

Linear and Quadratic Interpolation

1 (2 marks)

$$f(2.4) \approx 5 + \frac{2.4 - 2}{3 - 2} \times (9 - 5)$$

M1

$$= 6.6$$

A1

Answer: 6.6

2 marks

2 (2 marks)

$$\ln 2.5 \approx 0.6931 + \frac{2.5 - 2}{3 - 2} \times (1.0986 - 0.6931)$$

M1

$$= 0.6931 + 0.5 \times 0.4055 = 0.8959$$

A1

Answer: 0.8959

2 marks

3 (2 marks)

$$\theta \approx 23.5 + \frac{14 - 10}{20 - 10} \times (31.2 - 23.5)$$

M1

$$= 23.5 + 0.4 \times 7.7 = 26.58 \text{ }^{\circ}\text{C}$$

A1

Answer: 26.58 }^{\circ}\text{C}

2 marks

4 (2 marks)

$$\text{Gradient} = \frac{16 - 10}{6 - 4} = 3$$

M1

$$f(7) \approx 16 + 3 \times 1 = 19$$

A1

Answer: 19

2 marks

5 (2 marks)

$$\text{Error} = 4.92 - 4.8 = 0.12$$

M1

$$\text{Percentage error} = \frac{0.12}{4.8} \times 100 = 2.5\%$$

A1

Answer: 2.5%

2 marks

6 (3 marks)

(a) 35 lies between 10 and 50, so interpolation.

B1

(b) 60 lies outside the range of the data, so extrapolation

B1

Extrapolation is less reliable because the trend may not continue beyond the data.

E1

Answer: (a) interpolation (b) extrapolation, less reliable

3 marks

7 (3 marks)

x lies between 3 and 4 since $7.8 < 9.5 < 10.2$

B1

$$x \approx 3 + \frac{9.5 - 7.8}{10.2 - 7.8} \times (4 - 3)$$

M1

$$= 3 + \frac{1.7}{2.4} = 3.71$$

A1

Answer: 3.71

3 marks

8 (4 marks)

$$\text{(a) } \cos 44^{\circ} \approx 0.7660 + \frac{44 - 40}{50 - 40} \times (0.6428 - 0.7660)$$

M1

$$= 0.7660 - 0.4 \times 0.1232 = 0.7167$$

A1

Answer: 0.7167

2 marks

$$\text{(b) Error} = 0.7193 - 0.7167 = 0.0026$$

M1

$$\text{Percentage error} = \frac{0.0026}{0.7193} \times 100 = 0.36\%$$

A1

Answer: 0.36%

2 marks

9 (5 marks)

(a) Population $\approx 5.2 + \frac{4}{10} \times (6.1 - 5.2)$

M1

$= 5.56$ thousand

A1

Answer: 5.56 thousand

2 marks

(b) Population $\approx 5.2 + \frac{18}{10} \times 0.9 = 6.82$ thousand

M1 A1

It lies outside the range of the data, so we assume the trend continues unchanged.

E1

Answer: 6.82 thousand, extrapolation

3 marks

10 (4 marks)

(a) g changes sign on $[1, 2]$ and g is continuous

E1

Answer: sign change, continuous

1 mark

(b) Root $\approx 1 + \frac{2.3}{2.3 + 1.9} \times (2 - 1)$

M1 M1

$= 1.55$

A1

Answer: 1.55

3 marks

11 (5 marks)

(a) $a + b + c = 3$, $4a + 2b + c = 3$, $16a + 4b + c = 9$

M1

Subtracting: $3a + b = 0$ and $12a + 2b = 6$

M1

$a = 1$, $b = -3$

A1

$c = 5$

A1

Answer: a = 1, b = -3, c = 5

4 marks

(b) $p(3) = 5$

A1

Answer: 5

1 mark

12 (5 marks)

(a) The median is the 30th value, which lies between 30 and 40 since $15 < 30 < 38$

B1

median $\approx 30 + \frac{30 - 15}{38 - 15} \times (40 - 30)$

M1

$= 30 + \frac{150}{23} = 36.5$ mm

A1

Answer: 36.5 mm

3 marks

(b) Number below 35 $\approx 15 + \frac{35 - 30}{40 - 30} \times (38 - 15) = 26.5$

M1

Percentage $\approx \frac{26.5}{60} \times 100 = 44.2\%$

A1

Answer: 44.2%

2 marks

13 (5 marks)

(a) Use $x = 2$ and $x = 3$ (the values either side of 2.7)

$f(2.7) \approx 3.1 + 0.7 \times (5.0 - 3.1)$

M1

$= 4.43$

A1

Answer: 4.43

2 marks

(b) $f(5) \approx 8.2 + (8.2 - 5.0) = 11.4$

M1 A1

Outside the data range, so likely to be unreliable.

E1

Answer: 11.4, unreliable

3 marks

14 (6 marks)

(a) $f(2) = 8$, $f(3) = 27$

B1

$x \approx 2 + \frac{20 - 8}{27 - 8} \times (3 - 2)$

M1

$= 2 + \frac{12}{19} = 2.632$

A1

Answer: 2.632

3 marks

(b) $\sqrt[3]{20} = 2.7144$

B1

Percentage error $= \frac{2.7144 - 2.6316}{2.7144} \times 100 = 3.1\%$

M1 A1

Answer: 3.1%**3 marks****15 (7 marks)**

$$(a) \frac{\pi^2}{36}a + \frac{\pi}{6}b = \frac{1}{2} \text{ and } \frac{\pi^2}{4}a + \frac{\pi}{2}b = 1$$

M1

$$\text{Eliminating } b: -\frac{\pi^2}{6}a = \frac{1}{2}, \text{ so } a = -\frac{3}{\pi^2}$$

A1

$$\frac{\pi}{2}b = 1 + \frac{3}{4}, \text{ so } b = \frac{7}{2\pi}$$

A1

$$\text{Answer: } a = -\frac{3}{\pi^2}, b = \frac{7}{2\pi}$$

3 marks

$$(b) p\left(\frac{\pi}{4}\right) = -\frac{3}{16} + \frac{7}{8} = \frac{11}{16}$$

M1 A1

$$\text{Percentage error} = \frac{|0.6875 - 0.7071|}{0.7071} \times 100 = 2.8\%$$

M1 A1

$$\text{Answer: } \frac{11}{16}, 2.8\%$$

4 marks**16 (7 marks)**

$$(a) c = 4$$

B1

$$25a + 5b = 10 \text{ and } 100a + 10b = 16$$

M1

$$50a = -4, \text{ so } a = -0.08$$

M1

$$b = 2.4$$

A1

$$\text{Answer: } a = -0.08, b = 2.4, c = 4$$

4 marks

$$(b) v = -0.08 \times 49 + 2.4 \times 7 + 4 = 16.88$$

A1

$$\text{Answer: } 16.88 \text{ m s}^{-1}$$

1 mark

$$(c) -0.08t^2 + 2.4t - 12 = 0, \text{ so } t^2 - 30t + 150 = 0$$

M1

$$t = 15 - \sqrt{75} = 6.34 \text{ s (the other root, 23.7, is outside the range)}$$

A1

$$\text{Answer: } 6.34 \text{ s}$$

2 marks**17 (5 marks)**

$$(a) \text{ The line through } (a, f(a)) \text{ and } (b, f(b)) \text{ is } y - f(a) = \frac{f(b) - f(a)}{b - a}(x - a)$$

M1

$$\text{Set } y = 0: x - a = -\frac{f(a)(b - a)}{f(b) - f(a)}$$

M1

$$x = \frac{a(f(b) - f(a)) - f(a)(b - a)}{f(b) - f(a)} = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

A1

$$\text{Answer: } x = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

3 marks

$$(b) f(1) = -4, f(2) = 3$$

B1

$$x = \frac{1 \times 3 - 2 \times (-4)}{3 - (-4)} = \frac{11}{7}$$

M1

$$\text{Answer: } \frac{11}{7}$$

2 marks**18 (7 marks)**

$$(a) \sqrt{20} \approx 4 + \frac{20 - 16}{25 - 16} \times 1 = 4.4444$$

M1 A1

$$\text{Percentage error} = \frac{4.4721 - 4.4444}{4.4721} \times 100 = 0.62\%$$

M1 A1

$$\text{Answer: } 4.4444, 0.62\%$$

4 marks

$$(b) p(20) = -\frac{1}{504} \times 400 + \frac{97}{504} \times 20 + \frac{10}{7} = 4.4841$$

M1 A1

$$\text{Percentage error} = 0.27\%$$

A1

$$\text{Answer: } 4.4841, 0.27\%$$

3 marks**19 (5 marks)**

$$(a) h \approx 15 + 0.5 \times (20 - 15) = 17.5$$

M1 A1

$$\text{Answer: } 17.5 \text{ cm}$$

2 marks

$$(b) \text{Gradient} = 27 - 24 = 3 \text{ cm per week}$$

M1

$$h \approx 27 + 2 \times 3 = 33$$

Answer: 33 cm

A1

2 marks

(c) Weekly growth is slowing (5, 6, 5, 4, 3), so the rate keeps falling and the graph is levelling off; a constant gradient overestimates.

E1

Answer: growth slowing, so overestimate

1 mark

20 (5 marks)

(a) He used 3.5 and 5 instead of $3.5 - 3 = 0.5$ and $5 - 3 = 2$ in the fraction.

E1 E1

Answer: wrong fraction (should use 0.5/2)

2 marks

(b) $f(3.5) \approx 10 + \frac{0.5}{2} \times 8 = 12$

M1 A1

15.6 is greater than $f(4)$ would be on the line (14), so it is not sensible; the estimate must lie between 10 and 18 and be closer to 10.

E1

Answer: 12

3 marks

21 (9 marks)

(a) $p(x) = ax^2 + bx + c$: $a + b + c = 1$, $4a + 2b + c = \frac{1}{2}$, $16a + 4b + c = \frac{1}{4}$

$$3a + b = -\frac{1}{2} \text{ and } 12a + 2b = -\frac{1}{4}$$

$$a = \frac{1}{8}, b = -\frac{7}{8}$$

A1

$$c = \frac{7}{4}, \text{ so } p(x) = \frac{x^2 - 7x + 14}{8}$$

A1

Answer: $p(x) = \frac{x^2 - 7x + 14}{8}$

2 marks

(b) $p(3) = \frac{1}{4}$

M1

$$\text{Percentage error} = \frac{|\frac{1}{4} - \frac{1}{3}|}{\frac{1}{3}} \times 100 = 25\%$$

M1 A1

Answer: $\frac{1}{4}$, 25%

3 marks

(c) $f(3) \approx \frac{1}{2} + \frac{1}{2}(\frac{1}{4} - \frac{1}{2}) = \frac{3}{8}$

M1

$$\text{Percentage error} = \frac{|\frac{3}{8} - \frac{1}{3}|}{\frac{1}{3}} \times 100 = 12.5\%$$

M1 A1

Answer: $\frac{3}{8}$, 12.5%

3 marks

(d) The linear estimate is closer even though it uses less information, as the quadratic was fitted over $[1, 4]$ which is a wide interval where f varies rapidly.

E1

Answer: linear more accurate here

1 mark

22 (9 marks)

(a) $c = 2$

B1

$$4a + 2b = 1 \text{ and } 16a + 4b = 6$$

M1

$$8a = 4, a = \frac{1}{2}$$

A1

$$b = -\frac{1}{2}$$

A1

Answer: $a = \frac{1}{2}$, $b = -\frac{1}{2}$, $c = 2$

4 marks

(b) $\frac{1}{2}x^2 - \frac{1}{2}x + 2 = 4$, so $x^2 - x - 4 = 0$

M1 A1

$$x = \frac{1 \pm \sqrt{17}}{2} = 2.562 \text{ or } -1.56$$

M1

Only $x = 2.562$ is within the range $0 \leq x \leq 4$ (negative output is impossible)

E1

≈ 256 units

A1

Answer: about 256 units**5 marks****23 (8 marks)**

(a) $\text{Gradient} = \frac{b^2 - a^2}{b - a} = a + b$

M1

$y - a^2 = (a + b)(x - a)$, so $y = a^2 + (a + b)(x - a) = (a + b)x - ab$

M1 A1**Answer: $y = (a + b)x - ab$** **3 marks**

(b) $(a + b)x - ab - x^2 = -(x^2 - (a + b)x + ab)$

M1

$= -(x - a)(x - b) = (x - a)(b - x) > 0$

A1This is a downward quadratic in x with roots a and b , so it is greatest at $x = \frac{a + b}{2}$ **M1**

$\text{Maximum} = \frac{b - a}{2} \times \frac{b - a}{2} = \frac{(b - a)^2}{4}$

A1**Answer: overestimate $(x - a)(b - x)$; greatest overestimate $\frac{(b - a)^2}{4}$** **4 marks**

(c) $\frac{(0.2)^2}{4} = 0.01$

A1**Answer: 0.01****1 mark****24 (9 marks)**

(a) $f(1) = -1 < 0$ and $f(2) = 7 > 0$

B1

$x_1 = 1 + \frac{1}{8} \times 1 = 1.125$

M1 A1**Answer: $x_1 = 1.125$** **3 marks**

(b) $f(1.125) = -0.4512$

M1

$x_2 = \frac{1.125 \times 7 - 2 \times (-0.4512)}{7 - (-0.4512)}$

M1

$= 1.178$

A1**Answer: 1.178****3 marks**

(c) x_1 : 7.3%, x_2 : 2.9%

M1 A1

One endpoint stays at 2, far from the root, and the curve is convex, so each chord lies above the curve and the estimate approaches from one side only.

E1**Answer: x_1 : 7.3%, x_2 : 2.9%; one end stays at 2, so convergence is slow****3 marks****25 (9 marks)**

(a) $a + b + c = 15.1$, $4a + 2b + c = 20.4$, $9a + 3b + c = 15.9$

$3a + b = 5.3$ and $5a + b = -4.5$

M1

$a = -4.9$, $b = 20$

A1

$c = 0$

A1**Answer: $a = -4.9$, $b = 20$, $c = 0$** **3 marks**

(b) Greatest height at $t = -\frac{b}{2a} = \frac{20}{9.8} = 2.04$

M1 A1

$h = 20.4$ m

A1**Answer: 20.4 m at $t = 2.04$ s****3 marks**

(c) $-4.9t^2 + 20t = 0$ gives $t = 4.08$ s

A1Motion under constant acceleration gives $h = ut + \frac{1}{2}gt^2$, which is exactly quadratic in t (so with a**E1 E1**

$= -4.9 = -\frac{9.8}{2}$ the fit is exact).

Answer: 4.08 s**3 marks****26 (9 marks)**

(a) $f(2) \approx 3 + \frac{1}{2} \times 8 = 7$

M1 A1**Answer: 7****2 marks**

(b) $p(x) = ax^2 + bx + 2$: $a + b = 1$, $9a + 3b = 9$

M1

$a = 1$, $b = 0$, so $p(x) = x^2 + 2$

A1

$p(2) = 6$

A1

Answer: $p(x) = x^2 + 2$, $f(2) \approx 6$ **3 marks****(c) $p(4) = 18$** **B1**

$$\text{Percentage difference} = \frac{18.6 - 18}{18.6} \times 100 = 3.2\%$$

M1 A1**Answer: 18, 3.2%****3 marks**

(d) The quadratic: it uses curvature of the data (and its prediction at $x = 4$ is within about 3%), whereas the chord ignores curvature.

E1**Answer: the quadratic, because it allows for the curvature of the data****1 mark****27 (11 marks)****(a) $p(x) = ax^2 + bx + c$; $p(0) = y_1$ gives $c = y_1$** **B1**

$$y_2 - y_0 = 2bh, \text{ so } b = \frac{y_2 - y_0}{2h}$$

M1 A1

$$y_0 + y_2 = 2ah^2 + 2y_1, \text{ so } a = \frac{y_0 - 2y_1 + y_2}{2h^2}$$

M1 A1

$$\text{Answer: } p(x) = y_1 + \frac{y_2 - y_0}{2h}x + \frac{y_0 - 2y_1 + y_2}{2h^2}x^2$$

5 marks**(b) Odd term integrates to zero: $\int_{-h}^h p(x) dx = 2hy_1 + \frac{2h^3}{3} \times a$** **M1 M1**

$$= 2hy_1 + \frac{h}{3}(y_0 - 2y_1 + y_2) = \frac{h}{3}(y_0 + 4y_1 + y_2)$$

A1

$$\text{Answer: } \int_{-h}^h p(x) dx = \frac{h}{3}(y_0 + 4y_1 + y_2)$$

3 marks**(c) $h = 0.5$; $y = 1, \frac{2}{3}, \frac{1}{2}$** **B1**

$$\text{Integral} \approx \frac{0.5}{3} \left(1 + \frac{8}{3} + \frac{1}{2}\right) = 0.6944$$

M1 A1**Answer: 0.6944****3 marks****28 (11 marks)****(a) $f(4) = 2$, $f'(x) = \frac{1}{2\sqrt{x}}$ so $f'(4) = \frac{1}{4}$** **M1 A1**

$$p(x) = 2 + \frac{1}{4}(x - 4) + k(x - 4)^2$$

M1

$$p(9) = 3: 2 + \frac{5}{4} + 25k = 3$$

M1

$$k = -\frac{1}{100}$$

A1

$$\text{Answer: } k = -\frac{1}{100}$$

5 marks**(b) $p(6) = 2 + \frac{1}{2} - \frac{4}{100} = 2.46$** **M1 A1**

$$\sqrt{6} = 2.4495; \text{ percentage error} = 0.43\%$$

M1 A1**Answer: 2.46, 0.43%****4 marks****(c) $p(4) = f(4)$ and $p'(4) = f'(4)$** **M1**

So $p(x) - \sqrt{x}$ has a repeated root at 4: the graphs touch and share a tangent there, so p is a very good approximation near $x = 4$.

E1**Answer: $p(x) - \sqrt{x}$ has a double root at $x = 4$: the graphs touch and share a tangent****2 marks****29 (12 marks)****(a) $x \approx 1 + 0.5 \times \frac{1}{1.875} = 1.2667$** **M1 A1****Answer: 1.2667****2 marks****(b) Points (y, x) : $(-1, 1), (\frac{7}{8}, \frac{3}{2}), (5, 2)$** **B1**

$$A - B + C = 1, \frac{49}{64}A + \frac{7}{8}B + C = \frac{3}{2}, 25A + 5B + C = 2$$

M1

$$\text{Subtract the first equation from the second and third: } -\frac{15}{64}A + \frac{15}{8}B = \frac{1}{2} \text{ and } 24A + 6B = 1$$

From the second, $B = \frac{1 - 24A}{6}$; substituting: $-\frac{15}{64}A + \frac{5}{16}(1 - 24A) = \frac{1}{2}$, so $-\frac{495}{64}A = \frac{3}{16}$, $A = -\frac{4}{165}$ M1 A1

$B = \frac{29}{110}$, $C = \frac{85}{66}$ A1 A1

Answer: $A = -\frac{4}{165}$, $B = \frac{29}{110}$, $C = \frac{85}{66}$ 6 marks

(c) $x \approx C = 1.2879$ M1 A1

Percentage error = 2.8% M1 A1

Answer: 1.2879, 2.8% 4 marks

30 (10 marks)

(a) Line through (1, 1) and (2, $\frac{1}{2}$): gradient $-\frac{1}{2}$ M1

$y = 1 - \frac{1}{2}(x - 1) = \frac{3 - x}{2}$ A1

Answer: $y = \frac{3 - x}{2}$ 2 marks

(b) $\frac{3 - x}{2} - \frac{1}{x} = \frac{x(3 - x) - 2}{2x} = -\frac{(x - 1)(x - 2)}{2x}$ M1 A1

For $1 < x < 2$, $(x - 1)(x - 2) < 0$ so the difference is positive. E1

Answer: $\frac{3 - x}{2} - \frac{1}{x} = \frac{(x - 1)(2 - x)}{2x} > 0$ for $1 < x < 2$ 3 marks

(c) $E'(x) = -\frac{1}{2} + \frac{1}{x^2}$ M1

$E'(x) = 0$ gives $x = \sqrt{2}$ A1

$E(\sqrt{2}) = \frac{3 - \sqrt{2}}{2} - \frac{1}{\sqrt{2}} = \frac{3}{2} - \sqrt{2}$ M1 A1

≈ 0.0858 ; $E'' = -\frac{2}{x^3} < 0$ so this is a maximum E1

Answer: $x = \sqrt{2}$, $E = \frac{3}{2} - \sqrt{2} \approx 0.0858$ 5 marks

Lagrange and Newton Interpolation**1 (2 marks)**

$L_0(x) = \frac{x - 3}{1 - 3} = \frac{3 - x}{2}$ B1

$L_1(x) = \frac{x - 1}{3 - 1} = \frac{x - 1}{2}$ B1

Answer: $L_0(x) = \frac{3 - x}{2}$, $L_1(x) = \frac{x - 1}{2}$ 2 marks

2 (2 marks)

$L_1(x) = \frac{(x - 0)(x - 2)}{(1 - 0)(1 - 2)} = -x(x - 2)$ M1

$L_1(0.5) = -0.5 \times (-1.5) = 0.75$ A1

Answer: $L_1(x) = -x(x - 2)$, $L_1(0.5) = 0.75$ 2 marks

3 (2 marks)

$f[x_0, x_1] = \frac{10 - 4}{3 - 1} = 3$ M1

$p(x) = 4 + 3(x - 1)$ A1

Answer: 3; $p(x) = 4 + 3(x - 1)$ 2 marks

4 (3 marks)

First differences: 3, 5, 7

B1

Second differences: 2, 2; third difference: 0

B1

Second differences constant, so the polynomial has degree 2.

B1

Answer: Δ : 3, 5, 7; Δ^2 : 2, 2; Δ^3 : 0; degree 2

3 marks

5 (3 marks) $f[1, 2] = 2$ and $f[2, 4] = 3$

M1

$$f[1, 2, 4] = \frac{3 - 2}{4 - 1} = \frac{1}{3}$$

M1 A1

Answer: $\frac{1}{3}$

3 marks

6 (2 marks)

(a) Exactly one

B1

(b) 5

B1

Answer: (a) one (b) 5

2 marks

7 (5 marks)

$$(a) L_0(x) = \frac{(x-1)(x-3)}{3}, L_1(x) = -\frac{x(x-3)}{2}, L_2(x) = \frac{x(x-1)}{6}$$

B1 B1

$$\text{Answer: } L_0 = \frac{(x-1)(x-3)}{3}, L_1 = -\frac{x(x-3)}{2}, L_2 = \frac{x(x-1)}{6}$$

2 marks

$$(b) p(x) = 1 \cdot L_0 + 3 \cdot L_1 + 13 \cdot L_2$$

M1

$$x^2 \text{ coefficient: } \frac{1}{3} - \frac{3}{2} + \frac{13}{6} = 1; x \text{ coefficient: } -\frac{4}{3} + \frac{9}{2} - \frac{13}{6} = 1; \text{ constant: } 1$$

M1 A1

Answer: $p(x) = x^2 + x + 1$

3 marks

8 (5 marks)

$$(a) f[-1, 1] = \frac{2 - (-2)}{1 - (-1)} = 2, f[1, 2] = 5$$

A1

$$f[-1, 1, 2] = \frac{5 - 2}{2 - (-1)} = 1$$

A1

Answer: 2, 5, 1

2 marks

$$(b) p(x) = -2 + 2(x+1) + 1 \cdot (x+1)(x-1)$$

M1 A1

$$p(0) = -2 + 2 - 1 = -1$$

A1

Answer: $p(0) = -1$

3 marks

9 (5 marks)

$$(a) p_2(3) = 1 + 3 + 6 = 10$$

A1

Answer: 10

1 mark

(b) $p_3(x) = p_2(x) + kx(x-1)(x-2)$ keeps the first three values

M1

$$p_3(3) = 14: 10 + 6k = 14, \text{ so } k = \frac{2}{3}$$

M1 A1

The extra term vanishes at the first three nodes, so the earlier work is reused.

E1

Answer: $k = \frac{2}{3}$

4 marks

10 (5 marks)

$$(a) \Delta: 3, 7, 11; \Delta^2: 4, 4; \Delta^3: 0$$

A1

$$\Delta f_0 = 3, \Delta^2 f_0 = 4, \Delta^3 f_0 = 0$$

A1

Answer: 3, 4, 0

2 marks

$$(b) p(x) = 1 + 3x + 2x(x-1) = 2x^2 + x + 1$$

M1 A1

$$p(1.5) = 4.5 + 1.5 + 1 = 7$$

A1

Answer: $p(x) = 2x^2 + x + 1, f(1.5) \approx 7$

3 marks

11 (5 marks)

$$(a) P_{0,1}(3) = \frac{(3-2)(3) - (3-1)(7)}{1-2} = 11$$

M1 A1

$$P_{1,2}(3) = \frac{(3-4)(7) - (3-2)(21)}{2-4} = 14$$

M1

$$P_{0,1,2}(3) = \frac{(3-4)(11) - (3-1)(14)}{1-4} = 13$$

A1

Answer: $f(3) \approx 13$

4 marks

(b) $f[1, 2] = 4$, $f[2, 4] = 7$, $f[1, 2, 4] = 1$, so $p(x) = 3 + 4(x-1) + (x-1)(x-2) = x^2 + x + 1$,
 $p(3) = 13$

A1

Answer: $p(x) = x^2 + x + 1$, $p(3) = 13$

1 mark

12 (5 marks)

(a) First: $-1, 5, 26$

M1

Second: $3, 7$; third: 1

A1

$$p(x) = 1 - x + 3x(x-1) + 1 \cdot x(x-1)(x-2)$$

A1

Answer: $p(x) = 1 - x + 3x(x-1) + x(x-1)(x-2)$

3 marks

(b) $p(3) = 1 - 3 + 18 + 6 = 22$

M1 A1

Answer: 22

2 marks

13 (5 marks)

(a) $f[1, 2] = 0.6931$, $f[2, 4] = 0.3466$, $f[1, 2, 4] = -0.11550$

A1

$$p(3) = 0 + 0.6931 \times 2 + (-0.11550) \times 2 \times 1 = 1.1552$$

M1 A1

Answer: 1.1552

3 marks

$$(b) \frac{1.1552 - 1.0986}{1.0986} \times 100 = 5.2\%$$

M1 A1

Answer: 5.2%

2 marks

14 (5 marks)

(a) Let $r = p - q$. Then r has degree at most 2

B1

$r(x_0) = r(x_1) = r(x_2) = 0$, three distinct roots

M1

A non-zero polynomial of degree at most 2 has at most 2 roots, so r is identically zero, $p = q$.

E1

Answer: $p = q$ has degree at most 2 and three roots, so $p = q$

3 marks

(b) $p(x) = x + 1$ and $q(x) = x^2 + 1$ both pass through $(0, 1)$ and $(1, 2)$ but differ.

B1 B1

Answer: $x + 1$ and $x^2 + 1$

2 marks

15 (7 marks)

(a) Δ : $1, 11, 33, 67$

M1

Δ^2 : $10, 22, 34$; Δ^3 : $12, 12$

A1

Third differences are constant, so a cubic.

E1

Answer: cubic ($\Delta^3 = 12$)

3 marks

(b) Next $\Delta^3 = 12$, so $\Delta^2 = 46$, $\Delta = 113$

M1

$$f(5) = 115 + 113 = 228$$

A1

Answer: 228

2 marks

$$(c) f(x) = 3 + x + \frac{10}{2}x(x-1) + \frac{12}{6}x(x-1)(x-2)$$

M1

$$f(x) = 2x^3 - x^2 + 3$$

A1

Answer: $2x^3 - x^2 + 3$

2 marks

16 (5 marks)

(a) Δ : $2, 4, 6, 8, 12, 10, 14$

M1

Δ^2 : $2, 2, 2, 4, -2, 4$

A1

Δ^2 should be constant (2); the disturbance $2, -4, 2$ in a column of three centres on $f(5)$: $f(5)$ is wrong.

E1

Answer: $f(5)$ is wrong

3 marks

(b) A single error ϵ gives disturbances $\epsilon, -2\epsilon, \epsilon$; so $\epsilon = 2$

M1

$$\text{Correct } f(5) = 33 - 2 = 31$$

A1

Answer: 31 (error +2)**2 marks****17 (7 marks)**

$$(a) L_2(x) = \frac{(x+1)x(x-2)}{(2)(1)(-1)} = -\frac{x(x+1)(x-2)}{2}$$

M1 A1

$$\text{Answer: } -\frac{x(x+1)(x-2)}{2}$$

2 marks

$$(b) L_0 = -\frac{x(x-1)(x-2)}{6}, L_1 = \frac{(x+1)(x-1)(x-2)}{2}, L_3 = \frac{(x+1)x(x-1)}{6}$$

M1 L_2 is the polynomial found in part (a)

$$p = 3L_0 + L_1 - L_2 + 7L_3$$

M1

Expanding and collecting terms:

M1

$$p(x) = \frac{5}{3}x^3 - \frac{11}{3}x + 1$$

A1 A1

$$\text{Answer: } p(x) = \frac{5}{3}x^3 - \frac{11}{3}x + 1$$

5 marks**18 (6 marks)**(a) Nodes $y = 1, 3, 7$ with $x = 1, 2, 3$ **B1**

$$x \approx 1 \cdot \frac{(5-3)(5-7)}{(1-3)(1-7)} + 2 \cdot \frac{(5-1)(5-7)}{(3-1)(3-7)} + 3 \cdot \frac{(5-1)(5-3)}{(7-1)(7-3)}$$

M1 A1

$$= -\frac{1}{3} + 2 + 1 = \frac{8}{3}$$

A1

$$\text{Answer: } \frac{8}{3}$$

4 marks

$$(b) \frac{|2.6667 - 2.5616|}{2.5616} \times 100 = 4.1\%$$

M1 A1**Answer: 4.1%****2 marks****19 (7 marks)**

$$(a) f(1) = 2$$

$$f(3) = 2 + 6 = 8$$

A1

$$f(4) = 2 + 9 + 12 = 23$$

A1**Answer: 2, 8, 23****2 marks**

$$(b) f[3, 4] = \frac{23 - 8}{1} = 15$$

B1

$$f[1, 3, 4] = \frac{15 - 3}{4 - 1} = 4$$

M1 A1**Answer: $f[3, 4] = 15$ and $f[1, 3, 4] = 4$** **3 marks**

$$(c) p(x) = 2 + 3x - 3 + 4(x^2 - 4x + 3) = 4x^2 - 13x + 11$$

M1 A1**Answer: $4x^2 - 13x + 11$** **2 marks****20 (6 marks)**

$$(a) f[x_0, x_1] = \frac{f_1 - f_0}{h}, f[x_1, x_2] = \frac{f_2 - f_1}{h}$$

M1

$$f[x_0, x_1, x_2] = \frac{1}{2h} \left(\frac{f_2 - f_1}{h} - \frac{f_1 - f_0}{h} \right) = \frac{f_2 - 2f_1 + f_0}{2h^2}$$

M1 A1

$$\text{and } \Delta^2 f_0 = f_2 - 2f_1 + f_0$$

B1

$$\text{Answer: } f[x_0, x_1, x_2] = \frac{\Delta^2 f_0}{2h^2}$$

4 marks

$$(b) \Delta^2 f_0 = 3.0 - 3.8 + 1.2 = 0.4$$

M1

$$f[x_0, x_1, x_2] = \frac{0.4}{2 \times 0.25} = 0.8$$

A1**Answer: 0.8****2 marks**

21 (9 marks)

$$(a) \text{ Error} = \frac{f'''(\xi)}{3!}(x)(x - 0.2)(x - 0.4), |f'''| = |\cos \xi| \leq 1 \quad \text{M1 A1}$$

$$|x(x - 0.2)(x - 0.4)| \text{ at } 0.1 = 0.1 \times 0.1 \times 0.3 = 0.003 \quad \text{M1}$$

$$|\text{error}| \leq \frac{0.003}{6} = 0.0005 \quad \text{A1}$$

$$\text{Answer: } |\sin x - p(x)| \leq \frac{0.003}{6} = 0.0005 \quad \text{4 marks}$$

$$(b) f[0, 0.2] = 0.99335, f[0.2, 0.4] = 0.95375 \quad \text{A1}$$

$$f[0, 0.2, 0.4] = -0.09900 \quad \text{A1}$$

$$p(0.1) = 0.99335 \times 0.1 + (-0.09900)(0.1)(-0.1) = 0.10033 \quad \text{M1 A1}$$

$$\sin 0.1 = 0.09983, \text{ so the error is } 0.00049, \text{ within the bound} \quad \text{A1}$$

$$\text{Answer: } p(0.1) = 0.10033, \text{ error } 0.00049 \quad \text{5 marks}$$

22 (8 marks)

$$(a) |(x - a)(x - b)| \text{ is greatest at the midpoint, where it equals } \frac{h^2}{4} \quad \text{M1 A1}$$

$$|\text{error}| \leq \frac{M}{2} \times \frac{h^2}{4} = \frac{Mh^2}{8} \quad \text{A1}$$

$$\text{Answer: } |f(x) - p_1(x)| \leq \frac{Mh^2}{8} \quad \text{3 marks}$$

$$(b) f'' = -\frac{1}{x^2} \text{ so } M = 1 \text{ on } [1, 2] \quad \text{B1}$$

$$\frac{h^2}{8} < 5 \times 10^{-6} \quad \text{M1}$$

$$h < \sqrt{4 \times 10^{-5}} = 0.00632, \text{ so } h \leq 0.00632 \quad \text{A1 A1}$$

$$\text{Answer: } 0.00632 \quad \text{4 marks}$$

$$(c) \frac{1}{0.005} + 1 = 201 \quad \text{A1}$$

$$\text{Answer: } 201 \quad \text{1 mark}$$

23 (9 marks)

$$(a) f[1, 2] = 2, f[2, 4] = 4 \quad \text{M1 A1}$$

$$f[1, 2, 4] = \frac{2}{3} \quad \text{A1}$$

$$\text{Answer: } f[1, 2, 4] = \frac{2}{3} \quad \text{3 marks}$$

$$(b) f[4, 2] = 4, f[2, 1] = 2 \quad \text{M1 A1}$$

$$f[4, 2, 1] = \frac{2}{3} \quad \text{A1}$$

$$\text{Answer: } f[4, 2, 1] = \frac{2}{3} \quad \text{3 marks}$$

$$(c) \text{ First: } 3 + 2(x - 1) + \frac{2}{3}(x - 1)(x - 2) \quad \text{B1}$$

$$\text{Second: } 13 + 4(x - 4) + \frac{2}{3}(x - 4)(x - 2) \quad \text{B1}$$

Both interpolate the same three points with degree ≤ 2 , so by uniqueness they are the same polynomial. E1

Answer: equal by uniqueness 3 marks

24 (9 marks)

(a) The sum is the quadratic interpolating the data $y_k = 1$ at every node; the constant polynomial 1 also does, and by uniqueness they are equal. E1 E1

Answer: $L_0(x) + L_1(x) + L_2(x) = 1$ (the sum interpolates the constant 1) 2 marks

(b) This interpolates $f(x) = x$ at the nodes; since x has degree 1, it is its own interpolant by uniqueness. E1 E1

Answer: $x_0L_0 + x_1L_1 + x_2L_2 = x$ (the sum interpolates $f(x) = x$) 2 marks

$$(c) L_0(2) = \frac{(1)(-1)}{(-1)(-3)} = -\frac{1}{3}, L_1(2) = \frac{(2)(-1)}{(1)(-2)} = 1, L_2(2) = \frac{(2)(1)}{(3)(2)} = \frac{1}{3} \quad \text{M1 A1}$$

$$\text{Sum} = 1; 0 \times \left(-\frac{1}{3}\right) + 1 \times 1 + 3 \times \frac{1}{3} = 2$$

M1 A1

$$0 + 1 \times 1 + 9 \times \frac{1}{3} = 4$$

A1

Answer: $-\frac{1}{3}, 1, \frac{1}{3}$; sums 1, 2, 4

5 marks

25 (9 marks)

(a) $f[1.0, 1.1] = 0.497, f[1.1, 1.2] = 0.408$

M1 A1

$$f[1.0, 1.1, 1.2] = -0.445$$

A1

Answer: 0.497, 0.408, -0.445

3 marks

(b) $p(x) = 0.8415 + 0.497(x - 1) - 0.445(x - 1)(x - 1.1)$

B1

$$p'(x) = 0.497 - 0.445(2x - 2.1)$$

A1

$$p'(1.1) = 0.497 - 0.445 = 0.4525$$

A1

Answer: 0.4525

3 marks

(c) $p'' = 2 \times -0.445 = -0.890$, compared with $-\sin 1.1 = -0.891$

M1 A1

The derivative estimate is within 0.24% of $\cos 1.1$.

A1

Answer: 0.4525; -0.890

3 marks

26 (9 marks)

(a) $f[0, 1] = 1, f[1, 3] = \frac{5}{2}, f[0, 1, 3] = \frac{\frac{5}{2} - 1}{3} = \frac{1}{2}$

M1 A1

$$p(x) = 2 + x + \frac{1}{2}x(x - 1) = \frac{1}{2}x^2 + \frac{1}{2}x + 2$$

M1 A1

Answer: $p(x) = \frac{1}{2}x^2 + \frac{1}{2}x + 2$

4 marks

(b) $\int_0^3 p \, dx = \left[\frac{1}{6}x^3 + \frac{1}{4}x^2 + 2x\right]_0^3$

M1

$$= \frac{9}{2} + \frac{9}{4} + 6 = \frac{51}{4}$$

A1

Answer: $\frac{51}{4}$

2 marks

(c) $\frac{1}{2}(1)(2 + 3) + \frac{1}{2}(2)(3 + 8) = \frac{27}{2}$

M1 A1

p is convex, so the chords lie above the curve and the trapezium estimate is larger (not smaller):

E1

$$\frac{27}{2} > \frac{51}{4}$$

Answer: $\frac{27}{2}$

3 marks

27 (11 marks)

(a) Coefficient of f_0 : from $-f[x_0, x_1]$: $\frac{1}{(x_2 - x_0)(x_1 - x_0)} = \frac{1}{(x_0 - x_1)(x_0 - x_2)}$

M1 A1

Coefficient of f_2 : $\frac{1}{(x_2 - x_0)(x_2 - x_1)}$

A1

Coefficient of f_1 : $\frac{1}{x_2 - x_0} \left(-\frac{1}{x_2 - x_1} - \frac{1}{x_1 - x_0}\right)$

M1

$$= -\frac{1}{x_2 - x_0} \times \frac{(x_1 - x_0) + (x_2 - x_1)}{(x_2 - x_1)(x_1 - x_0)} = -\frac{1}{(x_2 - x_1)(x_1 - x_0)}$$

M1 A1

$$= \frac{1}{(x_1 - x_0)(x_1 - x_2)}$$

A1

Answer: $f[x_0, x_1, x_2] = \frac{f_0}{(x_0 - x_1)(x_0 - x_2)} + \frac{f_1}{(x_1 - x_0)(x_1 - x_2)} + \frac{f_2}{(x_2 - x_0)(x_2 - x_1)}$

7 marks

(b) The expression is a sum of three terms, each depending on one node and the product of differences with the others, so swapping any nodes only reorders the terms (each factor changes sign twice).

E1 E1

Answer: the expression is symmetric in the three nodes, so the order does not matter

2 marks

(c) $\frac{3}{(0 - 2)(0 - 5)} + \frac{1}{(2 - 0)(2 - 5)} + \frac{13}{(5 - 0)(5 - 2)}$

M1

$$= \frac{3}{10} - \frac{1}{6} + \frac{13}{15} = 1$$

Answer: 1

A1

2 marks

28 (11 marks)

(a) $w(a) = w(b) = 0$ and p interpolates f , so $g(a) = g(b) = 0$

B1

g has three zeros a, x, b

B1

By Rolle's theorem, g' has at least two zeros in (a, b)

M1 A1

and so g'' has at least one zero ξ

A1

$g'' = f'' - 2K$ since p is linear and $w'' = 2$, so $K = \frac{f''(\xi)}{2}$

M1

$g(x) = 0$ gives $f(x) - p(x) = Kw(x)$

A1

Answer: $f(x) - p(x) = \frac{f''(\xi)}{2}(x - a)(x - b)$

7 marks

(b) $|f''| \leq e$ on $[0, 1]$; $|w(0.5)| = 0.25$

M1

$|\text{error}| \leq \frac{e}{2} \times 0.25 = 0.340$

A1

Answer: ≤ 0.340

2 marks

(c) $p(0.5) = \frac{1+e}{2} = 1.8591$, $e^{0.5} = 1.6487$

M1

Actual error magnitude $0.210 < 0.340$

A1

Answer: 0.210

2 marks

29 (12 marks)

(a) f values 0, 1, 8; $f[0, 1] = 1$, $f[1, 2] = 7$

M1 A1

$f[0, 1, 2] = 3$

A1

$p(x) = x + 3x(x - 1) = 3x^2 - 2x$

A1

Answer: $3x^2 - 2x$

4 marks

(b) $x^3 - 3x^2 + 2x$

M1

$= x(x^2 - 3x + 2) = x(x - 1)(x - 2)$

A1

Answer: $x^3 - p(x) = x(x - 1)(x - 2)$

2 marks

(c) $f''' = 6$ for all ξ , so $\frac{f'''(\xi)}{3!}(x)(x - 1)(x - 2) = x(x - 1)(x - 2)$

M1 A1

f''' is constant, so the value of ξ does not matter.

E1

Answer: $\frac{f'''(\xi)}{3!} = 1$ for every ξ , so the error is $x(x - 1)(x - 2)$

3 marks

(d) $p(1.5) = 6.75 - 3 = 3.75$; $1.5^3 = 3.375$

M1

error $= -0.375 = w(1.5)$; percentage error $= \frac{0.375}{3.375} \times 100 = 11.1\%$

A1 A1

Answer: $-0.375, 11.1\%$

3 marks

30 (11 marks)

(a) For equal unit spacing the leading coefficient of the Newton forward form is $\frac{\Delta^4 f_0}{4!}$

B1

$\Delta: 1, 3, 5, y - 10$; $\Delta^2: 2, 2, y - 15$; $\Delta^3: 0, y - 17$; $\Delta^4: y - 17$

M1 A1

$\frac{y - 17}{24} = \frac{1}{12}$, so $y = 19$

M1 A1

Answer: $y = 19$

5 marks

(b) $p(x) = 1 + x + x(x - 1) + 0 + \frac{2}{24}x(x - 1)(x - 2)(x - 3)$

M1 A1

$= x^2 + 1 + \frac{1}{12}x(x - 1)(x - 2)(x - 3)$

A1

Answer: $p(x) = 1 + x + x(x - 1) + \frac{1}{12}x(x - 1)(x - 2)(x - 3)$

3 marks

(c) $p(2.5) = 7.25 + \frac{1}{12} \times 2.5 \times 1.5 \times 0.5 \times (-0.5) = \frac{459}{64}$

M1 A1

$p(5) = 26 + \frac{120}{12} = 36$

A1

Answer: $\frac{459}{64}$; 36

3 marks

Cubic Splines and Piecewise Interpolation

1 (3 marks)

On $[0, 2]$: gradient $\frac{5-1}{2} = 2$, $S(1) = 3$

B1

On $[2, 5]$: gradient $\frac{2-5}{3} = -1$, $S(3) = 5 - 1 = 4$

M1 A1

Answer: $S(1) = 3$, $S(3) = 4$

3 marks

2 (3 marks)

(a) 4 pieces, 16 coefficients

B1

(b) 8 value conditions + 3 for S' + 3 for S'' + 2 natural end conditions = 16

B1 B1

Answer: 4 pieces, 16 coefficients; $8 + 3 + 3 + 2 = 16$

3 marks

3 (2 marks)

 $S_1''(x) = 12x + 2b$

M1

Natural: $S_1''(0) = 0$, so $b = 0$

A1

Answer: $b = 0$

2 marks

4 (3 marks)

 $S_1(1) = 3 = S_2(1)$

B1

 $S_1'(1) = 3 + 2 = 5 = S_2'(1)$

B1

 $S_1''(1) = 6 = S_2''(1) = 2 \times 3$

B1

Answer: values 3, first derivatives 5, second derivatives 6 at $x = 1$

3 marks

5 (3 marks)

(a) 3 on $[0, 1]$ and $-\frac{1}{2}$ on $[1, 3]$

B1

(b) The gradients differ, so S' is not continuous at $x = 1$

B1

There is a corner in the graph at $x = 1$.

E1

Answer: 3 and $-\frac{1}{2}$; corner at $x = 1$

3 marks

6 (2 marks)

 $S(1.5) = 2.25$

B1

 $S(3) = 4 + 4 - 1 = 7$

B1

Answer: 2.25 and 7

2 marks

7 (5 marks)

(a) $S(0) = 0$; $S(1) = \frac{3}{2} - \frac{1}{2} = 1$ from the left, 1 from the right; $S(2) = 1 - \frac{3}{2} + \frac{1}{2} = 0$

B1 B1

Answer: $S(0) = 0$, $S(1) = 1$, $S(2) = 0$, and both pieces give $S(1) = 1$

2 marks

(b) Left: $S' = \frac{3}{2} - \frac{3}{2}x^2 = 0$ and $S'' = -3x = -3$ at $x = 1$

M1

Right: $S' = -3(x - 1) + \frac{3}{2}(x - 1)^2 = 0$ and $S'' = -3 + 3(x - 1) = -3$ at $x = 1$

A1

Answer: $S'(1) = 0$ and $S''(1) = -3$ from both pieces

2 marks

(c) $S''(0) = 0$ from $-3x$; $S''(2) = -3 + 3 = 0$

B1

Answer: $S''(0) = 0$ and $S''(2) = 0$

1 mark

8 (5 marks)

(a) Left: $S(1) = 0$, $S'(1) = 3 - 1 = 2$

B1

Right: $S(1) = 0$, $S'(1) = 2$

B1

Answer: $S(1) = 0$ and $S'(1) = 2$ from both pieces

2 marks

(b) $S''(x) = 6x$ on the left, so 6 at $x = 1$; right: $2b = 6$, $b = 3$

A1

$$S(2) = a + 3 + 2 = 4, \text{ so } a = -1$$

Answer: $b = 3, a = -1$

M1 A1

3 marks

9 (5 marks)

(a) $S_1(1) = 3, S_2(2) = 2, S_1'(1) = S_2'(1), S_1''(1) = S_2''(1), S_1''(0) = 0, S_2''(2) = 0$

B1

$$S_1''(0) = 2c_1 = 0 \text{ and } S_2''(2) = 2c_2 + 6d_2 = 0$$

A1

Answer: $c_1 = 0, c_2 = -3d_2$

2 marks

(b) $b_1 + d_1 = 2; S_1'(1) = b_1 + 3d_1 = b_2; S_1''(1) = 6d_1 = 2c_2$, so $c_2 = 3d_1$ and $d_2 = -d_1$

$$S_2(2): b_2 + c_2 + d_2 = -1, \text{ i.e. } b_1 + 3d_1 + 3d_1 - d_1 = -1$$

M1

$$\text{Subtract } b_1 + d_1 = 2: 4d_1 = -3, d_1 = -\frac{3}{4}$$

A1

$$b_1 = \frac{11}{4}, b_2 = \frac{1}{2}, c_2 = -\frac{9}{4}, d_2 = \frac{3}{4}$$

$$S_1(x) = -\frac{3}{4}x^3 + \frac{11}{4}x + 1, S_2(x) = \frac{3}{4}(x-1)^3 - \frac{9}{4}(x-1)^2 + \frac{1}{2}(x-1) + 3$$

A1

Answer: $S_1 = -\frac{3}{4}x^3 + \frac{11}{4}x + 1; S_2 = \frac{3}{4}(x-1)^3 - \frac{9}{4}(x-1)^2 + \frac{1}{2}(x-1) + 3$

3 marks

10 (5 marks)

(a) Between 1000 m and 3000 m: $96.7 + \frac{1}{2}(90.0 - 96.7)$

M1

$$= 93.35 \text{ }^\circ\text{C}$$

A1

Answer: 93.35 }^\circ\text{C}

2 marks

(b) 86 lies between 90.0 and 83.6, so use [3000, 5000]

$$\text{Altitude} = 3000 + \frac{90.0 - 86}{90.0 - 83.6} \times 2000$$

M1

$$= 4250 \text{ m}$$

A1

Answer: 4250 m

2 marks

(c) The true relationship is curved, so straight segments only approximate it between the data points.

E1

Answer: curve not straight

1 mark

11 (5 marks)

(a) $S(0.5) = 2 - \frac{7}{4} \times \frac{1}{2} + \frac{3}{4} \times \frac{1}{8}$

M1

$$= \frac{39}{32}$$

A1

Answer: $\frac{39}{32}$

2 marks

(b) $S'(x) = \frac{1}{2} + 2(\frac{9}{4})(x-1) + 3(-\frac{3}{4})(x-1)^2$

M1

$$S'(1.5) = \frac{35}{16}$$

A1

$$S''(1.5) = \frac{9}{4}$$

A1

Answer: $S'(1.5) = \frac{35}{16}, S''(1.5) = \frac{9}{4}$

3 marks

12 (5 marks)

(a) $Q_1: c = 1, a + b = 1, 4a + 2b = 4$ gives $a = 1, b = 0$

A1

$$Q_2 \text{ symmetric about } x = 3 \text{ with value } 6: Q_2 = 6 - (x-3)^2$$

A1

Answer: $Q_1 = x^2 + 1, Q_2 = 6 - (x-3)^2$

2 marks

(b) $Q_1'(2) = 4$ and $Q_2'(2) = -2(2-3) = 2$

M1 A1

Answer: gradients 4 and 2

2 marks

(c) $Q_2(3.5) = 6 - 0.25 = 5.75$

A1

Answer: 5.75

1 mark

13 (5 marks)

(a) $S_1''(1) = 6d_1 = S_2''(1) = 2c_2$, so $c_2 = 3d_1$

M1

$$S_2''(3) = 2c_2 + 12d_2 = 0 \text{ gives } d_2 = -\frac{c_2}{6} = -\frac{1}{2}d_1$$

A1

Answer: $c_2 = 3d_1$ and $d_2 = -\frac{1}{2}d_1$

2 marks

(b) $b_1 + d_1 = 2$ and $b_2 = b_1 + 3d_1$; $S_2(3) = 1$: $2b_2 + 4c_2 + 8d_2 = -1$

M1

$$2b_1 + 6d_1 + 12d_1 - 4d_1 = -1 \text{ gives } 2b_1 + 14d_1 = -1$$

M1

$$12d_1 = -5, \text{ so } d_1 = -\frac{5}{12} \text{ and } b_1 = \frac{29}{12}$$

A1

Answer: $d_1 = -\frac{5}{12}$, $b_1 = \frac{29}{12}$

3 marks

14 (7 marks)

(a) $4m_1 + m_2 = 6(0 - 4 + 3) = -6$

M1 A1

$$m_1 + 4m_2 = 6(2 - 6 + 1) = -18$$

M1

$$m_1 = -\frac{2}{5}, m_2 = -\frac{22}{5}$$

A1

Answer: $m_1 = -\frac{2}{5}$, $m_2 = -\frac{22}{5}$

4 marks

(b) $b_1 = 1 - \frac{2m_1 + m_2}{6} = \frac{28}{15}$

M1

$$S(1.5) = 2 + \frac{28}{15} \times \frac{1}{2} - \frac{1}{5} \times \frac{1}{4} - \frac{2}{3} \times \frac{1}{8} = \frac{14}{5}$$

M1 A1

Answer: $\frac{14}{5}$

3 marks

15 (7 marks)

(a) (i) $\frac{1}{2} + \frac{1}{2}\left(\frac{1}{3} - \frac{1}{2}\right) = \frac{5}{12}$

A1

(ii) $f[2, 3] = -\frac{1}{6}$, $f[3, 4] = -\frac{1}{12}$, $f[2, 3, 4] = \frac{1}{24}$

M1

$$p(2.5) = \frac{1}{2} - \frac{1}{6} \times \frac{1}{2} + \frac{1}{24} \times \frac{1}{2} \times \left(-\frac{1}{2}\right) = \frac{13}{32}$$

A1

(iii) $\frac{5}{12} - \frac{1}{16} \times \frac{1}{2} = \frac{37}{96}$

A1

Answer: $\frac{5}{12}$, $\frac{13}{32}$, $\frac{37}{96}$

4 marks

(b) Errors: 4.17%, 1.56%, 3.65%

M1 A1

The quadratic is the most accurate.

A1

Answer: 4.17%, 1.56%, 3.65%; quadratic best

3 marks

16 (6 marks)

(a) With $m_0 = m_2 = 0$: $4h m_1 = \frac{6}{h}(y_2 - 2y_1 + y_0)$

M1 A1

$$m_1 = \frac{3(y_0 - 2y_1 + y_2)}{2h^2}$$

A1

Answer: $m_1 = \frac{3(y_0 - 2y_1 + y_2)}{2h^2}$

3 marks

(b) $m_1 = \frac{3(1 - 5 + 2)}{2 \times 0.25} = -12$

M1 A1

$m_1 < 0$ so the spline is concave (curving downwards) at the middle knot.

E1

Answer: $m_1 = -12$, concave

3 marks

17 (7 marks)

(a) $S(1) = 1 + 2 + c + d = 4$, so $c + d = 1$

M1 A1

$$S'(1) = 2 + 2c + 3d = -1, \text{ so } 2c + 3d = -3$$

M1

$$d = -5, c = 6$$

A1

Answer: $c = 6$, $d = -5$

4 marks

(b) $S''(x) = 12 - 30x$, $S''(0) = 12 \neq 0$

Answer: $S''(0) = 12$

B1

1 mark

(c) $S'(x) = 2 + 12x - 15x^2 = 0$

M1

$x = \frac{12 + \sqrt{264}}{30} = 0.942$ (the other root is negative)

A1

Answer: 0.942

2 marks

18 (6 marks)

(a) Left $S(1) = 2$, right $S(1) = 2$

B1

Left $S'(1) = 3 - 2 = 1$; right $S'(1) = k$, so $k = 1$

M1 A1

Answer: $k = 1$

3 marks

(b) Left $S''(1) = 6$; right $S''(1) = 2 \times 3 = 6$

M1 A1

Answer: $S''(1) = 6$ from both pieces

2 marks

(c) S' jumps from 1 to 0, so the graph has a corner (not smooth) at $x = 1$.

E1

Answer: S' jumps from 1 to 0 at $x = 1$, so there is a corner

1 mark

19 (6 marks)

(a) $S'(x) = -\frac{1}{2} + \frac{9}{2}(x-1) - \frac{9}{4}(x-1)^2 = 0$

M1

Multiply by 4: $-2 + 18(x-1) - 9(x-1)^2 = 0$

M1

$x - 1 = 0.118$ (the other root exceeds 1), so $x = 1.12$

A1

Answer: 1.12

3 marks

(b) $S = -0.0289$

M1 A1

$S''(x) = 2c + 6d > 0$ at this t , so a local minimum (below the data value 0)

E1

Answer: -0.0289 , local minimum

3 marks

20 (5 marks)

(a) $S_2(2) = 3 + \frac{9}{2} - \frac{9}{2} + \frac{3}{2} = \frac{9}{2} \neq 0$

M1 A1

Answer: $S_2(2) = 9/2$

2 marks

(b) $S_1'(1) = \frac{9}{2} - \frac{9}{2} = 0$ but $S_2'(1) = \frac{9}{2}$: S' is not continuous

M1 A1

Correct: $b_2 = 0$; with $c_2 = 3d_1 = -\frac{9}{2}$, $d_2 = \frac{3}{2}$: $S_2 = 3 - \frac{9}{2}(x-1)^2 + \frac{3}{2}(x-1)^3$

A1

Answer: S' discontinuous; $S_2 = 3 - \frac{9}{2}(x-1)^2 + \frac{3}{2}(x-1)^3$

3 marks

21 (9 marks)

(a) $s_0 = 1$, $s_1 = -1$, $s_2 = 3$

B1

$6m_1 + 2m_2 = 6(-1 - 1) = -12$

A1

$2m_1 + 6m_2 = 6(3 + 1) = 24$

A1

Answer: $6m_1 + 2m_2 = -12$, $2m_1 + 6m_2 = 24$

3 marks

(b) From the first, $m_2 = -6 - 3m_1$; substituting: $2m_1 - 36 - 18m_1 = 24$

M1

$m_1 = -\frac{15}{4}$, $m_2 = \frac{21}{4}$

A1

Answer: $m_1 = -\frac{15}{4}$, $m_2 = \frac{21}{4}$

2 marks

(c) On $[1, 3]$, $h = 2$, $b_1 = -1 - \frac{2(2m_1 + m_2)}{6} = -\frac{1}{4}$

M1

$S(2) = 2 - \frac{1}{4} + \frac{m_1}{2} + \frac{3}{4} = \frac{5}{8}$

M1 A1

Answer: $\frac{5}{8}$

3 marks

(d) Linear: 1, so the spline estimate differs by $\frac{3}{8}$

A1

Answer: linear gives 1

1 mark

22 (8 marks)

$$(a) \int_0^1 S \, dx = 1 + \frac{7}{8} - \frac{3}{16} = \frac{27}{16}$$

M1 A1

$$\text{With } t = x - 1: \int_0^1 S \, dx = 2 - \frac{1}{4} - \frac{3}{4} + \frac{3}{16} = \frac{19}{16}$$

M1 A1

$$\text{Total} = \frac{23}{8}$$

A1

$$\text{Answer: } \frac{23}{8}$$

5 marks

$$(b) \text{Trapezium} = \frac{1}{2}(1 + 2) + \frac{1}{2}(2 + 0) = \frac{5}{2}$$

M1

$$\text{Difference} = \frac{3}{8}$$

A1

$$\text{Answer: } \frac{5}{2}, \text{ difference } \frac{3}{8}$$

2 marks

(c) The spline follows the curvature between the points, while the trapezium rule replaces the curve by chords.

E1

Answer: curvature

1 mark

23 (9 marks)

(a) $S'' = m_i + (m_{i+1} - m_i)t$, which equals m_i at $t = 0$ and m_{i+1} at $t = 1$

B1

$$y_{i+1} = y_i + b_i + \frac{m_i}{2} + \frac{m_{i+1} - m_i}{6}$$

M1

$$b_i = y_{i+1} - y_i - \frac{3m_i + m_{i+1} - m_i}{6} = y_{i+1} - y_i - \frac{2m_i + m_{i+1}}{6}$$

A1

$$\text{Answer: } b_i = y_{i+1} - y_i - \frac{2m_i + m_{i+1}}{6}$$

3 marks

$$(b) S'(1) = b_i + m_i + \frac{m_{i+1} - m_i}{2}$$

M1

$$= y_{i+1} - y_i - \frac{2m_i + m_{i+1}}{6} + \frac{m_i + m_{i+1}}{2} = y_{i+1} - y_i + \frac{m_i + 2m_{i+1}}{6}$$

A1

$$\text{Answer: } S'(x_{i+1}) = y_{i+1} - y_i + \frac{m_i + 2m_{i+1}}{6}$$

2 marks

$$(c) b_{i+1} = y_{i+2} - y_{i+1} - \frac{2m_{i+1} + m_{i+2}}{6}$$

B1

$$y_{i+1} - y_i + \frac{m_i + 2m_{i+1}}{6} = y_{i+2} - y_{i+1} - \frac{2m_{i+1} + m_{i+2}}{6}$$

M1

$$\frac{m_i + 4m_{i+1} + m_{i+2}}{6} = y_i - 2y_{i+1} + y_{i+2}$$

M1 A1

$$\text{Answer: } m_i + 4m_{i+1} + m_{i+2} = 6(y_i - 2y_{i+1} + y_{i+2})$$

4 marks

24 (8 marks)

(a) $y_{i-1} - 2y_i + y_{i+1} = 0$ for collinear equally spaced points

M1

$$4m_1 + m_2 = 0 \text{ and } m_1 + 4m_2 = 0, \text{ so } m_1 = m_2 = 0$$

M1 A1

$$\text{Answer: } m_1 = m_2 = 0$$

3 marks

(b) All $m_i = 0$, so $S'' = 0$ on each piece and S is linear on each piece

M1

Each piece passes through two points on the line, so each piece is the line $y = 1 + 2x$.

A1

$$\text{Answer: } S = 1 + 2x$$

2 marks

(c) The line satisfies all the spline conditions (continuous with zero second derivative, natural ends), and the natural spline is unique, so it is the spline.

E1 E1

Answer: the line satisfies every spline condition, so by uniqueness it is the spline

2 marks

(d) It adds no spurious oscillation for straight-line data; there is no wiggling away from the data.

E1

Answer: no spurious oscillation

1 mark

25 (9 marks)

$$(a) Q_1(1) = 3, \text{ so } 1 + \alpha = 3, \alpha = 2, Q_1 = 1 + 2x^2$$

M1 A1

$$\text{Answer: } Q_1 = 1 + 2x^2$$

2 marks

$$(b) Q_1'(1) = 4 \text{ so } \beta = 4$$

M1 A1

$$Q_2(2) = 2: 3 + 4 + \gamma = 2, \gamma = -5$$

A1

Answer: $\beta = 4, \gamma = -5$

3 marks

(c) $Q_2'(x) = 4 - 10(x - 1) = 0$ at $x = 1.4$

M1 A1

$$Q_2(1.4) = 3 + 1.6 - 0.8 = 3.8$$

A1

Answer: 3.8 at $x = 1.4$

3 marks

(d) $Q_1'' = 4$ but $Q_2'' = -10$

B1

Answer: 4 and -10

1 mark

26 (9 marks)

(a) $4m_1 + m_2 = 6, m_1 + 4m_2 + m_3 = -12, m_2 + 4m_3 = 6$

M1 A1

Symmetry: $m_3 = m_1$; so $4m_1 + m_2 = 6$ and $2m_1 + 4m_2 = -12$

M1

$$m_1 = \frac{18}{7}, m_2 = -\frac{30}{7}$$

A1 A1

Answer: $m_1 = m_3 = \frac{18}{7}, m_2 = -\frac{30}{7}$

5 marks

(b) Midpoint of $[-2, -1]$: $\frac{0+0}{2} - \frac{0 + \frac{18}{7}}{16}$

M1

$$= -\frac{9}{56} \approx -0.1607$$

A1

Answer: $-\frac{9}{56} \approx -0.1607$

2 marks

(c) S dips below zero near $x = -1.5$ although all the data are non-negative (overshoot)

E1

Piecewise linear interpolation never leaves the range of neighbouring data.

E1

Answer: overshoot; use piecewise linear

2 marks

27 (10 marks)

(a) Using the formula at x_1, x_2, x_3 : $6(0 - 2 + 3) = 6, 6(1 - 6 + 2) = -18, 6(3 - 4 + 4) = 18$

M1 A1 A1

Answer: $4m_1 + m_2 = 6, m_1 + 4m_2 + m_3 = -18, m_2 + 4m_3 = 18$

3 marks

(b) Eliminate m_1 using the first and m_3 using the third: $m_1 = \frac{6 - m_2}{4}, m_3 = \frac{18 - m_2}{4}$

M1

$$\frac{6 - m_2}{4} + 4m_2 + \frac{18 - m_2}{4} = -18$$

M1

$$\frac{24 - 2m_2}{4} + 4m_2 = -18 \text{ gives } m_2 = -\frac{48}{7}$$

A1

$$m_1 = \frac{45}{14}, m_3 = \frac{87}{14}$$

A1

Answer: $m_1 = \frac{45}{14}, m_2 = -\frac{48}{7}, m_3 = \frac{87}{14}$

4 marks

(c) $b_2 = \frac{1}{4}$

M1

$$S(2.5) = 3 + \frac{b_2}{2} + \frac{m_2}{8} + \frac{m_3 - m_2}{48} = \frac{569}{224}$$

M1 A1

Answer: $\frac{569}{224}$

3 marks

28 (12 marks)

(a) $f'' = S'' + g''$ and expand $(S'' + g'')^2$

M1 A1

Answer: $\int_0^2 (f'')^2 dx = \int_0^2 (S'')^2 dx + 2 \int_0^2 S''g'' dx + \int_0^2 (g'')^2 dx$

2 marks

(b) $u = S'', dv = g'' dx$, giving $[S''g'] - \int S'''g' dx$

M1 A1

Answer: $\int_{x_i}^{x_{i+1}} S''g'' dx = [S''g'] - \int_{x_i}^{x_{i+1}} S'''g' dx$

2 marks

(c) $\int_{x_i}^{x_{i+1}} S'''g' dx = k_i[g]_{x_i}^{x_{i+1}} = 0$

M1 A1

Summing the boundary terms, S'' and g' are continuous at $x = 1$ so the interior terms cancel, leaving $[S''g']_0^2$

M1 A1

Answer: $\int_0^2 S''g'' \, dx = [S'g'] \text{ from } 0 \text{ to } 2$

4 marks

(d) $S''(0) = S''(2) = 0$ (natural), so $[S'g']_0^2 = 0$

M1 A1

So $\int_0^2 (f'')^2 \, dx = \int_0^2 (S'')^2 \, dx + \int_0^2 (g'')^2 \, dx \geq \int_0^2 (S'')^2 \, dx$ since $(g'')^2 \geq 0$

A1 E1

Answer: $\int_0^2 (f'')^2 \, dx \geq \int_0^2 (S'')^2 \, dx$

4 marks

29 (12 marks)

(a) $c_0 + d_0 = 1$

M1

$b_1 = S_1'(1) = 1 + 2c_0 + 3d_0$

M1 A1

$2c_1 = S_1''(1) = 2c_0 + 6d_0$, so $c_1 = c_0 + 3d_0$

A1

Answer: $c_0 + d_0 = 1$, $b_1 = 1 + 2c_0 + 3d_0$, $c_1 = c_0 + 3d_0$

4 marks

(b) $S(2) = 1$: $b_1 + c_1 + d_1 = -1$

M1

$S'(2) = 0$: $b_1 + 2c_1 + 3d_1 = 0$

M1

Subtracting: $c_1 + 2d_1 = 1$; solve with $c_0 = 1 - d_0$: $d_0 = -\frac{5}{2}$, $c_0 = \frac{7}{2}$

M1 A1

$d_1 = \frac{5}{2}$, $c_1 = -4$, $b_1 = \frac{1}{2}$

A1

Answer: $S_1 = -\frac{5}{2}x^3 + \frac{7}{2}x^2 + x$; $S_2 = \frac{5}{2}(x-1)^3 - 4(x-1)^2 + \frac{1}{2}(x-1) + 2$

5 marks

(c) Clamped: $S(0.5) = \frac{1}{2} + \frac{7}{2} \times \frac{1}{4} - \frac{5}{2} \times \frac{1}{8} = \frac{17}{16}$

M1 A1

The natural spline has $S'(0) = \frac{11}{4}$, much steeper than the prescribed gradient 1, so the two curves differ near the end.

E1

Answer: $\frac{17}{16}$ (clamped) and $\frac{41}{32}$ (natural)

3 marks

30 (10 marks)

(a) $4m_1 + m_2 = 6(0 - 1.3862 + 1.0986) = -1.7256$

M1 A1

$m_1 + 4m_2 = 6(0.6931 - 2.1972 + 1.3863) = -0.7068$

M1

$m_1 = -0.4130$, $m_2 = -0.0734$

A1

Answer: $m_1 = -0.4130$, $m_2 = -0.0734$

4 marks

(b) $S(2.5) = \frac{0.6931 + 1.0986}{2} - \frac{(-0.4130) + (-0.0734)}{16}$

M1

$= 0.9263$

A1

Answer: 0.9263

2 marks

(c) Linear: 0.8959

B1

Errors: spline 1.1%, linear 2.2%

M1 A1

The spline estimate is more accurate: it accounts for the concavity of the logarithm.

E1

Answer: spline 0.9263 (1.1%), linear 0.8959 (2.2%)

4 marks

Approximating Functions

1 (2 marks)

$$e^x \approx 1 + x + \frac{x^2}{2}$$

B1

$$e^{0.2} \approx 1 + 0.2 + 0.02 = 1.22$$

A1

Answer: 1.22

2 marks

2 (2 marks)

$$\sin 0.1 \approx 0.1 - \frac{0.001}{6}$$

M1

$$= 0.099833$$

A1

Answer: 0.099833

2 marks

3 (3 marks)

$$f(1) = 0, f'(1) = 1, f''(1) = -1, \text{ so } \ln x \approx (x - 1) - \frac{(x - 1)^2}{2}$$

B1 B1

$$\ln 1.1 \approx 0.1 - 0.005 = 0.095$$

A1

Answer: 0.095

3 marks

4 (2 marks)

$$f(4) = 2, f'(x) = \frac{1}{2\sqrt{x}}, f'(4) = \frac{1}{4}$$

B1

$$\sqrt{x} \approx 2 + \frac{1}{4}(x - 4); \sqrt{4.2} \approx 2.05$$

A1

Answer: 2.05

2 marks

5 (2 marks)

$$\sqrt{1.04} \approx 1 + 0.02 - \frac{0.0016}{8}$$

M1

$$= 1.0198$$

A1

Answer: 1.0198

2 marks

6 (3 marks)

$$\text{Estimate} = 1 - 0.125 = 0.875$$

B1

$$\text{Percentage error} = \frac{0.87758 - 0.875}{0.87758} \times 100 = 0.29\%$$

M1 A1

Answer: 0.29%

3 marks

7 (5 marks)

$$(a) f = \cos x, f' = -\sin x, f'' = -\cos x, f''' = \sin x$$

$$\text{At } \frac{\pi}{3}: \frac{1}{2}, -\frac{\sqrt{3}}{2}, -\frac{1}{2}, \frac{\sqrt{3}}{2}$$

M1

$$P_3(x) = \frac{1}{2} - \frac{\sqrt{3}}{2}(x - \frac{\pi}{3}) - \frac{1}{4}(x - \frac{\pi}{3})^2 + \frac{\sqrt{3}}{12}(x - \frac{\pi}{3})^3$$

A1

$$\text{Answer: } P_3(x) = \frac{1}{2} - \frac{\sqrt{3}}{2}(x - \frac{\pi}{3}) - \frac{1}{4}(x - \frac{\pi}{3})^2 + \frac{\sqrt{3}}{12}(x - \frac{\pi}{3})^3$$

2 marks

$$(b) x - \frac{\pi}{3} = -0.04720$$

B1

$$P_3(1) = 0.540302$$

M1 A1

Answer: 0.540302

3 marks

8 (5 marks)

$$(a) \ln(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots$$

B1

$$\ln(1 + x) - \ln(1 - x): \text{even powers cancel, odd powers double}$$

A1

$$\text{Answer: } \ln\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \dots\right)$$

2 marks

$$(b) \frac{1+x}{1-x} = 3 \text{ gives } x = \frac{1}{2}$$

A1

$$\ln 3 \approx 2\left(0.5 + \frac{0.125}{3} + \frac{0.03125}{5}\right) = 1.0958$$

M1 A1

Answer: 1.0958

3 marks

9 (5 marks)

$$(a) e^{2x} = 1 + 2x + 2x^2 + \frac{4}{3}x^3, \cos x = 1 - \frac{x^2}{2}$$

B1 B1

$$\text{Product: } 1 + 2x + \left(2 - \frac{1}{2}\right)x^2 + \left(\frac{4}{3} - 1\right)x^3$$

M1

$$= 1 + 2x + \frac{3}{2}x^2 + \frac{1}{3}x^3$$

A1

$$\text{Answer: } 1 + 2x + \frac{3}{2}x^2 + \frac{1}{3}x^3$$

4 marks

(b) With $x = 0.1$: $1 + 0.2 + 1.5(0.01) + \frac{1}{3}(0.001) = 1.2153$

A1

Answer: 1.2153

1 mark

10 (5 marks)

(a) $e^u = 1 + u + \frac{u^2}{2}$, $u = -x^2$: $1 - x^2 + \frac{x^4}{2}$

M1 A1

Answer: $1 - x^2 + \frac{x^4}{2}$

2 marks

(b) $\int_0^{0.5} (1 - x^2 + \frac{x^4}{2}) dx = [x - \frac{x^3}{3} + \frac{x^5}{10}]_0^{0.5}$
 $= 0.46146$

M1

M1 A1

Answer: 0.46146

3 marks

11 (5 marks)

(a) $x^2 + 2x - 2 = 0$

M1

$x = -1 + \sqrt{3} = 0.7321$

M1 A1

Answer: $-1 + \sqrt{3} = 0.7321$

3 marks

(b) $\frac{0.7391 - 0.7321}{0.7391} \times 100 = 0.95\%$

M1 A1

Answer: 0.95%

2 marks

12 (5 marks)

(a) $1 + 0.5 + \frac{0.25}{2} + \frac{0.125}{6} = 1.64583$

M1 A1

Answer: 1.64583

2 marks

(b) $f^{(4)}(\xi) = e^\xi < 1.7$ for $0 < \xi < 0.5$

M1

$|R_3| < \frac{1.7}{24} \times 0.5^4 = 0.00443$

A1

Answer: < 0.00443

2 marks

(c) Actual error = $0.00289 < 0.00443$

A1

Answer: $0.00289 < 0.00443$

1 mark

13 (5 marks)

(a) $\cos 2x \approx 1 - \frac{(2x)^2}{2} = 1 - 2x^2$, so $1 - \cos 2x \approx 2x^2$

M1 A1

$x \sin x \approx x^2$, so the ratio $\approx \frac{2x^2}{x^2} = 2$

A1

Answer: $\frac{1 - \cos 2x}{x \sin x} \approx 2$

3 marks

(b) $\frac{1 - \cos 0.2}{0.1 \sin 0.1} = 1.99667$

M1

Percentage error = $\frac{2 - 1.99667}{1.99667} \times 100 = 0.17\%$

A1

Answer: 1.99667, error 0.17%

2 marks

14 (7 marks)

(a) $e^\xi < e < 3$

B1

$n = 6$: $\frac{3}{7!} = 0.000595 \geq 10^{-4}$

M1

$n = 7$: $\frac{3}{8!} = 0.0000744 < 10^{-4}$, so $n = 7$

M1 A1

Answer: $n = 7$

4 marks

(b) Estimate = $1 + 1 + \frac{1}{2} + \dots + \frac{1}{7!} = 2.718254$

M1 A1

Actual error = 0.0000279

A1

Answer: 2.718254, error 0.0000279

3 marks

15 (7 marks)

$$(a) f' = \frac{1}{x}, f'' = -\frac{1}{x^2}, f''' = \frac{2}{x^3}$$

$$\text{At } x = 2: \frac{1}{2}, -\frac{1}{4}, \frac{1}{4}$$

A1

$$P_3(x) = \ln 2 + \frac{1}{2}(x - 2) - \frac{1}{8}(x - 2)^2 + \frac{1}{24}(x - 2)^3$$

A1

$$\text{Answer: } \ln 2 + \frac{1}{2}(x - 2) - \frac{1}{8}(x - 2)^2 + \frac{1}{24}(x - 2)^3$$

2 marks

$$(b) P_3(2.2) = 0.693147 + 0.1 - 0.005 + \frac{0.008}{24}$$

M1

$$= 0.788481$$

A1

$$\text{Answer: } 0.788481$$

2 marks

$$(c) |R_3| = \frac{|f^{(4)}(\xi)|}{4!}(0.2)^4, |f^{(4)}| \leq \frac{6}{16} \text{ for } \xi \geq 2$$

M1 A1

$$|R_3| \leq 0.0000250 < 3 \times 10^{-5}$$

A1

$$\text{Answer: } \leq 0.0000250$$

3 marks

16 (7 marks)

$$(a) w = (x)(x - 1)(x - 2) = (t + 1)t(t - 1) = t(t^2 - 1)$$

A1

$$w' = 3t^2 - 1 = 0 \text{ at } t = \pm \frac{1}{\sqrt{3}}, |w| = \frac{1}{\sqrt{3}} \times \frac{2}{3} = 0.3849$$

A1

$$\text{Answer: } t(t^2 - 1), \text{ greatest } |w| = 0.3849$$

2 marks

$$(b) T_3 = t(4t^2 - 3) = 0: t = 0, \pm \frac{\sqrt{3}}{2}$$

M1 A1

$$w = t(t^2 - \frac{3}{4}) = \frac{1}{4}(4t^3 - 3t) = \frac{1}{4}T_3(t)$$

A1

$$\text{Answer: } t = 0, \pm \frac{\sqrt{3}}{2}; w = \frac{1}{4}T_3$$

3 marks

$$(c) \text{Chebyshev: } \leq \frac{1}{4} = 0.25, \text{ versus } 0.385 \text{ for equal spacing, a reduction of } 35\%$$

M1 A1

$$\text{Answer: } 0.25 \text{ against } 0.385$$

2 marks

17 (6 marks)

$$(a) \sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \dots; \text{ divide by } x$$

M1 A1

$$\text{Answer: } \frac{\sin x}{x} \approx 1 - \frac{x^2}{6} + \frac{x^4}{120}$$

2 marks

$$(b) \int_0^{0.5} (1 - \frac{x^2}{6} + \frac{x^4}{120}) dx = [x - \frac{x^3}{18} + \frac{x^5}{600}]_0^{0.5}$$

M1 A1

$$= 0.493108$$

A1

$$\text{Answer: } 0.493108$$

3 marks

$$(c) \text{The series alternates with decreasing terms, and the next term integrates to that value.}$$

E1

$$\text{Answer: alternating series}$$

1 mark

18 (7 marks)

$$(a) (1 + bx)(1 + x + \frac{x^2}{2}) = 1 + (1 + b)x + (\frac{1}{2} + b)x^2 + \dots$$

M1

$$\text{Match: } a = 1 + b \text{ and } \frac{1}{2} + b = 0$$

M1

$$b = -\frac{1}{2}, a = \frac{1}{2}$$

A1

$$\text{Answer: } a = \frac{1}{2}, b = -\frac{1}{2}$$

3 marks

$$(b) \text{Rational: } \frac{1.25}{0.75} = 1.66667; \text{ error } 1.1\%$$

A1

$$\text{Quadratic: } 1.625; \text{ error } 1.4\%$$

M1 A1

$$\text{The rational form is more accurate here.}$$

E1

$$\text{Answer: rational } 1.1\%, \text{ quadratic } 1.4\%$$

4 marks

19 (7 marks)

(a) $q(x) = 1 + (e - 1)x$

B1

$x = 0: 1, 1, 1; x = 0.5: 1.649, 1.625, 1.859$

A1

$x = 1: 2.718, 2.500, 2.718$

A1

Answer: $q(x) = 1 + (e - 1)x$ (values in the working)

3 marks

(b) Near 0: p (Taylor-type, matches the function and its derivatives at 0)

B1

Near 1: q (exact at $x = 1$; p has error 0.218)

B1

At 0.5: errors 0.0237 (p) and 0.210 (q), so p is better

B1 B1

Answer: p near 0 and at 0.5; q near 1

4 marks

20 (7 marks)

(a) $1 + x + \frac{x^2}{2} + x = 3$, so $x^2 + 4x - 4 = 0$

A1

$x = -2 + 2\sqrt{2} = 0.8284$

A1

Answer: $-2 + 2\sqrt{2} = 0.8284$

2 marks

(b) Percentage error = 4.6%

M1 A1

The polynomial is accurate only near 0; at $x \approx 0.8$ the neglected $x^3/6$ term is not small.

E1

Answer: 4.6%

3 marks

(c) $g(x) = \frac{x^3}{6} + \frac{x^2}{2} + 2x - 2$; $g(0.78) = -0.0567$, $g(0.8) = 0.00533$

M1

Sign change, so a root exists in (0.78, 0.8) (root 0.7983)

A1

Answer: root = 0.7983

2 marks

21 (9 marks)

(a) $\frac{ax}{1+bx} = ax(1 - bx + b^2x^2 - \dots) = ax - abx^2 + ab^2x^3$

A1

$a = 1$ and $ab = \frac{1}{2}$, so $b = \frac{1}{2}$

M1 A1

Answer: $a = 1$, $b = \frac{1}{2}$

3 marks

(b) $\frac{x}{1 + \frac{1}{2}x} = \frac{2x}{2 + x}$

B1

x^3 coefficient = $ab^2 = \frac{1}{4}$, compared with $\frac{1}{3}$

M1 A1

Answer: $\frac{1}{4}$ vs $\frac{1}{3}$

3 marks

(c) Rational: $\frac{2}{3} = 0.66667$, error 3.8%

A1

Polynomial: 0.5, error 28%

M1 A1

Answer: rational 3.8%, polynomial 28%

3 marks

22 (8 marks)

(a) Worst case $|x| = 1$: error $\leq \frac{1}{(n+2)!}$

M1

$n = 5: \frac{1}{7!} = 0.000198 > 5 \times 10^{-5}$

M1 A1

$n = 7: \frac{1}{9!} = 0.00000276 < 5 \times 10^{-5}$

A1

Answer: $n = 7$

4 marks

(b) $P = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$

B1

$P(1) = 0.841468$

M1 A1

Answer: 0.841468

3 marks

(c) Error = $0.00000273 < 5 \times 10^{-5}$

A1

Answer: 0.00000273

1 mark

23 (9 marks)

(a) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

M1 A1

Answer: $T_3(t) = 4t^3 - 3t$

2 marks

(b) $t^3 - \frac{1}{4}T_3(t) = t^3 - t^3 + \frac{3}{4}t = \frac{3}{4}t$

A1

Greatest error $= 2^{-2} = \frac{1}{4}$

A1

Answer: $\frac{3}{4}t$, error $\frac{1}{4}$

2 marks

(c) $t^3 - \frac{3}{4}t = \frac{1}{4}T_3(t)$: values at 1, -1, ...: $t = 1: \frac{1}{4}; t = \frac{1}{2}: -\frac{1}{4}; t = -\frac{1}{2}: \frac{1}{4}; t = -1: -\frac{1}{4}$

A1 A1

Answer: alternating $\pm \frac{1}{4}$

2 marks

(d) Zero polynomial: greatest error 1; $t^3 \approx t$: greatest $|t - t^3| = \frac{2}{3\sqrt{3}} = 0.385$

M1 A1

$\frac{3}{4}t$ gives $\frac{1}{4}$, far smaller.

E1

Answer: 1, 0.385, $\frac{1}{4}$

3 marks

24 (9 marks)

(a) Term by term: $1, \frac{1}{3}, \frac{1}{(2 \times 5)}, \frac{1}{(6 \times 7)}, \frac{1}{(24 \times 9)}$

M1 A1

Answer: $\int_0^1 e^{-x^2} dx = 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \dots$

2 marks

(b) Sum = 0.74286

M1 A1

Alternating decreasing series: $|\text{error}| < \frac{1}{216} = 0.00463$

M1 A1

Answer: 0.74286, error < 0.00463

4 marks

(c) Using five terms, the error is less than $\frac{1}{1320} = 0.000758 < 0.001$

M1 A1

Four terms only guarantee $\frac{1}{216} = 0.00463 > 0.001$, so five terms.

A1

Answer: 5 terms

3 marks

25 (9 marks)

(a) $f'(x) = \frac{1}{2}(1+x)^{-1/2}$, $f''(x) = -\frac{1}{4}(1+x)^{-3/2}$

A1

$f'(1) = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$, $f''(1) = -\frac{1}{4 \times 2\sqrt{2}} = -\frac{\sqrt{2}}{16}$

A1

$p = f'(1) = \frac{\sqrt{2}}{4}$ (= 0.3536), $q = \frac{f''(1)}{2} = -\frac{\sqrt{2}}{32}$ (= -0.0442)

A1

Answer: $p = \frac{\sqrt{2}}{4}$, $q = -\frac{\sqrt{2}}{32}$

3 marks

(b) $x = 0$: $f = 1$; $M = 1$ (0%); $T = 1.0165$ (1.6%)

B1

$x = 1.5$: $f = 1.5811$; $M = 1.4688$ (7.1%); $T = 1.5799$ (0.076%)

A1

$x = 3$: $f = 2$; $M = 1.3750$ (31%); $T = 1.9445$ (2.8%)

A1

Answer: errors M: 0%, 7.1%, 31%; T: 1.6%, 0.076%, 2.8% (values in working)

3 marks

(c) T: its largest error is 2.8% against 31% for M

M1 A1

T is centred near the middle of the interval, so no point is far from the centre; M is centred at the end $x = 0$ and degrades quickly towards $x = 3$.

E1

Answer: T (greatest error 2.8% against 31% for M)

3 marks

26 (9 marks)

(a) $\frac{\partial I}{\partial a} = -2 \int_0^1 (x^2 - a - bx) dx = 0$, so $\frac{1}{3} - a - \frac{b}{2} = 0$

M1 A1

$\frac{\partial I}{\partial b} = -2 \int_0^1 x(x^2 - a - bx) dx = 0$, so $\frac{1}{4} - \frac{a}{2} - \frac{b}{3} = 0$

A1

Answer: $a + \frac{b}{2} = \frac{1}{3}$, $\frac{a}{2} + \frac{b}{3} = \frac{1}{4}$

3 marks

(b) Multiply the second by 2: $a + \frac{2b}{3} = \frac{1}{2}$; subtracting the first: $\frac{b}{6} = \frac{1}{6}$

M1 A1

$$b = 1 \text{ and } a = \frac{1}{3} - \frac{1}{2} = -\frac{1}{6}$$

A1

Answer: $a = -\frac{1}{6}$, $b = 1$

3 marks

(c) $e(x) = x^2 - x + \frac{1}{6}$: $e(0) = e(1) = \frac{1}{6}$, $e(\frac{1}{2}) = -\frac{1}{12}$, so the greatest $|e| = \frac{1}{6}$

M1 A1

For $y = x$ the error $x - x^2$ has greatest value $\frac{1}{4}$ (at $x = \frac{1}{2}$), which is larger than $\frac{1}{6}$

B1

Answer: $\frac{1}{6}$ against $\frac{1}{4}$

3 marks

27 (12 marks)

(a) x : $1 - \frac{1}{2} = \frac{1}{2}$

M1 A1

$$x^2: \frac{1}{2} - \frac{1}{2} + \frac{1}{12} = \frac{1}{12}$$

A1

$$x^3: \frac{1}{6} - \frac{1}{4} + \frac{1}{12} = 0$$

A1

$$x^4: \frac{1}{24} - \frac{1}{12} + \frac{1}{24} = 0$$

A1

Answer: product = $1 + \frac{x}{2} + \frac{x^2}{12} + 0x^3 + 0x^4 + \dots$

5 marks

(b) $e^x(1 - \frac{x}{2} + \frac{x^2}{12}) = 1 + \frac{x}{2} + \frac{x^2}{12} + O(x^5)$, so dividing gives $R(x) = e^x + O(x^5)$

E1 E1

Answer: agrees to x^4

2 marks

$$(c) R(1) = \frac{1 + \frac{1}{2} + \frac{1}{12}}{1 - \frac{1}{2} + \frac{1}{12}} = \frac{19}{7} = 2.71429$$

M1 A1

R error: 0.15%; degree 4: $2.70833 = \frac{65}{24}$, error 0.37%

M1 A1

R is more accurate even though both have the same order of agreement.

E1

Answer: $\frac{19}{7}$; 0.15% vs 0.37%

5 marks

28 (12 marks)

(a) $|R_3| = \frac{|f^{(4)}(\xi)|}{4!}|x|^4 \leq \frac{e}{24}$

M1 A1

$$= 0.113$$

A1

Answer: ≤ 0.113

3 marks

(b) w is monic of degree 4 with the zeros of T_4 , and T_4 has leading coefficient 8, so $w = \frac{T_4}{8}$

M1 A1

$$|T_4| \leq 1, \text{ so } |w| \leq \frac{1}{8}$$

A1

$$|\text{error}| \leq \frac{e}{24} \times \frac{1}{8} = \frac{e}{192} = 0.0142$$

M1 A1

Answer: ≤ 0.0142

5 marks

(c) Both actual errors are within their bounds ($0.052 < 0.11$ and $0.0067 < 0.014$)

M1 A1

The interpolant error 0.0067 is about 7.8 times smaller than the Maclaurin error 0.052

M1

The interpolant spreads the error evenly across the interval, whereas the Maclaurin polynomial is concentrated at the ends; for an approximation on a whole interval, nodes chosen well are far better.

E1

Answer: 0.0067 vs 0.052

4 marks

29 (11 marks)

$$(a) \theta_0 = 0.2: 100(0.0025 + \frac{11 \times 0.0016}{3072}) = 0.25\%$$

M1 A1

$$\theta_0 = 0.5: 100(0.015625 + \frac{11 \times 0.0625}{3072}) = 1.6\%$$

M1 A1

Answer: 0.25% and 1.6%

4 marks

$$(b) \frac{\theta_0^2}{16} = 0.01$$

M1

$$\theta_0 = 0.4 \text{ rad}$$

A1

Answer: 0.4 rad

2 marks

$$(c) \frac{11u^2}{3072} + \frac{u}{16} = 0.01; \text{ multiply by } 3072: 11u^2 + 192u = 30.72$$

M1 A1

$$u = 0.1586$$

M1 A1

$$\theta_0 = 0.398$$

A1

Answer: 0.398

5 marks

30 (12 marks)

$$(a) y' = x + y^2, y'(0) = 1$$

B1

$$y'' = 1 + 2yy', y''(0) = 3$$

M1 A1

$$y''' = 2(y')^2 + 2yy'', y'''(0) = 2 + 6 = 8$$

M1 A1

$$y^{(4)} = 6y'y'' + 2yy''', y^{(4)}(0) = 18 + 16 = 34$$

A1

Answer: 1, 3, 8, 34

6 marks

$$(b) y \approx 1 + x + \frac{3}{2}x^2 + \frac{8}{6}x^3 + \frac{34}{24}x^4$$

M1

$$= 1 + x + \frac{3}{2}x^2 + \frac{4}{3}x^3 + \frac{17}{12}x^4$$

A1

$$\text{Answer: } 1 + x + \frac{3}{2}x^2 + \frac{4}{3}x^3 + \frac{17}{12}x^4$$

2 marks

$$(c) y(0.2) \approx 1 + 0.2 + 0.06 + \frac{4}{3} \times 0.008 + \frac{17}{12} \times 0.0016 = 1.2729$$

M1 A1

Difference 0.00063; the fifth derivative term is the first neglected.

A1

Answer: 1.2729 (numerical 1.2736)

3 marks

(d) The solution grows quickly (the equation is non-linear and y blows up at a finite x), so the polynomial converges only for small |x|, and x = 1 is outside this range.

E1

Answer: series valid only for small x

1 mark