

Matrix Notation and Operations

1 (2 marks)

A has 2 rows and 3 columns, so its order is 2×3

B1

The element in row 2, column 3 is 6

B1

Answer: 2×3 ; 6

2 marks

2 (2 marks)

$$A + B = \begin{pmatrix} 5 & 4 \\ 1 & 3 \end{pmatrix}$$

M1 A1

Answer: $\begin{pmatrix} 5 & 4 \\ 1 & 3 \end{pmatrix}$

2 marks

3 (2 marks)

$$3I = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

M1

$$A - 3I = \begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix}$$

A1

Answer: $\begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix}$

2 marks

4 (2 marks)

$$3A = \begin{pmatrix} 6 & -3 & 12 \\ 0 & 9 & -15 \end{pmatrix}$$

M1 A1

Answer: $\begin{pmatrix} 6 & -3 & 12 \\ 0 & 9 & -15 \end{pmatrix}$

2 marks

5 (2 marks)

$P + Q$ is defined because P and Q have the same order

B1

$P + R$ is not defined because the orders 2×3 and 3×2 differ

B1

Answer: $P + Q$ is defined; $P + R$ is not

2 marks

6 (2 marks)

$$A^T = \begin{pmatrix} 1 & -2 & 5 \\ 4 & 0 & 3 \end{pmatrix}$$

B1

A^T has order 2×3

B1

Answer: $A^T = \begin{pmatrix} 1 & -2 & 5 \\ 4 & 0 & 3 \end{pmatrix}$, order 2×3

2 marks

7 (3 marks)

$$2A = \begin{pmatrix} 4 & -2 \\ 0 & 8 \end{pmatrix} \text{ and } 3B = \begin{pmatrix} 3 & 9 \\ -6 & 15 \end{pmatrix}$$

M1

$$2A - 3B = \begin{pmatrix} 1 & -11 \\ 6 & -7 \end{pmatrix}$$

M1 A1

Answer: $\begin{pmatrix} 1 & -11 \\ 6 & -7 \end{pmatrix}$

3 marks

8 (3 marks)

$$A + B = \begin{pmatrix} b + 3 & 3 \\ a - 1 & c + 1 \end{pmatrix}; \text{equate corresponding elements}$$

M1

$3 + b = 0$ gives $b = -3$ and $-2 + 5 = d$ gives $d = 3$

A1

$a - 1 = 4$ gives $a = 5$ and $1 + c = -2$ gives $c = -3$

A1

Answer: $a = 5$, $b = -3$, $c = -3$, $d = 3$

3 marks

9 (3 marks)

$$3X = A - 2B = \begin{pmatrix} 2 & -2 \\ 12 & 6 \end{pmatrix}$$

M1 A1

$$X = \begin{pmatrix} \frac{2}{3} & -\frac{2}{3} \\ 4 & 2 \end{pmatrix}$$

A1

$$\text{Answer: } X = \begin{pmatrix} \frac{2}{3} & -\frac{2}{3} \\ 4 & 2 \end{pmatrix}$$

3 marks

10 (3 marks)

$$2A - I = \begin{pmatrix} 2k - 1 & 6 \\ 4 & 2k - 1 \end{pmatrix}$$

M1

$$2k - 1 = 9, \text{ so } k = 5$$

M1

The other elements, 6 and 4, agree, so $k = 5$

A1

Answer: $k = 5$

3 marks

11 (3 marks)

$$x + y = 7 \text{ and } x - y = 1$$

M1

$$\text{Adding: } 2x = 8, \text{ so } x = 4$$

A1

$$y = 3$$

A1

$$\text{Answer: } x = 4, y = 3$$

3 marks

12 (4 marks)

$$A^T = \begin{pmatrix} 2 & 4 & 1 \\ -1 & 0 & 7 \\ 3 & 5 & -2 \end{pmatrix}$$

B1

$$A + A^T = \begin{pmatrix} 4 & 3 & 4 \\ 3 & 0 & 12 \\ 4 & 12 & -4 \end{pmatrix}$$

M1 A1

 $A + A^T$ is symmetric (it equals its own transpose)

E1

$$\text{Answer: } A + A^T = \begin{pmatrix} 4 & 3 & 4 \\ 3 & 0 & 12 \\ 4 & 12 & -4 \end{pmatrix}, \text{ which is symmetric}$$

4 marks

13 (3 marks)

$$a_{11} = 2, a_{12} = 3, a_{13} = 4 \text{ and } a_{21} = 5, a_{22} = 6, a_{23} = 7$$

M1

$$A = \begin{pmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{pmatrix}$$

A1

$$3A = \begin{pmatrix} 6 & 9 & 12 \\ 15 & 18 & 21 \end{pmatrix}$$

B1

$$\text{Answer: } A = \begin{pmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{pmatrix}, 3A = \begin{pmatrix} 6 & 9 & 12 \\ 15 & 18 & 21 \end{pmatrix}$$

3 marks

14 (5 marks)

$$pA + qB = \begin{pmatrix} p + 2q & 2p - q \\ q & p + 3q \end{pmatrix}$$

M1

$$\text{Bottom left: } q = -2. \text{ Top left: } p + 2q = -1$$

M1 A1

$$\text{So } p = 3; \text{ check top right: } 2p - q = 8 \text{ (correct) and bottom right: } p + 3q = -3 \text{ (correct)}$$

M1 A1

$$\text{Answer: } p = 3, q = -2$$

5 marks

15 (5 marks)

Multiply the first equation by 3 and add: $7X = 3A + B$

M1

$$7X = \begin{pmatrix} 16 & 6 \\ -6 & 10 \end{pmatrix}$$

A1

$$X = \begin{pmatrix} \frac{16}{7} & \frac{6}{7} \\ -\frac{6}{7} & \frac{10}{7} \end{pmatrix}$$

A1

$$Y = A - 2X = \begin{pmatrix} \frac{3}{7} & -\frac{5}{7} \\ -\frac{2}{7} & \frac{8}{7} \end{pmatrix}$$

M1 A1

$$\text{Answer: } X = \begin{pmatrix} \frac{16}{7} & \frac{6}{7} \\ -\frac{6}{7} & \frac{10}{7} \end{pmatrix}, Y = \begin{pmatrix} \frac{3}{7} & -\frac{5}{7} \\ -\frac{2}{7} & \frac{8}{7} \end{pmatrix}$$

5 marks

16 (4 marks)

Take $S = \frac{1}{2}(A + A^T)$ and $K = \frac{1}{2}(A - A^T)$

M1

$$S = \begin{pmatrix} 3 & \frac{7}{2} & -1 \\ \frac{7}{2} & 4 & 1 \\ -1 & 1 & 2 \end{pmatrix}$$

A1

$$K = \begin{pmatrix} 0 & \frac{5}{2} & -1 \\ -\frac{5}{2} & 0 & 4 \\ 1 & -4 & 0 \end{pmatrix}$$

A1

$S + K = A$, $S^T = S$ and $K^T = -K$, as required

E1

$$\text{Answer: } S = \begin{pmatrix} 3 & \frac{7}{2} & -1 \\ \frac{7}{2} & 4 & 1 \\ -1 & 1 & 2 \end{pmatrix}, K = \begin{pmatrix} 0 & \frac{5}{2} & -1 \\ -\frac{5}{2} & 0 & 4 \\ 1 & -4 & 0 \end{pmatrix}$$

4 marks

17 (4 marks)

Let A have elements a_{ij} and B have elements b_{ij} . $A + B$ has element $a_{ij} + b_{ij}$ in row i , column j

M1

$(A + B)^T$ has $a_{ji} + b_{ji}$ in row i , column j

M1

A^T has a_{ji} and B^T has b_{ji} in row i , column j , so $A^T + B^T$ has element $a_{ji} + b_{ji}$

M1

The two matrices have identical elements, so $(A + B)^T = A^T + B^T$

E1

Answer: $(A + B)^T = A^T + B^T$

4 marks

18 (4 marks)

$$\text{Total sales} = P + Q = \begin{pmatrix} 2200 & 1770 & 910 \\ 1850 & 2100 & 1120 \end{pmatrix}$$

M1 A1

$$10\% \text{ increase: } 1.1(P + Q) = \begin{pmatrix} 2420 & 1947 & 1001 \\ 2035 & 2310 & 1232 \end{pmatrix}$$

M1 A1

$$\text{Answer: } P + Q = \begin{pmatrix} 2200 & 1770 & 910 \\ 1850 & 2100 & 1120 \end{pmatrix}; 1.1(P + Q) = \begin{pmatrix} 2420 & 1947 & 1001 \\ 2035 & 2310 & 1232 \end{pmatrix}$$

4 marks

19 (5 marks)

$$\text{Let } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \text{ so } A + 2A^T = \begin{pmatrix} 3a & b + 2c \\ 2b + c & 3d \end{pmatrix}$$

M1

$$3a = 5 \text{ and } 3d = 4, \text{ so } a = \frac{5}{3} \text{ and } d = \frac{4}{3}$$

M1 A1

$$b + 2c = 7 \text{ and } c + 2b = -1; \text{ solving gives } b = -3 \text{ and } c = 5$$

M1 A1

$$\text{Answer: } A = \begin{pmatrix} \frac{5}{3} & -3 \\ 5 & \frac{4}{3} \end{pmatrix}$$

5 marks**20 (6 marks)**

$$B^T = \begin{pmatrix} 1 & 4 \\ y & x \end{pmatrix}$$

B1

$$2A + B^T = \begin{pmatrix} 2x + 1 & 8 \\ 3y & x + 6 \end{pmatrix}$$

M1

$$\text{Top left: } 2x + 1 = 5, \text{ so } x = 2$$

M1 A1

$$\text{Bottom left: } 3y = -3, \text{ so } y = -1; \text{ the other two elements (8 and } 6 + x = 8) \text{ are then consistent}$$

M1 A1

$$\text{Answer: } x = 2, y = -1$$

6 marks**21 (7 marks)**

$$K^T = -K \text{ means } k_{ji} = -k_{ij} \text{ for every pair of elements}$$

M1

$$x = -2 \text{ (from } k_{21} = 2), y = 5 \text{ (from } k_{32} = -5), z = -3 \text{ (from } k_{13} = 3)$$

M1 A1 A1

$$K = \begin{pmatrix} 0 & -2 & 3 \\ 2 & 0 & 5 \\ -3 & -5 & 0 \end{pmatrix}$$

B1

$$K^T + 3K = -K + 3K = 2K = \begin{pmatrix} 0 & -4 & 6 \\ 4 & 0 & 10 \\ -6 & -10 & 0 \end{pmatrix}$$

M1 A1

$$\text{Answer: } x = -2, y = 5, z = -3; K^T + 3K = \begin{pmatrix} 0 & -4 & 6 \\ 4 & 0 & 10 \\ -6 & -10 & 0 \end{pmatrix}$$

7 marks**22 (6 marks)**

$$pA + qB = \begin{pmatrix} p + 2q & 2p + q \\ 3p + 4q & 4p + 3q \end{pmatrix}$$

M1

$$\text{Top row: } p + 2q = 1 \text{ and } 2p + q = 0, \text{ giving } p = -\frac{1}{3} \text{ and } q = \frac{2}{3}$$

M1 A1

$$\text{Bottom row: } 3p + 4q = -1 + \frac{8}{3} = \frac{5}{3}, \text{ but the required element is } 0$$

M1

$$\text{This is a contradiction, so no such } p \text{ and } q \text{ exist}$$

A1

$$\text{Two matrices give only a 2-parameter family of combinations, whereas four conditions have to be met}$$

E1

$$\text{Answer: No such } p, q \text{ exist}$$

6 marks

23 (6 marks)Transpose the first equation: $X^T + Y = A^T$

M1

Add to the second equation: $2X^T = A^T + B$

M1

$$X^T = \begin{pmatrix} 3 & -2 \\ \frac{1}{2} & 4 \end{pmatrix}, \text{ so } X = \begin{pmatrix} 3 & \frac{1}{2} \\ -2 & 4 \end{pmatrix}$$

A1 A1

$$Y = A^T - X^T = \begin{pmatrix} -1 & 1 \\ \frac{1}{2} & -1 \end{pmatrix}$$

M1 A1

$$\text{Answer: } X = \begin{pmatrix} 3 & \frac{1}{2} \\ -2 & 4 \end{pmatrix}, Y = \begin{pmatrix} -1 & 1 \\ \frac{1}{2} & -1 \end{pmatrix}$$

6 marks

24 (6 marks) $A^T = -A$ and $B^T = -B$

B1

 $(A + B)^T = A^T + B^T = -A - B = -(A + B)$

M1 A1

So $A + B$ is skew-symmetric, and it is symmetric only if it is the zero matrix

E1

Example: $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$, $A + B = \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix}$, which is skew-symmetric and not

B1 B1

zero

Answer: $A + B$ is skew-symmetric; claim false

6 marks

25 (6 marks)

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}$$

B1

$$B = \begin{pmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{pmatrix}$$

B1

$$X = B - 2A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 4 & 3 \\ 0 & 3 & 6 \end{pmatrix}$$

M1 A1

 $X^T = X$ because $B^T = B$ and $A^T = A$ ($a_{ij} = a_{ji}$ and $b_{ij} = b_{ji}$), so X is symmetric

E1 E1

$$\text{Answer: } X = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 4 & 3 \\ 0 & 3 & 6 \end{pmatrix}$$

6 marks

26 (6 marks)

$$pA + qB + rC = \begin{pmatrix} p+r & q-r \\ 2p+q & p+q+2r \end{pmatrix}$$

M1

Top left: $p + r = 1$. Top right: $q - r = 4$. Bottom left: $2p + q = 7$

M1 A1

Solving: $r = 1 - p$, $q = 4 + r = 5 - p$, so $2p + 5 - p = 7$, $p = 2$, $q = 3$, $r = -1$

M1 A1

Bottom right: $p + q + 2r = 2 + 3 - 2 = 3$, which equals 3 (correct)

B1

Answer: $p = 2$, $q = 3$, $r = -1$

6 marks

27 (10 marks)

$(A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$, so $A + A^T$ is symmetric M1 A1

$(A - A^T)^T = A^T - A = -(A - A^T)$, so $A - A^T$ is skew-symmetric M1 A1

$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$, a symmetric plus a skew-symmetric matrix M1 A1

Suppose $A = S_1 + K_1 = S_2 + K_2$ with S symmetric and K skew-symmetric. Then $D = S_1 - S_2 = K_2 - K_1$ M1

$D^T = D$ (a difference of symmetric matrices) and $D^T = -D$ (a difference of skew-symmetric matrices) M1

So $D = -D$, hence $D = 0$ and $S_1 = S_2$, $K_1 = K_2$ A1 E1

Answer: $A + A^T$ is symmetric, $A - A^T$ is skew-symmetric, and the decomposition $A = S + K$ is unique. 10 marks

28 (8 marks)

Let $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, so $X + X^T = \begin{pmatrix} 2a & b+c \\ b+c & 2d \end{pmatrix}$ M1

$2a = 4$, $2d = 6$ and $b + c = 2$, so $a = 2$, $d = 3$ and $b = 1 + t$, $c = 1 - t$ for a constant t M1 A1

Sum of squares: $4 + 9 + (1 + t)^2 + (1 - t)^2 = 23$ M1

$15 + 2t^2 = 23$, so $t^2 = 4$ M1 A1

$t = 2$ or $t = -2$ A1

$X = \begin{pmatrix} 2 & 3 \\ -1 & 3 \end{pmatrix}$ or $X = \begin{pmatrix} 2 & -1 \\ 3 & 3 \end{pmatrix}$ A1

Answer: $X = \begin{pmatrix} 2 & 3 \\ -1 & 3 \end{pmatrix}$ or $\begin{pmatrix} 2 & -1 \\ 3 & 3 \end{pmatrix}$ 8 marks

29 (8 marks)

Let P have diagonal elements p_1, p_2 with $p_1 + p_2 = 0$, and Q have q_1, q_2 with $q_1 + q_2 = 0$ B1

$P + Q$ has diagonal sum $(p_1 + q_1) + (p_2 + q_2) = 0$ and kP has diagonal sum $k(p_1 + p_2) = 0$, so both belong to T M1 A1

A general member of T is $\begin{pmatrix} s & u \\ v & -s \end{pmatrix}$ B1

$aE + bF + cG = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ M1

Equating elements gives $a = s$, $b = u$, $c = v$, and these are the only possible values M1 E1

For the given matrix: $a = 3$, $b = -2$, $c = 5$ A1

Answer: $a = 3$, $b = -2$, $c = 5$ 8 marks

30 (9 marks)

Take the transpose of the given equation: $A = 3A^T - I$ M1

Substitute $A^T = 3A - I$: $A = 3(3A - I) - I = 9A - 4I$ M1 A1

So $8A = 4I$ and $A = \frac{1}{2}I$ A1

Check: $A^T = \frac{1}{2}I$ and $3A - I = \frac{1}{2}I$, so the condition holds E1

$B + B^T = 2I$, so with $B = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ the diagonal elements satisfy $2p = 2$ and $2s = 2$, and $q + r = 0$ M1 A1

$p = s = 1$ and $r = -q$, so $B = \begin{pmatrix} 1 & t \\ -t & 1 \end{pmatrix}$ M1 A1

Answer: $A = \frac{1}{2}I$; $B = \begin{pmatrix} 1 & t \\ -t & 1 \end{pmatrix}$ 9 marks

Matrix Multiplication

1 (2 marks)

$$AB = \begin{pmatrix} 7 & 2 \\ 23 & 8 \end{pmatrix}$$

M1 A1

$$\text{Answer: } \begin{pmatrix} 7 & 2 \\ 23 & 8 \end{pmatrix}$$

2 marks

2 (2 marks)

AB is defined (columns of A = rows of B) and has order 2×4

B1

BA is not defined, since B has 4 columns but A has only 2 rows

B1

Answer: AB: 2×4 ; BA not defined

2 marks

3 (2 marks)

$$A^2 = \begin{pmatrix} 7 & -8 \\ 16 & 23 \end{pmatrix}$$

M1 A1

$$\text{Answer: } \begin{pmatrix} 7 & -8 \\ 16 & 23 \end{pmatrix}$$

2 marks

4 (2 marks)

$$\begin{pmatrix} -4 \\ -2 \end{pmatrix}$$

M1 A1

$$\text{Answer: } \begin{pmatrix} -4 \\ -2 \end{pmatrix}$$

2 marks

5 (2 marks)

Row times column: $4 - 2 + 6 = 8$, a 1×1 matrix

M1 A1

Answer: (8), order 1×1

2 marks

6 (3 marks)

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

B1

$$AI = \begin{pmatrix} 3 & -1 \\ 2 & 5 \end{pmatrix} \text{ and } IA = \begin{pmatrix} 3 & -1 \\ 2 & 5 \end{pmatrix}$$

M1

$AI = IA = A$: multiplying by the identity matrix leaves A unchanged

A1

$$\text{Answer: } I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; AI = IA = A$$

3 marks

7 (4 marks)

$$AB = \begin{pmatrix} 4 & -1 \\ 2 & 2 \end{pmatrix}$$

M1 A1

$$BA = \begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix}$$

A1

$$AB \neq BA$$

B1

$$\text{Answer: } AB = \begin{pmatrix} 4 & -1 \\ 2 & 2 \end{pmatrix}, BA = \begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix}, AB \neq BA$$

4 marks

8 (4 marks)

$$A^2 = \begin{pmatrix} 1 & -6 \\ 18 & 13 \end{pmatrix}$$

M1 A1

$$A^2 - 3A + 2I = \begin{pmatrix} -3 & -3 \\ 9 & 3 \end{pmatrix}$$

M1 A1

$$\text{Answer: } \begin{pmatrix} -3 & -3 \\ 9 & 3 \end{pmatrix}$$

4 marks

9 (3 marks)

$$\begin{pmatrix} -1 \\ -2 \\ 7 \end{pmatrix}$$

M1 A1 A1

Answer: $\begin{pmatrix} -1 \\ -2 \\ 7 \end{pmatrix}$

3 marks

10 (3 marks)

$$AB = \begin{pmatrix} k & 4 - k \\ 1 & 5 \end{pmatrix}$$

M1 A1

$$k = 4$$

A1

Answer: k = 4

3 marks

11 (4 marks)

$$A^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

M1 A1

$$A^3 = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$$

A1

$$A^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

A1

Answer: $A^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$

4 marks

12 (3 marks)

$$AB = \begin{pmatrix} 2 & 9 \\ 8 & -7 \end{pmatrix}$$

M1 A1

BA exists and has order 3×3

B1

Answer: $AB = \begin{pmatrix} 2 & 9 \\ 8 & -7 \end{pmatrix}$; BA has order 3×3

3 marks

13 (4 marks)

$$\begin{pmatrix} xy + 3 \\ 3y + 6 \end{pmatrix} = \begin{pmatrix} 5 \\ 9 \end{pmatrix}$$

M1

$$3 + xy = 5 \text{ and } 6 + 3y = 9$$

M1

$$y = 1 \text{ and then } x = 2$$

A1 A1

Answer: x = 2, y = 1

4 marks

14 (6 marks)

$$A^2 = \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix}$$

M1 A1

$$aA + bI = \begin{pmatrix} 2a + b & a \\ -a & b \end{pmatrix}, \text{ so } 2a + b = 3 \text{ and } a = 2$$

M1

$$a = 2 \text{ and } b = -1$$

A1

$$A^3 = A \cdot A^2 = 2A^2 - A = 2(2A - I) - A = 3A - 2I$$

M1 A1

Answer: a = 2, b = -1; $A^3 = 3A - 2I$

6 marks

15 (5 marks)

$$A + B = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}, \text{ so } (A + B)^2 = \begin{pmatrix} 11 & 14 \\ 7 & 18 \end{pmatrix} \quad \text{M1 A1}$$

$$A^2 = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}, AB = \begin{pmatrix} 4 & 6 \\ 1 & 3 \end{pmatrix}, B^2 = \begin{pmatrix} 4 & 0 \\ 5 & 9 \end{pmatrix} \quad \text{M1}$$

$$A^2 + 2AB + B^2 = \begin{pmatrix} 13 & 16 \\ 7 & 16 \end{pmatrix}, \text{ which differs from } (A + B)^2 \quad \text{A1}$$

$$(A + B)^2 = A^2 + AB + BA + B^2, \text{ since } AB \neq BA \text{ in general} \quad \text{B1}$$

$$\text{Answer: } (A+B)^2 = \begin{pmatrix} 11 & 14 \\ 7 & 18 \end{pmatrix}; A^2 + 2AB + B^2 = \begin{pmatrix} 13 & 16 \\ 7 & 16 \end{pmatrix} \quad \text{5 marks}$$

16 (4 marks)

$$AB = \begin{pmatrix} ap + bx & aq + by \\ cp + dx & cq + dy \end{pmatrix} \quad \text{M1}$$

$$(AB)^T = \begin{pmatrix} ap + bx & cp + dx \\ aq + by & cq + dy \end{pmatrix} \quad \text{A1}$$

$$B^T A^T = \begin{pmatrix} p & x \\ q & y \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} ap + bx & cp + dx \\ aq + by & cq + dy \end{pmatrix} \quad \text{M1}$$

$$\text{This equals } (AB)^T, \text{ as required} \quad \text{E1}$$

$$\text{Answer: } (AB)^T = B^T A^T \quad \text{4 marks}$$

17 (5 marks)

$$AB = \begin{pmatrix} a + c & b + d \\ c & d \end{pmatrix} \text{ and } BA = \begin{pmatrix} a & a + b \\ c & c + d \end{pmatrix} \quad \text{M1 A1}$$

$$\text{Equating elements: } a + c = a, b + d = a + b, c = c, d = c + d \quad \text{M1}$$

$$\text{So } c = 0 \text{ and } d = a \quad \text{A1}$$

$$B = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \text{ where } a \text{ and } b \text{ are arbitrary constants} \quad \text{A1}$$

$$\text{Answer: } B = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \quad \text{5 marks}$$

18 (4 marks)

$$P^2 = \begin{pmatrix} \frac{7}{10} & \frac{9}{20} \\ \frac{3}{10} & \frac{11}{20} \end{pmatrix} \quad \text{M1 A1}$$

$$P^2 \begin{pmatrix} 1000 \\ 500 \end{pmatrix} = \begin{pmatrix} 925 \\ 575 \end{pmatrix} \quad \text{M1}$$

$$\text{After two weeks 925 use bus A and 575 use bus B} \quad \text{A1}$$

$$\text{Answer: A: 925, B: 575} \quad \text{4 marks}$$

19 (5 marks)

$$p = \begin{pmatrix} 4 \\ 7 \\ 9 \end{pmatrix} \quad \text{B1}$$

$$Np = \begin{pmatrix} 605 \\ 685 \\ 590 \end{pmatrix} \quad \text{M1 A1}$$

$$\text{Each element is the total takings of one shop, in pounds} \quad \text{E1}$$

$$\text{Total takings} = 605 + 685 + 590 = \text{£1880} \quad \text{B1}$$

$$\text{Answer: } Np = \begin{pmatrix} 605 \\ 685 \\ 590 \end{pmatrix}; \text{ total £1880} \quad \text{5 marks}$$

20 (5 marks)

$$N^2 = \begin{pmatrix} 5 & 1 & 2 \\ 1 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix}$$

M1 A1 A1

Element (1, 3) of N^2 is the number of routes from A to C using exactly two roads (a road from A to a village, then a road from that village to C)

E1

There are 2 such routes

A1

Answer: $N^2 = \begin{pmatrix} 5 & 1 & 2 \\ 1 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix}$; 2 routes

5 marks

21 (7 marks)

$$(A + B)(A - B) = A^2 - AB + BA - B^2$$

M1

So the condition is $AB = BA$

A1

$$AB = \begin{pmatrix} 4p + q & 5p + 1 \\ 2q + 12 & 17 \end{pmatrix} \text{ and } BA = \begin{pmatrix} q + 2 & pq + 3 \\ 14 & 4p + 15 \end{pmatrix}$$

M1 A1

Bottom left: $2q = 2$ so $q = 1$

M1

Top left: $4p = 2$ so $p = \frac{1}{2}$; the remaining elements are then equal

M1 A1

Answer: $p = \frac{1}{2}, q = 1$

7 marks

22 (6 marks)

$$n = 1: \begin{pmatrix} 2^1 & 2^1 - 1 \\ 0 & 1 \end{pmatrix} = A, \text{ so the result is true for } n = 1$$

B1

$$\text{Assume } A^k = \begin{pmatrix} 2^k & 2^k - 1 \\ 0 & 1 \end{pmatrix} \text{ for some positive integer } k$$

M1

$$A^{k+1} = A^k A = \begin{pmatrix} 2^k & 2^k - 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$$

M1

$$= \begin{pmatrix} 2^{k+1} & 2^k + 2^k - 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2^{k+1} & 2^{k+1} - 1 \\ 0 & 1 \end{pmatrix}$$

M1 A1

True for $n = 1$ and, if true for $n = k$, true for $n = k + 1$, so true for all positive integers n

E1

Answer: $A^n = \begin{pmatrix} 2^n & 2^n - 1 \\ 0 & 1 \end{pmatrix}$

6 marks

23 (7 marks)

$$A^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I$$

M1 A1

$$A^4 = (A^2)^2 = I$$

A1

$$2026 = 4 \times 506 + 2, \text{ so } A^{2026} = (A^4)^{506} A^2 = A^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

M1 A1

$$A^{2027} = A^{2026} A = -A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

M1 A1

Answer: $A^{2026} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, A^{2027} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

7 marks

24 (7 marks)

$$A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

M1 A1

$$(I + A)^n = I^n + nI^{n-1}A + \frac{n(n-1)}{2}I^{n-2}A^2 + \dots$$

M1

Every term with A^2 or a higher power is zero, so $(I + A)^n = I + nA$

M1 A1

$$(I + A)^{10} = I + 10A = \begin{pmatrix} 21 & -40 \\ 10 & -19 \end{pmatrix}$$

M1 A1

$$\text{Answer: } (I + A)^{10} = \begin{pmatrix} 21 & -40 \\ 10 & -19 \end{pmatrix}$$

7 marks

25 (7 marks)

$$AB = \begin{pmatrix} a+2c & b+2d \\ 3a+4c & 3b+4d \end{pmatrix} \text{ and } BA = \begin{pmatrix} a+3b & 2a+4b \\ c+3d & 2c+4d \end{pmatrix}$$

M1 A1

Equating elements: $2c = 3b$, $b + 2d = 2a + 4b$ and $3a + 4c = c + 3d$, so $c = \frac{3b}{2}$ and $d = a + \frac{3b}{2}$

M1 A1

$$B = \begin{pmatrix} a & b \\ \frac{3b}{2} & a + \frac{3b}{2} \end{pmatrix}$$

A1

Writing $B = pI + qA$ gives $\begin{pmatrix} p+q & 2q \\ 3q & p+4q \end{pmatrix}$, so $q = \frac{b}{2}$ and $p = a - \frac{b}{2}$

M1

These values reproduce all four elements, so $B = pI + qA$

A1

$$\text{Answer: } B = \begin{pmatrix} a & b \\ \frac{3b}{2} & a + \frac{3b}{2} \end{pmatrix} = pI + qA$$

7 marks

26 (7 marks)

$$AB = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \text{ so } (AB)^2 = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$$

M1 A1

$$A^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, B^2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \text{ so } A^2B^2 = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$$

M1 A1

The results differ, so the statement is false

E1

$(AB)^2 = ABAB$, which equals $AABB = A^2B^2$ only if $BA = AB$ (cancel A on the left and B on the right)

M1 A1

Answer: False; $(AB)^2 = ABAB$, equal to A^2B^2 only when $AB = BA$

7 marks

27 (10 marks)

$$A^2 = \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix}$$

M1 A1

$$(a + d)A = \begin{pmatrix} a^2 + ad & ab + bd \\ ac + cd & ad + d^2 \end{pmatrix}$$

M1

$A^2 - (a + d)A + (ad - bc)I$ has top left $a^2 + bc - a^2 - ad + ad - bc = 0$; the off-diagonal elements cancel and the bottom right is $bc + d^2 - ad - d^2 + ad - bc = 0$

M1 A1

For T: $a + d = 3$ and $ad - bc = 0 + 2 = 2$, so $T^2 - 3T + 2I = 0$, giving $T^2 = 3T - 2I$

M1 A1

$$T^3 = 3T^2 - 2T = 3(3T - 2I) - 2T = 7T - 6I$$

M1

$$T^4 = 7T^2 - 6T = 7(3T - 2I) - 6T = 15T - 14I, \text{ so } p = 15 \text{ and } q = -14$$

M1 A1

Answer: $p = 15$, $q = -14$

10 marks

28 (8 marks)

$$n = 1: F^1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \text{ and } f_2 = 1, f_1 = 1, f_0 = 0, \text{ so true for } n = 1$$

B1

$$\text{Assume } F^k = \begin{pmatrix} f_{k+1} & f_k \\ f_k & f_{k-1} \end{pmatrix}$$

M1

$$F^{k+1} = F^k F = \begin{pmatrix} f_{k+1} + f_k & f_{k+1} \\ f_k + f_{k-1} & f_k \end{pmatrix}$$

M1

$$= \begin{pmatrix} f_{k+2} & f_{k+1} \\ f_{k+1} & f_k \end{pmatrix} \text{ using the Fibonacci recurrence}$$

A1

True for $n = 1$ and, if true for $n = k$, true for $n = k + 1$, so true for all positive integers n

E1

$$\text{Take determinants: } \det(F^n) = (\det F)^n = (-1)^n$$

M1

$$\text{So } f_{n+1}f_{n-1} - f_n^2 = (-1)^n$$

M1

$$\text{With } n = 6: f_7 f_5 - f_6^2 = 1 \text{ (check: } 13 \times 5 - 64 = 1)$$

A1

Answer: 1

8 marks

29 (9 marks)

$$X^2 = \begin{pmatrix} a^2 + bc & b(a + d) \\ c(a + d) & bc + d^2 \end{pmatrix} = I$$

M1 A1

$$\text{Off-diagonal: } b(a + d) = 0 \text{ and } c(a + d) = 0$$

M1

If $a + d \neq 0$ then $b = c = 0$, so $a^2 = d^2 = 1$ and $a + d \neq 0$ forces $a = d = 1$ or $a = d = -1$: $X = I$ or $X = -I$

M1 A1

$$\text{Otherwise } a + d = 0, d = -a \text{ and } a^2 + bc = 1$$

A1

$$\text{Here } a = 3, \text{ which is neither } 1 \text{ nor } -1, \text{ so } X \text{ is not } \pm I; \text{ hence } d = p = -3 \text{ and } 9 + bc = 1$$

M1 A1

$$b = -\frac{8}{c}, \text{ so } X = \begin{pmatrix} 3 & -\frac{8}{c} \\ c & -3 \end{pmatrix}$$

A1

$$\text{Answer: } X = I, -I, \text{ or } a + d = 0 \text{ with } a^2 + bc = 1; \text{ for the second part } X = \begin{pmatrix} 3 & -\frac{8}{c} \\ c & -3 \end{pmatrix}$$

9 marks

30 (10 marks)

$$N = A - 2I = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

B1

$$N^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

M1 A1

$$N^3 = N^2 N = 0$$

A1

$$A^n = (2I + N)^n = (2I)^n + n(2I)^{n-1}N + \frac{n(n-1)}{2}(2I)^{n-2}N^2 + \frac{n(n-1)(n-2)}{6}(2I)^{n-3}N^3 + \dots$$

M1

Terms containing N^3 or higher powers vanish

A1

$$\text{So } A^n = 2^n I + n2^{n-1}N + \frac{n(n-1)}{2}2^{n-2}N^2$$

E1

$$A^{10} = 1024I + 10 \times 512N + 45 \times 256N^2$$

M1

$$= 1024I + 5120N + 11520N^2 = \begin{pmatrix} 1024 & 5120 & 11520 \\ 0 & 1024 & 5120 \\ 0 & 0 & 1024 \end{pmatrix}$$

M1 A1

$$\text{Answer: } A^{10} = \begin{pmatrix} 1024 & 5120 & 11520 \\ 0 & 1024 & 5120 \\ 0 & 0 & 1024 \end{pmatrix}$$

10 marks

Determinants

1 (2 marks)

$$\det A = (3)(4) - (2)(5) = 2$$

Answer: 2

M1 A1
2 marks

2 (2 marks)

$$\cos \theta \times \cos \theta - (-\sin \theta)(\sin \theta) = \cos^2 \theta + \sin^2 \theta = 1$$

Answer: 1

M1
A1
2 marks

3 (3 marks)

$$\det A = 6 - 6 = 0 \text{ and } \det B = 8 - 6 = 2$$

A is singular because its determinant is zero

Answer: A is singular

M1 A1
B1
3 marks

4 (2 marks)

The determinant is 0

A row of zeros means every term in the expansion is zero

Answer: 0

B1
E1
2 marks

5 (2 marks)

The matrix is upper triangular, so the determinant is the product of the leading diagonal elements

$$(2)(3)(-2) = -12$$

Answer: -12

M1
A1
2 marks

6 (2 marks)

$$\det = 6k - 6 = 0$$

$$k = 1$$

Answer: k = 1

M1
A1
2 marks

7 (3 marks)

$$x(x - 1) - (2)(3) = x^2 - x - 6 = 0$$

$$(x - 3)(x + 2) = 0, \text{ so } x = 3 \text{ or } x = -2$$

Answer: x = 3 or x = -2

M1 A1
A1
3 marks

8 (3 marks)

$$\text{Expanding along the first row: } 2(4 - 0) - (-1)(1 + 10) + 3(0 - 20) = 8 + 11 - 60 = -41$$

Answer: -41

M1 A1
A1
3 marks

9 (3 marks)

$$\det A = 12 - 2 = 10$$

$$\text{Area of image} = |\det A| \times 7 = 70$$

Answer: 70 square units

M1 A1
A1
3 marks

10 (4 marks)

$$\det A = 5 \text{ and } \det B = 10$$

$$AB = \begin{pmatrix} 7 & -4 \\ 23 & -6 \end{pmatrix}, \text{ so } \det(AB) = 50$$

$$\det A \times \det B = 5 \times 10 = 50 = \det(AB)$$

Answer: det(AB) = 50

M1 A1
M1
A1
4 marks

11 (4 marks)

$$(a) \det(A^T) = \det A = 4$$

Answer: 4

B1
1 mark

$$(b) \det(A^2) = (\det A)^2 = 16$$

B1

Answer: 16**1 mark**

$$\begin{aligned} \text{(c) } \det(3A) &= 3^2 \det A \\ &= 36 \end{aligned}$$

M1**A1****Answer: 36****2 marks****12 (4 marks)**

$$\det A = x(x - 1) - 3(-2) = x^2 - x + 6$$

M1 A1

$$\text{The discriminant of } x^2 - x + 6 \text{ is } (-1)^2 - 4 \times 1 \times 6 = -23 < 0$$

M1So $\det A \neq 0$ for every real x , and A is non-singular**A1****Answer: $\det A = x^2 - x + 6$ has no real roots, so A is non-singular for all real x** **4 marks****13 (4 marks)**

$$\det A = 1(5k - 48) - 2(4k - 42) + 3(32 - 35)$$

M1 A1

$$= 5k - 48 - 8k + 84 - 9 = 27 - 3k$$

M1 A is singular when $27 - 3k = 0$, so $k = 9$ **A1****Answer: $k = 9$** **4 marks****14 (5 marks)**

$$\text{Expanding along the first column: } \det A = k(k - 0) - 0 + 1(-3 - 2k)$$

M1 A1

$$= k^2 - 2k - 3$$

A1 A is singular when $(k - 3)(k + 1) = 0$ **M1**

$$k = 3 \text{ or } k = -1$$

A1**Answer: $k = 3$ or $k = -1$** **5 marks****15 (4 marks)**Apply $R_3 \rightarrow R_3 - R_2$ and $R_2 \rightarrow R_2 - R_1$ (this does not change the determinant)**M1**

$$\text{The determinant becomes } \begin{vmatrix} 201 & 202 & 203 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{vmatrix}$$

A1

Two identical rows, so the determinant is 0

E1 A matrix with zero determinant is singular**B1****Answer: 0****4 marks****16 (5 marks)**

$$\det A = 3k - 2$$

B1

$$\text{Area scale factor} = |\det A| = \frac{20}{5} = 4$$

M1

$$3k - 2 = 4 \text{ or } 3k - 2 = -4$$

M1 A1

$$k = 2 \text{ or } k = -\frac{2}{3}$$

A1**Answer: $k = 2$ or $k = -\frac{2}{3}$** **5 marks****17 (4 marks)**

$$A + B = \begin{pmatrix} 3 & -1 \\ 8 & 4 \end{pmatrix}, \text{ so } \det(A + B) = 20$$

M1 A1

$$\det A = 5 \text{ and } \det B = 10, \text{ so } \det A + \det B = 15$$

M1

They are not equal, so the determinant of a sum is not in general the sum of the determinants

E1**Answer: $\det(A + B) = 20$, $\det A + \det B = 15$; not equal****4 marks****18 (5 marks)**

$$\text{Expanding: } x(x^2 - 1) - 1(x - 1) + 1(1 - x)$$

M1 A1

$$= x^3 - x - x + 1 + 1 - x = x^3 - 3x + 2$$

A1

$$(x - 1)^2(x + 2) = (x^2 - 2x + 1)(x + 2) = x^3 - 3x + 2$$

M1Solutions: $x = 1$ (repeated) and $x = -2$ **A1****Answer: $x = 1$ or $x = -2$** **5 marks**

19 (5 marks)

The columns of M are the images of $(1, 0)$ and $(0, 1)$, so $M = \begin{pmatrix} 3 & 1 \\ 1 & 4 \end{pmatrix}$

M1 A1

$$B' = M \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}, \text{ that is } (4, 5)$$

M1 A1

$$\text{Area} = |\det M| \times 1 = 12 - 1 = 11$$

A1

Answer: $M = \begin{pmatrix} 3 & 1 \\ 1 & 4 \end{pmatrix}$, $B' = (4, 5)$, area 11

5 marks

20 (5 marks)

$\det A = 4 - 4 = 0$, so A is singular

M1 A1

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 2y \\ 2x + 4y \end{pmatrix}$$

M1 A1

Every image point has the form $(s, 2s)$ with $s = x + 2y$, so every point of the plane is mapped onto the line $y = 2x$

A1

Answer: Singular; the plane is mapped onto the line $y = 2x$

5 marks

21 (6 marks)

$$R_2 \rightarrow R_2 - R_1 \text{ and } R_3 \rightarrow R_3 - R_1 \text{ give } \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}$$

M1 A1

Expanding along the first column: $(b-a)(c^2-a^2) - (c-a)(b^2-a^2)$

M1 A1

Factor out $(b-a)(c-a)$: $= (b-a)(c-a)[(c+a) - (b+a)]$

M1

$= (b-a)(c-a)(c-b)$

A1

Answer: $(b-a)(c-a)(c-b)$

6 marks

22 (8 marks)

$$(a) \det(AB) = \det A \times \det B = -6$$

M1 A1

Answer: -6

2 marks

$$(b) \det(2A) = 2^3 \det A = 16$$

M1 A1

Answer: 16

2 marks

$$(c) \det(A^T) = \det A = 2 \text{ and } \det(B^{-1}) = \frac{1}{\det B} = -\frac{1}{3}$$

M1

$$\det(A^T B^{-1}) = -\frac{2}{3}$$

A1

Answer: $-\frac{2}{3}$

2 marks

$$(d) \det(A^2 B^3) = (\det A)^2 (\det B)^3 = 4 \times (-27) = -108$$

M1 A1

Answer: -108

2 marks

23 (6 marks)

$\det(A^2) = (\det A)^2$, and $A^2 = A$, so $(\det A)^2 = \det A$

M1 A1

$\det A(\det A - 1) = 0$, so $\det A = 0$ or $\det A = 1$

A1

For example $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ has $A^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = A$ and $\det A = 0$ (not the identity)

M1 A1

$A = I$ has $A^2 = A$ and $\det A = 1$

B1

Answer: $\det A = 0$ or 1

6 marks

24 (6 marks)

$$A - \lambda I = \begin{pmatrix} 2 - \lambda & 0 & 1 \\ 0 & 3 - \lambda & 0 \\ 1 & 0 & 2 - \lambda \end{pmatrix}$$

B1Expanding along the second row: $\det(A - \lambda I) = (3 - \lambda)[(2 - \lambda)^2 - 1]$ **M1 A1**

$$(2 - \lambda)^2 - 1 = (1 - \lambda)(3 - \lambda)$$

M1

$$\det(A - \lambda I) = (3 - \lambda)^2(1 - \lambda) = 0$$

A1

$$\lambda = 3 \text{ (repeated) or } \lambda = 1$$

A1**Answer: $\lambda = 1$ or $\lambda = 3$** **6 marks****25 (7 marks)**

$$AB = \begin{pmatrix} ap + bx & aq + by \\ cp + dx & cq + dy \end{pmatrix}$$

M1 A1

$$\det(AB) = (ap + bx)(cq + dy) - (aq + by)(cp + dx)$$

M1

$$= acpq + adpy + bcqx + bdx - acpq - adqx - bcpx - bdx$$

M1 A1

$$= ad(py - qx) - bc(py - qx) = (ad - bc)(py - qx)$$

M1

$$= \det A \times \det B$$

E1**Answer: $\det(AB) = \det A \times \det B = (ad - bc)(py - qx)$** **7 marks****26 (7 marks)**

$$\det(K^T) = \det K$$

B1

$$\det(-K) = (-1)^3 \det K = -\det K$$

M1 A1

$$K^T = -K, \text{ so } \det K = -\det K, \text{ which gives } 2 \det K = 0 \text{ and } \det K = 0$$

M1 A1

$$\text{Expanding along the first row: } 0 - a(0 + bc) + b(ac - 0)$$

M1

$$= -abc + abc = 0$$

A1**Answer: $\det K = 0$** **7 marks****27 (10 marks)**

$$C_1 \rightarrow C_1 + C_2 + C_3 \text{ gives } \begin{vmatrix} a+b+c & b & c \\ a+b+c & c & a \\ a+b+c & a & b \end{vmatrix}$$

M1 A1

$$\text{Factor out } (a+b+c): (a+b+c) \begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix}$$

M1

$$R_2 \rightarrow R_2 - R_1, R_3 \rightarrow R_3 - R_1: (a+b+c) \begin{vmatrix} 1 & b & c \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix}$$

M1 A1

$$= (a+b+c)[(c-b)(b-c) - (a-c)(a-b)]$$

M1

$$(c-b)(b-c) - (a-c)(a-b) = -(b-c)^2 - (a^2 - ab - ac + bc) = -(a^2 + b^2 + c^2 - ab - bc - ca)$$

A1

$$\text{Singular when } a+b+c=0, \text{ or } a^2 + b^2 + c^2 - ab - bc - ca = 0$$

M1

$$\text{The second factor equals } \frac{1}{2}[(a-b)^2 + (b-c)^2 + (c-a)^2], \text{ which is 0 only if } a=b=c$$

M1

$$\text{So singular if and only if } a+b+c=0 \text{ or } a=b=c$$

A1**Answer: Singular iff $a+b+c=0$ or $a=b=c$** **10 marks****28 (8 marks)**

$$\text{Expanding along the first row: } (1+a)[(1+b)(1+c) - 1] - 1[(1+c) - 1] + 1[1 - (1+b)]$$

M1 A1

$$= (1+a)(b+c+bc) - c - b$$

M1

$$= b+c+bc+ab+ac+abc - b - c = abc + ab + bc + ca$$

A1

$$\text{If } a, b, c > 0 \text{ every term is positive, so the determinant is positive and non-zero}$$

M1 E1

$$\text{With } a=b=c=t \text{ the determinant is } t^3 + 3t^2 = t^2(t+3)$$

M1

$$\text{Singular when } t=0 \text{ or } t=-3$$

A1**Answer: $t=0$ or $t=-3$** **8 marks**

29 (10 marks)

$$D = 1(3 - 7) - 2(5 - 2) + 1(35 - 6) = -4 - 6 + 29 = 19$$

M1 A1

$$\text{Area} = \frac{1}{2} \times 19 = \frac{19}{2}$$

A1

$$D = \begin{vmatrix} 1 & 1 & 1 \\ a & a^2 & 1 \\ b & b^2 & 1 \end{vmatrix}$$

M1

$$\text{Column operations } C_1 \rightarrow C_1 - C_3, C_2 \rightarrow C_2 - C_3 \text{ give } \begin{vmatrix} 0 & 0 & 1 \\ a-1 & a^2-1 & 1 \\ b-1 & b^2-1 & 1 \end{vmatrix}$$

M1 A1

$$D = (a-1)(b^2-1) - (a^2-1)(b-1) = (a-1)(b-1)[(b+1) - (a+1)]$$

M1 A1

$$= (a-1)(b-1)(b-a), \text{ so the area is } \frac{1}{2}|(a-1)(b-1)(b-a)|$$

A1

For distinct points $a \neq 1$, $b \neq 1$ and $a \neq b$, so the area is non-zero and the points are not collinear (collinear points would give area 0)

E1

Answer: Area = $\frac{19}{2}$; area of ABC = $\frac{1}{2}|(a-1)(b-1)(b-a)|$, which is never 0 for distinct points 10 marks

30 (9 marks)

$$R_2 \rightarrow R_2 - R_1, R_3 \rightarrow R_3 - R_1, R_4 \rightarrow R_4 - R_1$$

M1

$$\text{The determinant becomes } \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & x & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & z \end{vmatrix}$$

A1

This is an upper triangular determinant, equal to $1 \times x \times y \times z = xyz$

M1 A1

With $x = 2$, $y = -1$, $z = 3$ the value is -6

B1

Singular when $x = 0$ or $y = 0$ or $z = 0$

B1

For the last determinant, subtract row 1 from rows 2, 3 and 4, so that the rows become $(3, 1, 1, 1)$, $(-2, 2, 0, 0)$, $(-2, 0, 2, 0)$, $(-2, 0, 0, 2)$

M1

Replace C_1 by $C_1 + C_2 + C_3 + C_4$: the first column becomes $(6, 0, 0, 0)$

M1

The determinant is $6 \times 2 \times 2 \times 2 = 48$

A1

Answer: xyz; -6 ; 48

9 marks

Inverse Matrices**1 (2 marks)**

$$\det A = 6 - 5 = 1$$

M1

$$A^{-1} = \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}$$

A1

$$\text{Answer: } \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}$$

2 marks

2 (2 marks)

$$\det C = 6 - 6 = 0$$

M1

C is singular, so its inverse does not exist

E1

Answer: det C = 0, so C is singular

2 marks

3 (2 marks)

$$D^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{5} \end{pmatrix}$$

B1 B1

$$\text{Answer: } \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{5} \end{pmatrix}$$

2 marks**4 (2 marks)**

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

M1

$$AB = I, \text{ so } B = A^{-1}$$

A1

$$\text{Answer: } AB = I, \text{ so } B = A^{-1}$$

2 marks**5 (2 marks)**

$$\det R = 0 - (1)(-1) = 1$$

M1

$$R^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

A1

$$\text{Answer: } \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

2 marks**6 (4 marks)**

$$\det A = -3 + 8 = 5$$

M1 A1

$$A^{-1} = \frac{1}{5} \begin{pmatrix} -1 & 2 \\ -4 & 3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{5} & \frac{2}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{pmatrix}$$

M1 A1

$$\text{Answer: } \begin{pmatrix} -\frac{1}{5} & \frac{2}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{pmatrix}$$

4 marks**7 (5 marks)**

$$\det A = k(k+1) - 6 = k^2 + k - 6$$

M1 A1

$$A^{-1} \text{ does not exist when } (k+3)(k-2) = 0, \text{ so } k = -3 \text{ or } k = 2$$

M1 A1

$$A^{-1} = \frac{1}{(k+3)(k-2)} \begin{pmatrix} k+1 & -2 \\ -3 & k \end{pmatrix}$$

A1

$$\text{Answer: } k = -3 \text{ or } k = 2; A^{-1} = \frac{1}{(k+3)(k-2)} \begin{pmatrix} k+1 & -2 \\ -3 & k \end{pmatrix}$$

5 marks**8 (4 marks)**

$$A^{-1} = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}$$

M1 A1

$$X = A^{-1}B \text{ (premultiply both sides by } A^{-1})$$

M1

$$X = \begin{pmatrix} 5 & -5 \\ -6 & 9 \end{pmatrix}$$

A1

$$\text{Answer: } X = \begin{pmatrix} 5 & -5 \\ -6 & 9 \end{pmatrix}$$

4 marks

9 (4 marks)

$$A^{-1} = \begin{pmatrix} 3 & -2 \\ -4 & 3 \end{pmatrix}$$

M1 A1

$$X = BA^{-1} \text{ (postmultiply both sides by } A^{-1})$$

M1

$$X = \begin{pmatrix} -17 & 13 \\ -6 & 4 \end{pmatrix}$$

A1

$$\text{Answer: } X = \begin{pmatrix} -17 & 13 \\ -6 & 4 \end{pmatrix}$$

4 marks

10 (3 marks)

$$B = (B^{-1})^{-1}$$

M1

$$\det(B^{-1}) = -2 + 3 = 1, \text{ so } B = \begin{pmatrix} 2 & -1 \\ 3 & -1 \end{pmatrix}$$

M1 A1

$$\text{Answer: } B = \begin{pmatrix} 2 & -1 \\ 3 & -1 \end{pmatrix}$$

3 marks

11 (5 marks)

$$A^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \text{ and } B^{-1} = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$$

M1 A1

$$AB = \begin{pmatrix} 7 & 3 \\ 2 & 1 \end{pmatrix}, \text{ so } (AB)^{-1} = \begin{pmatrix} 1 & -3 \\ -2 & 7 \end{pmatrix}$$

M1 A1

$$B^{-1}A^{-1} = \begin{pmatrix} 1 & -3 \\ -2 & 7 \end{pmatrix}, \text{ the same matrix}$$

A1

$$\text{Answer: } (AB)^{-1} = \begin{pmatrix} 1 & -3 \\ -2 & 7 \end{pmatrix}$$

5 marks

12 (4 marks)

$$AX = B - I = \begin{pmatrix} 0 & 0 \\ 4 & 0 \end{pmatrix}$$

M1 A1

$$X = A^{-1}(B - I) \text{ with } A^{-1} = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}$$

M1

$$X = \begin{pmatrix} -4 & 0 \\ 8 & 0 \end{pmatrix}$$

A1

$$\text{Answer: } X = \begin{pmatrix} -4 & 0 \\ 8 & 0 \end{pmatrix}$$

4 marks

13 (8 marks)

$$\det A = 6 - 1 = 5, \text{ so } A^{-1} = \frac{1}{5} \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}$$

M1 A1

$$A^{-1} \begin{pmatrix} 5 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \text{ so } (5, 5) \text{ comes from } (1, 2)$$

M1 A1

$$A^{-1} \begin{pmatrix} 10 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \text{ and } A^{-1} \begin{pmatrix} 10 \\ 10 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \text{ so the other vertices are } (3, 1) \text{ and } (2, 4)$$

M1 A1

$$\text{Area of } T' = |\det A| \times \text{area of } T, \text{ so area of } T = \frac{12.5}{5} = 2.5$$

M1 A1

$$\text{Answer: } (1, 2), (3, 1), (2, 4); \text{ area } 2.5$$

8 marks

14 (6 marks)

$$\det A = 2(2 - 0) - 1(0 - 4) + 3(0 - 1) = 5$$

M1 A1

$$\text{Matrix of cofactors} = \begin{pmatrix} 2 & 4 & -1 \\ -2 & 1 & 1 \\ 1 & -8 & 2 \end{pmatrix}$$

M1 A1

$$\text{Transpose and divide by the determinant: } A^{-1} = \frac{1}{5} \begin{pmatrix} 2 & -2 & 1 \\ 4 & 1 & -8 \\ -1 & 1 & 2 \end{pmatrix}$$

M1

$$A^{-1} = \begin{pmatrix} \frac{2}{5} & -\frac{2}{5} & \frac{1}{5} \\ \frac{4}{5} & \frac{1}{5} & -\frac{8}{5} \\ -\frac{1}{5} & \frac{1}{5} & \frac{2}{5} \end{pmatrix}$$

A1

$$\text{Answer: } A^{-1} = \begin{pmatrix} \frac{2}{5} & -\frac{2}{5} & \frac{1}{5} \\ \frac{4}{5} & \frac{1}{5} & -\frac{8}{5} \\ -\frac{1}{5} & \frac{1}{5} & \frac{2}{5} \end{pmatrix}$$

6 marks

15 (6 marks)

$$\det A = 1(1 - 0) - 0 + 2(2 - 0) = 5$$

M1 A1

$$\text{The matrix of cofactors, transposed, gives } \text{adj } A = \begin{pmatrix} 1 & 2 & -2 \\ -2 & 1 & 4 \\ 2 & -1 & 1 \end{pmatrix}$$

M1

$$A^{-1} = \frac{1}{5} \begin{pmatrix} 1 & 2 & -2 \\ -2 & 1 & 4 \\ 2 & -1 & 1 \end{pmatrix}$$

A1

$$P = A^{-1} \begin{pmatrix} 7 \\ 4 \\ 5 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 5 \\ 10 \\ 15 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

M1 A1

$$\text{Answer: } A^{-1} = \frac{1}{5} \begin{pmatrix} 1 & 2 & -2 \\ -2 & 1 & 4 \\ 2 & -1 & 1 \end{pmatrix}; P = (1, 2, 3)$$

6 marks

16 (6 marks)

$$A^2 = \begin{pmatrix} 5 & 5 \\ 5 & 10 \end{pmatrix}$$

M1 A1

$$5A - 5I = \begin{pmatrix} 5 & 5 \\ 5 & 10 \end{pmatrix} = A^2$$

A1

$$\text{Rearranging: } A^2 - 5A = -5I, \text{ so } A(A - 5I) = -5I, \text{ giving } A(5I - A) = 5I$$

M1

$$\text{Premultiplying by } A^{-1}: 5I - A = 5A^{-1}, \text{ so } A^{-1} = \frac{1}{5}(5I - A)$$

A1

$$A^{-1} = \begin{pmatrix} \frac{3}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{pmatrix}$$

A1

$$\text{Answer: } A^{-1} = \begin{pmatrix} \frac{3}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{pmatrix}$$

6 marks

17 (6 marks)Premultiply by P^{-1} and postmultiply by Q^{-1} : $X = P^{-1}RQ^{-1}$

M1 A1

$$P^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \text{ and } Q^{-1} = \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}$$

B1 B1

$$P^{-1}R = \begin{pmatrix} 2 & 2 \\ 1 & 0 \end{pmatrix}$$

M1

$$X = \begin{pmatrix} 2 & -4 \\ 1 & -3 \end{pmatrix}$$

A1

$$\text{Answer: } X = \begin{pmatrix} 2 & -4 \\ 1 & -3 \end{pmatrix}$$

6 marks

18 (5 marks)

$$A \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad - bc & -ab + ab \\ cd - cd & -bc + ad \end{pmatrix}$$

M1 A1

$$= (ad - bc)I$$

A1

$$\text{So } A \times \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = I, \text{ which proves the formula for } A^{-1}$$

M1

If $ad - bc = 0$ the formula would need division by zero; indeed $\det A = 0$ means A is singular and no matrix B satisfies $AB = I$

E1

$$\text{Answer: } A^{-1} = \frac{1}{(ad - bc)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}; \text{ no inverse when } ad - bc = 0$$

5 marks

19 (6 marks)

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

M1 A1

$$\text{Similarly } (B^{-1}A^{-1})(AB) = I, \text{ so } (AB)^{-1} = B^{-1}A^{-1}$$

E1

$$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$$

M1 A1

$$\text{So } (A^{-1})^T \text{ is the inverse of } A^T, \text{ that is } (A^T)^{-1} = (A^{-1})^T$$

E1

$$\text{Answer: } (AB)^{-1} = B^{-1}A^{-1} \text{ and } (A^T)^{-1} = (A^{-1})^T$$

6 marks

20 (5 marks)

$$\det = 3 - 2 = 1, \text{ so the decoding matrix is } \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}$$

M1 A1

$$\begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 97 \\ 39 \end{pmatrix} = \begin{pmatrix} 19 \\ 20 \end{pmatrix}$$

M1

$$\begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 41 \\ 18 \end{pmatrix} = \begin{pmatrix} 5 \\ 13 \end{pmatrix}$$

A1

19, 20, 5, 13 correspond to S, T, E, M, so the word is STEM

A1

Answer: STEM

5 marks

21 (9 marks)

$$\det A = 1(1 - 0) - 2(0 - 6) + k(0 - 2) = 13 - 2k$$

M1 A1

$$A \text{ is singular when } k = \frac{13}{2}$$

A1

$$\text{For } k = 5, \det A = 3 \text{ and the matrix of cofactors is } \begin{pmatrix} 1 & 6 & -2 \\ -2 & -9 & 4 \\ 1 & -3 & 1 \end{pmatrix}$$

M1 A1

$$A^{-1} = \frac{1}{3} \begin{pmatrix} 1 & -2 & 1 \\ 6 & -9 & -3 \\ -2 & 4 & 1 \end{pmatrix}$$

M1 A1

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ -7 \\ 3 \end{pmatrix}$$

M1 A1

$$\text{Answer: } k = \frac{13}{2}; (x, y, z) = (0, -7, 3)$$

9 marks

22 (5 marks)

Premultiply $AB = AC$ by A^{-1} : $A^{-1}AB = A^{-1}AC$ M1

$IB = IC$, so $B = C$ A1

Take $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ (singular), $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ B1

$$AB = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = AC$$

M1

but $B \neq C$

A1

Answer: $B = C$ when A is invertible; with A singular, $AB = AC$ does not give $B = C$

5 marks

23 (7 marks)

$$A^2 = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$$

M1 A1

$$A^3 = A^2A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

A1

$$A \cdot A^2 = I, \text{ so } A^{-1} = A^2 = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$$

M1 A1

$$100 = 3 \times 33 + 1, \text{ so } A^{100} = (A^3)^{33}A = A = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

M1 A1

$$\textbf{Answer: } A^{-1} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}; A^{100} = A$$

7 marks

24 (6 marks)

$$\text{adj}(AB) = \det(AB)(AB)^{-1}$$

M1

$$= \det A \det B \times B^{-1}A^{-1}$$

M1 A1

Scalars commute with matrices, so this is $(\det B B^{-1})(\det A A^{-1})$

M1

$$= \text{adj}(B) \text{adj}(A)$$

A1

Remark: the order of the factors reverses, as with inverses and transposes

E1

Answer: $\text{adj}(AB) = \text{adj}(B) \text{adj}(A)$

6 marks

25 (7 marks)

Premultiply $AB = BA$ by A^{-1} : $B = A^{-1}BA$ M1 A1

Postmultiply by A^{-1} : $BA^{-1} = A^{-1}BAA^{-1} = A^{-1}B$ M1 A1

So $A^{-1}B = BA^{-1}$ E1

Take inverses of both sides of $A^{-1}B = BA^{-1}$: $(A^{-1}B)^{-1} = B^{-1}A$ and $(BA^{-1})^{-1} = AB^{-1}$, so $B^{-1}A = AB^{-1}$ M1

Premultiply by A^{-1} and postmultiply by A^{-1} : $A^{-1}B^{-1} = B^{-1}A^{-1}$ A1

Answer: $A^{-1}B = BA^{-1}$ and $A^{-1}B^{-1} = B^{-1}A^{-1}$

7 marks

26 (6 marks)

$$A + B = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}, \text{ with determinant } 10, \text{ so } (A + B)^{-1} = \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{10} & \frac{3}{10} \end{pmatrix} \quad \text{M1 A1}$$

$$A^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \text{ and } B^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{1}{6} & \frac{1}{3} \end{pmatrix} \quad \text{M1}$$

$$A^{-1} + B^{-1} = \begin{pmatrix} \frac{3}{2} & -2 \\ -\frac{1}{6} & \frac{4}{3} \end{pmatrix} \neq (A + B)^{-1} \quad \text{A1}$$

Simple example: $A = B = I$, where $(A + B)^{-1} = (2I)^{-1} = \frac{1}{2}I$ but $A^{-1} + B^{-1} = 2I$ B1

In general the inverse is not additive (the inverse of a sum is not a sum of inverses), just as $\frac{1}{a+b}$ E1

$\neq \frac{1}{a} + \frac{1}{b}$ for numbers

Answer: Statement false

6 marks

27 (10 marks)

$\det V = 1(18 - 12) - 1(9 - 4) + 1(3 - 2) = 2$ M1 A1

Matrix of cofactors = $\begin{pmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{pmatrix}$ M1 A1

$$V^{-1} = \frac{1}{2} \begin{pmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -3 & 1 \\ -\frac{5}{2} & 4 & -\frac{3}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{pmatrix} \quad \text{M1 A1}$$

The conditions give $V \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 5 \\ 9 \\ 15 \end{pmatrix}$ M1

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = V^{-1} \begin{pmatrix} 5 \\ 9 \\ 15 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}, \text{ so } \alpha = 3, \beta = 1, \gamma = 1 \quad \text{M1 A1}$$

With two equal x-values, V would have two equal rows, so $\det V = 0$ and V^{-1} would not exist E1

Answer: $\alpha = 3, \beta = 1, \gamma = 1$

10 marks

28 (10 marks)

Rearrange: $4A - A^2 = 3I$, so $A(4I - A) = 3I$

M1 A1

Hence $A \times \frac{1}{3}(4I - A) = I$, so A is invertible with $A^{-1} = \frac{1}{3}(4I - A)$

A1

$$A^{-2} = (A^{-1})^2 = \frac{1}{9}(4I - A)^2 = \frac{1}{9}(16I - 8A + A^2)$$

M1 A1

$$= \frac{1}{9}(16I - 8A + 4A - 3I) = \frac{1}{9}(13I - 4A)$$

A1

$$A^2 = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} \text{ and } 4A - 3I = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}, \text{ which agree}$$

M1 A1

$$A^{-1} = \frac{1}{3}(4I - A) = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

A1

$$A^{-2} = \frac{1}{9}(13I - 4A) = \begin{pmatrix} \frac{5}{9} & -\frac{4}{9} \\ -\frac{4}{9} & \frac{5}{9} \end{pmatrix}$$

A1

$$\text{Answer: } A^{-1} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix}, A^{-2} = \begin{pmatrix} \frac{5}{9} & -\frac{4}{9} \\ -\frac{4}{9} & \frac{5}{9} \end{pmatrix}$$

10 marks

29 (10 marks)

$$\det A = 1, \text{ so } A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

M1 A1

$$A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

B1

Equating $A^{-1} = A^T$: $d = a$, $-b = c$, so $c = -b$

M1 A1

$\det A = a^2 + b^2 = 1$ (since $\det A = ad - bc = a^2 + b^2$)

M1 A1

A point (a, b) with $a^2 + b^2 = 1$ can be written $a = \cos \theta$, $b = -\sin \theta$ for some θ

M1

$$\text{So } A = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

A1

This is a rotation about the origin through angle θ anticlockwise

E1

Answer: A is a rotation about O through θ anticlockwise

10 marks

30 (12 marks)

Let $N = (I - AB)^{-1}$ and form the product $(I - BA)(I + BNA)$

M1

$$= I + BNA - BA - BABNA$$

M1 A1

$$= I - BA + B(N - ABN)A$$

M1

$$= I - BA + B(I - AB)NA$$

A1

$$= I - BA + BIA \text{ (since } (I - AB)N = I)$$

M1

$$= I$$

A1

A similar calculation of $(I + BNA)(I - BA)$ gives I as well (or note that a one-sided inverse of a square matrix is two-sided)

E1

$$I - AB = \begin{pmatrix} -2 & -1 \\ -6 & -1 \end{pmatrix} \text{ has inverse } N = \begin{pmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix}$$

B1

$$BNA = \begin{pmatrix} 3 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -\frac{5}{4} \end{pmatrix}$$

M1 A1

$$(I - BA)^{-1} = I + BNA = \begin{pmatrix} -\frac{1}{4} \end{pmatrix}$$

A1

$$\text{Answer: } (I - BA)^{-1} = \begin{pmatrix} -\frac{1}{4} \end{pmatrix}$$

12 marks

Solving Simultaneous Equations Using Matrices

1 (2 marks)

$$A = \begin{pmatrix} 3 & -2 \\ 1 & 4 \end{pmatrix}, x = \begin{pmatrix} x \\ y \end{pmatrix}, b = \begin{pmatrix} 5 \\ 11 \end{pmatrix}$$

B1 B1

$$\text{Answer: } \begin{pmatrix} 3 & -2 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 11 \end{pmatrix}$$

2 marks

2 (2 marks)

$$\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1} \begin{pmatrix} 5 \\ 8 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

M1 A1

$$\text{Answer: } x = 2, y = 1$$

2 marks

3 (3 marks)

$$\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}, \det = -3 \text{ and the inverse is } \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{pmatrix}$$

M1 A1

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

A1

$$\text{Answer: } x = 2, y = 3$$

3 marks

4 (3 marks)

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 3 \\ 3 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \\ 7 \end{pmatrix}$$

B1 B1 B1

$$\text{Answer: } \begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 3 \\ 3 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \\ 7 \end{pmatrix}$$

3 marks

5 (2 marks)

$$\text{(a) } \det \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix} = 12 - 12 = 0, \text{ so there is no unique solution}$$

B1

Answer: No unique solution (determinant 0)

1 mark

$$\text{(b) } \det \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} = 10 - 12 = -2 \neq 0, \text{ so there is a unique solution}$$

B1

Answer: Unique solution (determinant -2)

1 mark

6 (2 marks)

$$\det = 5 - 6 = -1, \text{ so the inverse is } \begin{pmatrix} -1 & 3 \\ 2 & -5 \end{pmatrix}$$

M1

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

A1

$$\text{Answer: } x = -1, y = 3$$

2 marks

7 (5 marks)

$$5 = m + c \text{ and } 11 = 3m + c, \text{ so } \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} m \\ c \end{pmatrix} = \begin{pmatrix} 5 \\ 11 \end{pmatrix}$$

M1 A1

$$\det = 1 - 3 = -2, \text{ so the inverse is } -\frac{1}{2} \begin{pmatrix} 1 & -1 \\ -3 & 1 \end{pmatrix}$$

M1

$$\begin{pmatrix} m \\ c \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} 1 & -1 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 11 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

M1 A1

$$\text{Answer: } m = 3, c = 2$$

5 marks

8 (3 marks)

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1} \begin{pmatrix} 4 \\ 7 \\ 1 \end{pmatrix} \quad \text{M1}$$

$$= \frac{1}{2} \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \quad \text{M1 A1}$$

Answer: $x = 1, y = 2, z = -1$

3 marks

9 (5 marks)

$$\det A = k(k + 1) - 6 = k^2 + k - 6 \quad \text{M1 A1}$$

No unique solution when $(k + 3)(k - 2) = 0$, so $k = -3$ or $k = 2$

A1

For $k = 1$, $\det A = -4$ and the equations are $\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ **M1**

$$\begin{pmatrix} x \\ y \end{pmatrix} = -\frac{1}{4} \begin{pmatrix} 2 & -2 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{7}{4} \end{pmatrix} \quad \text{A1}$$

Answer: $k = -3$ or $k = 2$; $x = -\frac{1}{2}, y = \frac{7}{4}$

5 marks

10 (6 marks)

(a) $\det \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix} = 12 - 12 = 0$ **B1**

The lines $2x + 3y = 5$ and $2x + 3y = \frac{7}{2}$ are parallel and distinct **M1**

There is no solution **A1**

Answer: Parallel distinct lines, no solution

3 marks

(b) $\det \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix} = 12 - 12 = 0$ **B1**

The second equation is twice the first, so the lines coincide **M1**

There are infinitely many solutions **A1**

Answer: Coincident lines, infinitely many solutions

3 marks

11 (4 marks)

Let an adult ticket cost £a and a child ticket £c: $\begin{pmatrix} 3 & 2 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} 47 \\ 68 \end{pmatrix}$ **M1 A1**

$\det = 15 - 4 = 11$ and the inverse is $\frac{1}{11} \begin{pmatrix} 5 & -2 \\ -2 & 3 \end{pmatrix}$ **M1**

$$\begin{pmatrix} a \\ c \end{pmatrix} = \frac{1}{11} \begin{pmatrix} 99 \\ 110 \end{pmatrix} = \begin{pmatrix} 9 \\ 10 \end{pmatrix} \quad \text{A1}$$

Answer: Adult £9, child £10

4 marks

12 (6 marks)

$$\text{Augmented matrix } \begin{pmatrix} 1 & 2 & 3 & 13 \\ 2 & -1 & 1 & 6 \\ 3 & 1 & -2 & 1 \end{pmatrix} \quad \text{B1}$$

$$R_2 - 2R_1 \text{ and } R_3 - 3R_1: \begin{pmatrix} 1 & 2 & 3 & 13 \\ 0 & -5 & -5 & -20 \\ 0 & -5 & -11 & -38 \end{pmatrix} \quad \text{M1 A1}$$

$$-\frac{1}{5}R_2, \text{ then } R_3 + 5R_2: \begin{pmatrix} 1 & 2 & 3 & 13 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & -6 & -18 \end{pmatrix} \quad \text{M1}$$

$$-\frac{1}{6}R_3, \text{ then } R_1 - 2R_2: \begin{pmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & 3 \end{pmatrix} \quad \text{M1}$$

$$R_1 - R_3 \text{ and } R_2 - R_3: \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{pmatrix}, \text{ so } x = 2, y = 1, z = 3 \quad \text{A1}$$

Answer: $x = 2, y = 1, z = 3$ 6 marks

13 (5 marks)

$$AB = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix} = 6I \quad \text{M1 A1}$$

$$\text{So } A^{-1} = \frac{1}{6}B = \frac{1}{6} \begin{pmatrix} 3 & -3 & -3 \\ 1 & 1 & 3 \\ 1 & 1 & -3 \end{pmatrix} \quad \text{A1}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1} \begin{pmatrix} 6 \\ 3 \\ -1 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 3 & -3 & -3 \\ 1 & 1 & 3 \\ 1 & 1 & -3 \end{pmatrix} \begin{pmatrix} 6 \\ 3 \\ -1 \end{pmatrix} \quad \text{M1}$$

$$= \frac{1}{6} \begin{pmatrix} 12 \\ 6 \\ 12 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \text{ so } x = 2, y = 1, z = 2 \quad \text{A1}$$

Answer: $AB = 6I; A^{-1} = \frac{1}{6}B; x = 2, y = 1, z = 2$ 5 marks

14 (7 marks)

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 3 \\ 3 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ 14 \\ 8 \end{pmatrix} \quad \text{B1}$$

$$\det A = 1(-1 - 3) - 2(2 - 9) - 1(2 + 3) = 5 \quad \text{M1 A1}$$

$$\text{Matrix of cofactors} = \begin{pmatrix} -4 & 7 & 5 \\ -3 & 4 & 5 \\ 5 & -5 & -5 \end{pmatrix} \quad \text{M1}$$

$$A^{-1} = \frac{1}{5} \begin{pmatrix} -4 & -3 & 5 \\ 7 & 4 & -5 \\ 5 & 5 & -5 \end{pmatrix} \quad \text{A1}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 10 \\ -5 \\ 15 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \quad \text{M1 A1}$$

Answer: $x = 2, y = -1, z = 3$ 7 marks

15 (6 marks)

$$\det A = k(k - 0) + 1(0 - 3) + 2(0 - k) = k^2 - 2k - 3$$

M1 A1

$$(k - 3)(k + 1) = 0, \text{ so } k = 3 \text{ or } k = -1$$

A1

$$\text{When } k = 3: 3x - y + 2z = 1, 3y + 3z = 2, x + z = 3$$

B1

$$\text{From the third equation } x = 3 - z, \text{ and the first then gives } 9 - 3z - y + 2z = 1, \text{ so } y = 8 - z$$

M1

Substituting into the second equation: $3(8 - z) + 3z = 24 = 2$, which is a contradiction, so the equations are inconsistent

A1

Answer: $k = 3$ or $k = -1$

6 marks

16 (6 marks)

$$R_2 \rightarrow R_2 - 2R_1 \text{ and } R_3 \rightarrow R_3 - R_1$$

M1

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -3 & -1 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ -9 \\ -4 \end{pmatrix}$$

A1

$$R_3 \rightarrow 3R_3 + R_2: -7z = -21, \text{ so } z = 3$$

M1 A1

$$\text{From } R_2: -3y - 3 = -9, \text{ so } y = 2$$

A1

$$\text{From } R_1: x = 6 - 2 - 3 = 1$$

A1

Answer: $x = 1, y = 2, z = 3$

6 marks

17 (5 marks)

Coefficient matrix $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 0 & 1 \end{pmatrix}$ has two proportional rows, so $\det = 0$ and there is no unique

M1 A1

solution

Doubling the first equation gives $2x + 4y + 6z = 8$, which contradicts $2x + 4y + 6z = 9$

M1 A1

The first two planes are parallel and distinct, and the third plane cuts both in parallel lines

E1

Answer: No solution; two parallel planes cut by a third

5 marks

18 (6 marks)

$$\text{Eliminate } x: R_2 - 2R_1: -3y + 3z = 0 \text{ and } R_3 - R_1: -3y + 3z = 0$$

M1 A1

These are identical, so there are only two independent equations and infinitely many solutions

E1

$$\text{Let } z = t: \text{ then } y = t \text{ and } x = 2 - 2y + z = 2 - t$$

M1 A1

$$r = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

A1

$$\text{Answer: } x = 2 - t, y = t, z = t; r = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

6 marks

19 (5 marks)

$$x + y + z = 100$$

B1

$$\text{Copper: } 0.6x + 0.4y + 0.2z = 40 \text{ and zinc: } 0.3x + 0.4y + 0.6z = 43, \text{ that is } 6x + 4y + 2z = 400 \text{ and}$$

B1

$$3x + 4y + 6z = 430$$

$$\begin{pmatrix} 6 & 4 & 2 \\ 3 & 4 & 6 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 400 \\ 430 \\ 100 \end{pmatrix}$$

M1

$$\text{Using a calculator, } \det A = -2 \text{ and } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1} \begin{pmatrix} 400 \\ 430 \\ 100 \end{pmatrix} = \begin{pmatrix} 30 \\ 40 \\ 30 \end{pmatrix}$$

M1 A1

Answer: $x = 30, y = 40, z = 30$

5 marks

20 (5 marks)

$$I_1 - I_2 - I_3 = 0, 2I_1 + 3I_2 = 12 \text{ and } 3I_2 - 6I_3 = 0$$

M1

$$\begin{pmatrix} 1 & -1 & -1 \\ 2 & 3 & 0 \\ 0 & 3 & -6 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 12 \\ 0 \end{pmatrix}$$

A1

$\det = -36 \neq 0$ so a unique solution exists

B1

$$\begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = A^{-1} \begin{pmatrix} 0 \\ 12 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

M1 A1

Answer: $I_1 = 3$ A, $I_2 = 2$ A, $I_3 = 1$ A

5 marks

21 (8 marks)

$$\det A = 1(1 - k) - k(2 - k) + 1(2 - 1)$$

M1 A1

$$= 1 - k - 2k + k^2 + 1 = k^2 - 3k + 2 = (k - 1)(k - 2)$$

A1

No unique solution when $k = 1$ or $k = 2$

B1

$$\text{For } k = 3, \det A = 2 \text{ and } A^{-1} = \begin{pmatrix} -1 & -1 & 4 \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{2} & 1 & -\frac{5}{2} \end{pmatrix}$$

M1 A1

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 9 \\ -1 \\ -5 \end{pmatrix}$$

M1 A1

Answer: $k = 1$ or $k = 2$; $x = 9$, $y = -1$, $z = -5$

8 marks

22 (8 marks)

$$\det A = 1(2k + 1) - 2(4 - 1) - 1(-2 - k)$$

M1 A1

$$= 2k + 1 - 6 + 2 + k = 3k - 3$$

A1

If $\det A \neq 0$, A^{-1} exists and the only solution is $x = y = z = 0$; non-trivial solutions need $\det A = 0$, that is $k = 1$

E1

For $k = 1$: $x + 2y - z = 0$ and $2x + y + z = 0$ (the third equation is then redundant)

M1

Adding them: $3x + 3y = 0$, so $y = -x$; then $z = x + 2y = -x$

M1 A1

General solution: $(x, y, z) = \lambda(1, -1, -1)$ for any real λ

A1

Answer: $k = 1$; $(x, y, z) = \lambda(1, -1, -1)$

8 marks

23 (6 marks)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix}, \text{ so } \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} e \\ f \end{pmatrix}$$

M1 A1

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{ad - bc} \begin{pmatrix} de - bf \\ -ce + af \end{pmatrix}$$

M1

$$\text{So } x = \frac{de - bf}{ad - bc} \text{ and } y = \frac{af - ce}{ad - bc}$$

A1

$$\text{Here } a = 4, b = -3, c = 5, d = 2, e = 10, f = 1: ad - bc = 8 + 15 = 23$$

M1

$$x = \frac{20 + 3}{23} = 1 \text{ and } y = \frac{4 - 50}{23} = -2$$

A1

Answer: $x = 1$, $y = -2$

6 marks

24 (9 marks)

$$\det A = 1(10 - 9) - 1(5 - 3) + 1(3 - 2) = 1 - 2 + 1 = 0$$

M1 A1

Row 3 = $2 \times$ row 2 - row 1 on the left, so the equations are consistent only if $c = 2(2) - 1 = 3$

M1 A1

With $c = 3$: $y + 2z = 1$ (from equations 2 - 1), so $y = 1 - 2z$ and $x = 1 - y - z = z$

M1 A1

$$r = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

A1

If $c \neq 3$ the equations are inconsistent: the planes meet in pairs in three parallel lines, forming a triangular prism

E1 E1

Answer: $c = 3$, $r = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$; otherwise a triangular prism

9 marks

25 (7 marks)

$$\det A = 2(6 - 1) - 1(2 - 0) = 8$$

M1 A1

$$A^{-1} = \frac{1}{8} \begin{pmatrix} 5 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 5 \end{pmatrix}$$

M1 A1

$$X = A^{-1}B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 3 \end{pmatrix}$$

M1 A1

The columns of X give the two solutions: $(x, y, z) = (1, 2, 0)$ and $(x, y, z) = (0, 1, 3)$

A1

Answer: $X = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 3 \end{pmatrix}$; $(1, 2, 0)$ and $(0, 1, 3)$

7 marks

26 (8 marks)

(a) The coefficient matrix is $\begin{pmatrix} 1 & 1 \\ 1 & 1.001 \end{pmatrix}$ with determinant $1.001 - 1 = 0.001$

M1 A1

The inverse is $\frac{1}{0.001} \begin{pmatrix} 1.001 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1001 & -1000 \\ -1000 & 1000 \end{pmatrix}$

M1

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1001 & -1000 \\ -1000 & 1000 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \text{ so } x = 2, y = 0$$

A1

Answer: $x = 2$, $y = 0$

4 marks

(b) The same coefficient matrix has inverse $\begin{pmatrix} 1001 & -1000 \\ -1000 & 1000 \end{pmatrix}$

M1

$$\begin{pmatrix} 1001 & -1000 \\ -1000 & 1000 \end{pmatrix} \begin{pmatrix} 2 \\ 2.001 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \text{ so } x = 1, y = 1$$

A1

Answer: $x = 1$, $y = 1$

2 marks

(c) A change of 0.001 in one constant changes the solution from $(2, 0)$ to $(1, 1)$, a very large change

E1

The determinant is very close to 0 (the lines are nearly parallel), so the system is very sensitive to small changes

E1

Answer: Large change in the solution; determinant close to 0

2 marks

27 (12 marks)

Coefficient matrix = $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & a \end{pmatrix}$, $\det = 1(2a - 9) - 1(a - 3) + 1(3 - 2) = a - 5$, so the matrix M1 A1

is singular when $a = 5$

$R_2 - R_1$: $y + 2z = 1$ and $R_3 - R_1$: $2y + (a - 1)z = b - 1$ M1

$R_3 - 2R_2$: $(a - 5)z = b - 3$ M1 A1

If $a \neq 5$ then $z = \frac{b-3}{a-5}$, so a unique solution exists A1

$y = 1 - 2z = \frac{a-2b+1}{a-5}$ and $x = 1 - y - z = z = \frac{b-3}{a-5}$ M1 A1

If $a = 5$ the last equation reads $0 = b - 3$: consistent only if $b = 3$ M1 A1

With $a = 5$, $b = 3$: $z = t$, $y = 1 - 2t$, $x = t$ A1

Three planes meeting in a line ($r = (0, 1, 0) + t(1, -2, 1)$); if $b \neq 3$ the planes form a triangular prism with no common point E1

Answer: $a = 5$; unique solution $x = z = \frac{b-3}{a-5}$, $y = \frac{a-2b+1}{a-5}$; $a = 5$, $b = 3$ gives $(t, 1 - 2t, t)$ 12 marks

28 (11 marks)

$R_2 - R_1$, $R_3 - R_1$ gives $\begin{vmatrix} 1 & 1 & 1 \\ 0 & a & 0 \\ 0 & 0 & c \end{vmatrix}$ M1 A1

This is triangular, so the determinant is $1 \times a \times c = ac$ A1

The reduced equations are $ay = k - 1$ and $cz = 2$ M1

$y = \frac{k-1}{a}$ and $z = \frac{2}{c}$ A1

$x = 1 - y - z = 1 - \frac{k-1}{a} - \frac{2}{c}$ M1 A1

With $a = 0$ the second reduced equation is $0 \times y = k - 1$, which is consistent only if $k = 1$ M1 A1

$c = 2$ gives $z = 1$ and then $x + y = 0$; y is free M1

Let $y = t$: $(x, y, z) = (-t, t, 1)$ A1

Answer: $y = \frac{k-1}{a}$, $z = \frac{2}{c}$, $x = 1 - \frac{k-1}{a} - \frac{2}{c}$; $k = 1$ and $(x, y, z) = (-t, t, 1)$ 11 marks

29 (10 marks)

$A(P + \lambda(Q - P)) = AP + \lambda(AQ - AP)$ M1

$= B + \lambda(B - B) = B$, so every such X is a solution A1

$P \neq Q$, so $Q - P \neq 0$ and the vectors $P + \lambda(Q - P)$ are different for different λ , giving infinitely many solutions M1 E1

$A(Q - P) = AQ - AP = B - B = 0$ with $Q - P \neq 0$ M1

If A were invertible, premultiplying by A^{-1} would give $Q - P = 0$, a contradiction M1 A1

So A is singular E1

Three equations in three unknowns have a square coefficient matrix. If there are two solutions, the first part shows that there are infinitely many, so exactly two is impossible E1 E1

Answer: Two different solutions give infinitely many solutions and A is singular, so exactly two solutions is impossible

30 (8 marks)

Cx is the output used up by the industries themselves, and d is the demand from outside, so the total output x must equal $Cx + d$

E1

Rearranging: $x - Cx = d$, so $(I - C)x = d$

M1

$$I - C = \begin{pmatrix} \frac{4}{5} & -\frac{1}{10} & -\frac{1}{5} \\ -\frac{1}{10} & \frac{4}{5} & -\frac{1}{10} \\ -\frac{1}{5} & -\frac{1}{10} & \frac{9}{10} \end{pmatrix}$$

M1 A1

$$(I - C)^{-1} = \frac{1}{523} \begin{pmatrix} 710 & 110 & 170 \\ 110 & 680 & 100 \\ 170 & 100 & 630 \end{pmatrix}$$

M1 A1

$$x = (I - C)^{-1}d = \begin{pmatrix} 100 \\ 150 \\ 200 \end{pmatrix}$$

M1

The outputs are 100, 150 and 200 units for industries 1, 2 and 3

A1

Answer: $x = \begin{pmatrix} 100 \\ 150 \\ 200 \end{pmatrix}$

8 marks**Matrix Transformations****1 (2 marks)**

The images of $(1, 0)$ and $(0, 1)$ are $(0, 1)$ and $(-1, 0)$, which form the columns

M1

$$\text{Matrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

A1

Answer: $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

2 marks**2 (2 marks)**

$(1, 0) \rightarrow (0, 1)$ and $(0, 1) \rightarrow (1, 0)$

M1

$$\text{Matrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

A1

Answer: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

2 marks**3 (2 marks)**

$$\text{Matrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

B1

Image of $(2, -1)$ is $(6, -3)$

B1

Answer: $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}; (6, -3)$

2 marks**4 (2 marks)**

$$\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$$

M1 A1

Answer: $(3, -3)$

2 marks**5 (2 marks)**

Every point (x, y) maps to $(-x, -y)$

M1

A rotation of 180° about the origin (equivalently an enlargement with scale factor -1 , centre O)

A1

Answer: Rotation of 180° about the origin

2 marks

6 (2 marks)

$$(1, 0) \rightarrow (2, 0) \text{ and } (0, 1) \rightarrow (0, 1)$$

M1

$$\text{Matrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

A1

$$\text{Answer: } \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

2 marks

7 (3 marks)

$$\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 1 \\ 1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 6 & 2 \\ 2 & 4 & 3 \end{pmatrix}$$

M1 A1

$$A'(2, 2), B'(6, 4), C'(2, 3)$$

A1

$$\text{Answer: } A'(2, 2), B'(6, 4), C'(2, 3)$$

3 marks

8 (6 marks)

$$(a) (x, y) \rightarrow (-x, y), \text{ a reflection in the } y\text{-axis}$$

B1 B1

$$\text{Answer: Reflection in the } y\text{-axis}$$

2 marks

$$(b) (x, y) \rightarrow (-y, -x), \text{ a reflection in the line } y = -x$$

B1 B1

$$\text{Answer: Reflection in the line } y = -x$$

2 marks

$$(c) \text{ An enlargement with scale factor 5 and centre the origin}$$

B1 B1

$$\text{Answer: Enlargement, scale factor 5, centre O}$$

2 marks

9 (4 marks)

$$\text{Matrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

M1 A1

$$\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 \\ 2\sqrt{3} \end{pmatrix} = \begin{pmatrix} -2 \\ 2\sqrt{3} \end{pmatrix}$$

M1

$$\text{Image is } (-2, 2\sqrt{3})$$

A1

$$\text{Answer: } \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}; (-2, 2\sqrt{3})$$

4 marks

10 (4 marks)

$$(1, 0) \rightarrow (0, -1) \text{ and } (0, 1) \rightarrow (-1, 0)$$

M1

$$\text{Matrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

A1

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}, \text{ so the image is } (2, -3)$$

M1 A1

$$\text{Answer: } \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}; (2, -3)$$

4 marks

11 (4 marks)

$$\text{The images of } (1, 0) \text{ and } (0, 1) \text{ form the columns: } \begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix}$$

M1 A1

$$\begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 12 \\ 10 \end{pmatrix}, \text{ so } (2, 3) \rightarrow (12, 10)$$

M1

$$\det = 12 + 2 = 14$$

A1

$$\text{Answer: } \begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix}; (12, 10); \det = 14$$

4 marks

12 (4 marks)

$(1, 0)$ is invariant and $(0, 1)$ maps to $(1, 1)$, so $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

M1 A1

A shear, with the x-axis invariant (shear factor 1 parallel to the x-axis)

B1

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$, so $(2, 3) \rightarrow (5, 3)$

A1

Answer: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, shear; $(5, 3)$

4 marks

13 (5 marks)

$(x, y) \rightarrow (X, Y) = (2x + y, x + y)$

M1

Solving: $x = X - Y$ and $y = 2Y - X$

M1 A1

Substituting into $x + 2y = 6$: $(X - Y) + 2(2Y - X) = 6$, so $-X + 3Y = 6$

M1

The image is the line $y = \frac{1}{3}x + 2$

A1

Answer: $y = \frac{1}{3}x + 2$

5 marks

14 (5 marks)

A point (x, y) maps to (x', y') where $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

M1

$x' = x + 2y$ and $y' = y$, so $x = x' - 2y'$ and $y = y'$

M1 A1

Substitute into $y = 2x + 1$: $y' = 2(x' - 2y') + 1$

M1

$5y' = 2x' + 1$, so the image is $y = \frac{2}{5}x + \frac{1}{5}$

A1

Answer: $y = \frac{2}{5}x + \frac{1}{5}$

5 marks

15 (5 marks)

$a + 2b = 8$ and $c + 2d = 3$

M1

$3a + b = 9$ and $3c + d = -1$

M1

Solving the first pair of equations for a and b : $a = 2$, $b = 3$

M1 A1

Solving the second pair for c and d : $c = -1$, $d = 2$

A1

Answer: $a = 2$, $b = 3$, $c = -1$, $d = 2$

5 marks

16 (4 marks)

An invariant point satisfies $\begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

M1

$3x - 2y = x$ and $x = y$

M1 A1

Both equations give $y = x$, so every point on the line $y = x$ is invariant

A1

Answer: $y = x$

4 marks

17 (5 marks)

$\det A = \frac{9}{25} + \frac{16}{25} = 1$

B1

The matrix has the form $\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ with $\cos \theta = \frac{3}{5}$ and $\sin \theta = \frac{4}{5}$

M1 A1

$\theta = 53.1^\circ$

M1

A rotation of 53.1° anticlockwise about the origin

A1

Answer: Rotation of 53.1° anticlockwise about O

5 marks

18 (4 marks)

Points on the mirror line are invariant: $B \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

M1

$$-\frac{1}{2}x + \frac{\sqrt{3}}{2}y = x, \text{ so } \frac{\sqrt{3}}{2}y = \frac{3}{2}x$$

M1 A1

$$y = \sqrt{3}x$$

A1

Answer: $y = \sqrt{3}x$

4 marks

19 (5 marks)

Reading from the diagram, $A' = (2, 1)$ and $C' = (-1, 2)$

B1

The columns of M are the images of $(1, 0)$ and $(0, 1)$: $M = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$

M1 A1

Check: $B' = M \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, which agrees with the diagram

M1

$$\text{Area} = |\det M| \times 1 = 4 + 1 = 5$$

A1

Answer: $M = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$; area 5

5 marks

20 (5 marks)

The image of $(1, 0)$ is $\begin{pmatrix} 0 \\ -1 \end{pmatrix} = (0, -1)$

M1 A1

$(1, 0)$ moves to $(0, -1)$, which is a rotation of 90° clockwise

A1

For 90° anticlockwise, $(1, 0) \rightarrow (0, 1)$ and $(0, 1) \rightarrow (-1, 0)$, so the matrix is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

M1 A1

Answer: Rotation 90° clockwise; correct matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

5 marks

21 (7 marks)

A point (x, mx) on the line maps to $\begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ mx \end{pmatrix} = \begin{pmatrix} (3 - 2m)x \\ x \end{pmatrix}$

M1 A1

The image lies on $y = mx$ when $x = m(3 - 2m)x$

M1

$$\text{So } m(3 - 2m) = 1, \text{ that is } 2m^2 - 3m + 1 = 0$$

A1

$$(2m - 1)(m - 1) = 0, \text{ so } m = \frac{1}{2} \text{ or } m = 1, \text{ giving } y = \frac{1}{2}x \text{ and } y = x$$

M1 A1

For $y = x$ the point (x, x) maps to (x, x) , so every point of $y = x$ is invariant (this is not true of $y = \frac{1}{2}x$)

E1

Answer: $y = x$ (invariant points) and $y = \frac{1}{2}x$

7 marks

22 (5 marks)

A point (x, y) on C maps to $(x', y') = (2x, 3y)$

M1

$$x = \frac{x'}{2} \text{ and } y = \frac{y'}{3}$$

A1

$$\text{Substituting into } x^2 + y^2 = 1: \frac{x'^2}{4} + \frac{y'^2}{9} = 1$$

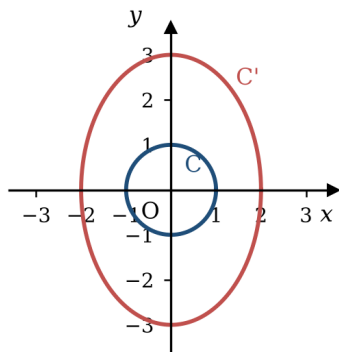
M1 A1

C' is an ellipse enclosing an area $\pi \times 2 \times 3 = 6\pi$, which equals $|\det| \times \pi$

A1

Answer: $\frac{x^2}{4} + \frac{y^2}{9} = 1$; area 6π

5 marks

**23 (7 marks)**

The reflection preserves lengths and the line OP makes angle θ with the mirror, so the image P' of P makes angle θ on the other side, that is an angle 2θ with the x-axis

M1 A1

$OP' = 1$, so $P' = (\cos 2\theta, \sin 2\theta)$

A1

OQ is at angle $90^\circ - \theta$ to the mirror, so Q' lies at angle $\theta - (90^\circ - \theta) = 2\theta - 90^\circ$ from the x-axis

M1 A1

$Q' = (\cos(2\theta - 90^\circ), \sin(2\theta - 90^\circ)) = (\sin 2\theta, -\cos 2\theta)$

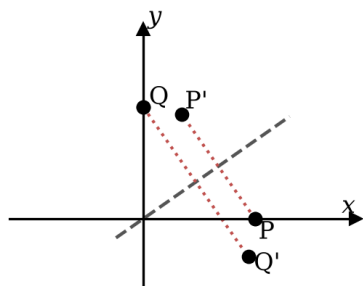
A1

The columns of R are the images P' and Q', giving the stated matrix

E1

Answer: $R = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$

7 marks

**24 (6 marks)**

Write $N \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 5 & -3 \end{pmatrix}$

M1

$N = \begin{pmatrix} 3 & 1 \\ 5 & -3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1}$

M1

$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

M1 A1

$N = \frac{1}{2} \begin{pmatrix} 4 & 2 \\ 2 & 8 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$

A1

$N \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 10 \\ 12 \end{pmatrix}$, so the image is (10, 12)

A1

Answer: $N = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}; (10, 12)$

6 marks

25 (7 marks)

$\det M = 2 - 1 = 1$ and $M^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$

M1 A1

A point (x, y) of the circle maps to (X, Y) with $x = X - Y$ and $y = -X + 2Y$

M1 A1

$(X - Y)^2 + (2Y - X)^2 = 1$

M1

$2X^2 - 6XY + 5Y^2 = 1$, so C has equation $2x^2 - 6xy + 5y^2 = 1$

A1

Area enclosed = $|\det M| \times \pi = \pi$

B1

Answer: $2x^2 - 6xy + 5y^2 = 1$; area π

7 marks

26 (8 marks)

$$R = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \text{ with } \cos \theta \text{ and } \sin \theta \text{ to be found: } \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \quad \text{M1}$$

$$3\cos \theta - 4\sin \theta = 5 \text{ and } 3\sin \theta + 4\cos \theta = 0 \quad \text{M1}$$

From the second, $\tan \theta = -\frac{4}{3}$; the length of (3, 4) is 5, so $\cos \theta = \frac{3}{5}$ and $\sin \theta = -\frac{4}{5}$ (these satisfy the first equation) M1 A1

$$R = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{pmatrix} \quad \text{A1}$$

$$R \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \text{ so the image is } (1, 2) \quad \text{M1 A1}$$

$\theta = -53.1^\circ$, that is a clockwise rotation of 53.1° A1

$$\text{Answer: } R = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{pmatrix}; (1, 2); 53.1^\circ \text{ clockwise} \quad \text{8 marks}$$

27 (10 marks)

$$\text{The point } (x, mx) \text{ maps to } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ mx \end{pmatrix} = \begin{pmatrix} (a + bm)x \\ (c + dm)x \end{pmatrix} \quad \text{M1 A1}$$

Since M is non-singular the image of the line is a line through O, and it is $y = mx$ exactly when the image point lies on $y = mx$ E1

$$(c + dm)x = m(a + bm)x, \text{ so } c + dm = am + bm^2 \quad \text{M1}$$

$$bm^2 + (a - d)m - c = 0 \quad \text{A1}$$

$$\text{For A: } a = 1, b = k, c = 2, d = 3, \text{ giving } km^2 - 2m - 2 = 0 \quad \text{M1 A1}$$

$$\text{Exactly one line: repeated root, so the discriminant is zero: } 4 + 8k = 0, \text{ so } k = -\frac{1}{2} \quad \text{M1 A1}$$

$$\text{Then } -\frac{1}{2}m^2 - 2m - 2 = 0, \text{ that is } (m + 2)^2 = 0, \text{ so } m = -2 \text{ and the line is } y = -2x \quad \text{A1}$$

$$\text{Answer: } k = -\frac{1}{2}; y = -2x \quad \text{10 marks}$$

28 (11 marks)

Each element is a multiple of 2: $N = 2 \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$

M1 A1

The second matrix is a rotation of 60° anticlockwise about the origin ($\cos 60^\circ = \frac{1}{2}$, $\sin 60^\circ = \frac{\sqrt{3}}{2}$)

A1

N is a rotation of 60° followed by an enlargement with scale factor 2, both about the origin (in either order)

E1

$N^6 = 2^6 Q^6$, and Q^6 is a rotation of 360° , which is I

M1

$$N^6 = 64I = \begin{pmatrix} 64 & 0 \\ 0 & 64 \end{pmatrix}$$

A1

$2025 = 6 \times 337 + 3$, so $N^{2025} = (N^6)^{337} N^3 = 64^{337} N^3$

M1

$N^3 = 2^3 Q^3$, a rotation of 180° scaled by 8, so $N^3 = -8I$

M1

$$N^{2025} = -8 \times 2^{2022} I = -2^{2025} I$$

A1

$N^{-1} = \frac{1}{2} Q^{-1}$, the inverse of an enlargement of scale factor 2 and a rotation of 60° clockwise: N^{-1}

M1 A1

$$= \begin{pmatrix} \frac{1}{4} & \frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & \frac{1}{4} \end{pmatrix}$$

Answer: $N = 2R(60^\circ)$; $N^6 = 64I$; $N^{2025} = -2^{2025}I$; $N^{-1} = \begin{pmatrix} \frac{1}{4} & \frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & \frac{1}{4} \end{pmatrix}$

11 marks**29 (11 marks)**

$M \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} p+q \\ r+s \end{pmatrix}$ must be parallel to $(1, 1)$, so $p+q = r+s$

M1 A1

$M \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} p-q \\ r-s \end{pmatrix}$ must be parallel to $(1, -1)$, so $p-q = -(r-s) = s-r$

M1

Adding: $2p = 2s$ and subtracting: $2q = 2r$, so $s = p$ and $r = q$

A1

$M \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} p \\ r \end{pmatrix}$, so $p = 4$ and $q = 2$: $M = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$

M1 A1

$\det M = 16 - 4 = 12$

B1

$M \begin{pmatrix} t \\ t \end{pmatrix} = \begin{pmatrix} 6t \\ 6t \end{pmatrix} = 6 \begin{pmatrix} t \\ t \end{pmatrix}$, so points of $y = x$ are mapped to points of $y = x$ with the distance from O

M1 A1

multiplied by 6

$M \begin{pmatrix} t \\ -t \end{pmatrix} = \begin{pmatrix} 2t \\ -2t \end{pmatrix} = 2 \begin{pmatrix} t \\ -t \end{pmatrix}$, so the scale factor along $y = -x$ is 2

A1

The transformation stretches by 6 along $y = x$ and by 2 along $y = -x$, which are perpendicular directions, so areas are multiplied by $6 \times 2 = 12 = \det M$

E1

Answer: $M = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$; scale factors 6 and 2

11 marks

30 (11 marks)

$$\det M = 6 - 1 = 5 \text{ and } M^{-1} = \frac{1}{5} \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}$$

M1 A1

If (X, Y) is a point of C' , then $(x, y) = M^{-1}(X, Y)$ lies on C , where $x = \frac{2X - Y}{5}$ and $y = \frac{3Y - X}{5}$

M1 A1

$$\frac{(2X - Y)^2 + (3Y - X)^2}{25} = 1$$

M1

$$5X^2 - 10XY + 10Y^2 = 25, \text{ so } X^2 - 2XY + 2Y^2 = 5$$

A1

$$(1, 0) \rightarrow (3, 1): 9 - 6 + 2 = 5, \text{ and } (0, 1) \rightarrow (1, 2): 1 - 4 + 8 = 5$$

B1

$$\text{x-axis } (y = 0): x^2 = 5, \text{ so } (\pm\sqrt{5}, 0)$$

M1 A1

$$\text{y-axis } (x = 0): 2y^2 = 5, \text{ so } y = \pm\sqrt{\frac{5}{2}}$$

A1

$$\text{Area} = |\det M| \times \pi = 5\pi$$

B1

$$\text{Answer: } x^2 - 2xy + 2y^2 = 5; (\pm\sqrt{5}, 0), (0, \pm\sqrt{\frac{5}{2}}); \text{ area } 5\pi$$

11 marks

Combinations of Transformations**1 (2 marks)**

The matrix for A followed by B is BA

M1

$$BA = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

A1

$$\text{Answer: } \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

2 marks

2 (3 marks)

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

M1 A1

A rotation of 180° about the origin

B1

$$\text{Answer: } \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}; \text{ rotation of } 180^\circ \text{ about O}$$

3 marks

3 (3 marks)

$$A^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

M1 A1

A rotation of 180° about the origin

B1

$$\text{Answer: } \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}; \text{ rotation of } 180^\circ \text{ about O}$$

3 marks

4 (3 marks)

$$B^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

M1 A1

Reflecting twice in the same line returns every point to its original position

E1

$$\text{Answer: } B^2 = I$$

3 marks

5 (2 marks)

The inverse is a rotation of 90° clockwise about the origin

B1

$$A^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

B1

$$\text{Answer: } \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

2 marks

6 (4 marks)

The reflection has matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and the rotation has matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ B1

$$\text{Combined matrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \quad \text{M1 A1}$$

A reflection in the line $y = -x$ A1

Answer: $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$; reflection in the line $y = -x$ 4 marks

7 (4 marks)

$$\text{Combined matrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{M1}$$

$$= \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \quad \text{A1}$$

$$\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -6 \end{pmatrix}, \text{ so the image is } (2, -6) \quad \text{M1 A1}$$

Answer: $\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$; (2, -6) 4 marks

8 (3 marks)

$$\text{Combined matrix} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{M1}$$

$$= \begin{pmatrix} 3 & 6 \\ 0 & 1 \end{pmatrix} \quad \text{A1}$$

$$\begin{pmatrix} 3 & 6 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 9 \\ 1 \end{pmatrix}, \text{ so the image is } (9, 1) \quad \text{A1}$$

Answer: $\begin{pmatrix} 3 & 6 \\ 0 & 1 \end{pmatrix}$; (9, 1) 3 marks

9 (5 marks)

$$BA = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \text{ (rotate, then reflect), a reflection in the line } y = -x \quad \text{M1 A1}$$

$$AB = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ (reflect, then rotate), a reflection in the line } y = x \quad \text{M1 A1}$$

$BA \neq AB$, so the order of the transformations matters E1

Answer: BA: reflection in $y = -x$; AB: reflection in $y = x$ 5 marks

10 (5 marks)

$$\text{Matrix of } T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & -3 \\ 3 & 0 \end{pmatrix} \quad \text{M1 A1}$$

$$T^{-1} = \begin{pmatrix} 0 & \frac{1}{3} \\ -\frac{1}{3} & 0 \end{pmatrix} \quad \text{M1 A1}$$

T^{-1} is a rotation of 90° clockwise followed by an enlargement with scale factor $\frac{1}{3}$ B1

$$\text{Answer: } T^{-1} = \begin{pmatrix} 0 & \frac{1}{3} \\ -\frac{1}{3} & 0 \end{pmatrix} \quad \text{5 marks}$$

11 (4 marks)

Matrices: reflection $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, rotation $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, stretch $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

B1

Combined = $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (last transformation on the left)

M1

$$= \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$$

A1

$$\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \text{ so the image is } (1, 4)$$

A1

Answer: $\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}; (1, 4)$

4 marks**12 (4 marks)**

The stretch has matrix $\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$ and the reflection has matrix $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

B1

Combined matrix = $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & -3 \\ -1 & 0 \end{pmatrix}$

M1 A1

$$\begin{pmatrix} 0 & -3 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \end{pmatrix}, \text{ so the image is } (-3, -2)$$

A1

Answer: $\begin{pmatrix} 0 & -3 \\ -1 & 0 \end{pmatrix}; (-3, -2)$

4 marks**13 (5 marks)**

$$A = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \text{ and } B = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

B1 B1

$$AB = \begin{pmatrix} \frac{1}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} \cdot \frac{1}{2} & -\frac{1}{2} \cdot \frac{1}{2} - \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{1}{2} \cdot \frac{1}{2} & -\frac{\sqrt{3}}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{\sqrt{3}}{2} \end{pmatrix}$$

M1

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

A1

Rotating through 30° and then 60° is the same as rotating through 90° , and this is the matrix of a 90° anticlockwise rotation

E1

Answer: $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

5 marks

14 (4 marks)

$$\text{Reflection in } l (\theta = 30^\circ): \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \quad \text{B1}$$

$$\text{Combined matrix} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{M1}$$

$$= \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \quad \text{A1}$$

A rotation of 60° anticlockwise about the origin (twice the angle between the mirror lines) A1

$$\text{Answer: } \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}; \text{ rotation of } 60^\circ \text{ anticlockwise about O} \quad \text{4 marks}$$

15 (5 marks)

$$U = TS, \text{ so } S = T^{-1}U \quad \text{M1 A1}$$

$$T^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \quad \text{M1 A1}$$

$$S = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{A1}$$

$$\text{Answer: } S = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{5 marks}$$

16 (6 marks)

$$\text{Matrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \quad \text{M1 A1}$$

$$(x, y) \rightarrow (X, Y) = (2y, -2x), \text{ so } x = -\frac{Y}{2} \text{ and } y = \frac{X}{2} \quad \text{M1 A1}$$

$$\text{Substituting into } y = 2x + 1: \frac{X}{2} = -Y + 1, \text{ so } X + 2Y = 2 \quad \text{M1}$$

$$\text{The image is the line } y = -\frac{1}{2}x + 1 \quad \text{A1}$$

$$\text{Answer: } y = -\frac{1}{2}x + 1 \quad \text{6 marks}$$

17 (5 marks)

$$\text{Combined matrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} k & 2 \\ 0 & 1 \end{pmatrix} \quad \text{M1 A1}$$

$$\det = k, \text{ so } |k| \times 3 = 12 \quad \text{M1}$$

$$k = 4 \text{ or } k = -4 \quad \text{A1}$$

$$\text{For } k = -4: \begin{pmatrix} -4 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \text{ so the image is } (-2, 1) \quad \text{A1}$$

$$\text{Answer: } k = \pm 4; (-2, 1) \quad \text{5 marks}$$

18 (4 marks)

$$\begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix} = 3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{M1}$$

A reflection in $y = x$ followed by an enlargement with scale factor 3 (centre the origin) A1

$\det = 0 - 9 = -9$, so areas are multiplied by 9 (and orientation is reversed) M1 A1

Answer: Reflection in $y = x$ with enlargement sf 3; area $\times 9$ 4 marks

19 (5 marks)

$$\text{Combined matrix} = \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 3 \\ 4 & 3 \end{pmatrix}$$

M1 A1

$$\det = 6 \times 3 - 3 \times 4 = 6$$

M1

$$\text{Area of image} = 6 \times 6 = 36 \text{ square units}$$

A1

$$\text{Check: } \det T_1 \times \det T_2 = 1 \times 6 = 6$$

B1

$$\text{Answer: } \begin{pmatrix} 6 & 3 \\ 4 & 3 \end{pmatrix}; \text{ area } 36$$

5 marks

20 (6 marks)

The reflection gives (y, x) and the translation then gives $x' = y + 3, y' = x - 1$

M1

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 3 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}, \text{ so } N = \begin{pmatrix} 0 & 1 & 3 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

A1

$$\begin{pmatrix} 0 & 1 & 3 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 1 \\ 1 \end{pmatrix}, \text{ so } (2, 5) \rightarrow (8, 1)$$

M1 A1

$$N^{-1} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -3 \\ 0 & 0 & 1 \end{pmatrix}$$

M1

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 6 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \\ 1 \end{pmatrix}, \text{ so } (7, 1) \text{ is mapped onto } (4, 6)$$

A1

$$\text{Answer: } N = \begin{pmatrix} 0 & 1 & 3 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix}; (8, 1); (7, 1)$$

6 marks

21 (6 marks)

$$R(\theta)R(\varphi) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix}$$

M1

$$= \begin{pmatrix} \cos \theta \cos \varphi - \sin \theta \sin \varphi & -\cos \theta \sin \varphi - \sin \theta \cos \varphi \\ \sin \theta \cos \varphi + \cos \theta \sin \varphi & -\sin \theta \sin \varphi + \cos \theta \cos \varphi \end{pmatrix}$$

M1 A1

$$R(\theta + \varphi) = \begin{pmatrix} \cos(\theta + \varphi) & -\sin(\theta + \varphi) \\ \sin(\theta + \varphi) & \cos(\theta + \varphi) \end{pmatrix}$$

B1

$$\text{Equating the elements in the first column: } \cos(\theta + \varphi) = \cos \theta \cos \varphi - \sin \theta \sin \varphi$$

A1

$$\sin(\theta + \varphi) = \sin \theta \cos \varphi + \cos \theta \sin \varphi$$

A1

$$\text{Answer: } \cos(\theta + \varphi) = \cos \theta \cos \varphi - \sin \theta \sin \varphi; \sin(\theta + \varphi) = \sin \theta \cos \varphi + \cos \theta \sin \varphi$$

6 marks

22 (7 marks)

$$\theta = 15^\circ: \cos 30^\circ = \frac{\sqrt{3}}{2}, \sin 30^\circ = \frac{1}{2}, \text{ so the first matrix is } \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$$

M1 A1

$$\theta = 60^\circ: \cos 120^\circ = -\frac{1}{2}, \sin 120^\circ = \frac{\sqrt{3}}{2}, \text{ so the second matrix is } \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

M1 A1

$$\text{Combined} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

M1 A1

A rotation of 90° anticlockwise about the origin

A1

$$\text{Answer: } \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}; \text{ rotation of } 90^\circ \text{ anticlockwise about O}$$

7 marks

23 (6 marks)

$$\text{Shear then stretch: } \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2k \\ 0 & 3 \end{pmatrix}$$

M1 A1

$$\text{Comparing with M: } 2k = 1, \text{ so } k = \frac{1}{2}$$

A1

$$\text{Stretch then shear: } \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 3c \\ 0 & 3 \end{pmatrix}$$

M1 A1

$$3c = 1, \text{ so } c = \frac{1}{3}$$

A1

$$\text{Answer: } k = \frac{1}{2}; c = \frac{1}{3}$$

6 marks

24 (6 marks)

$$\text{Combined matrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

M1

$$= \begin{pmatrix} \cos \theta & -\sin \theta \\ -\sin \theta & -\cos \theta \end{pmatrix}$$

M1 A1

$$\text{This has the form } \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix} \text{ with } 2\alpha = -\theta, \text{ since } \cos(-\theta) = \cos \theta \text{ and } \sin(-\theta) = -\sin \theta$$

M1 A1

So it is a reflection in the line through the origin at angle $\alpha = -\frac{\theta}{2}$ to the x-axis

A1

Answer: Reflection in the line at angle $-\frac{\theta}{2}$ to the x-axis

6 marks

25 (6 marks)

$$\text{Combined matrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$$

M1 A1

$$(x, y) \rightarrow (X, Y) = (2y, x), \text{ so } x = Y \text{ and } y = \frac{X}{2}$$

M1 A1

$$\text{Substituting into } y = 3x - 1: \frac{X}{2} = 3Y - 1$$

M1

$$X = 6Y - 2, \text{ so the image is the line } y = \frac{1}{6}x + \frac{1}{3}$$

A1

$$\text{Answer: } y = \frac{1}{6}x + \frac{1}{3}$$

6 marks

26 (6 marks)

Reflection in $y = x$ takes $(1, 0)$ to $(0, 1)$, and the stretch then leaves $(0, 1)$ unchanged, so the correct image of $(1, 0)$ is $(0, 1)$

M1 A1

$$\text{The student's matrix gives } \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} = (0, 2)$$

M1 A1

The images differ, so the matrix is wrong: when combining transformations the first transformation is written on the right, so the product should be (stretch)(reflection)

E1

$$\text{Correct matrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$$

A1

$$\text{Answer: Correct matrix } \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$$

6 marks

27 (8 marks)Combined matrix = $F(\beta)F(\alpha)$

M1

(1, 1) element: $\cos 2\beta \cos 2\alpha + \sin 2\beta \sin 2\alpha = \cos(2\beta - 2\alpha)$

M1 A1

(1, 2) element: $\cos 2\beta \sin 2\alpha - \sin 2\beta \cos 2\alpha = -\sin(2\beta - 2\alpha)$

A1

(2, 1) element: $\sin 2\beta \cos 2\alpha - \cos 2\beta \sin 2\alpha = \sin(2\beta - 2\alpha)$, and the (2, 2) element equals $\cos(2\beta - 2\alpha)$

A1

The matrix is $\begin{pmatrix} \cos 2(\beta - \alpha) & -\sin 2(\beta - \alpha) \\ \sin 2(\beta - \alpha) & \cos 2(\beta - \alpha) \end{pmatrix}$, a rotation anticlockwise through $2(\beta - \alpha)$

A1

The first mirror is the x-axis, so $\alpha = 0$, and we need $2\beta = 100^\circ$

M1

 $\beta = 50^\circ$, so the second mirror line is $y = x \tan 50^\circ$

A1

Answer: Rotation through $2(\beta - \alpha)$; mirror lines $y = 0$ then $y = x \tan 50^\circ$

8 marks

28 (11 marks) $NR = N(N^2) = N^3$ and $RN = N^2N = N^3$, so $NR = RN$

M1 A1

$$NR = \begin{pmatrix} b & -a \\ d & -c \end{pmatrix} \text{ and } RN = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix}$$

M1 A1

Equating: $b = -c$ and $-a = -d$, so $d = a$ and $c = -b$

M1

Writing $p = a$ and $q = c$ gives $N = \begin{pmatrix} p & -q \\ q & p \end{pmatrix}$

A1

$$N^2 = \begin{pmatrix} p^2 - q^2 & -2pq \\ 2pq & p^2 - q^2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

M1

$$p^2 = q^2 \text{ and } 2pq = 1, \text{ so } p = q \text{ and } 2p^2 = 1, p = q = \pm \frac{1}{\sqrt{2}}$$

M1 A1

$$N = \pm \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \pm \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$$

A1

 N is a rotation through 45° anticlockwise (with matrix $+$) or through 225° anticlockwise (with matrix $-$)

A1

Answer: $N = \pm \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$: rotations of 45° or 225° anticlockwise

11 marks

29 (10 marks)

Stretch $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ and rotation $\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$

B1

$$T = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \sqrt{3} & -\frac{1}{2} \\ 1 & \frac{\sqrt{3}}{2} \end{pmatrix}$$

M1 A1

det $T = 2$ (the stretch has determinant 2 and the rotation has determinant 1)

B1

$$T^{-1} = \begin{pmatrix} \frac{\sqrt{3}}{4} & \frac{1}{4} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

M1 A1

A point (X, Y) of the image comes from $(x, y) = T^{-1}(X, Y)$, so $x = \frac{\sqrt{3}X + Y}{4}$ and $y = \frac{-X + \sqrt{3}Y}{2}$

M1

$$x^2 + y^2 = \frac{(\sqrt{3}X + Y)^2}{16} + \frac{(-2X + 2\sqrt{3}Y)^2}{16} = 1$$

M1

$(3X^2 + 2\sqrt{3}XY + Y^2) + (4X^2 - 8\sqrt{3}XY + 12Y^2) = 16$, so the image is $7x^2 - 6\sqrt{3}xy + 13y^2 = 16$

A1

Area enclosed = $|\det T| \times \pi = 2\pi$

A1

Answer: $T = \begin{pmatrix} \sqrt{3} & -\frac{1}{2} \\ 1 & \frac{\sqrt{3}}{2} \end{pmatrix}$; **area 2π**

10 marks**30 (10 marks)**

$\tan \alpha = 2$ gives $\cos \alpha = \frac{1}{\sqrt{5}}$ and $\sin \alpha = \frac{2}{\sqrt{5}}$

B1 B1

The matrix is $\text{Rot}(\alpha) \times \text{Refl}_x \times \text{Rot}(-\alpha)$

M1

$$\text{Rot}(-\alpha) = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}$$

A1

$$\text{Refl}_x \text{Rot}(-\alpha) = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{pmatrix}$$

M1

$$\text{Rot}(\alpha) \times \text{this} = \frac{1}{5} \begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}$$

M1

$$= \begin{pmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}$$

A1

$$\begin{pmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{9}{5} \\ \frac{13}{5} \end{pmatrix}, \text{ so } (1, 3) \rightarrow \left(\frac{9}{5}, \frac{13}{5}\right)$$

M1 A1

With $\theta = \alpha$: $\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha = -\frac{3}{5}$ and $\sin 2\alpha = 2 \sin \alpha \cos \alpha = \frac{4}{5}$, which match the matrix

B1

Answer: $\begin{pmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}; \left(\frac{9}{5}, \frac{13}{5}\right)$

10 marks

Three-Dimensional Matrix Transformations

1 (2 marks)

$$(x, y, z) \rightarrow (x, y, -z)$$

M1

$$\text{Matrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

A1

$$\text{Answer: } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

2 marks

2 (2 marks)

$$\text{Matrix} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

B1

$$\text{Image of } (1, -2, 3) \text{ is } (4, -8, 12)$$

B1

$$\text{Answer: } \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}; (4, -8, 12)$$

2 marks

3 (2 marks)

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 1 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ 9 \\ 4 \end{pmatrix}$$

M1 A1

$$\text{Answer: } (7, 9, 4)$$

2 marks

4 (2 marks)

$$i \rightarrow j, j \rightarrow -i \text{ and } k \rightarrow k$$

M1

$$\text{Matrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

A1

$$\text{Answer: } \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2 marks

5 (2 marks)

$$i \rightarrow i, j \rightarrow j \text{ and } k \rightarrow 3k$$

M1

$$\text{Matrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

A1

$$\text{Answer: } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

2 marks

6 (3 marks)

$$i \rightarrow i, \text{ so the } x\text{-axis is invariant}$$

B1

$$j \rightarrow k \text{ and } k \rightarrow -j$$

M1

$$\text{A rotation of } 90^\circ \text{ anticlockwise about the } x\text{-axis (viewed from the positive } x\text{-axis)}$$

A1

$$\text{Answer: Rotation of } 90^\circ \text{ anticlockwise about the } x\text{-axis}$$

3 marks

7 (3 marks)

A reflection in $x = y$ swaps the x and y coordinates and leaves z unchanged

M1

$$\text{Matrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

A1

Image of $(3, -1, 5)$ is $(-1, 3, 5)$

B1

$$\text{Answer: } \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}; (-1, 3, 5)$$

3 marks

8 (4 marks)

$i \rightarrow -k, j \rightarrow j$ and $k \rightarrow i$

M1

$$\text{Matrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

A1

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}, \text{ so the image is } (3, 2, -1)$$

M1 A1

$$\text{Answer: } \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}; (3, 2, -1)$$

4 marks

9 (4 marks)

$$\det A = 2(12 - 0) - 0 + 0 = 24$$

M1 A1

Volume scale factor = $|\det A| = 24$, so the image has volume $24 \times 5 = 120$ cubic units

M1 A1

Answer: $\det A = 24$; volume 120

4 marks

10 (4 marks)

$$\text{Combined matrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

M1

$$= \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

A1

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix}, \text{ so the image is } (-2, 1, -3)$$

M1 A1

$$\text{Answer: } \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}; (-2, 1, -3)$$

4 marks

11 (4 marks)

$$\text{The images of } i, j \text{ and } k \text{ form the columns: } \begin{pmatrix} 2 & 0 & 1 \\ 1 & 3 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

M1 A1

$$\begin{pmatrix} 2 & 0 & 1 \\ 1 & 3 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}, \text{ so } (1, 1, 1) \rightarrow (3, 4, 2)$$

M1

$$\det = 2(3 - 0) - 0 + 1(1 - 0) = 7$$

A1

$$\text{Answer: } \begin{pmatrix} 2 & 0 & 1 \\ 1 & 3 & 0 \\ 0 & 1 & 1 \end{pmatrix}; (3, 4, 2); \det = 7$$

4 marks

12 (4 marks)**(a)** Reflecting twice in the same plane gives the identity, so the inverse is the same reflection:**B1 B1**

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Answer: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

2 marks**(b)** The inverse is a rotation of 90° clockwise about the z-axis**M1**

Matrix = $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

A1

Answer: $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

2 marks**13 (3 marks)**

The matrix is $2 \times \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

M1

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is a reflection in the plane $x = 0$ (the yz-plane)**A1**

The factor 2 is an enlargement with scale factor 2 and centre the origin

A1**Answer: A reflection in the plane $x = 0$ combined with an enlargement of scale factor 2, centre O****3 marks****14 (6 marks)**A point of l maps to $A(p + td) = Ap + tAd$ **M1**

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

M1 A1

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$$

M1 A1

The image has vector equation $r = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$

A1

Answer: $r = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$

6 marks**15 (5 marks)**A point (x, y, z) maps to $(X, Y, Z) = (x, x + y, z)$ **M1**So $x = X$, $y = Y - X$ and $z = Z$ **M1 A1**Substituting into $2x - y + z = 4$: $2X - (Y - X) + Z = 4$ **M1**The image is the plane $3x - y + z = 4$ **A1****Answer: $3x - y + z = 4$** **5 marks**

16 (5 marks)

$A^2 = I$ and $\det A = -1$, as for a reflection

M1 A1

Invariant points satisfy $A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$: $y = z$, $z = y$ and $x = x$

M1

So every point with $y = z$ is invariant, which is the plane $y = z$

A1

A reflection in the plane $y = z$

A1

Answer: Reflection in the plane $y = z$

5 marks

17 (4 marks)

$$\text{Matrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

M1 A1

$$\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ \sqrt{3} \\ 5 \end{pmatrix}$$

M1

Image is $(1, \sqrt{3}, 5)$

A1

$$\text{Answer: } \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}; (1, \sqrt{3}, 5)$$

4 marks

18 (5 marks)

$\det A = 2(k - 0) - 1(0 - 1) + 0 = 2k + 1$

M1 A1

$|\det A| \times 5 = 55$, so $|2k + 1| = 11$

M1

$2k + 1 = 11$ or $2k + 1 = -11$

M1

$k = 5$ or $k = -6$

A1

Answer: $k = 5$ or $k = -6$

5 marks

19 (5 marks)

The matrices are $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, $\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ respectively

B1

$$\text{Combined} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

M1

$$= \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

A1

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ -6 \end{pmatrix}, \text{ so the image is } (-2, 1, -6)$$

M1 A1

$$\text{Answer: } \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}; (-2, 1, -6)$$

5 marks

20 (4 marks)

(a) $(x, y, z) \rightarrow (-x, -y, z)$ and the z -axis is invariant

M1

A rotation of 180° about the z -axis

A1

Answer: Rotation of 180° about the z -axis

2 marks

(b) $(x, y, z) \rightarrow (y, x, z)$ and every point with $x = y$ is invariant

M1

A reflection in the plane $x = y$

A1

Answer: Reflection in the plane $x = y$

2 marks

21 (6 marks)

$$R = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

M1 A1

$$\begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \sqrt{2} \\ 1 \end{pmatrix}$$

M1

Image is $(0, \sqrt{2}, 1)$

A1

R^8 is a rotation of $8 \times 45^\circ = 360^\circ$ about the z-axis

M1

A rotation through a full turn leaves every point where it is, so $R^8 = I$

E1

Answer: $\begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}; (0, \sqrt{2}, 1)$

6 marks

22 (8 marks)

The reflections have matrices $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

B1 B1

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

M1 A1

$$C^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } C^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

M1 A1

$$C \begin{pmatrix} t \\ t \\ t \end{pmatrix} = \begin{pmatrix} t \\ t \\ t \end{pmatrix}, \text{ so the line } x = y = z \text{ consists of invariant points}$$

M1

$C^3 = I$, so the angle is 120° or 240° ; C sends $i \rightarrow k \rightarrow j \rightarrow i$, the opposite sense to an anticlockwise rotation (which would send $i \rightarrow j \rightarrow k \rightarrow i$)

A1

Answer: $C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}; \text{ rotation of } 120^\circ \text{ clockwise (} 240^\circ \text{ anticlockwise)}$

8 marks

23 (6 marks)

A point (x, y, z) maps to $(X, Y, Z) = (z, y, -x)$

M1 A1

So $x = -Z$, $y = Y$ and $z = X$

M1 A1

Substituting: $-Z + 2Y - X = 3$

M1

The image plane is $x - 2y + z = -3$

A1

Answer: $x - 2y + z = -3$

6 marks

24 (7 marks)

$$S \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + kz \\ y \\ z \end{pmatrix}$$

M1 A1

Invariant points need $kz = 0$, so $z = 0$: every point of the plane $z = 0$ is invariant

M1 A1

$$S^2 = \begin{pmatrix} 1 & 0 & 2k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ and in general } S^n = \begin{pmatrix} 1 & 0 & nk \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

M1 A1

 $\det S = 1$, so the unit cube maps to a solid of volume 1

A1

Answer: $z = 0$ invariant; S^n has nk in the top right

7 marks

25 (5 marks)

$$\text{Original volume} = |\det \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}| = 1 \times 2 \times 3 = 6$$

M1 A1

$$\det A = 1(1 - 0) - 1(0 - 1) + 0 = 2$$

M1 A1

$$\text{Volume after} = 2 \times 6 = 12$$

A1

Answer: Before 6, after 12

5 marks

26 (7 marks)

$$A^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

M1 A1

$$A^3 = A \cdot A^2 = I$$

A1

$$A^{-1} = A^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

A1

$$A \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \text{ and } A^2 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

M1 A1

A sends $i \rightarrow k$, $j \rightarrow i$ and $k \rightarrow j$ (the coordinates are cycled)

A1

$$\text{Answer: } A^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

7 marks

27 (8 marks)

The image of i is $i - 2(i \cdot n)n = i - \frac{2}{3}n = \begin{pmatrix} \frac{7}{9} \\ -\frac{4}{9} \\ -\frac{4}{9} \end{pmatrix}$ M1 A1

The image of j is $j - 2(j \cdot n)n = j - \frac{4}{3}n$; the image of k is $k - \frac{4}{3}n$ M1 A1

Placing these as columns gives $N = \begin{pmatrix} \frac{7}{9} & -\frac{4}{9} & -\frac{4}{9} \\ -\frac{4}{9} & \frac{1}{9} & -\frac{8}{9} \\ -\frac{4}{9} & -\frac{8}{9} & \frac{1}{9} \end{pmatrix}$ A1

$N^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$ M1

$\det N = -1$, as for a reflection M1

$N \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \\ -1 \end{pmatrix}$, so the image is $(1, -4, -1)$ A1

Answer: $N = \begin{pmatrix} \frac{7}{9} & -\frac{4}{9} & -\frac{4}{9} \\ -\frac{4}{9} & \frac{1}{9} & -\frac{8}{9} \\ -\frac{4}{9} & -\frac{8}{9} & \frac{1}{9} \end{pmatrix}$; image $(1, -4, -1)$ 8 marks

28 (9 marks)

Row 1 $\cdot$ row 1 $= \frac{1}{4} + \frac{1}{4} + \frac{1}{2} = 1$ and row 1 $\cdot$ row 2 $= \frac{1}{4} + \frac{1}{4} - \frac{1}{2} = 0$; the other pairs of rows M1 A1

work in the same way, so $QQ^T = I$

Expanding along the third column: $\det Q = \frac{\sqrt{2}}{2}(\frac{\sqrt{2}}{4} + \frac{\sqrt{2}}{4}) + \frac{\sqrt{2}}{2}(\frac{\sqrt{2}}{4} + \frac{\sqrt{2}}{4}) = 1$ M1 A1

$Q \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + \frac{1}{2} \\ \frac{1}{2} + \frac{1}{2} \\ -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, so every multiple of $(1, 1, 0)$ is invariant M1 A1

$Q \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \\ 0 \end{pmatrix}$ M1

$(0, 0, 1)$ is perpendicular to the axis, and so is its image; their dot product is 0, so $\cos \theta = 0$ and the angle of rotation is 90° A1 A1

Answer: Axis $(1, 1, 0)$; rotation of 90° 9 marks

29 (9 marks)

$$\det M = 3(4 - 1) = 9$$

M1 A1

$$M^{-1} = \frac{1}{9} \begin{pmatrix} 6 & -3 & 0 \\ -3 & 6 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & 0 \\ -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$$

M1 A1

The point (x, y, z) maps to (X, Y, Z) , where $(x, y, z) = M^{-1}(X, Y, Z)$: $x = \frac{2X - Y}{3}$, $y = \frac{-X + 2Y}{3}$,
 $z = \frac{Z}{3}$

M1

Substituting in $2x - y + 3z = 6$: $\frac{2(2X - Y)}{3} - \frac{(-X + 2Y)}{3} + Z = 6$

M1

$\frac{5X - 4Y}{3} + Z = 6$, so the image plane is $5x - 4y + 3z = 18$

A1

Volume scale factor 9, so $V = \frac{45}{9} = 5$

M1 A1

Answer: $\det M = 9$; $5x - 4y + 3z = 18$; $V = 5$

9 marks

30 (10 marks)

$$N = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

B1

$$N^2 = \begin{pmatrix} 0 & 0 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } N^3 = N^2 N = 0$$

M1 A1

$$(I + N)^n = I + nN + \frac{n(n-1)}{2}N^2 + (\text{terms containing } N^3 = 0)$$

M1 A1

$$B^n = \begin{pmatrix} 1 & 2n & 3n(n-1) \\ 0 & 1 & 3n \\ 0 & 0 & 1 \end{pmatrix}$$

A1

$$n = -1: B^{-1} = I - N + N^2 = \begin{pmatrix} 1 & -2 & 6 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}$$

M1

Check: $B B^{-1} = I$ (the product of the matrices equals the identity)

A1

$$B^{10} = \begin{pmatrix} 1 & 20 & 270 \\ 0 & 1 & 30 \\ 0 & 0 & 1 \end{pmatrix}$$

M1 A1

$$\text{Answer: } B^n \text{ as above; } B^{-1} = \begin{pmatrix} 1 & -2 & 6 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}; B^{10} = \begin{pmatrix} 1 & 20 & 270 \\ 0 & 1 & 30 \\ 0 & 0 & 1 \end{pmatrix}$$

10 marks

Eigenvalues and Eigenvectors**1 (3 marks)**

$$A \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \text{ so it is an eigenvector}$$

M1 A1

The eigenvalue is 3

B1

Answer: Eigenvalue 3

3 marks

2 (2 marks)

The matrix is upper triangular, so the eigenvalues are the diagonal elements

M1

4 and 2

A1

Answer: 4 and 2

2 marks

3 (3 marks)

$$(3 - \lambda)(2 - \lambda) - 2 = \lambda^2 - 5\lambda + 4 = 0$$

M1 A1

$$\lambda = 1 \text{ or } \lambda = 4$$

A1

Answer: $\lambda = 1, 4$

3 marks

4 (2 marks)

$$(A - 4I)\begin{pmatrix} x \\ y \end{pmatrix} = 0 \text{ gives } -x + y = 0$$

M1

$$\text{An eigenvector is } \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

A1

Answer: $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (or any multiple)

2 marks

5 (2 marks)

$$A\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

M1

$$\begin{pmatrix} 5 \\ 3 \end{pmatrix} \text{ is not a multiple of } \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \text{ so it is not an eigenvector}$$

A1

Answer: No

2 marks

6 (2 marks)

The matrix is diagonal; the eigenvalues are 2, -1 and 5

B1

$$\text{with eigenvectors } \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ respectively}$$

B1

Answer: 2, -1, 5 with i, j, k

2 marks

7 (6 marks)

$$\det(A - \lambda I) = (4 - \lambda)(3 - \lambda) - 2 = \lambda^2 - 7\lambda + 10 = 0$$

M1 A1

$$\lambda = 2 \text{ or } \lambda = 5$$

A1

$$\lambda = 2: 2x + 2y = 0, \text{ eigenvector } \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

M1 A1

$$\lambda = 5: -x + 2y = 0, \text{ eigenvector } \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

A1

Answer: $\lambda = 2$ with $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$; $\lambda = 5$ with $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

6 marks

8 (6 marks)

$$\lambda^2 + \lambda - 6 = 0, \text{ so } \lambda = 2 \text{ or } \lambda = -3$$

M1 A1

$$\lambda = 2: -x + 2y = 0, \text{ eigenvector } \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

M1 A1

$$\lambda = -3: 4x + 2y = 0, \text{ eigenvector } \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

A1

$$\text{Unit eigenvectors: } \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ and } \frac{1}{\sqrt{5}}\begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

A1

Answer: $\lambda = 2: \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$; $\lambda = -3: \frac{1}{\sqrt{5}}\begin{pmatrix} -1 \\ 2 \end{pmatrix}$

6 marks

9 (4 marks)

$$A\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2k + 6 \\ 9 \end{pmatrix}$$

M1

$$\text{This equals } \lambda\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \text{ so } 3\lambda = 9 \text{ and } 2\lambda = 2k + 6$$

M1

$$\lambda = 3 \text{ and } k = 0$$

A1 A1

Answer: $k = 0$, eigenvalue 3

4 marks

10 (5 marks)

If $A\mathbf{v} = \lambda\mathbf{v}$ then $A^2\mathbf{v} = \lambda^2\mathbf{v}$ M1

Eigenvalues of A^2 : 1 and 36 A1

Eigenvalues of A^{-1} : 1 and $\frac{1}{6}$ A1

Eigenvalues of $3A + 2I$: $3\lambda + 2 = 5$ and 20 M1 A1

Answer: A^2 : 1, 36; A^{-1} : 1, $\frac{1}{6}$; $3A + 2I$: 5, 20 5 marks

11 (4 marks)

A is upper triangular, so the eigenvalues are 3, 3 and 2 M1 A1

$(A - 2I)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$: $x + y = 0$ and $y + z = 0$ M1

An eigenvector is $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ A1

Answer: 3, 3, 2; $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ 4 marks

12 (4 marks)

$\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0$, so $\lambda = 2$ is repeated M1 A1

$(A - 2I)\begin{pmatrix} x \\ y \end{pmatrix} = 0$ gives $x + y = 0$ M1

Eigenvectors are multiples of $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ (only one independent eigenvector) A1

Answer: $\lambda = 2$ (repeated); eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ 4 marks

13 (4 marks)

$\det(A - \lambda I) = (2 - \lambda)^2 + 1 = \lambda^2 - 4\lambda + 5$ M1 A1

Discriminant = $16 - 20 = -4 < 0$, so there are no real roots (the eigenvalues are $2 \pm i$) M1 A1

Answer: No real eigenvalues; $2 \pm i$ 4 marks

14 (7 marks)

Expanding along the second row: $\det(A - \lambda I) = (3 - \lambda)[(1 - \lambda)(2 - \lambda) - 0]$ M1 A1

Eigenvalues: $\lambda = 3, 1$ and 2 A1

$\lambda = 1$: $2y = 0$ and $2x - 4y + z = 0$, giving the eigenvector $\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$ M1 A1

$\lambda = 2$: $-x + 2y = 0$, $y = 0$ and $2x - 4y = 0$, giving $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ M1

$\lambda = 3$: $-2x + 2y = 0$ and $2x - 4y - z = 0$, giving $\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$ A1

Answer: $\lambda = 1$: $\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$; $\lambda = 2$: $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$; $\lambda = 3$: $\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$ 7 marks

15 (5 marks)

$$(1 - \lambda)^2 + 4 = 0$$

M1 A1

$$\lambda = 1 \pm 2i$$

A1

For $\lambda = 1 + 2i$: $-2ix - 2y = 0$, so $y = -ix$

M1

An eigenvector is $\begin{pmatrix} 1 \\ -i \end{pmatrix}$

A1

Answer: $\lambda = 1 \pm 2i$; eigenvector $\begin{pmatrix} 1 \\ -i \end{pmatrix}$ for $1 + 2i$

5 marks

16 (5 marks)

Invariant lines have direction vectors that are eigenvectors

M1

$$\lambda^2 - 7\lambda + 10 = 0 \text{ gives } \lambda = 2 \text{ and } \lambda = 5$$

A1

$\lambda = 2$: eigenvector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$, line $y = -x$

M1 A1

$\lambda = 5$: eigenvector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, line $y = \frac{1}{2}x$

A1

Answer: $y = -x$ and $y = \frac{1}{2}x$

5 marks

17 (4 marks)

Sum of the eigenvalues = trace: $a + 1 = 3 + (-1) = 2$, so $a = 1$

M1 A1

Product of the eigenvalues = determinant: $a - 2b = -3$

M1

$$1 - 2b = -3, \text{ so } b = 2$$

A1

Answer: $a = 1$, $b = 2$

4 marks

18 (5 marks)

$$(a - \lambda)^2 - b^2 = 0$$

M1 A1

$$\lambda = a \pm b$$

A1

$\lambda = a + b$: $-bx + by = 0$, so $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

M1

$\lambda = a - b$: $bx + by = 0$, so $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

A1

Answer: $a + b$ with $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$; $a - b$ with $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

5 marks

19 (5 marks)

$$\det(A - \lambda I) = (a - \lambda)(d - \lambda) - bc$$

M1 A1

$$A^T - \lambda I = \begin{pmatrix} a - \lambda & c \\ b & d - \lambda \end{pmatrix}, \text{ with determinant } (a - \lambda)(d - \lambda) - cb$$

M1 A1

The characteristic equations are identical, so the eigenvalues are the same

E1

Answer: Same characteristic equation

5 marks

20 (5 marks)

$$\det(P - I) = \left(-\frac{1}{10}\right)\left(-\frac{2}{5}\right) - \left(\frac{2}{5}\right)\left(\frac{1}{10}\right) = 0, \text{ so } 1 \text{ is an eigenvalue}$$

M1 A1

$$(P - I)\begin{pmatrix} x \\ y \end{pmatrix} = 0: -\frac{1}{10}x + \frac{2}{5}y = 0, \text{ so } x = 4y \text{ and an eigenvector is } \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

M1 A1

In the long run the proportions form an eigenvector with $a + b = 1$, so $a = \frac{4}{5}$ and $b = \frac{1}{5}$

A1

Answer: $\frac{4}{5}$ in region A

5 marks

21 (8 marks)

$$\det(A - \lambda I) = (2 - \lambda)[(2 - \lambda)^2 - 1] - [(2 - \lambda) - 1] + [1 - (2 - \lambda)]$$

$$= (2 - \lambda)(1 - \lambda)(3 - \lambda) - 2(1 - \lambda) = (1 - \lambda)[(2 - \lambda)(3 - \lambda) - 2] = (1 - \lambda)(\lambda^2 - 5\lambda + 4) = (1 - \lambda)^2(4 - \lambda)$$

M1 A1

M1 A1

$$\lambda = 4: -2x + y + z = 0 \text{ and } x - 2y + z = 0 \text{ give } x = y = z, \text{ eigenvector } \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

M1 A1

$$\lambda = 1: \text{ the three equations all reduce to } x + y + z = 0$$

M1

$$\text{Two independent solutions are } \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \text{ (neither is a multiple of the other)}$$

A1

$$\text{Answer: } \lambda = 4: \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}; \lambda = 1: \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

8 marks

22 (8 marks)

$$\text{Expanding along the first row: } \det(A - \lambda I) = (2 - \lambda)[(3 - \lambda)^2 - 1] = (2 - \lambda)(2 - \lambda)(4 - \lambda)$$

M1 A1

$$\text{Eigenvalues: } 2 \text{ (repeated) and } 4$$

A1

$$\lambda = 2: x + y + z = 0 \text{ and } -x + y + z = 0 \text{ (from rows 2 and 3), so } x = 0 \text{ and } y = -z$$

M1 A1

$$\text{Only one independent eigenvector: } \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

A1

$$\lambda = 4: -2x = 0, x - y + z = 0, -x + y - z = 0, \text{ giving } \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

M1 A1

$$\text{Answer: } \lambda = 2: \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \text{ (only one); } \lambda = 4: \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

8 marks

23 (7 marks)

$$A \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a+1 \\ b+3 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \text{ so } a+1 = b+3$$

M1 A1

$$A \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} a+2 \\ b+6 \end{pmatrix} = \mu \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \text{ so } b+6 = 2(a+2)$$

M1 A1

$$\text{Solving: } b = 2a - 2 \text{ and } a + 1 = 2a + 1, \text{ so } a = 0 \text{ and } b = -2$$

M1 A1

$$\lambda = a + 1 = 1 \text{ and } \mu = a + 2 = 2$$

A1

$$\text{Answer: } a = 0, b = -2; \text{ eigenvalues } 1 \text{ and } 2$$

7 marks

24 (6 marks)

$$A \text{ and } B \text{ are triangular with eigenvalues } 1, 1 \text{ in each case; the claim predicts eigenvalues } 1 \text{ and } 1 \text{ for } AB$$

B1

$$AB = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

M1 A1

$$\det(AB - \lambda I) = (2 - \lambda)(1 - \lambda) - 1 = \lambda^2 - 3\lambda + 1$$

M1 A1

$$\lambda = \frac{3 \pm \sqrt{5}}{2}, \text{ which are not } 1 \text{ and } 1, \text{ so the claim is false}$$

A1

$$\text{Answer: Claim false: eigenvalues of } AB \text{ are } \frac{3 \pm \sqrt{5}}{2}$$

6 marks

25 (7 marks)

$$\det(F - \lambda I) = -(\cos 2\theta - \lambda)(\cos 2\theta + \lambda) - \sin^2 2\theta = \lambda^2 - \cos^2 2\theta - \sin^2 2\theta = \lambda^2 - 1$$

M1 A1

So $\lambda = 1$ or $\lambda = -1$

A1

$$F \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} = \begin{pmatrix} \cos 2\theta \cos \theta + \sin 2\theta \sin \theta \\ \sin 2\theta \cos \theta - \cos 2\theta \sin \theta \end{pmatrix} = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \text{ using the double angle formulae}$$

M1 A1

$$\text{Similarly } F \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix} = - \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix}$$

A1

Points on the mirror line are unchanged (eigenvalue 1); vectors perpendicular to it are reversed (eigenvalue -1)

E1

Answer: $\lambda = 1$ and -1

7 marks

26 (8 marks) $n = 1$ is the given result

B1

Assume $A^k \mathbf{v} = \lambda^k \mathbf{v}$. Then $A^{k+1} \mathbf{v} = A(\lambda^k \mathbf{v}) = \lambda^k A \mathbf{v} = \lambda^{k+1} \mathbf{v}$

M1 A1

True for $n = 1$ and, if true for k , true for $k + 1$, so true for all n

E1

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ since } -\alpha + 2\beta = 1 \text{ and } \alpha + \beta = 2 \text{ give } \alpha = \beta = 1$$

M1 A1

$$\text{The eigenvalues are 2 and 5, so } A^3 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 2^3 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + 5^3 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

M1

$$= \begin{pmatrix} -8 \\ 8 \end{pmatrix} + \begin{pmatrix} 250 \\ 125 \end{pmatrix} = \begin{pmatrix} 242 \\ 133 \end{pmatrix}$$

A1

$$\text{Answer: } \begin{pmatrix} 242 \\ 133 \end{pmatrix}$$

8 marks

27 (8 marks)The characteristic equation is $\lambda^2 - (a + d)\lambda + (ad - bc) = 0$

M1 A1

Since λ_1 and λ_2 are its roots, $(\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2$

M1

Comparing coefficients: $\lambda_1 + \lambda_2 = a + d$ and $\lambda_1\lambda_2 = ad - bc$

A1

For A: the trace is $k + 6 - k = 6$, so $\lambda + 2\lambda = 6$ and $\lambda = 2$, the eigenvalues being 2 and 4

M1 A1

The determinant is $k(6 - k) + 3 = 8$

M1

$$k^2 - 6k + 5 = 0, \text{ so } k = 1 \text{ or } k = 5$$

A1

Answer: $k = 1$ or 5 ; eigenvalues 2 and 4

8 marks

28 (11 marks)

$$(\lambda - 3)(\lambda^2 - 15\lambda + 54) = 0$$

M1 A1

 $(\lambda - 6)(\lambda - 9) = 0$, so the other eigenvalues are 6 and 9

M1 A1

$$\lambda = 3: 2x - 2y = 0 \text{ and } -2x + 3y + 2z = 0 \text{ give } \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}$$

M1 A1

$$\lambda = 6: -x - 2y = 0 \text{ and } 2y + z = 0 \text{ give } \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$

A1

$$\lambda = 9: -4x - 2y = 0 \text{ and } -2x - 3y + 2z = 0 \text{ give } \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$$

A1

Dot products: $(-2)(2) + (-2)(-1) + (1)(2) = 0$, $(-2)(-1) + (-2)(2) + (1)(2) = 0$ and $(2)(-1) + (-1)(2) + (2)(2) = 0$, so they are mutually perpendicular

M1 A1

$$\text{Unit eigenvector for } \lambda = 9: \frac{1}{3} \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$$

A1

$$\text{Answer: } \lambda = 3, 6, 9; \text{ unit eigenvector for } 9: \frac{1}{3} \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$$

11 marks

29 (10 marks)

Suppose $\alpha v_1 + \beta v_2 = 0$ (1)

B1

Multiply by A: $\alpha \lambda_1 v_1 + \beta \lambda_2 v_2 = 0$ (2)

M1 A1

Multiply (1) by λ_1 and subtract from (2): $\beta(\lambda_2 - \lambda_1)v_2 = 0$

M1 A1

$v_2 \neq 0$ (an eigenvector) and $\lambda_2 \neq \lambda_1$, so $\beta = 0$; then (1) gives $\alpha v_1 = 0$, so $\alpha = 0$

M1 A1

Two linearly independent vectors in the plane span the plane, so every w is $\alpha v_1 + \beta v_2$

E1

$A^n w = \alpha A^n v_1 + \beta A^n v_2 = \alpha \lambda_1^n v_1 + \beta \lambda_2^n v_2$

M1 A1

Answer: v_1 and v_2 are linearly independent; $A^n w = \alpha \lambda_1^n v_1 + \beta \lambda_2^n v_2$

10 marks

30 (10 marks)

$(2 - \lambda)^2 - k = 0$

M1 A1

$\lambda = 2 \pm \sqrt{k}$

A1

Two distinct real eigenvalues when $k > 0$

A1

$k = 4$: $\lambda = 4$ or $\lambda = 0$ ($\lambda = 2 \pm 2$)

B1

$\lambda = 4$: $-2x + 4y = 0$, eigenvector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, line $y = \frac{1}{2}x$

M1 A1

$\lambda = 0$: $2x + 4y = 0$, eigenvector $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$, line $y = -\frac{1}{2}x$

A1

$k = 0$: repeated eigenvalue 2 with a single independent eigenvector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$; $k < 0$: complex

A1 A1

eigenvalues $2 \pm i\sqrt{-k}$, so there are no real invariant lines through the origin

Answer: $\lambda = 2 \pm \sqrt{k}$; $k > 0$

10 marks

Diagonalisation of Matrices**1 (2 marks)**

Raise each diagonal element to the power 4: $D^4 = \begin{pmatrix} 16 & 0 \\ 0 & 81 \end{pmatrix}$

M1 A1

Answer: $\begin{pmatrix} 16 & 0 \\ 0 & 81 \end{pmatrix}$

2 marks

2 (2 marks)

$\det P = 2 - 1 = 1$

M1

$P^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$

A1

Answer: $\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$

2 marks

3 (2 marks)

The columns of P are the eigenvectors: $P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

B1

D has the eigenvalues in the same order: $D = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$

B1

Answer: $P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $D = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$

2 marks

4 (2 marks)

$$D^3 = \begin{pmatrix} 8 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 27 \end{pmatrix}$$

M1 A1

Answer: $\begin{pmatrix} 8 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 27 \end{pmatrix}$

2 marks

5 (2 marks)

The three eigenvalues are distinct, so there are three linearly independent eigenvectors so the matrix P of eigenvectors is invertible and the matrix can be diagonalised

M1

E1

Answer: Three distinct eigenvalues

2 marks

6 (2 marks)

$$AP = \begin{pmatrix} 3 & 1 \\ 3 & -1 \end{pmatrix}$$

M1

$$PD = \begin{pmatrix} 3 & 1 \\ 3 & -1 \end{pmatrix} = AP$$

A1

Answer: $AP = PD = \begin{pmatrix} 3 & 1 \\ 3 & -1 \end{pmatrix}$

2 marks

7 (4 marks)

$$P = \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix} \text{ and } D = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}$$

B1 B1

$$\det P = -1 - 2 = -3$$

M1

$$P^{-1} = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

A1

Answer: $P = \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix}, D = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}, P^{-1} = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}$

4 marks

8 (5 marks)

$$A^4 = PD^4P^{-1}$$

M1

$$D^4 = \begin{pmatrix} 81 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } P^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

B1 B1

$$A^4 = \begin{pmatrix} 41 & 40 \\ 40 & 41 \end{pmatrix}$$

M1 A1

Answer: $\begin{pmatrix} 41 & 40 \\ 40 & 41 \end{pmatrix}$

5 marks

9 (4 marks)

$$A = PDP^{-1} \text{ with } P = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \text{ and } D = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}$$

M1

$$P^{-1} = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$$

A1

$$PD = \begin{pmatrix} 4 & -2 \\ 4 & -3 \end{pmatrix}$$

M1

$$A = \begin{pmatrix} 14 & -10 \\ 15 & -11 \end{pmatrix}$$

A1

Answer: $A = \begin{pmatrix} 14 & -10 \\ 15 & -11 \end{pmatrix}$

4 marks

10 (5 marks)

$$\lambda = 9: -4x + 4y = 0, \text{ eigenvector } \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \lambda = 1: 4x + 4y = 0, \text{ eigenvector } \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

M1 A1

$$\text{Normalise: } \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

M1

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

A1

$$D = \begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix}$$

A1

$$\text{Answer: } P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, D = \begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix}$$

5 marks

11 (3 marks)

The matrix is upper triangular, so the eigenvalues are 2, 3 and 5

B1

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

B1

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

B1

$$\text{Answer: } P = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}, D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

3 marks

12 (5 marks)

$$\lambda = 2: x + 2y = 0, \begin{pmatrix} -2 \\ 1 \end{pmatrix}; \lambda = 5: -2x + 2y = 0, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

M1 A1

$$P = \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix}, D = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix} \text{ and } P^{-1} = -\frac{1}{3} \begin{pmatrix} 1 & -1 \\ -1 & -2 \end{pmatrix}$$

M1

$$A^3 = PD^3P^{-1} = \begin{pmatrix} 47 & 78 \\ 39 & 86 \end{pmatrix}$$

M1 A1

$$\text{Answer: } A^3 = \begin{pmatrix} 47 & 78 \\ 39 & 86 \end{pmatrix}$$

5 marks

13 (4 marks)

Eigenvalues 2 and 3 (upper triangular)

B1

$$\lambda = 2: \text{eigenvector } \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \lambda = 3: -x + y = 0, \text{ eigenvector } \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

M1 A1

$$P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

A1

$$\text{Answer: } P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

4 marks

14 (6 marks)

$$A^n = PD^nP^{-1} \text{ with } D^n = \begin{pmatrix} 2^n & 0 \\ 0 & 5^n \end{pmatrix} \text{ and } P^{-1} = \frac{1}{3} \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix}$$

M1 A1

$$PD^n = \begin{pmatrix} -2^n & 2 \times 5^n \\ 2^n & 5^n \end{pmatrix}$$

M1 A1

Multiplying by P^{-1} : top left $\frac{1}{3}(2^n + 2 \times 5^n)$, top right $\frac{1}{3}(-2 \times 2^n + 2 \times 5^n)$, and so on
which gives the stated matrix

M1

A1

$$\text{Answer: } A^n = \frac{1}{3} \begin{pmatrix} 2^n + 2 \times 5^n & -2 \times 2^n + 2 \times 5^n \\ -2^n + 5^n & 2 \times 2^n + 5^n \end{pmatrix}$$

6 marks

15 (7 marks)

$$A = \begin{pmatrix} 5 & -6 \\ 1 & 0 \end{pmatrix}$$

B1

Eigenvalues: $\lambda^2 - 5\lambda + 6 = 0$, so $\lambda = 2$ and $\lambda = 3$

M1 A1

So $x_n = a2^n + b3^n$

M1

 $x_0 = 1$: $a + b = 1$; $x_1 = 5$: $2a + 3b = 5$, so $b = 3$ and $a = -2$

M1 A1

 $x_n = 3^{n+1} - 2^{n+1}$

A1

Answer: $x_n = 3^{n+1} - 2^{n+1}$

7 marks

16 (5 marks) $(3 - \lambda)(1 - \lambda) + 1 = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$, so $\lambda = 2$ only

M1 A1

 $(A - 2I)\begin{pmatrix} x \\ y \end{pmatrix} = 0$ gives $x + y = 0$, so every eigenvector is a multiple of $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

M1 A1

Diagonalisation needs two linearly independent eigenvectors to form P, so A cannot be diagonalised

E1

Answer: Only one independent eigenvector

5 marks

17 (5 marks) $A = PDP^{-1}$ with $P = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$, $D = \begin{pmatrix} 8 & 0 \\ 0 & 1 \end{pmatrix}$ and $P^{-1} = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$

M1 A1

Take $C = PEP^{-1}$ with $E = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, since $E^3 = D$

M1

$$C = \begin{pmatrix} 4 & -2 \\ 3 & -1 \end{pmatrix}$$

M1 A1

Answer: $C = \begin{pmatrix} 4 & -2 \\ 3 & -1 \end{pmatrix}$

5 marks

18 (4 marks)

$$P = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

M1 A1

$$D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

A1

The repeated eigenvalue 1 has two independent eigenvectors, so there are three independent eigenvectors in total and P is invertible

E1

Answer: $P = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

4 marks

19 (6 marks)

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = T^n \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

M1

Write $\begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (solving $-\alpha + \beta = 0$ and $2\alpha + \beta = 3$ gives $\alpha = \beta = 1$)

M1 A1

$$T^n \begin{pmatrix} 0 \\ 3 \end{pmatrix} = 1^n \begin{pmatrix} -1 \\ 2 \end{pmatrix} + 4^n \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

M1

 $x_n = 4^n - 1$ and $y_n = 4^n + 2$

A1 A1

Answer: $x_n = 4^n - 1$, $y_n = 4^n + 2$

6 marks

20 (6 marks)

Eigenvalues 2 and 1 (lower triangular)

B1

$$\lambda = 2: 3x - y = 0, \text{ eigenvector } \begin{pmatrix} 1 \\ 3 \end{pmatrix}; \lambda = 1: x = 0, \text{ eigenvector } \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

M1 A1

$$P = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}, D = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } P^{-1} = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$$

A1

$$A^n = PD^nP^{-1} = \begin{pmatrix} 2^n & 0 \\ 3(2^n - 1) & 1 \end{pmatrix}$$

M1 A1

$$\text{Answer: } A^n = \begin{pmatrix} 2^n & 0 \\ 3(2^n - 1) & 1 \end{pmatrix}$$

6 marks**21 (8 marks)**

$$\lambda = 2: x - y = 0, \text{ eigenvector } \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}; \lambda = 4: x + y = 0, \text{ eigenvector } \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}; \lambda = 5: \text{ eigenvector } \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

M1 A1 A1

The eigenvectors are mutually perpendicular; normalise them

M1

$$P = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

A1

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

B1

$$A^3 = PD^3P^T, \text{ with } D^3 = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 64 & 0 \\ 0 & 0 & 125 \end{pmatrix}$$

M1

$$A^3 = \begin{pmatrix} 36 & -28 & 0 \\ -28 & 36 & 0 \\ 0 & 0 & 125 \end{pmatrix}$$

A1

$$\text{Answer: } P = \dots, D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix}; A^3 = \begin{pmatrix} 36 & -28 & 0 \\ -28 & 36 & 0 \\ 0 & 0 & 125 \end{pmatrix}$$

8 marks**22 (8 marks)** $n = 1$ is the given result**B1**Assume $A^k = PD^kP^{-1}$; then $A^{k+1} = A^kA = PD^kP^{-1}PDP^{-1} = PD^{k+1}P^{-1}$ **M1 A1**True for $n = 1$ and, if true for k , true for $k + 1$, so true for all n **E1** $\det A = \det P \det D \det P^{-1} = \det D$ (since $\det P^{-1} = 1/\det P$), the product of the diagonal elements**M1 A1** D^n has diagonal elements d_i^n , which tend to 0 when $|d_i| < 1$, so $D^n \rightarrow 0$ and $A^n = PD^nP^{-1} \rightarrow 0$ **M1 A1****Answer: $A^n = PD^nP^{-1}$; $\det A$ is the product of the diagonal elements of D ; $A^n \rightarrow 0$ when every $|d_i| < 1$** 8 marks

23 (8 marks)

$$M = PDP^{-1} \text{ with } P = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \text{ and } D = \begin{pmatrix} 1 & 0 \\ 0 & \frac{7}{10} \end{pmatrix} \quad \text{M1 A1}$$

$$P^{-1} = \frac{1}{3} \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \quad \text{B1}$$

$$M^n = P \begin{pmatrix} 1 & 0 \\ 0 & (\frac{7}{10})^n \end{pmatrix} P^{-1} \quad \text{M1}$$

$$M^n = \frac{1}{3} \begin{pmatrix} 2 + (\frac{7}{10})^n & 2 - 2(\frac{7}{10})^n \\ 1 - (\frac{7}{10})^n & 1 + 2(\frac{7}{10})^n \end{pmatrix} \quad \text{A1}$$

$$\text{As } n \rightarrow \infty, (\frac{7}{10})^n \rightarrow 0 \text{ and } M^n \rightarrow \begin{pmatrix} \frac{2}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix} \quad \text{M1 A1}$$

Starting from (c, s) with $c + s = 1$, the limit is $\frac{2}{3}(c + s) = \frac{2}{3}$ in the city A1

Answer: $M^n \rightarrow \begin{pmatrix} \frac{2}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}$; **long-term city proportion** $\frac{2}{3}$ 8 marks

24 (7 marks)

$$A = PDP^{-1} \text{ with } P = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \text{ and } D = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}, \text{ and } P^{-1} = -\frac{1}{3} \begin{pmatrix} -2 & -1 \\ -1 & 1 \end{pmatrix} \quad \text{M1 A1}$$

$$A = \begin{pmatrix} \frac{1}{3} & \frac{5}{3} \\ \frac{10}{3} & -\frac{4}{3} \end{pmatrix} \quad \text{M1 A1}$$

$$A^{-1} = PD^{-1}P^{-1} \text{ with } D^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} \end{pmatrix} \quad \text{M1}$$

$$A^{-1} = \begin{pmatrix} \frac{2}{9} & \frac{5}{18} \\ \frac{5}{9} & -\frac{1}{18} \end{pmatrix} \quad \text{A1}$$

The eigenvalues of A^{-1} are $\frac{1}{2}$ and $-\frac{1}{3}$, the reciprocals of those of A , with the same eigenvectors E1

Answer: $A = \begin{pmatrix} \frac{1}{3} & \frac{5}{3} \\ \frac{10}{3} & -\frac{4}{3} \end{pmatrix}$; $A^{-1} = \begin{pmatrix} \frac{2}{9} & \frac{5}{18} \\ \frac{5}{9} & -\frac{1}{18} \end{pmatrix}$ 7 marks

25 (7 marks)

$$A^2 = \begin{pmatrix} 1 & 3 \\ 0 & 4 \end{pmatrix}, \text{ but the student's formula gives } \begin{pmatrix} 1 & 1 \\ 0 & 4 \end{pmatrix} \quad \text{M1 A1}$$

These differ, so the student is wrong E1

$$\text{The eigenvalues are 1 and 2 with eigenvectors } \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \text{ so } P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{M1 A1}$$

$$A^n = PD^nP^{-1} = \begin{pmatrix} 1 & 2^n - 1 \\ 0 & 2^n \end{pmatrix} \quad \text{M1 A1}$$

Answer: $A^n = \begin{pmatrix} 1 & 2^n - 1 \\ 0 & 2^n \end{pmatrix}$ 7 marks

26 (8 marks)

$$(a - \lambda)^2 - b^2 = 0, \text{ so } \lambda = a \pm b$$

M1 A1

$$\lambda = a + b: \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \lambda = a - b: \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

M1 A1

$$P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, D = \begin{pmatrix} a+b & 0 \\ 0 & a-b \end{pmatrix}$$

A1

$$P^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

B1

$$G^n = PD^nP^{-1} = \frac{1}{2} \begin{pmatrix} (a+b)^n & (a-b)^n \\ (a+b)^n & -(a-b)^n \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

M1

Multiplying out gives the stated result

A1

$$\text{Answer: } P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, D = \begin{pmatrix} a+b & 0 \\ 0 & a-b \end{pmatrix}, \text{ and } G^n = \frac{1}{2} \begin{pmatrix} (a+b)^n + (a-b)^n & (a+b)^n - (a-b)^n \\ (a+b)^n - (a-b)^n & (a+b)^n + (a-b)^n \end{pmatrix} \quad 8 \text{ marks}$$

27 (10 marks)

$$\det(F - \lambda I) = (1 - \lambda)(-\lambda) - 1 = \lambda^2 - \lambda - 1 = 0$$

M1 A1

$$\lambda = \frac{1 \pm \sqrt{5}}{2}, \text{ that is } \varphi \text{ and } \psi$$

A1

$$(F - \varphi I) \begin{pmatrix} \varphi \\ 1 \end{pmatrix}: (1 - \varphi)\varphi + 1 = 0 \text{ and } \varphi - \varphi = 0, \text{ using } \varphi^2 = \varphi + 1, \text{ so } \begin{pmatrix} \varphi \\ 1 \end{pmatrix} \text{ is an eigenvector;}$$

M1 A1

similarly for ψ

$$\varphi - \psi = \sqrt{5}, \text{ so } \frac{1}{\sqrt{5}} \left(\begin{pmatrix} \varphi \\ 1 \end{pmatrix} - \begin{pmatrix} \psi \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

B1

$$F^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{5}} \left(\varphi^n \begin{pmatrix} \varphi \\ 1 \end{pmatrix} - \psi^n \begin{pmatrix} \psi \\ 1 \end{pmatrix} \right)$$

M1 A1

$$\text{The second component gives } f_n = \frac{\varphi^n - \psi^n}{\sqrt{5}}$$

A1

$$\varphi^{10} = \frac{123 + 55\sqrt{5}}{2} \text{ and } \psi^{10} = \frac{123 - 55\sqrt{5}}{2}, \text{ so } f_{10} = \frac{55\sqrt{5}}{\sqrt{5}} = 55$$

A1

$$\text{Answer: } f_n = \frac{\varphi^n - \psi^n}{\sqrt{5}}$$

10 marks

28 (11 marks)

$$A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, A \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \text{ and } A \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 6 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

M1 A1

Eigenvalues 1, 2 and 3

B1

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix} \text{ and } D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

B1 B1

$$P^{-1} = \begin{pmatrix} 1 & -1 & \frac{1}{2} \\ 0 & 1 & -1 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$

M1 A1

$$A^n = PD^nP^{-1}$$

M1

$$PD^n = \begin{pmatrix} 1 & 2^n & 3^n \\ 0 & 2^n & 2 \times 3^n \\ 0 & 0 & 2 \times 3^n \end{pmatrix}$$

M1

$$A^n = \begin{pmatrix} 1 & 2^n - 1 & \frac{3^n + 1}{2} - 2^n \\ 0 & 2^n & 3^n - 2^n \\ 0 & 0 & 3^n \end{pmatrix}$$

A1

$$n = 1 \text{ gives } \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix} = A$$

B1

$$\text{Answer: } A^n = \begin{pmatrix} 1 & 2^n - 1 & \frac{3^n + 1}{2} - 2^n \\ 0 & 2^n & 3^n - 2^n \\ 0 & 0 & 3^n \end{pmatrix}$$

11 marks

29 (8 marks)

$$u \cdot (Av) = u \cdot (\lambda_2 v) = \lambda_2 (u \cdot v)$$

M1 A1

$$(Au) \cdot v = (\lambda_1 u) \cdot v = \lambda_1 (u \cdot v)$$

M1 A1

$$\text{These are equal, so } (\lambda_1 - \lambda_2)(u \cdot v) = 0$$

M1

$$\lambda_1 \neq \lambda_2, \text{ so } u \cdot v = 0$$

A1

Normalising u and v gives two perpendicular unit vectors; these are the columns of P

M1

$$\text{Then } P^T P = I, \text{ so } P^{-1} = P^T \text{ and } A = P D P^{-1} = P D P^T$$

E1

Answer: $u \cdot v = 0$, so $A = P D P^T$ with P orthogonal

8 marks

30 (9 marks)Suppose $B^2 = S$. Then B commutes with S, so B maps each eigenvector of S onto a multiple of itself, and B has the same eigenvectors

M1 E1

$$\text{So } B = P E P^{-1} \text{ with } E = \text{diag}(e_1, e_2)$$

M1

$$B^2 = S \text{ requires } e_1^2 = 9 \text{ and } e_2^2 = 1$$

M1 A1

So $e_1 = \pm 3$ and $e_2 = \pm 1$, giving exactly four matrices

A1

$$\text{With } P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \text{ and } P^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \text{ the four matrices are } \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} -1 & -2 \\ -2 & -1 \end{pmatrix}$$

M1 A1

$$\text{and } \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix}$$

The matrix with $e_1 = 3$ and $e_2 = 1$ is $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, with eigenvalues 3 and 1 > 0, and it is symmetric

A1

$$\text{Answer: Four matrices: } \pm \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \text{ and } \pm \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}; \text{ the positive one is } \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

9 marks

Cayley-Hamilton Theorem

1 (3 marks)

$$\lambda^2 - (1 + 4)\lambda + (4 - 6) = 0, \text{ that is } \lambda^2 - 5\lambda - 2 = 0$$

M1 A1

The matrix satisfies its own characteristic equation: $A^2 - 5A - 2I = 0$

B1

Answer: $\lambda^2 - 5\lambda - 2 = 0$; $A^2 - 5A - 2I = 0$

3 marks

2 (3 marks)

$$A^2 = \begin{pmatrix} 4 & 5 \\ 0 & 9 \end{pmatrix} \text{ and } 5A = \begin{pmatrix} 10 & 5 \\ 0 & 15 \end{pmatrix}$$

M1 A1

$$A^2 - 5A + 6I = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

A1

Answer: $A^2 - 5A + 6I = 0$

3 marks

3 (2 marks)

The characteristic equation is $\lambda^2 - 7\lambda + 12 = 0$

B1

By the Cayley-Hamilton theorem $A^2 - 7A + 12I = 0$, so $A^2 = 7A - 12I$

B1

Answer: $\lambda^2 - 7\lambda + 12 = 0$; $A^2 = 7A - 12I$

2 marks

4 (3 marks)

$$A^2 = 4A - 3I$$

B1

Rearrange: $3I = 4A - A^2 = A(4I - A)$, so $A^{-1} = \frac{1}{3}(4I - A)$

M1 A1

Answer: $A^2 = 4A - 3I$; $A^{-1} = \frac{1}{3}(4I - A)$

3 marks

5 (2 marks)

The characteristic equation is $\lambda^2 - 8\lambda + 13 = 0$

M1

By Cayley-Hamilton $A^2 = 8A - 13I$, so $p = 8$ and $q = -13$

A1

Answer: $p = 8$, $q = -13$

2 marks

6 (2 marks)

$$A^3 = A \cdot A^2 = 4A^2 - 5A$$

M1

$$= 4(4A - 5I) - 5A = 11A - 20I$$

A1

Answer: $11A - 20I$

2 marks

7 (5 marks)

$$(2 - \lambda)(3 - \lambda) + 1 = \lambda^2 - 5\lambda + 7 = 0$$

M1 A1

$$A^2 = \begin{pmatrix} 3 & 5 \\ -5 & 8 \end{pmatrix}$$

M1

$$A^2 - 5A + 7I = \begin{pmatrix} 3 & 5 \\ -5 & 8 \end{pmatrix} - \begin{pmatrix} 10 & 5 \\ -5 & 15 \end{pmatrix} + \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix} = 0$$

M1 A1

Answer: $\lambda^2 - 5\lambda + 7 = 0$ and $A^2 - 5A + 7I = 0$

5 marks

8 (4 marks)

$$\lambda^2 - 5\lambda + 1 = 0, \text{ so } A^2 - 5A + I = 0$$

M1 A1

$$A^2 - 5A = -I, \text{ so } A(5I - A) = I \text{ and } A^{-1} = 5I - A$$

M1

$$A^{-1} = \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}$$

A1

Answer: $A^{-1} = \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}$

4 marks

9 (4 marks)

$$A^2 = 5A + 2I$$

B1

$$A^3 = 5A^2 + 2A = 5(5A + 2I) + 2A = 27A + 10I$$

M1 A1

$$A^3 = \begin{pmatrix} 37 & 54 \\ 81 & 118 \end{pmatrix}$$

A1

$$\text{Answer: } A^3 = 27A + 10I = \begin{pmatrix} 37 & 54 \\ 81 & 118 \end{pmatrix}$$

4 marks

10 (4 marks)

The eigenvalues are 1, 2 and 3, so the characteristic equation is $(\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$, that is $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

M1 A1

$$(D - I)(D - 2I)(D - 3I) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

M1

Each diagonal element of the product contains a zero factor, so the product is the zero matrix

A1

$$\text{Answer: } \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0 \text{ and } D^3 - 6D^2 + 11D - 6I = 0$$

4 marks

11 (5 marks)

$$\text{Trace} = 6$$

B1

$$\text{Sum of principal } 2 \times 2 \text{ minors: } (6 - 0) + (2 - 0) + (3 - 0) = 11$$

M1 A1

$$\det A = 2(3 - 0) - 1(0 - 1) + 0 = 7, \text{ giving } \lambda^3 - 6\lambda^2 + 11\lambda - 7 = 0$$

M1

$$A^3 - 6A^2 + 11A - 7I = 0$$

A1

$$\text{Answer: } A^3 - 6A^2 + 11A - 7I = 0$$

5 marks

12 (6 marks)

$$\text{Characteristic equation } \lambda^2 - 3\lambda + 2 = 0, \text{ so } A^2 = 3A - 2I$$

M1 A1

$$A^3 = 3A^2 - 2A = 7A - 6I$$

M1

$$A^4 = 7A^2 - 6A = 7(3A - 2I) - 6A = 15A - 14I$$

M1 A1

$$A^4 = \begin{pmatrix} -14 & 15 \\ -30 & 31 \end{pmatrix}$$

A1

$$\text{Answer: } A^4 = 15A - 14I = \begin{pmatrix} -14 & 15 \\ -30 & 31 \end{pmatrix}$$

6 marks

13 (4 marks)

$$A - 2I = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

B1

$$(A - 2I)^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

M1 A1

$$\text{Expanding: } A^2 - 4A + 4I = 0, \text{ so } A^2 = 4A - 4I$$

A1

$$\text{Answer: } A^2 = 4A - 4I$$

4 marks

14 (5 marks)

$$A^2 = \begin{pmatrix} a^2 + bc & b(a + d) \\ c(a + d) & d^2 + bc \end{pmatrix}$$

M1 A1

$$(a + d)A = \begin{pmatrix} a^2 + ad & ab + bd \\ ac + cd & ad + d^2 \end{pmatrix}$$

M1

$$A^2 - (a + d)A = \begin{pmatrix} bc - ad & 0 \\ 0 & bc - ad \end{pmatrix}$$

M1

Adding $(ad - bc)I$ gives the zero matrix

A1

$$\text{Answer: } A^2 - (a + d)A + (ad - bc)I = 0$$

5 marks

15 (6 marks)

$$A^3 - 6A^2 + 11A - 7I = 0$$

B1

$$7I = A^3 - 6A^2 + 11A = A(A^2 - 6A + 11I)$$

M1 A1

$$\text{So } A^{-1} = \frac{1}{7}(A^2 - 6A + 11I)$$

A1

$$A^2 = \begin{pmatrix} 4 & 5 & 1 \\ 1 & 9 & 4 \\ 3 & 1 & 1 \end{pmatrix}$$

M1

$$A^{-1} = \frac{1}{7} \begin{pmatrix} 3 & -1 & 1 \\ 1 & 2 & -2 \\ -3 & 1 & 6 \end{pmatrix} = \begin{pmatrix} \frac{3}{7} & -\frac{1}{7} & \frac{1}{7} \\ \frac{1}{7} & \frac{2}{7} & -\frac{2}{7} \\ -\frac{3}{7} & \frac{1}{7} & \frac{6}{7} \end{pmatrix}$$

A1

$$\text{Answer: } A^{-1} = \begin{pmatrix} \frac{3}{7} & -\frac{1}{7} & \frac{1}{7} \\ \frac{1}{7} & \frac{2}{7} & -\frac{2}{7} \\ -\frac{3}{7} & \frac{1}{7} & \frac{6}{7} \end{pmatrix}$$

6 marks**16 (6 marks)**

The characteristic equation of A is $\lambda^2 - (k+3)\lambda + (3k-2) = 0$

M1 A1

Comparing with $\lambda^2 - 7\lambda + 10 = 0$: $k+3 = 7$ and $3k-2 = 10$, so $k = 4$

M1 A1

$$A^{-1} = \frac{1}{10}(7I - A) = \begin{pmatrix} \frac{3}{10} & -\frac{1}{5} \\ -\frac{1}{10} & \frac{2}{5} \end{pmatrix}$$

M1 A1

$$\text{Answer: } k = 4; A^{-1} = \begin{pmatrix} \frac{3}{10} & -\frac{1}{5} \\ -\frac{1}{10} & \frac{2}{5} \end{pmatrix}$$

6 marks**17 (5 marks)**

$$A^3 = 5A^2 - 7A = 5(5A - 7I) - 7A = 18A - 35I$$

M1 A1

$$A^4 = 18A^2 - 35A = 18(5A - 7I) - 35A = 55A - 126I$$

M1 A1

$$A^4 = \begin{pmatrix} -16 & 55 \\ -55 & 39 \end{pmatrix}$$

A1

$$\text{Answer: } A^4 = 55A - 126I = \begin{pmatrix} -16 & 55 \\ -55 & 39 \end{pmatrix}$$

5 marks**18 (5 marks)**

Trace = 1 and determinant = $-2 + 2 = 0$, so $\lambda^2 - \lambda = 0$

M1 A1

$$A^2 - A = 0, \text{ so } A^2 = A$$

M1

By induction $A^n = A$ for all $n \geq 1$ ($A^{n+1} = A \cdot A^n = A \cdot A = A$)

M1

$$A^{100} = A = \begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix}$$

A1

$$\text{Answer: } A^{100} = \begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix}$$

5 marks

19 (6 marks)

$$\text{Trace} = 0 \text{ and determinant} = -4 + 4 = 0, \text{ so } \lambda^2 = 0$$

M1 A1

$$\text{By the Cayley-Hamilton theorem } N^2 = 0$$

A1

$$(I + N)(I - N) = I - N^2 = I, \text{ so } (I + N)^{-1} = I - N$$

M1 A1

$$I - N = \begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix}$$

A1

$$\text{Answer: } (I + N)^{-1} = \begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix}$$

6 marks

20 (6 marks)

$$\lambda^2 - 6\lambda + 5 = 0, \text{ so } A^2 = 6A - 5I$$

M1 A1

$$5I = 6A - A^2 = A(6I - A), \text{ so } A^{-1} = \frac{1}{5}(6I - A) = \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{3}{5} & \frac{4}{5} \end{pmatrix}$$

M1 A1

$$A^3 = 6A^2 - 5A = 31A - 30I \text{ and } A^4 = 31A^2 - 30A = 156A - 155I$$

M1

$$A^5 = 156A^2 - 155A = 781A - 780I$$

A1

$$\text{Answer: } A^{-1} = \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{3}{5} & \frac{4}{5} \end{pmatrix}; A^5 = 781A - 780I$$

6 marks

21 (8 marks)

$$A^3 = 6A^2 - 11A + 7I$$

M1 A1

$$A^4 = 6A^3 - 11A^2 + 7A$$

M1

$$= 6(6A^2 - 11A + 7I) - 11A^2 + 7A$$

M1

$$= 25A^2 - 59A + 42I$$

A1

$$A^2 = \begin{pmatrix} 4 & 5 & 1 \\ 1 & 9 & 4 \\ 3 & 1 & 1 \end{pmatrix}$$

B1

$$A^4 = \begin{pmatrix} 24 & 66 & 25 \\ 25 & 90 & 41 \\ 16 & 25 & 8 \end{pmatrix}$$

M1 A1

$$\text{Answer: } A^4 = 25A^2 - 59A + 42I = \begin{pmatrix} 24 & 66 & 25 \\ 25 & 90 & 41 \\ 16 & 25 & 8 \end{pmatrix}$$

8 marks

22 (7 marks)

$$(A - I)(A - 4I) = A^2 - 5A + 4I$$

M1 A1

$$= (A^2 - 5A + 7I) - 3I = -3I$$

M1

$$\text{So } (A - I)\left[-\frac{1}{3}(A - 4I)\right] = I, \text{ hence } A - I \text{ is invertible with inverse } -\frac{1}{3}(A - 4I)$$

E1

$$A - 4I = \begin{pmatrix} -2 & 1 \\ -1 & -1 \end{pmatrix}$$

B1

$$(A - I)^{-1} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

M1 A1

$$\text{Answer: } (A - I)^{-1} = -\frac{1}{3}(A - 4I) = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

7 marks

23 (8 marks)

By Cayley-Hamilton: $A^3 + pA^2 + qA + rI = 0$

B1

Rearranging: $A(A^2 + pA + qI) = -rI$

M1 A1

Since $r \neq 0$, $A \cdot \left[-\frac{1}{r}(A^2 + pA + qI)\right] = I$, so A^{-1} exists and equals $-\frac{1}{r}(A^2 + pA + qI)$

E1

Here $p = -3$, $q = 3$, $r = -13$, so $A^{-1} = \frac{1}{13}(A^2 - 3A + 3I)$

M1

$$A^2 = \begin{pmatrix} 1 & 4 & 6 \\ 6 & 1 & 6 \\ 4 & 4 & 1 \end{pmatrix}$$

B1

$$A^{-1} = \begin{pmatrix} \frac{1}{13} & -\frac{2}{13} & \frac{6}{13} \\ \frac{6}{13} & \frac{1}{13} & -\frac{3}{13} \\ -\frac{2}{13} & \frac{4}{13} & \frac{1}{13} \end{pmatrix}$$

M1 A1

$$\text{Answer: } A^{-1} = \begin{pmatrix} \frac{1}{13} & -\frac{2}{13} & \frac{6}{13} \\ \frac{6}{13} & \frac{1}{13} & -\frac{3}{13} \\ -\frac{2}{13} & \frac{4}{13} & \frac{1}{13} \end{pmatrix}$$

8 marks

24 (6 marks)

$\det(A - \lambda I)$ is a polynomial in λ with scalar coefficients; substituting $\lambda = A$ means replacing each power of λ by the same power of the matrix A

E1

For A , $\det(A - \lambda I) = \lambda^2 - 5\lambda - 2$, so the substitution gives $A^2 - 5A - 2I$, a matrix

M1 A1

The student's $\det(A - A) = \det(0) = 0$ is a scalar zero, not the zero matrix, and it was obtained by substituting into the matrix $A - \lambda I$ before taking the determinant, which is not the same operation

E1

The theorem claims the matrix $A^2 - 5A - 2I$ is the zero matrix, which has to be checked:

M1 A1

$$\begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix} - \begin{pmatrix} 5 & 10 \\ 15 & 20 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 0$$

Answer: The argument confuses a scalar with a matrix

6 marks

25 (8 marks)

Characteristic equation: $\lambda^2 - (-1)\lambda + 1 = 0$, that is $\lambda^2 + \lambda + 1 = 0$

M1 A1

So $A^2 + A + I = 0$ by the Cayley-Hamilton theorem

A1

$(A - I)(A^2 + A + I) = A^3 - I = 0$, so $A^3 = I$

M1 A1

$$100 = 3 \times 33 + 1, \text{ so } A^{100} = (A^3)^{33}A = A = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

M1 A1

$$A^{-1} = A^2 = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$$

A1

$$\text{Answer: } A^{100} = A; A^{-1} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$$

8 marks

26 (7 marks)

Trace = 0 and $\det B = -1 + 6 = 5$, so the characteristic equation is $\lambda^2 + 5 = 0$

M1 A1

By the Cayley-Hamilton theorem $B^2 + 5I = 0$, so $B^2 = -5I$

A1

$$B^{10} = (B^2)^5 = (-5I)^5 = -3125I$$

M1 A1

$$B^{11} = -3125B = \begin{pmatrix} -3125 & -6250 \\ 9375 & 3125 \end{pmatrix}$$

M1 A1

$$\text{Answer: } B^{10} = -3125I; B^{11} = \begin{pmatrix} -3125 & -6250 \\ 9375 & 3125 \end{pmatrix}$$

7 marks

27 (12 marks)

$$\text{Trace} = 2 + 1 + 1 = 4$$

B1

Principal 2×2 minors: $2 - 0 = 2$, $2 - 0 = 2$ and $1 - 0 = 1$, with sum 5

M1 A1

 $\det A = 2(1 - 0) - 0 + 1(2 - 0) = 4$, so the equation is $\lambda^3 - 4\lambda^2 + 5\lambda - 4 = 0$

M1 A1

$$A^3 = 4A^2 - 5A + 4I$$

B1

$$A^4 = 4A^3 - 5A^2 + 4A = 4(4A^2 - 5A + 4I) - 5A^2 + 4A = 11A^2 - 16A + 16I$$

M1 A1

$$A^2 = \begin{pmatrix} 4 & 2 & 3 \\ 3 & 1 & 1 \\ 2 & 4 & 1 \end{pmatrix}, \text{ so } A^4 = \begin{pmatrix} 28 & 22 & 17 \\ 17 & 11 & 11 \\ 22 & 12 & 11 \end{pmatrix}$$

M1 A1

$$4I = A^3 - 4A^2 + 5A = A(A^2 - 4A + 5I), \text{ so } A^{-1} = \frac{1}{4}(A^2 - 4A + 5I)$$

M1

$$A^{-1} = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{pmatrix}$$

A1

$$\text{Answer: } A^4 = 11A^2 - 16A + 16I = \begin{pmatrix} 28 & 22 & 17 \\ 17 & 11 & 11 \\ 22 & 12 & 11 \end{pmatrix}; A^{-1} = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{pmatrix}$$

12 marks

28 (9 marks)Trace $A = 29$ and $\det A = 198 - 98 = 100$, so $\lambda^2 - 29\lambda + 100 = (\lambda - 4)(\lambda - 25) = 0$

M1 A1

The eigenvalues of X^2 are the squares of the eigenvalues of X , so the squares of the eigenvalues of X are 4 and 25

M1

 X has positive eigenvalues, so they are 2 and 5

A1

The characteristic equation of X is $(\lambda - 2)(\lambda - 5) = \lambda^2 - 7\lambda + 10 = 0$

B1

By Cayley-Hamilton $X^2 - 7X + 10I = 0$, so $A = 7X - 10I$

M1

$$X = \frac{1}{7}(A + 10I)$$

A1

$$A + 10I = \begin{pmatrix} 21 & 7 \\ 14 & 28 \end{pmatrix}, \text{ so } X = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$$

M1 A1

$$\text{Answer: } X = \frac{1}{7}(A + 10I) = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$$

9 marks

29 (11 marks)If $A = kI$ then $A^2 = k^2I = -I$ would need $k^2 = -1$, which has no real solution

M1 E1

By Cayley-Hamilton $A^2 - (\text{tr } A)A + (\det A)I = 0$; with $A^2 = -I$ this gives $(\text{tr } A)A = (\det A - 1)I$

M1 A1

If $\text{tr } A \neq 0$ then A would be a scalar multiple of I , which is impossible; so $\text{tr } A = 0$ and then $\det A = 1$

M1 A1

Trace 0 means $d = -a$, and $\det A = -a^2 - bc = 1$, so $a^2 + bc = -1$

M1 A1

$$\text{With } a = 0 \text{ and } b = 2: 2c = -1, \text{ so } c = -\frac{1}{2} \text{ and } A = \begin{pmatrix} 0 & 2 \\ -\frac{1}{2} & 0 \end{pmatrix}$$

M1 A1

$$A^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I$$

A1

$$\text{Answer: } A = \begin{pmatrix} 0 & 2 \\ -\frac{1}{2} & 0 \end{pmatrix}$$

11 marks

30 (11 marks)

Trace 4 and determinant 4, so $\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0$

M1 A1

$$A^2 = 4A - 4I$$

B1

$$n = 1: 1 \times A - 0 = A$$

B1

Assume $A^k = k2^{k-1}A - (k-1)2^kI$; then $A^{k+1} = k2^{k-1}A^2 - (k-1)2^kA$

M1

$$= k2^{k-1}(4A - 4I) - (k-1)2^kA = [k2^{k+1} - (k-1)2^k]A - k2^{k+1}I$$

M1 A1

$= (k+1)2^kA - k2^{k+1}I$, the formula with $n = k+1$

A1

$$A^6 = 6 \times 32A - 5 \times 64I = \begin{pmatrix} 64 & 192 \\ 0 & 64 \end{pmatrix}$$

M1 A1

If A were diagonalisable, with the only eigenvalue 2, then $A = P(2I)P^{-1} = 2I$, which it is not; so A cannot be diagonalised

E1

Answer: $A^6 = \begin{pmatrix} 64 & 192 \\ 0 & 64 \end{pmatrix}$

11 marks