

Formulating Linear Programming Problems

- 1 A bakery makes x cakes and y loaves each day. The bakery can make at most 40 items in total each day.
Write down an inequality in x and y for this restriction. [1]
- 2 A bakery makes a profit of £12 on each cake and £9 on each loaf. It makes x cakes and y loaves each day.
Write down an expression for the daily profit, £P, which is to be maximised. [1]
- 3 In a linear programming problem, x and y are the numbers of items made.
Explain why the constraints $x \geq 0$ and $y \geq 0$ must be included. [1]
- 4 Each cake needs 3 minutes in an oven and each loaf needs 2 minutes in the oven. The oven is available for at most 120 minutes each day.
Write down a constraint for the oven time, in terms of x (cakes) and y (loaves). [1]
- 5 A bakery makes x cakes and y loaves each day. Write down an inequality for each of the following conditions.
(a) At least 15 loaves are made each day. [1]
(b) No more than 30 cakes are made each day. [1]
- 6 A constraint in a linear programming problem is $4x + 5y \leq 200$.
Write this constraint as an equation by introducing a slack variable s , and state the condition satisfied by s . [2]
- 7 A factory makes x tables and y chairs each week. The factory must make at least twice as many chairs as tables.
Write down an inequality in x and y for this condition. [2]
- 8 A bakery makes x cakes and y loaves each day. At most 60% of the items made each day are cakes.
Show that this condition can be written as $2x \leq 3y$. [3]
- 9 A workshop makes two types of garden gnome, Standard and Deluxe. A Standard gnome needs 20 minutes of painting and 30 minutes of baking. A Deluxe gnome needs 40 minutes of painting and 30 minutes of baking.
Each day there are at most 16 hours available for painting and at most 15 hours available for baking. The profit is £4 on a Standard gnome and £7 on a Deluxe gnome.
Let x be the number of Standard gnomes and y the number of Deluxe gnomes made each day.
Formulate this as a linear programming problem, stating the objective function and the constraints. [5]
- 10 A shop makes x litres of matt paint and y litres of gloss paint. Each litre of matt paint needs 5 minutes of mixing time and each litre of gloss paint needs 4 minutes. At most 600 minutes of mixing time are available. The constraint is $5x + 4y \leq 600$.
(a) Explain what the term $4y$ represents in this context. [1]
(b) Show that making 60 litres of matt paint and 75 litres of gloss paint uses all of the available mixing time. [1]
- 11 A farmer mixes two feeds, P and Q, for her animals. Feed P costs 30p per kg and feed Q costs 45p per kg. Each kg of P contains 4 units of protein and 2 units of vitamins. Each kg of Q contains 3 units of protein and 5 units of vitamins.
The mixture must contain at least 36 units of protein and at least 30 units of vitamins in total.
The farmer uses x kg of P and y kg of Q and wishes to minimise the cost.
Formulate this as a linear programming problem. [5]

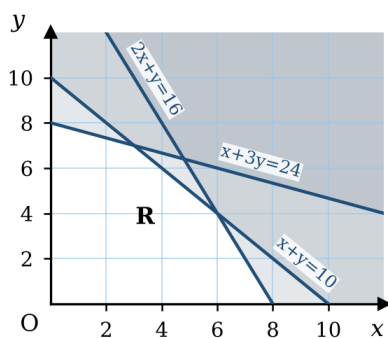
- 12** A graphical solution of a linear programming problem about hiring vehicles gives the optimal point as $x = 4.6$ coaches and $y = 7.3$ minibuses.
Explain why this cannot be the solution to the real problem, and state what must be done instead. [2]
- 13** The constraints of a linear programming problem are
 $2x + 3y \leq 18$, $x + 4y \leq 16$, $5x + y \leq 30$, $x \geq 0$, $y \geq 0$.
Write the three constraints as equations by introducing slack variables, and find the values of the slack variables when $x = 2$ and $y = 3$. [4]
- 14** A craft company makes small, medium and large candles. Each small candle uses 100 g of wax, each medium candle 250 g and each large candle 400 g. The labour time needed is 6, 10 and 15 minutes respectively. Each candle is packed in a box and at most 150 boxes are available.
Each week at most 20 kg of wax and at most 30 hours of labour are available. The profit is £1.20 on a small, £2.50 on a medium and £4 on a large candle.
Formulate a linear programming problem to maximise the weekly profit, defining your variables. [5]
- 15** A shop needs 100 loaves each morning, supplied by two bakeries. Bakery A can supply at most 70 loaves and bakery B at most 80 loaves. A loaf from A costs the shop 40p and a loaf from B costs 30p. At least 30% of the loaves must come from A.
Let x and y be the numbers of loaves bought from A and B. Formulate a linear programming problem to minimise the cost, writing the ratio condition in the form $ax \geq by$ with integers a and b . [6]
- 16** A pottery makes mugs and plates. Each mug uses 2 kg of clay and each plate uses 3 kg. At most 60 kg of clay is available and at least 5 mugs must be made. The profit is £3 per mug and £4 per plate.
A student formulates this, with x mugs and y plates, as:
Maximise $P = 3x + 4y$ subject to $2x + 3y \geq 60$, $x \leq 5$, $x \geq 0$, $y \geq 0$.
Identify the two errors in the student's formulation and write down the correct formulation. [3]
- 17** A company makes two products. The constraints and objective of its linear programming problem are
Maximise $P = 5x + 4y$ subject to $3x + 2y \leq 60$, $x + 3y \leq 48$, $y \leq 15$, $x \geq 0$, $y \geq 0$.
(a) Show that the point (10, 12) satisfies all of the constraints. [2]
(b) Find the value of P at (10, 12). [1]
(c) Find how much of the resource in the constraint $3x + 2y \leq 60$ is unused at (10, 12). [1]
- 18** A call centre employs x full-time and y part-time staff each day. A full-time worker works 8 hours and costs £90 per day; a part-time worker works 4 hours and costs £50 per day. At least 200 hours of cover are needed each day, at most 45 staff can be employed in total, and there must be at least 2 full-time staff for every 3 part-time staff.
Formulate a linear programming problem to minimise the daily cost, giving each constraint in its simplest form. [6]
- 19** A farmer plants wheat on x hectares and barley on y hectares. The total area planted is at most 80 hectares. The area of wheat must be at most 1.5 times the area of barley. Wheat needs 4 days of labour per hectare and barley needs 6 days per hectare, with at most 420 days of labour available. The profit is £350 per hectare of wheat and £420 per hectare of barley.
Formulate a linear programming problem to maximise the profit. [6]
- 20** A carpenter works for at most 40 hours each week. She spends x hours making chairs and y hours making tables. At least one quarter of the time she works must be spent making tables.
Write down the constraint on the total time worked, and show that the condition on tables can be written as $3y \geq x$. [3]

- 21** A studio makes x bowls and y vases each week. A bowl needs 2 hours of throwing and 1 hour of glazing. A vase needs 3 hours of throwing and 2 hours of glazing. At most 60 hours of throwing and 36 hours of glazing are available each week. At most 14 vases can be made, and at least as many bowls as vases must be made. The profit is £8 per bowl and £13 per vase.
- Formulate this as a linear programming problem to maximise the weekly profit. [5]
 - State a further restriction on x and y . [1]
 - Show that making 18 bowls and 6 vases is feasible, and find the profit, and the unused glazing time. [3]
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- 22** A chemical firm blends three liquids X, Y and Z to make 200 litres of a solvent. The liquids cost £3, £5 and £4 per litre. The solvent must contain at least 40% of X, and the volume of Y must be at most half the volume of Z. At most 90 litres of Z are available.
- Let x , y and z be the volumes in litres of X, Y and Z used.
- Formulate a linear programming problem to minimise the cost, giving each constraint in its simplest form with integer coefficients. [6]
 - Write the objective in a form suitable for maximisation. [1]
 - Explain why a model with the constraint $x + y + z \leq 200$ in place of $x + y + z = 200$ would give the wrong answer. [1]
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- 23** A foundry makes exactly 100 kg of an alloy from three scrap sources. Source 1 is 80% copper and 2% tin, source 2 is 50% copper and 12% tin, source 3 is 60% copper and 4% tin. The sources cost £4, £3 and £3.50 per kg.
- The alloy must contain at least 60 kg of copper and at most 6 kg of tin. Let x , y and z be the masses in kg used from sources 1, 2 and 3.
- Formulate a linear programming problem to minimise the cost, with integer coefficients in the constraints. [5]
 - Eliminate x to write the problem in terms of y and z only. [4]
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- 24** A firm makes products A, B and C in quantities x , y and z . The resources used are limited as follows. Machine time: $2x + y + 3z \leq 60$; assembly: $x + 2y + 2z \leq 50$; materials: $3x + y + z \leq 70$; storage: $x + y + z \leq 30$.
- Write the four constraints as equations using slack variables s_1 , s_2 , s_3 and s_4 . [2]
 - The firm makes 8 of A, 9 of B and 6 of C. Find the value of each slack variable. [3]
 - Explain which resource, if any, limits production at this plan. [1]
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- 25** A school hires x coaches (50 seats, £400 each) and y minibuses (16 seats, £160 each) for a trip for 400 students and 24 staff. Every person needs a seat. At most 10 drivers are available, so $x + y \leq 10$.
- Formulate a linear programming problem to minimise the cost, and explain why integer values are required. [5]
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- 26** A caterer prepares x trays of sandwiches and y trays of wraps. A tray of sandwiches needs 30 minutes of preparation and 2 kg of filling; a tray of wraps needs 20 minutes and 3 kg of filling. At most 10 hours of preparation time and 40 kg of filling are available. At least 4 trays of each must be made, and the number of trays of wraps must not exceed the number of trays of sandwiches by more than 6. Profit is £15 per tray of sandwiches and £14 per tray of wraps. Formulate a linear programming problem to maximise the profit. [7]
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- 27** A theatre sells three types of ticket: stalls (x), circle (y) and balcony (z), at £40, £30 and £20. There are 200 stalls, 150 circle and 100 balcony seats. At least 60% of all tickets sold must be for stalls or circle, and no more than 40% of the tickets sold can be for balcony. At least 250 tickets must be sold to cover costs.
- Formulate a linear programming problem to maximise the revenue, giving the percentage conditions as inequalities with integer coefficients. [7]

- 28** A refinery produces petrol (x thousand litres) and diesel (y thousand litres) from crude oil. Each thousand litres of petrol needs 3 units of crude and each thousand litres of diesel needs 2 units. At most 120 units of crude are available. Petrol sells at £900 and diesel at £700 per thousand litres, but at most 20 thousand litres of petrol can be produced. Because of contracts, the diesel produced must be at least half of the petrol produced. A manager says: "Since petrol earns more per thousand litres, only petrol should be made." Formulate the problem as a linear programme and use your formulation to explain whether the manager's plan satisfies all conditions. [6]
- 29** A company can choose to run production lines of two types. Type A produces 30 units per hour at a running cost of £20 per hour, type B produces 50 units per hour at £45 per hour. At least 2400 units must be produced in a week. Each line can run for at most 40 hours in the week, at most 60 hours of line-time are available in total because of staffing, and line A must run for at least as many hours as line B. Let x and y be the hours for which lines A and B run. Formulate a linear programming problem to minimise the weekly cost and show that the demand constraint can be written $3x + 5y \geq 240$. [5]
- 30** A linear programming problem is: maximise $P = 4x + 3y + 2z$ subject to $2x + y + z \leq 40$, $x + 3y + 2z \leq 60$, $x - y \geq 5$, $x, y, z \geq 0$. Explain why the standard simplex method cannot be applied directly, state what must be done to the third constraint to allow the problem to be solved, and write the full set of equations that would be used. [6]

Graphical Linear Programming

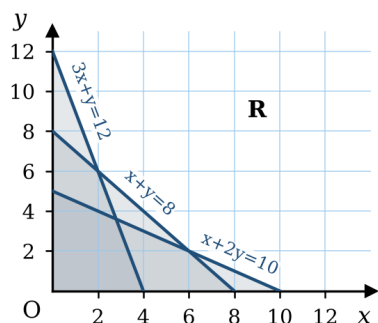
- 1** Find the coordinates of the points where the line $3x + 4y = 24$ crosses the x -axis and the y -axis. [2]
- 2** State, with a reason, whether the point $(4, 5)$ satisfies the constraint $2x + 3y \leq 24$. [1]
- 3** The objective function is $P = 3x + 2y$. Find the value of P at the point $(6, 4)$. [1]
- 4** Solve simultaneously $x + y = 10$ and $2x + y = 16$, to find the point where the two lines meet. [2]
- 5** In a graphical solution of a linear programming problem, state what is meant by the feasible region. [1]
- 6** The objective function is $P = 5x + 4y$. Find the gradient of the line $P = 40$, and state whether this gradient is positive or negative. [2]
- 7** The diagram shows the feasible region R for a linear programming problem, with the lines $x + y = 10$, $2x + y = 16$ and $x + 3y = 24$ drawn. The unwanted regions are shaded. Find the coordinates of the vertices of R . [5]



- 8** The feasible region R for a linear programming problem has vertices (0, 0), (8, 0), (6, 4), (3, 7) and (0, 8).
The objective is to maximise $P = 3x + 2y$. Find the maximum value of P and where it occurs. [3]

- 9** A workshop makes x Standard and y Deluxe gnomes. The constraints are $x + 2y \leq 48$ and $x + y \leq 30$, with $x \geq 0$ and $y \geq 0$, and the profit to be maximised is $P = 4x + 7y$ (£).
Find the vertices of the feasible region and hence the maximum profit. [5]

- 10** The diagram shows the feasible region R (unshaded) for the problem: minimise $C = 2x + 3y$ subject to $x + y \geq 8$, $x + 2y \geq 10$, $3x + y \geq 12$, $x, y \geq 0$.
Find the minimum value of C, and the values of x and y at which it occurs.



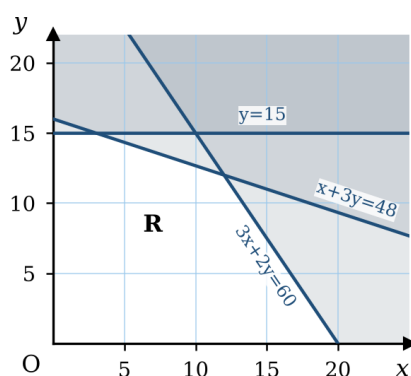
[4]

- 11** The objective function is $P = 6x + 4y$.
Write down the equation of the line $P = 24$ and state its gradient. Explain how a line of this type is used to find the optimal vertex. [4]

- 12** A feasible region is defined by $x \geq 2$, $y \geq 1$ and $x + y \leq 9$.
Find the vertices of the region and the maximum value of $3x + y$ over it. [4]

- 13** The feasible region of a problem has vertices (0, 0), (20, 0), (16, 8), (10, 14) and (0, 19). The objective is to maximise $P = 4x + 5y$.
Show that the optimal vertex is (10, 14) and find the maximum value of P. [3]

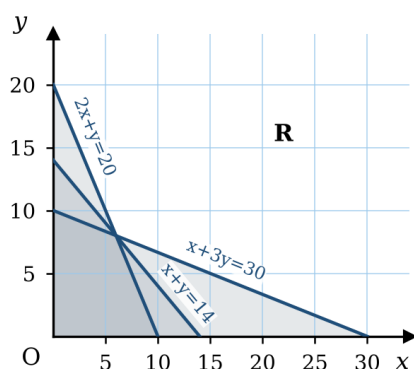
- 14** The feasible region R for the problem: maximise $P = 5x + 4y$ subject to $3x + 2y \leq 60$, $x + 3y \leq 48$, $y \leq 15$, $x, y \geq 0$ is shown unshaded in the diagram.
Find the coordinates of each vertex of R and hence the optimal solution.



[6]

- 15** Maximise $P = 4x + 5y$ subject to $2x + y \leq 40$, $x + 2y \leq 38$, $x + y \leq 24$, $x \geq 0$, $y \geq 0$.
Solve this problem graphically, and state how much of the resource in the constraint $2x + y \leq 40$ is unused at the optimum. [6]

- 16** Minimise $C = 6x + 5y$ subject to
 $2x + y \geq 20$, $x + 3y \geq 30$, $x + y \geq 14$, $x \geq 0$, $y \geq 0$.
 The feasible region is shown unshaded in the diagram. Find the minimum value of C .



[5]

- 17** Minimise $C = 3x + 5y$ subject to $x + 2y \geq 12$, $3x + y \geq 15$, $x + y \leq 20$, $x \geq 0$, $y \geq 0$, where x and y must be integers.
 Find the minimum of C ignoring the integer condition, then find the optimal integer solution.

[6]

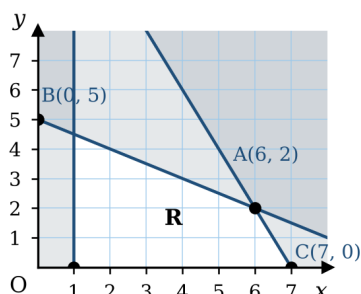
- 18** A region is defined by $x + y \geq 6$, $x + 3y \geq 12$, $x \geq 0$, $y \geq 0$.
 Explain why $2x + 5y$ has a minimum on this region but no maximum, and find the minimum.

[6]

- 19** Maximise $P = 5x + 3y$ subject to $4x + 3y \leq 120$, $x + y \leq 35$, $x \leq 25$, $x \geq 0$, $y \geq 0$.
 Use the objective line method to find the optimal point, giving the values of x , y and P .

[6]

- 20** The diagram shows the feasible region R of a linear programming problem. The boundary of R consists of the line through $B(0, 5)$ and $A(6, 2)$, the line through $A(6, 2)$ and $C(7, 0)$, and the coordinate axes, and the vertical line $x = 1$.
 Find three inequalities that, together with $x \geq 0$, $y \geq 0$, define R .



[6]

- 21** A company makes x kitchen units and y wall units each week. The constraints are $x + y \leq 20$, $3x + y \leq 48$, $x \geq 4$ and $y \geq 2$. The profit is $P = 7x + 4y$ (in hundreds of pounds).
 Find the vertices of the feasible region, and hence the maximum profit and the numbers of each unit made. Explain why making 16 of each unit is not possible.

[7]

- 22** For the feasible region R with vertices $(0, 0)$, $(20, 0)$, $(12, 12)$, $(3, 15)$ and $(0, 15)$, bounded by $3x + 2y \leq 60$, $x + 3y \leq 48$ and $y \leq 15$, the objective is to maximise $P = 5x + ky$, where $k > 0$.
 Find the range of values of k for which $(12, 12)$ is the optimal vertex.

[6]

- 23** A company hires x large lorries at £100c per day and y small lorries at £200 per day. The constraints are $x + y \geq 8$, $x + 2y \geq 10$, $3x + y \geq 12$, $x \geq 0$, $y \geq 0$, whose feasible region has vertices $(0, 12)$, $(2, 6)$, $(6, 2)$ and $(10, 0)$. The total cost, in hundreds of pounds, is $C = cx + 2y$, where c is a positive constant.
 Given that the minimum total cost is 18 (£1800), find the value of c .

[6]

- 24** The feasible region for maximising $P = 5x + 4y$ is bounded by $3x + 2y \leq k$, $x + 3y \leq 48$, $y \leq 15$, $x, y \geq 0$. When $k = 60$ the optimum is $(12, 12)$ with $P = 108$. Find the optimal point and P when $k = 66$, assuming that it lies on the lines $3x + 2y = k$ and $x + 3y = 48$. Hence find the increase in P per unit increase in k , and verify that the vertex is still feasible. [6]
- 25** A farmer mixes x kg of feed P (30p per kg) and y kg of feed Q (45p per kg). The mixture must provide at least 36 units of protein and 30 units of vitamins, where each kg of P gives 4 units of protein and 2 of vitamins, and each kg of Q gives 3 units of protein and 5 of vitamins. Solve the problem graphically to find the cheapest mixture, giving exact values. [7]
- 26** For the problem: maximise $P = 5x + 4y$ subject to $3x + 2y \leq 60$, $x + 3y \leq 48$, $y \leq 15$, $x \geq 0$, $y \geq 0$, a student writes:
'The point $(3, 15)$ has the largest y -value, so it gives the biggest P , $P = 5(3) + 4(15) = 75$, and that is the maximum.'
Identify the error in the student's reasoning, and give the correct solution. [6]
- 27** The problem: maximise $P = 3x + 2y$ subject to $x + y \leq 10$, $2x + y \leq 16$, $x + 3y \leq 24$, $x, y \geq 0$ has optimum $P = 26$ at $(6, 4)$. An extra constraint $x + 2y \leq k$ is added.
(i) Find the least value of k for which the new constraint does not change the feasible region.
(ii) Find the maximum value of P when $k = 12$. [8]
- 28** Maximise $P = x + ky$, where $k > 0$, over the region bounded by $x + y \leq 10$, $2x + y \leq 16$, $x + 3y \leq 24$, $x \geq 0$, $y \geq 0$, whose vertices are $(0, 0)$, $(8, 0)$, $(6, 4)$, $(3, 7)$ and $(0, 8)$. Find, in terms of k , the optimal vertex for every $k > 0$, stating the values of k for which the optimal solution is not unique. [8]
- 29** A factory maximises weekly profit $P = 5x + 4y$ (in pounds) subject to $3x + 2y \leq k$, $x + 3y \leq 48$, $y \leq 15$, $x \geq 0$, $y \geq 0$, where k is the number of machine hours available. At present $k = 60$, with optimum $(12, 12)$ and $P = 108$.
(a) Show that, while the optimum stays on the lines $3x + 2y = k$ and $x + 3y = 48$, the profit is $P = \frac{(11k + 96)}{7}$. [4]
(b) Find the range of k for which this vertex is the optimum. [3]
(c) The factory can hire 15 extra machine hours at £1.20 per hour. Decide whether this is worthwhile. [3]
- 30** A haulage firm hires x large vans (capacity 12 pallets, cost £50 per day) and y small vans (capacity 5 pallets, cost £19 per day). At least 100 pallets must be moved, at most 12 drivers are available, and at most 8 large vans can be hired. Formulate the problem, solve it graphically ignoring integers, and hence find the cheapest feasible integer solution, showing that rounding the graphical solution does not work. [10]

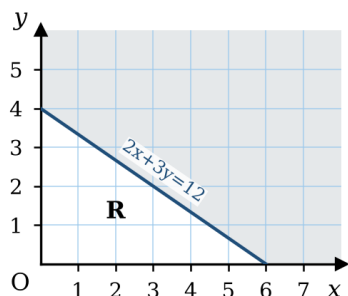
Feasible Regions

- 1** A feasible region is defined by $x + y \leq 8$, $2x + y \leq 12$, $x \geq 0$, $y \geq 0$. Show that the point $(3, 4)$ lies in the feasible region. [2]
- 2** The constraints are $x + y \leq 8$ and $x \geq 2$ (with $y \geq 0$). State which of the points $(1, 5)$, $(5, 3)$ and $(4, 4)$ are feasible. [2]
- 3** The line $2x + 3y = 12$ is drawn on a graph. Explain how to decide which side of the line represents $2x + 3y \leq 12$. [2]
- 4** Write down the inequalities that describe the region where x is at most 5 and y is at least 1. [2]

5 The line L passes through the points $(0, 6)$ and $(4, 0)$. Find the equation of L in the form $ax + by = c$, where a, b, c are integers with no common factor. [2]

6 Write down what it means for a linear programming problem if its feasible region is empty. [1]

7 The diagram shows a feasible region R (unshaded) bounded by the axes and one line which crosses the axes at $(6, 0)$ and $(0, 4)$. Write down the three inequalities that define R .



[3]

8 Find the vertices of the region $x + y \leq 6$, $x \leq 4$, $x \geq 0$, $y \geq 0$, and find its area. [4]

9 List how many points (x, y) with x and y both non-negative integers satisfy $x + y \leq 4$. [2]

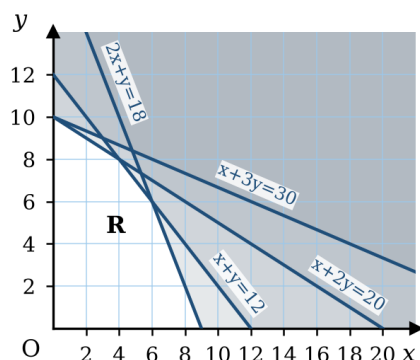
10 A linear programming problem has the constraints $x + y \leq 4$ and $x + y \geq 6$. Explain why the feasible region is empty. [2]

11 The constraints of a problem are $x + y = 10$, $x \leq 7$, $y \leq 8$, $x \geq 0$, $y \geq 0$. Explain why the feasible region is a line segment, and find its end points. [3]

12 A feasible region is defined by $x + y \leq 12$, $x + 2y \leq 20$, $2x + y \leq 18$ and $x, y \geq 0$. Show that the point $(4, 8)$ is a vertex of the region, and state which two constraints meet there. [3]

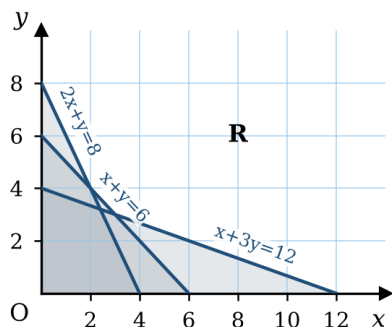
13 A feasible region is defined by $x \geq 1$, $y \geq 2$ and $x + y \leq 7$. Find the vertices of the region. [3]

14 The feasible region R is defined by $x + y \leq 12$, $x + 2y \leq 20$, $2x + y \leq 18$, $x + 3y \leq 30$, $x \geq 0$, $y \geq 0$ and is shown unshaded in the diagram. Show that one of the four constraints is redundant, state which one, and find the vertices of R .



[5]

- 15** The region R defined by $x + y \geq 6$, $x + 3y \geq 12$, $2x + y \geq 8$, $x \geq 0$, $y \geq 0$ is shown unshaded in the diagram.
- (i) Find the coordinates of the vertices of R .
- (ii) Explain why R is unbounded, and explain why a problem to maximise $x + y$ over R has no solution.



[5]

- 16** Find the vertices of the region defined by $x + 2y \leq 14$, $3x + y \leq 18$, $x + y \geq 3$, $x \geq 1$, $y \geq 0$, and find the number of points with integer coordinates in the region (including its boundary).

[7]

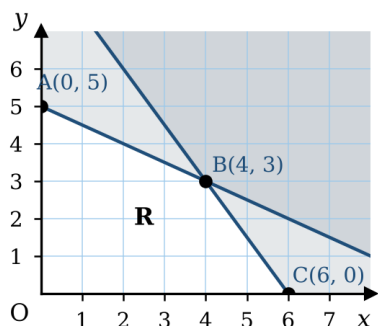
- 17** The constraints $2x + y \leq 16$, $x + 2y \leq 14$, $x \geq 0$, $y \geq 0$ are written as $2x + y + s = 16$ and $x + 2y + t = 14$ with $s, t \geq 0$. Find the values of s and t at each vertex of the feasible region, and state how many of x, y, s, t are zero at each vertex.

[6]

- 18** A feasible region is defined by $x \geq 3$, $y \geq 4$ and $x + y \leq k$, where k is a constant. Find the range of values of k for which the feasible region is not empty, and state what the region is when k takes its smallest possible value.

[4]

- 19** The diagram shows the feasible region R , bounded by the axes and by the lines AB and BC , where A is $(0, 5)$, B is $(4, 3)$ and C is $(6, 0)$. Find the inequalities that define R . A further constraint $2x + 3y \leq 15$ is then added; find the vertices of the new region.



[7]

- 20** Find the vertices of the region defined by $x + y \leq 10$, $2x + 5y \geq 20$, $x \leq 8$, $x \geq 0$, $y \geq 0$, and find its area.

[7]

- 21** The feasible region R is defined by $x + y \leq 12$, $x + 2y \leq 20$, $2x + y \leq 18$, $x \geq 0$, $y \geq 0$, with vertices $(0, 0)$, $(9, 0)$, $(6, 6)$, $(4, 8)$, $(0, 10)$. Find the area of R , and find the number of integer points (x, y) in R that lie on the line $x + y = 10$.

[7]

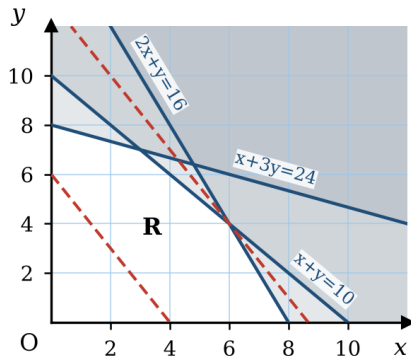
- 22** A three-variable feasible region is defined by $x + y + z \leq 6$, $x \leq 4$, $y \leq 3$, $x \geq 0$, $y \geq 0$, $z \geq 0$. A vertex is a feasible point at which three of the constraint boundaries meet. Find all vertices of the region, and show that the point $(2, 1, 3)$ is feasible but is not a vertex.

[6]

- 23** A feasible region is defined by $x + y \leq 8$, $y \leq 5$, $0 \leq x \leq k$ and $y \geq 0$, where $k > 0$ is a constant.
Find the number of vertices of the region for each possible value of k , and find the area of the region when $k = 6$. [7]
- 24** A linear programming problem has constraints $x + y \leq 8$, $2x + y \leq 12$, $x + 2y \leq 12$, $x \geq 0$, $y \geq 0$.
Show that the three boundary lines of the first three constraints all pass through the point (4, 4), write the constraints using slack variables s , t and u , and explain why the vertex (4, 4) is described as degenerate. [6]
- 25** A constraint is $ax + by \leq c$. The points $P(x_1, y_1)$ and $Q(x_2, y_2)$ both satisfy it.
Prove that every point on the line segment PQ also satisfies the constraint, and state why this means that a feasible region is convex. [6]
- 26** A student is asked to find the vertices of the region $x + y \leq 10$, $x + 2y \leq 16$, $3x + y \leq 24$, $x \geq 0$, $y \geq 0$. The student solves each pair of boundary lines and writes down the six intersection points (0, 0), (8, 0), (0, 8), (7, 3), (4, 6) and $(\frac{32}{5}, \frac{24}{5})$ as the vertices.
Explain the error and give the correct list of vertices. [6]
- 27** The feasible region R_k is defined by $x \geq 0$, $y \geq 0$, $x + y \leq 10$ and $x \leq k$, where $0 < k \leq 10$.
Show that the area of R_k is $10k - \frac{k^2}{2}$, and find the exact value of k for which the area is 40. [8]
- 28** A region S is defined by $|x - 3| + |y - 2| \leq 4$, $x \geq 0$ and $y \geq 0$.
Write the conditions defining S as linear inequalities, find the vertices of S , and find its area. [9]
- 29** For a positive integer n , let $N(n)$ be the number of points (x, y) , where x and y are non-negative integers, in the region $x + y \leq n$.
Prove that $N(n) = \frac{(n+1)(n+2)}{2}$, and hence find the value of n for which $N(n) = 190$. [8]
- 30** The points $A(3, 7)$ and $B(6, 4)$ are both vertices of the feasible region defined by $x + y \leq 10$, $2x + y \leq 16$, $x + 3y \leq 24$, $x \geq 0$, $y \geq 0$, and both give the same maximum value of $P = x + y$.
Show that every point on the line segment AB is feasible and also gives this maximum value. Hence state how many integer points are optimal solutions. [9]

Objective Functions

- 1** A carpenter makes x chairs and y tables. The profit is £5 on each chair and £8 on each table.
Write down the objective function, P , to be maximised. [1]
- 2** The objective function is $P = 4x + 7y$. Find the value of P at the point (12, 18). [1]
- 3** Find the gradient of the line $3x + 2y = 12$. [2]
- 4** A cost function $C = 2x + 3y$ is to be minimised. Write down an objective function P that should be maximised to give the same solution. [1]
- 5** For an objective function $P = 3x + 2y$, with $x, y \geq 0$, describe the direction in which a line of constant P should be moved to increase the value of P . [1]
- 6** State where in the feasible region the optimal solution of a linear programming problem is found. [1]

- 7** The feasible region of a problem has vertices $(0, 0)$, $(8, 0)$, $(6, 4)$, $(3, 7)$ and $(0, 8)$. The objective is to maximise $P = 2x + 5y$. Find the optimal solution. [3]
- 8** The feasible region of a problem has vertices $(0, 20)$, $(6, 8)$ and $(30, 0)$. The objective is to minimise $C = 6x + 5y$. Find the minimum value of C and the point at which it occurs. [3]
- 9** For the feasible region with vertices $(0, 0)$, $(8, 0)$, $(6, 4)$, $(3, 7)$ and $(0, 8)$, find the maximum and the minimum values of $P = x - y$. [3]
- 10** The objective function is $P = 3x + 2y$. Find the equation of the line of constant P that passes through the point $(6, 4)$, and write down where this line crosses the axes. [3]
- 11** A shop sells item A for £14, with material costs of £6 and labour costs of £3 per item, and item B for £10, with material costs of £4 and labour costs of £2 per item. The numbers sold are x and y . Show that the profit to be maximised is $P = 5x + 4y$. [3]
- 12** The objective function is $P = 5x + 3y$. Two production plans are $(6, 4)$ and $(3, 7)$. Find the value of P for each plan and state which is better, and by how much. [3]
- 13** The objective is to maximise $P = 4x + y$ over the region bounded by $x + y \leq 10$, $2x + y \leq 16$, $x + 3y \leq 24$, $x, y \geq 0$, which has vertices $(0, 0)$, $(8, 0)$, $(6, 4)$, $(3, 7)$ and $(0, 8)$. Find the optimal vertex and explain, using gradients, why the objective line last touches the region there. [4]
- 14** The diagram shows the feasible region R of a problem in which $P = 3x + 2y$ is to be maximised. The dashed lines are $3x + 2y = 12$ and $3x + 2y = 26$. Explain how the diagram shows that the maximum value of P occurs at $(6, 4)$, and state the maximum value.
- 
- [4]
- 15** Maximise $P = 2x + y$ over the region bounded by $x + y \leq 10$, $2x + y \leq 16$, $x + 3y \leq 24$, $x, y \geq 0$. Show that the optimal solution is not unique, and find all optimal solutions with integer coordinates. [6]
- 16** The objective function is $P = 3x + ky$, to be maximised over the region with vertices $(0, 0)$, $(8, 0)$, $(6, 4)$, $(3, 7)$, $(0, 8)$, bounded by $2x + y \leq 16$, $x + y \leq 10$ and $x + 3y \leq 24$. Find the range of values of k for which $(6, 4)$ is an optimal vertex. [5]
- 17** The profit on product Y is changed from £2 to £ c , so that the objective is $P = 3x + cy$, for the region with vertices $(0, 0)$, $(8, 0)$, $(6, 4)$, $(3, 7)$, $(0, 8)$ and boundary lines $2x + y = 16$, $x + y = 10$, $x + 3y = 24$. Find the values of c for which the optimal vertex is $(3, 7)$ rather than $(6, 4)$. [5]

- 18** The total profit is $P = 5x + 4y - 50$, where the £50 is a fixed weekly overhead. The feasible region has vertices $(0, 0)$, $(20, 0)$, $(12, 12)$, $(3, 15)$, $(0, 15)$. Find the maximum weekly profit and show that the overhead does not change the optimal vertex. [5]
- 19** The feasible region is bounded by $3x + 2y \leq 60$, $x + 3y \leq 48$, $y \leq 15$, $x \geq 0$, $y \geq 0$, with vertices $(0, 0)$, $(20, 0)$, $(12, 12)$, $(3, 15)$, $(0, 15)$. List the vertices in order of increasing value of $P = 5x + 4y$, and state the maximum and minimum values of P . [4]
- 20** The feasible region defined by $x + y \geq 8$, $x + 2y \geq 10$, $3x + y \geq 12$, $x \geq 0$, $y \geq 0$ has vertices $(0, 12)$, $(2, 6)$, $(6, 2)$ and $(10, 0)$. The cost $C = 3x + y$ is to be minimised. Find the minimum cost and describe all the points at which it is achieved. [5]
- 21** The feasible region for $P = 4x + ky$ ($k > 0$) is bounded by $2x + y \leq 40$, $x + y \leq 24$, $x + 2y \leq 38$, $x \geq 0$, $y \geq 0$, with vertices $(0, 0)$, $(20, 0)$, $(16, 8)$, $(10, 14)$ and $(0, 19)$. Find the range of values of k for which each of the vertices $(20, 0)$, $(16, 8)$, $(10, 14)$ and $(0, 19)$ is optimal. [8]
- 22** The region R is bounded by $x + y \leq 10$, $2x + y \leq 16$, $x + 3y \leq 24$, $x \geq 0$, $y \geq 0$ and the profit is $P = 2x + 5y$. The manager would like to make as many items ($x + y$) as possible while earning a profit of at least £35. Find the maximum number of items that can be made, and the range of values of x at which this is achieved. [7]
- 23** A company sells product A at £40 and product B at £60, in numbers x and y . The costs are 35% of the selling price for A and 30% of the selling price for B, plus a delivery cost of £2 per unit for each product. Find the profit objective function. The feasible region has vertices $(0, 0)$, $(8, 0)$, $(6, 4)$, $(3, 7)$, $(0, 8)$. Find the optimal vertex, and find whether it changes if the delivery cost is raised to £6 per unit. [7]
- 24** The feasible region R is bounded by $3x + 2y \leq 60$, $x + 3y \leq 48$, $y \leq 15$, $x \geq 0$, $y \geq 0$, with vertices $(0, 0)$, $(20, 0)$, $(12, 12)$, $(3, 15)$ and $(0, 15)$. Show that $P = 3x + 2y$ has more than one optimal vertex, and find all the optimal solutions in which x and y are both integers. [7]
- 25** A three-variable problem has feasible region with vertices $(0, 0, 0)$, $(4, 0, 0)$, $(0, 3, 0)$, $(4, 2, 0)$, $(3, 3, 0)$, $(0, 0, 6)$, $(4, 0, 2)$ and $(0, 3, 3)$. The objective is to maximise $P = 2x + 3y + 4z$. Find the value of P at each vertex and state the optimal solution. A student claims that, because y has a larger coefficient than x , the optimum must have y larger than x . Show that the claim is false. [6]
- 26** The cost $C = 6x + 5y$ is to be minimised over the region defined by $2x + y \geq 20$, $x + 3y \geq 30$, $x + y \geq 14$, $x \geq 0$, $y \geq 0$, which has vertices $(0, 20)$, $(6, 8)$ and $(30, 0)$. A student writes: 'The vertices on the axes are the cheapest because one of the variables is zero. C is 100 at $(0, 20)$ and 180 at $(30, 0)$, so the minimum is 100.' Explain what is wrong with the student's reasoning and give the correct minimum. [6]
- 27** The profits per unit on two products are £ p and £ q with $p + q = 10$, and $p, q \geq 0$. The feasible region has vertices $(0, 0)$, $(8, 0)$, $(6, 4)$, $(3, 7)$, $(0, 8)$, bounded by $2x + y \leq 16$, $x + y \leq 10$, $x + 3y \leq 24$, $x \geq 0$, $y \geq 0$. The objective is $P = px + qy$. Find the range of values of p for which each of $(8, 0)$, $(6, 4)$, $(3, 7)$ and $(0, 8)$ is optimal. [9]

- 28** The feasible region R is bounded by $x \geq 2$, $x + y \leq 10$, $y \leq 6$ and $y \geq 0$. Find the maximum value of the ratio $\frac{y}{x}$ over R , and the point at which it occurs. Explain why the problem of finding the greatest value of $\frac{y}{x}$ can be treated using lines through the origin. [8]
- 29** The feasible region R is bounded by $3x + 2y \leq 60$, $x + 3y \leq 48$, $y \leq 15$, $x \geq 0$, $y \geq 0$, with vertices $(0, 0)$, $(20, 0)$, $(12, 12)$, $(3, 15)$, $(0, 15)$. The profit is $P = ax + 4y$, where a is a positive constant. When $a = 5$ the optimum is $(12, 12)$.
(i) Find the range of values of a for which $(12, 12)$ remains optimal.
(ii) Find the maximum value of P when $a = 7$, and the optimal solution. [8]
- 30** The objective function $P = ax + by$ is replaced by $Q = \lambda P + \mu$, where λ and μ are constants and $\lambda \neq 0$.
Prove that the vertex at which P is maximised also maximises Q if $\lambda > 0$, and minimises Q if $\lambda < 0$. Hence explain how maximising $P = 2x + 3y$ gives the same solution as minimising $Q = 100 - 4x - 6y$. [8]

Simplex Algorithm

- 1** A linear programming problem is: maximise $P = 3x + 5y$ subject to $2x + 3y \leq 24$, $x + 4y \leq 20$, $x, y \geq 0$.
Write the problem in the form needed for the simplex algorithm, by introducing slack variables r and s . [2]
- 2** A problem is to maximise $P = 4x + 7y$. Write down the equation of the objective row of the initial simplex tableau in the form $P + \dots = 0$, and state the entries of the P row under x and y . [1]
- 3** The objective row of a simplex tableau for a maximisation problem is $(P: -3, -5, 0, 0, 0)$. State which variable should enter the basis, and why. [2]
- 4** In a simplex tableau the entering column has entries 2, 4 and 0 in the three constraint rows, and the Value column has entries 12, 20 and 15. Find the ratios and state which row gives the pivot. [2]
- 5** State the condition on the objective row that shows that a simplex tableau for a maximisation problem is optimal. [1]
- 6** The diagram shows the final tableau of a simplex solution. Read off the optimal values of x , y , t and P , and the value of the slack variable t .
- | b.v. | x | y | r | s | t | Value |
|------|-----|-----|-----|-----|-----|-------|
| x | 1 | 0 | 2 | -1 | 0 | 8 |
| y | 0 | 1 | -1 | 1 | 0 | 4 |
| t | 0 | 0 | -5 | 2 | 1 | 2 |
| P | 0 | 0 | 1 | 1 | 0 | 28 |
- [2]
- 7** Maximise $P = 2x + 3y$ subject to $x + y \leq 12$, $x + 2y \leq 16$, $3x + y \leq 30$, $x, y \geq 0$.
Introduce slack variables r , s and t and set up the initial simplex tableau, with basic variables r , s and t . [3]

- 8** The initial tableau for the problem: maximise $P = 2x + 3y$ subject to $x + y \leq 12$, $x + 2y \leq 16$, $3x + y \leq 30$, $x, y \geq 0$ is shown in the diagram. Choose the pivot and perform one iteration of the simplex algorithm, stating the row operations used.

b.v.	x	y	r	s	t	Value
r	1	1	1	0	0	12
s	1	2	0	1	0	16
t	3	1	0	0	1	30
P	-2	-3	0	0	0	0

[5]

- 9** Explain why, in the simplex algorithm, the pivot row is chosen as the row with the smallest non-negative ratio rather than the largest. [2]

- 10** After one iteration of the simplex algorithm for maximising $P = 2x + 3y$ the tableau has basic variables r, y and t with values $r = 4$, $y = 8$, $t = 22$ and $P = 24$, and the P row is $(-\frac{1}{2}, 0, 0, \frac{3}{2}, 0 | 24)$. State the current values of x, y and P, and explain whether the tableau is optimal. [3]

- 11** The final tableau for the problem in which $P = 2x + 3y$ is maximised subject to $x + y \leq 12$, $x + 2y \leq 16$, $3x + y \leq 30$ has slack variable $t = 2$ and slack variables $r = s = 0$ (non-basic). Interpret these values for the three constraints. [3]

- 12** The diagram shows the initial simplex tableau for a maximisation problem in x, y and z, with slack variables r, s and t. Write down the linear programming problem that the tableau represents, giving the objective function and the three constraints.

b.v.	x	y	z	r	s	t	Value
r	2	1	1	1	0	0	18
s	1	3	2	0	1	0	30
t	3	1	0	0	0	1	24
P	-4	-5	-3	0	0	0	0

[3]

- 13** A problem is to maximise $P = 6x + 8y + 5z$ subject to $2x + 3y + z \leq 30$, $x + y + z \leq 14$, $3x + y + 2z \leq 27$, $x, y, z \geq 0$. Write the constraints using slack variables r, s and t. State the basic feasible solution of the initial tableau and the value of P there. [3]

- 14** Maximise $P = 7x + 5y$ subject to $x \leq 6$, $2x + 3y \leq 24$, $x + y \leq 10$, $x, y \geq 0$. The initial tableau is shown in the diagram. Use the simplex algorithm to solve the problem, showing each pivot and the row operations.

b.v.	x	y	r	s	t	Value
r	1	0	1	0	0	6
s	2	3	0	1	0	24
t	1	1	0	0	1	10
P	-7	-5	0	0	0	0

[6]

- 15** After one iteration of the simplex algorithm on a problem, the tableau is shown in the diagram. Carry out the next iteration, stating the pivot, and state the new values of x , y and P .

b.v.	x	y	r	s	t	Value
r	0	1	1	0	-3	3
s	0	$\frac{3}{2}$	0	1	$-\frac{1}{2}$	$\frac{9}{2}$
x	1	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{15}{2}$
P	0	$-\frac{3}{2}$	0	0	$\frac{5}{2}$	$\frac{75}{2}$

[5]

- 16** The initial tableau for the problem: maximise $P = 3x + 4y$ subject to $2x + y \leq 18$, $x + 2y \leq 16$, $x + y \leq 11$, $x, y \geq 0$ is shown in the diagram, with slack variables r , s and t .
 (i) State, with a reason, which variable should enter the basis and find the pivot row.
 (ii) Show that using the row with the largest ratio as the pivot row would give a negative value for a basic variable.

b.v.	x	y	r	s	t	Value
r	2	1	1	0	0	18
s	1	2	0	1	0	16
t	1	1	0	0	1	11
P	-3	-4	0	0	0	0

[5]

- 17** Maximise $P = x + y$ subject to $x - y \leq 3$, $x - 2y \leq 4$, $x, y \geq 0$.
 Use the simplex algorithm to show that the problem is unbounded.

[6]

- 18** A bakery makes x cakes and y pies. Each cake uses 2 units of flour and 1 hour of oven time, and each pie uses 1 unit of flour and 2 hours of oven time. At most 10 units of flour, 7 hours of oven time are available and $x + 3y \leq 15$ on packaging. The profit is £3 per cake and £2 per pie.
 Formulate the problem, set up the initial tableau, and carry out the first iteration of the simplex algorithm, stating the values of x , y and P after it.

[7]

- 19** The tableau below is for a maximisation problem in x , y and z . The constraint rows are (x , y , z , r , s | value): row r : (1, 2, 0, 1, 0 | 8); row s : (2, 0, 4, 0, 1 | 24) and the P row is (-3 , -2 , -5 , 0, 0 | 0).
 Perform one iteration, then state the values of all variables and say whether the tableau is optimal.

[6]

- 20** Maximise $P = 6x + 8y + 5z$ subject to $2x + 3y + z \leq 30$, $x + y + z \leq 14$, $3x + y + 2z \leq 27$, $x, y, z \geq 0$. The initial tableau (with slack variables r , s , t) is shown in the diagram.
 Perform the first iteration, and state the values of the variables and of P afterwards.

b.v.	x	y	z	r	s	t	Value
r	2	3	1	1	0	0	30
s	1	1	1	0	1	0	14
t	3	1	2	0	0	1	27
P	-6	-8	-5	0	0	0	0

[5]

- 21** A firm makes three products with quantities x , y and z . The problem is: maximise $P = 6x + 8y + 5z$ subject to $2x + 3y + z \leq 30$, $x + y + z \leq 14$, $3x + y + 2z \leq 27$, $x, y, z \geq 0$.
- (a) Set up the initial tableau, using slack variables r , s and t . [2]
- (b) Perform the first iteration. [2]
- (c) Perform the second iteration and state whether the final tableau is optimal. [4]
- (d) State the optimal quantities and find which resource is not fully used. [1]

- 22** Maximise $P = 5x + 4y$ subject to $6x + 4y \leq 48$, $x + 2y \leq 12$, $2x + y \leq 15$, $x, y \geq 0$. Use the simplex algorithm, with the slack variables r , s and t . At the second iteration there is a tie in the ratio test: explain why the tie means that a basic variable becomes zero in the next tableau, and complete the solution. [9]

- 23** Maximise $P = 2x + 4y$ subject to $x + 2y \leq 10$, $x + y \leq 8$, $y \leq 4$, $x, y \geq 0$, using the simplex algorithm with slack variables r , s and t . Show that the optimal solution is not unique, find a second optimal solution, and describe the complete set of optimal solutions. [9]

- 24** Maximise $P = 3x + 2y$ subject to $2x + y \leq 10$, $x + y \leq 7$, $x + 3y \leq 15$, $x, y \geq 0$. Solve the problem using the simplex algorithm with slack variables r , s and t . State the values of x , y and P at each tableau, and explain how the sequence of tableaux corresponds to moving between vertices of the feasible region. Explain also why the final vertex is degenerate. [8]

- 25** The final tableau of the problem: maximise $P = 2x + 3y$ subject to $x + y \leq 12$, $x + 2y \leq 16$, $3x + y \leq 30$, $x, y \geq 0$ is shown in the diagram, with slack variables r , s and t .

b.v.	x	y	r	s	t	Value
x	1	0	2	-1	0	8
y	0	1	-1	1	0	4
t	0	0	-5	2	1	2
P	0	0	1	1	0	28

- (a) The first constraint is changed to $x + y \leq 12 + \delta$. The columns of the final tableau give the new values of the basic variables as $x = 8 + 2\delta$, $y = 4 - \delta$ and $t = 2 - 5\delta$. Find the largest value of δ for which these values remain valid (non-negative). [3]
- (b) Find the new optimal value of P when $\delta = \frac{2}{5}$, and state the increase in P per unit increase in the first constraint's limit. [3]

- 26** The problem: maximise $P = 4x + 3y + 2z$ subject to $2x + y + z \leq 40$, $x + 2y + z \leq 36$, $x + y + 3z \leq 42$, with x, y, z non-negative integers. The simplex algorithm, ignoring the integer condition, gives $x = \frac{44}{3}$, $y = \frac{32}{3}$, $z = 0$ and $P = \frac{272}{3}$. Find the optimal integer solution, and justify that it is optimal. [7]

- 27** A florist makes three types of bouquet: Spring (x), Summer (y) and Autumn (z) each week. Each Spring bouquet needs 1 metre of ribbon, 2 hours of arranging and 2 hours of packing; each Summer bouquet needs 2 metres, 1 hour and 2 hours; each Autumn bouquet needs 2 metres, 3 hours and 1 hour. At most 30 metres of ribbon, 40 hours of arranging and 36 hours of packing are available each week. The profits are £3, £2 and £4 per bouquet respectively.
- (a) Formulate the problem as a linear programming problem. [4]
- (b) Set up the initial tableau with slack variables r , s and t , and perform the first iteration. [4]
- (c) Explain which variable enters at the second iteration, and why. [2]
- (d) The simplex algorithm finally gives $x = \frac{74}{7}$, $y = \frac{36}{7}$, $z = \frac{32}{7}$ and $P = \frac{422}{7}$. Find the best weekly plan using whole bouquets, justifying that it is the best. [2]

- 28** The final tableau of the problem: maximise $P = 6x + 8y + 5z$ subject to $2x + 3y + z \leq 30$, $x + y + z \leq 14$, $3x + y + 2z \leq 27$ is shown in the diagram, with slack variables r , s and t .

b.v.	x	y	z	r	s	t	Value
y	$\frac{1}{2}$	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	8
z	$\frac{1}{2}$	0	1	$-\frac{1}{2}$	$\frac{3}{2}$	0	6
t	$\frac{3}{2}$	0	0	$\frac{1}{2}$	$-\frac{5}{2}$	1	7
P	$\frac{1}{2}$	0	0	$\frac{3}{2}$	$\frac{7}{2}$	0	94

- Write down the shadow price of each of the three resources and interpret the one for the first resource. [3]
- The first resource limit is changed to $30 + \delta$. Use the r column to find the new values of y , z and t , and the range of δ for which the same basic variables remain valid. [4]
- The first resource is increased to 36. Find the new optimal plan and profit. [3]
- Explain why x remains zero in the optimal plan for every δ in the range. [1]

- 29** In a simplex tableau for a maximisation problem the basic variables have values $b_i \geq 0$. The pivot column has entries a_i in the constraint rows, and the pivot row r is chosen with $a_r > 0$ and $\frac{b_r}{a_r} \leq \frac{b_i}{a_i}$ for every i with $a_i > 0$.
Prove that, after the iteration, every basic variable still has a non-negative value, and that the value of P does not decrease when the objective row entry $p < 0$ for the entering column. [9]

- 30** Consider the problem: maximise $P = 3x + ky$ subject to $2x + y \leq 10$, $x + 3y \leq 15$, $x, y \geq 0$, where k is a positive constant.
- For the initial tableau with slack variables r and s , state the P row and find the range of k for which x , rather than y , enters first. [3]
 - Take $k = 5$. Solve the problem using the simplex algorithm. [5]
 - For this vertex to remain optimal, the entries under r and s in the final P row must be non-negative. Show that these entries are $\frac{(9-k)}{5}$ and $\frac{(2k-3)}{5}$, and find the range of k for which $(3, 4)$ is optimal. [4]

Big-M Method

- A linear programming problem is: maximise $P = 3x + 2y$ subject to $x + y \leq 10$, $x + 3y \leq 18$, $x \geq 2$, $x, y \geq 0$.
Explain why the simplex algorithm cannot be started in the usual way with the basic variables equal to the slack variables. [2]
- Write the constraint $x + y \geq 8$ as an equation by introducing a surplus variable s , and state the condition on s . [2]
- The constraint $x \geq 2$ is written as $x - s + t = 2$, where s is a surplus variable and t is an artificial variable.
Explain the purpose of the artificial variable t . [2]
- State what M represents in the big-M method. [1]
- The objective is to maximise $P = 3x + 2y$ and an artificial variable t is added to a constraint. Write down the modified objective function used in the big-M method, and write it as an equation with all variables on the left. [2]

6 State which of the constraint types $\leq$, $\geq$ and $=$ need an artificial variable in the big-M method. [1]

7 Maximise $P = 3x + 2y$ subject to $x + y \leq 10$, $x + 3y \leq 18$, $x \geq 2$, $x, y \geq 0$. Write the constraints as equations using slack variables s_1 and s_2 , a surplus variable s_3 and an artificial variable t , and write the modified objective equation. [3]

8 For the equations $x + y + s_1 = 10$, $x + 3y + s_2 = 18$, $x - s_3 + t = 2$, state the initial basic variables and their values, and explain why this is not a feasible solution of the original problem. [3]

9 In the big-M method for maximising $P = 3x + 2y$ with the constraint $x - s_3 + t = 2$, show how the objective row is obtained in the form with t removed. [3]

10 Explain how the equation $x - s_3 + t = 2$ is obtained from the constraint $x \geq 2$. [2]

11 The initial tableau for the problem: maximise $P = 3x + 2y$ subject to $x + y \leq 10$, $x + 3y \leq 18$, $x \geq 2$ is shown in the diagram. State which variable enters, which leaves, and the pivot, giving the ratios.

b.v.	x	y	s_1	s_2	s_3	t	Value
s_1	1	1	1	0	0	0	10
s_2	1	3	0	1	0	0	18
t	1	0	0	0	-1	1	2
P	-3	-M	-2	0	0	M	-2M

[4]

12 After one iteration of the big-M method the tableau is shown in the diagram, with basic variables s_1 , s_2 and x . State the current values of x , y and P , and say whether the artificial variable has been removed from the basis. State whether the tableau is optimal.

b.v.	x	y	s_1	s_2	s_3	t	Value
s_1	0	1	1	0	1	-1	8
s_2	0	3	0	1	1	-1	16
x	1	0	0	0	-1	1	2
P	0	-2	0	0	-3	3 + M	6

[4]

13 A big-M solution ends with the optimality condition satisfied but with the artificial variable t basic with value 2. State what this tells you about the original problem. [2]

14 The initial tableau of the big-M method for a problem is shown in the diagram. Perform the first iteration, stating the pivot, the ratios and the row operations, and write down the new values of x , y and P .

b.v.	x	y	s_1	s_2	s_3	t	Value
s_1	1	1	1	0	0	0	10
s_2	1	3	0	1	0	0	18
t	1	0	0	0	-1	1	2
P	-3	-M	-2	0	0	M	-2M

[4]

- 15** The tableau shown is the initial tableau for: maximise $P = 3x + 2y$ subject to $x + y \leq 10$, $x + 3y \leq 18$, $x \geq 2$. Use the big-M method to solve the problem, stating the pivots and the row operations.

b.v.	x	y	s_1	s_2	s_3	t	Value
s_1	1	1	1	0	0	0	10
s_2	1	3	0	1	0	0	18
t	1	0	0	0	-1	1	2
P	$-3 - M$	-2	0	0	M	0	$-2M$

[6]

- 16** Minimise $C = 3x + 5y$ subject to $x + 2y \geq 10$, $3x + y \geq 15$, $x + y \leq 20$, $x, y \geq 0$. Write the problem as a maximisation problem in P, introduce slack, surplus and artificial variables s_1, s_2, s_3, t_1, t_2 , and find the P row of the initial tableau after t_1 and t_2 have been removed from it.

[5]

- 17** Maximise $P = 3x + 2y$ subject to $x + y = 8$, $2x + y \leq 12$, $x, y \geq 0$. Use the big-M method with an artificial variable t in the equality constraint and a slack variable s in the second constraint. The initial tableau, after eliminating t from the P row, has P row $(-3 - M, -2 - M, 0, 0 | -8M)$.

[6]

- 18** Maximise $P = 2x + 3y$ subject to $x + y \leq 12$, $x + 2y \geq 8$, $2x + y \leq 18$, $x, y \geq 0$. Introduce slack variables s_1 and s_3 , a surplus variable s_2 and an artificial variable t, and write down the initial big-M tableau (with t eliminated from the P row).

[5]

- 19** Maximise $P = 3x + 2y$ subject to $x + y \leq 10$, $x + 3y \leq 18$, $x \geq 2$, $x, y \geq 0$. The final tableau has basic variables s_3, s_2, x with $s_3 = 8$, $s_2 = 8$, $x = 10$, $y = 0$ and $P = 30$. Interpret the value of s_3 in this context, and state which of the three constraints are binding.

[5]

- 20** Maximise $P = x + y$ subject to $x + y \leq 4$, $x + y \geq 6$, $x, y \geq 0$. Use the big-M method to show that the problem has no feasible solution. Use slack variable s_1 , surplus variable s_2 and artificial variable t.

[6]

- 21** Maximise $P = 2x + 3y$ subject to $x + y \leq 12$, $x + 2y \geq 8$, $2x + y \leq 18$, $x, y \geq 0$. The initial big-M tableau is shown in the diagram, with slack variables s_1 and s_3 , surplus variable s_2 and artificial variable t. Use the big-M method to solve the problem. State the optimal plan, and the amount by which the second constraint is exceeded.

b.v.	x	y	s_1	s_2	s_3	t	Value
s_1	1	1	1	0	0	0	12
t	1	2	0	-1	0	1	8
s_3	2	1	0	0	1	0	18
P	$-2 - M$	$-3 - 2M$	0	M	0	0	$-8M$

[7]

- 22** Minimise $C = 3x + 5y$ subject to $x + 2y \geq 10$, $3x + y \geq 15$, $x + y \leq 20$, $x, y \geq 0$. The problem has been converted to a maximisation problem with $P = -C$ and the initial big-M tableau (surplus variables s_1, s_2 , slack variable s_3 and artificial variables t_1, t_2) is shown in the diagram.

Solve the problem, state the minimum cost, and find how much of the third constraint's capacity is unused.

b.v.	x	y	s_1	s_2	s_3	t_1	t_2	Value
t_1	1	2	-1	0	0	1	0	10
t_2	3	1	0	-1	0	0	1	15
s_3	1	1	0	0	1	0	0	20
P	$3 - 4M$	$-5 - 3M$	M	M	0	0	0	$-25M$

[7]

- 23** A farmer minimises the cost $C = 30x + 45y$ (pence) of a feed mix subject to $4x + 3y \geq 36$ and $2x + 5y \geq 30$, $x, y \geq 0$. The problem is converted to maximise $P = -C$. The initial big-M tableau with surplus variables s_1, s_2 and artificial variables t_1, t_2 is shown in the diagram.

Use the big-M method to find the cheapest mix and its cost, giving exact values.

b.v.	x	y	s_1	s_2	t_1	t_2	Value
t_1	4	3	-1	0	1	0	36
t_2	2	5	0	-1	0	1	30
P	$30 - 6M$	$-45 - 8M$	M	M	0	0	$-66M$

[7]

- 24** Maximise $P = 5x + 4y$ subject to $x + y \leq 14$, $x \geq 3$, $y \geq 2$, $2x + y \leq 24$, $x, y \geq 0$. The big-M method with slack variables s_1, s_4 , surplus variables s_2, s_3 and artificial variables t_1, t_2 has been applied for two iterations, giving the tableau in the diagram ($x = 3, y = 2$).

Complete the solution, stating the pivots and the row operations, and the optimal values of x, y and P .

b.v.	x	y	s_1	s_2	s_3	s_4	t_1	t_2	Value
s_1	0	0	1	1	1	0	-1	-1	9
x	1	0	0	-1	0	0	1	0	3
y	0	1	0	0	-1	0	0	1	2
s_4	0	0	0	2	1	1	-2	-1	16
P	0	0	0	-5	-4	0	$5 + M$	$4 + M$	23

[7]

- 25** Maximise $P = 2x + y + 3z$ subject to $x + y + z = 12$, $2x + y \leq 16$, $y + 2z \geq 6$, $x, y, z \geq 0$. Slack variable s_1 is used for the second constraint, surplus s_2 for the third, and artificial variables t_1 and t_2 for the first and third. The initial tableau is shown in the diagram. Perform three iterations of the big-M method, and state the values of x, y, z, t_1, t_2 and P after the third iteration.

b.v.	x	y	z	s_1	s_2	t_1	t_2	Value
t_1	1	1	1	0	0	1	0	12
s_1	2	1	0	1	0	0	0	16
t_2	0	1	2	0	-1	0	1	6
P	$-2 - M$	$-1 - 2M$	$-3M$	0	M	0	0	$-18M$

[7]

- 26** For the problem: maximise $P = 3x + 2y$ subject to $x + y \leq 10$, $x + 3y \leq 18$, $x \geq 2$, a student sets up the initial big-M tableau with P row $(-3 + M, -2, 0, 0, -M, 0 | 2M)$. Identify the errors in the student's P row, show the correct derivation, and explain the effect the error would have on the choice of entering variable.

[6]

- 27** A school canteen buys two ingredients. Ingredient X costs 6 pence per unit and ingredient Y costs 5 pence per unit. A recipe must contain at least 20 units of vitamin A, at least 30 units of vitamin B and at least 14 units of fibre. Each unit of X has 2 units of vitamin A, 1 of vitamin B and 1 of fibre, and each unit of Y has 1, 3 and 1 respectively. The canteen buys x units of X and y units of Y and wishes to minimise the cost C pence.
- Formulate the problem as a linear programming problem. [3]
 - Explain how the big-M method deals with the $\geq$ constraints, and show that the P row of the initial tableau for maximising $P = -C$, after eliminating the artificial variables t_1, t_2, t_3 , is $(6 - 4M, 5 - 5M, M, M, M, 0, 0, 0 \mid -64M)$. [4]
 - Perform the first iteration. [3]
 - After further iterations the optimal tableau gives $x = 6, y = 8$ with all artificial variables zero. State the minimum cost and say what is special about the point $(6, 8)$ for the three constraints. [2]
-
- 28** Maximise $P = 5x + 2y$ subject to $x + y \leq 10, y \geq 3, x, y \geq 0$, using the big-M method with objective $P = 5x + 2y - Mt$, where t is the artificial variable in the constraint $y - s + t = 3$.
- Write down the true optimal solution.
 - By comparing the value of the modified objective at $(x, y, t) = (10, 0, 3)$ with its value at the true optimum, find the range of M for which the big-M method gives the correct answer, and explain why M must be large. [9]
-
- 29** A maximisation problem has objective $P = c_1x_1 + c_2x_2$ and a single $\geq$ constraint $a_1x_1 + a_2x_2 \geq b$, with $b > 0$, and no other constraint (apart from non-negativity). Write the constraint with a surplus variable s and an artificial variable t , and use the big-M method to prove that the P row of the initial tableau is $(-c_1 - Ma_1, -c_2 - Ma_2, M, 0 \mid -Mb)$. Hence state the condition on c_1, c_2, a_1, a_2 that makes x_1 the entering variable. [9]
-
- 30** The big-M method is used for: maximise $P = 5x + 4y$ subject to $x + y \leq 14, x \geq 3, y \geq 2, 2x + y \leq 24$. The successive tableaux give the values $(x, y, t_1, t_2) = (0, 0, 3, 2), (3, 0, 0, 2), (3, 2, 0, 0), (11, 2, 0, 0)$ and $(10, 4, 0, 0)$, where t_1 and t_2 are the artificial variables for $x \geq 3$ and $y \geq 2$. Explain what happens during the first two iterations in terms of the feasible region, and find the values of P at the last three tableaux. Find the value of the modified objective $P = 5x + 4y - M(t_1 + t_2)$ at the first tableau. [9]

Two-Stage Simplex Method

- State the purpose of Phase 1 of the two-stage simplex method. [1]
- A problem uses two artificial variables t_1 and t_2 . Write down the objective of Phase 1 as a function to be minimised. [1]
- State the value of I at the end of Phase 1 that shows that the original problem has a feasible solution. [1]
- State what is done with the artificial variable columns and the I row at the end of Phase 1 when $I = 0$. [1]
- During Phase 1 the original objective row P is carried along through the row operations. State whether the P row is used to choose the pivot column in Phase 1. [1]
- State which of the constraint types $\leq, \geq$ and $=$ require an artificial variable in the two-stage method. [1]

7 Maximise $P = 3x + 2y$ subject to $x + y \leq 10$, $x + 3y \leq 18$, $x \geq 2$, $x, y \geq 0$.
Write the constraints as equations using slack variables s_1 and s_2 , a surplus variable s_3 and an artificial variable t , and write down the objective of Phase 1. [3]

8 For the equations $x + y + s_1 = 10$, $x + 3y + s_2 = 18$, $x - s_3 + t = 2$, the Phase 1 objective is to maximise $I = -t$.
Show that the I row of the initial tableau is $(-1, 0, 0, 0, 1, 0 \mid -2)$, in the order x, y, s_1, s_2, s_3, t . [3]

9 The initial two-stage tableau for the problem in which $P = 3x + 2y$ is maximised subject to $x + y \leq 10$, $x + 3y \leq 18$, $x \geq 2$ is shown in the diagram. Carry out the first iteration of Phase 1, and state whether Phase 1 is complete.

b.v.	x	y	s_1	s_2	s_3	t	Value
s_1	1	1	1	0	0	0	10
s_2	1	3	0	1	0	0	18
t	1	0	0	0	-1	1	2
P	-3	-2	0	0	0	0	0
I	-1	0	0	0	1	0	-2

[3]

10 At the end of Phase 1 for the problem in which $P = 3x + 2y$ is maximised subject to $x + y \leq 10$, $x + 3y \leq 18$, $x \geq 2$, the basic variables are $s_1 = 8$, $s_2 = 16$ and $x = 2$.
State the basic feasible solution that Phase 2 starts from, and the value of P there. [2]

11 At the end of Phase 1 of the two-stage method for a problem, the I row has no negative entries but the value of I is -2 .
State what this shows about the original problem, with a reason. [2]

12 Give one advantage of the two-stage method over the big-M method. [2]

13 In Phase 1 of a two-stage tableau the P row is $(-3, -2, 0, 0, 0, 0 \mid 0)$ and the I row is $(-1, 0, 0, 0, 1, 0 \mid -2)$. State which column is the pivot column and explain why the P row is not used to decide this.
The ratios for that column are 10, 18 and 2. State the pivot row. [3]

14 Use the two-stage simplex method to solve: maximise $P = 3x + 2y$ subject to $x + y \leq 10$, $x + 3y \leq 18$, $x \geq 2$, $x, y \geq 0$. The initial tableau is shown in the diagram. State the pivots and row operations in both phases.

b.v.	x	y	s_1	s_2	s_3	t	Value
s_1	1	1	1	0	0	0	10
s_2	1	3	0	1	0	0	18
t	1	0	0	0	-1	1	2
P	-3	-2	0	0	0	0	0
I	-1	0	0	0	1	0	-2

[6]

15 Maximise $P = 2x + 3y$ subject to $x + y \leq 12$, $x + 2y \geq 8$, $2x + y \leq 18$, $x, y \geq 0$. Use slack variables s_1 and s_3 , a surplus variable s_2 and an artificial variable t .
Write down the initial tableau for the two-stage method, with rows for P and I. [6]

16 Maximise $P = 3x + 2y$ subject to $x + y = 8$, $2x + y \leq 12$, $x, y \geq 0$. Use the two-stage method with an artificial variable t in the equality constraint and a slack variable s in the second constraint.
Show that Phase 1 leaves the tableau optimal for Phase 2, and state the solution. [5]

- 17 Maximise $P = x + y$ subject to $x + y \leq 4$, $x + y \geq 6$, $x, y \geq 0$. Carry out Phase 1 of the two-stage method, with slack variable s_1 , surplus variable s_2 and artificial variable t , and explain what the result shows. [6]

- 18 Minimise $C = 3x + 5y$ subject to $x + 2y \geq 10$, $3x + y \geq 15$, $x + y \leq 20$, $x, y \geq 0$. Write the problem as a maximisation problem in $P = -C$, with surplus variables s_1, s_2 , slack variable s_3 and artificial variables t_1, t_2 , and set up the initial tableau for the two-stage method. [6]

- 19 The initial tableau of the two-stage method for the problem: minimise $C = 3x + 5y$ subject to $x + 2y \geq 10$, $3x + y \geq 15$, $x + y \leq 20$ is shown in the diagram. Carry out Phase 1 and state the resulting basic feasible solution.

b.v.	x	y	s_1	s_2	s_3	t_1	t_2	Value
t_1	1	2	-1	0	0	1	0	10
t_2	3	1	0	-1	0	0	1	15
s_3	1	1	0	0	1	0	0	20
P	3	5	0	0	0	0	0	0
I	-4	-3	1	1	0	0	0	-25

[6]

- 20 At the end of Phase 1 for the problem: minimise $C = 3x + 5y$ subject to $x + 2y \geq 10$, $3x + y \geq 15$, $x + y \leq 20$, the basic variables are y, x and s_3 with $x = 4$, $y = 3$, and the P row is $(0, 0, \frac{12}{5}, \frac{1}{5}, 0 \mid -27)$ for x, y, s_1, s_2, s_3 after deleting the artificial columns. Show that this is the optimal tableau, state the minimum value of C , and interpret s_3 . [5]

- 21 Maximise $P = 5x + 4y$ subject to $x + y \leq 14$, $x \geq 3$, $y \geq 2$, $2x + y \leq 24$, $x, y \geq 0$. The initial two-stage tableau, with slack variables s_1, s_4 , surplus variables s_2, s_3 and artificial variables t_1, t_2 , is shown in the diagram.

b.v.	x	y	s_1	s_2	s_3	s_4	t_1	t_2	Value
s_1	1	1	1	0	0	0	0	0	14
t_1	1	0	0	-1	0	0	1	0	3
t_2	0	1	0	0	-1	0	0	1	2
s_4	2	1	0	0	0	1	0	0	24
P	-5	-4	0	0	0	0	0	0	0
I	-1	-1	0	1	1	0	0	0	-5

- (a) Carry out Phase 1. [4]
 (b) State the basic feasible solution at the end of Phase 1, and the value of P there. [1]
 (c) Carry out Phase 2 and state the optimal solution. [4]

- 22 A farmer minimises the cost $C = 30x + 45y$ (pence) of a feed mix subject to $4x + 3y \geq 36$ and $2x + 5y \geq 30$, $x, y \geq 0$. The problem is converted to maximise $P = -C$, and the initial two-stage tableau (surplus variables s_1, s_2 , artificial variables t_1, t_2) is shown in the diagram. Use the two-stage method to find the cheapest mix and its cost, giving exact values.

b.v.	x	y	s_1	s_2	t_1	t_2	Value
t_1	4	3	-1	0	1	0	36
t_2	2	5	0	-1	0	1	30
P	30	45	0	0	0	0	0
I	-6	-8	1	1	0	0	-66

[7]

- 23** Minimise $C = 6x + 5y$ subject to $2x + y \geq 20$, $x + 3y \geq 30$, $x + y \geq 14$, $x, y \geq 0$. After conversion to maximise $P = -C$ with surplus variables s_1, s_2, s_3 and artificial variables t_1, t_2, t_3 , the initial two-stage tableau is shown in the diagram.
- Perform Phase 1, and show that at the end of Phase 1 the basic solution is $x = 6, y = 8$. Explain why the third iteration has a zero ratio, and state what this means geometrically.

b.v.	x	y	s_1	s_2	s_3	t_1	t_2	t_3	Value
t_1	2	1	-1	0	0	1	0	0	20
t_2	1	3	0	-1	0	0	1	0	30
t_3	1	1	0	0	-1	0	0	1	14
P	6	5	0	0	0	0	0	0	0
I	-4	-5	1	1	1	0	0	0	-64

[9]

- 24** Maximise $P = 2x + y + 3z$ subject to $x + y + z = 12$, $2x + y \leq 16$, $y + 2z \geq 6$, $x, y, z \geq 0$. Use slack variable s_1 for the second constraint, surplus variable s_2 for the third, and artificial variables t_1 (first constraint) and t_2 (third constraint). The initial two-stage tableau is shown in the diagram.
- Carry out Phase 1 and state the basic feasible solution with which Phase 2 begins.

b.v.	x	y	z	s_1	s_2	t_1	t_2	Value
t_1	1	1	1	0	0	1	0	12
s_1	2	1	0	1	0	0	0	16
t_2	0	1	2	0	-1	0	1	6
P	-2	-1	-3	0	0	0	0	0
I	-1	-2	-3	0	1	0	0	-18

[8]

- 25** Explain why Phase 1 of the two-stage method, which minimises $I = t_1 + t_2 + \dots + t_k$ over the artificial variables, ends with $I = 0$ if and only if the original problem has a feasible solution. Assume all the artificial variables are non-negative.

[6]

- 26** A student solves: minimise $C = 3x + 5y$ subject to $x + 2y \geq 10$, $3x + y \geq 15$, $x + y \leq 20$, using the two-stage method on $P = -C$. After the first iteration of Phase 1 the student's tableau has P row $(0, 4, 0, 1, 0, 0, -1 \mid -15)$ (for $x, y, s_1, s_2, s_3, t_1, t_2$) and I row $(0, -\frac{5}{3}, 1, -\frac{1}{3}, 0, 0, \frac{4}{3} \mid -5)$.

The student writes: 'The only negative entry in the P row is under t_2 , which is an artificial variable, so the tableau is optimal and $C = 15$.' Explain two errors in this.

[6]

- 27** A company makes three products in quantities x , y and z . The profit to be maximised is $P = 3x + 4y + 2z$. The constraints are $x + y + z \leq 20$, $x + 2y \geq 6$, $2x + z \geq 8$ and $y + z \leq 12$. Slack variables s_1 and s_4 , surplus variables s_2 and s_3 and artificial variables t_1 and t_2 (for the second and third constraints) are introduced.

b.v.	x	y	z	s_1	s_2	s_3	s_4	t_1	t_2	Value
s_1	1	1	1	1	0	0	0	0	0	20
t_1	1	2	0	0	-1	0	0	1	0	6
t_2	2	0	1	0	0	-1	0	0	1	8
s_4	0	1	1	0	0	0	1	0	0	12
P	-3	-4	-2	0	0	0	0	0	0	0
I	-3	-2	-1	0	1	1	0	0	0	-14

- (a) Write the constraints as equations and write down the objective of Phase 1. [2]
 (b) Perform Phase 1 using the initial tableau in the diagram and state the basic feasible solution reached. [4]
 (c) Perform Phase 2 and state the optimal solution. [4]
 (d) Find which constraints are binding at the optimum, and show that the optimal point is feasible by checking all four constraints. [2]

- 28** Maximise $P = x + y$ subject to $x + y \leq 10$ and $x \geq k$, $x, y \geq 0$, where k is a positive constant. The two-stage method is used with slack variable s_1 , surplus variable s_2 and artificial variable t .

Write down the initial constraint equations and I row, and by performing the first iteration of Phase 1 with the pivot in the x column, show that Phase 1 ends with $I = 0$ if $k \leq 10$ and with $I = -(k - 10)$ if $k > 10$. Hence state the values of k for which the problem is feasible. [10]

- 29** At the end of Phase 1 in the two-stage solution of the problem: maximise $P = 5x + 4y$ subject to $x + y \leq 14$, $x \geq 3$, $y \geq 2$, $2x + y \leq 24$, the basic feasible solution is $x = 3$, $y = 2$, with $s_2 = s_3 = 0$ (surplus variables) and the artificial variables deleted.

Show that this basic feasible solution is feasible, that it is a vertex of the feasible region, and explain why Phase 2 can start from it. State the value of the original objective there. [9]

- 30** In Phase 1 for the problem: maximise $P = 3x + 2y$ subject to $x + y \leq 10$, $x + 3y \leq 18$, $x \geq 2$, the P row is carried through the row operations. The I row pivot is the 1 in the x column of the third row, and the initial P row is $(-3, -2, 0, 0, 0, 0 | 0)$ for x, y, s_1, s_2, s_3, t .

- (i) Show that, after the iteration, the P row is $(0, -2, 0, 0, -3, 3 | 6)$, and check this by substituting for x directly. (ii) Delete the artificial column and complete Phase 2. [9]