) = kx$ for $x = 1, 2, 3, 4$, where k is a constant.
Find the value of k and hence find $M_X(t)$. [4]
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- 20** The random variable X has a geometric distribution with $P(X = x) = q^{x-1}p$ for $x = 1, 2, 3, \dots$, where $q = 1 - p$.
A student works out the moment generating function as follows.
$$M_X(t) = \sum_{x=1}^{\infty} e^{tx} q^{x-1} p = p \sum_{x=1}^{\infty} (qe^t)^{x-1} = \frac{p}{1 - qe^t}$$
- (a) Explain the error in the student's working. [2]
- (b) Write down the correct moment generating function. [2]
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- 21** The standard normal random variable Z has probability density function $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$ for all real z , and $\int_{-\infty}^{\infty} \phi(z) dz = 1$.
- (a) Show that $M_Z(t) = e^{\frac{1}{2}t^2}$ for all real t , by completing the square in the exponent. [5]
- (b) The random variable $X = \mu + \sigma Z$, where $\sigma > 0$. Use $M_{aY+b}(t) = e^{bt} M_Y(at)$ to find $M_X(t)$. [2]

- 22** The continuous random variable X has probability density function $f(x) = x$ for $0 \leq x \leq 1$, $f(x) = 2 - x$ for $1 < x \leq 2$, and $f(x) = 0$ otherwise.
You may use the result $\int x e^{tx} dx = e^{tx} \left(\frac{x}{t} - \frac{1}{t^2} \right) + c$.
Show that $M_X(t) = \frac{(e^t - 1)^2}{t^2}$ for $t \neq 0$. [7]
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- 23** The discrete random variable X has probability distribution $P(X = x) = \frac{e^{-\lambda} \lambda^x}{x!(1 - e^{-\lambda})}$ for $x = 1, 2, 3, \dots$, where $\lambda > 0$ (a Poisson distribution conditioned on being positive).
Show that $M_X(t) = \frac{e^{\lambda(e^t - 1)} - e^{-\lambda}}{1 - e^{-\lambda}}$ and state the range of t for which it exists. [6]
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- 24** A random variable X is 0 with probability $\frac{1}{3}$. Otherwise, with probability $\frac{2}{3}$, X has an exponential distribution with rate 2.
The moment generating function of an exponential distribution with rate λ is $\frac{\lambda}{\lambda - t}$ for $t < \lambda$.
Find $M_X(t)$, simplifying your answer into a single fraction, and state the range of t for which it exists. [6]
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- 25** The continuous random variable X has probability density function $f(x) = \frac{2}{x^3}$ for $x \geq 1$ and $f(x) = 0$ otherwise.
(Do not attempt to find $M_X(t)$ in closed form.)
(a) Show that $E(e^{tX})$ is finite for all $t \leq 0$. [2]
(b) Show that $E(e^{tX})$ does not exist for any $t > 0$. You may use $e^u > \frac{u^3}{6}$ for $u > 0$. [3]
(c) $E(X) = 2$ exists. Explain why the moment generating function cannot be used to find $E(X)$ here. [1]
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- 26** The gamma distribution with parameters $\alpha > 0$ and $\lambda > 0$ has probability density function $f(x) = \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}$ for $x > 0$, where $\Gamma(\alpha) = \int_0^\infty u^{\alpha-1} e^{-u} du$.
Show that $M_X(t) = \left(\frac{\lambda}{\lambda - t} \right)^\alpha$ for $t < \lambda$, by using the substitution $u = (\lambda - t)x$. [6]
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- 27** In a sequence of independent trials, each succeeds with probability p . Let $q = 1 - p$ and let X be the number of trials needed to obtain the r -th success, so $P(X = x) = {}^{x-1}C_{r-1} p^r q^{x-r}$ for $x = r, r + 1, r + 2, \dots$
You may use the series $(1 - u)^{-r} = \sum_{k=0}^\infty {}^{k+r-1}C_{r-1} u^k$ for $|u| < 1$.
(a) Show that $M_X(t) = \left(\frac{pe^t}{1 - qe^t} \right)^r$, and state the range of t for which it exists. [6]
(b) Let $Y = X - r$ be the number of failures before the r -th success. Write down $M_Y(t)$. [2]
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- 28** Let $Z \sim N(0, 1)$ and $Y = Z^2$.
You may use the result $\int_{-\infty}^\infty e^{-z^2/(2s)} dz = \sqrt{2\pi s}$ for $s > 0$.
(a) Show that $M_Y(t) = (1 - 2t)^{-1/2}$ for $t < \frac{1}{2}$. [6]
(b) Hence find the exact value of $E(e^{-Z^2/2})$. [1]
(c) Explain why $M_Y(t)$ does not exist for $t \geq \frac{1}{2}$. [1]

- 29** The continuous random variable X has probability density function $f(x) = \frac{1}{2} \sin x$ for $0 \leq x \leq \pi$ and $f(x) = 0$ otherwise.
- You may use the result $\int e^{tx} \sin x \, dx = \frac{e^{tx}(t \sin x - \cos x)}{1 + t^2} + c$.
- (a) Show that $M_X(t) = \frac{1 + e^{\pi t}}{2(1 + t^2)}$. [4]
- (b) Verify that $M_X(0) = 1$. [1]
- (c) The distribution is symmetrical about $x = \frac{\pi}{2}$. Show that the moment generating function of $X - \frac{\pi}{2}$ is $\frac{\cosh(\frac{\pi t}{2})}{1 + t^2}$, and explain why this function is even. [3]
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- 30** The random variable U is uniformly distributed on $(0, 1)$ and $Y = -\ln U$.
(Standard results: the exponential distribution with rate 1 has distribution function $1 - e^{-y}$ for $y \geq 0$.)
- (a) Find $P(Y \leq y)$ for $y \geq 0$, and hence identify the distribution of Y . [3]
- (b) By writing $M_Y(t) = E(U^{-t})$, show that $M_Y(t) = \frac{1}{1 - t}$ for $t < 1$. [2]
- (c) U_1, U_2, U_3 are independent copies of U . Find the exact value of $E(((U_1 U_2 U_3))^{-1/2})$. [3]

Moments from MGFs

- 1** The random variable X has moment generating function $M_X(t) = 0.3 + 0.7e^t$.
Find $E(X)$ by differentiating $M_X(t)$. [2]
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- 2** The random variable X has moment generating function $M_X(t) = e^{t + t^2}$.
Find $M_X'(t)$ and hence find $E(X)$. [2]
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- 3** The Maclaurin series of the moment generating function of X begins $M_X(t) = 1 + 4t + 9t^2 + \dots$
Write down the values of $E(X)$ and $E(X^2)$. [2]
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- 4** The random variable X has moment generating function $M_X(t) = (0.5 + 0.5e^t)^4$.
Find $E(X)$. [2]
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- 5** The random variable X has moment generating function $M_X(t) = \frac{5}{5 - t}$ for $t < 5$.
Find $E(X)$. [2]
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- 6** For a random variable X , $M_X'(0) = 3$ and $M_X''(0) = 13$.
Find $\text{Var}(X)$. [2]
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- 7** The random variable X has moment generating function $M_X(t) = \frac{4}{4 - t}$ for $t < 4$.
Find $E(X)$ and $\text{Var}(X)$. [4]
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- 8** The random variable X has moment generating function $M_X(t) = e^{\lambda(e^t - 1)}$, where $\lambda > 0$.
Use differentiation to show that $E(X) = \lambda$ and $\text{Var}(X) = \lambda$. [5]
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- 9** The random variable X has moment generating function $M_X(t) = (1 - t)^{-2}$ for $t < 1$.
Expand $M_X(t)$ as a series in ascending powers of t up to and including the term in t^3 , and hence find $E(X)$, $E(X^2)$ and $E(X^3)$, and $\text{Var}(X)$. [4]

- 10** The random variable $X \sim B(n, p)$ has moment generating function $M_X(t) = (q + pe^t)^n$, where $q = 1 - p$.
Use this to show that $E(X) = np$ and $\text{Var}(X) = npq$. [5]
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- 11** The random variable X has moment generating function $M_X(t) = 0.2e^{-t} + 0.5 + 0.3e^{2t}$.
Find $E(X)$ and $\text{Var}(X)$. [4]
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- 12** The random variable X has moment generating function $M_X(t) = e^{t + \frac{1}{2}t^2}$.
By expanding $M_X(t)$ as a series up to the term in t^3 , find $E(X^3)$. [4]
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- 13** The random variable X has moment generating function $M_X(t) = \frac{1}{3}(e^t + e^{2t} + e^{3t})$.
Find $E(X)$ and $\text{Var}(X)$ by differentiating $M_X(t)$, and state the distribution of X . [4]
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- 14** The random variable X has a geometric distribution with moment generating function $M_X(t) = \frac{pe^t}{1 - qe^t}$, where $q = 1 - p$.
Given that $M_X'(t) = \frac{pe^t}{(1 - qe^t)^2}$, show that $\text{Var}(X) = \frac{q}{p^2}$. [6]
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- 15** The random variable X has moment generating function $M_X(t) = \frac{e^t - 1}{t}$ for $t \neq 0$, with $M_X(0) = 1$.
Use the series expansion of e^t to find $E(X)$, $E(X^2)$ and $E(X^3)$, and deduce $\text{Var}(X)$. [5]
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- 16** The random variable X has moment generating function $M_X(t) = (1 - 2t)^{-3}$ for $t < \frac{1}{2}$.
Use the binomial series $(1 + u)^{-3} = 1 - 3u + 6u^2 - 10u^3 + \dots$ to find $E(X)$, $\text{Var}(X)$ and $E(X^3)$. [6]
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- 17** For a random variable X with moment generating function $M_X(t)$, the function $K(t) = \ln M_X(t)$ is called the cumulant generating function.
(a) Show that $E(X) = K'(0)$ and $\text{Var}(X) = K''(0)$. [3]
(b) The random variable $Y \sim B(10, 0.4)$ has $M_Y(t) = (0.6 + 0.4e^t)^{10}$. Use $K(t)$ to find $E(Y)$ and $\text{Var}(Y)$. [3]
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- 18** The random variable X has moment generating function $M_X(t) = \frac{k}{(2 - t)^2}$ for $t < 2$, where k is a constant.
Find the value of k , and hence find $E(X)$ and $\text{Var}(X)$. [5]
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- 19** The random variable $X \sim \text{Po}(2)$ has moment generating function $M_X(t) = e^{2(e^t - 1)}$.
A student states: " $M_X''(0) = 6$, so $\text{Var}(X) = 6$."
(a) Explain the error and state the correct value of $\text{Var}(X)$. [3]
(b) Find $E(X^3)$. [2]
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- 20** The random variable X has moment generating function $M_X(t) = \frac{1}{1 - t^2}$ for $|t| < 1$.
Use the expansion of $M_X(t)$ to find $E(X)$, $E(X^2)$, $E(X^3)$ and $E(X^4)$. Write down the value of $E(X^{2k})$ in terms of k . [5]
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- 21** The random variable $X \sim N(\mu, \sigma^2)$ has moment generating function $M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$.
- (a) Show, by differentiation, that $E(X) = \mu$ and $\text{Var}(X) = \sigma^2$. [4]
- (b) Find $E(X^3)$ in terms of μ and σ . [3]
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- 22** The random variable X has an exponential distribution with moment generating function $M_X(t) = \frac{\lambda}{\lambda - t}$ for $t < \lambda$.
- (a) Expand $M_X(t)$ as a power series in t and hence show that $E(X^n) = \frac{n!}{\lambda^n}$ for $n = 1, 2, 3, \dots$ [4]
- (b) Hence show that the third central moment $E[(X - \mu)^3]$ equals $\frac{2}{\lambda^3}$, where $\mu = E(X)$. [3]
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- 23** For a random variable X , the factorial moment $E[X(X - 1)]$ is defined where X takes non-negative integer values.
- The random variable $X \sim B(n, p)$ has moment generating function $M_X(t) = (1 - p + pe^t)^n$.
- (a) Show that $E[X(X - 1)] = M_X''(0) - M_X'(0)$ for any random variable with a moment generating function. [2]
- (b) Use differentiation of $M_X(t)$ to show that $E[X(X - 1)] = n(n - 1)p^2$. [3]
- (c) Hence show that $\text{Var}(X) = np(1 - p)$. [2]
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- 24** The standard normal random variable Z has moment generating function $M_Z(t) = e^{\frac{1}{2}t^2}$.
- (a) Expand $M_Z(t)$ in ascending powers of t as far as the term in t^6 . [2]
- (b) Hence find $E(Z^4)$ and $E(Z^6)$. [2]
- (c) Find $\text{Var}(Z^2)$. [2]
- (d) Find $E((Z^2 - 1)^3)$. [2]
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- 25** The random variable X takes the values 0, 1 and 2 only. Its moment generating function is $M_X(t) = a + be^t + ce^{2t}$, where a, b and c are constants.
- Given that $E(X) = 1.2$ and $E(X^2) = 2$, find a, b and c , and hence find $\text{Var}(X)$. [7]
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- 26** The random variable $X \sim N(0, 1)$ has moment generating function $M_X(t) = e^{\frac{1}{2}t^2}$. The random variable $Y = e^X$ has a log-normal distribution.
- Use the fact that $E(Y^n) = E(e^{nX}) = M_X(n)$ to find $E(Y)$ and $\text{Var}(Y)$, giving exact answers. [6]
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- 27** The random variable X , the number of trials needed to obtain the r -th success where each trial succeeds with probability p , has moment generating function $M_X(t) = \left(\frac{pe^t}{1 - qe^t}\right)^r$, where $q = 1 - p$.
- Use the cumulant generating function $K(t) = \ln M_X(t)$ to show that $E(X) = \frac{r}{p}$ and $\text{Var}(X) = \frac{rq}{p^2}$. (You may use $E(X) = K'(0)$ and $\text{Var}(X) = K''(0)$.) [8]
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- 28** The random variable Y has moment generating function $M_Y(t) = (1 - 2t)^{-1/2}$ for $t < \frac{1}{2}$.
- (a) Use the binomial expansion of $(1 + u)^{-1/2}$ to expand $M_Y(t)$ in ascending powers of t up to and including the term in t^3 . [3]
- (b) Hence find $E(Y)$, $\text{Var}(Y)$ and $E(Y^3)$. [3]
- (c) Verify $E(Y)$ and $E(Y^2)$ by differentiating $M_Y(t)$ twice. [2]

- 29** The random variable X is uniformly distributed on $[-1, 1]$, so its moment generating function is $M_X(t) = \frac{\sinh t}{t}$ for $t \neq 0$.
 You may use $\sinh t = t + \frac{t^3}{3!} + \frac{t^5}{5!} + \dots$
- (a) Show that $E(X^{2k}) = \frac{1}{2k+1}$ for $k = 0, 1, 2, \dots$ and that $E(X^{2k+1}) = 0$. [4]
- (b) Hence find $\text{Var}(X)$ and $E(X^4)$, and show that $E[(X^2 - \frac{1}{3})^2] = \frac{4}{45}$. [4]
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- 30** The random variable $X \sim \text{Po}(\lambda)$ has moment generating function $M_X(t) = e^{\lambda(e^t - 1)}$. Let $\mu = E(X) = \lambda$.
- (a) Write down the moment generating function of $X - \lambda$, and show that it can be written as $\exp[\lambda(e^t - 1 - t)]$. [2]
- (b) Expand $\lambda(e^t - 1 - t)$ up to the term in t^4 , and hence show that $E[(X - \lambda)^3] = \lambda$. [3]
- (c) Show that $E[(X - \lambda)^4] = 3\lambda^2 + \lambda$. [3]
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Sums, Transformations and Uniqueness

- 1** The random variable X has moment generating function $M_X(t) = 0.5 + 0.5e^t$.
 Write down the moment generating function of $2X$. [2]
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- 2** The random variable X has moment generating function $M_X(t) = \frac{1}{1-t}$ for $t < 1$.
 Write down the moment generating function of $X + 3$. [2]
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- 3** X and Y are independent random variables with moment generating functions $M_X(t) = (0.5 + 0.5e^t)^2$ and $M_Y(t) = (0.5 + 0.5e^t)^3$.
 Find the moment generating function of $X + Y$ and state the distribution of $X + Y$. [2]
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- 4** The random variable X has moment generating function $M_X(t) = e^{3(e^t - 1)}$.
 Identify the distribution of X and find $P(X = 2)$, giving your answer to 3 significant figures. [2]
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- 5** The random variable X has moment generating function $M_X(t) = (0.7 + 0.3e^t)^6$.
 Identify the distribution of X and find $P(X = 2)$. [2]
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- 6** The random variable X has moment generating function $M_X(t) = e^{5t + t^2}$.
 Identify the distribution of X , giving its parameters. [2]
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- 7** X is a random variable with moment generating function $M_X(t)$, and a and b are constants.
 Show that $M_{aX+b}(t) = e^{bt}M_X(at)$. [3]
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- 8** The random variable X has an exponential distribution with moment generating function $M_X(t) = \frac{3}{3-t}$ for $t < 3$.
 Find the moment generating function of $Y = 2X + 1$, and state the range of t for which it exists. [3]
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- 9** X and Y are independent random variables with $X \sim \text{Po}(2)$ and $Y \sim \text{Po}(5)$. The moment generating function of $\text{Po}(\lambda)$ is $e^{\lambda(e^t - 1)}$.
 Find the moment generating function of $X + Y$ and hence state its distribution. [3]
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- 10** X and Y are independent, $X \sim N(1, 4)$ and $Y \sim N(2, 9)$. The moment generating function of $N(\mu, \sigma^2)$ is $e^{\mu t + \sigma^2 t^2/2}$.
Use moment generating functions to find the distribution of $X + Y$. [4]
- 11** The random variable X has moment generating function $M_X(t) = \frac{0.4e^t}{1 - 0.6e^t}$.
Identify the distribution of X and find $P(X = 3)$. [3]
- 12** The random variable X has moment generating function $M_X(t) = 0.1e^{-t} + 0.4 + 0.5e^{2t}$.
Write down the probability distribution of X and find $E(X)$. [3]
- 13** The random variable $X \sim N(\mu, \sigma^2)$ has moment generating function $e^{\mu t + \sigma^2 t^2/2}$.
Let $Z = \frac{X - \mu}{\sigma}$. Show that $M_Z(t) = e^{t^2/2}$ and state the distribution of Z. [4]
- 14** $X \sim B(n, p)$ and $Y \sim B(m, p)$ are independent. The moment generating function of $B(n, p)$ is $(q + pe^t)^n$, where $q = 1 - p$.
(a) Use moment generating functions to prove that $X + Y \sim B(n + m, p)$. [3]
(b) Explain why the result does not hold if $Y \sim B(m, p')$ with $p' \neq p$. [1]
- 15** $X \sim N(2, 1)$ and $Y \sim N(-1, 4)$ are independent. The moment generating function of $N(\mu, \sigma^2)$ is $e^{\mu t + \sigma^2 t^2/2}$.
Find the distribution of $W = 3X - 2Y + 5$ using moment generating functions. [5]
- 16** X_1, X_2, X_3, X_4 are independent random variables, each distributed $N(\mu, \sigma^2)$. Let $\bar{X} = \frac{1}{4}(X_1 + X_2 + X_3 + X_4)$.
Use moment generating functions to show that $\bar{X} \sim N(\mu, \frac{\sigma^2}{4})$. [5]
- 17** The random variable Z has moment generating function $M_Z(t) = e^{-2t}(0.3 + 0.7e^t)^4$.
(a) By considering $M_{Z+2}(t)$, identify the distribution of $Y = Z + 2$. [2]
(b) Find $E(Z)$. [2]
(c) Find $P(Z = 0)$. [2]
- 18** A random variable X is distributed $Po(2)$ with probability $\frac{1}{3}$ and $Po(5)$ with probability $\frac{2}{3}$.
Show that $M_X(t) = \frac{1}{3}e^{2(e^t-1)} + \frac{2}{3}e^{5(e^t-1)}$, and hence find $E(X)$ and $Var(X)$. [6]
- 19** $X \sim Po(3)$ and $Y \sim Po(2)$ are independent, and $D = X - Y$.
Find the moment generating function of D, and use it to find $E(D)$ and $Var(D)$. Explain why D does not have a Poisson distribution. [5]
- 20** $X \sim B(2, 0.5)$ and $Y \sim B(1, 0.2)$ are independent. The moment generating function of $B(n, p)$ is $(1 - p + pe^t)^n$.
(a) Write down $M_{X+Y}(t)$ and find $P(X + Y = 3)$ by considering its expansion. [4]
(b) Show that $X + Y$ does not have a binomial distribution. [2]
- 21** A gift box contains three jars and one bottle. The masses, in grams, of the jars are independent and each distributed $N(200, 25)$; the mass of the bottle is independent of the jars and distributed $N(50, 9)$. The moment generating function of $N(\mu, \sigma^2)$ is $e^{\mu t + \sigma^2 t^2/2}$.
Use moment generating functions to find the distribution of the total mass T of the contents, and hence find $P(T > 660)$, giving your answer to 3 significant figures. [6]

- 22** X_1 and X_2 are independent geometric random variables, each with $P(X_i = x) = 0.6^{x-1} \times 0.4$ for $x = 1, 2, 3, \dots$ and moment generating function $\frac{0.4e^t}{1 - 0.6e^t}$. Let $S = X_1 + X_2$.
- (a) Write down $M_S(t)$. [2]
- (b) Using $(1 - u)^{-2} = 1 + 2u + 3u^2 + 4u^3 + \dots$, expand $M_S(t)$ in powers of e^t and hence find $P(S = 4)$. [3]
- (c) Verify your value of $P(S = 4)$ by a direct calculation using the distribution of X_1 and X_2 . [2]
-
- 23** The moment generating function of a random variable X is $M_X(t) = \frac{e^{4t} - e^{-2t}}{6t}$ for $t \neq 0$. The moment generating function of the uniform distribution on $[a, b]$ is $\frac{e^{bt} - e^{at}}{(b - a)t}$.
- (a) Identify the distribution of X and find $P(X > 1)$. [3]
- (b) Let $Y = \frac{X + 2}{6}$. Find $M_Y(t)$ and hence identify the distribution of Y . [3]
-
- 24** X_1, X_2, X_3, X_4 are independent exponential random variables, each with rate 2 and moment generating function $\frac{2}{2 - t}$. Let $\bar{X} = \frac{1}{4}(X_1 + X_2 + X_3 + X_4)$.
- The gamma distribution with parameters α and λ has moment generating function $(\frac{\lambda}{\lambda - t})^\alpha$, mean $\frac{\alpha}{\lambda}$ and variance $\frac{\alpha}{\lambda^2}$.
- (a) Show that $M_{\bar{X}}(t) = (\frac{8}{8 - t})^4$ and identify the distribution of $\bar{X}$. [4]
- (b) Write down $E(\bar{X})$ and $\text{Var}(\bar{X})$, and check these against the values obtained from the mean and variance of a single X_i . [2]
-
- 25** X and Y are independent exponential random variables, each with rate 1 (moment generating function $\frac{1}{1 - t}$ for $t < 1$). Let $D = X - Y$.
- The continuous random variable W with probability density function $f(w) = \frac{1}{2}e^{-|w|}$ has moment generating function $\frac{1}{1 - t^2}$ for $|t| < 1$.
- (a) Show that $M_D(t) = \frac{1}{1 - t^2}$ for $|t| < 1$. [3]
- (b) Hence state the probability density function of D and find $P(D > 2)$. [4]
-
- 26** The random variable X is distributed $N(0, 1)$ with probability $\frac{1}{2}$, and $N(0, 4)$ with probability $\frac{1}{2}$. The moment generating function of $N(0, \sigma^2)$ is $e^{\sigma^2 t^2/2}$.
- (a) Show that $M_X(t) = \frac{1}{2}e^{t^2/2} + \frac{1}{2}e^{2t^2}$. [2]
- (b) Expand $M_X(t)$ up to the term in t^4 , and hence find $\text{Var}(X)$ and $E(X^4)$. [4]
- (c) A normal distribution with variance σ^2 has $E(X^4) = 3\sigma^4$. Hence show that X is not normally distributed. [2]
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- 27** $X \sim \text{Po}(a)$ and $Y \sim \text{Po}(b)$ are independent, and $D = X - Y$. The cumulant generating function of a random variable is $K(t) = \ln M(t)$, and $E(D) = K'(0)$, $\text{Var}(D) = K''(0)$. The third central moment $E[(D - E(D))^3]$ equals $K'''(0)$.
- (a) Show that $M_D(t) = \exp[a(e^t - 1) + b(e^{-t} - 1)]$. [2]
- (b) Show that $E(D) = a - b$ and $\text{Var}(D) = a + b$. [3]
- (c) Find the third central moment of D and state, with a reason, whether D is positively or negatively skewed when $a > b$. [2]
- (d) Show that when $a = b$ the distribution of D is symmetrical about 0. [2]

- 28** X and Y are independent random variables, each uniformly distributed on $[0, 1]$.
The continuous random variable W with probability density function $f(w) = w$ for $0 \leq w \leq 1$, $f(w) = 2 - w$ for $1 < w \leq 2$ and $f(w) = 0$ otherwise has moment generating function $\frac{(e^t - 1)^2}{t^2}$ for $t \neq 0$.
- Show that $M_X(t) = \frac{e^t - 1}{t}$ for $t \neq 0$. [2]
 - Find $M_{X+Y}(t)$. [2]
 - Hence state the probability density function of $X + Y$. [2]
 - Find $P(X + Y > 1.5)$. [2]
-
- 29** $Z_1, Z_2, \dots$ are independent standard normal random variables. The moment generating function of Z_1^2 is $(1 - 2t)^{-1/2}$ for $t < \frac{1}{2}$. Let $Y_k = Z_1^2 + \dots + Z_k^2$.
- Write down $M_{Y_3}(t)$. [2]
 - Use differentiation to find $E(Y_3)$ and $\text{Var}(Y_3)$. [2]
 - W is independent of Y_3 and has the same distribution as Y_5 . Show that $Y_3 + W$ has the same distribution as Y_8 . [2]
 - Show that $M_{Y_2}(t) = \frac{1}{1 - 2t}$, and use the exponential distribution with rate $\frac{1}{2}$ (distribution function $1 - e^{-x/2}$) to find $P(Y_2 > 4)$. [2]
-
- 30** The discrete random variable X has moment generating function $M_X(t) = \frac{1}{2 - e^t}$.
 X_1, X_2 and X_3 are independent copies of X, and $Y = X_1 + X_2 + X_3$.
You may use the series $(1 - u)^{-3} = \sum_{k=0}^{\infty} \binom{k+2}{2} u^k$.
- Show that X has a geometric distribution on $0, 1, 2, \dots$ with $P(X = x) = \left(\frac{1}{2}\right)^{x+1}$, by expanding $M_X(t)$ as a series in e^t . [3]
 - Show that $M_Y(t) = \frac{1}{8} \left(1 - \frac{e^t}{2}\right)^{-3}$. [2]
 - Find a formula for $P(Y = k)$, and evaluate $P(Y = 2)$. [2]
 - Find $E(Y)$ and $\text{Var}(Y)$ by first differentiating $M_X(t)$ twice. [3]

Applications and Limits

- 1** $X_n \sim B(n, \frac{3}{n})$. The moment generating function of $B(n, p)$ is $(1 - p + pe^t)^n$.
Write down $M_{X_n}(t)$ in the form $(1 + \frac{c}{n})^n$, where c is to be found in terms of t, and state the limit of $M_{X_n}(t)$ as $n \rightarrow \infty$, using $(1 + \frac{x}{n})^n \rightarrow e^x$. [2]
-
- 2** $X \sim \text{Po}(2)$, with moment generating function $M_X(t) = e^{2(e^t - 1)}$. For $t > 0$, $P(X \geq a) \leq e^{-ta} M_X(t)$.
Use $t = 1$ to show that $P(X \geq 6) \leq e^{2e - 8}$, and evaluate this bound to 3 significant figures. [2]
-
- 3** X and Y are independent exponential random variables, each with rate 2 and moment generating function $\frac{2}{2 - t}$.
Write down the moment generating function of $X + Y$. [2]

- 4** The independent random variables $X_1, X_2, \dots, X_{100}$ each have mean 3 and variance 4. Let $S = X_1 + X_2 + \dots + X_{100}$.
Write down $E(S)$ and $\text{Var}(S)$, and use the central limit theorem to state an approximate distribution for S . [2]
- 5** The random variable $S = X_1 + \dots + X_N$, where the X_i are independent and each exponential with rate 2, and N is independent of the X_i with $P(N = 0) = 0.3$ and $P(N = 1) = 0.7$. For such a random sum $M_S(t) = G_N(M_X(t))$, where $G_N(s) = E(s^N)$.
Find $M_S(t)$. [2]
- 6** Explain why $e^{tX} \geq e^{ta}$ whenever $X \geq a$ and $t > 0$, and say what happens to the inequality when $t < 0$. [2]
- 7** $X_n \sim B(n, \frac{\lambda}{n})$, where $\lambda > 0$ is fixed. The moment generating function of $B(n, p)$ is $(1 - p + pe^t)^n$.
Prove that as $n \rightarrow \infty$ the moment generating function of X_n tends to that of $\text{Po}(\lambda)$. You may use $(1 + \frac{x}{n})^n \rightarrow e^x$. [4]
- 8** X is exponentially distributed with rate 1, so $M_X(t) = \frac{1}{1-t}$ for $t < 1$. For $0 < t < 1$, $P(X \geq a) \leq e^{-ta}M_X(t)$.
For $a = 5$, find the value of t in $(0, 1)$ which gives the smallest bound, and show that the best bound is $5e^{-4}$. Compare it with the exact value of $P(X \geq 5)$. [5]
- 9** The number of claims N received by an insurer in a day has a Poisson distribution with mean 4. Each claim is for 1 unit with probability $\frac{1}{2}$ and for 2 units with probability $\frac{1}{2}$, independently. S is the total claimed in a day.
Given that $M_S(t) = \exp[\lambda(M_X(t) - 1)]$ where $N \sim \text{Po}(\lambda)$ and X is the size of one claim, find $E(S)$ and $\text{Var}(S)$ using the cumulant generating function $K(t) = \ln M_S(t)$. [5]
- 10** $X_1, \dots, X_n$ are independent exponential random variables, each with rate λ and moment generating function $\frac{\lambda}{\lambda - t}$. Let $S = X_1 + \dots + X_n$.
Show that $M_S(t) = (\frac{\lambda}{\lambda - t})^n$, and use $K(t) = \ln M_S(t)$ to show that $E(S) = \frac{n}{\lambda}$ and $\text{Var}(S) = \frac{n}{\lambda^2}$. [4]
- 11** The random variable X has mean μ and variance σ^2 , and $Y = \frac{X - \mu}{\sigma}$. The moment generating function of Y has a Maclaurin series $M_Y(t) = 1 + E(Y)t + \frac{E(Y^2)}{2}t^2 + \dots$
Show that $M_Y(t) = 1 + \frac{1}{2}t^2 + (\text{terms in } t^3 \text{ and higher powers})$, and write down the corresponding expression for $M_Y(\frac{t}{\sqrt{n}})$ in terms of n . [4]
- 12** $Z \sim N(0, 1)$ has moment generating function $e^{t^2/2}$. For $t > 0$, $P(Z \geq a) \leq e^{-ta}M_Z(t)$.
Show that the choice $t = a$ gives $P(Z \geq a) \leq e^{-a^2/2}$, and evaluate this bound for $a = 2$. Compare it with the exact value 0.02275. [4]
- 13** X is a random variable with moment generating function $M_X(t)$. Prove that for $t > 0$, $P(X \geq a) \leq e^{-ta}M_X(t)$. (You may assume that $E(Y) \geq k P(Y \geq k)$ for a non-negative random variable Y and $k > 0$.) [4]

- 14** X_1 and X_2 are independent exponential random variables, each with rate $\lambda = 1.5$. A gamma distribution with parameters α and λ has pdf $\frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}$ for $x > 0$ and moment generating function $(\frac{\lambda}{\lambda - t})^\alpha$.
Let $S = X_1 + X_2$. Show that S has pdf $\lambda^2 s e^{-\lambda s}$ for $s > 0$, and find $P(S > 1)$, giving your answer to 3 significant figures. [5]
- 15** $X \sim B(20, 0.5)$, with moment generating function $M_X(t) = (0.5 + 0.5e^t)^{20}$. For $t > 0$, $P(X \geq 15) \leq e^{-15t} M_X(t)$.
Find the value of e^t which minimises this bound, and hence show that $P(X \geq 15) \leq \frac{2^{20}}{3^{15}}$.
Evaluate the bound and compare it with the exact probability. [6]
- 16** The number of eggs N laid by an insect is $Po(\lambda)$. Each egg independently hatches with probability p . Let Y be the number of eggs which hatch. The moment generating function of $B(n, p)$ is $(q + pe^t)^n$, where $q = 1 - p$, and the probability generating function of $Po(\lambda)$ is $G_N(s) = e^{\lambda(s-1)}$.
By conditioning on N , show that $M_Y(t) = G_N(q + pe^t)$, and hence prove that $Y \sim Po(\lambda p)$. [6]
- 17** $X \sim Po(3)$, with moment generating function $e^{3(e^t - 1)}$. For $t > 0$, $P(X \geq a) \leq e^{-ta} M_X(t)$.
Show that choosing $e^t = \frac{a}{3}$ gives $P(X \geq a) \leq e^{a-3} (\frac{3}{a})^a$, and find the value of this bound for $a = 9$. Compare it with the exact value of $P(X \geq 9)$. [5]
- 18** $X \sim Po(10)$, with moment generating function $M_X(t) = e^{10(e^t - 1)}$.
(a) Show that for $t > 0$, $P(X \leq a) \leq e^{ta} M_X(-t)$. [2]
(b) Show that the best choice has $e^{-t} = \frac{a}{10}$, and evaluate the resulting bound on $P(X \leq 4)$.
Compare it with the exact value. [3]
- 19** $X_n \sim B(n, \frac{\lambda}{n})$, where $\lambda > 0$ is fixed and n is large. A student writes:
" $M_{X_n}(t) = (1 + \frac{\lambda(e^t - 1)}{n})^n$. As $n \rightarrow \infty$, $\frac{\lambda(e^t - 1)}{n} \rightarrow 0$, so the bracket tends to 1 and the limit is $1^n = 1$. So X_n tends to the constant 1."
(a) Identify the error in the student's argument. [2]
(b) State the correct limit of $M_{X_n}(t)$, and identify the limiting distribution. [3]
- 20** Z_1, Z_2, Z_3, Z_4 are independent standard normal random variables, and $Y = Z_1^2 + Z_2^2 + Z_3^2 + Z_4^2$. The moment generating function of Z_i^2 is $(1 - 2t)^{-1/2}$ for $t < \frac{1}{2}$. The gamma distribution with shape 2 and rate λ has pdf $\lambda^2 y e^{-\lambda y}$ for $y > 0$ and moment generating function $(\frac{\lambda}{\lambda - t})^2$.
(a) Show that $M_Y(t) = (1 - 2t)^{-2}$. [2]
(b) Show that Y has pdf $\frac{y}{4} e^{-y/2}$ for $y > 0$. [2]
(c) Find $P(Y > 6)$. [2]

- 21** $X_1, X_2, \dots$ are independent identically distributed random variables with mean μ , variance $\sigma^2 > 0$ and a moment generating function M which exists near $t = 0$. Let $Y_i = \frac{X_i - \mu}{\sigma}$ and $Z_n = \frac{Y_1 + \dots + Y_n}{\sqrt{n}}$. Near 0, $M_Y(s) = 1 + \frac{1}{2}s^2 + o(s^2)$, where $o(s^2)/s^2 \rightarrow 0$ as $s \rightarrow 0$.
- (a) Show that $M_{Z_n}(t) = [M_Y(\frac{t}{\sqrt{n}})]^n$. [2]
- (b) Show that $\ln M_{Z_n}(t) \rightarrow \frac{1}{2}t^2$ as $n \rightarrow \infty$. You may use $\ln(1 + u) = u + O(u^2)$ for small u . [3]
- (c) Hence state the limiting distribution of Z_n , naming the theorem and the result about moment generating functions that you use. [2]
-
- 22** $X_1, X_2, \dots$ are independent exponential random variables, each with rate 1 and moment generating function $\frac{1}{1-t}$. Let $S_n = X_1 + \dots + X_n$ and $Z_n = \frac{S_n - n}{\sqrt{n}}$. You may use $\ln(1 - u) = -u - \frac{u^2}{2} - \frac{u^3}{3} - \dots$ for $|u| < 1$.
- (a) Show that $M_{Z_n}(t) = e^{-t\sqrt{n}}(1 - \frac{t}{\sqrt{n}})^{-n}$. [3]
- (b) Show that $\ln M_{Z_n}(t) = \frac{1}{2}t^2 + \frac{t^3}{3\sqrt{n}} + (\text{terms which tend to 0 faster as } n \rightarrow \infty)$. [3]
- (c) Hence state the limit of $M_{Z_n}(t)$ as $n \rightarrow \infty$ and interpret it. [1]
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- 23** Y_r is the number of failures before the r -th success in independent trials, each with success probability $p_r = \frac{r}{r + \mu}$, where $\mu > 0$ is fixed. Y_r has moment generating function $(1 - q_r e^t)^{-r} p_r^r$, where $q_r = 1 - p_r$. Show that $M_{Y_r}(t) = (1 - \frac{\mu(e^t - 1)}{r})^{-r}$, and prove that as $r \rightarrow \infty$ it tends to the moment generating function of $\text{Po}(\mu)$. You may use $(1 + \frac{x}{r})^r \rightarrow e^x$. [6]
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- 24** A random number N of independent exponential random variables X_i (rate λ) are added, where N has a geometric distribution with $P(N = n) = q^{n-1}p$ for $n = 1, 2, 3, \dots$ and $q = 1 - p$. Let $S = X_1 + \dots + X_N$. For a random sum $M_S(t) = G_N(M_X(t))$, where $G_N(s) = E(s^N) = \frac{ps}{1 - qs}$, and $M_X(t) = \frac{\lambda}{\lambda - t}$.
- (a) Show that $M_S(t) = \frac{p\lambda}{p\lambda - t}$, and hence identify the distribution of S . [5]
- (b) For $p = 0.2$ and $\lambda = 3$ find $P(S > 5)$. [2]
-
- 25** $X_\lambda \sim \text{Po}(\lambda)$ has moment generating function $e^{\lambda(e^t - 1)}$. Let $Z_\lambda = \frac{X_\lambda - \lambda}{\sqrt{\lambda}}$. Show that $M_{Z_\lambda}(t) = \exp[\lambda(e^{t/\sqrt{\lambda}} - 1) - t\sqrt{\lambda}]$, and by expanding the exponential show that $M_{Z_\lambda}(t) \rightarrow e^{t^2/2}$ as $\lambda \rightarrow \infty$. [6]
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- 26** $X_1, X_2, \dots$ are independent identically distributed random variables with mean μ and moment generating function M , which exists near $t = 0$, with $M(s) = 1 + \mu s + o(s)$ as $s \rightarrow 0$. Let $\bar{X}_n = \frac{1}{n}(X_1 + \dots + X_n)$. Show that $M_{\bar{X}_n}(t) \rightarrow e^{\mu t}$ as $n \rightarrow \infty$, and explain what this tells you about $\bar{X}_n$. You may use $\ln(1 + u) = u + O(u^2)$. [6]

- 27** $X_1, \dots, X_n$ are independent, each taking the values $+1$ and -1 with probability $\frac{1}{2}$. Let $S_n = X_1 + \dots + X_n$.
- (a) Show that $M_{X_i}(t) = \cosh t$. [1]
- (b) By comparing the series for $\cosh t$ and $e^{t^2/2}$, show that $\cosh t \leq e^{t^2/2}$ for all real t . (Hint: compare terms in t^{2k} , using $(2k)! \geq 2^k k!$.) [3]
- (c) Hence prove that for $a > 0$, $P(S_n \geq a) \leq e^{-a^2/2n}$. [3]
- (d) Find the bound for $n = 100$ and $a = 30$. [1]
-
- 28** $X = I_1 + \dots + I_n$, where the I_k are independent and I_k takes the value 1 with probability p_k and 0 otherwise. Let $\mu = E(X) = \sum p_k$. You may use $1 + x \leq e^x$ for all real x .
- (a) Show that $M_X(t) \leq e^{\mu(e^t - 1)}$. [3]
- (b) Hence show that for $\delta > 0$, $P(X \geq (1 + \delta)\mu) \leq \left(\frac{e^\delta}{(1 + \delta)^{1+\delta}}\right)^\mu$. [4]
- (c) A fair-coin game has $n = 20$ independent trials, so $\mu = 10$. Use the bound with $\delta = 1$ to bound $P(X \geq 20)$, and give the value of the bound to 3 significant figures. [2]
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- 29** $S = X_1 + \dots + X_N$, where $X_1, X_2, \dots$ are independent identically distributed with moment generating function $M_X(t)$, and N is independent of the X_i with moment generating function $M_N(t)$. For a random sum $M_S(t) = M_N(\ln M_X(t))$. Let $K_X(t) = \ln M_X(t)$ and $K_N(t) = \ln M_N(t)$, with $K'(0) = \text{mean}$ and $K''(0) = \text{variance}$.
- (a) Show that $K_S(t) = K_N(K_X(t))$. [2]
- (b) Differentiate to show that $E(S) = E(N)E(X)$. [2]
- (c) Show that $\text{Var}(S) = E(N)\text{Var}(X) + \text{Var}(N)(E(X))^2$. [4]
- (d) $N \sim \text{Po}(\lambda)$. Show that $\text{Var}(S) = \lambda E(X^2)$. [1]
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- 30** The number of claims N an insurer receives in a year is $\text{Po}(3)$. Each claim (in thousands of pounds) is independent and exponentially distributed with rate 1. S is the total claimed in a year. For a Poisson random sum, $M_S(t) = \exp[\lambda(M_X(t) - 1)]$. For $t > 0$, $P(S \geq a) \leq e^{-ta}M_S(t)$.
- (a) Show that $M_S(t) = \exp\left[\frac{3t}{1-t}\right]$ for $t < 1$. [2]
- (b) Use $K(t) = \ln M_S(t)$ to find $E(S)$ and $\text{Var}(S)$. [3]
- (c) Find the value of t in $(0, 1)$ which gives the best bound on $P(S \geq 12)$, show that it is a minimum, and give the bound. [4]