

Formulating Linear Programming Problems

1 (1 mark)

Total number of items is $x + y$, so $x + y \leq 40$

Answer: $x + y \leq 40$

B1

1 mark

2 (1 mark)

$P = 12x + 9y$

Answer: $P = 12x + 9y$

B1

1 mark

3 (1 mark)

The numbers of items made cannot be negative

Answer: A negative number of items cannot be made

E1

1 mark

4 (1 mark)

Oven time used is $3x + 2y$ minutes, so $3x + 2y \leq 120$

Answer: $3x + 2y \leq 120$

B1

1 mark

5 (2 marks)

(a) $y \geq 15$

Answer: $y \geq 15$

B1

1 mark

(b) $x \leq 30$

Answer: $x \leq 30$

B1

1 mark

6 (2 marks)

$4x + 5y + s = 200$

$s \geq 0$

Answer: $4x + 5y + s = 200, s \geq 0$

B1

B1

2 marks

7 (2 marks)

Number of chairs $\geq 2 \times$ number of tables

$y \geq 2x$

Answer: $y \geq 2x$

M1

A1

2 marks

8 (3 marks)

$x \leq 0.6(x + y)$

$0.4x \leq 0.6y$

Multiply by 5: $2x \leq 3y$

Answer: $2x \leq 3y$

M1

M1

A1

3 marks

9 (5 marks)

Maximise $P = 4x + 7y$

Painting: $20x + 40y \leq 960$, so $x + 2y \leq 48$

Baking: $30x + 30y \leq 900$, so $x + y \leq 30$

$x \geq 0, y \geq 0$

Answer: Maximise $P = 4x + 7y$ subject to $x + 2y \leq 48, x + y \leq 30, x, y \geq 0$

B1

M1 A1

A1

B1

5 marks

10 (2 marks)

(a) The mixing time, in minutes, used to make y litres of gloss paint

Answer: Minutes of mixing time used for gloss paint

E1

1 mark

(b) $5(60) + 4(75) = 300 + 300 = 600$

Answer: $5(60) + 4(75) = 600$

B1

1 mark

11 (5 marks)Minimise $C = 30x + 45y$

B1

Protein: $4x + 3y \geq 36$

M1 A1

Vitamins: $2x + 5y \geq 30$

A1

 $x \geq 0, y \geq 0$

B1

Answer: Minimise $C = 30x + 45y$ subject to $4x + 3y \geq 36, 2x + 5y \geq 30, x, y \geq 0$

5 marks

12 (2 marks)

The numbers of vehicles must be whole numbers (integers)

E1

Check integer points near the optimum that satisfy every constraint, and choose the best one

E1

(rounding may break a constraint)

Answer: Values must be integers; test nearby feasible integer points

2 marks

13 (4 marks) $2x + 3y + s_1 = 18, x + 4y + s_2 = 16, 5x + y + s_3 = 30$, with $s_1, s_2, s_3 \geq 0$

B1 B1

 $s_1 = 18 - (4 + 9) = 5$

M1

 $s_2 = 16 - (2 + 12) = 2$ and $s_3 = 30 - (10 + 3) = 17$

A1

Answer: $s_1 = 5, s_2 = 2, s_3 = 17$

4 marks

14 (5 marks)Let x, y, z be the numbers of small, medium and large candles. Maximise $P = 1.2x + 2.5y + 4z$

B1

Wax: $100x + 250y + 400z \leq 20\,000$, so $2x + 5y + 8z \leq 400$

M1 A1

Labour: $6x + 10y + 15z \leq 1800$

A1

Boxes: $x + y + z \leq 150$, and $x, y, z \geq 0$

B1

Answer: Maximise $P = 1.2x + 2.5y + 4z$ with $2x + 5y + 8z \leq 400, 6x + 10y + 15z \leq 1800, x + y + z \leq 150, x, y, z \geq 0$

5 marks

15 (6 marks)Minimise $C = 40x + 30y$ (pence)

B1

 $x + y \geq 100$

B1

 $x \leq 70, y \leq 80$

B1

 $x \geq 0.3(x + y)$

M1

 $0.7x \geq 0.3y$, so $7x \geq 3y$

A1

 $x \geq 0, y \geq 0$

B1

Answer: Minimise $C = 40x + 30y$ subject to $x + y \geq 100, x \leq 70, y \leq 80, 7x \geq 3y, x, y \geq 0$

6 marks

16 (3 marks)Clay is limited, so the inequality should be $2x + 3y \leq 60$, not $\geq$

E1

At least 5 mugs means $x \geq 5$, not $x \leq 5$

E1

Correct: maximise $P = 3x + 4y$ subject to $2x + 3y \leq 60, x \geq 5, y \geq 0$

B1

Answer: $2x + 3y \leq 60$ and $x \geq 5$

3 marks

17 (4 marks)(a) $3(10) + 2(12) = 54 \leq 60$ and $10 + 3(12) = 46 \leq 48$

M1

 $12 \leq 15$ and $x, y \geq 0$, so every constraint is satisfied

A1

Answer: All satisfied

2 marks

(b) $P = 50 + 48 = 98$

B1

Answer: 98

1 mark

(c) $60 - 54 = 6$

B1

Answer: 6

1 mark

18 (6 marks)Minimise $C = 90x + 50y$

B1

 $8x + 4y \geq 200$, so $2x + y \geq 50$

M1 A1

 $x + y \leq 45$

B1

 $\frac{x}{y} \geq \frac{2}{3}$, so $3x \geq 2y$

B1

 $x \geq 0, y \geq 0$

B1

Answer: Minimise $C = 90x + 50y$ subject to $2x + y \geq 50, x + y \leq 45, 3x \geq 2y, x, y \geq 0$

6 marks

19 (6 marks)

Maximise $P = 350x + 420y$

B1

$x + y \leq 80$

B1

$x \leq 1.5y$, so $2x \leq 3y$

M1 A1

$4x + 6y \leq 420$, so $2x + 3y \leq 210$

B1

$x \geq 0, y \geq 0$

B1

Answer: Maximise $P = 350x + 420y$ subject to $x + y \leq 80, 2x \leq 3y, 2x + 3y \leq 210, x, y \geq 0$

6 marks

20 (3 marks)

$x + y \leq 40$

B1

$y \geq \frac{1}{4}(x + y)$

M1

$4y \geq x + y$, so $3y \geq x$

A1

Answer: $x + y \leq 40; 3y \geq x$

3 marks

21 (9 marks)

(a) Maximise $P = 8x + 13y$

B1

$2x + 3y \leq 60$

M1 A1

$x + 2y \leq 36$

A1

$y \leq 14, x \geq y, x, y \geq 0$

B1

Answer: Maximise $P = 8x + 13y$ subject to $2x + 3y \leq 60, x + 2y \leq 36, y \leq 14, x \geq y, x, y \geq 0$

5 marks

(b) x and y must be integers

B1

Answer: x, y are integers

1 mark

(c) $36 + 18 = 54 \leq 60, 18 + 12 = 30 \leq 36, 6 \leq 14, 18 \geq 6$

M1

$P = 8(18) + 13(6) = 222$

A1

Unused glazing time = $36 - 30 = 6$ hours

B1

Answer: Feasible; profit £222; 6 hours unused

3 marks

22 (8 marks)

(a) Minimise $C = 3x + 5y + 4z$

B1

$x + y + z = 200$

B1

$x \geq 0.4 \times 200$, so $x \geq 80$

M1 A1

$y \leq 0.5z$, so $2y \leq z$

B1

$z \leq 90$, and $x, y, z \geq 0$

B1

Answer: Minimise $C = 3x + 5y + 4z$ subject to $x + y + z = 200, x \geq 80, 2y \leq z, z \leq 90, x, y, z \geq 0$

6 marks

(b) Maximise $P = -C = -3x - 5y - 4z$

B1

Answer: Maximise $P = -3x - 5y - 4z$

1 mark

(c) Cost is minimised, so it would use no liquid at all (zero cost); the total must be exactly 200

E1

Answer: Minimising cost would choose to make nothing

1 mark

23 (9 marks)

(a) Minimise $C = 4x + 3y + 3.5z$

B1

$x + y + z = 100$

B1

Copper: $0.8x + 0.5y + 0.6z \geq 60$, so $8x + 5y + 6z \geq 600$

M1 A1

Tin: $0.02x + 0.12y + 0.04z \leq 6$, so $x + 6y + 2z \leq 300$, and $x, y, z \geq 0$

B1

Answer: Minimise $C = 4x + 3y + 3.5z$ with $x + y + z = 100, 8x + 5y + 6z \geq 600, x + 6y + 2z \leq 300$

5 marks

(b) Substitute $x = 100 - y - z$

M1

Copper: $800 - 3y - 2z \geq 600$, so $3y + 2z \leq 200$

A1

Tin: $100 + 5y + z \leq 300$, so $5y + z \leq 200$; and $x \geq 0$ gives $y + z \leq 100$

A1

$C = 400 - y - 0.5z$, to be minimised, with $y, z \geq 0$

A1

Answer: Minimise $C = 400 - y - 0.5z$ subject to $3y + 2z \leq 200, 5y + z \leq 200, y + z \leq 100, y, z \geq 0$

4 marks

24 (6 marks)

(a) $2x + y + 3z + s_1 = 60, x + 2y + 2z + s_2 = 50$

B1

$3x + y + z + s_3 = 70, x + y + z + s_4 = 30$, with $s_1, s_2, s_3, s_4 \geq 0$

B1

Answer: $2x + y + 3z + s_1 = 60, x + 2y + 2z + s_2 = 50, 3x + y + z + s_3 = 70, x + y + z + s_4 = 30$

2 marks

(b) Machine: $16 + 9 + 18 = 43$, so $s_1 = 17$

M1

Assembly: $8 + 18 + 12 = 38$, so $s_2 = 12$; **materials:** $24 + 9 + 6 = 39$, so $s_3 = 31$

A1

Storage: 23, so $s_4 = 7$

A1

Answer: $s_1 = 17$, $s_2 = 12$, $s_3 = 31$, $s_4 = 7$

3 marks

(c) No slack variable is zero, so no resource is fully used at this plan

E1

Answer: None: every slack is positive

1 mark

25 (5 marks)

Minimise $C = 400x + 160y$

B1

Seats: $50x + 16y \geq 424$, so $25x + 8y \geq 212$

M1 A1

Drivers: $x + y \leq 10$, and $x, y \geq 0$

A1

Vehicles cannot be part-hired, so x and y must be integers

E1

Answer: Minimise $C = 400x + 160y$ subject to $25x + 8y \geq 212$, $x + y \leq 10$, $x, y \geq 0$, integers

5 marks

26 (7 marks)

Maximise $P = 15x + 14y$

B1

$30x + 20y \leq 600$, so $3x + 2y \leq 60$

M1 A1

$2x + 3y \leq 40$

A1

$x \geq 4$, $y \geq 4$

B1

$y - x \leq 6$

B1

Integers (whole trays)

B1

Answer: Maximise $P = 15x + 14y$ with $3x + 2y \leq 60$, $2x + 3y \leq 40$, $x \geq 4$, $y \geq 4$, $y - x \leq 6$, x, y integers

27 (7 marks)

Maximise $R = 40x + 30y + 20z$

B1

$x \leq 200$, $y \leq 150$, $z \leq 100$

B1

$x + y + z \geq 250$

B1

$x + y \geq 0.6(x + y + z)$

M1

$2x + 2y \geq 3z$

A1

$z \leq 0.4(x + y + z)$, so $3z \leq 2x + 2y$

M1

$x, y, z \geq 0$ (and the two percentage conditions give the same inequality)

A1

Answer: Maximise $R = 40x + 30y + 20z$ with $x \leq 200$, $y \leq 150$, $z \leq 100$, $x + y + z \geq 250$, $2x + 2y \geq 3z$

28 (6 marks)

Maximise $R = 900x + 700y$

B1

$3x + 2y \leq 120$

B1

$x \leq 20$

B1

$y \geq 0.5x$, so $2y \geq x$

B1

Only petrol means $y = 0$, so $2y \geq x$ gives $x \leq 0$

M1

So $x = 0$ as well: the manager's plan is not feasible (it makes nothing)

E1

Answer: Maximise $R = 900x + 700y$ with $3x + 2y \leq 120$, $x \leq 20$, $2y \geq x$; manager's plan infeasible

6 marks

29 (5 marks)

Minimise $C = 20x + 45y$

B1

$30x + 50y \geq 2400$, so $3x + 5y \geq 240$

M1 A1

$x + y \leq 60$, $x \geq y$, $x, y \geq 0$

B1

(the 40 hour week applies to each line: $x \leq 40$, $y \leq 40$)

B1

Answer: Minimise $C = 20x + 45y$ subject to $3x + 5y \geq 240$, $x + y \leq 60$, $x \geq y$, $x \leq 40$, $y \leq 40$, $x, y \geq 0$

5 marks

30 (6 marks)

The third constraint is a $\geq$ constraint, so the origin is not feasible and there is no obvious starting basic feasible solution E1

Subtract a surplus variable s_3 : $x - y - s_3 = 5$ M1

Add an artificial variable t : $x - y - s_3 + t = 5$ (Big-M or two-stage method) A1

$2x + y + z + s_1 = 40$ B1

$x + 3y + 2z + s_2 = 60$ B1

$P - 4x - 3y - 2z + Mt = 0$ (Big-M), with all variables ≥ 0 B1

Answer: Add surplus s_3 and artificial t to the third constraint; use Big-M or two-stage 6 marks

Graphical Linear Programming

1 (2 marks)

x-axis: $y = 0$ so $x = 8$ B1

y-axis: $x = 0$ so $y = 6$ B1

Answer: (8, 0) and (0, 6) 2 marks

2 (1 mark)

$2(4) + 3(5) = 23 \leq 24$, so it satisfies the constraint B1

Answer: Yes ($23 \leq 24$) 1 mark

3 (1 mark)

$P = 18 + 8 = 26$ B1

Answer: 26 1 mark

4 (2 marks)

Subtract: $x = 6$ M1

$y = 10 - 6 = 4$, so the point is (6, 4) A1

Answer: (6, 4) 2 marks

5 (1 mark)

The set of points (x, y) that satisfy all of the constraints (including non-negativity) B1

Answer: Points satisfying every constraint 1 mark

6 (2 marks)

$5x + 4y = 40$ gives $y = 10 - \frac{5}{4}x$ M1

Gradient = $-\frac{5}{4}$, which is negative A1

Answer: $-\frac{5}{4}$, negative 2 marks

7 (5 marks)

Axes: (0, 0), (8, 0) from $2x + y = 16$, (0, 8) from $x + 3y = 24$ B1 B1

$2x + y = 16$ and $x + y = 10$ give (6, 4) M1 A1

$x + y = 10$ and $x + 3y = 24$ give (3, 7) A1

Answer: (0, 0), (8, 0), (6, 4), (3, 7), (0, 8) 5 marks

8 (3 marks)

Evaluate P at each vertex: 0, 24, 26, 23, 16 M1 A1

The maximum is $P = 26$ at (6, 4) A1

Answer: $P = 26$ at (6, 4) 3 marks

9 (5 marks)

Vertices: (0, 0), (30, 0), (0, 24)

B1

 $x + 2y = 48$ and $x + y = 30$ give (12, 18)

M1 A1

 $P = 0, 120, 168, 174$ at the vertices

M1

Maximum profit is £174 at (12, 18)

A1

Answer: £174 with 12 Standard and 18 Deluxe

5 marks

10 (4 marks)

Vertices of R: (0, 12), (2, 6), (6, 2), (10, 0)

M1 A1

 $C = 36, 22, 18, 20$ respectively

M1

Minimum $C = 18$ when $x = 6, y = 2$

A1

Answer: $C = 18$ at (6, 2)

4 marks

11 (4 marks) $6x + 4y = 24$ has gradient $-\frac{3}{2}$

B1 B1

Lines of constant P are parallel to this one

M1

Slide the line away from the origin (maximise) to the last point of the feasible region it touches

A1

Answer: $6x + 4y = 24$, gradient $-\frac{3}{2}$; slide to the last vertex touched

4 marks

12 (4 marks)

Vertices: (2, 1), (8, 1), (2, 7)

B1 B1

 $3x + y = 7, 25, 13$

M1

Maximum 25 at (8, 1)

A1

Answer: Maximum 25 at (8, 1)

4 marks

13 (3 marks) P at vertices: 0, 80, 104, 110, 95

M1 A1

Maximum is $P = 110$ at (10, 14)

A1

Answer: $P = 110$ at (10, 14)

3 marks

14 (6 marks)

(0, 0), (20, 0) and (0, 15) from the axes

B1

 $3x + 2y = 60$ and $x + 3y = 48$: $y = 12, x = 12$, giving (12, 12)

M1 A1

 $y = 15$ and $x + 3y = 48$ give (3, 15)

A1

 $P = 0, 100, 108, 75, 60$ at (0, 0), (20, 0), (12, 12), (3, 15), (0, 15)

M1

Optimal: $P = 108$ at $x = 12, y = 12$

A1

Answer: $P = 108$ at (12, 12)

6 marks

15 (6 marks)Draw $2x + y = 40, x + 2y = 38$ and $x + y = 24$ and identify R

M1

Vertices (0, 0), (20, 0), (16, 8), (10, 14), (0, 19)

A1

 $P = 0, 80, 104, 110, 95$

M1

Maximum $P = 110$ at (10, 14)

A1

 $2x + y = 34$ at the optimum, so $40 - 34 = 6$ is unused

M1 A1

Answer: $P = 110$ at (10, 14); 6 unused

6 marks

16 (5 marks)

Vertices of R: (0, 20), (6, 8), (30, 0)

B1 B1

 $x + y = 14$ meets $x + 3y = 30$ at (6, 8); $2x + y = 20$ meets $x + y = 14$ at (6, 8) as well

M1

 $C = 100, 76, 180$

M1

Minimum $C = 76$ at (6, 8)

A1

Answer: $C = 76$ at (6, 8)

5 marks

17 (6 marks)

The lines $x + 2y = 12$ and $3x + y = 15$ meet at $(\frac{18}{5}, \frac{21}{5})$

M1 A1

$C = \frac{54}{5} + 21 = \frac{159}{5} = 31.8$ is the minimum without the integer condition

A1

Test integer points close to it: (4, 4) satisfies all three constraints ($12 \geq 12$, $16 \geq 15$, $8 \leq 20$), giving $C = 32$

M1

(3, 5) fails $3x + y \geq 15$ and nothing nearby is smaller

A1

Optimal integer solution: $x = 4$, $y = 4$, $C = 32$

A1

Answer: $x = 4$, $y = 4$, $C = 32$

6 marks

18 (6 marks)

The region is unbounded (extends without limit away from the origin)

E1

$2x + 5y$ increases without limit as x or y increases, so no maximum

E1

Vertices: (0, 6), (3, 3), (12, 0)

M1 A1

Values: 30, 21, 24, so the minimum is 21 at (3, 3)

M1 A1

Answer: No maximum; minimum 21 at (3, 3)

6 marks

19 (6 marks)

Draw the objective line, e.g. $5x + 3y = 75$, gradient $-\frac{5}{3}$

M1 A1

Move parallel to itself away from the origin until it last touches R

M1

That vertex is on $x = 25$ and $4x + 3y = 120$, so $y = \frac{20}{3}$

M1 A1

$P = 125 + 20 = 145$

A1

Answer: $x = 25$, $y = \frac{20}{3}$, $P = 145$

6 marks

20 (6 marks)

BA has gradient $-\frac{1}{2}$ and y-intercept 5: $x + 2y \leq 10$

M1 A1

AC has gradient -2 and passes through (7, 0): $2x + y \leq 14$

M1 A1

The vertical boundary gives $x \geq 1$

B1

Test (2, 2): $6 \leq 10$, $6 \leq 14$, so the inequalities are the correct way round

A1

Answer: $x + 2y \leq 10$, $2x + y \leq 14$, $x \geq 1$

6 marks

21 (7 marks)

Vertices: (4, 2) and (4, 16) from $x = 4$

M1 A1

$3x + y = 48$ with $y = 2$ gives $(\frac{46}{3}, 2)$; with $x + y = 20$ gives (14, 6)

M1 A1

$P = 36, 92, 122$ at (4, 2), (4, 16), (14, 6); at $(\frac{46}{3}, 2)$ $P = \frac{322}{3} + 8 \approx 115.3$

M1

Maximum $P = 122$ at (14, 6): profit £12 200

A1

16 of each gives $x + y = 32 > 20$, violating the first constraint

B1

Answer: 122 (hundreds) at (14, 6)

7 marks

22 (6 marks)

The optimum at (12, 12) requires the gradient of the objective line to lie between those of the two boundary lines meeting there

M1

Gradients: $3x + 2y = 60$ gives $-\frac{3}{2}$; $x + 3y = 48$ gives $-\frac{1}{3}$; objective gives $-\frac{5}{k}$

A1

$-\frac{3}{2} \leq -\frac{5}{k} \leq -\frac{1}{3}$

M1

So $\frac{1}{3} \leq \frac{5}{k} \leq \frac{3}{2}$

A1

Hence $\frac{10}{3} \leq k \leq 15$

A1 A1

Answer: $\frac{10}{3} \leq k \leq 15$

6 marks

23 (6 marks)

Costs at the vertices: 24 , $2c + 12$, $6c + 4$ and $10c$

M1 A1

The minimum is 18 , so one of these equals 18 and the others are not smaller

M1

$2c + 12 = 18$ gives $c = 3$, and then the costs are 24 , 18 , 22 , 30 , so 18 is the minimum

A1

$6c + 4 = 18$ gives $c = \frac{7}{3}$, but then $2c + 12 = \frac{50}{3} < 18$, so this is rejected

M1

$10c = 18$ gives $c = \frac{9}{5}$, but then $6c + 4 = \frac{74}{5} < 18$, so this is rejected; hence $c = 3$

A1

Answer: $c = 3$

6 marks

24 (6 marks)

Solve $3x + 2y = 66$ and $x + 3y = 48$: From the second equation $x = 48 - 3y$; substituting gives
 $144 - 7y = 66$

M1

$7y = 78$, so $y = \frac{78}{7}$, $x = 48 - \frac{234}{7} = \frac{102}{7}$

A1

$P = \frac{510}{7} + \frac{312}{7} = \frac{822}{7}$

A1

Increase in $P = \frac{822}{7} - 108 = \frac{66}{7}$ for 6 more units of k , i.e. $\frac{11}{7}$ per unit

M1 A1

$y = \frac{78}{7} \approx 11.1 < 15$ and $x, y > 0$, so still feasible

B1

Answer: $P = \frac{822}{7}$ at $(\frac{102}{7}, \frac{78}{7})$; P rises by $\frac{11}{7}$ per unit of k

6 marks

25 (7 marks)

Minimise $C = 30x + 45y$ subject to $4x + 3y \geq 36$, $2x + 5y \geq 30$, $x, y \geq 0$

M1 A1

Vertices: $(0, 12)$, $(15, 0)$ and the meeting point of the lines

A1

$4x + 3y = 36$ and $2x + 5y = 30$ give $x = \frac{45}{7}$, $y = \frac{24}{7}$

M1 A1

$C = 360, 450$, and $\frac{1350}{7} + \frac{1080}{7} = \frac{2430}{7} \approx 347.1$

M1

Cheapest: $\frac{45}{7}$ kg of P and $\frac{24}{7}$ kg of Q at a cost of $\frac{2430}{7}$ pence $\approx 347p$

A1

Answer: $x = \frac{45}{7}$, $y = \frac{24}{7}$, cost $\approx 347p$

7 marks

26 (6 marks)

The student has not compared the values of P at all vertices; the largest y -value does not guarantee the largest P

E1

P at $(3, 15)$ is 75 , and at $(0, 15)$ is 60

B1

At $(12, 12)$: $P = 60 + 48 = 108$

M1

At $(20, 0)$: $P = 100$

M1

The maximum is 108 at $(12, 12)$

A1

Check $(12, 12)$: $36 + 24 = 60$ and $12 + 36 = 48$, so it is feasible and on two constraints

E1

Answer: Error: only one vertex checked; maximum is 108 at $(12, 12)$

6 marks

27 (8 marks)

The extra constraint is redundant when the greatest value of $x + 2y$ over the region is at most k

M1

Values of $x + 2y$ at the vertices $(0, 0)$, $(8, 0)$, $(6, 4)$, $(3, 7)$, $(0, 8)$: $0, 8, 14, 17, 16$

M1 A1

Least $k = 17$

A1

For $k = 12$ the vertices of the new region are $(0, 0)$, $(8, 0)$, $(0, 6)$ and the meeting point of $x + 2y = 12$ and $2x + y = 16$

M1

Solving gives $x = \frac{20}{3}$, $y = \frac{8}{3}$, which satisfies $x + y \leq 10$ and $x + 3y \leq 24$

A1

$P = 20 + \frac{16}{3} = \frac{76}{3}$ at this point; $P = 0, 24, 12$ elsewhere

M1

Maximum $P = \frac{76}{3}$

A1

Answer: (i) $k = 17$; (ii) $P = \frac{76}{3}$

8 marks

28 (8 marks)

Gradient of objective is (1, k); the optimum at a vertex needs it to lie between the normals of the two boundaries meeting there M1

At (8, 0), normals (0, 1) edge x-axis and (2, 1): optimal when $k \leq \frac{1}{2}$ M1 A1

At (6, 4), normals (2, 1) and (1, 1): optimal for $\frac{1}{2} \leq k \leq 1$ M1 A1

At (3, 7), normals (1, 1) and (1, 3): optimal for $1 \leq k \leq 3$ A1

At (0, 8): optimal for $k \geq 3$ A1

At $k = \frac{1}{2}$, 1 and 3 the objective line is parallel to an edge, so two adjacent vertices give the same maximum and the solution is not unique E1

Answer: (8, 0): $k \leq 1/2$; (6, 4): $1/2 \leq k \leq 1$; (3, 7): $1 \leq k \leq 3$; (0, 8): $k \geq 3$; ties at $k = 1/2, 1, 3$ 8 marks

29 (10 marks)

(a) From $x + 3y = 48$, $x = 48 - 3y$; then $3x + 2y = k$ gives $144 - 7y = k$, so $y = \frac{(144 - k)}{7}$ M1 A1

$x = 48 - \frac{3(144 - k)}{7} = \frac{(3k - 96)}{7}$ A1

$P = 5x + 4y = \frac{(15k - 480 + 576 - 4k)}{7} = \frac{(11k + 96)}{7}$ M1

Answer: $P = \frac{(11k + 96)}{7}$ 4 marks

(b) Need $y \leq 15$: $144 - k \leq 105$ so $k \geq 39$ M1 A1

Need $x \geq 0$, $y \geq 0$: $k \geq 32$ and $k \leq 144$, so $39 \leq k \leq 144$ A1

Answer: $39 \leq k \leq 144$ 3 marks

(c) $k = 75$ is within the range B1

$P = \frac{(11(75) + 96)}{7} = \frac{921}{7} \approx 131.57$, an increase of ≈ 23.57 M1

Cost $15 \times 1.20 = £18 < £23.57$, so it is worthwhile A1

Answer: Yes: gain £23.57 against cost £18 3 marks

30 (10 marks)

Minimise $C = 50x + 19y$ subject to $12x + 5y \geq 100$, $x + y \leq 12$, $x \leq 8$, $x, y \geq 0$ B1 M1 A1

Vertices: $(8, \frac{4}{5})$, $(8, 4)$, $(\frac{40}{7}, \frac{44}{7})$ M1 A1

$C = 415.2, 476, \frac{2836}{7} \approx 405.1$, so the LP minimum is at $(\frac{40}{7}, \frac{44}{7})$ M1 A1

Rounding up to (6, 7) gives 13 vehicles, more than 12 drivers, so infeasible E1

Test (6, 6): $72 + 30 = 102 \geq 100$ and 12 vans, feasible, $C = 414$ M1

(5, 7) gives 95 pallets, (8, 1) costs £419 and (7, 4) costs £426, so the cheapest integer solution is 6 A1

large and 6 small vans at £414

Answer: 6 large and 6 small vans, cost £414 10 marks

Feasible Regions**1 (2 marks)**

$3 + 4 = 7 \leq 8$ and $6 + 4 = 10 \leq 12$ M1

$x = 3 \geq 0$ and $y = 4 \geq 0$, so (3, 4) is feasible A1

Answer: (3, 4) satisfies every constraint 2 marks

2 (2 marks)

(1, 5): $x = 1 < 2$, not feasible B1

(5, 3) and (4, 4): sums 8 and $8 \leq 8$ and $x \geq 2$, both feasible B1

Answer: (5, 3) and (4, 4) 2 marks

3 (2 marks)

Substitute a test point not on the line, e.g. the origin: $0 \leq 12$ is true

M1

So the side containing the origin satisfies $2x + 3y \leq 12$

A1

Answer: The side containing the origin

2 marks

4 (2 marks)

$$x \leq 5$$

B1

$$y \geq 1$$

B1

Answer: $x \leq 5$ and $y \geq 1$

2 marks

5 (2 marks)

Intercept form $\frac{x}{4} + \frac{y}{6} = 1$

M1

$$3x + 2y = 12$$

A1

Answer: $3x + 2y = 12$

2 marks

6 (1 mark)

No point satisfies all the constraints, so the problem has no solution (it is infeasible)

B1

Answer: No feasible solution exists

1 mark

7 (3 marks)

The line has equation $2x + 3y = 12$

M1

R lies on the origin side: $2x + 3y \leq 12$

A1

$$x \geq 0 \text{ and } y \geq 0$$

B1

Answer: $2x + 3y \leq 12$, $x \geq 0$, $y \geq 0$

3 marks

8 (4 marks)

Vertices: (0, 0), (4, 0), (0, 6)

B1

$x = 4$ meets $x + y = 6$ at (4, 2)

B1

Trapezium with parallel sides 6 and 2 and width 4: area = $\frac{1}{2}(6 + 2)(4)$

M1

Area = 16

A1

Answer: Vertices (0, 0), (4, 0), (4, 2), (0, 6); area 16

4 marks

9 (2 marks)

$x = 0$: $y = 0$ to 4 (5 points); $x = 1$: 4 points; $x = 2$: 3; $x = 3$: 2; $x = 4$: 1

M1

Total $5 + 4 + 3 + 2 + 1 = 15$

A1

Answer: 15

2 marks

10 (2 marks)

$x + y$ cannot be both at most 4 and at least 6

M1

No point satisfies both, so there is no feasible region

E1

Answer: Contradictory constraints

2 marks

11 (3 marks)

Every feasible point lies on the line $x + y = 10$, so the region is part of a line

E1

$x \leq 7$ gives $y \geq 3$; $y \leq 8$ gives $x \geq 2$

M1

End points (2, 8) and (7, 3)

A1

Answer: Segment from (2, 8) to (7, 3)

3 marks

12 (3 marks)

$x + y = 12$ gives $12 \leq 12$ and $x + 2y = 20$ gives $20 \leq 20$: both hold with equality

M1

$2x + y = 16 \leq 18$, so (4, 8) is feasible

A1

It is a vertex where $x + y = 12$ and $x + 2y = 20$ meet

A1

Answer: Vertex of $x + y = 12$ and $x + 2y = 20$

3 marks

13 (3 marks)Vertex of $x = 1$ and $y = 2$: (1, 2)

B1

 $x + y = 7$ with $y = 2$: (5, 2); with $x = 1$: (1, 6)

B1 B1

Answer: (1, 2), (5, 2), (1, 6)

3 marks

14 (5 marks)Intersection of $x + y = 12$ and $2x + y = 18$: (6, 6)

M1 A1

 $x + 3y$ at (6, 6) is $24 \leq 30$ and at (4, 8) is $28 \leq 30$, so $x + 3y \leq 30$ holds at every vertex of the region defined by the others

M1

Hence $x + 3y \leq 30$ is redundant

A1

Vertices (0, 0), (9, 0), (6, 6), (4, 8), (0, 10)

A1

Answer: $x + 3y \leq 30$ is redundant; vertices (0, 0), (9, 0), (6, 6), (4, 8), (0, 10)

5 marks

15 (5 marks)

Vertices on the axes: (0, 8) and (12, 0)

B1

 $x + y = 6$ meets $2x + y = 8$ at (2, 4); $x + y = 6$ meets $x + 3y = 12$ at (3, 3)

M1 A1

R extends indefinitely to the right and upwards (no constraint of the form $ax + by \leq c$)

E1

 $x + y$ can be made as large as we like, so there is no maximum

E1

Answer: (0, 8), (2, 4), (3, 3), (12, 0); unbounded

5 marks

16 (7 marks)

Vertices: (1, 2), (3, 0), (6, 0)

B1 B1

 $3x + y = 18$ meets $x + 2y = 14$ at $(\frac{22}{5}, \frac{24}{5})$

M1 A1

 $x = 1$ meets $x + 2y = 14$ at $(1, \frac{13}{2})$

A1

Counting integer points column by column: $x = 1$: $y = 2$ to 6 (5); $x = 2$: $y = 1$ to 6 (6); $x = 3$: $y = 0$ to 5 (6); $x = 4$: $y = 0$ to 5 (6); $x = 5$: $y = 0$ to 3 (4); $x = 6$: $y = 0$ (1)

M1

Total $5 + 6 + 6 + 6 + 4 + 1 = 28$

A1

Answer: 28 integer points

7 marks

17 (6 marks)

Vertices: (0, 0), (8, 0), (6, 4), (0, 7)

B1 B1

(0, 0): $s = 16$, $t = 14$; (8, 0): $s = 0$, $t = 6$

M1 A1

(6, 4): $s = 16 - 16 = 0$ and $t = 14 - 14 = 0$; (0, 7): $s = 9$, $t = 0$

A1

At every vertex exactly two of x , y , s , t are zero

E1

Answer: (16, 14), (0, 6), (0, 0), (9, 0) for (s, t); two variables are zero at each vertex

6 marks

18 (4 marks)The smallest value of $x + y$ in the region is $3 + 4 = 7$

M1

Region non-empty when $k \geq 7$

A1

When $k = 7$: the only feasible point is (3, 4)

A1

For $k > 7$ the region is a triangle with vertices (3, 4), $(k - 4, 4)$, $(3, k - 3)$

B1

Answer: $k \geq 7$; single point (3, 4) when $k = 7$

4 marks

19 (7 marks)AB: gradient $-\frac{1}{2}$, through (0, 5): $x + 2y = 10$; BC: through (4, 3), (6, 0): $3x + 2y = 18$

M1 A1

R: $x + 2y \leq 10$, $3x + 2y \leq 18$, $x \geq 0$, $y \geq 0$

A1

 $2x + 3y$ at B is $17 > 15$, so B is cut off; at A it is 15 and at C it is 12

M1

 $2x + 3y = 15$ meets $3x + 2y = 18$ at $(\frac{24}{5}, \frac{9}{5})$

M1 A1

New vertices: (0, 0), (6, 0), $(\frac{24}{5}, \frac{9}{5})$, (0, 5)

A1

Answer: $x + 2y \leq 10$, $3x + 2y \leq 18$; new vertices (0, 0), (6, 0), $(\frac{24}{5}, \frac{9}{5})$, (0, 5)

7 marks

20 (7 marks)

$2x + 5y = 20$ meets the axes at $(10, 0)$ and $(0, 4)$, but $x \leq 8$ and $x + y \leq 10$ restrict the region M1

$x = 8$ meets $2x + 5y = 20$ at $(8, \frac{4}{5})$ and $x + y = 10$ at $(8, 2)$ M1 A1

$x + y = 10$ meets the y-axis at $(0, 10)$ and $2x + 5y = 20$ meets it at $(0, 4)$ A1

Vertices: $(0, 4), (8, \frac{4}{5}), (8, 2), (0, 10)$ A1

Shoelace gives area = $\frac{144}{5}$ M1 A1

Answer: Vertices $(0, 4), (8, \frac{4}{5}), (8, 2), (0, 10)$; area $\frac{144}{5}$ 7 marks

21 (7 marks)

Shoelace with the vertices in order: sum of $(x_i y_{i+1} - x_{i+1} y_i)$ M1

$(0 - 0) + (54 - 0) + (48 - 24) + (40 - 0) + (0 - 0) = 118$ A1

Area = $\frac{1}{2}(118) = 59$ A1

On $x + y = 10$: $y = 10 - x$, so $x + 2y = 20 - x \leq 20$ always and $2x + y = 10 + x \leq 18$ gives $x \leq 8$ M1 A1

$x + y = 10 \leq 12$ holds; x and $y \geq 0$ gives $0 \leq x \leq 10$; so $x = 0, 1, 2, \dots, 8$ A1

That is 9 points A1

Answer: Area 59; 9 integer points 7 marks

22 (6 marks)

Vertices are feasible points where three boundary planes meet; check all combinations M1

On the base $z = 0$: $(0, 0, 0), (4, 0, 0), (0, 3, 0)$, and $(4, 2, 0), (3, 3, 0)$ from $x + y = 6$ with $x = 4$ and with $y = 3$ A1

Above the base: $(0, 0, 6), (4, 0, 2), (0, 3, 3)$ A1

$(4, 3, -1)$ is not feasible ($z < 0$), so it is rejected M1

Eight vertices in total A1

$(2, 1, 3)$: $6 \leq 6, 2 \leq 4, 1 \leq 3$ and all non-negative, so feasible; only $x + y + z = 6$ is an equality, so only one boundary plane meets there, so it is not a vertex E1

Answer: 8 vertices; $(2, 1, 3)$ lies on only one boundary 6 marks

23 (7 marks)

The corner $(k, 5)$ is feasible only if $k + 5 \leq 8$, that is $k \leq 3$: then the region is a rectangle with 4 vertices M1 A1

For $3 < k < 8$, $x = k$ cuts $x + y = 8$: vertices $(0, 0), (k, 0), (k, 8 - k), (3, 5), (0, 5)$, so 5 vertices M1 A1

For $k \geq 8$, $x \leq k$ is redundant: vertices $(0, 0), (8, 0), (3, 5), (0, 5)$, so 4 vertices A1

$k = 6$: a 6 by 5 rectangle (area 30) minus the triangle with vertices $(6, 5), (6, 2), (3, 5)$, of area M1

$$\frac{1}{2}(3)(3) = \frac{9}{2}$$

Area = $30 - \frac{9}{2} = \frac{51}{2}$ A1

Answer: 4 vertices for $k \leq 3$, 5 for $3 < k < 8$, 4 for $k \geq 8$; area $\frac{51}{2}$ 7 marks

24 (6 marks)

$x + y = 8, 2x + y = 12$ and $x + 2y = 12$ all hold at $(4, 4)$: $8 = 8, 12 = 12, 12 = 12$ M1 A1

$x + y + s = 8, 2x + y + t = 12, x + 2y + u = 12$, with $s, t, u \geq 0$ B1

At $(4, 4)$: $s = t = u = 0$ A1

Five variables with three equations: a vertex normally has two variables equal to zero M1

Here three variables are zero, so a basic variable equals zero: the vertex is degenerate (more than two boundaries meet there) E1

Answer: Degenerate: three boundaries meet at $(4, 4), s = t = u = 0$ 6 marks

25 (6 marks)

A point on PQ is $(x, y) = ((1 - t)x_1 + tx_2, (1 - t)y_1 + ty_2)$ for $0 \leq t \leq 1$ M1

$$ax + by = (1 - t)(ax_1 + by_1) + t(ax_2 + by_2)$$
 M1 A1

$$\leq (1 - t)c + tc = c, \text{ since } 1 - t \geq 0 \text{ and } t \geq 0$$
 M1 A1

So the segment lies in the region of every constraint, hence in their intersection: the feasible region is convex E1

Answer: $ax + by \leq c$ holds on the whole segment 6 marks

26 (6 marks)

A vertex must be a feasible point as well as an intersection of two boundaries E1

$$(\frac{32}{5}, \frac{24}{5}): x + y = \frac{56}{5} = 11.2 > 10, \text{ so it breaks } x + y \leq 10$$
 M1 A1

It must be discarded; the other five points satisfy all constraints M1

$(8, 0): 3x + y = 24$ and $(0, 8): x + 2y = 16$ are feasible, as are $(7, 3)$ and $(4, 6)$ A1

Correct vertices: $(0, 0), (8, 0), (7, 3), (4, 6), (0, 8)$ A1

Answer: $(\frac{32}{5}, \frac{24}{5})$ is infeasible; vertices $(0, 0), (8, 0), (7, 3), (4, 6), (0, 8)$ 6 marks

27 (8 marks)

R_k is a trapezium with parallel sides 10 (on $x = 0$) and $10 - k$ (on $x = k$) and width k M1 A1

$$\text{Area} = \frac{1}{2}(10 + 10 - k)k$$
 M1

$$= 10k - \frac{k^2}{2}$$
 A1

$$10k - \frac{k^2}{2} = 40 \text{ gives } k^2 - 20k + 80 = 0$$
 M1

$$k = 10 \pm \sqrt{20} = 10 \pm 2\sqrt{5}$$
 M1 A1

Since $k \leq 10$, $k = 10 - 2\sqrt{5}$ (≈ 5.53) A1

Answer: $k = 10 - 2\sqrt{5}$ 8 marks

28 (9 marks)

The modulus condition is equivalent to the four inequalities $\pm(x - 3) \pm (y - 2) \leq 4$ M1

$$x + y \leq 9, x - y \leq 5, y - x \leq 3, x + y \geq 1$$
 A1 A1

Vertices: $(7, 2)$ and $(3, 6)$ are corners of the diamond with centre $(3, 2)$ B1

$x + y = 1$ meets the axes at $(1, 0)$ and $(0, 1)$; $x - y = 5$ meets $y = 0$ at $(5, 0)$; $y - x = 3$ meets $x = 0$ at $(0, 3)$ M1 A1

Vertices: $(1, 0), (5, 0), (7, 2), (3, 6), (0, 3), (0, 1)$ A1

Area of the full diamond is $2(4)^2 = 32$, minus the two corner triangles outside the axes, of areas 4 and 1 M1

$$\text{Area} = 32 - 5 = 27$$
 A1

Answer: Vertices $(1, 0), (5, 0), (7, 2), (3, 6), (0, 3), (0, 1)$; area 27 9 marks

29 (8 marks)

For each $x = 0, 1, \dots, n$, y takes the values $0, 1, \dots, n - x$, which is $n - x + 1$ values M1 A1

$$N(n) = \sum_{x=0}^n (n - x + 1) = (n + 1) + n + \dots + 1$$
 M1

This is an arithmetic series with $n + 1$ terms: $N(n) = \frac{1}{2}(n + 1)(n + 2)$ M1 A1

$$\frac{(n + 1)(n + 2)}{2} = 190 \text{ gives } n^2 + 3n + 2 = 380$$
 M1

$$n^2 + 3n - 378 = 0, (n - 18)(n + 21) = 0$$
 M1

$n = 18$ (as n is positive) A1

Answer: $N(n) = \frac{(n + 1)(n + 2)}{2}$; $n = 18$ 8 marks

30 (9 marks)

A general point on AB: $(3 + 3t, 7 - 3t)$ for $0 \leq t \leq 1$

M1 A1

$$x + y = 10 \leq 10$$

B1

$$2x + y = 13 + 3t \leq 16$$

M1

$$x + 3y = 24 - 6t \leq 24, \text{ and } x, y \geq 0$$

M1 A1

So all points on AB are feasible, with $P = 10$ throughout

A1

Integer points on AB: $(3, 7), (4, 6), (5, 5), (6, 4)$, so 4

M1 A1

Answer: All points on AB are optimal ($P = 10$); 4 integer points

9 marks

Objective Functions**1 (1 mark)**

$$P = 5x + 8y$$

B1

Answer: $P = 5x + 8y$

1 mark

2 (1 mark)

$$P = 48 + 126 = 174$$

B1

Answer: 174

1 mark

3 (2 marks)

$$y = 6 - \frac{3}{2}x$$

M1

$$\text{Gradient} = -\frac{3}{2}$$

A1

Answer: $-\frac{3}{2}$

2 marks

4 (1 mark)

Minimising C is the same as maximising $-C$: $P = -2x - 3y$

B1

Answer: $P = -2x - 3y$

1 mark

5 (1 mark)

Away from the origin (parallel to itself)

B1

Answer: Away from the origin

1 mark

6 (1 mark)

At a vertex of the feasible region (or along an edge joining two optimal vertices)

B1

Answer: At a vertex

1 mark

7 (3 marks)

P at the vertices: 0, 16, 32, 41, 40

M1 A1

Maximum $P = 41$ at $(3, 7)$

A1

Answer: $P = 41$ at $(3, 7)$

3 marks

8 (3 marks)

$C = 100, 76, 180$ at the three vertices

M1 A1

Minimum $C = 76$ at $(6, 8)$

A1

Answer: $C = 76$ at $(6, 8)$

3 marks

9 (3 marks)

P at the vertices: 0, 8, 2, -4 , -8

M1 A1

Maximum 8 at $(8, 0)$; minimum -8 at $(0, 8)$

A1

Answer: Maximum 8 at $(8, 0)$; minimum -8 at $(0, 8)$

3 marks

10 (3 marks)

$P = 18 + 8 = 26$, so the line is $3x + 2y = 26$

M1 A1

It crosses the x-axis at $(\frac{26}{3}, 0)$ and the y-axis at $(0, 13)$

A1

Answer: $3x + 2y = 26$; $(\frac{26}{3}, 0)$ and $(0, 13)$

3 marks

11 (3 marks)

Profit on A = $14 - 6 - 3 = £5$

M1 A1

Profit on B = $10 - 4 - 2 = £4$, so $P = 5x + 4y$

A1

Answer: $P = 5x + 4y$

3 marks

12 (3 marks)

$(6, 4)$: $30 + 12 = 42$

B1

$(3, 7)$: $15 + 21 = 36$

B1

$(6, 4)$ is better, by 6

B1

Answer: $(6, 4)$ by 6

3 marks

13 (4 marks)

$P = 0, 32, 28, 19, 8$, so the maximum is 32 at $(8, 0)$

M1 A1

Objective lines have gradient -4 , steeper than the edge $2x + y = 16$ (gradient -2)

M1

Moving the line away from the origin, it leaves the region at $(8, 0)$

A1

Answer: $P = 32$ at $(8, 0)$

4 marks

14 (4 marks)

Lines of constant P are parallel; P increases as the line moves away from the origin

E1

$3x + 2y = 12$ passes through R, so larger values are possible

M1

$3x + 2y = 26$ passes through the vertex $(6, 4)$ and no point of R lies beyond it

A1

Maximum $P = 18 + 8 = 26$

B1

Answer: $P = 26$ at $(6, 4)$

4 marks

15 (6 marks)

P at the vertices $(0, 0), (8, 0), (6, 4), (3, 7), (0, 8)$: 0, 16, 16, 13, 8

M1 A1

The objective lines $2x + y = c$ are parallel to the edge $2x + y = 16$, so every point of the edge between $(8, 0)$ and $(6, 4)$ is optimal

E1

Maximum $P = 16$

A1

Integer points on the edge: $(8, 0), (7, 2), (6, 4)$

M1 A1

Answer: Maximum 16 along the edge; integer points $(8, 0), (7, 2), (6, 4)$

6 marks

16 (5 marks)

$(6, 4)$ lies on $2x + y = 16$ and $x + y = 10$, with gradients -2 and -1

M1

The objective line must have gradient between these: $-2 \leq -\frac{3}{k} \leq -1$

M1 A1

$1 \leq \frac{3}{k} \leq 2$, so $\frac{3}{2} \leq k \leq 3$

A1 A1

Answer: $\frac{3}{2} \leq k \leq 3$

5 marks

17 (5 marks)

At $(6, 4)$ and $(3, 7)$ the shared boundary is $x + y = 10$ with gradient -1

M1

The optimum moves to $(3, 7)$ when the objective line is flatter than gradient -1 : $\frac{3}{c} < 1$

M1 A1

So $c > 3$ (and when $c = 3$, both vertices are optimal)

A1 A1

Answer: $c > 3$

5 marks

18 (5 marks)

Without the overhead, $5x + 4y$ at the vertices: 0, 100, 108, 75, 60

M1 A1

Maximum is at (12, 12): $P = 108 - 50 = 58$

A1

A constant added to every value of the objective does not change the ranking of the vertices

E1

So the optimal vertex is unchanged and the maximum weekly profit is £58

A1

Answer: (12, 12); $P = 58$

5 marks

19 (4 marks)

P : (0, 0) = 0; (20, 0) = 100; (12, 12) = 108; (3, 15) = 75; (0, 15) = 60

M1 A1

Order: (0, 0), (0, 15), (3, 15), (20, 0), (12, 12)

A1

Maximum 108 and minimum 0

A1

Answer: Maximum 108 at (12, 12), minimum 0 at (0, 0)

4 marks

20 (5 marks)

C at the vertices: 12, 12, 20, 30

M1 A1

Minimum $C = 12$ at both (0, 12) and (2, 6)

A1

The line $C = 12$ is the boundary $3x + y = 12$, so every point on the segment from (0, 12) to (2, 6) gives the minimum

E1 A1

Answer: $C = 12$ along the segment from (0, 12) to (2, 6)

5 marks

21 (8 marks)

Gradients of the boundaries: $2x + y = 40$: -2 ; $x + y = 24$: -1 ; $x + 2y = 38$: $-\frac{1}{2}$; the objective has gradient $-\frac{4}{k}$

M1 A1

(20, 0): between the x -axis (0) and -2 : $\frac{4}{k} \geq 2$, so $k \leq 2$

M1 A1

(16, 8): between -2 and -1 : $1 \leq \frac{4}{k} \leq 2$, so $2 \leq k \leq 4$

M1 A1

(10, 14): between -1 and $-\frac{1}{2}$: $4 \leq k \leq 8$

A1

(0, 19): $\frac{4}{k} \leq \frac{1}{2}$, so $k \geq 8$

A1

Answer: (20, 0): $k \leq 2$; (16, 8): $2 \leq k \leq 4$; (10, 14): $4 \leq k \leq 8$; (0, 19): $k \geq 8$

8 marks

22 (7 marks)

Maximum of $x + y$ over R is 10, attained on the edge from (6, 4) to (3, 7) ($x + y = 10$)

M1 A1

On $x + y = 10$: $y = 10 - x$, so $P = 2x + 50 - 5x = 50 - 3x$

M1 A1

$P \geq 35$ gives $x \leq 5$

A1

The edge has $3 \leq x \leq 6$, so $3 \leq x \leq 5$ (with $y = 10 - x$)

A1

Maximum number of items is 10

B1

Answer: 10 items, $3 \leq x \leq 5$

7 marks

23 (7 marks)

Profit on A = $40 - 14 - 2 = £24$

M1 A1

Profit on B = $60 - 18 - 2 = £40$, so $P = 24x + 40y$

A1

P at the vertices: 0, 192, 304, 352, 320

M1

Optimal vertex (3, 7), $P = 352$

A1

Delivery £6: profits 20 and 36; $P = 20x + 36y$ gives 0, 160, 264, 312, 288

M1

Same vertex (3, 7), $P = 312$

A1

Answer: $P = 24x + 40y$; (3, 7), $P = 352$; unchanged with delivery £6 ($P = 312$)

7 marks

24 (7 marks)

P at the vertices: 0, 60, 60, 39, 30

M1 A1

(20, 0) and (12, 12) both give 60, so the objective line is parallel to the edge $3x + 2y = 60$

E1

Every point on that edge is optimal, with maximum 60

A1

On the edge, $y = 30 - 1.5x$, which is an integer when x is even: $x = 12, 14, 16, 18, 20$

M1 A1

Points: (12, 12), (14, 9), (16, 6), (18, 3), (20, 0)

A1

Answer: Maximum 60 along the edge; (12, 12), (14, 9), (16, 6), (18, 3), (20, 0)

7 marks

25 (6 marks)

P: 0, 8, 9, 14, 15, 24, 16, 21 in the order given

M1 A1 A1

Maximum $P = 24$ at $(0, 0, 6)$

A1

At the optimum $x = y = 0$, so y is not larger than x

M1

The optimum must be found by comparing P at every vertex, not by comparing single coefficients

E1

Answer: $P = 24$ at $(0, 0, 6)$; claim false

6 marks

26 (6 marks)

The student has not checked every vertex: the point $(6, 8)$ has been left out

E1

Having a variable equal to zero does not make a vertex cheaper

E1

C at $(6, 8) = 36 + 40 = 76$

M1 A1

$76 < 100$, so the minimum is 76 at $(6, 8)$

A1

$(6, 8)$ is feasible: $20 \geq 20$, $30 \geq 30$, $14 \geq 14$

B1

Answer: Minimum 76 at $(6, 8)$

6 marks

27 (9 marks)

Boundary normals at the vertices: $(8, 0)$: $(0, 1)$ and $(2, 1)$; $(6, 4)$: $(2, 1)$, $(1, 1)$; $(3, 7)$: $(1, 1)$, $(1, 3)$; $(0, 8)$: $(1, 3)$, $(1, 0)$

M1

The objective vector (p, q) must lie between the normals of the two boundaries at the optimal vertex

M1

$(8, 0)$: $\frac{q}{p} \leq \frac{1}{2}$, so $10 - p \leq \frac{p}{2}$ and $p \geq \frac{20}{3}$

M1 A1

$(6, 4)$: $\frac{1}{2} \leq \frac{q}{p} \leq 1$, so $5 \leq p \leq \frac{20}{3}$

M1 A1

$(3, 7)$: $1 \leq \frac{q}{p} \leq 3$, so $\frac{5}{2} \leq p \leq 5$

M1 A1

$(0, 8)$: $\frac{q}{p} \geq 3$, so $p \leq \frac{5}{2}$

A1

Answer: $(8, 0)$: $p \geq 20/3$; $(6, 4)$: $5 \leq p \leq 20/3$; $(3, 7)$: $5/2 \leq p \leq 5$; $(0, 8)$: $p \leq 5/2$

9 marks

28 (8 marks)

The ratio y/x is the gradient of the line from the origin to (x, y)

E1

Lines $y = mx$ through the origin sweep the region; m is greatest for the steepest line that still touches R

M1

Vertices of R : $(2, 0)$, $(10, 0)$, $(4, 6)$, $(2, 6)$

M1 A1

Gradients from the origin: $0, 0, \frac{3}{2}, 3$

M1 A1

Maximum $\frac{y}{x} = 3$ at $(2, 6)$

A1

(Minimum is 0 along $y = 0$)

E1

Answer: Maximum 3 at $(2, 6)$

8 marks

29 (8 marks)

At $(12, 12)$ the boundaries are $3x + 2y = 60$ (gradient $-\frac{3}{2}$) and $x + 3y = 48$ (gradient $-\frac{1}{3}$)

M1

Objective gradient $-\frac{a}{4}$ must satisfy $-\frac{3}{2} \leq -\frac{a}{4} \leq -\frac{1}{3}$

M1 A1

$\frac{1}{3} \leq \frac{a}{4} \leq \frac{3}{2}$, so $\frac{4}{3} \leq a \leq 6$

A1 A1

$a = 7$ is outside the range, and the objective line is steeper than $3x + 2y = 60$, so the optimum moves to $(20, 0)$

M1 A1

$P = 7(20) + 0 = 140$; check $(12, 12)$: $84 + 48 = 132 < 140$

A1

Answer: $\frac{4}{3} \leq a \leq 6$; $P = 140$ at $(20, 0)$

8 marks

30 (8 marks)

Let V be the vertex maximising P : $P(V) \geq P(W)$ for every feasible W M1
 If $\lambda > 0$: $Q(V) - Q(W) = \lambda(P(V) - P(W)) \geq 0$, so V maximises Q M1 A1
 If $\lambda < 0$: $Q(V) - Q(W) = \lambda(P(V) - P(W)) \leq 0$, so V minimises Q M1 A1
 μ cancels in the difference, so has no effect E1
 $Q = 100 - 4x - 6y = -2(2x + 3y) + 100 = -2P + 100$, with $\lambda = -2 < 0$ M1
 So minimising Q is equivalent to maximising P A1
Answer: $Q(V) - Q(W) = \lambda(P(V) - P(W))$; $\lambda = -2$ so minimising $Q \equiv$ maximising P 8 marks

Simplex Algorithm**1 (2 marks)**

$2x + 3y + r = 24$ B1
 $x + 4y + s = 20$, and $P - 3x - 5y = 0$, with $r, s \geq 0$ B1
Answer: $2x + 3y + r = 24$, $x + 4y + s = 20$, $P - 3x - 5y = 0$ 2 marks

2 (1 mark)

$P - 4x - 7y = 0$ B1
Answer: $P - 4x - 7y = 0$; entries -4 and -7 1 mark

3 (2 marks)

y enters: it has the most negative entry (-5) in the P row B1 B1
Answer: y , the most negative entry 2 marks

4 (2 marks)

Ratios: $\frac{12}{2} = 6$ and $\frac{20}{4} = 5$; the row with entry 0 is not used M1
 The smallest ratio is 5, so the second row is the pivot row A1
Answer: Ratios 6, 5; pivot row 2 2 marks

5 (1 mark)

No negative entries in the objective row B1
Answer: No negative entries in the P row 1 mark

6 (2 marks)

$x = 8$, $y = 4$ (basic variables read from the Value column) B1
 $t = 2$ and $P = 28$; $r = s = 0$ (non-basic) B1
Answer: $x = 8$, $y = 4$, $P = 28$, $t = 2$ 2 marks

7 (3 marks)

$x + y + r = 12$, $x + 2y + s = 16$, $3x + y + t = 30$ B1
 The three constraint rows are $(1, 1, 1, 0, 0 | 12)$, $(1, 2, 0, 1, 0 | 16)$, $(3, 1, 0, 0, 1 | 30)$ B1
 The P row is $(-2, -3, 0, 0, 0 | 0)$ B1
Answer: $(1, 1, 1, 0, 0, 12)$ $(1, 2, 0, 1, 0, 16)$ $(3, 1, 0, 0, 1, 30)$ $P: (-2, -3, 0, 0, 0, 0)$ 3 marks

b.v.	x	y	r	s	t	Value
r	1	1	1	0	0	12
s	1	2	0	1	0	16
t	3	1	0	0	1	30
P	-2	-3	0	0	0	0

8 (5 marks)

The most negative entry in the P row is -3 , so y enters; ratios $\frac{12}{1} = 12$, $\frac{16}{2} = 8$, $\frac{30}{1} = 30$

M1 A1

Least ratio 8: s leaves and the pivot is 2

A1

Row operations: $R_2 \div 2 \rightarrow R_2$; $R_1 - R_2 \rightarrow R_1$; $R_3 - R_2 \rightarrow R_3$; $P + 3 R_2 \rightarrow P$

M1

New tableau as shown: basic variables r, y, t with $P = 24$

A1

Answer: Pivot 2 in the y column; after the iteration $y = 8$, $P = 24$

5 marks

Initial tableau (pivot boxed)

b.v.	x	y	r	s	t	Value
r	1	1	1	0	0	12
s	1	2	0	1	0	16
t	3	1	0	0	1	30
P	-2	-3	0	0	0	0

After one iteration

b.v.	x	y	r	s	t	Value
r	$\frac{1}{2}$	0	1	$-\frac{1}{2}$	0	4
y	$\frac{1}{2}$	1	0	$\frac{1}{2}$	0	8
t	$\frac{5}{2}$	0	0	$-\frac{1}{2}$	1	22
P	$-\frac{1}{2}$	0	0	$\frac{3}{2}$	0	24

9 (2 marks)

Choosing the smallest ratio ensures that the value of the entering variable is as large as it can be while every basic variable stays non-negative

E1

A larger ratio would make the basic variable in a row with a smaller ratio negative, so the solution would no longer be feasible

E1

Answer: Smallest ratio keeps all basic variables non-negative (feasible)

2 marks

10 (3 marks)

The non-basic variables x and s are zero, so $x = 0$, $y = 8$, $P = 24$

B1 B1

The P row has a negative entry $-\frac{1}{2}$ (under x), so P can be increased

E1

Answer: $x = 0$, $y = 8$, $P = 24$; not optimal

3 marks

11 (3 marks)

$r = 0$ and $s = 0$: the first two constraints are binding (fully used)

B1 B1

$t = 2$: 2 units of the third resource are unused ($3x + y = 28$ compared with 30)

B1

Answer: First two binding; 2 units unused in the third

3 marks

12 (3 marks)

Maximise $P = 4x + 5y + 3z$

B1

$2x + y + z \leq 18$ and $x + 3y + 2z \leq 30$

B1

$3x + y \leq 24$, with $x, y, z \geq 0$

B1

Answer: Maximise $P = 4x + 5y + 3z$ with $2x + y + z \leq 18$, $x + 3y + 2z \leq 30$, $3x + y \leq 24$

3 marks

13 (3 marks)

$2x + 3y + z + r = 30$, $x + y + z + s = 14$, $3x + y + 2z + t = 27$

B1

Non-basic: $x = y = z = 0$, so $r = 30$, $s = 14$, $t = 27$

B1

$P = 0$

B1

Answer: $r = 30$, $s = 14$, $t = 27$, $x = y = z = 0$, $P = 0$

3 marks

14 (6 marks)

Iteration 1: most negative entry in the P row is -7 , so x enters; ratios 6, 12, 10, least is 6, so r leaves and the pivot is 1

M1

Row operations: $R_2 - 2 R_1 \rightarrow R_2$; $R_3 - R_1 \rightarrow R_3$; $P + 7 R_1 \rightarrow P$ (new tableau in the diagram)

A1

Iteration 2: most negative entry in the P row is -5 , so y enters; ratios $-$, 4, 4, least is 4, so s leaves and the pivot is 3

M1

Row operations: $R_2 \div 3 \rightarrow R_2$; $R_3 - R_2 \rightarrow R_3$; $P + 5 R_2 \rightarrow P$ (new tableau in the diagram)

A1

No negative entries remain in the P row, so the solution is optimal: $x = 6$, $y = 4$ with $P = 62$

A1 A1

Answer: $x = 6$, $y = 4$, $P = 62$

6 marks

Initial

b.v.	x	y	r	s	t	Value
r	1	0	1	0	0	6
s	2	3	0	1	0	24
t	1	1	0	0	1	10
P	-7	-5	0	0	0	0

Iteration 1

b.v.	x	y	r	s	t	Value
x	1	0	1	0	0	6
s	0	3	-2	1	0	12
t	0	1	-1	0	1	4
P	0	-5	7	0	0	42

Iteration 2

b.v.	x	y	r	s	t	Value
x	1	0	1	0	0	6
y	0	1	$-\frac{2}{3}$	$\frac{1}{3}$	0	4
t	0	0	$-\frac{1}{3}$	$-\frac{1}{3}$	1	0
P	0	0	$\frac{11}{3}$	$\frac{5}{3}$	0	62

15 (5 marks)

Iteration 2: most negative entry in the P row is $-\frac{3}{2}$, so y enters; ratios 3, 3, 15, least is 3, so r leaves and the pivot is 1

M1 A1

Row operations: $R_2 - \frac{3}{2} R_1 \rightarrow R_2$; $R_3 - \frac{1}{2} R_1 \rightarrow R_3$; $P + \frac{3}{2} R_1 \rightarrow P$ (new tableau in the diagram)

M1 A1

New tableau (in the diagram): $x = 6$, $y = 3$, $P = 42$

A1

Answer: $x = 6$, $y = 3$, $P = 42$

5 marks

Given tableau (pivot boxed)

b.v.	x	y	r	s	t	Value
r	0	1	1	0	-3	3
s	0	$\frac{3}{2}$	0	1	$-\frac{1}{2}$	$\frac{9}{2}$
x	1	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{15}{2}$
P	0	$-\frac{3}{2}$	0	0	$\frac{5}{2}$	$\frac{75}{2}$

After the next iteration

b.v.	x	y	r	s	t	Value
y	0	1	1	0	-3	3
s	0	0	$-\frac{3}{2}$	1	4	0
x	1	0	$-\frac{1}{2}$	0	2	6
P	0	0	$\frac{3}{2}$	0	-2	42

16 (5 marks)

y enters, as -4 is the most negative entry in the P row

B1

Ratios: $\frac{18}{1} = 18$, $\frac{16}{2} = 8$, $\frac{11}{1} = 11$

M1

The least ratio is 8, so the pivot row is the s row and the pivot is 2

A1

Using the r row (largest ratio 18) as the pivot row gives $y = 18$, and then $s = 16 - 2(18) = -20$

M1

A negative value of s is not feasible, so the least ratio must be used

A1

Answer: y enters; pivot row s (ratio 8); the largest ratio gives $s = -20$

5 marks

17 (6 marks)

$x - y + r = 3$, $x - 2y + s = 4$; P row $(-1, -1, 0, 0 | 0)$

B1

Choose x (tie): ratios 3 and 4, pivot 1 in row 1; after the iteration the P row is $(0, -2, 1, 0 | 3)$

M1 A1

y enters; the y column has entries -1 and -1 in the constraint rows

M1

There is no positive entry, so no ratio can be formed: y can be increased without limit, so P is unbounded

E1 A1

Answer: Unbounded

6 marks

18 (7 marks)

Maximise $P = 3x + 2y$ subject to $2x + y \leq 10$, $x + 2y \leq 7$, $x + 3y \leq 15$

B1 B1

Slack form: $2x + y + r = 10$, $x + 2y + s = 7$, $x + 3y + t = 15$, $P - 3x - 2y = 0$

B1

x enters (-3); ratios $\frac{10}{2} = 5$, $\frac{7}{1} = 7$, $\frac{15}{1} = 15$; pivot 2 in row 1

M1 A1

After the iteration $x = 5$, $y = 0$, $P = 15$

A1 A1

Answer: $x = 5$, $y = 0$, $P = 15$ after one iteration

7 marks

19 (6 marks)

z enters (-5); ratios: row r has 0 in the z column so is ignored; row s: $\frac{24}{4} = 6$

M1 A1

Pivot 4: $R_2 \div 4$ gives $(1/2, 0, 1, 0, 1/4 | 6)$

M1

$P + 5R_2$: P row $(-\frac{1}{2}, -2, 0, 0, \frac{5}{4} | 30)$

A1

$x = y = 0$, $z = 6$, $r = 8$, $s = 0$, $P = 30$

A1

Negative entries remain $(-\frac{1}{2}$ and $-2)$, so it is not optimal

A1

Answer: $z = 6$, $r = 8$, $P = 30$; not optimal

6 marks

20 (5 marks)

Iteration 1: most negative entry in the P row is -8 , so y enters; ratios 10, 14, 27, least is 10, so r leaves and the pivot is 3

M1 A1

Row operations: $R_1 \div 3 \rightarrow R_1$; $R_2 - R_1 \rightarrow R_2$; $R_3 - R_1 \rightarrow R_3$; $P + 8 R_1 \rightarrow P$ (new tableau in the diagram)

M1 A1

After the iteration: $y = 10$, $s = 4$, $t = 17$, $x = z = r = 0$, $P = 80$

A1

Answer: $y = 10$, $s = 4$, $t = 17$, $P = 80$

5 marks

Initial (pivot boxed)

b.v.	x	y	z	r	s	t	Value
r	2	3	1	1	0	0	30
s	1	1	1	0	1	0	14
t	3	1	2	0	0	1	27
P	-6	-8	-5	0	0	0	0

After iteration 1

b.v.	x	y	z	r	s	t	Value
y	$\frac{2}{3}$	1	$\frac{1}{3}$	$\frac{1}{3}$	0	0	10
s	$\frac{1}{3}$	0	$\frac{2}{3}$	$-\frac{1}{3}$	1	0	4
t	$\frac{7}{3}$	0	$\frac{5}{3}$	$-\frac{1}{3}$	0	1	17
P	$-\frac{2}{3}$	0	$-\frac{7}{3}$	$\frac{8}{3}$	0	0	80

21 (9 marks)

(a) Rows (x, y, z, r, s, t | value): (2, 3, 1, 1, 0, 0 | 30), (1, 1, 1, 0, 1, 0 | 14), (3, 1, 2, 0, 0, 1 | 27)

B1

P row (-6, -8, -5, 0, 0, 0 | 0)

B1

Answer: Initial tableau as described

2 marks

(b) Iteration 1: most negative entry in the P row is -8 , so y enters; ratios 10, 14, 27, least is 10, so r leaves and the pivot is 3

M1

Row operations: $R_1 \div 3 \rightarrow R_1$; $R_2 - R_1 \rightarrow R_2$; $R_3 - R_1 \rightarrow R_3$; $P + 8 R_1 \rightarrow P$ (new tableau in the diagram)

A1

Answer: $y = 10$, $s = 4$, $t = 17$, $P = 80$

2 marks

(c) Iteration 2: most negative entry in the P row is $-\frac{7}{3}$, so z enters; ratios 30, 6, $\frac{51}{5}$, least is 6, so s leaves and the pivot is $\frac{2}{3}$

M1

Row operations: $R_2 \div \frac{2}{3} \rightarrow R_2$; $R_1 - \frac{1}{3} R_2 \rightarrow R_1$; $R_3 - \frac{5}{3} R_2 \rightarrow R_3$; $P + \frac{7}{3} R_2 \rightarrow P$ (new tableau in the diagram)

A1

No negative entries remain in the P row, so the solution is optimal: $x = 0$, $y = 8$, $z = 6$ with $P = 94$

A1 A1

Answer: $x = 0$, $y = 8$, $z = 6$, $P = 94$; optimal

4 marks

(d) Optimal plan: 0 of x , 8 of y , 6 of z ; the slack $t = 7$, so 7 units of the third resource are unused while $r = s = 0$

B1

Answer: $y = 8$, $z = 6$; 7 units of resource 3 unused

1 mark

Initial

b.v.	x	y	z	r	s	t	Value
r	2	3	1	1	0	0	30
s	1	1	1	0	1	0	14
t	3	1	2	0	0	1	27
P	-6	-8	-5	0	0	0	0

After iteration 1

b.v.	x	y	z	r	s	t	Value
y	$\frac{2}{3}$	1	$\frac{1}{3}$	$\frac{1}{3}$	0	0	10
s	$\frac{1}{3}$	0	$\frac{2}{3}$	$-\frac{1}{3}$	1	0	4
t	$\frac{7}{3}$	0	$\frac{5}{3}$	$-\frac{1}{3}$	0	1	17
P	$-\frac{2}{3}$	0	$-\frac{7}{3}$	$\frac{8}{3}$	0	0	80

After iteration 2

b.v.	x	y	z	r	s	t	Value
y	$\frac{1}{2}$	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	8
z	$\frac{1}{2}$	0	1	$-\frac{1}{2}$	$\frac{3}{2}$	0	6
t	$\frac{3}{2}$	0	0	$\frac{1}{2}$	$-\frac{5}{2}$	1	7
P	$\frac{1}{2}$	0	0	$\frac{3}{2}$	$\frac{7}{2}$	0	94

22 (9 marks)

Iteration 1: most negative entry in the P row is -5 , so x enters; ratios 8, 12, $\frac{15}{2}$, least is $\frac{15}{2}$, so t leaves and the pivot is 2

M1

Row operations: $R_3 \div 2 \rightarrow R_3$; $R_1 - 6 R_3 \rightarrow R_1$; $R_2 - R_3 \rightarrow R_2$; $P + 5 R_3 \rightarrow P$ (new tableau in the diagram)

A1

Iteration 2 has a tie: two rows give the same least ratio (the ratios are 3, 3, 15)

E1

Iteration 2: most negative entry in the P row is $-\frac{3}{2}$, so y enters; ratios 3, 3, 15, least is 3, so r leaves and the pivot is 1

M1

Row operations: $R_2 - \frac{3}{2} R_1 \rightarrow R_2$; $R_3 - \frac{1}{2} R_1 \rightarrow R_3$; $P + \frac{3}{2} R_1 \rightarrow P$ (new tableau in the diagram)

A1

Iteration 3: most negative entry in the P row is -2 , so t enters; ratios $-\infty$, 0, 3, least is 0, so s leaves and the pivot is 4

M1

Row operations: $R_2 \div 4 \rightarrow R_2$; $R_1 + 3 R_2 \rightarrow R_1$; $R_3 - 2 R_2 \rightarrow R_3$; $P + 2 R_2 \rightarrow P$ (new tableau in the diagram)

A1

The tied row that does not leave keeps a basic variable at value 0, so the solution is degenerate

E1

No negative entries remain in the P row, so the solution is optimal: $x = 6$, $y = 3$ with $P = 42$

A1

Answer: $x = 6$, $y = 3$, $P = 42$

9 marks

b.v.	x	y	r	s	t	Value
r	6	4	1	0	0	48
s	1	2	0	1	0	12
t	2	1	0	0	1	15
P	-5	-4	0	0	0	0

b.v.	x	y	r	s	t	Value
r	0	1	1	0	-3	3
s	0	$\frac{3}{2}$	0	1	$-\frac{1}{2}$	$\frac{9}{2}$
x	1	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{15}{2}$
P	0	$-\frac{3}{2}$	0	0	$\frac{5}{2}$	$\frac{75}{2}$

b.v.	x	y	r	s	t	Value
y	0	1	1	0	-3	3
s	0	0	$-\frac{3}{2}$	1	4	0
x	1	0	$-\frac{1}{2}$	0	2	6
P	0	0	$\frac{3}{2}$	0	-2	42

b.v.	x	y	r	s	t	Value
y	0	1	$-\frac{1}{8}$	$\frac{3}{4}$	0	3
t	0	0	$-\frac{3}{8}$	$\frac{1}{4}$	1	0
x	1	0	$\frac{1}{4}$	$-\frac{1}{2}$	0	6
P	0	0	$\frac{3}{4}$	$\frac{1}{2}$	0	42

23 (9 marks)

Iteration 1: most negative entry in the P row is -4 , so y enters; ratios 5, 8, 4, least is 4, so t leaves and the pivot is 1

M1

Row operations: $R_1 - 2R_3 \rightarrow R_1$; $R_2 - R_3 \rightarrow R_2$; $P + 4R_3 \rightarrow P$ (new tableau in the diagram)

A1

Iteration 2: most negative entry in the P row is -2 , so x enters; ratios 2, 4, $-$, least is 2, so r leaves and the pivot is 1

M1

Row operations: $R_2 - R_1 \rightarrow R_2$; $P + 2R_1 \rightarrow P$ (new tableau in the diagram)

A1

Optimal tableau: $x = 2$, $y = 4$, $s = 2$, $P = 20$

A1

The P row has a 0 under the non-basic variable t, so entering t does not change P: another optimal solution exists

E1

Entering t: the x row has -2 (ignored); ratios $\frac{2}{1} = 2$ (s row) and $\frac{4}{1} = 4$ (y row), so s leaves

M1

New solution: $x = 6$, $y = 2$, $t = 2$, $P = 20$

A1

Every point on the segment between (2, 4) and (6, 2) (the line $x + 2y = 10$) is optimal

A1

Answer: (2, 4) and (6, 2), $P = 20$; all points of the segment

9 marks

Initial

b.v.	x	y	r	s	t	Value
r	1	2	1	0	0	10
s	1	1	0	1	0	8
t	0	1	0	0	1	4
P	-2	-4	0	0	0	0

Iteration 1

b.v.	x	y	r	s	t	Value
r	1	0	1	0	-2	2
s	1	0	0	1	-1	4
y	0	1	0	0	1	4
P	-2	0	0	0	4	16

Iteration 2 (optimal)

b.v.	x	y	r	s	t	Value
x	1	0	1	0	-2	2
s	0	0	-1	1	1	2
y	0	1	0	0	1	4
P	0	0	2	0	0	20

Entering t

b.v.	x	y	r	s	t	Value
x	1	0	-1	2	0	6
t	0	0	-1	1	1	2
y	0	1	1	-1	0	2
P	0	0	2	0	0	20

24 (8 marks)

Iteration 1: most negative entry in the P row is -3 , so x enters; ratios 5, 7, 15, least is 5, so r leaves and the pivot is 2

M1

Row operations: $R_1 \div 2 \rightarrow R_1$; $R_2 - R_1 \rightarrow R_2$; $R_3 - R_1 \rightarrow R_3$; $P + 3 R_1 \rightarrow P$ (new tableau in the diagram)

A1

Iteration 2: most negative entry in the P row is $-\frac{1}{2}$, so y enters; ratios 10, 4, 4, least is 4, so s leaves and the pivot is $\frac{1}{2}$

M1

Row operations: $R_2 \div \frac{1}{2} \rightarrow R_2$; $R_1 - \frac{1}{2} R_2 \rightarrow R_1$; $R_3 - \frac{5}{2} R_2 \rightarrow R_3$; $P + \frac{1}{2} R_2 \rightarrow P$ (new tableau in the diagram)

A1

Tableaux: $(x, y, P) = (0, 0, 0)$, then $(5, 0, 15)$, then $(3, 4, 17)$

A1

The simplex algorithm moves along edges from one vertex of the feasible region to an adjacent vertex, increasing P each time: $(0, 0)$, $(5, 0)$, $(3, 4)$

E1

At $(3, 4)$: $2x + y = 10$, $x + y = 7$ and $x + 3y = 15$ all hold with equality, so three boundary lines pass through one vertex

E1

so $r = s = t = 0$ with only two non-zero variables: a basic variable is zero (degenerate)

A1

Answer: $(0, 0)$, $(5, 0)$, $(3, 4)$; $P = 0, 15, 17$; degenerate at $(3, 4)$

8 marks

Initial

b.v.	x	y	r	s	t	Value
r	2	1	1	0	0	10
s	1	1	0	1	0	7
t	1	3	0	0	1	15
P	-3	-2	0	0	0	0

Iteration 1

b.v.	x	y	r	s	t	Value
x	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	5
s	0	$\frac{1}{2}$	$-\frac{1}{2}$	1	0	2
t	0	$\frac{5}{2}$	$-\frac{1}{2}$	0	1	10
P	0	$-\frac{1}{2}$	$\frac{3}{2}$	0	0	15

Iteration 2

b.v.	x	y	r	s	t	Value
x	1	0	1	-1	0	3
y	0	1	-1	2	0	4
t	0	0	2	-5	1	0
P	0	0	1	1	0	17

25 (6 marks)

(a) $t = 2 - 5\delta \geq 0$ gives $\delta \leq \frac{2}{5}$

M1 A1

$y = 4 - \delta \geq 0$ gives $\delta \leq 4$, so the largest value is $\frac{2}{5}$

A1

Answer: $\delta = \frac{2}{5}$

3 marks

(b) $x = \frac{44}{5}$, $y = \frac{18}{5}$, $P = 2x + 3y = \frac{88}{5} + \frac{54}{5} = \frac{142}{5} = 28.4$

M1 A1

Increase in $P = 0.4$ for $\delta = \frac{2}{5}$, so 1 per unit (the entry 1 under r in the P row)

A1

Answer: $P = 28.4$; 1 per unit

3 marks

26 (7 marks)

$P = \frac{272}{3} \approx 90.67$ is an upper bound for any feasible point

B1

All coefficients of P are integers and x, y, z are integers, so P is an integer and $P \leq 90$

M1 A1

Test $(15, 10, 0)$: $30 + 10 = 40 \leq 40$, $15 + 20 = 35 \leq 36$, $15 + 10 = 25 \leq 42$

M1 A1

$P = 60 + 30 = 90$, which equals the bound, so it is optimal

A1 B1

Answer: $x = 15$, $y = 10$, $z = 0$, $P = 90$

7 marks

27 (12 marks)

(a) Maximise $P = 3x + 2y + 4z$

B1

Ribbon: $x + 2y + 2z \leq 30$

B1

Arranging: $2x + y + 3z \leq 40$

B1

Packing: $2x + 2y + z \leq 36$, $x, y, z \geq 0$

B1

Answer: Maximise $P = 3x + 2y + 4z$ subject to $x + 2y + 2z \leq 30$, $2x + y + 3z \leq 40$, $2x + 2y + z \leq 36$ 4 marks

(b) Rows $(x, y, z, r, s, t \mid \text{value})$: $(1, 2, 2, 1, 0, 0 \mid 30)$, $(2, 1, 3, 0, 1, 0 \mid 40)$, $(2, 2, 1, 0, 0, 1 \mid 36)$; P

B1

row $(-3, -2, -4, 0, 0, 0 \mid 0)$

Iteration 1: most negative entry in the P row is -4 , so z enters; ratios $15, \frac{40}{3}, 36$, least is $\frac{40}{3}$, so s

M1 A1

leaves and the pivot is 3

Row operations: $R_2 \div 3 \rightarrow R_2$; $R_1 - 2R_2 \rightarrow R_1$; $R_3 - R_2 \rightarrow R_3$; $P + 4R_2 \rightarrow P$ (new tableau in the diagram)

A1

Answer: $z = \frac{40}{3}$, $P = \frac{160}{3}$ after the first iteration

4 marks

(c) After the first iteration the P row is $(-\frac{1}{3}, -\frac{2}{3}, 0, 0, \frac{4}{3}, 0)$

B1

y has the most negative entry $(-\frac{2}{3})$, so y enters

E1

Answer: y enters (most negative entry $-\frac{2}{3}$)

2 marks

(d) $P \leq \frac{422}{7} \approx 60.3$, and P is an integer for whole bouquets (integer coefficients), so $P \leq 60$

M1

(12, 4, 4): ribbon $28 \leq 30$, arranging $40 \leq 40$, packing $36 \leq 36$, $P = 36 + 8 + 16 = 60$, so it is optimal

A1

Answer: 12 Spring, 4 Summer, 4 Autumn, $P = £60$

2 marks

Initial tableau (pivot boxed)

b.v.	x	y	z	r	s	t	Value
r	1	2	2	1	0	0	30
s	2	1	3	0	1	0	40
t	2	2	1	0	0	1	36
P	-3	-2	-4	0	0	0	0

After iteration 1

b.v.	x	y	z	r	s	t	Value
r	$-\frac{1}{3}$	$\frac{4}{3}$	0	1	$-\frac{2}{3}$	0	$\frac{10}{3}$
z	$\frac{2}{3}$	$\frac{1}{3}$	1	0	$\frac{1}{3}$	0	$\frac{40}{3}$
t	$\frac{4}{3}$	$\frac{5}{3}$	0	0	$-\frac{1}{3}$	1	$\frac{68}{3}$
P	$-\frac{1}{3}$	$-\frac{2}{3}$	0	0	$\frac{4}{3}$	0	$\frac{160}{3}$

28 (11 marks)

(a) P row entries under r, s, t: $\frac{3}{2}, \frac{7}{2}, 0$

M1 A1

An extra unit of the first resource increases P by $\frac{3}{2}$, while the third resource has slack so extra units are worth nothing

E1

Answer: $\frac{3}{2}, \frac{7}{2}, 0$

3 marks

(b) $y = 8 + \frac{\delta}{2}$, $z = 6 - \frac{\delta}{2}$, $t = 7 + \frac{\delta}{2}$

M1 A1

$z \geq 0$ gives $\delta \leq 12$; $t \geq 0$ gives $\delta \geq -14$; $y \geq 0$ gives $\delta \geq -16$

M1

So $-14 \leq \delta \leq 12$

A1

Answer: $-14 \leq \delta \leq 12$

4 marks

(c) $\delta = 6$ is in the range

B1

$y = 11$, $z = 3$, $t = 10$

M1

$P = 94 + \frac{3}{2}(6) = 103$

A1

Answer: $y = 11$, $z = 3$, $P = 103$

3 marks

(d) The entry under x in the P row is $\frac{1}{2} > 0$, so it is not beneficial to make x

E1

Answer: P row entry for x is positive

1 mark

29 (9 marks)

After the row operations the new value in row r is $b_r' = \frac{b_r}{a_r} \geq 0$

M1 A1

For $i \neq r$: $b_i' = b_i - a_i \frac{b_r}{a_r}$

M1

If $a_i \leq 0$, then $b_i' \geq b_i \geq 0$

A1

If $a_i > 0$, then $\frac{b_r}{a_r} \leq \frac{b_i}{a_i}$ so $a_i \frac{b_r}{a_r} \leq b_i$, so $b_i' \geq 0$

M1 A1

The new value of P is $P' = P - p \frac{b_r}{a_r}$

M1

Since $p < 0$ and $\frac{b_r}{a_r} \geq 0$, $P' \geq P$

A1

Equality only if $b_r = 0$ (degenerate case)

E1

Answer: $b_i' \geq 0$ and $P' \geq P$

9 marks**30 (12 marks)**

(a) P row $(-3, -k, 0, 0 | 0)$

B1

x enters first if $3 > k$, that is $0 < k < 3$

M1 A1

Answer: P row $(-3, -k, 0, 0 | 0)$; $0 < k < 3$

3 marks

(b) Iteration 1: most negative entry in the P row is -5 , so y enters; ratios 10, 5, least is 5, so s leaves and the pivot is 3

M1

Row operations: $R_2 \div 3 \rightarrow R_2$; $R_1 - R_2 \rightarrow R_1$; $P + 5 R_2 \rightarrow P$ (new tableau in the diagram)

A1

Iteration 2: most negative entry in the P row is $-\frac{4}{3}$, so x enters; ratios 3, 15, least is 3, so r leaves

M1

and the pivot is $\frac{5}{3}$

Row operations: $R_1 \div \frac{5}{3} \rightarrow R_1$; $R_2 - \frac{1}{3} R_1 \rightarrow R_2$; $P + \frac{4}{3} R_1 \rightarrow P$ (new tableau in the diagram)

A1

No negative entries remain in the P row, so the solution is optimal: $x = 3$, $y = 4$ with $P = 29$

A1

Answer: $x = 3$, $y = 4$, $P = 29$

5 marks

(c) The final basis is x, y ; the entries are the multipliers u, v with $2u + v = 3$ and $u + 3v = k$

M1

Solving: $u = \frac{(9-k)}{5}$, $v = \frac{(2k-3)}{5}$

A1

Both non-negative: $k \leq 9$ and $k \geq \frac{3}{2}$

M1

So $\frac{3}{2} \leq k \leq 9$

A1

($k = 5$ lies in this range, consistent with part (b))

Answer: $\frac{3}{2} \leq k \leq 9$

4 marks

k = 5: initial

b.v.	x	y	r	s	Value
r	2	1	1	0	10
s	1	3	0	1	15
P	-3	-5	0	0	0

Iteration 1

b.v.	x	y	r	s	Value
r	$\frac{5}{3}$	0	1	$-\frac{1}{3}$	5
y	$\frac{1}{3}$	1	0	$\frac{1}{3}$	5
P	$-\frac{4}{3}$	0	0	$\frac{5}{3}$	25

Iteration 2

b.v.	x	y	r	s	Value
x	1	0	$\frac{3}{5}$	$-\frac{1}{5}$	3
y	0	1	$-\frac{1}{5}$	$\frac{2}{5}$	4
P	0	0	$\frac{4}{5}$	$\frac{7}{5}$	29

Big-M Method

1 (2 marks)

The origin ($x = 0, y = 0$) does not satisfy $x \geq 2$, so it is not feasible

E1

A slack variable for $x \geq 2$ would have to be negative, so there is no obvious starting basic feasible solution

E1

Answer: The origin is infeasible ($x \geq 2$ fails)

2 marks

2 (2 marks)

$$x + y - s = 8$$

B1

$$s \geq 0$$

B1

Answer: $x + y - s = 8, s \geq 0$

2 marks

3 (2 marks)

It gives an initial basic variable ($t = 2$) for the constraint, so that a starting basic solution exists

E1

It must be zero in any solution of the original problem

E1

Answer: Provides a starting basic variable; must end at zero

2 marks

4 (1 mark)

A very large positive number (larger than any other number in the problem), used as a penalty

B1

Answer: A very large positive constant

1 mark

5 (2 marks)

$$P = 3x + 2y - Mt$$

B1

$$P - 3x - 2y + Mt = 0$$

B1

Answer: $P = 3x + 2y - Mt$, i.e. $P - 3x - 2y + Mt = 0$

2 marks

6 (1 mark)

The $\geq$ and $=$ constraints

B1

Answer: $\geq$ and $=$

1 mark

7 (3 marks)

$$x + y + s_1 = 10 \text{ and } x + 3y + s_2 = 18$$

B1

$$x - s_3 + t = 2$$

B1

$$P - 3x - 2y + Mt = 0$$

B1

$$\text{Answer: } x + y + s_1 = 10, x + 3y + s_2 = 18, x - s_3 + t = 2, P - 3x - 2y + Mt = 0$$

3 marks

8 (3 marks)

$$\text{Initial basic variables: } s_1 = 10, s_2 = 18, t = 2$$

B1

The non-basic variables x , y and s_3 are 0

B1

$$t = 2 > 0, \text{ which means that } x \geq 2 \text{ is violated (} x = 0 \text{)}$$

E1

$$\text{Answer: } s_1 = 10, s_2 = 18, t = 2; \text{ infeasible because } t > 0$$

3 marks

9 (3 marks)

$$\text{From the constraint, } t = 2 - x + s_3$$

B1

$$P = 3x + 2y - M(2 - x + s_3) = (3 + M)x + 2y - Ms_3 - 2M$$

M1

$$P - (3 + M)x - 2y + Ms_3 = -2M, \text{ so the P row is } (-3 - M, -2, 0, 0, M, 0 \mid -2M)$$

A1

$$\text{Answer: } P - (3 + M)x - 2y + Ms_3 = -2M$$

3 marks

10 (2 marks)

$$\text{Subtract a surplus variable } s_3 \geq 0 \text{ to turn the inequality into an equation: } x - s_3 = 2$$

E1

$$\text{Add an artificial variable } t \geq 0 \text{ so that there is a basic variable: } x - s_3 + t = 2$$

E1

$$\text{Answer: } x - s_3 = 2 \text{ then add artificial } t$$

2 marks

11 (4 marks)

The most negative entry in the P row is $-3 - M$, since M is large, so x enters

M1 A1

$$\text{Ratios: } \frac{10}{1} = 10, \frac{18}{1} = 18, \frac{2}{1} = 2; \text{ least is 2 so } t \text{ leaves and the pivot is 1}$$

M1 A1

$$\text{Answer: } x \text{ enters, } t \text{ leaves, pivot 1 (ratios 10, 18, 2)}$$

4 marks

12 (4 marks)

$$x = 2, y = 0 \text{ and } P = 6$$

B1 B1

t is no longer basic, so $t = 0$ and the solution is feasible

E1

The P row has a negative entry -3 (under s_3), so it is not optimal

E1

$$\text{Answer: } x = 2, y = 0, P = 6; t \text{ has left; not optimal}$$

4 marks

Initial tableau (pivot boxed)

b.v.	x	y	s_1	s_2	s_3	t	Value
s_1	1	1	1	0	0	0	10
s_2	1	3	0	1	0	0	18
t	1	0	0	0	-1	1	2
P	$-3 - M$	-2	0	0	M	0	$-2M$

After iteration 1

b.v.	x	y	s_1	s_2	s_3	t	Value
s_1	0	1	1	0	1	-1	8
s_2	0	3	0	1	1	-1	16
x	1	0	0	0	-1	1	2
P	0	-2	0	0	-3	$3 + M$	6

13 (2 marks)

An artificial variable is positive, so at least one original constraint cannot be satisfied

E1

The original problem has no feasible solution

E1

$$\text{Answer: No feasible solution}$$

2 marks

14 (4 marks)

Iteration 1: most negative entry in the P row is $-3 - M$, so x enters; ratios 10, 18, 2, least is 2, so t leaves and the pivot is 1

M1

Row operations: $R_1 - R_3 \rightarrow R_1$; $R_2 - R_3 \rightarrow R_2$; $P + (3 + M) R_3 \rightarrow P$ (new tableau in the diagram)

A1

After the iteration: $x = 2$, $y = 0$, $P = 6$ ($t = 0$)

A1

The tableau is shown in the diagram

B1

Answer: $x = 2$, $y = 0$, $P = 6$

4 marks

Initial tableau (pivot boxed)

b.v.	x	y	s_1	s_2	s_3	t	Value
s_1	1	1	1	0	0	0	10
s_2	1	3	0	1	0	0	18
t	1	0	0	0	-1	1	2
P	$-3 - M$	-2	0	0	M	0	$-2M$

After iteration 1

b.v.	x	y	s_1	s_2	s_3	t	Value
s_1	0	1	1	0	1	-1	8
s_2	0	3	0	1	1	-1	16
x	1	0	0	0	-1	1	2
P	0	-2	0	0	-3	$3 + M$	6

15 (6 marks)

Iteration 1: most negative entry in the P row is $-3 - M$, so x enters; ratios 10, 18, 2, least is 2, so t leaves and the pivot is 1

M1

Row operations: $R_1 - R_3 \rightarrow R_1$; $R_2 - R_3 \rightarrow R_2$; $P + (3 + M) R_3 \rightarrow P$ (new tableau in the diagram)

A1

Iteration 2: most negative entry in the P row is -3 , so s_3 enters; ratios 8, 16, -, least is 8, so s_1 leaves and the pivot is 1

M1

Row operations: $R_2 - R_1 \rightarrow R_2$; $R_3 + R_1 \rightarrow R_3$; $P + 3 R_1 \rightarrow P$ (new tableau in the diagram)

A1

No negative entries remain in the P row and the artificial variables are zero: optimal, $x = 10$, $y = 0$ with $P = 30$

A1 A1

Answer: $x = 10$, $y = 0$, $P = 30$

6 marks

Initial

b.v.	x	y	s_1	s_2	s_3	t	Value
s_1	1	1	1	0	0	0	10
s_2	1	3	0	1	0	0	18
t	1	0	0	0	-1	1	2
P	$-3 - M$	-2	0	0	M	0	$-2M$

Iteration 1

b.v.	x	y	s_1	s_2	s_3	t	Value
s_1	0	1	1	0	1	-1	8
s_2	0	3	0	1	1	-1	16
x	1	0	0	0	-1	1	2
P	0	-2	0	0	-3	$3 + M$	6

Iteration 2

b.v.	x	y	s_1	s_2	s_3	t	Value
s_3	0	1	1	0	1	-1	8
s_2	0	2	-1	1	0	0	8
x	1	1	1	0	0	0	10
P	0	1	3	0	0	M	30

16 (5 marks)

$P = -C$: maximise $P = -3x - 5y$

B1

$x + 2y - s_1 + t_1 = 10$, $3x + y - s_2 + t_2 = 15$, $x + y + s_3 = 20$

B1

$P = -3x - 5y - Mt_1 - Mt_2$, so $P + 3x + 5y + Mt_1 + Mt_2 = 0$

B1

Subtract $M \times$ (first row + second row) to eliminate t_1 and t_2

M1

P row: $(3 - 4M, 5 - 3M, M, M, 0, 0, 0 | -25M)$

A1

Answer: P row $(3 - 4M, 5 - 3M, M, M, 0, 0, 0 | -25M)$

5 marks

17 (6 marks)

$x + y + t = 8$ and $2x + y + s = 12$

B1

Iteration 1: most negative entry in the P row is $-3 - M$, so x enters; ratios 8, 6, least is 6, so s leaves and the pivot is 2

M1

Row operations: $R_2 \div 2 \rightarrow R_2$; $R_1 - R_2 \rightarrow R_1$; $P + (3 + M) R_2 \rightarrow P$ (new tableau in the diagram)

A1

Iteration 2: most negative entry in the P row is $-\frac{1}{2} - \frac{1}{2}M$, so y enters; ratios 4, 12, least is 4, so t leaves and the pivot is $\frac{1}{2}$

M1

leaves and the pivot is $\frac{1}{2}$

Row operations: $R_1 \div \frac{1}{2} \rightarrow R_1$; $R_2 - \frac{1}{2} R_1 \rightarrow R_2$; $P + (\frac{1}{2} + \frac{1}{2}M) R_1 \rightarrow P$ (new tableau in the diagram)

A1

No negative entries remain in the P row and the artificial variables are zero: optimal, $x = 4$, $y = 4$ with $P = 20$

A1

Answer: $x = 4$, $y = 4$, $P = 20$

6 marks

b.v.	x	y	s	t	Value
t	1	1	0	1	8
s	2	1	1	0	12
P	$-3 - M$	$-2 - M$	0	0	$-8M$

b.v.	x	y	s	t	Value
t	0	$\frac{1}{2}$	$-\frac{1}{2}$	1	2
x	1	$\frac{1}{2}$	$\frac{1}{2}$	0	6
P	0	$-\frac{1}{2} - \frac{1}{2}M$	$\frac{1}{2} + \frac{1}{2}M$	0	$18 - 2M$

b.v.	x	y	s	t	Value
y	0	1	-1	2	4
x	1	0	1	-1	4
P	0	0	1	$1 + M$	20

18 (5 marks)

$x + y + s_1 = 12$, $x + 2y - s_2 + t = 8$, $2x + y + s_3 = 18$

B1

Rows (x, y, s_1, s_2, s_3, t | value): $(1, 1, 1, 0, 0, 0, 12)$ $(1, 2, 0, -1, 0, 1, 8)$ $(2, 1, 0, 0, 1, 0, 18)$

B1 B1

$P - 2x - 3y + Mt = 0$ and $t = 8 - x - 2y + s_2$: P row $(-2 - M, -3 - 2M, 0, M, 0, 0, -8M)$

M1 A1

Answer: P row $(-2 - M, -3 - 2M, 0, M, 0, 0, -8M)$

5 marks

b.v.	x	y	s ₁	s ₂	s ₃	t	Value
s ₁	1	1	1	0	0	0	12
t	1	2	0	-1	0	1	8
s ₃	2	1	0	0	1	0	18
P	$-2 - M$	$-3 - 2M$	0	M	0	0	$-8M$

19 (5 marks)

s_3 is the surplus in $x - s_3 = 2$ ($t = 0$): $x = 10$ exceeds the minimum 2 by 8

E1 B1

s_1 is non-basic, so $x + y \leq 10$ is binding ($10 + 0 = 10$)

M1 A1

$x + 3y = 10 < 18$: $s_2 = 8$, so not binding; $x \geq 2$ is not binding

E1

Answer: $s_3 = 8$: x is 8 above its minimum; only $x + y \leq 10$ is binding

5 marks

20 (6 marks)

$x + y + s_1 = 4$, $x + y - s_2 + t = 6$, with initial P row $(-1 - M, -1 - M, 0, M, 0 \mid -6M)$

B1 B1

Iteration 1: most negative entry in the P row is $-1 - M$, so x enters; ratios 4, 6, least is 4, so s_1 leaves and the pivot is 1

M1

Row operations: $R_2 - R_1 \rightarrow R_2$; $P + (1 + M) R_1 \rightarrow P$ (new tableau in the diagram)

A1

No negative entries remain in the P row, but $t = 2$ is still basic with value 2 ($P = 4 - 2M$)

M1

A positive artificial variable at the optimum means that the original constraints cannot all be satisfied: no feasible solution

E1

Answer: No feasible solution

6 marks

b.v.	x	y	s_1	s_2	t	Value
s_1	1	1	1	0	0	4
t	1	1	0	-1	1	6
P	$-1 - M$	$-1 - M$	0	M	0	$-6M$

b.v.	x	y	s_1	s_2	t	Value
x	1	1	1	0	0	4
t	0	0	-1	-1	1	2
P	0	0	$1 + M$	M	0	$4 - 2M$

21 (7 marks)

Iteration 1: most negative entry in the P row is $-3 - 2M$, so y enters; ratios 12, 4, 18, least is 4, so t leaves and the pivot is 2

M1

Row operations: $R_2 \div 2 \rightarrow R_2$; $R_1 - R_2 \rightarrow R_1$; $R_3 - R_2 \rightarrow R_3$; $P + (3 + 2M) R_2 \rightarrow P$ (new tableau in the diagram)

A1

Iteration 2: most negative entry in the P row is $-\frac{3}{2}$, so s_2 enters; ratios 16, $-\infty$, 28, least is 16, so s_1

M1

leaves and the pivot is $\frac{1}{2}$

Row operations: $R_1 \div \frac{1}{2} \rightarrow R_1$; $R_2 + \frac{1}{2} R_1 \rightarrow R_2$; $R_3 - \frac{1}{2} R_1 \rightarrow R_3$; $P + \frac{3}{2} R_1 \rightarrow P$ (new tableau in the diagram)

A1

No negative entries remain in the P row and the artificial variables are zero: optimal, $x = 0$, $y = 12$ with $P = 36$

A1 A1

$s_2 = 16$: $x + 2y = 24$ exceeds 8 by 16

B1

Answer: $x = 0$, $y = 12$, $P = 36$; the second constraint is exceeded by 16

7 marks

b.v.	x	y	s ₁	s ₂	s ₃	t	Value
s ₁	1	1	1	0	0	0	12
t	1	2	0	-1	0	1	8
s ₃	2	1	0	0	1	0	18
P	-2 - M	3 - 2M	0	M	0	0	-8M

b.v.	x	y	s ₁	s ₂	s ₃	t	Value
s ₁	$\frac{1}{2}$	0	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	8
y	$\frac{1}{2}$	1	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	4
s ₃	$\frac{3}{2}$	0	0	$\frac{1}{2}$	1	$-\frac{1}{2}$	14
P	$-\frac{1}{2}$	0	0	$-\frac{3}{2}$	0	$\frac{3}{2} + M$	12

b.v.	x	y	s ₁	s ₂	s ₃	t	Value
s ₂	1	0	2	1	0	-1	16
y	1	1	1	0	0	0	12
s ₃	1	0	-1	0	1	0	6
P	1	0	3	0	0	M	36

22 (7 marks)

Iteration 1: most negative entry in the P row is $3 - 4M$, so x enters; ratios 10, 5, 20, least is 5, so t_2 leaves and the pivot is 3

M1

Row operations: $R_2 \div 3 \rightarrow R_2$; $R_1 - R_2 \rightarrow R_1$; $R_3 - R_2 \rightarrow R_3$; $P + (-3 + 4M) R_2 \rightarrow P$ (new tableau in the diagram)

A1

Iteration 2: most negative entry in the P row is $4 - \frac{5}{3}M$, so y enters; ratios 3, 15, $\frac{45}{2}$, least is 3, so t_1 leaves and the pivot is $\frac{5}{3}$

M1

Row operations: $R_1 \div \frac{5}{3} \rightarrow R_1$; $R_2 - \frac{1}{3} R_1 \rightarrow R_2$; $R_3 - \frac{2}{3} R_1 \rightarrow R_3$; $P + (-4 + \frac{5}{3}M) R_1 \rightarrow P$

A1

(new tableau in the diagram)

No negative entries in the P row and the artificial variables are zero: optimal, $x = 4$, $y = 3$ with $C = 27$

A1 A1

$s_3 = 13$, so $x + y = 7$ is 13 below the limit 20

B1

Answer: $x = 4$, $y = 3$, $C = 27$; 13 unused

7 marks

b.v.	x	y	s ₁	s ₂	s ₃	t ₁	t ₂	Value
t ₁	1	2	-1	0	0	1	0	10
t ₂	3	1	0	-1	0	0	1	15
s ₃	1	1	0	0	1	0	0	20
P	3 - 4M	4 - 3M	M	M	0	0	0	-25M

b.v.	x	y	s ₁	s ₂	s ₃	t ₁	t ₂	Value
t ₁	0	$\frac{5}{3}$	-1	$\frac{1}{3}$	0	1	$-\frac{1}{3}$	5
x	1	$\frac{1}{3}$	0	$-\frac{1}{3}$	0	0	$\frac{1}{3}$	5
s ₃	0	$\frac{2}{3}$	0	$\frac{1}{3}$	1	0	$-\frac{1}{3}$	15
P	0	$4 - \frac{5}{3}M$	M	$1 - \frac{1}{3}M$	0	0	$-1 + \frac{4}{3}M$	$15 - 5M$

b.v.	x	y	s ₁	s ₂	s ₃	t ₁	t ₂	Value
y	0	1	$-\frac{3}{5}$	$\frac{1}{5}$	0	$\frac{3}{5}$	$-\frac{1}{5}$	3
x	1	0	$\frac{1}{5}$	$-\frac{2}{5}$	0	$-\frac{1}{5}$	$\frac{2}{5}$	4
s ₃	0	0	$\frac{2}{5}$	$\frac{1}{5}$	1	$-\frac{2}{5}$	$-\frac{1}{5}$	13
P	0	0	$\frac{12}{5}$	$\frac{1}{5}$	0	$-\frac{12}{5} + M$	$\frac{1}{5} + M$	-27

23 (7 marks)

Iteration 1: most negative entry in the P row is $45 - 8M$, so y enters; ratios 12, 6, least is 6, so t₂ leaves and the pivot is 5

M1

Row operations: $R_2 \div 5 \rightarrow R_2$; $R_1 - 3 R_2 \rightarrow R_1$; $P + (-45 + 8M) R_2 \rightarrow P$ (new tableau in the diagram)

A1

Iteration 2: most negative entry in the P row is $12 - \frac{14}{5}M$, so x enters; ratios $\frac{45}{7}$, 15, least is $\frac{45}{7}$, so t₁ leaves and the pivot is $\frac{14}{5}$

M1

Row operations: $R_1 \div \frac{14}{5} \rightarrow R_1$; $R_2 - \frac{2}{5} R_1 \rightarrow R_2$; $P + (-12 + \frac{14}{5}M) R_1 \rightarrow P$ (new tableau in the diagram)

A1

No negative entries in the P row and the artificial variables are zero: optimal, $x = \frac{45}{7}$, $y = \frac{24}{7}$ with

A1 A1

$$C = \frac{2430}{7}$$

Cost = $2430/7 \approx 347.1$ pence; both nutrient constraints are exactly met

B1

Answer: $x = \frac{45}{7}$, $y = \frac{24}{7}$, $C = \frac{2430}{7}$

7 marks

b.v.	x	y	s ₁	s ₂	t ₁	t ₂	Value
t ₁	4	3	-1	0	1	0	36
t ₂	2	5	0	-1	0	1	30
P	30 - 6M	45 - 8M	M	M	0	0	-66M

b.v.	x	y	s ₁	s ₂	t ₁	t ₂	Value
t ₁	$\frac{14}{5}$	0	-1	$\frac{3}{5}$	1	$-\frac{3}{5}$	18
y	$\frac{2}{5}$	1	0	$-\frac{1}{5}$	0	$\frac{1}{5}$	6
P	$12 - \frac{14}{5}M$	0	M	$9 - \frac{3}{5}M$	0	$-9 + \frac{8}{5}M$	$70 - 18M$

b.v.	x	y	s ₁	s ₂	t ₁	t ₂	Value
x	1	0	$-\frac{5}{14}$	$\frac{3}{14}$	$\frac{5}{14}$	$-\frac{3}{14}$	$\frac{45}{7}$
y	0	1	$\frac{1}{7}$	$-\frac{2}{7}$	$-\frac{1}{7}$	$\frac{2}{7}$	$\frac{24}{7}$
P	0	0	$\frac{30}{7}$	$\frac{45}{7} - \frac{30}{7}M + \frac{15}{7}M$	$-\frac{24}{7} + \frac{15}{7}M$	$-\frac{24}{7} + \frac{30}{7}M$	$\frac{2430}{7}$

24 (7 marks)

Iteration 3: most negative entry in the P row is -5, so s₂ enters; ratios 9, -, -, 8, least is 8, so s₄ leaves and the pivot is 2

M1

Row operations: $R_4 \div 2 \rightarrow R_4$; $R_1 - R_4 \rightarrow R_1$; $R_2 + R_4 \rightarrow R_2$; $P + 5 R_4 \rightarrow P$ (new tableau in the diagram)

A1

Iteration 4: most negative entry in the P row is $-\frac{3}{2}$, so s₃ enters; ratios 2, 22, -, 16, least is 2, so s₁ leaves and the pivot is $\frac{1}{2}$

M1

Row operations: $R_1 \div \frac{1}{2} \rightarrow R_1$; $R_2 - \frac{1}{2} R_1 \rightarrow R_2$; $R_3 + R_1 \rightarrow R_3$; $R_4 - \frac{1}{2} R_1 \rightarrow R_4$; $P + \frac{3}{2} R_1 \rightarrow P$ (new tableau in the diagram)

A1

No negative entries remain in the P row and the artificial variables are zero: optimal, x = 10, y = 4 with P = 66

A1 A1

Both $x \geq 3$ and $y \geq 2$ are satisfied with room to spare ($s_2 = 7$, $s_3 = 2$)

B1

Answer: x = 10, y = 4, P = 66

7 marks

Given (after iteration 2)

b.v.	x	y	s ₁	s ₂	s ₃	s ₄	t ₁	t ₂	Value
s ₁	0	0	1	1	1	0	-1	-1	9
x	1	0	0	-1	0	0	1	0	3
y	0	1	0	0	-1	0	0	1	2
s ₄	0	0	0	2	1	1	-2	-1	16
P	0	0	0	-5	-4	0	5 + M	4 + M	23

Iteration 3

b.v.	x	y	s ₁	s ₂	s ₃	s ₄	t ₁	t ₂	Value
s ₁	0	0	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	1
x	1	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	11
y	0	1	0	0	-1	0	0	1	2
s ₂	0	0	0	1	$\frac{1}{2}$	$\frac{1}{2}$	-1	$-\frac{1}{2}$	8
P	0	0	0	0	$-\frac{3}{2}$	$\frac{5}{2}$	M	$\frac{3}{2} + M$	63

Iteration 4

b.v.	x	y	s ₁	s ₂	s ₃	s ₄	t ₁	t ₂	Value
s ₃	0	0	2	0	1	-1	0	-1	2
x	1	0	-1	0	0	1	0	0	10
y	0	1	2	0	0	-1	0	0	4
s ₂	0	0	-1	1	0	1	-1	0	7
P	0	0	3	0	0	1	M	M	66

25 (7 marks)

Iteration 1: most negative entry in the P row is $-3 - 3M$, so z enters; ratios 12, -, 3, least is 3, so t_2 leaves and the pivot is 2

M1

Row operations: $R_3 \div 2 \rightarrow R_3$; $R_1 - R_3 \rightarrow R_1$; $P + (3 + 3M) R_3 \rightarrow P$ (new tableau in the diagram)

A1

Iteration 2: most negative entry in the P row is $-2 - M$, so x enters; ratios 9, 8, -, least is 8, so s_1 leaves and the pivot is 2

M1

Row operations: $R_2 \div 2 \rightarrow R_2$; $R_1 - R_2 \rightarrow R_1$; $P + (2 + M) R_2 \rightarrow P$ (new tableau in the diagram)

A1

Iteration 3: most negative entry in the P row is $-\frac{3}{2} - \frac{1}{2}M$, so s_2 enters; ratios 2, -, -, least is 2, so t_1 leaves and the pivot is $\frac{1}{2}$

M1

Row operations: $R_1 \div \frac{1}{2} \rightarrow R_1$; $R_3 + \frac{1}{2} R_1 \rightarrow R_3$; $P + (\frac{3}{2} + \frac{1}{2}M) R_1 \rightarrow P$ (new tableau in the diagram)

A1

After iteration 3: $x = 8$, $y = 0$, $z = 4$, $t_1 = t_2 = 0$, $P = 28$

A1

Answer: $x = 8$, $y = 0$, $z = 4$, $t_1 = t_2 = 0$, $P = 28$

7 marks

Initial

b.v.	x	y	z	s ₁	s ₂	t ₁	t ₂	Value
t ₁	1	1	1	0	0	1	0	12
s ₁	2	1	0	1	0	0	0	16
t ₂	0	1	2	0	-1	0	1	6
P	-2 - M	-2 - 2M	-3 - 3M	0	M	0	0	-18M

Iteration 1

b.v.	x	y	z	s ₁	s ₂	t ₁	t ₂	Value
t ₁	1	$\frac{1}{2}$	0	0	$\frac{1}{2}$	1	$-\frac{1}{2}$	9
s ₁	2	1	0	1	0	0	0	16
z	0	$\frac{1}{2}$	1	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	3
P	$-2 - \frac{1}{2}M$	$-\frac{1}{2}M$	0	0	$-\frac{3}{2} - \frac{1}{2}M$	0	$\frac{3}{2} + \frac{1}{2}M$	$9 - 9M$

Iteration 2

b.v.	x	y	z	s ₁	s ₂	t ₁	t ₂	Value
t ₁	0	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	1	$-\frac{1}{2}$	1
x	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	0	8
z	0	$\frac{1}{2}$	1	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	3
P	0	$\frac{3}{2}$	0	$1 + \frac{1}{2}M$	$-\frac{1}{2}M$	0	$\frac{3}{2} + \frac{1}{2}M$	$25 - M$

Iteration 3

b.v.	x	y	z	s ₁	s ₂	t ₁	t ₂	Value
s ₂	0	0	0	-1	1	2	-1	2
x	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	0	8
z	0	$\frac{1}{2}$	1	$-\frac{1}{2}$	0	1	0	4
P	0	$\frac{3}{2}$	0	$-\frac{1}{2}$	0	$3 + M$	M	28

26 (6 marks)

The student has used +M where -M is needed (and the reverse for s₃), and the value should be -2M

E1

From $x - s_3 + t = 2$: $t = 2 - x + s_3$

M1

$P = 3x + 2y - Mt = (3 + M)x + 2y - Ms_3 - 2M$

M1 A1

Correct row: $(-3 - M, -2, 0, 0, M, 0 \mid -2M)$

A1

With the student's row the x entry $-3 + M$ is positive for large M, so x would not be chosen and the algorithm would wrongly look for another column (y), failing to remove t first

E1

Answer: Correct P row $(-3 - M, -2, 0, 0, M, 0 \mid -2M)$

6 marks

27 (12 marks)

(a) Minimise $C = 6x + 5y$

B1

$2x + y \geq 20$ and $x + 3y \geq 30$

B1

$x + y \geq 14$, $x, y \geq 0$

B1

Answer: Minimise $C = 6x + 5y$ subject to $2x + y \geq 20$, $x + 3y \geq 30$, $x + y \geq 14$

3 marks

(b) Surplus variables s₁, s₂, s₃ and artificial variables t₁, t₂, t₃ are added: $2x + y - s_1 + t_1 = 20$, etc.

B1

$P = -6x - 5y - M(t_1 + t_2 + t_3)$, so $P + 6x + 5y + M(t_1 + t_2 + t_3) = 0$

M1

Subtract M times each constraint row: x: $6 - M(2 + 1 + 1) = 6 - 4M$; y: $5 - M(1 + 3 + 1) = 5 - 5M$

M1

s₁, s₂, s₃: +M each; value $-M(20 + 30 + 14) = -64M$

A1

Answer: P row ($6 - 4M, 5 - 5M, M, M, M, 0, 0, 0 \mid -64M$)

4 marks

(c) Iteration 1: most negative entry in the P row is $5 - 5M$, so y enters; ratios 20, 10, 14, least is 10, so t_2 leaves and the pivot is 3

M1 A1

Row operations: $R_2 \div 3 \rightarrow R_2$; $R_1 - R_2 \rightarrow R_1$; $R_3 - R_2 \rightarrow R_3$; $P + (-5 + 5M) R_2 \rightarrow P$ (new tableau in the diagram)

A1

After the iteration $y = 10$ and t_2 has left the basis; $P = -50 - 14M$

Answer: $y = 10, t_2$ leaves

3 marks

(d) Minimum cost $6(6) + 5(8) = 76$ pence

B1

At (6, 8): $12 + 8 = 20$, $6 + 24 = 30$, $6 + 8 = 14$, so all three constraints hold with equality (surplus variables all zero)

E1

Answer: 76 pence; all three constraints are exactly met

2 marks

Initial (pivot boxed)

b.v.	x	y	s_1	s_2	s_3	t_1	t_2	t_3	Value
t_1	2	1	-1	0	0	1	0	0	20
t_2	1	3	0	-1	0	0	1	0	30
t_3	1	1	0	0	-1	0	0	1	14
P	$6 - 4M$	$5 - 5M$	M	M	M	0	0	0	$-64M$

After iteration 1

b.v.	x	y	s_1	s_2	s_3	t_1	t_2	t_3	Value
t_1	$\frac{5}{3}$	0	-1	$\frac{1}{3}$	0	1	$-\frac{1}{3}$	0	10
y	$\frac{1}{3}$	1	0	$-\frac{1}{3}$	0	0	$\frac{1}{3}$	0	10
t_3	$\frac{2}{3}$	0	0	$\frac{1}{3}$	-1	0	$-\frac{1}{3}$	1	4
P	$\frac{13}{3} - \frac{7}{3}M$	0	M	$\frac{5}{3} - \frac{2}{3}M$	M	0	$-\frac{5}{3} + \frac{5}{3}M$	0	$-50 - 14M$

28 (9 marks)

True problem: $y \geq 3$, so $x + y \leq 10$ and $P = 5x + 2y$ is largest when $y = 3, x = 7$: $P = 35 + 6 = 41$

M1 A1

The point $x = 10, y = 0, t = 3$ (so $y - s + t = 3$ is satisfied with $y = 0$) is feasible for the modified problem with modified objective $50 - 3M$

M1 A1

At the true optimum $t = 0$, the objective is 41

B1

The big-M method gives the correct answer only if $41 > 50 - 3M$

M1

$3M > 9$, so $M > 3$

A1

If $M < 3$ the penalty is too small: the algorithm would keep $t > 0$ and wrongly report (10, 0), which violates $y \geq 3$; so M must exceed the possible gain from using the artificial variable

E1 A1

Answer: (i) $x = 7, y = 3, P = 41$; (ii) $M > 3$

9 marks

29 (9 marks)

$a_1x_1 + a_2x_2 - s + t = b$, so $t = b - a_1x_1 - a_2x_2 + s$

M1 A1

$P = c_1x_1 + c_2x_2 - Mt$

B1

$= (c_1 + Ma_1)x_1 + (c_2 + Ma_2)x_2 - Ms - Mb$

M1 A1

So $P - (c_1 + Ma_1)x_1 - (c_2 + Ma_2)x_2 + Ms = -Mb$

A1

P row ($-c_1 - Ma_1, -c_2 - Ma_2, M, 0 \mid -Mb$)

A1

x_1 enters when $c_1 + Ma_1 \geq c_2 + Ma_2$ (and is positive); for large M this means $a_1 > a_2$, or if $a_1 =$

M1 A1

$a_2, c_1 > c_2$

Answer: P row ($-c_1 - Ma_1, -c_2 - Ma_2, M, 0 \mid -Mb$); x_1 enters if $a_1 > a_2$ (or $a_1 = a_2$ and $c_1 > c_2$)

9 marks

30 (9 marks)

The origin does not satisfy $x \geq 3$ or $y \geq 2$, so the starting point is infeasible ($t_1 = 3$, $t_2 = 2$)

E1

Iteration 1 makes $x = 3$, so $x \geq 3$ is satisfied and $t_1 = 0$

M1 A1

Iteration 2 makes $y = 2$, so $t_2 = 0$: the point (3, 2) is the first feasible vertex

A1

$P = 15 + 8 = 23$ at (3, 2); $P = 55 + 8 = 63$ at (11, 2); $P = 50 + 16 = 66$ at (10, 4)

M1 A1 A1

The algorithm is mainly reaching feasibility in the first two iterations, with $5x + 4y$ equal to 0, 15 and then 23

E1

At the first tableau the modified objective is $0 - M(3 + 2) = -5M$

B1

Answer: (3, 2) is the first feasible vertex; $P = 23, 63, 66$; modified objective $-5M$

9 marks

Two-Stage Simplex Method

1 (1 mark)

To find a basic feasible solution of the original problem (or show there is none) by minimising the sum of the artificial variables

B1

Answer: To find a starting feasible solution

1 mark

2 (1 mark)

Minimise $I = t_1 + t_2$

B1

Answer: Minimise $I = t_1 + t_2$

1 mark

3 (1 mark)

$I = 0$ (all artificial variables are zero)

B1

Answer: $I = 0$

1 mark

4 (1 mark)

They are deleted, and the simplex algorithm continues on the P row (Phase 2)

B1

Answer: Delete them; continue with the P row

1 mark

5 (1 mark)

No: the pivot column in Phase 1 is chosen using the I row

B1

Answer: No, the I row is used

1 mark

6 (1 mark)

The $\geq$ and $=$ constraints

B1

Answer: $\geq$ and $=$

1 mark

7 (3 marks)

$x + y + s_1 = 10$ and $x + 3y + s_2 = 18$

B1

$x - s_3 + t = 2$

B1

Phase 1: minimise $I = t$ (equivalently maximise $I = -t$)

B1

Answer: $x + y + s_1 = 10$, $x + 3y + s_2 = 18$, $x - s_3 + t = 2$; minimise $I = t$

3 marks

8 (3 marks)

$t = 2 - x + s_3$

B1

$I = -t = -2 + x - s_3$

M1

$I - x + s_3 = -2$, so the row is $(-1, 0, 0, 0, 1, 0 \mid -2)$

A1

Answer: I row $(-1, 0, 0, 0, 1, 0 \mid -2)$

3 marks

9 (3 marks)

Iteration 1: most negative entry in the I row is -1 , so x enters; ratios 10, 18, 2, least is 2, so t leaves and the pivot is 1

M1

Row operations: $R_1 - R_3 \rightarrow R_1$; $R_2 - R_3 \rightarrow R_2$; $P + 3 R_3 \rightarrow P$; $I + R_3 \rightarrow I$ (new tableau in the diagram)

A1

$I = 0$ in the new tableau, so Phase 1 is complete ($t = 0$, $x = 2$)

B1

Answer: Pivot 1 in the x column, t leaves; $I = 0$ so Phase 1 is complete

3 marks

Initial (pivot boxed)

b.v.	x	y	s ₁	s ₂	s ₃	t	Value
s ₁	1	1	1	0	0	0	10
s ₂	1	3	0	1	0	0	18
t	1	0	0	0	-1	1	2
P	-3	-2	0	0	0	0	0
I	-1	0	0	0	1	0	-2

After Phase 1

b.v.	x	y	s ₁	s ₂	s ₃	t	Value
s ₁	0	1	1	0	1	-1	8
s ₂	0	3	0	1	1	-1	16
x	1	0	0	0	-1	1	2
P	0	-2	0	0	-3	3	6
I	0	0	0	0	0	1	0

10 (2 marks)

$$x = 2, y = 0, s_1 = 8, s_2 = 16, s_3 = 0$$

B1

$$P = 3(2) + 0 = 6$$

B1

Answer: x = 2, y = 0, P = 6

2 marks

11 (2 marks)

The sum of the artificial variables cannot be reduced to zero: $I = -2$ means that $t_1 + t_2 = 2 > 0$

E1

So the original constraints cannot all be satisfied: there is no feasible solution

E1

Answer: No feasible solution

2 marks

12 (2 marks)

It avoids the use of a large number M, so there is no need to choose M and no arithmetic with M

E1

Each phase uses numerical entries only

E1

Answer: No M needed

2 marks

13 (3 marks)

Column x: the only negative entry in the I row is -1 (under x)

B1

Phase 1 aims to remove the artificial variables, so it is the I row that decides

E1

Smallest ratio 2, so the third row (t) is the pivot row

B1

Answer: Column x, row t

3 marks

14 (6 marks)

Iteration 1: most negative entry in the I row is -1 , so x enters; ratios 10, 18, 2, least is 2, so t leaves and the pivot is 1

M1

Row operations: $R_1 - R_3 \rightarrow R_1$; $R_2 - R_3 \rightarrow R_2$; $P + 3 R_3 \rightarrow P$; $I + R_3 \rightarrow I$ (new tableau in the diagram)

A1

$I = 0$, so all artificial variables are zero and the problem is feasible; delete the artificial columns and the I row

E1

Iteration 1: most negative entry in the P row is -3 , so s_3 enters; ratios 8, 16, $-$, least is 8, so s_1 leaves and the pivot is 1

M1

Row operations: $R_2 - R_1 \rightarrow R_2$; $R_3 + R_1 \rightarrow R_3$; $P + 3 R_1 \rightarrow P$ (new tableau in the diagram)

A1

No negative entries in the P row: optimal, $x = 10$, $y = 0$ with $P = 30$

A1

Answer: x = 10, y = 0, P = 30

6 marks

Phase 1: initial

b.v.	x	y	s ₁	s ₂	s ₃	t	Value
s ₁	1	1	1	0	0	0	10
s ₂	1	3	0	1	0	0	18
t	1	0	0	0	-1	1	2
P	-3	-2	0	0	0	0	0
I	-1	0	0	0	1	0	-2

Phase 1: end

b.v.	x	y	s ₁	s ₂	s ₃	t	Value
s ₁	0	1	1	0	1	-1	8
s ₂	0	3	0	1	1	-1	16
x	1	0	0	0	-1	1	2
P	0	-2	0	0	-3	3	6
I	0	0	0	0	0	1	0

Phase 2: start (after deleting t and I)

b.v.	x	y	s ₁	s ₂	s ₃	Value
s ₁	0	1	1	0	1	8
s ₂	0	3	0	1	1	16
x	1	0	0	0	-1	2
P	0	-2	0	0	-3	6

Phase 2: end

b.v.	x	y	s ₁	s ₂	s ₃	Value
s ₃	0	1	1	0	1	8
s ₂	0	2	-1	1	0	8
x	1	1	1	0	0	10
P	0	1	3	0	0	30

15 (6 marks)

$$x + y + s_1 = 12, x + 2y - s_2 + t = 8, 2x + y + s_3 = 18$$

B1

Constraint rows (x, y, s₁, s₂, s₃, t | value): (1, 1, 1, 0, 0, 0, 12) (1, 2, 0, -1, 0, 1, 8) (2, 1, 0, 0, 1, 0, 18)

B1 B1

P row (-2, -3, 0, 0, 0, 0, 0)

B1

I row (-1, -2, 0, 1, 0, 0, -8) (from $I = -t = -8 + x + 2y - s_2$)

M1 A1

Answer: I row (-1, -2, 0, 1, 0, 0, -8)

6 marks

b.v.	x	y	s ₁	s ₂	s ₃	t	Value
s ₁	1	1	1	0	0	0	12
t	1	2	0	-1	0	1	8
s ₃	2	1	0	0	1	0	18
P	-2	-3	0	0	0	0	0
I	-1	-2	0	1	0	0	-8

16 (5 marks)

$$x + y + t = 8, 2x + y + s = 12; \text{ P row } (-3, -2, 0, 0 | 0); \text{ I row } (-1, -1, 0, 0 | -8)$$

B1

Iteration 1: most negative entry in the I row is -1, so x enters; ratios 8, 6, least is 6, so s leaves and the pivot is 2

M1

Row operations: $R_2 \div 2 \rightarrow R_2$; $R_1 - R_2 \rightarrow R_1$; $P + 3 R_2 \rightarrow P$; $I + R_2 \rightarrow I$ (new tableau in the diagram)

A1

$I = 0$, so all artificial variables are zero and the problem is feasible; delete the artificial columns and the I row

E1

No negative entries in the P row: optimal, $x = 4, y = 4$ with $P = 20$

A1

Answer: $x = 4, y = 4, P = 20$

5 marks

b.v.	x	y	s	t	Value
t	1	1	0	1	8
s	2	1	1	0	12
P	-3	-2	0	0	0
I	-1	-1	0	0	-8

b.v.	x	y	s	t	Value
t	0	$\frac{1}{2}$	$-\frac{1}{2}$	1	2
x	1	$\frac{1}{2}$	$\frac{1}{2}$	0	6
P	0	$-\frac{1}{2}$	$\frac{3}{2}$	0	18
I	0	$-\frac{1}{2}$	$\frac{1}{2}$	0	-2

b.v.	x	y	s	t	Value
y	0	1	-1	2	4
x	1	0	1	-1	4
P	0	0	1	1	20
I	0	0	0	1	0

17 (6 marks)

$x + y + s_1 = 4$, $x + y - s_2 + t = 6$; I row $(-1, -1, 0, 1, 0 \mid -6)$

B1 B1

Iteration 1: most negative entry in the I row is -1 , so x enters; ratios 4, 6, least is 4, so s_1 leaves and the pivot is 1

M1

Row operations: $R_2 - R_1 \rightarrow R_2$; $P + R_1 \rightarrow P$; $I + R_1 \rightarrow I$ (new tableau in the diagram)

A1

At the end of Phase 1 the I row has no negative entries but $I = -2 \neq 0$

M1

The artificial variable is still positive: no feasible solution

E1

Answer: No feasible solution

6 marks

b.v.	x	y	s_1	s_2	t	Value
s_1	1	1	1	0	0	4
t	1	1	0	-1	1	6
P	-1	-1	0	0	0	0
I	-1	-1	0	1	0	-6

b.v.	x	y	s_1	s_2	t	Value
x	1	1	1	0	0	4
t	0	0	-1	-1	1	2
P	0	0	1	0	0	4
I	0	0	1	1	0	-2

18 (6 marks)

Maximise $P = -3x - 5y$

B1

$x + 2y - s_1 + t_1 = 10$, $3x + y - s_2 + t_2 = 15$, $x + y + s_3 = 20$

B1

Constraint rows: $(1, 2, -1, 0, 0, 1, 0, 10)$ $(3, 1, 0, -1, 0, 0, 1, 15)$ $(1, 1, 0, 0, 1, 0, 0, 20)$

B1

P row $(3, 5, 0, 0, 0, 0, 0, 0)$

B1

$I = -t_1 - t_2$: I row $(-4, -3, 1, 1, 0, 0, 0, -25)$

M1 A1

Answer: P row $(3, 5, 0, 0, 0, 0, 0, 0)$; I row $(-4, -3, 1, 1, 0, 0, 0, -25)$

6 marks

b.v.	x	y	s ₁	s ₂	s ₃	t ₁	t ₂	Value
t ₁	1	2	-1	0	0	1	0	10
t ₂	3	1	0	-1	0	0	1	15
s ₃	1	1	0	0	1	0	0	20
P	3	5	0	0	0	0	0	0
I	-4	-3	1	1	0	0	0	-25

19 (6 marks)

Iteration 1: most negative entry in the I row is -4, so x enters; ratios 10, 5, 20, least is 5, so t₂ leaves and the pivot is 3

M1

Row operations: $R_2 \div 3 \rightarrow R_2$; $R_1 - R_2 \rightarrow R_1$; $R_3 - R_2 \rightarrow R_3$; $P - 3R_2 \rightarrow P$; $I + 4R_2 \rightarrow I$ (new tableau in the diagram)

A1

Iteration 2: most negative entry in the I row is $-\frac{5}{3}$, so y enters; ratios 3, 15, $\frac{45}{2}$, least is 3, so t₁

M1

leaves and the pivot is $\frac{5}{3}$

Row operations: $R_1 \div \frac{5}{3} \rightarrow R_1$; $R_2 - \frac{1}{3}R_1 \rightarrow R_2$; $R_3 - \frac{2}{3}R_1 \rightarrow R_3$; $P - 4R_1 \rightarrow P$; $I + \frac{5}{3}R_1 \rightarrow I$

A1

(new tableau in the diagram)

I = 0, so all artificial variables are zero and the problem is feasible; delete the artificial columns and the I row

E1

Basic feasible solution: x = 4, y = 3, s₃ = 13 (s₁ = s₂ = 0)

B1

Answer: x = 4, y = 3, s₃ = 13

6 marks**Initial**

b.v.	x	y	s ₁	s ₂	s ₃	t ₁	t ₂	Value
t ₁	1	2	-1	0	0	1	0	10
t ₂	3	1	0	-1	0	0	1	15
s ₃	1	1	0	0	1	0	0	20
P	3	5	0	0	0	0	0	0
I	-4	-3	1	1	0	0	0	-25

Iteration 1

b.v.	x	y	s ₁	s ₂	s ₃	t ₁	t ₂	Value
t ₁	0	$\frac{5}{3}$	-1	$\frac{1}{3}$	0	1	$-\frac{1}{3}$	5
x	1	$\frac{1}{3}$	0	$-\frac{1}{3}$	0	0	$\frac{1}{3}$	5
s ₃	0	$\frac{2}{3}$	0	$\frac{1}{3}$	1	0	$-\frac{1}{3}$	15
P	0	4	0	1	0	0	-1	-15
I	0	$-\frac{5}{3}$	1	$-\frac{1}{3}$	0	0	$\frac{4}{3}$	-5

Phase 1 end

b.v.	x	y	s ₁	s ₂	s ₃	t ₁	t ₂	Value
y	0	1	$-\frac{3}{5}$	$\frac{1}{5}$	0	$\frac{3}{5}$	$-\frac{1}{5}$	3
x	1	0	$\frac{1}{5}$	$-\frac{2}{5}$	0	$-\frac{1}{5}$	$\frac{2}{5}$	4
s ₃	0	0	$\frac{2}{5}$	$\frac{1}{5}$	1	$-\frac{2}{5}$	$-\frac{1}{5}$	13
P	0	0	$\frac{12}{5}$	$\frac{1}{5}$	0	$-\frac{12}{5}$	$-\frac{1}{5}$	-27
I	0	0	0	0	0	1	1	0

20 (5 marks)

The P row has no negative entries, so it is optimal

E1

$P = -27$, so $C = 27$

B1

$s_3 = 20 - (4 + 3) = 13$; the constraint $x + y \leq 20$ has 13 spare

M1 A1

$P = -27 - \frac{12}{5}s_1 - \frac{1}{5}s_2 \leq -27$ so C cannot fall below 27

E1

Answer: $C = 27$; $s_3 = 13$

5 marks

21 (9 marks)

(a) Iteration 1: most negative entry in the I row is -1 , so x enters; ratios 14, 3, $-$, 12, least is 3, so t_1 leaves and the pivot is 1

M1

Row operations: $R_1 - R_2 \rightarrow R_1$; $R_4 - 2 R_2 \rightarrow R_4$; $P + 5 R_2 \rightarrow P$; $I + R_2 \rightarrow I$ (new tableau in the diagram)

A1

Iteration 2: most negative entry in the I row is -1 , so y enters; ratios 11, $-$, 2, 18, least is 2, so t_2 leaves and the pivot is 1

M1

Row operations: $R_1 - R_3 \rightarrow R_1$; $R_4 - R_3 \rightarrow R_4$; $P + 4 R_3 \rightarrow P$; $I + R_3 \rightarrow I$ (new tableau in the diagram)

A1

Answer: $x = 3$, $y = 2$, $I = 0$

4 marks

(b) $x = 3$, $y = 2$, $P = 15 + 8 = 23$

B1

Answer: $x = 3$, $y = 2$, $P = 23$

1 mark

(c) Iteration 1: most negative entry in the P row is -5 , so s_2 enters; ratios 9, $-$, $-$, 8, least is 8, so s_4 leaves and the pivot is 2

M1

Row operations: $R_4 \div 2 \rightarrow R_4$; $R_1 - R_4 \rightarrow R_1$; $R_2 + R_4 \rightarrow R_2$; $P + 5 R_4 \rightarrow P$ (new tableau in the diagram)

A1

Iteration 2: most negative entry in the P row is $-\frac{3}{2}$, so s_3 enters; ratios 2, 22, $-$, 16, least is 2, so

M1

s_1 leaves and the pivot is $\frac{1}{2}$

Row operations: $R_1 \div \frac{1}{2} \rightarrow R_1$; $R_2 - \frac{1}{2} R_1 \rightarrow R_2$; $R_3 + R_1 \rightarrow R_3$; $R_4 - \frac{1}{2} R_1 \rightarrow R_4$; $P + \frac{3}{2} R_1 \rightarrow$

A1

P (new tableau in the diagram)

No negative entries in the P row: optimal, $x = 10$, $y = 4$ with $P = 66$

Answer: $x = 10$, $y = 4$, $P = 66$

4 marks

b.v.	x	y	s ₁	s ₂	s ₃	s ₄	t ₁	t ₂	Value
s ₁	1	1	1	0	0	0	0	0	14
t ₁	1	0	0	-1	0	0	1	0	3
t ₂	0	1	0	0	-1	0	0	1	2
s ₄	2	1	0	0	0	1	0	0	24
P	-5	-4	0	0	0	0	0	0	0
I	-1	-1	0	1	1	0	0	0	-5

b.v.	x	y	s ₁	s ₂	s ₃	s ₄	t ₁	t ₂	Value
s ₁	0	1	1	1	0	0	-1	0	11
x	1	0	0	-1	0	0	1	0	3
t ₂	0	1	0	0	-1	0	0	1	2
s ₄	0	1	0	2	0	1	-2	0	18
P	0	-4	0	-5	0	0	5	0	15
I	0	-1	0	0	1	0	1	0	-2

b.v.	x	y	s ₁	s ₂	s ₃	s ₄	t ₁	t ₂	Value
s ₁	0	0	1	1	1	0	-1	-1	9
x	1	0	0	-1	0	0	1	0	3
y	0	1	0	0	-1	0	0	1	2
s ₄	0	0	0	2	1	1	-2	-1	16
P	0	0	0	-5	-4	0	5	4	23
I	0	0	0	0	0	0	1	1	0

b.v.	x	y	s ₁	s ₂	s ₃	s ₄	Value
s ₁	0	0	1	1	1	0	9
x	1	0	0	-1	0	0	3
y	0	1	0	0	-1	0	2
s ₄	0	0	0	2	1	1	16
P	0	0	0	-5	-4	0	23

b.v.	x	y	s ₁	s ₂	s ₃	s ₄	Value
s ₁	0	0	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	1
x	1	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	11
y	0	1	0	0	-1	0	2
s ₂	0	0	0	1	$\frac{1}{2}$	$\frac{1}{2}$	8
P	0	0	0	0	$-\frac{3}{2}$	$\frac{5}{2}$	63

b.v.	x	y	s ₁	s ₂	s ₃	s ₄	Value
s ₃	0	0	2	0	1	-1	2
x	1	0	-1	0	0	1	10
y	0	1	2	0	0	-1	4
s ₂	0	0	-1	1	0	1	7
P	0	0	3	0	0	1	66

22 (7 marks)

Iteration 1: most negative entry in the I row is -8 , so y enters; ratios $12, 6$, least is 6 , so t_2 leaves and the pivot is 5

M1

Row operations: $R_2 \div 5 \rightarrow R_2$; $R_1 - 3 R_2 \rightarrow R_1$; $P - 45 R_2 \rightarrow P$; $I + 8 R_2 \rightarrow I$ (new tableau in the diagram)

A1

Iteration 2: most negative entry in the I row is $-\frac{14}{5}$, so x enters; ratios $\frac{45}{7}, 15$, least is $\frac{45}{7}$, so t_1 leaves and the pivot is $\frac{14}{5}$

M1

Row operations: $R_1 \div \frac{14}{5} \rightarrow R_1$; $R_2 - \frac{2}{5} R_1 \rightarrow R_2$; $P - 12 R_1 \rightarrow P$; $I + \frac{14}{5} R_1 \rightarrow I$ (new tableau in the diagram)

A1

$I = 0$, so all artificial variables are zero and the problem is feasible; delete the artificial columns and the I row

E1

The P row at the end of Phase 1 has no negative entries, so no Phase 2 iteration is needed

E1

No negative entries in the P row: optimal, $x = \frac{45}{7}$, $y = \frac{24}{7}$ with $C = \frac{2430}{7}$

A1

Answer: $x = \frac{45}{7}$, $y = \frac{24}{7}$, $C = \frac{2430}{7}$

7 marks

b.v.	x	y	s_1	s_2	t_1	t_2	Value
t_1	4	3	-1	0	1	0	36
t_2	2	5	0	-1	0	1	30
P	30	45	0	0	0	0	0
I	-6	-8	1	1	0	0	-66

b.v.	x	y	s_1	s_2	t_1	t_2	Value
t_1	$\frac{14}{5}$	0	-1	$\frac{3}{5}$	1	$-\frac{3}{5}$	18
y	$\frac{2}{5}$	1	0	$-\frac{1}{5}$	0	$\frac{1}{5}$	6
P	12	0	0	9	0	-9	-270
I	$-\frac{14}{5}$	0	1	$-\frac{3}{5}$	0	$\frac{8}{5}$	-18

b.v.	x	y	s_1	s_2	t_1	t_2	Value
x	1	0	$-\frac{5}{14}$	$\frac{3}{14}$	$\frac{5}{14}$	$-\frac{3}{14}$	$\frac{45}{7}$
y	0	1	$\frac{1}{7}$	$-\frac{2}{7}$	$-\frac{1}{7}$	$\frac{2}{7}$	$\frac{24}{7}$
P	0	0	$\frac{30}{7}$	$\frac{45}{7}$	$-\frac{30}{7}$	$-\frac{45}{7}$	$-\frac{2430}{7}$
I	0	0	0	0	1	1	0

23 (9 marks)

Iteration 1: most negative entry in the I row is -5 , so y enters; ratios 20, 10, 14, least is 10, so t_2 **M1**

leaves and the pivot is 3

Row operations: $R_2 \div 3 \rightarrow R_2$; $R_1 - R_2 \rightarrow R_1$; $R_3 - R_2 \rightarrow R_3$; $P - 5 R_2 \rightarrow P$; $I + 5 R_2 \rightarrow I$ (new tableau in the diagram) **A1**

Iteration 2: most negative entry in the I row is $-\frac{7}{3}$, so x enters; ratios 6, 30, 6, least is 6, so t_1 **M1**

leaves and the pivot is $\frac{5}{3}$

Row operations: $R_1 \div \frac{5}{3} \rightarrow R_1$; $R_2 - \frac{1}{3} R_1 \rightarrow R_2$; $R_3 - \frac{2}{3} R_1 \rightarrow R_3$; $P - \frac{13}{3} R_1 \rightarrow P$; $I + \frac{7}{3} R_1 \rightarrow I$ (new tableau in the diagram) **A1**

Iteration 3: most negative entry in the I row is $-\frac{2}{5}$, so s_1 enters; ratios $-, 40, 0$, least is 0, so t_3 **M1**

leaves and the pivot is $\frac{2}{5}$

Row operations: $R_3 \div \frac{2}{5} \rightarrow R_3$; $R_1 + \frac{3}{5} R_3 \rightarrow R_1$; $R_2 - \frac{1}{5} R_3 \rightarrow R_2$; $P - \frac{13}{5} R_3 \rightarrow P$; $I + \frac{2}{5} R_3 \rightarrow I$ (new tableau in the diagram) **A1**

At the end of Phase 1, $I = 0$ and $x = 6$, $y = 8$, so the solution is feasible **B1**

The zero ratio arises because t_3 is basic with value 0 after iteration 2: the point (6, 8) already satisfies $x + y \geq 14$ exactly **E1**

Geometrically three constraint boundaries $2x + y = 20$, $x + 3y = 30$ (and $x + y = 14$) all pass through (6, 8): the vertex is degenerate **E1**

Answer: $x = 6$, $y = 8$ at the end of Phase 1; degenerate vertex

9 marks

b.v.	x	y	s ₁	s ₂	s ₃	t ₁	t ₂	t ₃	Value
t ₁	2	1	-1	0	0	1	0	0	20
t ₂	1	3	0	-1	0	0	1	0	30
t ₃	1	1	0	0	-1	0	0	1	14
P	6	5	0	0	0	0	0	0	0
I	-4	-5	1	1	1	0	0	0	-64

b.v.	x	y	s ₁	s ₂	s ₃	t ₁	t ₂	t ₃	Value
t ₁	$\frac{5}{3}$	0	-1	$\frac{1}{3}$	0	1	$-\frac{1}{3}$	0	10
y	$\frac{1}{3}$	1	0	$-\frac{1}{3}$	0	0	$\frac{1}{3}$	0	10
t ₃	$\frac{2}{3}$	0	0	$\frac{1}{3}$	-1	0	$-\frac{1}{3}$	1	4
P	$\frac{13}{3}$	0	0	$\frac{5}{3}$	0	0	$-\frac{5}{3}$	0	-50
I	$-\frac{7}{3}$	0	1	$-\frac{2}{3}$	1	0	$\frac{5}{3}$	0	-14

b.v.	x	y	s ₁	s ₂	s ₃	t ₁	t ₂	t ₃	Value
x	1	0	$-\frac{3}{5}$	$\frac{1}{5}$	0	$\frac{3}{5}$	$-\frac{1}{5}$	0	6
y	0	1	$\frac{1}{5}$	$-\frac{2}{5}$	0	$-\frac{1}{5}$	$\frac{2}{5}$	0	8
t ₃	0	0	$\frac{2}{5}$	$\frac{1}{5}$	-1	$-\frac{2}{5}$	$-\frac{1}{5}$	1	0
P	0	0	$\frac{13}{5}$	$\frac{4}{5}$	0	$-\frac{13}{5}$	$-\frac{4}{5}$	0	-76
I	0	0	$-\frac{2}{5}$	$-\frac{1}{5}$	1	$\frac{7}{5}$	$\frac{6}{5}$	0	0

b.v.	x	y	s ₁	s ₂	s ₃	t ₁	t ₂	t ₃	Value
x	1	0	0	$\frac{1}{2}$	$-\frac{3}{2}$	0	$-\frac{1}{2}$	$\frac{3}{2}$	6
y	0	1	0	$-\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	8
s ₁	0	0	1	$\frac{1}{2}$	$-\frac{5}{2}$	-1	$-\frac{1}{2}$	$\frac{5}{2}$	0
P	0	0	0	$-\frac{1}{2}$	$\frac{13}{2}$	0	$\frac{1}{2}$	$-\frac{13}{2}$	-76
I	0	0	0	0	0	1	1	1	0

24 (8 marks)

Iteration 1: most negative entry in the I row is -3 , so z enters; ratios $12, -, 3$, least is 3 , so t_2 leaves and the pivot is 2

M1

Row operations: $R_3 \div 2 \rightarrow R_3$; $R_1 - R_3 \rightarrow R_1$; $P + 3 R_3 \rightarrow P$; $I + 3 R_3 \rightarrow I$ (new tableau in the diagram)

A1

Iteration 2: most negative entry in the I row is -1 , so x enters; ratios $9, 8, -$, least is 8 , so s_1 leaves and the pivot is 2

M1

Row operations: $R_2 \div 2 \rightarrow R_2$; $R_1 - R_2 \rightarrow R_1$; $P + 2 R_2 \rightarrow P$; $I + R_2 \rightarrow I$ (new tableau in the diagram)

A1

Iteration 3: most negative entry in the I row is $-\frac{1}{2}$, so s_2 enters; ratios $2, -, -$, least is 2 , so t_1 leaves and the pivot is $\frac{1}{2}$

M1

Row operations: $R_1 \div \frac{1}{2} \rightarrow R_1$; $R_3 + \frac{1}{2} R_1 \rightarrow R_3$; $P + \frac{3}{2} R_1 \rightarrow P$; $I + \frac{1}{2} R_1 \rightarrow I$ (new tableau in the diagram)

A1

$I = 0$, so the problem is feasible

B1

Phase 2 starts from $x = 8, y = 0, z = 4$, with $s_1 = 0, s_2 = 2$, and $P = 16 + 12 = 28$ (check: $8 + 4 = 12, 2(8) = 16 \leq 16, y + 2z = 8 \geq 6$)

B1

Answer: $x = 8, y = 0, z = 4, P = 28$

8 marks

b.v.	x	y	z	s_1	s_2	t_1	t_2	Value
t_1	1	1	1	0	0	1	0	12
s_1	2	1	0	1	0	0	0	16
t_2	0	1	2	0	-1	0	1	6
P	-2	-1	-3	0	0	0	0	0
I	-1	-2	-3	0	1	0	0	-18

b.v.	x	y	z	s_1	s_2	t_1	t_2	Value
t_1	1	$\frac{1}{2}$	0	0	$\frac{1}{2}$	1	$-\frac{1}{2}$	9
s_1	2	1	0	1	0	0	0	16
z	0	$\frac{1}{2}$	1	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	3
P	-2	$\frac{1}{2}$	0	0	$-\frac{3}{2}$	0	$\frac{3}{2}$	9
I	-1	$-\frac{1}{2}$	0	0	$-\frac{1}{2}$	0	$\frac{3}{2}$	-9

b.v.	x	y	z	s_1	s_2	t_1	t_2	Value
t_1	0	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	1	$-\frac{1}{2}$	1
x	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	0	8
z	0	$\frac{1}{2}$	1	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	3
P	0	$\frac{3}{2}$	0	1	$-\frac{3}{2}$	0	$\frac{3}{2}$	25
I	0	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{3}{2}$	-1

b.v.	x	y	z	s_1	s_2	t_1	t_2	Value
s_2	0	0	0	-1	1	2	-1	2
x	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	0	8
z	0	$\frac{1}{2}$	1	$-\frac{1}{2}$	0	1	0	4
P	0	$\frac{3}{2}$	0	$-\frac{1}{2}$	0	3	0	28
I	0	0	0	0	0	1	1	0

25 (6 marks)

Each $t_j \geq 0$, so $I \geq 0$ and its minimum is at least 0

E1

If the original problem has a feasible solution, give every artificial variable the value 0 and every other variable its feasible values: all the equations hold with $I = 0$, so the minimum is 0

M1 A1

Conversely, if the minimum is $I = 0$ then every $t_j = 0$ (sum of non-negative terms)

M1

Then the equations are the original constraints (with slack and surplus variables), so the values of x give a feasible solution

A1

If the minimum is greater than 0 then every choice has some $t_j > 0$, so no feasible solution exists

E1

Answer: $I_{\min} = 0$ iff a feasible solution exists

6 marks

26 (6 marks)

During Phase 1 optimality is judged from the I row, not the P row

E1

The I row still has a negative entry $-\frac{5}{3}$, and $I = -5 \neq 0$, so Phase 1 is not complete

M1 A1

The artificial variable t_1 is still positive (basic with value 5), so $(x, y) = (5, 0)$ violates $x + 2y \geq 10$: the solution is infeasible

M1 A1

The value $C = 15$ therefore does not belong to the feasible region: the true minimum is 27

E1

Answer: Optimality must use the I row; $I = -5 \neq 0$ so the point is infeasible

6 marks

27 (12 marks)

(a) $x + y + z + s_1 = 20$, $x + 2y - s_2 + t_1 = 6$, $2x + z - s_3 + t_2 = 8$, $y + z + s_4 = 12$

B1

Phase 1: maximise $I = -t_1 - t_2$; the I row after eliminating the artificial variables is $(-3, -2, -1, 0, 1, 1, 0, 0, -14)$

B1

Answer: $I = -t_1 - t_2$, with I row $(-3, -2, -1, 0, 1, 1, 0, 0, -14)$

2 marks

(b) Iteration 1: most negative entry in the I row is -3 , so x enters; ratios 20, 6, 4, $-$, least is 4, so t_2 leaves and the pivot is 2

M1

Row operations: $R_3 \div 2 \rightarrow R_3$; $R_1 - R_3 \rightarrow R_1$; $R_2 - R_3 \rightarrow R_2$; $P + 3 R_3 \rightarrow P$; $I + 3 R_3 \rightarrow I$ (new tableau in the diagram)

A1

Iteration 2: most negative entry in the I row is -2 , so y enters; ratios 16, 1, $-$, 12, least is 1, so t_1 leaves and the pivot is 2

M1

Row operations: $R_2 \div 2 \rightarrow R_2$; $R_1 - R_2 \rightarrow R_1$; $R_4 - R_2 \rightarrow R_4$; $P + 4 R_2 \rightarrow P$; $I + 2 R_2 \rightarrow I$ (new tableau in the diagram)

A1

Answer: $x = 4$, $y = 1$, $z = 0$

4 marks

(c) Iteration 1: most negative entry in the P row is -2 , so s_2 enters; ratios 30, $-$, $-$, 22, least is 22, so s_4 leaves and the pivot is $\frac{1}{2}$

M1

Row operations: $R_4 \div \frac{1}{2} \rightarrow R_4$; $R_1 - \frac{1}{2} R_4 \rightarrow R_1$; $R_2 + \frac{1}{2} R_4 \rightarrow R_2$; $P + 2 R_4 \rightarrow P$ (new tableau in the diagram)

A1

Iteration 2: most negative entry in the P row is $-\frac{3}{2}$, so s_3 enters; ratios 8, $-$, $-$, $-$, least is 8, so s_1 leaves and the pivot is $\frac{1}{2}$

M1

Row operations: $R_1 \div \frac{1}{2} \rightarrow R_1$; $R_3 + \frac{1}{2} R_1 \rightarrow R_3$; $R_4 + \frac{1}{2} R_1 \rightarrow R_4$; $P + \frac{3}{2} R_1 \rightarrow P$ (new tableau in the diagram)

A1

No negative entries in the P row: optimal, $x = 8$, $y = 12$, $z = 0$ with $P = 72$

Answer: $x = 8$, $y = 12$, $z = 0$, $P = 72$

4 marks

(d) $x + y + z = 20$ (binding); $y + z = 12$ (binding)

B1

$x + 2y = 32 \geq 6$ and $2x + z = 16 \geq 8$, so these are not binding

B1

Answer: Constraints 1 and 4 binding

2 marks

b.v.	x	y	z	s ₁	s ₂	s ₃	s ₄	t ₁	t ₂	Value
s ₁	1	1	1	1	0	0	0	0	0	20
t ₁	1	2	0	0	-1	0	0	1	0	6
t ₂	2	0	1	0	0	-1	0	0	1	8
s ₄	0	1	1	0	0	0	1	0	0	12
P	-3	-4	-2	0	0	0	0	0	0	0
I	-3	-2	-1	0	1	1	0	0	0	-14

b.v.	x	y	z	s ₁	s ₂	s ₃	s ₄	t ₁	t ₂	Value
s ₁	0	1	$\frac{1}{2}$	1	0	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	16
t ₁	0	2	$-\frac{1}{2}$	0	-1	$\frac{1}{2}$	0	1	$-\frac{1}{2}$	2
x	1	0	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	4
s ₄	0	1	1	0	0	0	1	0	0	12
P	0	-4	$-\frac{1}{2}$	0	0	$-\frac{3}{2}$	0	0	$\frac{3}{2}$	12
I	0	-2	$\frac{1}{2}$	0	1	$-\frac{1}{2}$	0	0	$\frac{3}{2}$	-2

b.v.	x	y	z	s ₁	s ₂	s ₃	s ₄	t ₁	t ₂	Value
s ₁	0	0	$\frac{3}{4}$	1	$\frac{1}{2}$	$\frac{1}{4}$	0	$-\frac{1}{2}$	$-\frac{1}{4}$	15
y	0	1	$-\frac{1}{4}$	0	$-\frac{1}{2}$	$\frac{1}{4}$	0	$\frac{1}{2}$	$-\frac{1}{4}$	1
x	1	0	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	4
s ₄	0	0	$\frac{5}{4}$	0	$\frac{1}{2}$	$-\frac{1}{4}$	1	$-\frac{1}{2}$	$\frac{1}{4}$	11
P	0	0	$-\frac{3}{2}$	0	-2	$-\frac{1}{2}$	0	2	$\frac{1}{2}$	16
I	0	0	0	0	0	0	0	1	1	0

b.v.	x	y	z	s ₁	s ₂	s ₃	s ₄	Value
s ₁	0	0	$\frac{3}{4}$	1	$\frac{1}{2}$	$\frac{1}{4}$	0	15
y	0	1	$-\frac{1}{4}$	0	$-\frac{1}{2}$	$\frac{1}{4}$	0	1
x	1	0	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	0	4
s ₄	0	0	$\frac{5}{4}$	0	$\frac{1}{2}$	$-\frac{1}{4}$	1	11
P	0	0	$-\frac{3}{2}$	0	-2	$-\frac{1}{2}$	0	16

b.v.	x	y	z	s ₁	s ₂	s ₃	s ₄	Value
s ₁	0	0	$-\frac{1}{2}$	1	0	$\frac{1}{2}$	-1	4
y	0	1	1	0	0	0	1	12
x	1	0	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	0	4
s ₂	0	0	$\frac{5}{2}$	0	1	$-\frac{1}{2}$	2	22
P	0	0	$\frac{7}{2}$	0	0	$-\frac{3}{2}$	4	60

b.v.	x	y	z	s ₁	s ₂	s ₃	s ₄	Value
s ₃	0	0	-1	2	0	1	-2	8
y	0	1	1	0	0	0	1	12
x	1	0	0	1	0	0	-1	8
s ₂	0	0	2	1	1	0	1	26
P	0	0	2	3	0	0	1	72

28 (10 marks)

$x + y + s_1 = 10$, $x - s_2 + t = k$; $I = -t = -k + x - s_2$, so I row $(-1, 0, 0, 1, 0 \mid -k)$

B1 B1

x enters (I row entry -1); ratios $\frac{10}{1} = 10$ and $\frac{k}{1} = k$

M1

If $k \leq 10$: the second row (t) gives the least ratio, t leaves, and I becomes 0 ($x = k$, $t = 0$)

A1 A1

If $k > 10$: the first row (s_1) leaves; row 2 becomes $R_2 - R_1$: $-y - s_1 - s_2 + t = k - 10$

M1 A1

I row becomes $I + R_1 = (0, 1, 1, 1, 0 \mid 10 - k)$

A1

No negative entries remain, so Phase 1 ends with $I = -(k - 10) < 0$

M1

So the problem is feasible exactly when $k \leq 10$

A1

Answer: Feasible iff $k \leq 10$ ($I = -(k - 10)$ for $k > 10$)

10 marks

29 (9 marks)

$x + y = 5 \leq 14$, $x = 3 \geq 3$, $y = 2 \geq 2$, $2x + y = 8 \leq 24$: all constraints hold

M1 A1

The surplus variables are 0, so $x \geq 3$ and $y \geq 2$ hold with equality: the point lies on the boundary lines $x = 3$ and $y = 2$

B1

These two lines meet at $(3, 2)$, so it is the intersection of two boundaries of the feasible region, which makes it a vertex

E1

Phase 2 needs a basic feasible solution to start from; this is one (with $s_1 = 9$, $s_4 = 16$ and the other basic variables x and y)

E1 B1

$P = 5(3) + 4(2) = 23$

B1

Phase 2 will then move along edges, increasing P : 23, 63 and finally 66 at $(10, 4)$

E1 A1

Answer: $(3, 2)$ is a vertex; $P = 23$

9 marks

30 (9 marks)

The pivot row is R_3 : $(1, 0, 0, 0, -1, 1 \mid 2)$

B1

$P + 3R_3$: $(-3 + 3, -2, 0, 0, -3, 3 \mid 6) = (0, -2, 0, 0, -3, 3 \mid 6)$

M1 A1

Check: from $x - s_3 + t = 2$, $x = 2 + s_3 - t$, so $P = 3x + 2y = 6 + 3s_3 - 3t + 2y$, giving $P - 2y - 3s_3 + 3t = 6$

M1 A1

Phase 2 tableau: delete t ; s_3 enters (-3) , x row has -1 and the other rows have ratios $\frac{8}{1} = 8$ and

M1

$\frac{16}{1} = 16$

s_1 leaves, pivot 1

A1

$P + 3R_1$: P row $(0, 1, 3, 0, 0 \mid 30)$

M1

Optimal: $x = 10$, $y = 0$, $P = 30$

A1

Answer: $x = 10$, $y = 0$, $P = 30$

9 marks