

Linear and Quadratic Interpolation

- 1** Given that $f(2) = 5$ and $f(3) = 9$, use linear interpolation to estimate $f(2.4)$. [2]
- 2** Given that $\ln 2 = 0.6931$ and $\ln 3 = 1.0986$ (to 4 decimal places), use linear interpolation to estimate $\ln 2.5$, giving your answer to 4 decimal places. [2]
- 3** The temperature of a metal bar is 23.5°C after 10 minutes and 31.2°C after 20 minutes. Use linear interpolation to estimate its temperature after 14 minutes. [2]
- 4** Two values of a function are $f(4) = 10$ and $f(6) = 16$. Use linear extrapolation to estimate $f(7)$. [2]
- 5** The true value of a quantity is 4.8 and an estimate of it is 4.92. Find the percentage error of the estimate. [2]
- 6** A function is known at $x = 10, 20, 30, 40, 50$.
 (a) State whether an estimate of $f(35)$ is found by interpolation or by extrapolation.
 (b) State whether an estimate of $f(60)$ is found by interpolation or by extrapolation, and say which of the two estimates is likely to be less reliable. [3]
- 7** The table shows values of a function f .
 Use inverse linear interpolation to estimate the value of x for which $f(x) = 9.5$. Give your answer to 3 significant figures.
- | | | |
|--------|----------|----------|
| x | 3 | 4 |
| $f(x)$ | 7.8 | 10.2 |
- [3]
- 8** Given that $\cos 40^\circ = 0.7660$ and $\cos 50^\circ = 0.6428$ (to 4 decimal places):
 (a) use linear interpolation to estimate $\cos 44^\circ$, giving your answer to 4 decimal places. [2]
 (b) Given that the true value of $\cos 44^\circ$ is 0.7193 (4 d.p.), find the percentage error in your estimate, to 2 significant figures. [2]
- 9** The population of a town was 5.2 thousand in the year 2000 and 6.1 thousand in the year 2010.
 (a) Use linear interpolation to estimate the population in 2004. [2]
 (b) Use linear extrapolation to estimate the population in 2018, and explain why this estimate is less reliable than the one in part (a). [3]
- 10** A continuous function g satisfies $g(1) = -2.3$ and $g(2) = 1.9$.
 (a) Explain why the equation $g(x) = 0$ has a root between $x = 1$ and $x = 2$. [1]
 (b) Use linear interpolation to estimate this root, giving your answer to 3 significant figures. [3]
- 11** A quadratic $p(x) = ax^2 + bx + c$ passes through the points (1, 3), (2, 3) and (4, 9).
 (a) Find the values of a , b and c . [4]
 (b) Hence estimate the value of $f(3)$, assuming that f is approximated by p . [1]

- 12** The cumulative frequency table shows the lengths l (in mm) of 60 leaves.

l (mm)	≤ 20	≤ 30	≤ 40	≤ 50
Cumulative frequency	4	15	38	60

- (a) Use linear interpolation to estimate the median length, giving your answer to 3 significant figures. [3]
- (b) Estimate the percentage of leaves shorter than 35 mm, giving your answer to 3 significant figures. [2]

- 13** The table shows some values of a function f .

x	0	1	2	3	4
$f(x)$	1.0	1.8	3.1	5.0	8.2

- (a) Use linear interpolation between the two most suitable values to estimate $f(2.7)$. [2]
- (b) Use linear extrapolation to estimate $f(5)$, and comment on its reliability. [3]

- 14** Values of $f(x) = x^3$ are known at $x = 2$ and $x = 3$.

- (a) Use inverse linear interpolation to estimate the solution of $x^3 = 20$, giving your answer to 4 significant figures. [3]
- (b) Find the percentage error in this estimate, given that the true solution is $\sqrt[3]{20}$, to 2 significant figures. [3]

- 15** The quadratic $p(x) = ax^2 + bx$ passes through the three points on the curve $y = \sin x$ with $x = 0$, $x = \frac{\pi}{6}$ and $x = \frac{\pi}{2}$.

- (a) Find the exact values of a and b . [3]
- (b) Use p to estimate $\sin \frac{\pi}{4}$, and find the percentage error given that $\sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$. [4]

- 16** The speed v m s⁻¹ of a cyclist is recorded as 4 at $t = 0$ s, 14 at $t = 5$ s and 20 at $t = 10$ s. A model $v = at^2 + bt + c$ is fitted through these three points.

- (a) Find a , b and c . [4]
- (b) Estimate the speed at $t = 7$ s. [1]
- (c) Find the time between $t = 0$ and $t = 10$ at which the model gives a speed of 16 m s⁻¹, to 3 significant figures. [2]

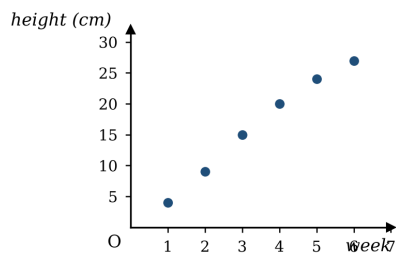
- 17** A function f has $f(a) < 0 < f(b)$, where f is continuous on $[a, b]$.

- (a) Show that the root of $f(x) = 0$ estimated by linear interpolation is $x = \frac{a f(b) - b f(a)}{f(b) - f(a)}$. [3]
- (b) Use this result with $f(x) = x^3 - 5$, $a = 1$, $b = 2$ to estimate the root, giving your answer as an exact fraction. [2]

- 18** Values of $\sqrt{x}$ are known at $x = 9$, 16 and 25.

- (a) Use linear interpolation between $x = 16$ and $x = 25$ to estimate $\sqrt{20}$, and find the percentage error given that $\sqrt{20} = 4.4721$ (to 4 d.p.). [4]
- (b) The quadratic through (9, 3), (16, 4) and (25, 5) is $p(x) = -\frac{1}{504}x^2 + \frac{97}{504}x + \frac{10}{7}$.
Use p to estimate $\sqrt{20}$ and find the percentage error. [3]

- 19** The graph shows the height of a plant measured at the end of each of the first six weeks: 4, 9, 15, 20, 24 and 27 cm.



- (a) Use linear interpolation to estimate the height after 3.5 weeks. [2]
 (b) Use linear extrapolation from the last two readings to estimate the height after 8 weeks. [2]
 (c) Explain, using the pattern in the data, why the estimate in part (b) is likely to be too large. [1]
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- 20** Ravi estimates $f(3.5)$ using $f(3) = 10$ and $f(5) = 18$. His working is:

$$f(3.5) \approx 10 + \frac{3.5}{5} \times (18 - 10) = 15.6$$

 (a) Identify the error in his working. [2]
 (b) Find the correct estimate and say whether Ravi's value could have been sensible. [3]
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- 21** The function $f(x) = \frac{1}{x}$ has values $f(1) = 1$, $f(2) = \frac{1}{2}$ and $f(4) = \frac{1}{4}$.
 (a) Find the quadratic $p(x)$ passing through $(1, 1)$, $(2, \frac{1}{2})$ and $(4, \frac{1}{4})$. [2]
 (b) Use p to estimate $f(3)$ and find the percentage error. [3]
 (c) Use linear interpolation between $x = 2$ and $x = 4$ to estimate $f(3)$, and find the percentage error. [3]
 (d) Comment on the relative accuracy of the two methods here. [1]
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- 22** A manufacturer records the cost £C (in thousands) of making x hundred units as 2 when $x = 0$, 3 when $x = 2$ and 8 when $x = 4$.
 (a) Fit a quadratic $C = ax^2 + bx + c$ through the three values and find a , b and c . [4]
 (b) Use the model to estimate the number of units that can be made for a cost of £4000, justifying which root you use, giving your answer to the nearest unit. [5]
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- 23** The line L joins the points (a, a^2) and (b, b^2) on the curve $y = x^2$, where $a < b$.
 (a) Show that the equation of L is $y = (a + b)x - ab$. [3]
 (b) Hence show that linear interpolation overestimates x^2 for $a < x < b$ by $(x - a)(b - x)$, and that the greatest overestimate is $\frac{(b - a)^2}{4}$. [4]
 (c) Use this to find the maximum possible error when x^2 is estimated by linear interpolation between $x = 3$ and $x = 3.2$. [1]
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- 24** Let $f(x) = x^3 + x - 3$.
 (a) Show that $f(x) = 0$ has a root between 1 and 2 and use linear interpolation to find a first estimate x_1 . [3]
 (b) Evaluate $f(x_1)$, and use linear interpolation between x_1 and 2 to find a second estimate x_2 , to 3 decimal places. [3]
 (c) The root is 1.213 to 3 d.p. Find the percentage error of x_1 and x_2 , and explain why linear interpolation converges slowly here. [3]

- 25** Take $g = 9.8 \text{ m s}^{-2}$. A ball is thrown upwards and its height h metres is recorded as 15.1 at $t = 1 \text{ s}$, 20.4 at $t = 2 \text{ s}$ and 15.9 at $t = 3 \text{ s}$. A quadratic $h = at^2 + bt + c$ is fitted through these three readings.
- (a) Find a , b and c . [3]
- (b) Use the model to estimate the greatest height reached and the time at which it occurs, to 3 significant figures. [3]
- (c) Estimate when the ball returns to its starting height, and explain why a quadratic model is exact for this motion. [3]

- 26** The table gives four values of a function f , which are believed to follow a smooth curve.

x	0	1	3	4
$f(x)$	2	3	11	18.6

- (a) Use linear interpolation between $x = 1$ and $x = 3$ to estimate $f(2)$. [2]
- (b) Find the quadratic through $(0, 2)$, $(1, 3)$ and $(3, 11)$ and use it to estimate $f(2)$. [3]
- (c) Use p to predict $f(4)$, and find the percentage difference between this and the tabulated value. [3]
- (d) State, with a reason, which estimate of $f(2)$ you would trust more. [1]
- 27** A quadratic $p(x)$ passes through $(-h, y_0)$, $(0, y_1)$ and (h, y_2) , where $h > 0$.
- (a) Show that $p(x) = y_1 + \frac{y_2 - y_0}{2h}x + \frac{y_0 - 2y_1 + y_2}{2h^2}x^2$. [5]
- (b) Hence show that $\int_{-h}^h p(x) \, dx = \frac{h}{3}(y_0 + 4y_1 + y_2)$. [3]
- (c) Use this result to estimate $\int_1^2 \frac{1}{x} \, dx$ using the values of $\frac{1}{x}$ at $x = 1, 1.5$ and 2 , giving your answer to 4 decimal places. [3]

- 28** A quadratic $p(x)$ is to approximate $f(x) = \sqrt{x}$ on $[4, 9]$. It is required that $p(4) = f(4)$, $p'(4) = f'(4)$ and $p(9) = f(9)$.
- (a) Show that $p(x) = 2 + \frac{1}{4}(x - 4) + k(x - 4)^2$ for some constant k , and find k . [5]
- (b) Use p to estimate $\sqrt{6}$, and find the percentage error to 2 significant figures. [4]
- (c) Show that $p(x) - \sqrt{x}$ has a double root at $x = 4$, and explain what this means for the graphs of $y = p(x)$ and $y = \sqrt{x}$ at $x = 4$. [2]

- 29** The function $f(x) = x^3 - x - 1$ has a root between 1 and 2. Values are $f(1) = -1$, $f(1.5) = 0.875$ and $f(2) = 5$.
- (a) Use linear interpolation between $x = 1$ and $x = 1.5$ to estimate the root. [2]
- (b) Inverse quadratic interpolation fits x as a quadratic in $y = f(x)$, $x = Ay^2 + By + C$, through the three points. Find A , B and C as exact fractions. [6]
- (c) Hence estimate the root as the value of x when $y = 0$, and compare with the true root 1.3247 by finding the percentage error. [4]

- 30** Linear interpolation between $x = 1$ and $x = 2$ is used to estimate values of $f(x) = \frac{1}{x}$.
- (a) Show that the interpolating line is $y = \frac{3-x}{2}$. [2]
- (b) Show that the line lies above the curve for $1 < x < 2$. [Hint: consider $x(3-x) - 2$.] [3]
- (c) Find the exact value of x at which the overestimate $E(x) = \frac{3-x}{2} - \frac{1}{x}$ is greatest, and the exact greatest overestimate. [5]

Lagrange and Newton Interpolation

- 1 Write down the Lagrange basis polynomials $L_0(x)$ and $L_1(x)$ for the nodes $x_0 = 1$ and $x_1 = 3$. [2]
- 2 The nodes are $x_0 = 0$, $x_1 = 1$ and $x_2 = 2$. Find the Lagrange basis polynomial $L_1(x)$ and evaluate $L_1(0.5)$. [2]
- 3 Find the divided difference $f[x_0, x_1]$ for the data $(1, 4)$ and $(3, 10)$, and write down the linear Newton interpolating polynomial $p(x)$. [2]
- 4 The values of f at $x = 0, 1, 2, 3$ are $2, 5, 10, 17$. Construct the forward difference table up to the third differences and state the degree of the interpolating polynomial. [3]
- 5 Calculate the second divided difference $f[x_0, x_1, x_2]$ for the data $(1, 3), (2, 5), (4, 11)$. [3]
- 6 (a) State how many different polynomials of degree at most n pass through $n + 1$ data points with distinct x -values.
(b) State the greatest possible degree of the polynomial through six such points. [2]
- 7 A quadratic passes through $(0, 1), (1, 3)$ and $(3, 13)$.
(a) Write down the three Lagrange basis polynomials for the nodes $0, 1, 3$. [2]
(b) Hence find the quadratic in the form $ax^2 + bx + c$. [3]
- 8 A function f has $f(-1) = -2$, $f(1) = 2$ and $f(2) = 7$.
(a) Construct a divided difference table. [2]
(b) Write the interpolating polynomial in Newton form and use it to estimate $f(0)$. [3]
- 9 The quadratic through $(0, 1), (1, 2)$ and $(2, 5)$ is $p_2(x) = 1 + x + x(x - 1)$ in Newton form. A fourth data point $(3, 14)$ is then added.
(a) Find $p_2(3)$. [1]
(b) Show that the cubic through all four points is $p_3(x) = p_2(x) + \frac{2}{3}x(x - 1)(x - 2)$, explaining why only one new term is needed. [4]
- 10 For equally spaced data with step h , the Newton forward difference formula is $p(x) = f_0 + s\Delta f_0 + \frac{s(s-1)}{2!}\Delta^2 f_0 + \dots$, where $s = \frac{x - x_0}{h}$.
A function takes the values $1, 4, 11, 22$ at $x = 0, 1, 2, 3$.
(a) Find the first, second and third forward differences at x_0 . [2]
(b) Find $p(x)$ and use it to estimate $f(1.5)$. [3]
- 11 Neville's algorithm builds an estimate from two earlier ones. For nodes x_i and x_j with estimates $P_i(x)$ and $P_j(x)$, the combined estimate is $\frac{(x - x_j)P_i(x) - (x - x_i)P_j(x)}{x_i - x_j}$.
The data are $f(1) = 3$, $f(2) = 7$ and $f(4) = 21$.
(a) Use Neville's algorithm to estimate $f(3)$, first combining the nodes 1 and 2 , then the nodes 2 and 4 , and finally combining these two results. [4]
(b) Check your answer by finding the quadratic through the three points and evaluating it at $x = 3$. [1]

- 12** The table gives four values of a function f .

x	0	1	2	4
$f(x)$	1	0	5	57

- (a) Construct the divided difference table and write down the Newton interpolating cubic. [3]
 (b) Hence estimate $f(3)$. [2]

- 13** Given that $\ln 1 = 0$, $\ln 2 = 0.6931$ and $\ln 4 = 1.3863$ (4 d.p.), a quadratic p is fitted through the points $(1, \ln 1)$, $(2, \ln 2)$ and $(4, \ln 4)$.

- (a) Use Newton's divided differences to estimate $\ln 3$ as $p(3)$. [3]
 (b) Given that $\ln 3 = 1.0986$, find the percentage error. [2]

- 14** Let x_0, x_1, x_2 be distinct. Suppose p and q are polynomials of degree at most 2 with $p(x_i) = q(x_i) = y_i$ for $i = 0, 1, 2$.

- (a) Prove that $p(x) = q(x)$ for all x . [3]
 (b) Show by example that the result fails if only two points are given, for polynomials of degree at most 2. Use the points $(0, 1)$ and $(1, 2)$. [2]

- 15** The values of a function f at $x = 0, 1, 2, 3, 4$ are 3, 4, 15, 48, 115.

- (a) Construct a forward difference table and explain why the data come from a cubic. [3]
 (b) Extend the table to predict $f(5)$. [2]
 (c) Find $f(x)$ in the form $ax^3 + bx^2 + cx + d$. [2]

- 16** The table shows values of a function that should be exactly quadratic. One of the values has been recorded wrongly.

x	0	1	2	3	4	5	6	7
$f(x)$	1	3	7	13	21	33	43	57

- (a) Construct the first and second difference columns and use them to locate the wrong value. [3]
 (b) Find the error in $f(5)$ and the correct value. [2]

- 17** A cubic is to pass through $(-1, 3)$, $(0, 1)$, $(1, -1)$ and $(2, 7)$.

- (a) Write down the Lagrange basis polynomial $L_2(x)$ associated with the node $x_2 = 1$. [2]
 (b) Find the cubic in expanded form. [5]

- 18** A function f is tabulated: $f(1) = 1$, $f(2) = 3$, $f(3) = 7$. Use Lagrange interpolation, treating x as a function of $y = f(x)$, to estimate the value of x for which $f(x) = 5$. The function is $f(x) = x^2 - x + 1$, whose positive solution of $f(x) = 5$ is $x = 2.562$ (4 s.f.).

- (a) Estimate x when $f(x) = 5$. [4]
 (b) Find the percentage error. [2]

- 19** The Newton form of an interpolating polynomial for a function f at the nodes 1, 3, 4 is $p(x) = 2 + 3(x - 1) + 4(x - 1)(x - 3)$.

- (a) Find $f(1)$, $f(3)$ and $f(4)$. [2]
 (b) Verify that $f[3, 4] = 15$ and hence confirm that $f[1, 3, 4] = 4$. [3]
 (c) Write $p(x)$ in the form $ax^2 + bx + c$. [2]

- 20** Equally spaced nodes $x_0, x_0 + h, x_0 + 2h$ have function values f_0, f_1, f_2 .

- (a) Show that $f[x_0, x_1, x_2] = \frac{\Delta^2 f_0}{2h^2}$. [4]
 (b) Use this to find $f[x_0, x_1, x_2]$ when $h = 0.5$ and the values are 1.2, 1.9, 3.0. [2]

- 21** The interpolation error for a polynomial p_n through $n + 1$ nodes is $f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)\dots(x - x_n)$ for some ξ in the interval. A quadratic p is fitted to $f(x) = \sin x$ at the nodes 0, 0.2, 0.4.
- (a) Show that $|\sin x - p(x)| \leq 0.0005$ at $x = 0.1$. [4]
- (b) Given $\sin 0.2 = 0.19867$ and $\sin 0.4 = 0.38942$, use Newton's divided differences to find $p(0.1)$ and the actual error. Give $p(0.1)$ to 5 decimal places. [5]
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- 22** For linear interpolation of f between $x = a$ and $x = b = a + h$ the error is $f(x) - p_1(x) = \frac{f''(\xi)}{2}(x - a)(x - b)$ for some ξ in (a, b) .
- (a) Show that if $|f''| \leq M$ on $[a, b]$ then $|f(x) - p_1(x)| \leq \frac{Mh^2}{8}$. [3]
- (b) A table of $\ln x$ for $1 \leq x \leq 2$ uses equally spaced values with step h and linear interpolation. Find the largest h for which the error is guaranteed to be below 5×10^{-6} . Give h to 3 s.f. [4]
- (c) A table with step 0.005 is used. State how many entries it contains (including $x = 1$ and $x = 2$). [1]
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- 23** Consider the data (1, 3), (2, 5), (4, 13).
- (a) Construct the divided difference table taking the nodes in the order 1, 2, 4. [3]
- (b) Repeat with the nodes in the order 4, 2, 1. [3]
- (c) Write both Newton polynomials and explain why they are equal. [3]
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- 24** Let L_0, L_1, L_2 be the Lagrange basis polynomials for distinct nodes x_0, x_1, x_2 .
- (a) Explain why $L_0(x) + L_1(x) + L_2(x) = 1$ for all x . [2]
- (b) Explain why $x_0L_0(x) + x_1L_1(x) + x_2L_2(x) = x$ for all x . [2]
- (c) For the nodes 0, 1, 3 find $L_0(2), L_1(2), L_2(2)$ and verify that the two identities hold at $x = 2$, and that $x_0^2L_0(2) + x_1^2L_1(2) + x_2^2L_2(2) = 4$. [5]
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- 25** Values of $f(x) = \sin x$ are 0.8415, 0.8912 and 0.9320 at $x = 1.0, 1.1$ and 1.2 . The quadratic p through these points is used to estimate derivatives.
- (a) Find the Newton divided differences $f[1.0, 1.1], f[1.1, 1.2]$ and $f[1.0, 1.1, 1.2]$. [3]
- (b) Write $p(x)$ in Newton form and differentiate to show that $p'(x) = f[1.0, 1.1] + f[1.0, 1.1, 1.2](2x - 2.1)$. Hence estimate $f'(1.1)$. [3]
- (c) Compare with $\cos 1.1 = 0.4536$ and use p'' to estimate $f''(1.1)$. [3]
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- 26** A function f takes the values 2, 3 and 8 at $x = 0, 1$ and 3 .
- (a) Find the quadratic Newton interpolating polynomial $p(x)$ and write it in the form $ax^2 + bx + c$. [4]
- (b) Estimate $\int_0^3 f(x) dx$ by integrating p . [2]
- (c) Find the trapezium rule estimate using the same three points and explain why it is larger than your answer to (b). [3]
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- 27** Let $f[x_0, x_1] = \frac{f_1 - f_0}{x_1 - x_0}$ and $f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$, where $f_k = f(x_k)$.
- (a) Show that $f[x_0, x_1, x_2] = \frac{f_0}{(x_0 - x_1)(x_0 - x_2)} + \frac{f_1}{(x_1 - x_0)(x_1 - x_2)} + \frac{f_2}{(x_2 - x_0)(x_2 - x_1)}$. [7]
- (b) Hence explain why the value of $f[x_0, x_1, x_2]$ does not depend on the order of the nodes. [2]
- (c) Use the formula to evaluate $f[x_0, x_1, x_2]$ for the data (0, 3), (2, 1), (5, 13). [2]

- 28** Let p be the linear polynomial interpolating f at $x = a$ and $x = b$, and let x be fixed with $a < x < b$. Define $w(t) = (t - a)(t - b)$ and $g(t) = f(t) - p(t) - Kw(t)$, where K is chosen so that $g(x) = 0$.
- (a) Show that $g(a) = g(b) = 0$ and hence that g'' has a zero ξ in (a, b) . Deduce that $f(x) - p(x) = \frac{f''(\xi)}{2}(x - a)(x - b)$. [7]
- (b) Use the result to bound the error when $f(x) = e^x$ is linearly interpolated between $x = 0$ and $x = 1$, evaluated at $x = 0.5$. [2]
- (c) Find the actual error at $x = 0.5$ and verify it is within the bound. [2]
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- 29** The cubic $f(x) = x^3$ is interpolated by a quadratic p at the nodes $x = 0, 1, 2$.
- (a) Use divided differences to find $p(x)$. [4]
- (b) Show that $x^3 - p(x) = x(x - 1)(x - 2)$. [2]
- (c) Show that this agrees with the interpolation error formula, and explain why ξ does not need to be known. [3]
- (d) Find the actual error when $x = 1.5$ is estimated by p , and the percentage error. [3]
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- 30** A function f has $f(0) = 1$, $f(1) = 2$, $f(2) = 5$, $f(3) = 10$ and $f(4) = y$, where y is unknown. The polynomial p of degree at most 4 interpolating these five values has x^4 coefficient equal to $\frac{1}{12}$.
- (a) Show that the x^4 coefficient equals $\frac{\Delta^4 f_0}{24}$ and hence find y . [5]
- (b) Write down $p(x)$ in Newton form. [3]
- (c) Hence estimate $f(2.5)$, and $f(5)$ by extrapolation. [3]

Cubic Splines and Piecewise Interpolation

- 1** A piecewise linear interpolant S passes through $(0, 1)$, $(2, 5)$ and $(5, 2)$. Find $S(1)$ and $S(3)$. [3]
- 2** A cubic spline is to pass through 5 data points, with a separate cubic on each interval between neighbouring points.
- (a) State the number of cubic pieces and the total number of coefficients.
- (b) Explain how these are determined by the conditions: values at the ends of each piece, continuity of S' and S'' at interior knots, and natural end conditions. [3]
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- 3** The first piece of a natural cubic spline on $0 \leq x \leq 1$ is $S_1(x) = 2x^3 + bx^2 + x + 1$. Find b . [2]
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- 4** A spline has $S_1(x) = x^3 + 2x$ for $0 \leq x \leq 1$ and $S_2(x) = 3 + 5(x - 1) + 3(x - 1)^2 + (x - 1)^3$ for $1 \leq x \leq 2$. Show that the values, first derivatives and second derivatives of the two pieces agree at $x = 1$. [3]
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- 5** A piecewise linear interpolant passes through $(0, 2)$, $(1, 5)$ and $(3, 4)$.
- (a) State the gradient on each piece.
- (b) State whether S' is continuous at $x = 1$ and what this means for the graph. [3]
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- 6** A function is defined by $S(x) = x^2$ for $0 \leq x \leq 2$ and $S(x) = 4 + 4(x - 2) - (x - 2)^2$ for $2 \leq x \leq 4$. Find $S(1.5)$ and $S(3)$. [2]
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- 7** A natural cubic spline through $(0, 0)$, $(1, 1)$ and $(2, 0)$ is claimed to be $S(x) = \frac{3}{2}x - \frac{1}{2}x^3$ for $0 \leq x \leq 1$ and $S(x) = 1 - \frac{3}{2}(x - 1)^2 + \frac{1}{2}(x - 1)^3$ for $1 \leq x \leq 2$.
- (a) Verify that S passes through the three points and that S is continuous. [2]
- (b) Verify that S' and S'' are continuous at $x = 1$. [2]
- (c) Verify that the natural end conditions hold. [1]

- 8** The function $S(x) = x^3 - x$ for $0 \leq x \leq 1$, and $S(x) = a(x - 1)^3 + b(x - 1)^2 + 2(x - 1)$ for $1 \leq x \leq 2$, is to be a cubic spline passing through (2, 4).
- (a) Show that S and S' are continuous at $x = 1$. [2]
- (b) Find b so that S'' is continuous at $x = 1$, and then find a using $S(2) = 4$. [3]

- 9** A natural cubic spline S passes through (0, 1), (1, 3) and (2, 2). On $[0, 1]$ it is $S_1(x) = 1 + b_1x + c_1x^2 + d_1x^3$, and on $[1, 2]$ it is $S_2(x) = 3 + b_2(x - 1) + c_2(x - 1)^2 + d_2(x - 1)^3$.
- (a) Write down the conditions that determine the eight unknowns and use the natural end conditions to show $c_1 = 0$ and $c_2 = -3d_2$. [2]
- (b) Solve for the coefficients and write down S_1 and S_2 . [3]

- 10** The boiling point of water ($^{\circ}\text{C}$) at various altitudes is shown in the table. The values are joined by straight line segments (piecewise linear interpolation).

Altitude (m)	0	1000	3000	5000
Boiling point ($^{\circ}\text{C}$)	100	96.7	90.0	83.6

- (a) Estimate the boiling point at an altitude of 2000 m. [2]
- (b) Estimate the altitude at which water boils at 86°C . [2]
- (c) Explain why the answers are only estimates. [1]

- 11** A natural cubic spline S through (0, 2), (1, 1) and (2, 3) is $S(x) = \frac{3}{4}x^3 - \frac{7}{4}x + 2$ for $0 \leq x \leq 1$ and $S(x) = -\frac{3}{4}(x - 1)^3 + \frac{9}{4}(x - 1)^2 + \frac{1}{2}(x - 1) + 1$ for $1 \leq x \leq 2$.
- (a) Find $S(0.5)$. [2]
- (b) Find $S'(1.5)$ and $S''(1.5)$. [3]

- 12** Piecewise quadratic interpolation is used through five points. The first quadratic Q_1 passes through (0, 1), (1, 2) and (2, 5), and the second Q_2 passes through (2, 5), (3, 6) and (4, 5).
- (a) Show that $Q_1(x) = x^2 + 1$ and find $Q_2(x)$. [2]
- (b) Show that the piecewise function is not smooth at $x = 2$ by comparing gradients. [2]
- (c) Estimate the value at $x = 3.5$. [1]

- 13** A natural cubic spline passes through (0, 0), (1, 2) and (3, 1), with $S_1(x) = b_1x + d_1x^3$ on $[0, 1]$ (since $S''(0) = 0$ and $S(0) = 0$) and $S_2(x) = 2 + b_2(x - 1) + c_2(x - 1)^2 + d_2(x - 1)^3$ on $[1, 3]$.
- (a) Use the conditions at $x = 1$ and the natural condition at $x = 3$ to show that $c_2 = 3d_1$ and $d_2 = -\frac{1}{2}d_1$. [2]
- (b) Hence find d_1 and b_1 . [3]

- 14** A natural cubic spline S is fitted through (0, 0), (1, 2), (2, 3) and (3, 1). Let $m_i = S''(x_i)$, so $m_0 = m_3 = 0$. For unit spacing, continuity of S' at x_i gives $m_{i-1} + 4m_i + m_{i+1} = 6(y_{i-1} - 2y_i + y_{i+1})$, and on $[x_i, x_{i+1}]$ the cubic is $S = y_i + b_it + \frac{m_i}{2}t^2 + \frac{m_{i+1} - m_i}{6}t^3$, where $t = x - x_i$ and $b_i = y_{i+1} - y_i - \frac{2m_i + m_{i+1}}{6}$.
- (a) Find m_1 and m_2 . [4]
- (b) Hence find $S(1.5)$. [3]

- 15** The function $f(x) = \frac{1}{x}$ is tabulated at $x = 1, 2, 3, 4$ with values $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$. Three methods are used to estimate $f(2.5)$: (i) linear interpolation between $x = 2$ and $x = 3$, (ii) the quadratic through $x = 2, 3, 4$, and (iii) the natural cubic spline through all four points, for which $m_1 = \frac{1}{2}$ and $m_2 = 0$ (where $m_i = S''(x_i)$).

For unit spacing, the spline on $[x_i, x_{i+1}]$ has midpoint value $\frac{y_i + y_{i+1}}{2} - \frac{m_i + m_{i+1}}{16}$.

x	1	2	3	4
$f(x)$	1	1/2	1/3	1/4

- (a) Find the three estimates of $f(2.5)$. [4]
 (b) Find the percentage error of each estimate (true value 0.4), to 3 significant figures, and say which is most accurate. [3]

- 16** A natural cubic spline passes through $(0, y_0)$, (h, y_1) and $(2h, y_2)$. Let $m_i = S''(x_i)$, so $m_0 = m_2 = 0$.

The continuity of S' at $x = h$ gives $h m_0 + 4h m_1 + h m_2 = 6\left(\frac{y_2 - y_1}{h} - \frac{y_1 - y_0}{h}\right)$.

- (a) Show that $m_1 = \frac{3(y_0 - 2y_1 + y_2)}{2h^2}$. [3]
 (b) Find m_1 when $h = 0.5$ and y_0, y_1, y_2 are 1, 2.5, 2, and state what the sign of m_1 tells you about the curve at the middle knot. [3]

- 17** A cubic $S(x) = 1 + 2x + cx^2 + dx^3$ on $[0, 1]$ satisfies $S(0) = 1$, $S'(0) = 2$, $S(1) = 4$ and $S'(1) = -1$ (a clamped end condition at both ends).

- (a) Find c and d . [4]
 (b) Show that this cubic does not satisfy the natural spline condition at $x = 0$. [1]
 (c) Find the x -coordinate of the maximum of S on $[0, 1]$, to 3 significant figures. [2]

- 18** The function $S(x) = x^3 - 2x + 3$ for $0 \leq x \leq 1$ is joined to $S(x) = 2 + k(x - 1) + 3(x - 1)^2 + (x - 1)^3$ for $1 \leq x \leq 2$.

- (a) Show that S is continuous at $x = 1$ for every k , and find the value of k for which S' is also continuous. [3]
 (b) Show that, with this k , S'' is also continuous at $x = 1$. [2]
 (c) Describe the graph at $x = 1$ if $k = 0$ is used instead. [1]

- 19** The natural cubic spline S through $(0, 2)$, $(1, 0)$ and $(2, 1)$ has, on $[1, 2]$, $S(x) = -\frac{3}{4}(x - 1)^3 + \frac{9}{4}(x - 1)^2 - \frac{1}{2}(x - 1)$.

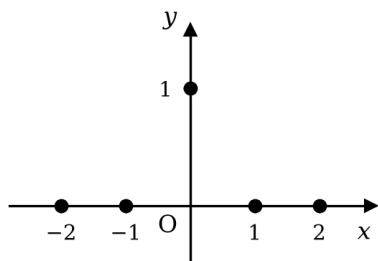
- (a) Find the x -coordinate of the stationary point of S on $(1, 2)$, giving your answer to 3 significant figures. [3]
 (b) Find the value of S at this point, to 3 significant figures, and describe the stationary point. [3]

- 20** A student finds the natural cubic spline through $(0, 0)$, $(1, 3)$ and $(2, 0)$. They obtain $S_1(x) = \frac{9}{2}x - \frac{3}{2}x^3$ for $0 \leq x \leq 1$ and $S_2(x) = 3 + \frac{9}{2}(x - 1) - \frac{9}{2}(x - 1)^2 + \frac{3}{2}(x - 1)^3$ for $1 \leq x \leq 2$.

- (a) Show that S_2 does not pass through the point $(2, 0)$. [2]
 (b) Identify which spline condition the student has violated and find the correct S_2 . [3]

- 21** A natural cubic spline S is fitted to $(0, 1)$, $(1, 2)$, $(3, 0)$ and $(4, 3)$. Let $m_i = S''(x_i)$, so $m_0 = m_3 = 0$.
For knots with spacings $h_i = x_{i+1} - x_i$ and gradients $s_i = \frac{y_{i+1} - y_i}{h_i}$, continuity of S' gives $h_{i-1}m_{i-1} + 2(h_{i-1} + h_i)m_i + h_im_{i+1} = 6(s_i - s_{i-1})$. On $[x_i, x_{i+1}]$ the cubic is $S = y_i + b_it + \frac{m_i}{2}t^2 + \frac{m_{i+1} - m_i}{6h_i}t^3$ with $t = x - x_i$ and $b_i = s_i - \frac{h_i(2m_i + m_{i+1})}{6}$.
- Find s_0, s_1, s_2 and write down the two equations for m_1 and m_2 . [3]
 - Solve for m_1 and m_2 . [2]
 - Find $S(2)$. [3]
 - Compare with the linear interpolation estimate at $x = 2$. [1]
-
- 22** The natural cubic spline through $(0, 1)$, $(1, 2)$ and $(2, 0)$ is $S(x) = -\frac{3}{4}x^3 + \frac{7}{4}x + 1$ on $[0, 1]$ and $S(x) = \frac{3}{4}(x - 1)^3 - \frac{9}{4}(x - 1)^2 - \frac{1}{2}(x - 1) + 2$ on $[1, 2]$.
- Find the exact value of $\int_0^2 S(x) dx$. [5]
 - Find the trapezium rule estimate using the three data points and the difference between it and your answer to (a). [2]
 - Explain why the spline integral is the more reliable estimate if the data come from a smooth curve. [1]
-
- 23** A cubic spline has unit knot spacing. Let $m_i = S''(x_i)$. On $[x_i, x_i + 1]$, $S''(x)$ is linear, so $S(x) = y_i + b_it + \frac{m_i}{2}t^2 + \frac{m_{i+1} - m_i}{6}t^3$, where $t = x - x_i$.
- Show that $S''(x_i) = m_i$ and $S''(x_{i+1}) = m_{i+1}$, and use $S(x_{i+1}) = y_{i+1}$ to show $b_i = y_{i+1} - y_i - \frac{2m_i + m_{i+1}}{6}$. [3]
 - Show that $S'(x_{i+1})$ from the left is $y_{i+1} - y_i + \frac{m_i + 2m_{i+1}}{6}$. [2]
 - By equating this with $S'(x_{i+1})$ from the right, which is b_{i+1} , show that $m_i + 4m_{i+1} + m_{i+2} = 6(y_i - 2y_{i+1} + y_{i+2})$. [4]
-
- 24** The points $(0, 1)$, $(1, 3)$, $(2, 5)$, $(3, 7)$ all lie on the line $y = 1 + 2x$. A natural cubic spline S is fitted through them. Let $m_i = S''(x_i)$, so that $m_{i-1} + 4m_i + m_{i+1} = 6(y_{i-1} - 2y_i + y_{i+1})$ at each interior knot.
- Show that $m_1 = m_2 = 0$. [3]
 - Deduce that $S(x) = 1 + 2x$ for $0 \leq x \leq 3$. [2]
 - Explain, without calculation, why the same must hold for any number of collinear points. [2]
 - Comment on what this shows about the behaviour of a natural cubic spline compared with a high-degree interpolating polynomial. [1]
-
- 25** A piecewise quadratic function S is to pass through $(0, 1)$, $(1, 3)$ and $(2, 2)$, with S continuous, S' continuous at $x = 1$ and $S'(0) = 0$. On $[0, 1]$ let $Q_1(x) = 1 + \alpha x^2$ (since $Q_1(0) = 1$ and $Q_1'(0) = 0$) and on $[1, 2]$ let $Q_2(x) = 3 + \beta(x - 1) + \gamma(x - 1)^2$.
- Find α and Q_1 . [2]
 - Find β and γ . [3]
 - Show that S exceeds 3.5 somewhere on $[1, 2]$, and find its greatest value there. [3]
 - Show that S'' is not continuous at $x = 1$. [1]

- 26** A natural cubic spline S is fitted to the five points shown: $(-2, 0)$, $(-1, 0)$, $(0, 1)$, $(1, 0)$, $(2, 0)$. Let $m_i = S''(x_i)$; at each interior knot $m_{i-1} + 4m_i + m_{i+1} = 6(y_{i-1} - 2y_i + y_{i+1})$ and $m_0 = m_4 = 0$. On $[x_i, x_{i+1}]$ the midpoint value is $S = \frac{y_i + y_{i+1}}{2} - \frac{m_i + m_{i+1}}{16}$.



- (a) Write down the three equations for m_1 , m_2 , m_3 , and use the symmetry of the data to find them. [5]
- (b) Find $S(-1.5)$. [2]
- (c) The data represent a quantity that cannot be negative. Comment on the suitability of S , and name a better alternative. [2]
-
- 27** A natural cubic spline S is fitted to $(0, 0)$, $(1, 1)$, $(2, 3)$, $(3, 2)$, $(4, 4)$. Let $m_i = S''(x_i)$, with $m_0 = m_4 = 0$; for unit spacing $m_{i-1} + 4m_i + m_{i+1} = 6(y_{i-1} - 2y_i + y_{i+1})$. On $[x_i, x_{i+1}]$, $S = y_i + b_i t + \frac{m_i}{2} t^2 + \frac{m_{i+1} - m_i}{6} t^3$ with $b_i = y_{i+1} - y_i - \frac{2m_i + m_{i+1}}{6}$.
- (a) Show that the equations for m_1 , m_2 , m_3 are $4m_1 + m_2 = 6$, $m_1 + 4m_2 + m_3 = -18$, $m_2 + 4m_3 = 18$. [3]
- (b) Solve these equations. [4]
- (c) Find $S(2.5)$. [3]
-
- 28** Let S be the natural cubic spline on the knots $0, 1, 2$ through given data, and let f be any twice continuously differentiable function with $f(0) = S(0)$, $f(1) = S(1)$, $f(2) = S(2)$. Let $g = f - S$. You are to show that $\int_0^2 (f'')^2 dx \geq \int_0^2 (S'')^2 dx$.
- (a) Show that $\int_0^2 (f'')^2 dx = \int_0^2 (S'')^2 dx + 2 \int_0^2 S''g'' dx + \int_0^2 (g'')^2 dx$. [2]
- (b) On the interval $[x_i, x_{i+1}]$, integrate by parts to show that $\int_{x_i}^{x_{i+1}} S''g'' dx = [S''g']_{x_i}^{x_{i+1}} - \int_{x_i}^{x_{i+1}} S'''g' dx$. [2]
- (c) Use the fact that S''' is a constant k_i on each piece and $g(0) = g(1) = g(2) = 0$ to show that $\int_0^2 S''g'' dx = [S''g']_0^2$. [4]
- (d) Hence complete the proof, stating where the natural end conditions are used. [4]
-
- 29** A cubic spline S through $(0, 0)$, $(1, 2)$ and $(2, 1)$ has the clamped end conditions $S'(0) = 1$ and $S'(2) = 0$. Let $S = x + c_0 x^2 + d_0 x^3$ on $[0, 1]$ and $S = 2 + b_1(x - 1) + c_1(x - 1)^2 + d_1(x - 1)^3$ on $[1, 2]$.
- (a) Use $S(1) = 2$ to find an equation for c_0 and d_0 , and use continuity of S' and S'' at $x = 1$ to find b_1 and c_1 in terms of c_0 and d_0 . [4]
- (b) Use $S(2) = 1$ and $S'(2) = 0$ to find c_0 , d_0 , d_1 , and hence give S on each interval. [5]
- (c) Find $S(0.5)$ for this clamped spline and for the natural spline through the same points (for which $S(0.5) = \frac{41}{32}$). [3]
- Comment on the difference.

- 30** The values $\ln 1 = 0$, $\ln 2 = 0.6931$, $\ln 3 = 1.0986$ and $\ln 4 = 1.3863$ are given to 4 decimal places. The natural cubic spline S through the four points has $m_i = S''(x_i)$ satisfying $m_0 = m_3 = 0$ and $m_{i-1} + 4m_i + m_{i+1} = 6(y_{i-1} - 2y_i + y_{i+1})$; the midpoint value on $[x_i, x_{i+1}]$ is $\frac{y_i + y_{i+1}}{2} - \frac{m_i + m_{i+1}}{16}$.
- Find m_1 and m_2 , to 4 decimal places. [4]
 - Use the spline to estimate $\ln 2.5$, to 4 decimal places. [2]
 - Find the linear interpolation estimate of $\ln 2.5$, and compare the percentage errors of the two estimates given that $\ln 2.5 = 0.9163$. [4]

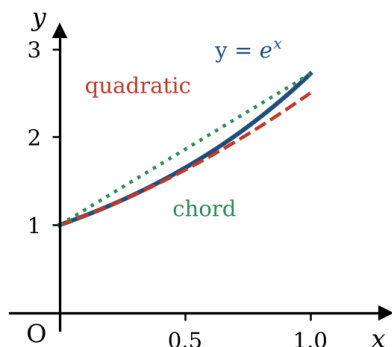
Approximating Functions

- Write down the Maclaurin polynomial of degree 2 for e^x and use it to estimate $e^{0.2}$. [2]
- Use the approximation $\sin x \approx x - \frac{x^3}{6}$ to estimate $\sin 0.1$, giving your answer to 5 significant figures. [2]
- Find the Taylor polynomial of degree 2 for $\ln x$ about $x = 1$, and use it to estimate $\ln 1.1$. [3]
- Find the linear Taylor approximation to $\sqrt{x}$ about $x = 4$ and use it to estimate $\sqrt{4.2}$. [2]
- Use the approximation $\sqrt{1+x} \approx 1 + \frac{x}{2} - \frac{x^2}{8}$ with $x = 0.04$ to estimate $\sqrt{1.04}$, giving your answer to 5 significant figures. [2]
- The approximation $\cos x \approx 1 - \frac{x^2}{2}$ is used at $x = 0.5$, where the true value of $\cos 0.5$ is 0.87758 (5 s.f.). Find the estimate and its percentage error, to 2 significant figures. [3]
- Let $f(x) = \cos x$.
 - Find the Taylor polynomial of degree 3 for f about $x = \frac{\pi}{3}$. [2]
 - Use P_3 to estimate $\cos 1$ and compare with $\cos 1 = 0.540302$ (to 6 s.f.). [3]
- The Maclaurin series for $\ln(1+x)$ is $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$ for $-1 < x \leq 1$.
 - By writing down the series for $\ln(1-x)$, show that $\ln\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \dots\right)$. [2]
 - By choosing a suitable value of x , use the first three terms of this series to estimate $\ln 3$ to 4 decimal places. [3]
- Use the standard Maclaurin series for e^{2x} and $\cos x$.
 - Find the Maclaurin series for $e^{2x} \cos x$ up to and including the term in x^3 . [4]
 - Use it to estimate $e^{0.2} \cos 0.1$ to 5 s.f. [1]
- Use the Maclaurin series for e^{-x^2} up to the term in x^4 to estimate $\int_0^{0.5} e^{-x^2} dx$.
 - Write down the first three terms of the Maclaurin series of e^{-x^2} . [2]
 - Hence estimate the integral to 5 decimal places. [3]

- 11** The equation $\cos x = x$ has a root near $x = 0.7$. Use the approximation $\cos x \approx 1 - \frac{x^2}{2}$ to estimate the root.
- (a) Solve $1 - \frac{x^2}{2} = x$ to find an estimate for the positive root, in exact form and then to 4 decimal places. [3]
- (b) Given that the root is 0.7391 to 4 d.p., find the percentage error. [2]
-
- 12** The Maclaurin polynomial of degree 3 for e^x is used to estimate $e^{0.5}$. The Lagrange form of the remainder is $R_3(x) = \frac{f^{(4)}(\xi)}{4!}x^4$ for some ξ between 0 and x . You may use $e^{0.5} < 1.7$.
- (a) Find the estimate of $e^{0.5}$. [2]
- (b) Find an upper bound for $|R_3(0.5)|$. [2]
- (c) Verify the bound using the actual value $e^{0.5} = 1.64872$ (6 s.f.). [1]
-
- 13** For small x , $\sin x \approx x$ and $\cos x \approx 1 - \frac{x^2}{2}$.
- (a) Use these approximations to show that $\frac{1 - \cos 2x}{x \sin x} \approx 2$ when x is small. [3]
- (b) Evaluate $\frac{1 - \cos 2x}{x \sin x}$ exactly (calculator) at $x = 0.1$, and find the percentage error of the approximation 2, to 2 significant figures. [2]
-
- 14** The Maclaurin polynomial of degree n for e^x is used to estimate $e = e^1$. The remainder is $R_n(1) = \frac{e^\xi}{(n+1)!}$ for some ξ in $(0, 1)$.
- (a) Show that $|R_n(1)| < \frac{3}{(n+1)!}$ and find the smallest n for which the error is guaranteed to be less than 10^{-4} . [4]
- (b) Calculate this estimate of e and the actual error, to 3 significant figures. [3]
-
- 15** Let $f(x) = \ln x$.
- (a) Find the Taylor polynomial of degree 3 for f about $x = 2$. [2]
- (b) Estimate $\ln 2.2$ to 6 decimal places. [2]
- (c) Use $f^{(4)}(x) = -\frac{6}{x^4}$ to show that the error is less than 3×10^{-5} . [3]
-
- 16** A quadratic is to interpolate a function on $[0, 2]$. The error is $\frac{f'''(\xi)}{3!}w(x)$ where $w(x) = (x - x_0)(x - x_1)(x - x_2)$. The Chebyshev polynomial $T_3(t) = 4t^3 - 3t$ satisfies $|T_3(t)| \leq 1$ on $[-1, 1]$.
- (a) The three nodes are the equally spaced values 0, 1, 2. Show that $w(x) = t(t^2 - 1)$ where $t = x - 1$, and find the greatest value of $|w|$ on $[0, 2]$. [2]
- (b) The Chebyshev nodes are the zeros of T_3 . Find them for $[-1, 1]$ and then for $t = x - 1$, and show that $w(x) = \frac{1}{4}T_3(t)$. [3]
- (c) Compare the greatest values of $|w|$ for the two choices of nodes. [2]
-
- 17** Use the Maclaurin series for $\sin x$ to show that $\frac{\sin x}{x} \approx 1 - \frac{x^2}{6} + \frac{x^4}{120}$, and hence estimate $\int_0^{0.5} \frac{\sin x}{x} dx$ to 6 decimal places.
- (a) Show the series. [2]
- (b) Estimate the integral. [3]
- (c) Explain why the error is smaller than $\frac{0.5^7}{(7 \times 5040)}$. [1]

- 18** A rational approximation $e^x \approx \frac{1+ax}{1+bx}$ is to agree with the Maclaurin series of e^x up to and including the x^2 term.
- (a) Show that $1+ax = (1+bx)(1+x+\frac{x^2}{2})$ up to x^2 , and hence find a and b . [3]
- (b) Evaluate the rational approximation at $x = 0.5$ and compare its percentage error with that of the quadratic Maclaurin polynomial. ($e^{0.5} = 1.64872$) [4]

- 19** The graph shows $y = e^x$ on $[0, 1]$ with two approximations: the Maclaurin polynomial $p(x) = 1 + x + \frac{x^2}{2}$ and the straight line $q(x)$ joining $(0, 1)$ to $(1, e)$.



- (a) Write down $q(x)$ and calculate the values of e^x , $p(x)$ and $q(x)$ at $x = 0, 0.5$ and 1 , to 4 significant figures. [3]
- (b) Hence state which approximation is better near $x = 0$, which is better near $x = 1$, and which has the smaller error in the middle of the interval. [4]
- 20** The equation $e^x + x = 3$ has a root near $x = 0.8$. Its value is 0.7921 to 4 decimal places.
- (a) Use $e^x \approx 1 + x + \frac{x^2}{2}$ to form a quadratic equation and solve it exactly for the positive root. [2]
- (b) Find the percentage error of this root, and explain why a better estimate would use more terms. [3]
- (c) With the cubic Maclaurin polynomial the equation becomes $\frac{x^3}{6} + \frac{x^2}{2} + 2x - 2 = 0$. Show that the root now lies between 0.78 and 0.8. [2]

- 21** The Maclaurin series for $\ln(1+x)$ is $x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$. A rational approximation $\ln(1+x) \approx \frac{ax}{1+bx}$ is to match the first two terms, x and x^2 , of this series.
- (a) Expand $\frac{ax}{1+bx}$ as a series in x up to x^3 and hence find a and b . [3]
- (b) Show that the approximation is $\frac{2x}{2+x}$ and compare its x^3 coefficient with the true value $\frac{1}{3}$. [3]
- (c) Evaluate the rational approximation and the two-term Maclaurin polynomial $x - \frac{x^2}{2}$ at $x = 1$, and find the percentage error of each given $\ln 2 = 0.69315$ (5 s.f.). [3]

- 22** The Maclaurin polynomial of odd degree n for $\sin x$ is used for $-1 \leq x \leq 1$. The error is no larger than the modulus of the first neglected term, that is $\frac{|x|^{n+2}}{(n+2)!}$.
- (a) Find the smallest odd n for which the error is less than 5×10^{-5} for all x in $[-1, 1]$. [4]
- (b) Write down this polynomial and use it to estimate $\sin 1$ to 6 decimal places. [3]
- (c) The true value is $\sin 1 = 0.841471$ (6 d.p.). Verify that the actual error satisfies the bound. [1]

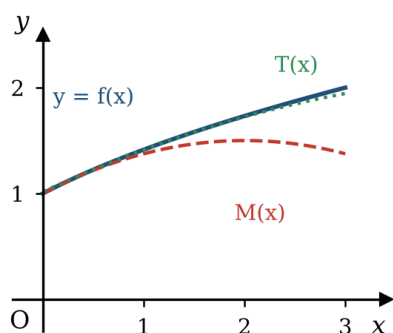
- 23** The Chebyshev polynomials satisfy $T_n(\cos \theta) = \cos n\theta$, so $|T_n(t)| \leq 1$ on $[-1, 1]$ and T_n takes the values ± 1 alternately $n + 1$ times on $[-1, 1]$. It is known that, among all polynomials of degree at most $n - 1$, the one that approximates t^n with the smallest greatest error on $[-1, 1]$ is $t^n - 2^{1-n}T_n(t)$, with greatest error 2^{1-n} .

- Show that $T_3(t) = 4t^3 - 3t$. [2]
- Hence find the quadratic (in fact linear) polynomial that best approximates t^3 on $[-1, 1]$, and its greatest error. [2]
- Show that the error $t^3 - \frac{3}{4}t$ reaches $\pm \frac{1}{4}$ at the points $t = -1, -\frac{1}{2}, \frac{1}{2}, 1$ with alternating signs. [2]
- Compare the greatest error with that of the best approximation $t^3 \approx 0$ given by the zero polynomial, and with the linear approximation $t^3 \approx t$. [3]

- 24** The series $e^{-x^2} = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24} - \dots$ is integrated term by term from 0 to 1.

- Show that $\int_0^1 e^{-x^2} dx = 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \dots$ [2]
- Use the first four terms to estimate the integral to 5 decimal places and give a bound for the error. [4]
- Find the least number of terms guaranteeing an error less than 0.001, given that the next term after $\frac{1}{216}$ is $-\frac{1}{1320}$. [3]

- 25** The function $f(x) = \sqrt{1+x}$ is to be approximated on the interval $[0, 3]$ by a quadratic polynomial. The Maclaurin polynomial is $M(x) = 1 + \frac{x}{2} - \frac{x^2}{8}$.



- Find the Taylor polynomial of degree 2 about $x = 1$, in the form $T(x) = \sqrt{2} + p(x - 1) + q(x - 1)^2$, with exact values of p and q . [3]
- Complete a table of f , M and T at $x = 0, 1.5$ and 3 (4 decimal places) and the percentage error of each approximation. [3]
- Which is the better approximation on the whole interval $[0, 3]$? Explain why. [3]

- 26** A straight line $y = a + bx$ is chosen to approximate $y = x^2$ on $[0, 1]$ by minimising the integral $I = \int_0^1 (x^2 - a - bx)^2 dx$.

- Show that a and b satisfy $a + \frac{b}{2} = \frac{1}{3}$ and $\frac{a}{2} + \frac{b}{3} = \frac{1}{4}$. [3]
- Solve these equations to find a and b . [3]
- Find the greatest value of $|x^2 - a - bx|$ on $[0, 1]$ for this line, and compare it with the greatest error of the line $y = x$ joining $(0, 0)$ and $(1, 1)$. [3]

27

A rational approximation to e^x is $R(x) = \frac{1 + \frac{x}{2} + \frac{x^2}{12}}{1 - \frac{x}{2} + \frac{x^2}{12}}$.

- (a) By multiplying out $(1 - \frac{x}{2} + \frac{x^2}{12})(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24})$, show that the product equals $1 + \frac{x}{2} + \frac{x^2}{12}$ up to and including the term in x^4 . [5]
- (b) Hence explain why $R(x)$ agrees with e^x up to the x^4 term. [2]
- (c) Evaluate $R(1)$ as an exact fraction and compare its percentage error with that of the Maclaurin polynomial of degree 4 at $x = 1$ ($e = 2.71828$). [5]

28

Two cubic approximations to $f(x) = e^x$ on $[-1, 1]$ are compared: the Maclaurin cubic, and the cubic interpolating e^x at the four Chebyshev nodes (the zeros of $T_4(x) = 8x^4 - 8x^2 + 1$). Use that $|f^{(4)}(\xi)| \leq e$ on $[-1, 1]$.

- (a) Find the Lagrange bound for the error of the Maclaurin cubic on $[-1, 1]$. [3]
- (b) Show that for the Chebyshev nodes, $w(x) = (x - x_0)(x - x_1)(x - x_2)(x - x_3) = \frac{1}{8}T_4(x)$, and deduce that the interpolation error is at most $\frac{e}{192}$. [5]
- (c) The greatest actual error of the Maclaurin cubic on $[-1, 1]$ is $e - (1 + 1 + \frac{1}{2} + \frac{1}{6}) = 0.0516$ (at $x = 1$), and that of the Chebyshev interpolant is found numerically to be 0.0067. Compare these with the bounds in (a) and (b), and say what the comparison suggests about the choice of approximation on an interval. [4]

29

The period of a simple pendulum swinging through amplitude θ_0 radians is $T = T_0(1 + \frac{\theta_0^2}{16} + \frac{11\theta_0^4}{3072} + \dots)$, where T_0 is the period given by the small angle approximation.

- (a) Find the percentage by which T exceeds T_0 when $\theta_0 = 0.2$ and when $\theta_0 = 0.5$, using the first three terms. Give your answers to 2 significant figures. [4]
- (b) Using only the first two terms, find the amplitude for which the small angle approximation underestimates the period by exactly 1% of T_0 . [2]
- (c) Using all three terms, show that this amplitude satisfies $11u^2 + 192u - 30.72 = 0$, where $u = \theta_0^2$, and solve for θ_0 to 3 significant figures. [5]

30

A function $y(x)$ satisfies $\frac{dy}{dx} = x + y^2$ with $y(0) = 1$.

- (a) Find $y'(0)$, $y''(0)$, $y'''(0)$ and $y^{(4)}(0)$. [6]
- (b) Hence write down the Maclaurin polynomial of degree 4 for y . [2]
- (c) Estimate $y(0.2)$ to 5 significant figures, and compare it with the numerical value 1.2736. [3]
- (d) Explain why the polynomial of part (b) would be unreliable for estimating $y(1)$. [1]