

Matrix Notation and Operations

- 1 The matrix A is given by $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$.
State the order of A and write down the element in row 2, column 3. [2]
- 2 Given $A = \begin{pmatrix} 3 & -1 \\ 4 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 5 \\ -3 & 1 \end{pmatrix}$, find $A + B$. [2]
- 3 Given $A = \begin{pmatrix} 5 & -2 \\ 1 & 4 \end{pmatrix}$ and I is the 2×2 identity matrix, find $A - 3I$. [2]
- 4 Given $A = \begin{pmatrix} 2 & -1 & 4 \\ 0 & 3 & -5 \end{pmatrix}$, find $3A$. [2]
- 5 The matrix P has order 2×3 , the matrix Q has order 2×3 and the matrix R has order 3×2 .
State, with a reason, which of $P + Q$ and $P + R$ is defined. [2]
- 6 Write down the transpose of the matrix $A = \begin{pmatrix} 1 & 4 \\ -2 & 0 \\ 5 & 3 \end{pmatrix}$ and state the order of A^T . [2]
- 7 Given $A = \begin{pmatrix} 2 & -1 \\ 0 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 3 \\ -2 & 5 \end{pmatrix}$, find $2A - 3B$. [3]
- 8 The matrices A, B and C are given by $A = \begin{pmatrix} 3 & -2 \\ a & 1 \end{pmatrix}$, $B = \begin{pmatrix} b & 5 \\ -1 & c \end{pmatrix}$ and $C = \begin{pmatrix} 0 & d \\ 4 & -2 \end{pmatrix}$,
where a, b, c and d are constants.
Given that $A + B = C$, find the values of a, b, c and d. [3]
- 9 Given $A = \begin{pmatrix} 4 & -2 \\ 6 & 10 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ -3 & 2 \end{pmatrix}$, find the matrix X such that $3X + 2B = A$. [3]
- 10 The matrix A is given by $A = \begin{pmatrix} k & 3 \\ 2 & k \end{pmatrix}$, where k is a constant.
Given that $2A - I = \begin{pmatrix} 9 & 6 \\ 4 & 9 \end{pmatrix}$, where I is the 2×2 identity matrix, find the value of k. [3]
- 11 Find the values of x and y such that $\begin{pmatrix} x+y & 2 \\ 3 & x-y \end{pmatrix} = \begin{pmatrix} 7 & 2 \\ 3 & 1 \end{pmatrix}$. [3]
- 12 The matrix A is given by $A = \begin{pmatrix} 2 & -1 & 3 \\ 4 & 0 & 5 \\ 1 & 7 & -2 \end{pmatrix}$.
Write down A^T , find $A + A^T$, and state what is special about the matrix $A + A^T$. [4]
- 13 The 2×3 matrix A has elements $a_{ij} = i^2 + j$, where i is the row number and j is the column number.
Write down the matrices A and $3A$. [3]

14 Given $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$, find the constants p and q such that

$$pA + qB = \begin{pmatrix} -1 & 8 \\ -2 & -3 \end{pmatrix}.$$

[5]

15 The 2×2 matrices X and Y satisfy $2X + Y = \begin{pmatrix} 5 & 1 \\ -2 & 4 \end{pmatrix}$ and $X - 3Y = \begin{pmatrix} 1 & 3 \\ 0 & -2 \end{pmatrix}$.

Find X and Y .

[5]

16 The matrix A is given by $A = \begin{pmatrix} 3 & 6 & -2 \\ 1 & 4 & 5 \\ 0 & -3 & 2 \end{pmatrix}$.

Express A in the form $S + K$, where S is a symmetric matrix and K is a matrix satisfying $K^T = -K$.

[4]

17 The matrices A and B are both 2×2 matrices.

Show that $(A + B)^T = A^T + B^T$.

[4]

18 A shop sells three products in two branches. The matrices P and Q show the number of units sold in January and February respectively; the rows represent the branches and the columns represent the products.

$$P = \begin{pmatrix} 1200 & 850 & 430 \\ 900 & 1100 & 520 \end{pmatrix}, Q = \begin{pmatrix} 1000 & 920 & 480 \\ 950 & 1000 & 600 \end{pmatrix}.$$

Find the matrix showing the total sales over the two months, and the matrix showing the sales that would result if every total rose by 10%.

[4]

19 The 2×2 matrix A satisfies $A + 2A^T = \begin{pmatrix} 5 & 7 \\ -1 & 4 \end{pmatrix}$. Find A .

[5]

20 The matrices A and B are given by $A = \begin{pmatrix} x & 2 \\ y & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & y \\ 4 & x \end{pmatrix}$, where x and y are constants.

Given that $2A + B^T = \begin{pmatrix} 5 & 8 \\ -3 & 8 \end{pmatrix}$, find the values of x and y .

[6]

21 A square matrix K is skew-symmetric if $K^T = -K$.

The matrix K is given by $K = \begin{pmatrix} 0 & x & 3 \\ 2 & 0 & y \\ z & -5 & 0 \end{pmatrix}$.

Given that K is skew-symmetric, find x , y and z , and write down $K^T + 3K$.

[7]

22 The matrices A and B are given by $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$.

Show that there are no constants p and q such that $pA + qB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and explain why this result is not surprising.

[6]

23 The 2×2 matrices X and Y satisfy $X + Y^T = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$ and $X^T - Y = \begin{pmatrix} 4 & -3 \\ 0 & 5 \end{pmatrix}$.

Find X and Y .

[6]

- 24** A student claims: "If A and B are both skew-symmetric matrices of the same order, then A + B is symmetric."
Show that this claim is false by proving what kind of matrix A + B must be, and give a numerical example with 2×2 matrices in which A + B is not the zero matrix. [6]
- 25** The 3×3 matrices A and B have elements $a_{ij} = |i - j|$ and $b_{ij} = i + j$ respectively.
Write down A and B, and find the matrix X such that $X + 2A = B$. State, with a reason, whether X is symmetric. [6]
- 26** The matrices A, B, C and D are $A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 4 \\ 7 & 3 \end{pmatrix}$.
Find constants p, q and r such that $pA + qB + rC = D$, and verify that your values satisfy all four element equations. [6]
- 27** Prove that, for any square matrix A, the matrix $A + A^T$ is symmetric and the matrix $A - A^T$ is skew-symmetric.
Hence show that every square matrix can be written as the sum of a symmetric matrix and a skew-symmetric matrix, and prove that this decomposition is unique. [10]
- 28** The 2×2 matrix X satisfies $X + X^T = \begin{pmatrix} 4 & 2 \\ 2 & 6 \end{pmatrix}$.
Show that X can be written in the form $X = \begin{pmatrix} 2 & 1+t \\ 1-t & 3 \end{pmatrix}$, where t is a constant.
Given also that the sum of the squares of the four elements of X is 23, find the possible matrices X. [8]
- 29** Let T be the set of all 2×2 matrices whose elements on the leading diagonal sum to zero (that is, with trace 0).
Show that if P and Q belong to T then so does P + Q, and so does kP for any constant k.
Show that every matrix in T can be written as $aE + bF + cG$, where $E = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $F = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $G = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and that the constants a, b, c are unique.
Find a, b, c for the matrix $\begin{pmatrix} 3 & -2 \\ 5 & -3 \end{pmatrix}$. [8]
- 30** The 2×2 matrix A satisfies $A^T = 3A - I$, where I is the 2×2 identity matrix.
Show that the only possible matrix A is $\frac{1}{2}I$.
The 2×2 matrix B satisfies $B^T = 2I - B$. Find the general form of B, in terms of one arbitrary constant t. [9]

Matrix Multiplication

- 1** Given $A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 5 & 2 \end{pmatrix}$, find AB. [2]
- 2** The matrix A has order 2×3 and the matrix B has order 3×4 .
State whether each of AB and BA is defined and, where it is, give its order. [2]
- 3** Given $A = \begin{pmatrix} 3 & -1 \\ 2 & 5 \end{pmatrix}$, find A^2 . [2]

- 4** Find $\begin{pmatrix} 2 & -1 & 4 \\ 1 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$. [2]
-
- 5** Find the product $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$ and state its order. [2]
-
- 6** Write down the 2×2 identity matrix I .
Given $A = \begin{pmatrix} 3 & -1 \\ 2 & 5 \end{pmatrix}$, find AI and IA , and state what you notice. [3]
-
- 7** Given $A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix}$, find AB and BA , and state whether $AB = BA$. [4]
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- 8** Given $A = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}$, find the matrix $A^2 - 3A + 2I$, where I is the 2×2 identity matrix. [4]
-
- 9** Find $\begin{pmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$. [3]
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- 10** The matrices A and B are given by $A = \begin{pmatrix} k & 2 \\ 1 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$, where k is a constant.
Given that $AB = \begin{pmatrix} 4 & 0 \\ 1 & 5 \end{pmatrix}$, find the value of k . [3]
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- 11** Given $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, find A^2 and A^3 , and write down A^n for a positive integer n . [4]
-
- 12** Given $A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 \\ 0 & 4 \\ 1 & -3 \end{pmatrix}$, find AB and state the order of BA . [3]
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- 13** Given that $\begin{pmatrix} 1 & x \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 9 \end{pmatrix}$, find the values of x and y . [4]
-
- 14** The matrix A is given by $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$.
Find the constants a and b such that $A^2 = aA + bI$, where I is the 2×2 identity matrix.
Hence find A^3 in the form $pA + qI$. [6]
-
- 15** Given $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$, find $(A + B)^2$ and $A^2 + 2AB + B^2$, and hence show that the two are not equal.
State the correct expansion of $(A + B)^2$ for general square matrices. [5]
-
- 16** The matrices A and B are given by $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} p & q \\ x & y \end{pmatrix}$.
Show that $(AB)^T = B^T A^T$. [4]
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- 17** The matrix A is given by $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.
Find all 2×2 matrices $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $AB = BA$. [5]
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- 18** In a town, each week 20% of the people who use bus A switch to bus B, and 30% of the people who use bus B switch to bus A; nobody else changes. The matrix $P = \begin{pmatrix} \frac{4}{5} & \frac{3}{10} \\ \frac{1}{5} & \frac{7}{10} \end{pmatrix}$ gives the proportions, where the columns correspond to the current service (A then B) and the rows to the service next week (A then B).
Initially 1000 people use bus A and 500 use bus B. Find P^2 , and use it to find how many people use each bus after two weeks. [4]
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- 19** A bakery has three shops. The matrix $N = \begin{pmatrix} 40 & 25 & 30 \\ 35 & 20 & 45 \\ 50 & 30 & 20 \end{pmatrix}$ shows the numbers of loaves, pies and cakes sold by the shops on one day (one row for each shop). The prices, in pounds, of a loaf, a pie and a cake are 4, 7 and 9 respectively.
Write the prices as a column matrix p , and find Np . Explain what each element of Np represents, and find the total takings of the three shops. [5]
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- 20** Three villages A, B and C are joined by roads. The matrix $N = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ shows the number of direct roads between each pair of villages (rows and columns in the order A, B, C).
Find N^2 . Explain what the element in row 1, column 3 of N^2 represents, and state how many different routes from A to C use exactly two roads. [5]
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- 21** The matrices A and B are given by $A = \begin{pmatrix} 1 & p \\ 2 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} q & 1 \\ 4 & 5 \end{pmatrix}$, where p and q are constants.
Given that $(A + B)(A - B) = A^2 - B^2$, find the values of p and q . [7]
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- 22** The matrix A is given by $A = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$.
Prove by induction that, for all positive integers n , $A^n = \begin{pmatrix} 2^n & 2^n - 1 \\ 0 & 1 \end{pmatrix}$. [6]
-
- 23** The matrix A is given by $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.
Find A^2 and A^4 , and hence find A^{2026} and A^{2027} . [7]
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- 24** The matrix A is given by $A = \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}$.
Show that $A^2 = 0$, the zero matrix.
Use the binomial theorem for matrices (valid because I commutes with A) to show that $(I + A)^n = I + nA$ for all positive integers n , and hence find $(I + A)^{10}$. [7]
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- 25** The matrix A is given by $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.
Find all matrices $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $AB = BA$, and show that every such matrix can be written in the form $B = pI + qA$ for constants p and q . [7]
- 26** A student writes: "For any two square matrices A and B , $(AB)^2 = A^2B^2$."
Show that this is false by considering $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.
Find the correct expansion of $(AB)^2$ in terms of A and B , and state the condition on A and B for which the student's statement is true. [7]
- 27** The matrix A is given by $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.
Show that $A^2 - (a + d)A + (ad - bc)I = 0$, where I is the 2×2 identity matrix and 0 is the zero matrix.
The matrix T is given by $T = \begin{pmatrix} 3 & 1 \\ -2 & 0 \end{pmatrix}$. Use the result from the first part to show that $T^2 = 3T - 2I$, and hence find constants p and q such that $T^4 = pT + qI$. [10]
- 28** The matrix F is given by $F = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, and the Fibonacci numbers are defined by $f_1 = f_2 = 1$ and $f_{n+2} = f_{n+1} + f_n$.
Prove by induction that $F^n = \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix}$ for all positive integers n . (Take $f_0 = 0$.)
Hence find the value of $f_5f_7 - f_6^2$. [8]
- 29** The real 2×2 matrix $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfies $X^2 = I$, where I is the 2×2 identity matrix.
Show that either $X = I$, or $X = -I$, or $a + d = 0$ and $a^2 + bc = 1$.
Hence find all matrices X of the form $\begin{pmatrix} 3 & b \\ c & p \end{pmatrix}$ with $X^2 = I$, where c is a non-zero constant, giving b in terms of c . [9]
- 30** The matrix A is given by $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$.
Write $A = 2I + N$, where N is a matrix to be found, and show that $N^3 = 0$ (the zero matrix), where I is the 3×3 identity matrix.
Use the binomial theorem for matrices (which applies because $2I$ commutes with N) to show that $A^n = 2^nI + n2^{n-1}N + \frac{n(n-1)}{2}2^{n-2}N^2$ for $n \geq 2$, and hence find A^{10} . [10]

Determinants

- 1** Find the determinant of the matrix $A = \begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix}$. [2]
- 2** Evaluate $\begin{vmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{vmatrix}$, simplifying your answer. [2]

- 3** The matrices A and B are given by $A = \begin{pmatrix} 2 & 6 \\ 1 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 6 \\ 1 & 4 \end{pmatrix}$.
State, with a reason, which of the two matrices is singular. [3]
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- 4** Write down the value of the determinant $\begin{vmatrix} 0 & 0 & 0 \\ 4 & 1 & 7 \\ 2 & 5 & 3 \end{vmatrix}$, giving a reason for your answer. [2]
-
- 5** Find the determinant of the matrix $\begin{pmatrix} 2 & 5 & -1 \\ 0 & 3 & 4 \\ 0 & 0 & -2 \end{pmatrix}$. [2]
-
- 6** Find the value of k for which the matrix $\begin{pmatrix} k & 3 \\ 2 & 6 \end{pmatrix}$ is singular. [2]
-
- 7** Solve the equation $\begin{vmatrix} x & 2 \\ 3 & x-1 \end{vmatrix} = 0$. [3]
-
- 8** Evaluate the determinant $\begin{vmatrix} 2 & -1 & 3 \\ 1 & 4 & -2 \\ 5 & 0 & 1 \end{vmatrix}$. [3]
-
- 9** The matrix $A = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$ represents a linear transformation of the plane.
A triangle T has area 7 square units. Find the area of the image of T under A. [3]
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- 10** The matrices A and B are given by $A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -2 \\ 5 & 0 \end{pmatrix}$.
Find $\det A$, $\det B$ and $\det(AB)$, and verify that $\det(AB) = \det A \times \det B$. [4]
-
- 11** The 2×2 matrix A has determinant 4. Write down the value of
(a) $\det(A^T)$, [1]
(b) $\det(A^2)$, [1]
(c) $\det(3A)$. [2]
-
- 12** The matrix A is given by $A = \begin{pmatrix} x & 3 \\ -2 & x-1 \end{pmatrix}$, where x is a real constant.
Show that A is non-singular for every real value of x. [4]
-
- 13** The matrix A is given by $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & k \end{pmatrix}$.
Find the value of k for which A is singular. [4]
-
- 14** The matrix A is given by $A = \begin{pmatrix} k & -1 & 2 \\ 0 & k & 3 \\ 1 & 0 & 1 \end{pmatrix}$.
Show that $\det A = k^2 - 2k - 3$, and hence find the values of k for which A is singular. [5]
-
- 15** Without expanding the determinant, show that $\begin{vmatrix} 201 & 202 & 203 \\ 204 & 205 & 206 \\ 207 & 208 & 209 \end{vmatrix} = 0$, explaining each step,
and deduce that the matrix is singular. [4]
-

- 16** The matrix $A = \begin{pmatrix} k & 2 \\ 1 & 3 \end{pmatrix}$, where k is a constant, represents a transformation of the plane that maps a region of area 5 square units onto a region of area 20 square units. Find the possible values of k . [5]
- 17** The matrices A and B are given by $A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -2 \\ 5 & 0 \end{pmatrix}$. Find $\det(A + B)$ and $\det A + \det B$, and comment on the result. [4]
- 18** Show that $\begin{vmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{vmatrix} = (x - 1)^2(x + 2)$, and hence solve the equation $\begin{vmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{vmatrix} = 0$. [5]
- 19** The unit square $OABC$, with $O(0, 0)$, $A(1, 0)$, $B(1, 1)$ and $C(0, 1)$, is mapped by the matrix M onto the parallelogram $OA'B'C'$, where A' is the point $(3, 1)$ and C' is the point $(1, 4)$. Write down the matrix M , find the coordinates of B' , and find the area of the parallelogram $OA'B'C'$. [5]
- 20** The matrix A is given by $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$. Show that A is singular. Find the image of the general point (x, y) under A , and describe geometrically what A does to the plane. [5]
- 21** Show that $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (b - a)(c - a)(c - b)$. [6]
- 22** The 3×3 matrices A and B satisfy $\det A = 2$ and $\det B = -3$. Find the value of
 (a) $\det(AB)$, [2]
 (b) $\det(2A)$, [2]
 (c) $\det(A^T B^{-1})$, [2]
 (d) $\det(A^2 B^3)$. [2]
- 23** The 2×2 matrix A satisfies $A^2 = A$. Show that $\det A = 0$ or $\det A = 1$. Find a matrix $A \neq I$ which satisfies $A^2 = A$ and has $\det A = 0$, and a matrix A which satisfies $A^2 = A$ and has $\det A = 1$. [6]
- 24** The matrix A is given by $A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix}$. Find the values of λ for which the matrix $A - \lambda I$ is singular, where I is the 3×3 identity matrix. [6]
- 25** The matrices A and B are given by $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} p & q \\ x & y \end{pmatrix}$. Prove that $\det(AB) = \det A \times \det B$. [7]
- 26** A 3×3 matrix K is skew-symmetric, so that $K^T = -K$. By considering $\det(K^T)$, prove that $\det K = 0$. Verify this result for the general skew-symmetric matrix $\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$. [7]

27

Show that $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a+b+c)(a^2+b^2+c^2-ab-bc-ca).$

Hence find all real values of a , b and c for which the matrix $\begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}$ is singular. [10]

28

Show that $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc + ab + bc + ca.$

Hence show that the matrix is non-singular whenever a , b and c are all positive.

Find the values of t for which the matrix is singular when $a = b = c = t$. [8]

29

The area of a triangle with vertices (x_1, y_1) , (x_2, y_2) and (x_3, y_3) is $\frac{1}{2}|D|$, where $D =$

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

Use this result to find the area of the triangle with vertices $(1, 2)$, $(5, 3)$ and $(2, 7)$.

The points $A(1, 1)$, $B(a, a^2)$ and $C(b, b^2)$ lie on the curve $y = x^2$. Show that the area of triangle

ABC is $\frac{1}{2}|(a-1)(b-1)(b-a)|$.

Hence explain why no three distinct points on the curve $y = x^2$ can be collinear. [10]

30

Show that $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1+x & 1 & 1 \\ 1 & 1 & 1+y & 1 \\ 1 & 1 & 1 & 1+z \end{vmatrix} = xyz.$

Hence find the value of the determinant when $x = 2$, $y = -1$ and $z = 3$, and state the conditions on x , y and z for the matrix to be singular.

Use a similar method to evaluate the 4×4 determinant in which every element on the leading diagonal is 3 and every other element is 1. [9]

Inverse Matrices

1

Find the inverse of the matrix $A = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$. [2]

2

Explain why the matrix $C = \begin{pmatrix} 6 & 3 \\ 2 & 1 \end{pmatrix}$ has no inverse. [2]

3

Write down the inverse of the matrix $D = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}$. [2]

4

The matrices A and B are given by $A = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$.
Show that B is the inverse of A . [2]

5

Find the inverse of the matrix $R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. [2]

- 6** Find the inverse of the matrix $A = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}$, giving the elements as fractions where necessary. [4]
-
- 7** The matrix A is given by $A = \begin{pmatrix} k & 2 \\ 3 & k+1 \end{pmatrix}$, where k is a constant.
Find the values of k for which A^{-1} does not exist, and find A^{-1} in terms of k when it does exist. [5]
-
- 8** Given $A = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & -1 \\ 7 & 2 \end{pmatrix}$, find the matrix X such that $AX = B$. [4]
-
- 9** Given $A = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 5 \\ -2 & 0 \end{pmatrix}$, find the matrix X such that $XA = B$. [4]
-
- 10** The matrix B satisfies $B^{-1} = \begin{pmatrix} -1 & 1 \\ -3 & 2 \end{pmatrix}$. Find the matrix B . [3]
-
- 11** The matrices A and B are given by $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$.
Find A^{-1} , B^{-1} and $(AB)^{-1}$, and verify that $(AB)^{-1} = B^{-1}A^{-1}$. [5]
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- 12** Given $A = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$, find the matrix X such that $AX + I = B$, where I is the 2×2 identity matrix. [4]
-
- 13** The triangle T is mapped by the matrix $A = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$ onto the triangle T' , whose vertices are $(5, 5)$, $(10, 5)$ and $(10, 10)$.
Find the coordinates of the vertices of T .
Given that the area of T' is 12.5 square units, find the area of T . [8]
-
- 14** Find the inverse of the matrix $A = \begin{pmatrix} 2 & 1 & 3 \\ 0 & 1 & 4 \\ 1 & 0 & 2 \end{pmatrix}$. [6]
-
- 15** The 3×3 matrix $A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$.
The point P is mapped onto the point $Q(7, 4, 5)$ by the transformation represented by A .
Find A^{-1} and hence find the coordinates of P . [6]
-
- 16** The matrix A is given by $A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$.
Show that $A^2 = 5A - 5I$, where I is the 2×2 identity matrix.
Hence show that $A^{-1} = \frac{1}{5}(5I - A)$ and write down A^{-1} . [6]
-
- 17** The matrices P , Q and R are given by $P = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, $Q = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$ and $R = \begin{pmatrix} 5 & 4 \\ 3 & 2 \end{pmatrix}$.
Find the matrix X such that $PXQ = R$. [6]
-

- 18** The matrix A is given by $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $ad - bc \neq 0$.
Show, by multiplication, that $A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.
Explain why A has no inverse when $ad - bc = 0$. [5]
-
- 19** The matrices A and B are invertible and of the same order.
Prove that $(AB)^{-1} = B^{-1}A^{-1}$, and that $(A^T)^{-1} = (A^{-1})^T$. [6]
-
- 20** A message is coded by writing it as pairs of numbers ($A = 1, B = 2, \dots, Z = 26$) and replacing each pair $\begin{pmatrix} x \\ y \end{pmatrix}$ by the pair given by $\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.
The coded message is the pairs (97, 39) and (41, 18). Find the inverse of $\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$ and use it to decode the message, giving the four-letter word. [5]
-
- 21** The matrix A is given by $A = \begin{pmatrix} 1 & 2 & k \\ 0 & 1 & 3 \\ 2 & 0 & 1 \end{pmatrix}$, where k is a constant.
Find the value of k for which A is singular.
Given that $k = 5$, find A^{-1} and hence solve $\begin{pmatrix} 1 & 2 & 5 \\ 0 & 1 & 3 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. [9]
-
- 22** A, B and C are square matrices of the same order and A is invertible.
Prove that if $AB = AC$ then $B = C$.
Show, by giving a counterexample with 2×2 matrices, that this result need not hold if A is singular. [5]
-
- 23** The matrix A is given by $A = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$.
Show that $A^3 = I$, where I is the 2×2 identity matrix, and hence write down A^{-1} in terms of A .
Find A^{-1} as a matrix, and find A^{100} . [7]
-
- 24** For an invertible 2×2 matrix M , the adjugate matrix is $\text{adj } M = (\det M)M^{-1}$.
Use the result $\det(AB) = \det A \det B$, together with $(AB)^{-1} = B^{-1}A^{-1}$, to show that $\text{adj}(AB) = \text{adj}(B) \text{adj}(A)$ for invertible 2×2 matrices A and B . [6]
-
- 25** The square matrices A and B satisfy $AB = BA$, and A is invertible.
Show that $A^{-1}B = BA^{-1}$.
Given also that B is invertible, deduce that $A^{-1}B^{-1} = B^{-1}A^{-1}$. [7]
-
- 26** A student writes: "If A, B and $A + B$ are all invertible, then $(A + B)^{-1} = A^{-1} + B^{-1}$."
Show that the student is wrong by calculating both sides when $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$.
Give a very simple pair of invertible matrices A and B for which the statement fails, and explain why. [6]
-

27

The matrix V is given by $V = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$.

Show that $\det V = 2$ and find V^{-1} .

The quadratic curve $y = \alpha + \beta x + \gamma x^2$ passes through the points $(1, 5)$, $(2, 9)$ and $(3, 15)$. By writing the conditions as a matrix equation involving V , find α , β and γ .

Explain why the same method would fail for three points two of which have the same x -coordinate.

[10]

28

The square matrix A satisfies $A^2 = 4A - 3I$, where I is the identity matrix of the same order.

Show that A is invertible and that $A^{-1} = \frac{1}{3}(4I - A)$.

Hence show that $A^{-2} = \frac{1}{9}(13I - 4A)$.

Verify that the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ satisfies $A^2 = 4A - 3I$, and use the formula to write down A^{-1} and A^{-2} as matrices.

[10]

29

The real 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has $\det A = 1$ and satisfies $A^{-1} = A^T$.

Show that $d = a$ and $c = -b$, and that $a^2 + b^2 = 1$.

Hence show that A is a matrix of the form $\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ for some angle θ , and describe the transformation represented by A .

[10]

30

A and B are square matrices of the same order and $I - AB$ is invertible, where I is the identity matrix.

Show that $I - BA$ is also invertible, with inverse $I + B(I - AB)^{-1}A$.

The result also holds when A and B are not square, provided that AB and BA are both defined

(with identity matrices of the appropriate sizes). Take $A = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 1 \end{pmatrix}$, so that AB is

2×2 and BA is 1×1 . Use the result, given that $(I - AB)^{-1} = \begin{pmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix}$, to find $(I -$

$BA)^{-1}$.

[12]

Solving Simultaneous Equations Using Matrices

1 Write the simultaneous equations $3x - 2y = 5$ and $x + 4y = 11$ in the form $Ax = b$, where A is a 2×2 matrix and x and b are column vectors. [2]

2 The matrix $A = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$ has inverse $A^{-1} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$.
Use A^{-1} to solve the simultaneous equations $2x + y = 5$ and $3x + 2y = 8$. [2]

3 Use matrices to solve the simultaneous equations $x + y = 5$ and $2x - y = 1$. [3]

4 Write the equations $x + 2y - z = 4$, $2x - y + 3z = -1$ and $3x + y + z = 7$ as a matrix equation. [3]

- 5** For each of the following pairs of equations, state whether there is a unique solution, giving a reason.
 (a) $2x + 3y = 1$ and $4x + 6y = 5$ [1]
 (b) $2x + 3y = 1$ and $4x + 5y = 5$ [1]
-
- 6** Solve $5x + 3y = 4$ and $2x + y = 1$ by using an inverse matrix. [2]
-
- 7** The line $y = mx + c$ passes through the points (1, 5) and (3, 11).
 Use matrices to find the values of m and c . [5]
-
- 8** The matrix $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ has inverse $A^{-1} = \begin{pmatrix} -\frac{5}{2} & \frac{3}{2} & \frac{1}{2} \\ \frac{3}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$.
 Use A^{-1} to solve the simultaneous equations $x + 2y + z = 4$, $2x + 3y + z = 7$ and $x + y + 2z = 1$. [3]
-
- 9** The simultaneous equations $kx + 2y = 3$ and $3x + (k + 1)y = 2$ have a coefficient matrix $A = \begin{pmatrix} k & 2 \\ 3 & k + 1 \end{pmatrix}$, where k is a constant.
 Find the values of k for which the equations do not have a unique solution, and solve the equations when $k = 1$. [5]
-
- 10** In each case show that the coefficient matrix of the simultaneous equations is singular, and describe geometrically what the two equations represent and how many solutions there are.
 (a) $2x + 3y = 5$ and $4x + 6y = 7$ [3]
 (b) $2x + 3y = 5$ and $4x + 6y = 10$ [3]
-
- 11** At a theatre, 3 adult tickets and 2 child tickets cost £47 in total, and 2 adult tickets and 5 child tickets cost £68 in total.
 Form a matrix equation and use it to find the price of an adult ticket and the price of a child ticket. [4]
-
- 12** Solve the simultaneous equations $x + 2y + 3z = 13$, $2x - y + z = 6$ and $3x + y - 2z = 1$ by reducing the augmented matrix to reduced row echelon form, showing the matrix after each stage. [6]
-
- 13** The matrices A and B are given by $A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 2 \\ 0 & 1 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & -3 & -3 \\ 1 & 1 & 3 \\ 1 & 1 & -3 \end{pmatrix}$.
 Find AB and hence write down A^{-1} .
 Hence solve the simultaneous equations $x + 2y + z = 6$, $-x + y + 2z = 3$ and $y - z = -1$. [5]
-
- 14** Solve the simultaneous equations
 $x + 2y - z = -3$,
 $2x - y + 3z = 14$,
 $3x + y + z = 8$,
 by using the inverse of a 3×3 matrix. [7]
-
- 15** The simultaneous equations $kx - y + 2z = 1$, $ky + 3z = 2$ and $x + z = 3$ are written as $Ax = b$,
 where $A = \begin{pmatrix} k & -1 & 2 \\ 0 & k & 3 \\ 1 & 0 & 1 \end{pmatrix}$.
 Find the values of k for which the equations do not have a unique solution.
 Show that when $k = 3$ the equations are inconsistent. [6]

- 16** Use row operations on the equations $x + y + z = 6$, $2x - y + z = 3$ and $x + 2y - z = 2$ to solve them, writing down the system after each stage in matrix form. [6]
-
- 17** Show that the simultaneous equations $x + 2y + 3z = 4$, $2x + 4y + 6z = 9$ and $x + z = 1$ have no solution, and describe the geometrical arrangement of the three planes. [5]
-
- 18** Show that the equations $x + 2y - z = 2$, $2x + y + z = 4$ and $x - y + 2z = 2$ have infinitely many solutions, and find the solution in terms of a parameter. Hence give the vector equation of the line in which the three planes meet. [6]
-
- 19** Three alloys A, B and C contain copper, zinc and tin. Alloy A is 60% copper and 30% zinc, alloy B is 40% copper and 40% zinc, and alloy C is 20% copper and 60% zinc; the rest of each alloy is tin.
A foundry mixes x kg of A, y kg of B and z kg of C to make 100 kg of alloy containing 40 kg of copper and 43 kg of zinc.
Write down three equations for x , y and z , write them as a matrix equation, and solve it. [5]
-
- 20** In an electrical circuit the currents I_1 , I_2 and I_3 , measured in amperes, satisfy $I_1 = I_2 + I_3$, $2I_1 + 3I_2 = 12$ and $3I_2 = 6I_3$.
Write these equations as a matrix equation and solve it to find the three currents. [5]
-
- 21** The simultaneous equations $x + ky + z = 1$, $2x + y + kz = 2$ and $x + y + z = 3$ have coefficient matrix $A = \begin{pmatrix} 1 & k & 1 \\ 2 & 1 & k \\ 1 & 1 & 1 \end{pmatrix}$, where k is a constant.
Show that $\det A = (k - 1)(k - 2)$.
State the values of k for which the equations do not have a unique solution, and find the solution when $k = 3$. [8]
-
- 22** The homogeneous system $x + 2y - z = 0$, $2x + ky + z = 0$ and $x - y + 2z = 0$ has coefficient matrix $A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & k & 1 \\ 1 & -1 & 2 \end{pmatrix}$.
Show that the system has a non-trivial solution (a solution other than $x = y = z = 0$) if and only if $k = 1$.
Find the general solution when $k = 1$. [8]
-
- 23** The equations $ax + by = e$ and $cx + dy = f$, where $ad - bc \neq 0$, are written as a matrix equation.
Use the inverse of the coefficient matrix to show that $x = \frac{de - bf}{ad - bc}$ and $y = \frac{af - ce}{ad - bc}$.
Use these formulae to solve $4x - 3y = 10$ and $5x + 2y = 1$. [6]
-
- 24** The three planes with equations $x + y + z = 1$, $x + 2y + 3z = 2$ and $x + 3y + 5z = c$ are written as $Ax = b$, with $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{pmatrix}$.
Show that $\det A = 0$.
Find the value of c for which the planes meet in a line, and find a vector equation of this line.
Describe the geometrical arrangement of the planes when c takes any other value. [9]
-

25

The matrix A is given by $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix}$.

Find A^{-1} .

Use your answer to solve both of the following systems at once, by finding the 3×2 matrix X

such that $AX = \begin{pmatrix} 4 & 1 \\ 7 & 6 \\ 2 & 7 \end{pmatrix}$.

Write down the solutions of $2x + y = 4$, $x + 3y + z = 7$, $y + 2z = 2$ and of $2x + y = 1$, $x + 3y + z = 6$, $y + 2z = 7$.

[7]**26**

Use matrices to solve each of the following pairs of simultaneous equations, and then comment on the effect of the very small change in one of the constants.

(a) $x + y = 2$ and $x + 1.001y = 2$

[4]

(b) $x + y = 2$ and $x + 1.001y = 2.001$

[2]

(c) Comment on how the solution changes when the constant 2 in the second equation of the first pair is replaced by 2.001, and relate this to the determinant of the coefficient matrix.

[2]**27**

The simultaneous equations $x + y + z = 1$, $x + 2y + 3z = 2$ and $x + 3y + az = b$, where a and b are constants, are written as a matrix equation.

Find the value of a for which the coefficient matrix is singular.

Show that, when a is not this value, the equations have a unique solution, and find it in terms of a and b .

For the singular value of a , find the value of b for which the equations are consistent, and find the general solution in this case. Describe the geometrical meaning of this solution, and of the situation when b takes any other value.

[12]**28**

The simultaneous equations $x + y + z = 1$, $x + (1 + a)y + z = k$ and $x + y + (1 + c)z = 3$, where a , c and k are constants, are written as a matrix equation.

Show that the coefficient matrix has determinant ac .

Given that $ac \neq 0$, solve the equations for x , y and z in terms of a , c and k .

Given that $a = 0$ and $c = 2$, find the value of k for which the equations are consistent, and find the general solution in this case.

[11]**29**

A is an $n \times n$ matrix and X and B are column vectors with n rows. Suppose that $AX = B$ has two different solutions $X = P$ and $X = Q$.

Show that, for every real number λ , $X = P + \lambda(Q - P)$ is also a solution, and deduce that the system has infinitely many solutions.

Prove that A is singular.

Deduce that a system of three linear equations in three unknowns cannot have exactly two solutions.

[10]

- 30** A simple economy has three industries. To produce one unit of output, industry j uses c_{ij} units

of the output of industry i , where the consumption matrix is $C = \begin{pmatrix} \frac{1}{5} & \frac{1}{10} & \frac{1}{5} \\ \frac{1}{10} & \frac{1}{5} & \frac{1}{10} \\ \frac{1}{5} & \frac{1}{10} & \frac{1}{10} \end{pmatrix}$.

The total outputs of the industries are given by the column vector x , and the external demand

for the three industries is $d = \begin{pmatrix} 25 \\ 90 \\ 145 \end{pmatrix}$ (in suitable units).

Explain why $x = Cx + d$, and show that $(I - C)x = d$.

Given that the determinant of $I - C$ is $\frac{523}{1000}$, find the matrix $I - C$ and use a calculator to find its inverse. Hence find the total output required from each industry.

[8]

Matrix Transformations

- 1** Write down the 2×2 matrix that represents a rotation of 90° anticlockwise about the origin. [2]

- 2** Write down the matrix that represents a reflection in the line $y = x$. [2]

- 3** Write down the matrix that represents an enlargement with scale factor 3 and centre the origin, and find the image of the point $(2, -1)$ under this enlargement. [2]

- 4** Find the image of the point $(2, -1)$ under the transformation represented by the matrix $\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$. [2]

- 5** Describe the transformation represented by the matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. [2]

- 6** Write down the matrix that represents a stretch with scale factor 2 parallel to the x -axis, in which the y -axis is invariant. [2]

- 7** The triangle ABC has vertices $A(1, 1)$, $B(3, 1)$ and $C(1, 2)$. It is transformed by the matrix $T = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$. Find the coordinates of the vertices of the image triangle $A'B'C'$. [3]

- 8** Describe fully the single transformation represented by each of these matrices.
(a) $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ [2]

- (b) $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ [2]

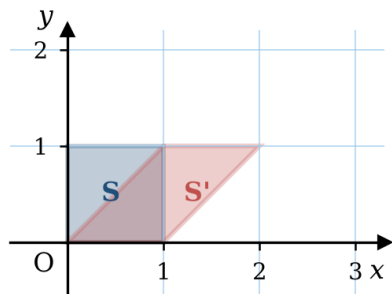
- (c) $\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$ [2]

- 9** Write down the matrix that represents a rotation of 60° anticlockwise about the origin, and find the image of the point $(2, 2\sqrt{3})$ under this rotation. [4]

- 10** Find the matrix that represents a reflection in the line $y = -x$, and use it to find the image of the point $(3, -2)$. [4]

- 11** A linear transformation T maps $(1, 0)$ to $(3, -1)$ and maps $(0, 1)$ to $(2, 4)$. Write down the matrix of T , find the image of the point $(2, 3)$, and find the determinant of this matrix. [4]

- 12** The diagram shows the unit square S , with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$ and $(0, 1)$, and its image S' under a transformation represented by the matrix A . Write down the matrix A , name the type of transformation, and find the image of the point $(2, 3)$.



- 13** Find the equation of the image of the line $x + 2y = 6$ under the transformation represented by the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, giving your answer in the form $y = mx + c$. [5]

- 14** The matrix $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ represents a shear. Find the equation of the image of the line $y = 2x + 1$ under this shear, giving your answer in the form $y = mx + c$. [5]

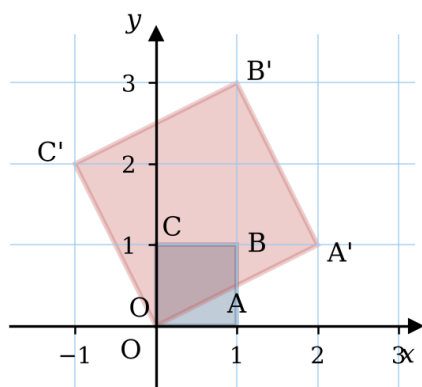
- 15** The matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is such that M maps the point $(1, 2)$ to $(8, 3)$ and maps the point $(3, 1)$ to $(9, -1)$. Find the values of a , b , c and d . [5]

- 16** The matrix $M = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}$ represents a transformation of the plane. Show that M has a line of invariant points through the origin, and find its equation. [4]

- 17** The matrix $A = \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}$ represents a single transformation. Show that $\det A = 1$ and describe the transformation fully, giving any angle to 1 decimal place. [5]

- 18** The matrix $B = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$ represents a reflection in a line through the origin. Find the equation of the mirror line, giving your answer in the form $y = mx$. [4]

- 19** The diagram shows the unit square OABC, where A is (1, 0), B is (1, 1) and C is (0, 1), and its image OA'B'C' under the transformation represented by the matrix M. Use the diagram to find M, and calculate the area of OA'B'C'.



[5]

- 20** A student states that the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ represents a rotation of 90° anticlockwise about the origin.

Show that the student is wrong, say what transformation the matrix does represent, and write down the correct matrix for a 90° anticlockwise rotation.

[5]

- 21** The matrix $M = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}$ represents a transformation of the plane.

Show that a line $y = mx$ through the origin is mapped onto itself if and only if $2m^2 - 3m + 1 = 0$.

Hence find the equations of the two invariant lines through the origin, and state which of them consists of invariant points.

[7]

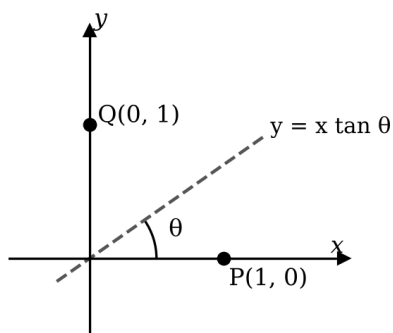
- 22** The unit circle C has equation $x^2 + y^2 = 1$. It is transformed by the matrix $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ onto the curve C'.

Find the Cartesian equation of C', and state the area enclosed by C'.

[5]

- 23** The line L through the origin makes an angle θ with the positive x-axis, as shown in the diagram, where P is the point (1, 0) and Q is the point (0, 1). A reflection in L is represented by the matrix R. By finding the images of P and Q, show that

$$R = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}.$$



[7]

- 24** A linear transformation T is represented by the matrix N. T maps the point (1, 1) to (3, 5) and the point (1, -1) to (1, -3).

Find N, and find the image of the point (4, 2) under T.

[6]

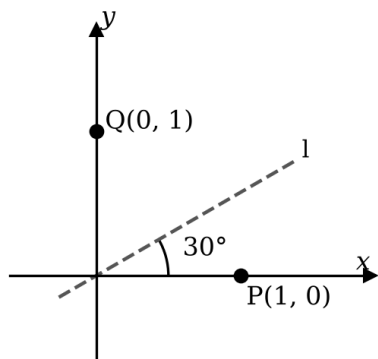
- 25** The matrix $M = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ represents a transformation of the plane. The unit circle $x^2 + y^2 = 1$ is mapped by M onto the curve C .
Find the Cartesian equation of C , and state the area enclosed by C . [7]
- 26** The matrix R represents a rotation about the origin which maps the point $(3, 4)$ onto the point $(5, 0)$.
Find the matrix R , giving its elements as fractions, and find the image of the point $(-1, 2)$ under this rotation.
State the angle and sense of the rotation, to 1 decimal place. [8]
- 27** The matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where a, b, c and d are constants and $ad - bc \neq 0$, represents a transformation of the plane.
Show that the line $y = mx$ is mapped onto itself if and only if $bm^2 + (a - d)m - c = 0$.
The matrix $A = \begin{pmatrix} 1 & k \\ 2 & 3 \end{pmatrix}$, where k is a non-zero constant with $k \neq \frac{3}{2}$, represents a transformation. Find the values of k for which A has exactly one invariant line through the origin of the form $y = mx$, and state its equation for this value of k . [10]
- 28** The matrix N is given by $N = \begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$.
Show that $N = pQ$, where p is a constant and Q is the matrix of a rotation, and hence describe the transformation represented by N as a combination of two simple transformations.
Find N^6 , and hence find N^{2025} in the form qI , where q is a constant.
Find N^{-1} . [11]
- 29** The matrix $M = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ is such that each of the lines $y = x$ and $y = -x$ is mapped onto itself (not necessarily point by point).
Show that $s = p$ and $r = q$.
Given also that M maps the point $(1, 0)$ to $(4, 2)$, find M , and show that $\det M = 12$.
Show that M maps each point of $y = x$ to a point of $y = x$ at a distance 6 times as far from O , and find the corresponding scale factor for $y = -x$. Explain why the product of these scale factors is $\det M$. [11]
- 30** The unit circle C with equation $x^2 + y^2 = 1$ is transformed by the matrix $M = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$ onto the curve C' .
Show that C' has equation $x^2 - 2xy + 2y^2 = 5$.
Verify that the images of $(1, 0)$ and $(0, 1)$ lie on C' , and find the exact coordinates of the points where C' crosses the x -axis and the y -axis.
Find the area enclosed by C' . [11]

Combinations of Transformations

- 1** The matrix $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ represents a rotation of 90° anticlockwise about the origin, and $B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ represents a reflection in the x -axis.
Find the matrix that represents A followed by B . [2]

- 2** A reflection in the x-axis is followed by a reflection in the y-axis. Find the matrix of the combined transformation and describe it as a single transformation. [3]
-
- 3** The matrix $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ represents a rotation of 90° anticlockwise about the origin. Find A^2 and state what transformation it represents. [3]
-
- 4** The matrix $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ represents a reflection in the line $y = x$. Find B^2 and explain the result geometrically. [3]
-
- 5** The matrix $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ represents a rotation of 90° anticlockwise about the origin. Write down the matrix of the inverse transformation, and describe this inverse transformation. [2]
-
- 6** A reflection in the line $y = x$ is followed by a rotation of 180° about the origin. Find the matrix of the combined transformation, and describe it as a single transformation. [4]
-
- 7** An enlargement with scale factor 2 and centre the origin is followed by a rotation of 90° clockwise about the origin. Find the matrix of the combined transformation, and the image of the point (3, 1). [4]
-
- 8** A shear with matrix $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ is followed by a stretch with matrix $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$. Find the matrix of the combined transformation, and the image of the point (1, 1). [3]
-
- 9** A is the matrix of a rotation of 90° anticlockwise about the origin and B is the matrix of a reflection in the x-axis. Find BA and AB, and describe each as a single transformation. What does this show about the order in which transformations are combined? [5]
-
- 10** The transformation T is an enlargement with scale factor 3 and centre the origin followed by a rotation of 90° anticlockwise about the origin. Find the matrix of T and the matrix of T^{-1} , and describe T^{-1} . [5]
-
- 11** A transformation consists of a reflection in the x-axis, followed by a rotation of 90° anticlockwise about the origin, followed by a stretch of scale factor 2 parallel to the y-axis (x-axis invariant). Find the matrix of the combined transformation, and the image of the point (2, 1). [4]
-
- 12** A stretch with scale factor 3 parallel to the y-axis (x-axis invariant) is followed by a reflection in the line $y = -x$. Find the matrix of the combined transformation, and the image of the point (2, 1). [4]
-
- 13** The matrix A represents a rotation of 60° anticlockwise about the origin and the matrix B represents a rotation of 30° anticlockwise about the origin. Write down A and B, find AB, and explain why the result is what you expect. [5]
-

- 14** The line l passes through the origin and makes an angle of 30° with the positive x -axis, as shown in the diagram. A reflection in the x -axis is followed by a reflection in l . Given that the matrix of a reflection in the line through O at angle θ to the x -axis is $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$, find the matrix of the combined transformation and describe it as a single transformation.



[4]

- 15** The transformation S is followed by the shear T with matrix $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. The combined transformation has matrix $U = \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix}$. Find the matrix of S .

[5]

- 16** A rotation of 90° clockwise about the origin is followed by an enlargement with scale factor 2, centre the origin. Find the equation of the image of the line $y = 2x + 1$ under this combined transformation.

[6]

- 17** A stretch with matrix $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$, where k is a constant, is followed by a shear with matrix $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. Find the matrix of the combined transformation. A triangle of area 3 is mapped to a triangle of area 12; find the possible values of k , and the image of the point $(1, 1)$ for the negative value of k .

[5]

- 18** The matrix $\begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$ represents a combination of a reflection in the line $y = x$ and an enlargement with centre the origin. Describe the combination, giving the scale factor, and state the factor by which areas are multiplied.

[4]

- 19** A triangle of area 6 square units is transformed first by the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and then by the matrix $\begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}$. Find the matrix of the combined transformation and the area of the final image.

[5]

- 20** A transformation T of the plane is a reflection in the line $y = x$ followed by a translation through the vector $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$.
- T maps (x, y) to (x', y') . Find the 3×3 matrix N such that $\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = N \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$.
- Find the image of the point $(2, 5)$ under T , and find the point that T maps onto $(4, 6)$. [6]
-
- 21** $R(\theta)$ denotes the matrix $\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ of a rotation through θ anticlockwise about the origin.
- By considering the product $R(\theta)R(\varphi)$, and the fact that a rotation through φ followed by a rotation through θ is a rotation through $\theta + \varphi$, prove that $\cos(\theta + \varphi) = \cos \theta \cos \varphi - \sin \theta \sin \varphi$ and $\sin(\theta + \varphi) = \sin \theta \cos \varphi + \cos \theta \sin \varphi$. [6]
-
- 22** A reflection in the line $y = x \tan 15^\circ$ is followed by a reflection in the line $y = x \tan 60^\circ$.
- Using the matrix $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$ for a reflection in the line through O at angle θ to the x -axis, find the matrix of the combined transformation and describe it as a single transformation. [7]
-
- 23** The matrix $M = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$.
- Show that M can be written as a shear with matrix $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ followed by a stretch with matrix $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, and find the value of k .
- Show also that M can be written as the stretch followed by a shear with matrix $\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$, and find the value of the constant c . [6]
-
- 24** A rotation through θ anticlockwise about the origin is followed by a reflection in the x -axis. Show that the combined transformation is a reflection in a line through the origin, and find the angle that this line makes with the positive x -axis in terms of θ . [6]
-
- 25** A reflection in the line $y = x$ is followed by a stretch with scale factor 2 parallel to the x -axis (y -axis invariant).
- Find the matrix of the combined transformation, and hence find the equation of the image of the line $y = 3x - 1$. [6]
-
- 26** A student is asked to find the matrix of a reflection in the line $y = x$ followed by a stretch with scale factor 2 parallel to the x -axis. The student writes the matrix as $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$.
- By considering the image of the point $(1, 0)$, show that the student's matrix is wrong, explain the mistake, and write down the correct matrix. [6]

- 27** The matrix $F(\theta) = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$ represents a reflection in the line through the origin at angle θ to the positive x-axis.
A reflection in the line at angle α is followed by a reflection in the line at angle β . Show that the combined transformation is a rotation about the origin, and find its angle in terms of α and β .
Hence express the rotation through 100° anticlockwise about the origin as the product of two reflections, the first of which is in the x-axis, and give the equation of the second mirror line in the form $y = x \tan \theta$. [8]
-
- 28** The matrix $R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ represents a rotation of 90° anticlockwise about the origin. The matrix N is such that $N^2 = R$.
Show that $NR = RN$.
Hence, by writing $N = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and comparing elements in $NR = RN$, show that N has the form $\begin{pmatrix} p & -q \\ q & p \end{pmatrix}$.
Find all possible matrices N , and describe the transformations they represent. [11]
-
- 29** The unit circle with equation $x^2 + y^2 = 1$ is transformed by a stretch with scale factor 2 parallel to the x-axis (y-axis invariant), followed by a rotation of 30° anticlockwise about the origin.
Find the matrix T of the combined transformation.
Show that the image of the circle has equation $7x^2 - 6\sqrt{3}xy + 13y^2 = 16$, and find the area that it encloses. [10]
-
- 30** The line l has equation $y = 2x$, and the angle α that l makes with the positive x-axis satisfies $\tan \alpha = 2$.
Write down $\cos \alpha$ and $\sin \alpha$.
A reflection in l can be achieved by a rotation through $-\alpha$, followed by a reflection in the x-axis, followed by a rotation through α . Use this to find the matrix of the reflection in l .
Hence find the image of the point $(1, 3)$ under the reflection, and verify that the matrix has the form $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$ for a suitable angle θ . [10]

Three-Dimensional Matrix Transformations

- 1** Write down the 3×3 matrix that represents a reflection in the plane $z = 0$ (the xy-plane). [2]
-
- 2** Write down the matrix that represents an enlargement with scale factor 4 and centre the origin, and find the image of the point $(1, -2, 3)$ under this enlargement. [2]
-
- 3** Find the image of the point $(1, 2, 3)$ under the transformation represented by the matrix $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 1 \\ 2 & 1 & 0 \end{pmatrix}$. [2]
-
- 4** Write down the matrix that represents a rotation of 90° anticlockwise about the z-axis, when viewed from the positive z-axis looking towards the origin. [2]
-
- 5** Write down the matrix that represents a stretch with scale factor 3 parallel to the z-axis, in which the points of the xy-plane are invariant. [2]

- 6** Describe fully the transformation represented by the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$. [3]
-
- 7** Find the matrix that represents a reflection in the plane $x = y$, and find the image of the point $(3, -1, 5)$. [3]
-
- 8** Write down the matrix that represents a rotation of 90° anticlockwise about the y -axis (viewed from the positive y -axis), and find the image of the point $(1, 2, 3)$. [4]
-
- 9** The matrix $A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 4 \end{pmatrix}$ represents a transformation of three-dimensional space. Find $\det A$, and hence the volume of the image of a solid of volume 5 cubic units. [4]
-
- 10** A rotation of 90° anticlockwise about the z -axis is followed by a reflection in the plane $z = 0$. Find the matrix of the combined transformation, and the image of the point $(1, 2, 3)$. [4]
-
- 11** A transformation T maps i to $(2, 1, 0)$, j to $(0, 3, 1)$ and k to $(1, 0, 1)$. Write down the matrix of T , find the image of the point $(1, 1, 1)$, and find the determinant of the matrix. [4]
-
- 12** Find the matrix of the inverse of each of these transformations.
 (a) A reflection in the plane $y = 0$. [2]
 (b) A rotation of 90° anticlockwise about the z -axis. [2]
-
- 13** Describe the transformation represented by the matrix $\begin{pmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ as a combination of two simple transformations. [3]
-
- 14** The matrix $A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ represents a transformation of three-dimensional space. The line l has vector equation $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. Find a vector equation of the image of l under A . [6]
-
- 15** The plane Π has equation $2x - y + z = 4$. It is transformed by the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Find the Cartesian equation of the image of Π . [5]
-
- 16** The matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ represents a transformation of three-dimensional space. Show that A represents a reflection in a plane through the origin, and find the equation of this plane. [5]
-
- 17** Write down the matrix that represents a rotation of 60° anticlockwise about the z -axis, giving exact values, and find the image of the point $(2, 0, 5)$. [4]
-

18

The matrix $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & k & 1 \\ 1 & 0 & 1 \end{pmatrix}$, where k is a constant, represents a transformation of

three-dimensional space. A tetrahedron of volume 5 cubic units is transformed by A into a tetrahedron of volume 55 cubic units.

Find the possible values of k .

[5]

19

A transformation consists of a reflection in the plane $z = 0$, followed by a rotation of 90° anticlockwise about the z -axis, followed by a stretch with scale factor 2 parallel to the z -axis (points of the xy -plane invariant).

Find the matrix of the combined transformation, and the image of the point $(1, 2, 3)$.

[5]

20

Describe fully the transformation represented by each matrix.

(a) $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

[2]

(b) $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

[2]

21

The matrix R represents a rotation of 45° anticlockwise about the z -axis.

Write down R in exact form and find the image of the point $(1, 1, 1)$ under R .

Show that $R^8 = I$, and explain this geometrically.

[6]

22

A reflection in the plane $x = y$ is followed by a reflection in the plane $y = z$.

Find the matrix C of the combined transformation, and show that $C^3 = I$.

Show that every point (t, t, t) is invariant under C . Given that C represents a rotation about the line through O and $(1, 1, 1)$, find the angle and sense of this rotation, as viewed from the point $(1, 1, 1)$ looking towards the origin.

[8]

23

The plane Π has equation $x + 2y - z = 3$. It is rotated through 90° anticlockwise about the

y -axis, with matrix $R = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$.

Find the Cartesian equation of the image of Π .

[6]

24

The matrix $S = \begin{pmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ represents a shear in three-dimensional space, where k is a

non-zero constant.

Find the set of invariant points, and find the image of the point (x, y, z) .

Find S^n for a positive integer n , and the volume of the image of the unit cube.

[7]

25

The matrix $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ represents a transformation of three-dimensional space.

The parallelepiped with edges from the origin along the vectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$ is

transformed by A . Find its volume before and after the transformation.

[5]

26

The matrix $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$.

Show that $A^3 = I$, find A^{-1} , and find the image of the point $(1, 2, 3)$ under A and under A^2 .
Describe the effect of A on the unit vectors i, j and k .

[7]

27

The plane Π passes through the origin and has unit normal vector $n = \frac{1}{3} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$.

A reflection in Π maps a position vector r to $r - 2(r \cdot n)n$. By finding the images of i, j and k ,

show that the matrix of the reflection is $\begin{pmatrix} \frac{7}{9} & -\frac{4}{9} & -\frac{4}{9} \\ -\frac{4}{9} & \frac{1}{9} & -\frac{8}{9} \\ -\frac{4}{9} & -\frac{8}{9} & \frac{1}{9} \end{pmatrix}$.

Verify that this matrix, N , satisfies $N^2 = I$ and $\det N = -1$, and find the image of the point $(3, 0, 3)$.

[8]

28

The matrix $Q = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{\sqrt{2}}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{pmatrix}$ represents a rotation about an axis through the origin.

Show that $QQ^T = I$, by checking one row of the product against itself and against another row, and show that $\det Q = 1$.

Show that every point on the line through the origin with direction $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is invariant, so this line is the axis of rotation.

Find the image of the point $(0, 0, 1)$, and hence find the angle of rotation.

[9]

29

The matrix $M = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ represents a transformation of three-dimensional space.

Find $\det M$ and M^{-1} .

The plane Π has equation $2x - y + 3z = 6$. Find the equation of the image of Π under M .

A solid of volume V is mapped by M onto a solid of volume 45. Find V .

[9]

30

The matrix $B = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$.

Write $B = I + N$ and show that $N^3 = 0$, where 0 is the zero matrix.

Use the binomial theorem for matrices to show that $B^n = I + nN + \frac{n(n-1)}{2}N^2$, and write B^n as a single matrix.

Find B^{-1} by using B^n with $n = -1$ (check that $B^{-1}B = I$), and find B^{10} .

[10]

Eigenvalues and Eigenvectors

- 1 Show that $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector of the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, and state the corresponding eigenvalue. [3]
- 2 Write down the eigenvalues of the matrix $\begin{pmatrix} 4 & 1 \\ 0 & 2 \end{pmatrix}$, giving a reason. [2]
- 3 Find the eigenvalues of the matrix $A = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$. [3]
- 4 The matrix $A = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$ has an eigenvalue 4. Find a corresponding eigenvector. [2]
- 5 Determine whether $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector of the matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}$. [2]
- 6 Write down the eigenvalues and corresponding eigenvectors of the matrix $\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{pmatrix}$. [2]
- 7 Find the eigenvalues and corresponding eigenvectors of the matrix $A = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$. [6]
- 8 The matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}$.
Find the eigenvalues of A and a unit eigenvector corresponding to each. [6]
- 9 The matrix $A = \begin{pmatrix} k & 2 \\ 3 & 1 \end{pmatrix}$ has $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ as an eigenvector. Find the value of k and the corresponding eigenvalue. [4]
- 10 The matrix $A = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$ has eigenvalues 1 and 6. Write down the eigenvalues of A^2 , and find the eigenvalues of A^{-1} and of $3A + 2I$. [5]
- 11 The matrix $A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{pmatrix}$.
Write down the eigenvalues of A , and find an eigenvector corresponding to the eigenvalue 2. [4]
- 12 The matrix $A = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$.
Show that A has a repeated eigenvalue, and find the corresponding eigenvectors. [4]
- 13 Show that the matrix $A = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$ has no real eigenvalues. [4]
- 14 Find the eigenvalues of the matrix $A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 2 & -4 & 2 \end{pmatrix}$, and find an eigenvector for each eigenvalue. [7]

- 15** The matrix $A = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$.
Find the eigenvalues of A , and an eigenvector corresponding to the eigenvalue with positive imaginary part. [5]
-
- 16** The matrix $A = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$ represents a transformation of the plane.
Find the equations of the two lines through the origin that are mapped onto themselves by the transformation. [5]
-
- 17** The matrix $A = \begin{pmatrix} a & 2 \\ b & 1 \end{pmatrix}$ has eigenvalues 3 and -1 . Find the values of a and b . [4]
-
- 18** The matrix $A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$, where $b \neq 0$.
Show that A has eigenvalues $a + b$ and $a - b$, and find corresponding eigenvectors. [5]
-
- 19** The matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.
Show that A and its transpose A^T have the same eigenvalues. [5]
-
- 20** A population is split between two regions A and B . Each year 10% of the people in A move to B , and 40% of the people in B move to A . The proportions in the two regions in successive years are related by $\begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} = P \begin{pmatrix} a_n \\ b_n \end{pmatrix}$, where $P = \begin{pmatrix} \frac{9}{10} & \frac{2}{5} \\ \frac{1}{10} & \frac{3}{5} \end{pmatrix}$.
Show that 1 is an eigenvalue of P , and find the long-term proportion of the population in region A . [5]
-
- 21** The matrix $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$.
Show that the characteristic equation of A is $(\lambda - 1)^2(\lambda - 4) = 0$.
Find the eigenvectors corresponding to each eigenvalue, and show that the eigenvalue 1 has two linearly independent eigenvectors. [8]
-
- 22** The matrix $A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{pmatrix}$.
Show that A has a repeated eigenvalue, and that this eigenvalue has only one linearly independent eigenvector. Find the other eigenvalue and a corresponding eigenvector. [8]
-
- 23** The matrix $A = \begin{pmatrix} a & 1 \\ b & 3 \end{pmatrix}$ has $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ as eigenvectors, where a and b are constants.
Find the values of a and b , and the two eigenvalues. [7]
-
- 24** A student claims that if A and B are 2×2 matrices then the eigenvalues of AB are the products of the eigenvalues of A with those of B .
Test this claim for $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, and state whether the claim is correct. [6]
-

- 25** The matrix $F = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$ represents a reflection in the line through the origin at angle θ to the x-axis.
Show that the eigenvalues of F are 1 and -1 , and verify that $\begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$ and $\begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix}$ are corresponding eigenvectors. Explain geometrically why these results are expected. [7]
- 26** The matrix $A = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$ has eigenvectors $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.
Prove by induction that if $Av = \lambda v$ then $A^n v = \lambda^n v$ for all positive integers n .
Express $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ as a combination of the two eigenvectors, and hence find $A^3 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. [8]
- 27** The matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has eigenvalues λ_1 and λ_2 .
Show that $\lambda_1 + \lambda_2 = a + d$ and $\lambda_1 \lambda_2 = ad - bc$.
The matrix $A = \begin{pmatrix} k & 3 \\ -1 & 6 - k \end{pmatrix}$, where k is a constant, has two eigenvalues one of which is twice the other. Find the possible values of k and the eigenvalues. [8]
- 28** The symmetric matrix $A = \begin{pmatrix} 5 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 7 \end{pmatrix}$ has characteristic equation $\lambda^3 - 18\lambda^2 + 99\lambda - 162 = 0$.
Given that $\lambda = 3$ is a root, find the other eigenvalues.
Find an eigenvector for each eigenvalue, and show that the three eigenvectors are mutually perpendicular. Write down a unit eigenvector for $\lambda = 9$. [11]
- 29** The 2×2 matrix A has eigenvalues λ_1 and λ_2 with $\lambda_1 \neq \lambda_2$, and corresponding eigenvectors v_1 and v_2 .
Prove that v_1 and v_2 are linearly independent, that is, that $\alpha v_1 + \beta v_2 = 0$ only if $\alpha = \beta = 0$.
Deduce that every vector in the plane can be written as a combination of v_1 and v_2 , and that $A^n w = \alpha \lambda_1^n v_1 + \beta \lambda_2^n v_2$ when $w = \alpha v_1 + \beta v_2$. [10]
- 30** The matrix $A = \begin{pmatrix} 2 & k \\ 1 & 2 \end{pmatrix}$, where k is a real constant.
Find the eigenvalues of A in terms of k , and state the range of values of k for which A has two distinct real eigenvalues.
For $k = 4$, find the eigenvectors of A and the equations of the invariant lines through the origin. State what happens when $k = 0$, and when $k < 0$. [10]

Diagonalisation of Matrices

- 1** Find D^4 for $D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$. [2]
- 2** Find the inverse of the matrix $P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. [2]
- 3** The matrix A has eigenvalue 3 with eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and eigenvalue 1 with eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.
Write down a matrix P and a diagonal matrix D such that $P^{-1}AP = D$. [2]

- 4** Find D^3 for $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$. [2]
-
- 5** A 3×3 matrix has eigenvalues 1, 4 and 6. Explain why it can be diagonalised. [2]
-
- 6** Given $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, $P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and $D = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$, show that $AP = PD$. [2]
-
- 7** The matrix $A = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$ has eigenvalues 2 and 5 with eigenvectors $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ respectively.
Write down matrices P and D such that $A = PDP^{-1}$, and find P^{-1} . [4]
-
- 8** The matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ can be written as $A = PDP^{-1}$, where $P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and $D = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$.
Use this to find A^4 . [5]
-
- 9** The matrix A has eigenvalues 4 and -1 with corresponding eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$. Find A . [4]
-
- 10** The symmetric matrix $A = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$ has eigenvalues 9 and 1.
Find an orthogonal matrix P and a diagonal matrix D such that $P^TAP = D$. [5]
-
- 11** The matrix $A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 5 \end{pmatrix}$.
Write down the eigenvalues of A . Given that the eigenvectors are $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ (in order of increasing eigenvalue), write down P and D with $P^{-1}AP = D$. [3]
-
- 12** The matrix $A = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$ has eigenvalues 2 and 5. Find P and D with $A = PDP^{-1}$, and use them to find A^3 . [5]
-
- 13** Diagonalise the matrix $A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$, giving P and D such that $P^{-1}AP = D$. [4]
-
- 14** The matrix $A = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$.
Given that $A = PDP^{-1}$ with $P = \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}$, show that $A^n = \frac{1}{3} \begin{pmatrix} 2^n + 2 \times 5^n & -2 \times 2^n + 2 \times 5^n \\ -2^n + 5^n & 2 \times 2^n + 5^n \end{pmatrix}$. [6]
-
- 15** A sequence is defined by $x_0 = 1$, $x_1 = 5$ and $x_{n+1} = 5x_n - 6x_{n-1}$.
By writing $\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = A \begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix}$ for a suitable matrix A , or otherwise, find a formula for x_n in terms of n . [7]
-

- 16** The matrix $A = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$.
Show that A has only one linearly independent eigenvector, and explain why A cannot be diagonalised. [5]
-
- 17** The matrix $A = \begin{pmatrix} 22 & -14 \\ 21 & -13 \end{pmatrix}$ has eigenvalues 8 and 1 with eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ respectively.
Find a matrix C such that $C^3 = A$. [5]
-
- 18** The matrix $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ has eigenvalues 4, 1 and 1. The eigenvalue 4 has eigenvector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and the eigenvalue 1 has the independent eigenvectors $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.
Write down P and D such that $A = PDP^{-1}$, and explain why A can be diagonalised even though it has a repeated eigenvalue. [4]
-
- 19** Two quantities x_n and y_n satisfy $x_{n+1} = 3x_n + y_n$ and $y_{n+1} = 2x_n + 2y_n$, with $x_0 = 0$ and $y_0 = 3$.
Given that the matrix $\begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$ has eigenvalues 1 and 4 with eigenvectors $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, find x_n and y_n in terms of n . [6]
-
- 20** The matrix $A = \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix}$. Find the matrices P and D such that $A = PDP^{-1}$, where D is diagonal with the larger eigenvalue first, and hence find A^n . [6]
-
- 21** The symmetric matrix $A = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$ has eigenvalues 2, 4 and 5.
Find an orthogonal matrix P and a diagonal matrix D such that $P^TAP = D$.
Use your answer to find A^3 . [8]
-
- 22** The square matrix A can be written as $A = PDP^{-1}$, where D is a diagonal matrix.
Prove by induction that $A^n = PD^nP^{-1}$ for all positive integers n .
Hence show that $\det A =$ the product of the diagonal elements of D , and that $A^n \rightarrow 0$ as $n \rightarrow \infty$ if every diagonal element of D has modulus less than 1. [8]
-
- 23** Each year, 10% of the people living in a city move to the suburbs and 20% of the people living in the suburbs move to the city, so the proportions (c_n, s_n) in successive years satisfy

$$\begin{pmatrix} c_{n+1} \\ s_{n+1} \end{pmatrix} = M \begin{pmatrix} c_n \\ s_n \end{pmatrix}, \text{ where } M = \begin{pmatrix} \frac{9}{10} & \frac{1}{5} \\ \frac{1}{10} & \frac{4}{5} \end{pmatrix}.$$

 Given that M has eigenvalues 1 and $\frac{7}{10}$ with eigenvectors $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$, find M^n and the limit of M^n as $n \rightarrow \infty$.
 Hence find the long-term proportion of people living in the city, whatever the starting proportions. [8]

- 24** The matrix A has eigenvalues 2 and -3 with corresponding eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$.
Find A , and find A^{-1} . Explain how the eigenvalues of A^{-1} are related to those of A . [7]
-
- 25** A student calculates A^n for $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ by raising each element to the power n , obtaining $\begin{pmatrix} 1 & 1 \\ 0 & 2^n \end{pmatrix}$.
Show that the student is wrong by considering A^2 , and find the correct formula for A^n . [7]
-
- 26** The matrix $G = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$, where $b \neq 0$.
Show that G has eigenvalues $a + b$ and $a - b$, and find P and D with $G = PDP^{-1}$, where P has integer elements.
Hence show that $G^n = \frac{1}{2} \begin{pmatrix} (a+b)^n + (a-b)^n & (a+b)^n - (a-b)^n \\ (a+b)^n - (a-b)^n & (a+b)^n + (a-b)^n \end{pmatrix}$. [8]
-
- 27** The Fibonacci numbers satisfy $f_0 = 0$, $f_1 = 1$ and $f_{n+1} = f_n + f_{n-1}$. The matrix $F = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ satisfies $\begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix} = F \begin{pmatrix} f_n \\ f_{n-1} \end{pmatrix}$.
Show that the eigenvalues of F are $\varphi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$, with eigenvectors $\begin{pmatrix} \varphi \\ 1 \end{pmatrix}$ and $\begin{pmatrix} \psi \\ 1 \end{pmatrix}$.
Use the fact that $\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{5}} \left(\begin{pmatrix} \varphi \\ 1 \end{pmatrix} - \begin{pmatrix} \psi \\ 1 \end{pmatrix} \right)$ to show that $f_n = \frac{\varphi^n - \psi^n}{\sqrt{5}}$, and use this to confirm that $f_{10} = 55$. [10]
-
- 28** The matrix $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$.
Show that A has eigenvectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$, and write down the corresponding eigenvalues.
Find $A = PDP^{-1}$ with D diagonal, and hence find A^n as a single matrix. Check your formula for $n = 1$. [11]
-
- 29** A is a real symmetric 2×2 matrix with two distinct eigenvalues λ_1 and λ_2 , and corresponding eigenvectors u and v .
Show that $u \cdot v = 0$, that is, the eigenvectors are perpendicular. (You may use the fact that $u \cdot (Av) = (Au) \cdot v$ for a symmetric matrix A .)
Hence explain why A can be written as $A = PDP^T$ where P is an orthogonal matrix. [8]
-
- 30** The matrix $S = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$ has eigenvalues 9 and 1 with eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.
Show that there are exactly four matrices B such that $B^2 = S$, and find them.
Say which of these matrices has only positive eigenvalues, and show that it is symmetric. [9]

Cayley-Hamilton Theorem

- 1 Find the characteristic equation of the matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and state what the Cayley-Hamilton theorem says about A. [3]
- 2 Verify that the matrix $A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$ satisfies $A^2 - 5A + 6I = 0$. [3]
- 3 The 2×2 matrix A has trace 7 and determinant 12.
Write down the characteristic equation of A, and hence express A^2 in terms of A and I. [2]
- 4 The 2×2 matrix A satisfies $A^2 - 4A + 3I = 0$. Express A^2 in terms of A and I, and show that $A^{-1} = \frac{1}{3}(4I - A)$. [3]
- 5 Find the values of p and q such that the matrix $A = \begin{pmatrix} 5 & 2 \\ 1 & 3 \end{pmatrix}$ satisfies $A^2 = pA + qI$. [2]
- 6 The 2×2 matrix A satisfies $A^2 = 4A - 5I$. Express A^3 in the form $pA + qI$. [2]
- 7 The matrix $A = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$.
Find the characteristic equation of A and verify the Cayley-Hamilton theorem for A. [5]
- 8 The matrix $A = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$.
Use the Cayley-Hamilton theorem to find A^{-1} . [4]
- 9 The matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.
Use the Cayley-Hamilton theorem to express A^3 in the form $pA + qI$, and hence write down A^3 as a matrix. [4]
- 10 The matrix $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.
Write down the characteristic equation of D, and verify that D satisfies it. [4]
- 11 The matrix $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \\ 1 & 0 & 1 \end{pmatrix}$.
Show that the characteristic equation of A is $\lambda^3 - 6\lambda^2 + 11\lambda - 7 = 0$, and write down the corresponding matrix equation. [5]
- 12 The matrix $A = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}$.
Show that $A^2 = 3A - 2I$, and use this to express A^4 in the form $pA + qI$. Hence find A^4 . [6]
- 13 The matrix $A = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$.
Show that $(A - 2I)^2 = 0$, and deduce that $A^2 = 4A - 4I$. [4]

- 14** The matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.
Prove the Cayley-Hamilton theorem for A , that is, show that $A^2 - (a + d)A + (ad - bc)I = 0$. [5]
-
- 15** The matrix $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ has characteristic equation $\lambda^3 - 6\lambda^2 + 11\lambda - 7 = 0$.
Use the Cayley-Hamilton theorem to show that $A^{-1} = \frac{1}{7}(A^2 - 6A + 11I)$, and hence find A^{-1} . [6]
-
- 16** The matrix $A = \begin{pmatrix} k & 2 \\ 1 & 3 \end{pmatrix}$, where k is a constant, satisfies $A^2 - 7A + 10I = 0$.
Find the value of k , and find A^{-1} . [6]
-
- 17** The matrix $A = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$ satisfies $A^2 = 5A - 7I$.
Express A^4 in the form $pA + qI$, and hence find A^4 as a matrix. [5]
-
- 18** The matrix $A = \begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix}$.
Show that the characteristic equation of A is $\lambda^2 - \lambda = 0$, and use the Cayley-Hamilton theorem to find A^{100} . [5]
-
- 19** The matrix $N = \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}$.
Show that the characteristic equation of N is $\lambda^2 = 0$, and deduce that $N^2 = 0$.
Hence show that $(I + N)^{-1} = I - N$, and find this matrix. [6]
-
- 20** The matrix $A = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$.
Use the Cayley-Hamilton theorem to find A^{-1} and to express A^5 in the form $pA + qI$. [6]
-
- 21** The matrix $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ has characteristic equation $\lambda^3 - 6\lambda^2 + 11\lambda - 7 = 0$.
Express A^4 in the form $pA^2 + qA + rI$, and use your answer to find A^4 as a matrix. [8]
-
- 22** The matrix $A = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$ satisfies $A^2 - 5A + 7I = 0$.
By considering the product $(A - I)(A - 4I)$, show that $A - I$ is invertible, and find $(A - I)^{-1}$ as a matrix. [7]
-
- 23** The 3×3 matrix A has characteristic equation $\lambda^3 + p\lambda^2 + q\lambda + r = 0$, where $r \neq 0$.
Show that A is invertible, and that $A^{-1} = -\frac{1}{r}(A^2 + pA + qI)$.
Use this to find A^{-1} for the matrix $A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 2 & 0 & 1 \end{pmatrix}$, whose characteristic equation is $\lambda^3 - 3\lambda^2 + 3\lambda - 13 = 0$. [8]

- 24** A student tries to prove the Cayley-Hamilton theorem as follows: "The characteristic equation is $\det(A - \lambda I) = 0$. Replacing λ by A gives $\det(A - A) = \det(0) = 0$, so A satisfies its own characteristic equation."
- Explain why this argument is wrong, using the matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ to illustrate. [6]
-
- 25** The matrix $A = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$.
- Show that $A^2 + A + I = 0$, and deduce that $A^3 = I$.
Hence find A^{100} and A^{-1} . [8]
-
- 26** The matrix $B = \begin{pmatrix} 1 & 2 \\ -3 & -1 \end{pmatrix}$.
- Show that $B^2 = -5I$, and hence find B^{10} and B^{11} . [7]
-
- 27** The matrix $A = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$.
- Show that the characteristic equation of A is $\lambda^3 - 4\lambda^2 + 5\lambda - 4 = 0$.
Use the Cayley-Hamilton theorem to express A^4 in the form $pA^2 + qA + rI$, and hence find A^4 as a matrix.
Find A^{-1} by using the Cayley-Hamilton theorem. [12]
-
- 28** The matrix $A = \begin{pmatrix} 11 & 7 \\ 14 & 18 \end{pmatrix}$.
- Show that A has eigenvalues 4 and 25.
The 2×2 matrix X has positive eigenvalues and satisfies $X^2 = A$. Show that the eigenvalues of X are 2 and 5, and use the Cayley-Hamilton theorem for X to show that $X = \frac{1}{7}(A + 10I)$.
Hence find X . [9]
-
- 29** The real 2×2 matrix A satisfies $A^2 = -I$, where I is the 2×2 identity matrix.
Show that A cannot be a scalar multiple of I .
Use the Cayley-Hamilton theorem to show that the trace of A is 0 and its determinant is 1,
and deduce that $A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ where $a^2 + bc = -1$.
Find the matrix A of this form with $a = 0$ and $b = 2$, and verify that $A^2 = -I$. [11]
-
- 30** The matrix $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$.
- Show that the characteristic equation of A is $(\lambda - 2)^2 = 0$, and write down the matrix equation given by the Cayley-Hamilton theorem.
Prove by induction that $A^n = n2^{n-1}A - (n - 1)2^nI$ for all $n \geq 1$.
Hence find A^6 , and show that A cannot be diagonalised. [11]
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