

Hyperbolic Sine, Cosine and Tangent

1 (3 marks)

$$\sinh x = \frac{e^x - e^{-x}}{2}, \cosh x = \frac{e^x + e^{-x}}{2} \quad \text{B1}$$

$$\sinh 0 = \frac{1 - 1}{2} = 0 \text{ and } \cosh 0 = \frac{1 + 1}{2} = 1 \quad \text{B1}$$

$$\tanh 0 = \frac{\sinh 0}{\cosh 0} = 0 \quad \text{B1}$$

Answer: $\sinh 0 = 0$, $\cosh 0 = 1$, $\tanh 0 = 0$ 3 marks

2 (2 marks)

$$\sinh 2 = \frac{e^2 - e^{-2}}{2} = 3.63 \quad \text{B1}$$

$$\cosh 2 = \frac{e^2 + e^{-2}}{2} = 3.76 \quad \text{B1}$$

Answer: $\sinh 2 = 3.63$, $\cosh 2 = 3.76$ 2 marks

3 (2 marks)

$$\sinh(\ln 3) = \frac{e^{\ln 3} - e^{-\ln 3}}{2} = \frac{3 - \frac{1}{3}}{2} \quad \text{M1}$$

$$= \frac{\frac{8}{3}}{2} = \frac{4}{3} \quad \text{A1}$$

Answer: $\sinh(\ln 3) = \frac{4}{3}$ 2 marks

4 (2 marks)

Range of $\cosh x$: $f(x) \geq 1$ B1

Range of $\tanh x$: $-1 < g(x) < 1$ B1

Answer: $f(x) \geq 1$; $-1 < g(x) < 1$ 2 marks

5 (2 marks)

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \quad \text{M1}$$

Multiply numerator and denominator by e^x : $\frac{e^{2x} - 1}{e^{2x} + 1}$ A1

Answer: $\tanh x = \frac{e^{2x} - 1}{e^{2x} + 1}$ 2 marks

6 (3 marks)

$$\operatorname{sech} 1 = \frac{1}{\cosh 1} = 0.648 \quad \text{B1}$$

$$\operatorname{cosech} 1 = \frac{1}{\sinh 1} = 0.851 \quad \text{B1}$$

$$\operatorname{coth} 1 = \frac{1}{\tanh 1} = 1.31 \quad \text{B1}$$

Answer: 0.648, 0.851, 1.31 3 marks

7 (3 marks)

$$\cosh(\ln 2) = \frac{2 + \frac{1}{2}}{2} = \frac{5}{4} \quad \text{B1}$$

$$\sinh(\ln 2) = \frac{2 - \frac{1}{2}}{2} = \frac{3}{4} \quad \text{B1}$$

$$\tanh(\ln 2) = \frac{3}{4} \div \frac{5}{4} = \frac{3}{5} \quad \text{B1}$$

Answer: $\cosh(\ln 2) = \frac{5}{4}$, $\tanh(\ln 2) = \frac{3}{5}$ 3 marks

8 (3 marks)

$$\sinh(-x) = \frac{e^{-x} - e^x}{2}$$

M1

$$= -\frac{e^x - e^{-x}}{2} = -\sinh x$$

A1

$$\cosh(-x) = \frac{e^{-x} + e^x}{2} = \cosh x$$

B1

Answer: $\sinh(-x) = -\sinh x$ and $\cosh(-x) = \cosh x$

3 marks

9 (3 marks)

$$\sinh x = 0: e^x = e^{-x}, \text{ so } e^{2x} = 1$$

M1

$$x = 0$$

A1

$$\cosh x = 1: e^x + e^{-x} = 2, \text{ so } (e^x - 1)^2 = 0 \text{ (after multiplying by } e^x), \text{ giving } x = 0$$

B1

Answer: $x = 0$ in both cases

3 marks

10 (3 marks)

$$\tanh x = \frac{1 - e^{-2x}}{1 + e^{-2x}} \rightarrow 1 \text{ as } x \rightarrow \infty, \text{ so } \coth x \rightarrow 1$$

M1 A1

Asymptotes of $y = \tanh x$: $y = 1$ and (by symmetry) $y = -1$; $y = \coth x$ has the vertical asymptote $x = 0$ because $\tanh 0 = 0$

B1

Answer: $\tanh x \rightarrow 1$, $\coth x \rightarrow 1$; $y = \pm 1$; $x = 0$

3 marks

11 (4 marks)

$$\frac{e^x - e^{-x}}{2} = \frac{3}{4}, \text{ so } e^x - e^{-x} = \frac{3}{2}$$

M1

$$\text{Multiply by } 2e^x: 2e^{2x} - 3e^x - 2 = 0$$

M1

$$(2e^x + 1)(e^x - 2) = 0, \text{ and } e^x > 0, \text{ so } e^x = 2$$

A1

$$x = \ln 2$$

A1

Answer: $x = \ln 2$

4 marks

12 (3 marks)

The graph is the cosh curve translated up by 2: U-shaped and symmetrical about the y-axis

B1

Crosses the y-axis at (0, 3)

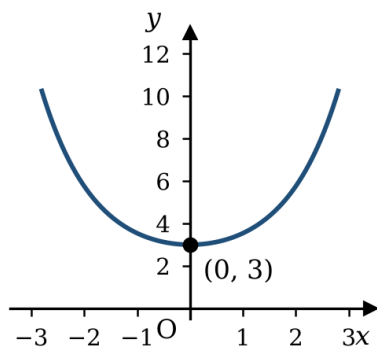
B1

Minimum point (0, 3); the curve does not meet the x-axis

B1

Answer: Minimum and y-intercept both at (0, 3)

3 marks

**13 (4 marks)**

$$\text{Lowest point at } x = 0: h = 5 \cosh 0 = 5 \text{ m}$$

B1

$$\text{At the supports } x = \pm 10: h = 5 \cosh 2$$

M1

$$\cosh 2 = \frac{e^2 + e^{-2}}{2} = 3.76$$

A1

$$= 18.8 \text{ m at each support}$$

A1

Answer: Lowest: 5 m; supports: 18.8 m

4 marks

14 (4 marks)

$$\sinh(\ln k) = \frac{k - \frac{1}{k}}{2}$$

M1

$$= \frac{k^2 - 1}{2k}$$

A1

$$\sinh(\ln 5) = \frac{24}{10} = \frac{12}{5}$$

B1

$$\cosh(\ln 5) = \frac{5 + \frac{1}{5}}{2} = \frac{13}{5}$$

B1

$$\text{Answer: } \sinh(\ln 5) = \frac{12}{5}, \cosh(\ln 5) = \frac{13}{5}$$

4 marks

15 (4 marks)

$$e^x + e^{-x} = 4, \text{ so } e^{2x} - 4e^x + 1 = 0$$

M1

$$e^x = \frac{4 \pm \sqrt{12}}{2} = 2 \pm \sqrt{3}$$

M1 A1

$$x = \ln(2 + \sqrt{3}) \text{ or } x = \ln(2 - \sqrt{3}) = -\ln(2 + \sqrt{3})$$

A1

$$\text{Answer: } x = \pm \ln(2 + \sqrt{3})$$

4 marks

16 (5 marks)

$$\cosh w = \frac{1}{\operatorname{sech} w} = \frac{13}{5}$$

B1

$$\sinh^2 w = \cosh^2 w - 1 = \frac{169}{25} - 1 = \frac{144}{25}, \text{ and } \sinh w > 0 \text{ since } w > 0, \text{ so } \sinh w = \frac{12}{5}$$

M1 A1

$$\tanh w = \frac{\sinh w}{\cosh w} = \frac{12}{13}$$

B1

$$e^w = \cosh w + \sinh w = \frac{13}{5} + \frac{12}{5} = 5, \text{ so } w = \ln 5$$

A1

$$\text{Answer: } \cosh w = \frac{13}{5}, \sinh w = \frac{12}{5}, \tanh w = \frac{12}{13}, w = \ln 5$$

5 marks

17 (4 marks)

$$(e^{x/2} - e^{-x/2})^2 = e^x - 2 + e^{-x}$$

M1

$$\text{So } \frac{(e^{x/2} - e^{-x/2})^2}{2} = \frac{e^x + e^{-x}}{2} - 1 = \cosh x - 1$$

A1

A square is non-negative, so $\cosh x - 1 \geq 0$

B1

Equality holds when $e^{x/2} = e^{-x/2}$, that is $x = 0$

B1

Answer: $\cosh x \geq 1$, equality at $x = 0$

4 marks

18 (4 marks)

$x = 0$ gives $y = 2 \times 0 - 1 = -1$, so the y-intercept is $(0, -1)$

B1

Horizontal asymptotes $y = 2(1) - 1 = 1$ and $y = 2(-1) - 1 = -3$

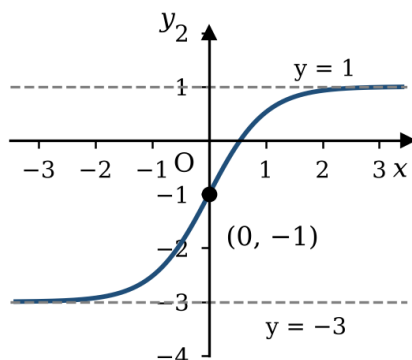
B1 B1

x-intercept where $\tanh x = \frac{1}{2}$: $x = \frac{1}{2} \ln 3 \approx 0.549$, so $(0.549, 0)$; the curve is increasing

B1

Answer: Asymptotes $y = 1$ and $y = -3$; intercepts $(0, -1)$ and $(\frac{1}{2} \ln 3, 0)$

4 marks



19 (5 marks)

$$\cosh x + \sinh x = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2}$$

M1

$$= e^x$$

A1

$$(\cosh x + \sinh x)^n = (e^x)^n = e^{nx}$$

M1 A1

$$\text{By the same result with } x \text{ replaced by } nx, e^{nx} = \cosh nx + \sinh nx$$

A1

$$\text{Answer: } \cosh x + \sinh x = e^x$$

5 marks

20 (5 marks)

$$\text{At ground level } 25 - 5 \cosh\left(\frac{x}{10}\right) = 0, \text{ so } \cosh\left(\frac{x}{10}\right) = 5$$

M1

$$\text{Let } e^{x/10} = u: u + \frac{1}{u} = 10, \text{ so } u^2 - 10u + 1 = 0$$

M1

$$u = 5 \pm \sqrt{24}; \text{ so } \frac{x}{10} = \pm \ln(5 + \sqrt{24}) \text{ (using } \ln(5 - \sqrt{24}) = -\ln(5 + \sqrt{24}))$$

A1

$$\text{Width} = 2 \times 10 \ln(5 + \sqrt{24}) = 20 \ln(5 + 2\sqrt{6})$$

A1

$$= 45.8 \text{ m}$$

A1

$$\text{Answer: } 20 \ln(5 + 2\sqrt{6}) = 45.8 \text{ m}$$

5 marks

21 (6 marks)

$$e^x + e^{-x} = \frac{5}{2}, \text{ so } 2e^{2x} - 5e^x + 2 = 0$$

M1

$$(2e^x - 1)(e^x - 2) = 0, \text{ so } e^x = 2 \text{ or } e^x = \frac{1}{2}$$

A1

$$e^x = 2: \sinh x = \frac{2 - \frac{1}{2}}{2} = \frac{3}{4}, \tanh x = \frac{3}{5}$$

M1 A1

$$e^x = \frac{1}{2}: \sinh x = -\frac{3}{4}, \tanh x = -\frac{3}{5}$$

A1

$$\text{Check: } \cosh x = \frac{2 + \frac{1}{2}}{2} = \frac{5}{4} \text{ for both values}$$

B1

$$\text{Answer: } e^x = 2 \text{ or } \frac{1}{2}; \sinh x = \pm \frac{3}{4}; \tanh x = \pm \frac{3}{5}$$

6 marks

22 (6 marks)

$$\text{(a) } \tanh x = \frac{e^{2x} - 1}{e^{2x} + 1}$$

M1

$$= \frac{(e^{2x} + 1) - 2}{e^{2x} + 1} = 1 - \frac{2}{e^{2x} + 1}$$

A1

$$\text{Answer: } \tanh x = 1 - \frac{2}{e^{2x} + 1}$$

2 marks

$$\text{(b) } e^{2x} + 1 > 1, \text{ so } 0 < \frac{2}{e^{2x} + 1} < 2$$

M1 A1

$$\text{So } -1 < \tanh x < 1$$

A1

$$\text{As } x \text{ increases, } e^{2x} + 1 \text{ increases, so } \frac{2}{e^{2x} + 1} \text{ decreases and } \tanh x \text{ increases}$$

E1

$$\text{Answer: } -1 < \tanh x < 1 \text{ for all } x, \text{ and that } \tanh x \text{ increases as } x \text{ increases}$$

4 marks

23 (6 marks)

$$\text{(a) Lowest point at } x = 0: y = 15 \cosh 0 = 15 \text{ m}$$

B1

$$\text{Answer: } 15 \text{ m}$$

1 mark

$$\text{(b) } 15 \cosh\left(\frac{x}{15}\right) = 25, \text{ so } \cosh\left(\frac{x}{15}\right) = \frac{5}{3}$$

M1

$$e^{x/15} + e^{-x/15} = \frac{10}{3}, \text{ giving } 3u^2 - 10u + 3 = 0 \text{ with } u = e^{x/15}, \text{ so } (3u - 1)(u - 3) = 0$$

M1

$$u = 3 \text{ or } \frac{1}{3}, \text{ so } \frac{x}{15} = \pm \ln 3, x = \pm 15 \ln 3$$

A1

$$\text{Answer: the poles are at } x = \pm 15 \ln 3$$

3 marks

(c) Distance = $30 \ln 3$
 $= 33.0 \text{ m (3 s.f.)}$

M1

A1

Answer: 33.0 m

2 marks

24 (6 marks)

The algebra is correct, but $e^{2x} > 0$ for all real x , so $e^{2x} = -3$ has no solution; $\ln(-3)$ is undefined

E1

So $\tanh x = 2$ has no real solution (consistent with $-1 < \tanh x < 1$)

E1

$\tanh x = k$: $\frac{e^{2x} - 1}{e^{2x} + 1} = k$, so $e^{2x} - 1 = k e^{2x} + k$

M1

$e^{2x}(1 - k) = 1 + k$, so $e^{2x} = \frac{1 + k}{1 - k}$

A1

A solution requires $\frac{1 + k}{1 - k} > 0$, which gives $-1 < k < 1$

M1

Then $x = \frac{1}{2} \ln \left(\frac{1 + k}{1 - k} \right)$

A1

Answer: No real solution; valid for $-1 < k < 1$

6 marks

25 (6 marks)

(a) $\cosh x \geq 1$, so $3 \cosh x \geq 3$

M1

$f(x) \geq -1$

A1

Answer: $f(x) \geq -1$

2 marks

(b) $3 \cosh x = 9$, so $\cosh x = 3$

B1

$e^{2x} - 6e^x + 1 = 0$

M1

$e^x = 3 \pm 2\sqrt{2}$

A1

$x = \pm \ln(3 + 2\sqrt{2})$

A1

Answer: $x = \pm \ln(3 + 2\sqrt{2})$

4 marks

26 (7 marks)

(a) $\coth(\ln k) = \frac{\cosh(\ln k)}{\sinh(\ln k)} = \frac{k + \frac{1}{k}}{k - \frac{1}{k}}$

M1 A1

Multiply numerator and denominator by k : $\frac{k^2 + 1}{k^2 - 1}$

A1

Answer: $\coth(\ln k) = \frac{k^2 + 1}{k^2 - 1}$

3 marks

(b) $\coth(\ln 3) = \frac{10}{8} = \frac{5}{4}$

B1

$\frac{k^2 + 1}{k^2 - 1} = \frac{13}{12}$, so $12k^2 + 12 = 13k^2 - 13$

M1

$k^2 = 25$

A1

$k = 5$ ($k > 1$)

A1

Answer: $\frac{5}{4}$; $k = 5$

4 marks

27 (8 marks)

$$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 \quad \text{M1}$$

$$= \frac{(e^{2x} + 2 + e^{-2x}) - (e^{2x} - 2 + e^{-2x})}{4} = 1 \quad \text{A1}$$

$$\tan \theta = \sinh x, \text{ so } \sec^2 \theta = 1 + \tanh^2 \theta = 1 + \sinh^2 x \quad \text{M1}$$

$$= \cosh^2 x \text{ (using the result above)} \quad \text{A1}$$

$$\theta \text{ is acute so } \sec \theta > 0 \text{ and } \cosh x > 0, \text{ giving } \sec \theta = \cosh x \quad \text{M1}$$

$$\text{So } \cos \theta = \frac{1}{\cosh x} \quad \text{A1}$$

$$\sin \theta = \tan \theta \cos \theta = \sinh x \times \frac{1}{\cosh x} \quad \text{M1}$$

$$= \tanh x \quad \text{A1}$$

$$\text{Answer: } \cos \theta = \frac{1}{\cosh x}, \sin \theta = \tanh x \quad \text{8 marks}$$

28 (10 marks)

$$\text{(a) } 5 \cosh x + 3 \sinh x = \frac{5(e^x + e^{-x}) + 3(e^x - e^{-x})}{2} \quad \text{M1}$$

$$= \frac{8e^x + 2e^{-x}}{2} = 4e^x + e^{-x} \quad \text{A1}$$

$$\text{Answer: } 5 \cosh x + 3 \sinh x = 4e^x + e^{-x} \quad \text{2 marks}$$

$$\text{(b) Let } u = e^x > 0: 4u + \frac{1}{u} = k, \text{ so } 4u^2 - ku + 1 = 0 \quad \text{M1 A1}$$

$$\text{Real } u \text{ requires } k^2 - 16 \geq 0, \text{ so } k \geq 4 \text{ or } k \leq -4 \quad \text{M1}$$

$$\text{The roots have product } \frac{1}{4} > 0 \text{ and sum } \frac{k}{4}; \text{ for positive roots } k > 0 \quad \text{M1}$$

$$\text{So solutions exist if and only if } k \geq 4 \quad \text{A1}$$

$$\text{Answer: } k \geq 4 \quad \text{5 marks}$$

$$\text{(c) } 4u^2 - 5u + 1 = 0, \text{ so } (4u - 1)(u - 1) = 0 \quad \text{M1}$$

$$u = 1 \text{ or } u = \frac{1}{4} \quad \text{A1}$$

$$x = 0 \text{ or } x = -\ln 4 = -2 \ln 2 \quad \text{A1}$$

$$\text{Answer: } x = 0 \text{ or } x = -2 \ln 2 \quad \text{3 marks}$$

29 (9 marks)

$$\text{Let } t = x + \sqrt{x^2 + 1}. \text{ Then } \frac{1}{t} = \frac{1}{x + \sqrt{x^2 + 1}} \quad \text{M1}$$

$$\text{Rationalise: } \frac{1}{t} = \frac{\sqrt{x^2 + 1} - x}{(x^2 + 1) - x^2} = \sqrt{x^2 + 1} - x \quad \text{M1 A1}$$

$$\sinh(\ln t) = \frac{t - \frac{1}{t}}{2} = \frac{(x + \sqrt{x^2 + 1}) - (\sqrt{x^2 + 1} - x)}{2} \quad \text{M1}$$

$$= x \quad \text{A1}$$

$$\text{Now let } s = x + \sqrt{x^2 - 1}; \text{ then } \frac{1}{s} = \frac{1}{x + \sqrt{x^2 - 1}} = x - \sqrt{x^2 - 1} \quad \text{M1 A1}$$

$$\cosh(\ln s) = \frac{s + \frac{1}{s}}{2} = \frac{(x + \sqrt{x^2 - 1}) + (x - \sqrt{x^2 - 1})}{2} \quad \text{M1}$$

$$= x \quad \text{A1}$$

$$\text{Answer: } \sinh(\ln(x + \sqrt{x^2 + 1})) = x \text{ for all real } x \quad \text{9 marks}$$

30 (8 marks)

$$\text{(a) } \tanh a = \frac{e^{2a} - 1}{e^{2a} + 1} = \frac{1}{2}, \text{ so } 2e^{2a} - 2 = e^{2a} + 1, \text{ giving } e^{2a} = 3 \quad \text{M1 A1}$$

$$\tanh b = \frac{1}{3}: 3e^{2b} - 3 = e^{2b} + 1, \text{ so } e^{2b} = 2 \quad \text{A1}$$

$$\text{Answer: } e^{2a} = 3 \text{ and } e^{2b} = 2 \quad \text{3 marks}$$

$$(b) e^{2(a+b)} = e^{2a} \times e^{2b} = 6$$

M1

$$\tanh(a+b) = \frac{e^{2(a+b)} - 1}{e^{2(a+b)} + 1}$$

M1

$$= \frac{5}{7}$$

A1

$$\text{Answer: } \frac{5}{7}$$

3 marks

$$(c) \text{ From part (a), } e^{2a} = 3 \text{ and } e^{2b} = 2, \text{ so } e^{2(a+b)} = 6$$

M1

$$2(a+b) = \ln 6, \text{ so } a+b = \frac{1}{2} \ln 6$$

A1

$$\text{Answer: } a+b = \frac{1}{2} \ln 6$$

2 marks

Hyperbolic Identities

1 (2 marks)

$$\cosh^2 x = 1 + \sinh^2 x = 1 + \frac{9}{16} = \frac{25}{16}$$

M1

$$\cosh x > 0, \text{ so } \cosh x = \frac{5}{4}$$

A1

$$\text{Answer: } \frac{5}{4}$$

2 marks

2 (2 marks)

$$\frac{\cosh^2 x}{\cosh^2 x} - \frac{\sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}$$

M1

$$1 - \tanh^2 x = \operatorname{sech}^2 x$$

A1

$$\text{Answer: } 1 - \tanh^2 x = \operatorname{sech}^2 x$$

2 marks

3 (2 marks)

$$(a) \sinh 2A = 2 \sinh A \cosh A \text{ with } A = 3x \text{ gives } \sinh 6x$$

B1

$$\text{Answer: } \sinh 6x$$

1 mark

$$(b) \cosh 2A = \cosh^2 A + \sinh^2 A \text{ with } A = 4x \text{ gives } \cosh 8x$$

B1

$$\text{Answer: } \cosh 8x$$

1 mark

4 (2 marks)

$$\operatorname{sech}^2 x = 1 - \tanh^2 x = 1 - \frac{9}{25} = \frac{16}{25}$$

M1

$$\cosh^2 x = \frac{25}{16}, \text{ so } \cosh x = \frac{5}{4}$$

A1

$$\text{Answer: } \frac{5}{4}$$

2 marks

5 (2 marks)

$$\cosh 2x = 2 \times 3^2 - 1$$

M1

$$= 17$$

A1

$$\text{Answer: } 17$$

2 marks

6 (3 marks)

$$\text{LHS} = \sinh(\ln 6) = \frac{6 - \frac{1}{6}}{2} = \frac{35}{12}$$

B1

$$\sinh(\ln 2) = \frac{3}{4}, \cosh(\ln 2) = \frac{5}{4}, \sinh(\ln 3) = \frac{4}{3}, \cosh(\ln 3) = \frac{5}{3}$$

B1

$$\text{RHS} = \frac{3}{4} \times \frac{5}{3} + \frac{5}{4} \times \frac{4}{3} = \frac{5}{4} + \frac{5}{3} = \frac{35}{12}$$

B1

$$\text{Answer: Both sides equal } \frac{35}{12}$$

3 marks

7 (3 marks)

$$\cosh^2 x = \frac{e^{2x} + 2 + e^{-2x}}{4} \text{ and } \sinh^2 x = \frac{e^{2x} - 2 + e^{-2x}}{4}$$

M1 A1

$$\text{Subtracting: } \frac{4}{4} = 1$$

A1

$$\text{Answer: } \cosh^2 x - \sinh^2 x = 1$$

3 marks

8 (3 marks)

$$2 \sinh x \cosh x = 2 \times \frac{e^x - e^{-x}}{2} \times \frac{e^x + e^{-x}}{2}$$

M1

$$= \frac{(e^x - e^{-x})(e^x + e^{-x})}{2}$$

A1

$$= \frac{e^{2x} - e^{-2x}}{2} = \sinh 2x$$

A1

$$\text{Answer: } \sinh 2x = 2 \sinh x \cosh x$$

3 marks

9 (4 marks)

$$1 - \tanh x = \frac{\cosh x - \sinh x}{\cosh x}$$

M1

$$\cosh x - \sinh x = \frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2} = e^{-x}$$

M1 A1

$$\text{So } -\ln(1 - \tanh x) = -\ln\left(\frac{e^{-x}}{\cosh x}\right) = -(-x - \ln(\cosh x)) = x + \ln(\cosh x)$$

A1

$$\text{Answer: } -\ln(1 - \tanh x) = x + \ln(\cosh x)$$

4 marks

10 (5 marks)

$$\text{Let } u = e^x: 5 \times \frac{u - \frac{1}{u}}{2} - \frac{u + \frac{1}{u}}{2} = 1, \text{ so } 2u - \frac{3}{u} = 1$$

M1 A1

$$2u^2 - u - 3 = 0, \text{ so } (2u - 3)(u + 1) = 0$$

M1

$$u = e^x > 0, \text{ so } e^x = \frac{3}{2}$$

A1

$$\tanh x = \frac{e^{2x} - 1}{e^{2x} + 1} = \frac{\frac{9}{4} - 1}{\frac{9}{4} + 1} = \frac{5}{13}$$

A1

$$\text{Answer: } \tanh x = \frac{5}{13}$$

5 marks

11 (4 marks)

$$\tanh 2x = \frac{\sinh 2x}{\cosh 2x} = \frac{2 \sinh x \cosh x}{\cosh^2 x + \sinh^2 x}$$

M1 A1

$$\text{Divide numerator and denominator by } \cosh^2 x$$

M1

$$= \frac{2 \tanh x}{1 + \tanh^2 x}$$

A1

$$\text{Answer: } \tanh 2x = \frac{2 \tanh x}{1 + \tanh^2 x}$$

4 marks

12 (4 marks)

$$(\cosh x + \sinh x)(\cosh x - \sinh x) = \cosh^2 x - \sinh^2 x = 1$$

M1

$$\text{So } \cosh x - \sinh x = \frac{1}{3}$$

A1

$$\text{Adding: } 2 \cosh x = 3 + \frac{1}{3}, \text{ so } \cosh x = \frac{5}{3}$$

M1

$$\text{Subtracting: } 2 \sinh x = 3 - \frac{1}{3}, \text{ so } \sinh x = \frac{4}{3}$$

A1

$$\text{Answer: } \cosh x = \frac{5}{3}, \sinh x = \frac{4}{3}$$

4 marks

13 (4 marks)

$$2\cosh^2 x - 1 = 3, \text{ so } \cosh^2 x = 2 \quad \text{M1}$$

$$\cosh x = \sqrt{2} \quad (\cosh x \geq 1) \quad \text{A1}$$

$$e^{2x} - 2\sqrt{2}e^x + 1 = 0, \text{ so } e^x = \sqrt{2} \pm 1 \quad \text{M1}$$

$$x = \pm \ln(\sqrt{2} + 1) \quad \text{A1}$$

$$\text{Answer: } x = \pm \ln(1 + \sqrt{2})$$

4 marks

14 (4 marks)

$$\sinh A \cosh B + \cosh A \sinh B = \frac{(e^A - e^{-A})(e^B + e^{-B}) + (e^A + e^{-A})(e^B - e^{-B})}{4} \quad \text{M1 A1}$$

$$\text{Numerator} = e^{A+B} + e^{A-B} - e^{-A+B} - e^{-A-B} + e^{A+B} - e^{A-B} + e^{-A+B} - e^{-A-B} \quad \text{M1}$$

$$= 2e^{A+B} - 2e^{-(A+B)}, \text{ so the expression} = \frac{e^{A+B} - e^{-(A+B)}}{2} = \sinh(A + B) \quad \text{A1}$$

$$\text{Answer: } \sinh(A + B) = \sinh A \cosh B + \cosh A \sinh B$$

4 marks

15 (7 marks)

$$\text{(a) } \cosh A \cosh B + \sinh A \sinh B = \frac{(e^A + e^{-A})(e^B + e^{-B}) + (e^A - e^{-A})(e^B - e^{-B})}{4} \quad \text{M1}$$

$$\text{Numerator} = 2e^{A+B} + 2e^{-A-B} \text{ after cancelling} \quad \text{A1}$$

$$\text{So the expression} = \frac{e^{A+B} + e^{-(A+B)}}{2} = \cosh(A + B) \quad \text{A1}$$

$$\text{Answer: } \cosh(A + B) = \cosh A \cosh B + \sinh A \sinh B \text{ using the definitions in terms of exponentials} \quad 3 \text{ marks}$$

$$\text{(b) } \cosh(x + \ln 2) = \cosh x \cosh(\ln 2) + \sinh x \sinh(\ln 2) = \frac{5}{4} \cosh x + \frac{3}{4} \sinh x \quad \text{M1 A1}$$

$$\frac{5}{4} \cosh x + \frac{3}{4} \sinh x = 3 \sinh x, \text{ so } 5 \cosh x = 9 \sinh x \quad \text{M1}$$

$$\tanh x = \frac{5}{9}, \text{ so } e^{2x} = \frac{1+5}{9/1-5/9} = \frac{7}{2}, x = \frac{1}{2} \ln\left(\frac{7}{2}\right) \quad \text{A1}$$

$$\text{Answer: } x = \frac{1}{2} \ln\left(\frac{7}{2}\right)$$

4 marks

16 (6 marks)

$$\cosh 2x = 2\cosh^2 x - 1, \text{ so } 2\cosh^2 x + 2 = 5 \cosh x \quad \text{M1}$$

$$2\cosh^2 x - 5 \cosh x + 2 = 0, \text{ so } (2 \cosh x - 1)(\cosh x - 2) = 0 \quad \text{M1 A1}$$

$$\cosh x = \frac{1}{2} \text{ is impossible since } \cosh x \geq 1 \quad \text{E1}$$

$$\cosh x = 2: e^x = 2 \pm \sqrt{3} \quad \text{M1}$$

$$x = \pm \ln(2 + \sqrt{3}) \quad \text{A1}$$

$$\text{Answer: } x = \pm \ln(2 + \sqrt{3})$$

6 marks

17 (5 marks)

$$\text{sech}^2 x = 1 - \tanh^2 x, \text{ so } 2 - 2 \tanh^2 x + 3 \tanh x = 3 \quad \text{M1}$$

$$2 \tanh^2 x - 3 \tanh x + 1 = 0, \text{ so } (2 \tanh x - 1)(\tanh x - 1) = 0 \quad \text{M1 A1}$$

$$\tanh x < 1 \text{ for all } x, \text{ so } \tanh x = \frac{1}{2} \quad \text{E1}$$

$$x = \frac{1}{2} \ln 3 \quad \text{A1}$$

$$\text{Answer: } x = \frac{1}{2} \ln 3$$

5 marks

18 (5 marks)

$$\sinh 3x = \sinh(2x + x) = \sinh 2x \cosh x + \cosh 2x \sinh x \quad \text{M1}$$

$$= 2 \sinh x \cosh^2 x + (1 + 2 \sinh^2 x) \sinh x \quad \text{M1 A1}$$

$$\cosh^2 x = 1 + \sinh^2 x, \text{ so } = 2 \sinh x + 2 \sinh^3 x + \sinh x + 2 \sinh^3 x \quad \text{M1}$$

$$= 3 \sinh x + 4 \sinh^3 x \quad \text{A1}$$

$$\text{Answer: } \sinh 3x = 3 \sinh x + 4 \sinh^3 x$$

5 marks

19 (4 marks)

$$\text{Let } X = P + Q \text{ and } Y = P - Q, \text{ so } P = \frac{X+Y}{2} \text{ and } Q = \frac{X-Y}{2}$$

M1

$$\cosh(P+Q) + \cosh(P-Q) = (\cosh P \cosh Q + \sinh P \sinh Q) + (\cosh P \cosh Q - \sinh P \sinh Q) = 2 \cosh P \cosh Q$$

M1 A1

$$\text{With } X = 5x \text{ and } Y = x: \cosh 5x + \cosh x = 2 \cosh 3x \cosh 2x$$

A1

$$\text{Answer: } 2 \cosh 3x \cosh 2x$$

4 marks

20 (5 marks)

$$(a) \cosh 2t = 1 + 2 \sinh^2 t = 1 + 2\left(\frac{x}{2}\right)^2$$

M1

$$y = 1 + \frac{x^2}{2}$$

A1

$$\text{Answer: } y = 1 + \frac{x^2}{2}$$

2 marks

$$(b) \cosh t = \frac{x}{3} \text{ and } \sinh t = \frac{y}{2}$$

B1

$$\text{Use } \cosh^2 t - \sinh^2 t = 1: \frac{x^2}{9} - \frac{y^2}{4} = 1$$

M1 A1

$$\text{Answer: } \frac{x^2}{9} - \frac{y^2}{4} = 1$$

3 marks

21 (8 marks)

$$(a) \cosh 3x = \cosh(2x + x) = \cosh 2x \cosh x + \sinh 2x \sinh x$$

M1

$$= (2\cosh^2 x - 1) \cosh x + 2 \sinh^2 x \cosh x$$

M1 A1

$$= 2\cosh^3 x - \cosh x + 2(\cosh^2 x - 1)\cosh x = 4\cosh^3 x - 3 \cosh x$$

A1

$$\text{Answer: } \cosh 3x = 4\cosh^3 x - 3 \cosh x$$

4 marks

$$(b) 4\cosh^3 x - 3 \cosh x = 13 \cosh x, \text{ so } 4 \cosh x (\cosh^2 x - 4) = 0$$

M1 A1

$$\cosh x \geq 1, \text{ so } \cosh^2 x = 4 \text{ and } \cosh x = 2$$

A1

$$x = \pm \ln(2 + \sqrt{3})$$

A1

$$\text{Answer: } x = \pm \ln(2 + \sqrt{3})$$

4 marks

22 (6 marks)

$$\cosh 2x = 2\cosh^2 x - 1, \text{ so } \cosh^2 x = \frac{1}{2}(\cosh 2x + 1)$$

M1 A1

$$\cosh^4 x = \frac{1}{4}(\cosh 2x + 1)^2 = \frac{1}{4}(\cosh^2 2x + 2 \cosh 2x + 1)$$

M1

$$\cosh^2 2x = \frac{1}{2}(\cosh 4x + 1)$$

M1

$$\cosh^4 x = \frac{1}{4}\left[\frac{1}{2} \cosh 4x + \frac{1}{2} + 2 \cosh 2x + 1\right]$$

A1

$$= \frac{1}{8} \cosh 4x + \frac{1}{2} \cosh 2x + \frac{3}{8}, \text{ so } a = \frac{1}{8}, b = \frac{1}{2}, c = \frac{3}{8}$$

A1

$$\text{Answer: } a = \frac{1}{8}, b = \frac{1}{2}, c = \frac{3}{8}$$

6 marks

23 (9 marks)

$$(a) \sinh x = 2 \sinh\left(\frac{x}{2}\right) \cosh\left(\frac{x}{2}\right) = 2 \tanh\left(\frac{x}{2}\right) \cosh^2\left(\frac{x}{2}\right)$$

M1

$$\cosh^2\left(\frac{x}{2}\right) = \frac{1}{1 - t^2} \text{ since } 1 - \tanh^2 = \text{sech}^2$$

M1

$$\sinh x = \frac{2t}{1 - t^2}$$

A1

$$\cosh x = \cosh^2\left(\frac{x}{2}\right) + \sinh^2\left(\frac{x}{2}\right) = \cosh^2\left(\frac{x}{2}\right)(1 + t^2) = \frac{1 + t^2}{1 - t^2}$$

A1

$$\text{Answer: } \sinh x = \frac{2t}{1 - t^2} \text{ and } \cosh x = \frac{1 + t^2}{1 - t^2}$$

4 marks

$$(b) 5(1 + t^2) - 8t = 3(1 - t^2)$$

M1 A1

$$8t^2 - 8t + 2 = 0, \text{ so } (2t - 1)^2 = 0, t = \frac{1}{2}$$

M1 A1

$$\tanh\left(\frac{x}{2}\right) = \frac{1}{2} \text{ gives } \frac{x}{2} = \frac{1}{2} \ln 3, \text{ so } x = \ln 3$$

A1

Answer: $x = \ln 3$

5 marks

24 (6 marks)At $x = 0$: $\cosh 0 = 1$ but $2\cosh^2 0 + 1 = 3$, so the identity is wrong

B1

The correct identity is $\cosh 2x = 2\cosh^2 x - 1$

E1

 $2\cosh^2 x - 1 = 7$, so $\cosh^2 x = 4$ and $\cosh x = 2$

M1 A1

$$e^x = 2 \pm \sqrt{3}$$

M1

$$x = \pm \ln(2 + \sqrt{3})$$

A1

Answer: Correct identity $\cosh 2x = 2\cosh^2 x - 1$; $x = \pm \ln(2 + \sqrt{3})$

6 marks

25 (7 marks)

$$\sinh 3x + \sinh x = \sinh(2x + x) + \sinh(2x - x)$$

M1

$$= (\sinh 2x \cosh x + \cosh 2x \sinh x) + (\sinh 2x \cosh x - \cosh 2x \sinh x)$$

M1 A1

$$= 2 \sinh 2x \cosh x$$

A1

$$2 \sinh 2x \cosh x = 4 \sinh 2x, \text{ so } \sinh 2x (\cosh x - 2) = 0$$

M1

$$\sinh 2x = 0 \text{ gives } x = 0$$

A1

$$\cosh x = 2 \text{ gives } x = \pm \ln(2 + \sqrt{3})$$

A1

Answer: $x = 0, \pm \ln(2 + \sqrt{3})$

7 marks

26 (7 marks)

$$\tanh 3x = \frac{\sinh 3x}{\cosh 3x} = \frac{3 \sinh x + 4 \sinh^3 x}{4 \cosh^3 x - 3 \cosh x}$$

M1 A1

Divide numerator and denominator by $\cosh^3 x$

M1

$$\text{Numerator: } 3 \tanh x \operatorname{sech}^2 x + 4 \tanh^3 x$$

M1

$$\text{Denominator: } 4 - 3 \operatorname{sech}^2 x$$

M1

$$\text{With } \operatorname{sech}^2 x = 1 - \tanh^2 x: \text{ numerator} = 3 \tanh x - 3 \tanh^3 x + 4 \tanh^3 x = 3 \tanh x + \tanh^3 x$$

A1 A1

$$\text{and denominator} = 1 + 3 \tanh^2 x$$

$$\text{Answer: } \tanh 3x = \frac{3 \tanh x + \tanh^3 x}{1 + 3 \tanh^2 x}$$

7 marks

27 (8 marks)Use $\cosh 5x + \cosh x = 2 \cosh 3x \cosh 2x$ (sum to product)

M1 A1

$$\text{Let } c = \cosh x. \cosh 3x = 4c^3 - 3c \text{ and } \cosh 2x = 2c^2 - 1$$

B1

$$\cosh 5x = 2(4c^3 - 3c)(2c^2 - 1) - c$$

M1

$$(4c^3 - 3c)(2c^2 - 1) = 8c^5 - 4c^3 - 6c^3 + 3c = 8c^5 - 10c^3 + 3c$$

M1 A1

$$\cosh 5x = 16c^5 - 20c^3 + 6c - c$$

M1

$$= 16c^5 - 20c^3 + 5c$$

A1

Answer: $\cosh 5x = 16\cosh^5 x - 20\cosh^3 x + 5 \cosh x$

8 marks

28 (9 marks)**(a)** $\cosh(A + B) - \cosh(A - B) = 2 \sinh A \sinh B$ using the addition formulae

M1 A1

With $A = rx$ and $B = \frac{x}{2}$, this gives the result

A1

$$\text{Answer: } 2 \sinh\left(\frac{x}{2}\right) \sinh(rx) = \cosh\left(\left(r + \frac{1}{2}\right)x\right) - \cosh\left(\left(r - \frac{1}{2}\right)x\right)$$

3 marks

(b) Sum the result of (a) for $r = 1$ to n : the sum telescopes

M1

$$2 \sinh\left(\frac{x}{2}\right) \sum_{r=1}^n \sinh(rx) = \cosh\left(\left(n + \frac{1}{2}\right)x\right) - \cosh\left(\frac{x}{2}\right)$$

A1

$$\cosh P - \cosh Q = 2 \sinh\left(\frac{P+Q}{2}\right) \sinh\left(\frac{P-Q}{2}\right)$$

M1

With $P = (n + \frac{1}{2})x$ and $Q = \frac{x}{2}$: $\frac{P+Q}{2} = \frac{(n+1)x}{2}$ and $\frac{P-Q}{2} = \frac{nx}{2}$

M1 A1

Divide by $2 \sinh(\frac{x}{2})$ to obtain the result

A1

Answer: $\sum_{r=1}^n \sinh(rx) = \frac{\sinh(\frac{nx}{2}) \sinh(\frac{(n+1)x}{2})}{\sinh(\frac{x}{2})}$

6 marks

29 (8 marks)

Treat as simultaneous equations in $\cosh t$ and $\sinh t$

M1

Subtract: $2y - x = (2 \cosh t + 4 \sinh t) - (2 \cosh t + 3 \sinh t)$

M1

So $\sinh t = 2y - x$

A1

$\cosh t = y - 2 \sinh t = y - 2(2y - x) = 2x - 3y$

M1 A1

$\cosh^2 t - \sinh^2 t = 1$: $(2x - 3y)^2 - (2y - x)^2 = 1$

M1

$4x^2 - 12xy + 9y^2 - 4y^2 + 4xy - x^2 = 1$, that is $3x^2 - 8xy + 5y^2 = 1$

A1

$3x^2 - 8xy + 5y^2 = (3x - 5y)(x - y)$, checked by expanding: $3x^2 - 3xy - 5xy + 5y^2$

A1

Answer: $3x^2 - 8xy + 5y^2 = 1 = (3x - 5y)(x - y)$

8 marks

30 (9 marks)

$\sinh 5x + \sinh x = 2 \sinh 3x \cosh 2x$ (sum to product, from $\sinh(3x + 2x) + \sinh(3x - 2x)$)

M1 A1

So the sum $= \sinh 3x + 2 \sinh 3x \cosh 2x = \sinh 3x (1 + 2 \cosh 2x)$

A1

Equation: $\sinh 3x (1 + 2 \cosh 2x) = 4 \sinh 3x$, so $\sinh 3x (2 \cosh 2x - 3) = 0$

M1

$\sinh 3x = 0$ gives $x = 0$

A1

$\cosh 2x = \frac{3}{2}$: $e^{2x} = \frac{3 \pm \sqrt{5}}{2}$

M1 A1

Note $\frac{3 + \sqrt{5}}{2} = (\frac{1 + \sqrt{5}}{2})^2$

M1

$2x = \pm \ln(\frac{3 + \sqrt{5}}{2}) = \pm 2 \ln(\frac{1 + \sqrt{5}}{2})$, so $x = \pm \ln(\frac{1 + \sqrt{5}}{2})$

A1

Answer: $x = 0, \pm \ln(\frac{1 + \sqrt{5}}{2})$

9 marks

Inverse Hyperbolic Functions

1 (3 marks)

$\operatorname{arsinh} 2 = 1.44$

B1

$\operatorname{arcosh} 3 = 1.76$

B1

$\operatorname{artanh} \frac{1}{2} = 0.549$

B1

Answer: 1.44, 1.76, 0.549

3 marks

2 (3 marks)

$\operatorname{arsinh} x$: all real x

B1

$\operatorname{arcosh} x$: $x \geq 1$

B1

$\operatorname{artanh} x$: $-1 < x < 1$

B1

Answer: all real numbers; $x \geq 1$; $-1 < x < 1$

3 marks

3 (3 marks)

$\sinh 0 = 0$, so $\operatorname{arsinh} 0 = 0$

B1

$\cosh 0 = 1$, so $\operatorname{arcosh} 1 = 0$

B1

$\tanh 0 = 0$, so $\operatorname{artanh} 0 = 0$

B1

Answer: 0, 0, 0

3 marks

4 (3 marks)

$$\sinh y = \frac{4}{3}, \text{ so } \cosh^2 y = 1 + \frac{16}{9} = \frac{25}{9}$$

M1

$$\cosh y > 0, \text{ so } \cosh y = \frac{5}{3}$$

A1

$$\tanh y = \frac{\sinh y}{\cosh y} = \frac{4}{5}$$

B1

$$\text{Answer: } \cosh y = \frac{5}{3}, \tanh y = \frac{4}{5}$$

3 marks

5 (2 marks)

$$\operatorname{artanh}\left(\frac{3}{5}\right) = \frac{1}{2} \ln\left(\frac{1 + \frac{3}{5}}{1 - \frac{3}{5}}\right) = \frac{1}{2} \ln 4$$

M1

$$= \ln 2$$

A1

$$\text{Answer: } \operatorname{artanh}\left(\frac{3}{5}\right) = \ln 2$$

2 marks

6 (2 marks)

$$\sqrt{\frac{9}{16} + 1} = \sqrt{\frac{25}{16}} = \frac{5}{4}$$

M1

$$\operatorname{arsinh}\left(\frac{3}{4}\right) = \ln\left(\frac{3}{4} + \frac{5}{4}\right) = \ln 2$$

A1

$$\text{Answer: } \operatorname{arsinh}\left(\frac{3}{4}\right) = \ln 2$$

2 marks

7 (4 marks)

$$x = \sinh y = \frac{e^y - e^{-y}}{2}, \text{ so } e^{2y} - 2xe^y - 1 = 0$$

M1 A1

$$e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2} = x \pm \sqrt{x^2 + 1}$$

M1

$$e^y > 0 \text{ and } \sqrt{x^2 + 1} > x, \text{ so } e^y = x + \sqrt{x^2 + 1}, \text{ giving } y = \ln(x + \sqrt{x^2 + 1})$$

A1

$$\text{Answer: } \operatorname{arsinh} x = \ln(x + \sqrt{x^2 + 1})$$

4 marks

8 (4 marks)

$$x = \cosh y = \frac{e^y + e^{-y}}{2}, \text{ so } e^{2y} - 2xe^y + 1 = 0$$

M1 A1

$$e^y = x \pm \sqrt{x^2 - 1}$$

M1

$$y \geq 0 \text{ means } e^y \geq 1, \text{ so the larger root is taken: } e^y = x + \sqrt{x^2 - 1}, \text{ giving } y = \ln(x + \sqrt{x^2 - 1})$$

A1

$$\text{Answer: } \operatorname{arcosh} x = \ln(x + \sqrt{x^2 - 1})$$

4 marks

9 (4 marks)

$$x = \frac{e^{2y} - 1}{e^{2y} + 1}$$

M1

$$x e^{2y} + x = e^{2y} - 1, \text{ so } e^{2y}(1 - x) = 1 + x$$

M1 A1

$$e^{2y} = \frac{1+x}{1-x}, \text{ so } y = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$$

A1

$$\text{Answer: } \operatorname{artanh} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$$

4 marks

10 (3 marks)

$$x = \sinh(\ln 3)$$

M1

$$= \frac{3 - \frac{1}{3}}{2}$$

A1

$$= \frac{4}{3}$$

A1

$$\text{Answer: } x = \frac{4}{3}$$

3 marks

11 (3 marks)

Let $y = \operatorname{arsinh} 3$, so $\sinh y = 3$ and $\cosh^2 y = 1 + 9 = 10$, $\cosh y = \sqrt{10}$
 $\sinh 2y = 2 \sinh y \cosh y = 2 \times 3 \times \sqrt{10} = 6\sqrt{10}$

M1 A1

A1

3 marks

Answer: $6\sqrt{10}$ **12 (3 marks)**

$$x = \tanh(\ln 5) = \frac{5 - \frac{1}{5}}{5 + \frac{1}{5}}$$

M1 A1

$$= \frac{24}{26} = \frac{12}{13}$$

A1

Answer: $x = \frac{12}{13}$

3 marks

13 (3 marks)

$$\operatorname{arcosh}\left(\frac{13}{5}\right) = \ln\left(\frac{13}{5} + \sqrt{\frac{169}{25} - 1}\right)$$

M1

$$\sqrt{\frac{144}{25}} = \frac{12}{5}$$

A1

$$\ln\left(\frac{13}{5} + \frac{12}{5}\right) = \ln 5$$

A1

Answer: $\operatorname{arcosh}\left(\frac{13}{5}\right) = \ln 5$

3 marks

14 (4 marks)

$$\operatorname{arsinh}\left(\frac{3}{4}\right) = \ln 2$$

B1

$$\operatorname{arsinh} x = 2 \ln 2 = \ln 4$$

M1

$$x = \sinh(\ln 4) = \frac{4 - \frac{1}{4}}{2}$$

M1

$$= \frac{15}{8}$$

A1

Answer: $x = \frac{15}{8}$

4 marks

15 (4 marks)Domain $x \geq 1$

B1

Range $y \geq 0$

B1

Intercept (1, 0) only, with a vertical tangent there

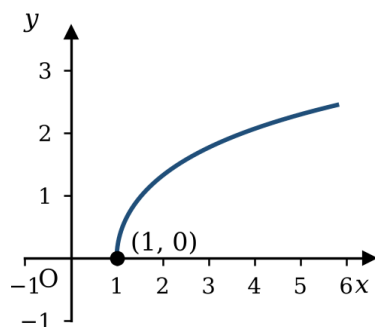
B1

Shape: the reflection of the right-hand branch of $y = \cosh x$ in the line $y = x$, increasing and concave

B1

Answer: Domain $x \geq 1$, range $y \geq 0$, intercept (1, 0)

4 marks

**16 (5 marks)**(a) Domain $\mathbb{R}$, range $\mathbb{R}$

B1

Answer: Domain all real numbers, range all real numbers

1 mark

(b) Domain $x \geq 1$

B1

Range $g(x) \geq 1$

B1

Answer: $x \geq 1$; $g(x) \geq 1$

2 marks

(c) $-1 < \frac{x}{2} < 1$, so domain $-2 < x < 2$

B1

Range $\mathbb{R}$

B1

Answer: $-2 < x < 2$; range all real numbers

2 marks

17 (4 marks)

$$\operatorname{artanh} y = \frac{1}{2} \ln \left(\frac{1+y}{1-y} \right) \text{ with } y = \frac{x^2-1}{x^2+1}$$

M1

$$1+y = \frac{2x^2}{x^2+1} \text{ and } 1-y = \frac{2}{x^2+1}$$

M1 A1

$$\frac{1}{2} \ln(x^2) = \ln x \quad (x > 0)$$

A1

Answer: for $x > 0$, $\operatorname{artanh} \left(\frac{x^2-1}{x^2+1} \right) = \ln x$

4 marks

18 (4 marks) $2x - 3 \geq 1$, so the domain is $x \geq 2$

B1

$$2x - 3 = \cosh(\ln(2 + \sqrt{3}))$$

M1

$$\cosh(\ln(2 + \sqrt{3})) = \frac{2 + \sqrt{3} + 2 - \sqrt{3}}{2} = 2$$

A1

$$2x - 3 = 2, \text{ so } x = \frac{5}{2}$$

A1

Answer: Domain $x \geq 2$; $x = \frac{5}{2}$

4 marks

19 (5 marks)Let $a = \operatorname{artanh} x$ and $b = \operatorname{artanh} y$, so $\tanh a = x$ and $\tanh b = y$

B1

$$\tanh(a+b) = \frac{\tanh a + \tanh b}{1 + \tanh a \tanh b} \quad (\text{addition formula})$$

M1 A1

$$= \frac{x+y}{1+xy}$$

A1

So $a+b = \operatorname{artanh} \left(\frac{x+y}{1+xy} \right)$ (the argument lies in $(-1, 1)$ since $|\tanh(a+b)| < 1$)

A1

Answer: for $-1 < x < 1$ and $-1 < y < 1$, $\operatorname{artanh} x + \operatorname{artanh} y = \operatorname{artanh} \left(\frac{x+y}{1+xy} \right)$

5 marks

20 (4 marks)

$$(a) \tanh \left(\frac{1}{2} \ln k \right) = \frac{e^{\ln k} - 1}{e^{\ln k} + 1}$$

M1

$$= \frac{k-1}{k+1}$$

A1

Answer: $\tanh \left(\frac{1}{2} \ln k \right) = \frac{k-1}{k+1}$ for $k > 0$

2 marks

$$(b) x = \tanh \left(\frac{1}{2} \ln 7 \right)$$

M1

$$= \frac{7-1}{7+1} = \frac{3}{4}$$

A1

Answer: $x = \frac{3}{4}$

2 marks

21 (6 marks)

$$y = 2 \sinh 3x - 1, \text{ so } \sinh 3x = \frac{y+1}{2}$$

M1

$$3x = \operatorname{arsinh} \left(\frac{y+1}{2} \right)$$

M1

$$\operatorname{arsinh} \left(\frac{y+1}{2} \right) = \ln \left(\frac{y+1}{2} + \sqrt{\frac{(y+1)^2}{4} + 1} \right) = \ln \left(\frac{y+1 + \sqrt{(y+1)^2 + 4}}{2} \right)$$

M1 A1

$$f^{-1}(x) = \frac{1}{3} \ln \left(\frac{x+1 + \sqrt{x^2 + 2x + 5}}{2} \right)$$

A1

Domain $\mathbb{R}$ and range $\mathbb{R}$

B1

Answer: $f^{-1}(x) = \frac{1}{3} \ln \left(\frac{x+1 + \sqrt{x^2 + 2x + 5}}{2} \right)$; domain and range all real numbers

6 marks

22 (7 marks)

(a) $\operatorname{arsinh} x = \ln (x + \sqrt{x^2 + 1})$ with $x = \tan \theta$

M1

$\sqrt{\tan^2 \theta + 1} = \sqrt{\sec^2 \theta} = \sec \theta$, since $\sec \theta > 0$ on this interval

M1 A1

Answer: $\operatorname{arsinh} (\tan \theta) = \ln (\sec \theta + \tan \theta)$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

3 marks

(b) $\sec \theta + \tan \theta = 3$

M1

$\sec \theta = 3 - \tan \theta$; square: $1 + \tan^2 \theta = 9 - 6 \tan \theta + \tan^2 \theta$

M1

$\tan \theta = \frac{4}{3}$

A1

$\theta = 0.927$

A1

Answer: $\theta = 0.927$

4 marks

23 (6 marks)

Let $y = \frac{2x}{1+x^2}$. Then $1+y = \frac{(1+x)^2}{1+x^2}$

M1 A1

$1-y = \frac{(1-x)^2}{1+x^2}$

A1

$\frac{1+y}{1-y} = \left(\frac{1+x}{1-x} \right)^2$

M1

$\operatorname{artanh} y = \frac{1}{2} \ln \left(\frac{1+y}{1-y} \right) = \frac{1}{2} \times 2 \ln \left(\frac{1+x}{1-x} \right)$

M1

$= 2 \times \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right) = 2 \operatorname{artanh} x$

A1

Answer: for $-1 < x < 1$, $\operatorname{artanh} \left(\frac{2x}{1+x^2} \right) = 2 \operatorname{artanh} x$

6 marks

24 (6 marks)

$x = \cosh y = \cosh \left(2 \times \frac{y}{2} \right)$

B1

$\cosh 2A = 1 + 2 \sinh^2 A$ with $A = \frac{y}{2}$: $x = 1 + 2 \sinh^2 \left(\frac{y}{2} \right)$

M1 A1

$\sinh^2 \left(\frac{y}{2} \right) = \frac{x-1}{2}$

A1

$y \geq 0$ so $\sinh \left(\frac{y}{2} \right) \geq 0$, giving $\sinh \left(\frac{y}{2} \right) = \sqrt{\frac{x-1}{2}}$

M1

$\frac{y}{2} = \operatorname{arsinh} \sqrt{\frac{x-1}{2}}$, so $y = 2 \operatorname{arsinh} \sqrt{\frac{x-1}{2}}$

A1

Answer: $y = 2 \operatorname{arsinh} \sqrt{\frac{x-1}{2}}$

6 marks

25 (6 marks)

(a) $\operatorname{arcosh} (\cosh 1) = \ln (\cosh 1 + \sinh 1) = \ln e = 1$, not -1

M1 A1

arcosh is defined as the non-negative inverse of $\cosh$ on $x \geq 1$, so its values are never negative:

E1

$\cosh (-1) = \cosh 1$ does not mean $\operatorname{arcosh} (\cosh 1) = -1$

Answer: $\operatorname{arcosh} x = -1$ has no solution

3 marks

$$(b) \operatorname{arcosh} x \geq 0, \text{ so } k \geq 0$$

B1

$$x = \cosh k$$

M1

$$= \frac{e^k + e^{-k}}{2}, \text{ valid for } k \geq 0$$

A1

$$\text{Answer: } k \geq 0; x = \cosh k$$

3 marks

26 (7 marks)

$$\operatorname{arsinh} \left(\frac{4}{3} \right) = \ln \left(\frac{4}{3} + \frac{5}{3} \right) = \ln 3$$

M1 A1

$$\operatorname{arsinh} \left(\frac{15}{8} \right) = \ln \left(\frac{15}{8} + \frac{17}{8} \right) = \ln 4$$

M1 A1

$$\operatorname{arsinh} x = \ln 4 - \ln 3 = \ln \left(\frac{4}{3} \right)$$

M1

$$x = \sinh \left(\ln \left(\frac{4}{3} \right) \right) = \frac{\frac{4}{3} - \frac{3}{4}}{2}$$

M1

$$= \frac{7}{24}$$

A1

$$\text{Answer: } x = \frac{7}{24}$$

7 marks

27 (9 marks)

$$(a) \operatorname{artanh} \left(\frac{1}{2r} \right) = \frac{1}{2} \ln \left(\frac{1 + \frac{1}{2r}}{1 - \frac{1}{2r}} \right)$$

M1

$$= \frac{1}{2} \ln \left(\frac{2r+1}{2r-1} \right) \text{ (multiply numerator and denominator by } 2r \text{)}$$

A1

$$\text{Answer: } \operatorname{artanh} \left(\frac{1}{2r} \right) = \frac{1}{2} \ln \left(\frac{2r+1}{2r-1} \right) \text{ for every integer } r \geq 1$$

2 marks

$$(b) \text{ Sum: } \frac{1}{2} \sum_{r=1}^n [\ln (2r+1) - \ln (2r-1)]$$

M1

The terms telescope: $\ln 3 - \ln 1, \ln 5 - \ln 3, \dots, \ln (2n+1) - \ln (2n-1)$

M1 A1

$$= \frac{1}{2} \ln (2n+1) - \frac{1}{2} \ln 1 = \frac{1}{2} \ln (2n+1)$$

A1

$$\text{Answer: } \sum_{r=1}^n \operatorname{artanh} \left(\frac{1}{2r} \right) = \frac{1}{2} \ln (2n+1)$$

4 marks

$$(c) \frac{1}{2} \ln (2n+1) > 2, \text{ so } 2n+1 > e^4 \approx 54.6$$

M1 A1

$$2n+1 > 54.6 \text{ means } n > 26.8, \text{ so } n = 27$$

A1

$$\text{Answer: } n = 27$$

3 marks

28 (9 marks)

$$(a) x = \sinh y, \text{ so } \cosh^2 y = 1 + \sinh^2 y = 1 + x^2$$

M1 A1

$$\cosh y = \sqrt{1+x^2} \text{ (positive), and } y > 0, \text{ so } y = \operatorname{arcosh} \sqrt{1+x^2}$$

A1

$$\text{Answer: } y = \operatorname{arcosh} \sqrt{1+x^2}$$

3 marks

$$(b) \tanh y = \frac{\sinh y}{\cosh y} = \frac{x}{\sqrt{1+x^2}}$$

M1 A1

$$y = \operatorname{artanh} \left(\frac{x}{\sqrt{1+x^2}} \right)$$

A1

$$\text{Answer: } y = \operatorname{artanh} \left(\frac{x}{\sqrt{1+x^2}} \right)$$

3 marks

$$(c) \text{ Apply (a) and (b) with } x = 2: \operatorname{arsinh} 2 = \operatorname{arcosh} \sqrt{5} = \operatorname{artanh} \left(\frac{2}{\sqrt{5}} \right)$$

M1

$$\operatorname{arsinh} 2 = \ln (2 + \sqrt{5})$$

A1

$$\text{Common value} = \ln (2 + \sqrt{5})$$

A1

$$\text{Answer: } \ln (2 + \sqrt{5})$$

3 marks

29 (8 marks)

$$\text{Adding: } 2 \operatorname{arcosh} x = \ln 6 + \ln \left(\frac{3}{2}\right) = \ln 9$$

M1 A1

$$\operatorname{arcosh} x = \ln 3$$

A1

$$x = \cosh (\ln 3) = \frac{3 + \frac{1}{3}}{2} = \frac{5}{3}$$

M1 A1

$$\text{Subtracting: } 2 \operatorname{arcosh} y = \ln 6 - \ln \left(\frac{3}{2}\right) = \ln 4$$

M1

$$\operatorname{arcosh} y = \ln 2$$

A1

$$y = \cosh (\ln 2) = \frac{2 + \frac{1}{2}}{2} = \frac{5}{4}$$

A1

$$\text{Answer: } x = \frac{5}{3}, y = \frac{5}{4}$$

8 marks

30 (9 marks)

$$\ln (x + \sqrt{x^2 - 1}) - \ln (x + \sqrt{x^2 + 1}) = -\ln 2$$

M1

$$\frac{x + \sqrt{x^2 - 1}}{x + \sqrt{x^2 + 1}} = \frac{1}{2}$$

A1

$$2x + 2\sqrt{x^2 - 1} = x + \sqrt{x^2 + 1}, \text{ so } x + 2\sqrt{x^2 - 1} = \sqrt{x^2 + 1}$$

M1

$$\text{Square: } x^2 + 4x\sqrt{x^2 - 1} + 4x^2 - 4 = x^2 + 1, \text{ so } 4x\sqrt{x^2 - 1} = 5 - 4x^2$$

M1 A1

$$\text{Square again: } 16x^2(x^2 - 1) = 25 - 40x^2 + 16x^4$$

M1

$$-16x^2 = 25 - 40x^2, \text{ so } x^2 = \frac{25}{24}$$

A1

$$x = \frac{5}{2\sqrt{6}} = \frac{5\sqrt{6}}{12} \quad (x \geq 1)$$

A1

$$\text{Check: } 5 - 4x^2 = \frac{5}{6} > 0, \text{ so squaring introduced no false root, and } x \approx 1.02 \geq 1$$

E1

$$\text{Answer: } x = \frac{5\sqrt{6}}{12}$$

9 marks

Hyperbolic Differentiation**1 (2 marks)**

$$(a) \frac{d}{dx}(\sinh 5x) = 5 \cosh 5x$$

B1

$$\text{Answer: } 5 \cosh 5x$$

1 mark

$$(b) \frac{d}{dx}(\cosh (\frac{x}{2})) = \frac{1}{2} \sinh (\frac{x}{2})$$

B1

$$\text{Answer: } \frac{1}{2} \sinh (\frac{x}{2})$$

1 mark

2 (2 marks)

$$\frac{d}{dx}(\tanh kx) = k \operatorname{sech}^2 kx$$

M1

$$\frac{dy}{dx} = 2 \operatorname{sech}^2 2x$$

A1

$$\text{Answer: } 2 \operatorname{sech}^2 2x$$

2 marks

3 (2 marks)

$$\text{Product rule: } \frac{dy}{dx} = 2x \cosh x + x^2 \sinh x$$

M1 A1

$$\text{Answer: } 2x \cosh x + x^2 \sinh x$$

2 marks

4 (2 marks)

$$\frac{d}{dx}(\operatorname{arsinh} u) = \frac{1}{\sqrt{u^2 + 1}} \frac{du}{dx} \text{ with } u = 3x$$

M1

$$\frac{dy}{dx} = \frac{3}{\sqrt{9x^2 + 1}}$$

A1

$$\text{Answer: } \frac{3}{\sqrt{9x^2 + 1}}$$

2 marks

5 (2 marks)

$$\frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x - (-e^{-x})}{2}$$

M1

$$= \frac{e^x + e^{-x}}{2} = \cosh x$$

A1

$$\text{Answer: } \frac{d}{dx}(\sinh x) = \cosh x$$

2 marks

6 (2 marks)

$$\frac{dy}{dx} = \sinh x$$

M1

$$\text{At } x = \ln 3: \sinh(\ln 3) = \frac{3 - \frac{1}{3}}{2} = \frac{4}{3}$$

A1

$$\text{Answer: } \frac{4}{3}$$

2 marks

7 (3 marks)

$$\tanh x = \frac{\sinh x}{\cosh x}; \text{ quotient rule: } \frac{\cosh x \cosh x - \sinh x \sinh x}{\cosh^2 x}$$

M1 A1

$$= \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x, \text{ using } \cosh^2 x - \sinh^2 x = 1$$

A1

$$\text{Answer: } \frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$$

3 marks

8 (3 marks)

$$\operatorname{sech} x = (\cosh x)^{-1}, \text{ so } \frac{d}{dx}(\operatorname{sech} x) = -(\cosh x)^{-2} \sinh x$$

M1 A1

$$= -\frac{1}{\cosh x} \times \frac{\sinh x}{\cosh x} = -\operatorname{sech} x \tanh x$$

A1

$$\text{Answer: } \frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

3 marks

9 (3 marks)

$$\text{Product rule with } u = e^x \text{ and } v = \sinh 2x$$

M1

$$\frac{dy}{dx} = e^x \sinh 2x + 2e^x \cosh 2x$$

M1 A1

$$\text{Answer: } e^x(\sinh 2x + 2 \cosh 2x)$$

3 marks

10 (3 marks)

$$(a) \frac{\sinh x}{\cosh x} = \tanh x$$

B1

$$\text{Answer: } \tanh x$$

1 mark

$$(b) \frac{3 \cosh 3x}{\sinh 3x}$$

M1

$$= 3 \coth 3x$$

A1

$$\text{Answer: } 3 \coth 3x$$

2 marks

11 (3 marks)

$$\frac{dy}{dx} = 2 \cosh x \times \sinh x \text{ (chain rule)}$$

M1 A1

$$= \sinh 2x, \text{ using } \sinh 2x = 2 \sinh x \cosh x$$

A1

$$\text{Answer: } \frac{dy}{dx} = \sinh 2x$$

3 marks

12 (4 marks)

$$(a) \frac{1}{\sqrt{(2x)^2 - 1}} \times 2$$

M1

$$= \frac{2}{\sqrt{4x^2 - 1}}$$

A1

$$\text{Answer: } \frac{2}{\sqrt{4x^2 - 1}}$$

2 marks

$$(b) \frac{1}{1 - (\frac{x}{2})^2} \times \frac{1}{2}$$

M1

$$= \frac{1}{2} \div (1 - \frac{x^2}{4}) = \frac{2}{4 - x^2}$$

A1

$$\text{Answer: } \frac{2}{4 - x^2}$$

2 marks

13 (4 marks)

$$\frac{dy}{dx} = \cosh x, \text{ so the gradient at } x = \ln 2 \text{ is } \cosh (\ln 2) = \frac{5}{4}$$

M1 A1

$$y = \sinh (\ln 2) = \frac{3}{4}$$

B1

$$y - \frac{3}{4} = \frac{5}{4}(x - \ln 2), \text{ that is } y = \frac{5}{4}x + \frac{3}{4} - \frac{5}{4} \ln 2$$

A1

$$\text{Answer: } y = \frac{5}{4}x + \frac{3}{4} - \frac{5}{4} \ln 2$$

4 marks

14 (4 marks)

$$x = \sinh y, \text{ so } \frac{dx}{dy} = \cosh y$$

M1 A1

$$\frac{dy}{dx} = \frac{1}{\cosh y}$$

M1

$$\cosh y = \sqrt{1 + \sinh^2 y} = \sqrt{1 + x^2} \text{ (cosh } y > 0), \text{ so } \frac{dy}{dx} = \frac{1}{\sqrt{x^2 + 1}}$$

A1

$$\text{Answer: } \frac{dy}{dx} = \frac{1}{\sqrt{x^2 + 1}}$$

4 marks

15 (4 marks)

$$x = \tanh y, \text{ so } \frac{dx}{dy} = \text{sech}^2 y$$

M1 A1

$$\frac{dy}{dx} = \frac{1}{\text{sech}^2 y}$$

M1

$$\text{sech}^2 y = 1 - \tanh^2 y = 1 - x^2, \text{ so } \frac{dy}{dx} = \frac{1}{1 - x^2}$$

A1

$$\text{Answer: } \frac{dy}{dx} = \frac{1}{1 - x^2}$$

4 marks

16 (6 marks)

$$\frac{dy}{dx} = 5 \sinh x - 4 \cosh x \quad \text{M1}$$

Stationary when $5 \sinh x = 4 \cosh x$, so $\tanh x = \frac{4}{5}$ M1

$$x = \frac{1}{2} \ln \left(\frac{1+4}{5/1-4/5} \right) = \frac{1}{2} \ln 9 = \ln 3 \quad \text{A1}$$

$$y = 5 \times \frac{5}{3} - 4 \times \frac{4}{3} = 3, \text{ so the point is } (\ln 3, 3) \quad \text{A1}$$

$$\frac{d^2y}{dx^2} = 5 \cosh x - 4 \sinh x = y = 3 > 0, \text{ so it is a minimum} \quad \text{A1 A1}$$

Answer: (ln 3, 3), minimum

6 marks

17 (5 marks)

Product rule: $\frac{d}{dx}(x \operatorname{arsinh} x) = \operatorname{arsinh} x + \frac{x}{\sqrt{x^2+1}}$ M1 A1

$$\frac{d}{dx}(\sqrt{x^2+1}) = \frac{x}{\sqrt{x^2+1}} \quad \text{B1}$$

$$\frac{dy}{dx} = \operatorname{arsinh} x + \frac{x}{\sqrt{x^2+1}} - \frac{x}{\sqrt{x^2+1}} \quad \text{M1}$$

$$= \operatorname{arsinh} x \quad \text{A1}$$

Answer: $\frac{dy}{dx} = \operatorname{arsinh} x$

5 marks

18 (5 marks)

$$\frac{dy}{dx} = \operatorname{sech}^2(\ln x) \times \frac{1}{x} \quad \text{M1 A1}$$

$$\cosh(\ln x) = \frac{x + \frac{1}{x}}{2} = \frac{x^2 + 1}{2x} \quad \text{M1}$$

$$\operatorname{sech}^2(\ln x) = \frac{4x^2}{(x^2 + 1)^2} \quad \text{A1}$$

$$\frac{dy}{dx} = \frac{4x^2}{(x^2 + 1)^2} \times \frac{1}{x} = \frac{4x}{(x^2 + 1)^2} \quad \text{A1}$$

Answer: $\frac{dy}{dx} = \frac{4x}{(x^2 + 1)^2}$

5 marks

19 (5 marks)

$$\frac{dx}{dt} = 1 - \operatorname{sech}^2 t = \tanh^2 t \quad \text{M1 A1}$$

$$\frac{dy}{dt} = -\operatorname{sech} t \tanh t \quad \text{B1}$$

$$\frac{dy}{dx} = \frac{-\operatorname{sech} t \tanh t}{\tanh^2 t} = -\frac{\operatorname{sech} t}{\tanh t} \quad \text{M1}$$

$$= -\frac{1}{\cosh t} \times \frac{\cosh t}{\sinh t} = -\operatorname{cosech} t \quad \text{A1}$$

Answer: $\frac{dy}{dx} = -\operatorname{cosech} t$

5 marks

20 (7 marks)

$$\frac{dy}{dx} = -\operatorname{sech} x \tanh x$$

M1 A1

$$\frac{d^2y}{dx^2} = -(-\operatorname{sech} x \tanh x) \tanh x - \operatorname{sech} x \times \operatorname{sech}^2 x = \operatorname{sech} x (\tanh^2 x - \operatorname{sech}^2 x)$$

M1 A1

Points of inflection: $\tanh^2 x = \operatorname{sech}^2 x = 1 - \tanh^2 x$, so $\tanh^2 x = \frac{1}{2}$

M1

$$x = \pm \operatorname{artanh} \left(\frac{1}{\sqrt{2}} \right) = \pm \frac{1}{2} \ln \left(\frac{1+1}{\sqrt{2}/1 - 1/\sqrt{2}} \right) = \pm \frac{1}{2} \ln \left(\frac{\sqrt{2}+1}{\sqrt{2}-1} \right) = \pm \ln(1 + \sqrt{2})$$

A1

$$y = \operatorname{sech} x = \frac{1}{\sqrt{2}} (\cosh(\ln(1 + \sqrt{2}))) = \sqrt{2}; \text{ points } (\pm \ln(1 + \sqrt{2}), \frac{1}{\sqrt{2}})$$

A1

Answer: $(\pm \ln(1 + \sqrt{2}), \frac{1}{\sqrt{2}})$

7 marks

21 (9 marks)

$$\frac{dy}{dx} = 7 - \sinh x - 5 \cosh x$$

M1 A1

In exponentials: $7 - 3e^x - 2e^{-x} = 0$, so $3u^2 - 7u + 2 = 0$ with $u = e^x$

M1 A1

$(3u - 1)(u - 2) = 0$, so $u = \frac{1}{3}$ or 2 , giving $x = -\ln 3$ or $x = \ln 2$

M1 A1

$x = \ln 2$: $y = 7 \ln 2 - \frac{5}{4} - \frac{15}{4} = 7 \ln 2 - 5$

A1

$x = -\ln 3$: $y = -7 \ln 3 - \frac{5}{3} + \frac{20}{3} = 5 - 7 \ln 3$

A1

$\frac{d^2y}{dx^2} = -\cosh x - 5 \sinh x$: it is -5 at $x = \ln 2$ (maximum) and 5 at $x = -\ln 3$ (minimum)

A1

Answer: $(\ln 2, 7 \ln 2 - 5)$ maximum; $(-\ln 3, 5 - 7 \ln 3)$ minimum

9 marks

22 (7 marks)

(a) $\frac{dy}{dx} = \sinh \left(\frac{x}{20} \right)$

M1

$x = 15$: gradient = $\sinh 0.75 = 0.822$

A1

Angle = $\arctan(0.822) = 39.4^\circ$

A1

Answer: 0.822; 39.4°

3 marks

(b) $\sinh \left(\frac{x}{20} \right) = 1$, so $\frac{x}{20} = \operatorname{arsinh} 1 = \ln(1 + \sqrt{2})$

M1 A1

$x = 20 \ln(1 + \sqrt{2})$

A1

$\cosh^2 = 1 + \sinh^2 = 2$, so $y = 20\sqrt{2}$

A1

Answer: $x = 20 \ln(1 + \sqrt{2})$, $y = 20\sqrt{2}$

4 marks

23 (9 marks)

(a) $\frac{d}{dx}[\arctan u] = \frac{1}{1+u^2} \frac{du}{dx}$ with $u = \sinh x$

M1

$$= \frac{\cosh x}{1 + \sinh^2 x}$$

A1

$$= \frac{\cosh x}{\cosh^2 x} = \operatorname{sech} x$$

A1

Answer: $\frac{d}{dx}[\arctan(\sinh x)] = \operatorname{sech} x$

3 marks

(b) $\frac{2e^x}{1+e^{2x}}$

M1 A1

$$= \frac{2}{e^{-x} + e^x} = \frac{1}{\cosh x} = \operatorname{sech} x$$

A1

Answer: $\frac{d}{dx}[2 \arctan(e^x)] = \operatorname{sech} x$

3 marks

(c) The two functions have equal derivatives, so they differ by a constant

M1

$x = 0$: $\arctan 0 = 0$ and $2 \arctan 1 = \frac{\pi}{2}$

M1

$$c = -\frac{\pi}{2}$$

A1

Answer: $c = -\frac{\pi}{2}$

3 marks

24 (7 marks)

$$\frac{dy}{dx} = 2 \operatorname{arsinh} x \times \frac{1}{\sqrt{1+x^2}}$$

M1 A1

So $\sqrt{1+x^2} \frac{dy}{dx} = 2 \operatorname{arsinh} x$

A1

Differentiate both sides with respect to x (product rule on the left)

M1

$$\frac{x}{\sqrt{1+x^2}} \frac{dy}{dx} + \sqrt{1+x^2} \frac{d^2y}{dx^2} = \frac{2}{\sqrt{1+x^2}}$$

M1 A1

Multiply by $\sqrt{1+x^2}$: $x \frac{dy}{dx} + (1+x^2) \frac{d^2y}{dx^2} = 2$

A1

Answer: $(1+x^2) \frac{d^2y}{dx^2} + x \frac{dy}{dx} = 2$

7 marks

25 (6 marks)

(i) The chain rule has been omitted: the derivative is $3 \sinh 3x$

E1 B1

(ii) The derivative of $\tanh x$ is $\operatorname{sech}^2 x$, not $\operatorname{sech} x$

E1 B1

(iii) The product rule has been omitted: the derivative is $\sinh x + x \cosh x$

E1 B1

Answer: $3 \sinh 3x$; $\operatorname{sech}^2 x$; $\sinh x + x \cosh x$

6 marks

26 (7 marks)

$f(x) = \tanh x$ and $f'(x) = \operatorname{sech}^2 x$

B1

$f''(x) = -2 \operatorname{sech}^2 x \tanh x$

M1 A1

$f'''(x) = 4 \operatorname{sech}^2 x \tanh^2 x - 2 \operatorname{sech}^4 x$

A1

$f(0) = 0$, $f'(0) = 0$, $f''(0) = 1$, $f'''(0) = 0$, $f^{(4)}(0) = -2$

M1

Maclaurin: $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \dots$

M1

$= \frac{x^2}{2} - \frac{x^4}{12} + \dots$

A1

Answer: $f(x) = \tanh x$, $f'(x) = \operatorname{sech}^2 x$, $f''(x) = -2 \operatorname{sech}^2 x \tanh x$, $f'''(x) = 4 \operatorname{sech}^2 x \tanh^2 x - 2 \operatorname{sech}^4 x$, $f^{(4)}(x) = 8 \operatorname{sech}^2 x \tanh^3 x - 8 \operatorname{sech}^4 x \tanh x$

27 (11 marks)

(a) $\frac{dy}{dx} = 4 \operatorname{sech} x - 4x \operatorname{sech} x \tanh x$

M1 A1

$= 4 \operatorname{sech} x (1 - x \tanh x)$

A1

$\operatorname{sech} x \neq 0$, so $\frac{dy}{dx} = 0$ when $x \tanh x = 1$

A1

Answer: C has a stationary point where $x \tanh x = 1$

4 marks

(b) $f(x) = x \tanh x - 1$: $f(1) = \tanh 1 - 1 = -0.238 < 0$

M1

$f(1.5) = 1.5 \tanh 1.5 - 1 = 0.358 > 0$; a sign change, so a root exists

A1

Answer: the equation $x \tanh x = 1$ has a root between 1 and 1.5

2 marks

(c) $f'(x) = \tanh x + x \operatorname{sech}^2 x$; $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

M1 A1

$x_1 = 1.1997$, $x_2 = 1.1997$, $x_3 = 1.1997$

M1

Root = 1.200

A1

$y = 4 \times 1.1997 \times \operatorname{sech} 1.1997 = 2.65$

A1

Answer: $x = 1.200$; $y = 2.65$

5 marks

28 (10 marks)

$$(a) \frac{dy}{dx} = \frac{3 \cosh t}{2 \sinh t} = \frac{3}{2} \coth t \quad \text{M1 A1}$$

$$\text{Tangent at P: } y - 3 \sinh p = \frac{3 \cosh p}{2 \sinh p} (x - 2 \cosh p) \quad \text{M1}$$

$$2y \sinh p - 6 \sinh^2 p = 3x \cosh p - 6 \cosh^2 p; \text{ using } \cosh^2 p - \sinh^2 p = 1 \text{ this gives } 3x \cosh p - 2y \sinh p = 6 \quad \text{A1}$$

Answer: the tangent to H at P has equation $3x \cosh p - 2y \sinh p = 6$ 4 marks

$$(b) \text{ On } y = \frac{3}{2}x: 3x \cosh p - 3x \sinh p = 6, \text{ so } x = \frac{2}{\cosh p - \sinh p} = 2e^p \quad \text{M1 A1}$$

$$A = (2e^p, 3e^p) \quad \text{A1}$$

$$\text{On } y = -\frac{3}{2}x: 3x \cosh p + 3x \sinh p = 6, \text{ so } x = 2e^{-p}, \text{ and } B = (2e^{-p}, -3e^{-p}) \quad \text{A1}$$

Answer: $A = (2e^p, 3e^p)$, $B = (2e^{-p}, -3e^{-p})$ 4 marks

$$(c) \text{ Midpoint} = \left(\frac{2e^p + 2e^{-p}}{2}, \frac{3e^p - 3e^{-p}}{2} \right) \quad \text{M1}$$

$$= (2 \cosh p, 3 \sinh p) = P \quad \text{A1}$$

Answer: P is the midpoint of AB 2 marks

29 (8 marks)

$$(a) \frac{dy}{dx} = \sinh\left(\frac{x}{a}\right) \text{ and } \frac{d^2y}{dx^2} = \frac{1}{a} \cosh\left(\frac{x}{a}\right) \quad \text{M1 A1}$$

$$a \frac{d^2y}{dx^2} = \cosh\left(\frac{x}{a}\right) = \sqrt{1 + \sinh^2\left(\frac{x}{a}\right)} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \quad \text{A1}$$

$$\text{Answer: } a \frac{d^2y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \quad \text{3 marks}$$

$$(b) 5 \cosh\left(\frac{x}{5}\right) = 13, \text{ so } \cosh\left(\frac{x}{5}\right) = \frac{13}{5} \quad \text{M1}$$

$$\sinh^2\left(\frac{x}{5}\right) = \frac{169}{25} - 1 = \frac{144}{25}, \text{ so } \sinh\left(\frac{x}{5}\right) = \pm \frac{12}{5} \quad \text{M1 A1}$$

$$\text{Gradient} = \sinh\left(\frac{x}{5}\right) = \pm \frac{12}{5} \quad \text{A1}$$

$$\text{Answer: } \pm \frac{12}{5} \quad \text{4 marks}$$

$$(c) \arctan\left(\frac{12}{5}\right) = 67.4^\circ \quad \text{B1}$$

Answer: 67.4° 1 mark

30 (11 marks)

$$(a) g'(x) = (\sinh x \sin x + \cosh x \cos x) + (\cosh x \cos x - \sinh x \sin x) = 2 \cosh x \cos x \quad \text{M1 A1}$$

Answer: $g'(x) = 2 \cosh x \cos x$ 3 marks

$$(b) g''(x) = 2(\sinh x \cos x - \cosh x \sin x) \quad \text{M1 A1}$$

$$g'''(x) = 2(\cosh x \cos x - \sinh x \sin x - \sinh x \sin x - \cosh x \cos x) = -4 \sinh x \sin x \quad \text{M1 A1}$$

$$g''''(x) = -4(\cosh x \sin x + \sinh x \cos x) = -4g(x) \quad \text{A1}$$

Answer: $g''''(x) = -4g(x)$ 5 marks

$$(c) g(0) = 0, g'(0) = 2, g''(0) = 0, g'''(0) = 0, g''''(0) = 0, g^{(5)}(0) = -4g'(0) = -8 \quad \text{M1 A1}$$

$$g(x) = 2x + \frac{-8}{120}x^5 = 2x - \frac{x^5}{15} + \dots \quad \text{A1}$$

$$\text{Answer: } 2x - \frac{x^5}{15} \quad \text{3 marks}$$

Hyperbolic Integration

1 (2 marks)

(a) $\frac{1}{3} \cosh 3x + c$

B1

Answer: $\frac{1}{3} \cosh 3x + c$

1 mark

(b) $\frac{1}{2} \sinh (2x + 1) + c$

B1

Answer: $\frac{1}{2} \sinh (2x + 1) + c$

1 mark

2 (2 marks)

(a) $\tanh x + c$

B1

Answer: $\tanh x + c$

1 mark

(b) $4 \times \frac{1}{2} \tanh 2x + c = 2 \tanh 2x + c$

B1

Answer: $2 \tanh 2x + c$

1 mark

3 (2 marks)

$[\sinh x]_0^1 = \sinh 1 - \sinh 0$
 $= 1.18$

M1

A1

Answer: 1.18

2 marks

4 (2 marks)

$\int \frac{\sinh x}{\cosh x} dx$, which has the form $\int \frac{f'(x)}{f(x)} dx$ with $f(x) = \cosh x$
 $= \ln (\cosh x) + c$, with no modulus needed since $\cosh x > 0$

M1

A1

Answer: $\int \tanh x dx = \ln (\cosh x) + c$

2 marks

5 (2 marks)

(a) $\operatorname{arsinh} x + c$

B1

Answer: $\operatorname{arsinh} x + c$

1 mark

(b) $\operatorname{arsinh} \left(\frac{x}{2}\right) + c$

B1

Answer: $\operatorname{arsinh} \left(\frac{x}{2}\right) + c$

1 mark

6 (2 marks)

$[\cosh x]_0^{\ln 2} = \cosh (\ln 2) - \cosh 0$
 $= \frac{5}{4} - 1 = \frac{1}{4}$

M1

A1

Answer: $\int_0^{\ln 2} \sinh x dx = \frac{1}{4}$

2 marks

7 (3 marks)

$\left[\frac{1}{2} \sinh 2x\right]_0^{\ln 3}$

M1

$= \frac{1}{2} \sinh (\ln 9) = \frac{1}{2} \times \frac{9 - \frac{1}{9}}{2}$

M1

$= \frac{1}{2} \times \frac{40}{9} = \frac{20}{9}$

A1

Answer: $\int_0^{\ln 3} \cosh 2x dx = \frac{20}{9}$

3 marks

8 (3 marks)

$$\cosh 2x = 2\cosh^2 x - 1, \text{ so } \cosh^2 x = \frac{1}{2}(\cosh 2x + 1)$$

M1 A1

$$\int = \frac{\sinh 2x}{4} + \frac{x}{2} + c$$

A1

$$\text{Answer: } \frac{x}{2} + \frac{\sinh 2x}{4} + c$$

3 marks

9 (3 marks)

$$\cosh 2x = 1 + 2 \sinh^2 x, \text{ so } \sinh^2 x = \frac{1}{2}(\cosh 2x - 1)$$

M1

$$\int = \left[\frac{\sinh 2x}{4} - \frac{x}{2} \right]_0^1$$

A1

$$= \frac{\sinh 2}{4} - \frac{1}{2}$$

A1

$$\text{Answer: } \frac{\sinh 2}{4} - \frac{1}{2}$$

3 marks

10 (3 marks)

$$\int \frac{1}{\sqrt{x^2 + 16}} dx = \operatorname{arsinh} \left(\frac{x}{4} \right)$$

M1

$$\left[\operatorname{arsinh} \left(\frac{x}{4} \right) \right]_0^3 = \operatorname{arsinh} \left(\frac{3}{4} \right) - 0$$

A1

$$\operatorname{arsinh} \left(\frac{3}{4} \right) = \ln \left(\frac{3}{4} + \frac{5}{4} \right) = \ln 2$$

A1

$$\text{Answer: } \int_0^3 \frac{1}{\sqrt{x^2 + 16}} dx = \ln 2$$

3 marks

11 (4 marks)

$$\int \frac{1}{\sqrt{x^2 - 9}} dx = \operatorname{arcosh} \left(\frac{x}{3} \right) = \ln (x + \sqrt{x^2 - 9}) + c$$

M1 A1

$$\left[\ln (x + \sqrt{x^2 - 9}) \right]_5^{15} = \ln (15 + \sqrt{216}) - \ln (5 + 4)$$

M1

$$= \ln \left(\frac{15 + 6\sqrt{6}}{9} \right) = \ln \left(\frac{5 + 2\sqrt{6}}{3} \right)$$

A1

$$\text{Answer: } \ln \left(\frac{5 + 2\sqrt{6}}{3} \right)$$

4 marks

12 (3 marks)

$$\int \frac{1}{1 - x^2} dx = \operatorname{artanh} x$$

M1

$$\left[\operatorname{artanh} x \right]_0^{1/2} = \operatorname{artanh} \left(\frac{1}{2} \right)$$

A1

$$= \frac{1}{2} \ln \left(\frac{1 + \frac{1}{2}}{1 - \frac{1}{2}} \right) = \frac{1}{2} \ln 3$$

A1

$$\text{Answer: } \int_0^{1/2} \frac{1}{1 - x^2} dx = \frac{1}{2} \ln 3$$

3 marks

13 (4 marks)

$$\text{Area} = \int_0^{\ln 2} \sinh 2x dx = \left[\frac{1}{2} \cosh 2x \right]_0^{\ln 2}$$

M1 A1

$$\cosh (2 \ln 2) = \cosh (\ln 4) = \frac{4 + \frac{1}{4}}{2} = \frac{17}{8}$$

M1

$$\text{Area} = \frac{1}{2} \left(\frac{17}{8} - 1 \right) = \frac{9}{16}$$

A1

$$\text{Answer: } \frac{9}{16}$$

4 marks

14 (4 marks)

$$u = x, \frac{dv}{dx} = \sinh x; \frac{du}{dx} = 1, v = \cosh x$$

M1

$$\int x \sinh x \, dx = x \cosh x - \int \cosh x \, dx$$

$$= x \cosh x - \sinh x + c$$

M1 A1

A1

Answer: $x \cosh x - \sinh x + c$

4 marks

15 (4 marks)

$$x = \sinh u \text{ gives } \frac{dx}{du} = \cosh u, \text{ and } \sqrt{x^2 + 1} = \sqrt{1 + \sinh^2 u} = \cosh u$$

M1 A1

$$\int \frac{\cosh u}{\cosh u} \, du = \int 1 \, du = u + c$$

M1

$$u = \operatorname{arsinh} x, \text{ so the integral is } \operatorname{arsinh} x + c$$

A1

$$\textbf{Answer: } \int \frac{1}{\sqrt{x^2 + 1}} \, dx = \operatorname{arsinh} x + c$$

4 marks

16 (5 marks)

$$\text{Separate the variables: } \int \frac{1}{\sqrt{1+y^2}} \, dy = \int \cosh x \, dx$$

M1

$$\operatorname{arsinh} y = \sinh x + c$$

M1 A1

$$y = 0 \text{ when } x = 0: \operatorname{arsinh} 0 = \sinh 0 + c, \text{ so } c = 0$$

M1

$$y = \sinh (\sinh x)$$

A1

Answer: $y = \sinh (\sinh x)$

5 marks

17 (5 marks)

$$\text{By parts: } u = x, \frac{dv}{dx} = \cosh x$$

M1

$$\int x \cosh x \, dx = x \sinh x - \int \sinh x \, dx = x \sinh x - \cosh x$$

M1 A1

$$[x \sinh x - \cosh x]_0^{\ln 2} = \ln 2 \times \frac{3}{4} - \frac{5}{4} - (0 - 1)$$

M1

$$= \frac{3}{4} \ln 2 - \frac{1}{4}$$

A1

$$\textbf{Answer: } \frac{3}{4} \ln 2 - \frac{1}{4}$$

5 marks

18 (5 marks)

$$\frac{1}{x^2 - 1} = \frac{1}{(x-1)(x+1)} = \frac{1}{2} \left(\frac{1}{x-1} - \frac{1}{x+1} \right)$$

M1 A1

$$\int = \frac{1}{2} \ln \left| \frac{x-1}{x+1} \right|$$

A1

$$\left[\frac{1}{2} \ln \left(\frac{x-1}{x+1} \right) \right]_2^3 = \frac{1}{2} \ln \left(\frac{2}{4} \right) - \frac{1}{2} \ln \left(\frac{1}{3} \right)$$

M1

$$= \frac{1}{2} \ln \left(\frac{1}{2} \times 3 \right) = \frac{1}{2} \ln \left(\frac{3}{2} \right)$$

A1

$$\textbf{Answer: } \int_2^3 \frac{1}{x^2 - 1} \, dx = \frac{1}{2} \ln \left(\frac{3}{2} \right)$$

5 marks

19 (5 marks)

$$\operatorname{sech} x = \frac{2}{e^x + e^{-x}}$$

M1

$$\text{Multiply numerator and denominator by } e^x: \frac{2e^x}{e^{2x} + 1}$$

A1

$$\text{Let } u = e^x, \text{ so } \frac{du}{dx} = e^x: \int \frac{2}{u^2 + 1} \, du$$

M1

$$= 2 \arctan u + c$$

A1

$$= 2 \arctan (e^x) + c$$

A1

Answer: $2 \arctan (e^x) + c$

5 marks

20 (5 marks)

$$\text{By parts: } u = x, \frac{dv}{dx} = \operatorname{sech}^2 x, v = \tanh x$$

M1

$$\int x \operatorname{sech}^2 x \, dx = x \tanh x - \int \tanh x \, dx$$

M1 A1

$$= x \tanh x - \ln (\cosh x)$$

A1

$$[]_0^1 = \tanh 1 - \ln (\cosh 1)$$

A1

$$\text{Answer: } \tanh 1 - \ln (\cosh 1)$$

5 marks

21 (6 marks)

$$\text{Complete the square: } x^2 + 4x + 13 = (x + 2)^2 + 9$$

M1 A1

$$\int \frac{1}{\sqrt{(x+2)^2 + 9}} \, dx = \operatorname{arsinh} \left(\frac{x+2}{3} \right)$$

M1 A1

$$[]_{-2}^1 = \operatorname{arsinh} 1 - \operatorname{arsinh} 0$$

M1

$$\operatorname{arsinh} 1 = \ln (1 + \sqrt{2}), \operatorname{arsinh} 0 = 0, \text{ giving } \ln (1 + \sqrt{2})$$

A1

$$\text{Answer: } \int_{-2}^1 \frac{1}{\sqrt{x^2 + 4x + 13}} \, dx = \ln (1 + \sqrt{2})$$

6 marks

22 (7 marks)

$$x = \sinh u, \frac{dx}{du} = \cosh u, \sqrt{x^2 + 1} = \cosh u$$

B1

$$\int \cosh^2 u \, du$$

M1

$$= \int \frac{1}{2} (\cosh 2u + 1) \, du = \frac{1}{4} \sinh 2u + \frac{u}{2}$$

M1 A1

$$\sinh 2u = 2 \sinh u \cosh u = 2x\sqrt{x^2 + 1}$$

M1

$$\frac{1}{4} \sinh 2u = \frac{1}{2} x \sqrt{x^2 + 1}$$

A1

$$u = \operatorname{arsinh} x, \text{ so the integral is } \frac{1}{2} x \sqrt{x^2 + 1} + \frac{1}{2} \operatorname{arsinh} x + c$$

A1

$$\text{Answer: } \int \sqrt{x^2 + 1} \, dx = \frac{1}{2} x \sqrt{x^2 + 1} + \frac{1}{2} \operatorname{arsinh} x + c$$

7 marks

23 (6 marks)

$$V = \pi \int_0^1 \cosh^2 x \, dx$$

M1

$$= \pi \int_0^1 \frac{1}{2} (\cosh 2x + 1) \, dx$$

M1 A1

$$= \pi \left[\frac{\sinh 2x}{4} + \frac{x}{2} \right]_0^1$$

A1

$$= \pi \left(\frac{\sinh 2}{4} + \frac{1}{2} \right)$$

A1

$$= 4.42$$

A1

$$\text{Answer: } \pi \left(\frac{\sinh 2}{4} + \frac{1}{2} \right) = 4.42$$

6 marks

24 (6 marks)

$$\frac{dy}{dx} = \sinh x$$

B1

$$1 + \left(\frac{dy}{dx} \right)^2 = 1 + \sinh^2 x = \cosh^2 x$$

M1 A1

$$\sqrt{\cosh^2 x} = \cosh x \text{ since } \cosh x > 0$$

A1

$$s = [\sinh x]_0^{\ln 2}$$

M1

$$= \sinh (\ln 2) = \frac{3}{4}$$

A1

$$\text{Answer: } s = \frac{3}{4}$$

6 marks

25 (6 marks)

$$\text{Take } u = \operatorname{arsinh} x \text{ and } \frac{dv}{dx} = 1: \frac{du}{dx} = \frac{1}{\sqrt{x^2 + 1}}, v = x \quad \text{M1}$$

$$\int \operatorname{arsinh} x \, dx = x \operatorname{arsinh} x - \int \frac{x}{\sqrt{x^2 + 1}} \, dx \quad \text{M1 A1}$$

$$= x \operatorname{arsinh} x - \sqrt{x^2 + 1} + c \quad \text{A1}$$

$$\left[\right]_0^{3/4} = \frac{3}{4} \operatorname{arsinh} \left(\frac{3}{4} \right) - \frac{5}{4} - (0 - 1) \quad \text{M1}$$

$$= \frac{3}{4} \ln 2 - \frac{1}{4} \quad \text{A1}$$

$$\text{Answer: } \int \operatorname{arsinh} x \, dx = x \operatorname{arsinh} x - \sqrt{x^2 + 1} + c, \text{ and } \int_0^{3/4} \operatorname{arsinh} x \, dx = \frac{3}{4} \ln 2 - \frac{1}{4} \quad \text{6 marks}$$

26 (6 marks)

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{\cosh x}{\cosh^2 x} = \frac{\cosh x}{1 + \sinh^2 x} \quad \text{M1 A1}$$

$$\text{Let } u = \sinh x, \frac{du}{dx} = \cosh x: \int \frac{1}{1 + u^2} \, du = \arctan u + c \quad \text{M1}$$

$$= \arctan (\sinh x) + c \quad \text{A1}$$

$$\sinh (\ln (1 + \sqrt{2})) = \frac{(1 + \sqrt{2}) - (\sqrt{2} - 1)}{2} = 1 \quad \text{M1}$$

$$\text{Value} = \arctan 1 - \arctan 0 = \frac{\pi}{4} \quad \text{A1}$$

$$\text{Answer: } \int \operatorname{sech} x \, dx = \arctan (\sinh x) + c \quad \text{6 marks}$$

27 (10 marks)

$$\text{(a) } I_n = \int \cosh^{n-1} x \cosh x \, dx; \text{ by parts } u = \cosh^{n-1} x, \frac{dv}{dx} = \cosh x \quad \text{M1}$$

$$I_n = [\cosh^{n-1} x \sinh x] - \int (n-1) \cosh^{n-2} x \sinh^2 x \, dx \quad \text{M1 A1}$$

$$\sinh^2 x = \cosh^2 x - 1, \text{ so the integral} = (n-1)(I_n - I_{n-2}) \quad \text{M1}$$

$$I_n = [\cosh^{n-1} x \sinh x] - (n-1)I_n + (n-1)I_{n-2}, \text{ so } n I_n = [\cosh^{n-1} x \sinh x]_0^{\ln 2} + (n-1)I_{n-2} \quad \text{A1}$$

$$\text{Answer: } n I_n = [\cosh^{n-1} x \sinh x]_0^{\ln 2} + (n-1) I_{n-2} \text{ for } n \geq 2 \quad \text{5 marks}$$

$$\text{(b) At } x = \ln 2: \cosh = \frac{5}{4}, \sinh = \frac{3}{4}; \text{ at } x = 0 \text{ the bracket is } 0 \quad \text{B1}$$

$$I_0 = \ln 2 \quad \text{B1}$$

$$2I_2 = \frac{5}{4} \times \frac{3}{4} + I_0, \text{ so } I_2 = \frac{15}{32} + \frac{1}{2} \ln 2 \quad \text{M1 A1}$$

$$4I_4 = \left(\frac{5}{4}\right)^3 \left(\frac{3}{4}\right) + 3I_2 = \frac{375}{256} + \frac{45}{32} + \frac{3}{2} \ln 2 = \frac{735}{256} + \frac{3}{2} \ln 2, \text{ so } I_4 = \frac{735}{1024} + \frac{3}{8} \ln 2 \quad \text{A1}$$

$$\text{Answer: } I_4 = \frac{735}{1024} + \frac{3}{8} \ln 2 \quad \text{5 marks}$$

28 (9 marks)

$$\text{(a) } \cosh x + 3 = \frac{1+t^2}{1-t^2} + 3 = \frac{4-2t^2}{1-t^2} \quad \text{M1 A1}$$

$$dx = \frac{2}{1-t^2} \, dt \quad \text{B1}$$

$$\text{Integrand: } \frac{1-t^2}{4-2t^2} \times \frac{2}{1-t^2} \, dt = \frac{1}{2-t^2} \, dt \quad \text{M1}$$

$$\text{Limits: } x = 0 \text{ gives } t = 0; x = \ln 3 \text{ gives } t = \tanh \left(\frac{1}{2} \ln 3 \right) = \frac{3-1}{3+1} = \frac{1}{2} \quad \text{A1}$$

$$\text{Answer: } J = \int_0^{1/2} \frac{1}{2-t^2} \, dt \quad \text{5 marks}$$

$$\text{(b) } \frac{1}{2-t^2} = \frac{1}{2\sqrt{2}} \left(\frac{1}{\sqrt{2}-t} + \frac{1}{\sqrt{2}+t} \right) \quad \text{M1 A1}$$

$$J = \frac{1}{2\sqrt{2}} \ln \left(\frac{\sqrt{2}+t}{\sqrt{2}-t} \right) \quad \text{A1}$$

$$[]_0^{1/2} = \frac{1}{2\sqrt{2}} \ln \left(\frac{\sqrt{2} + \frac{1}{2}}{\sqrt{2} - \frac{1}{2}} \right) = \frac{1}{2\sqrt{2}} \ln \left(\frac{2\sqrt{2} + 1}{2\sqrt{2} - 1} \right)$$

A1

Answer: J = $\frac{1}{2\sqrt{2}} \ln \left(\frac{2\sqrt{2} + 1}{2\sqrt{2} - 1} \right)$

4 marks

29 (11 marks)

(a) $\cosh x = 3 - \sinh x$, so $\cosh x + \sinh x = 3$

M1

$\cosh x + \sinh x = e^x$, so $e^x = 3$ and $x = \ln 3$

A1

$y = \cosh(\ln 3) = \frac{5}{3}$, so $A = (\ln 3, \frac{5}{3})$

A1

Answer: A = $(\ln 3, \frac{5}{3})$

3 marks

(b) Area = $\int_0^{\ln 3} (3 - \sinh x - \cosh x) dx = \int_0^{\ln 3} (3 - e^x) dx$

M1 A1

= $[3x - e^x]_0^{\ln 3} = 3 \ln 3 - 3 + 1 = 3 \ln 3 - 2$

A1

Answer: $3 \ln 3 - 2$

3 marks

(c) $V = \pi \int_0^{\ln 3} [(3 - \sinh x)^2 - \cosh^2 x] dx$

M1

$(3 - \sinh x)^2 - \cosh^2 x = 9 - 6 \sinh x + \sinh^2 x - \cosh^2 x = 8 - 6 \sinh x$

M1 A1

$V = \pi [8x - 6 \cosh x]_0^{\ln 3}$

A1

= $\pi(8 \ln 3 - 10 + 6) = \pi(8 \ln 3 - 4)$

A1

Answer: $\pi(8 \ln 3 - 4)$

5 marks

30 (8 marks)

$\sinh x + 2 \cosh x = \frac{e^x - e^{-x}}{2} + e^x + e^{-x} = \frac{3e^x + e^{-x}}{2}$

M1 A1

Integrand = $\frac{2}{3e^x + e^{-x}} = \frac{2e^x}{3e^{2x} + 1}$

M1 A1

Let $u = e^x$: $\int \frac{2}{3u^2 + 1} du = \frac{2}{\sqrt{3}} \arctan(\sqrt{3} u)$

M1 A1

Limits: $u = 1$ to 2 : $\frac{2}{\sqrt{3}} [\arctan(2\sqrt{3}) - \arctan(\sqrt{3})]$

M1

= $\frac{2}{\sqrt{3}} \arctan\left(\frac{2\sqrt{3} - \sqrt{3}}{1 + 6}\right) = \frac{2}{\sqrt{3}} \arctan\left(\frac{\sqrt{3}}{7}\right)$

A1

Answer: $\int_0^{\ln 2} \frac{1}{\sinh x + 2 \cosh x} dx = \frac{2}{\sqrt{3}} \arctan\left(\frac{\sqrt{3}}{7}\right)$

8 marks

Solving Hyperbolic Equations**1 (2 marks)**

$x = \operatorname{artanh}\left(\frac{3}{5}\right) = \frac{1}{2} \ln\left(\frac{1 + \frac{3}{5}}{1 - \frac{3}{5}}\right)$

M1

= $\frac{1}{2} \ln 4 = \ln 2$

A1

Answer: $x = \ln 2$

2 marks

2 (2 marks)

$e^{2x} - \frac{5}{2}e^x + 1 = 0$, so $(2e^x - 1)(e^x - 2) = 0$

M1

$e^x = 2$ or $\frac{1}{2}$, so $x = \pm \ln 2$

A1

Answer: $x = \pm \ln 2$

2 marks

3 (2 marks)

$$\sinh x = 1.5$$

B1

$$x = \operatorname{arsinh} 1.5 = 1.19$$

B1

Answer: 1.19

2 marks

4 (2 marks)

$$\cosh x = \frac{e^x + e^{-x}}{2} \text{ and } e^x + e^{-x} \geq 2 \text{ for all } x \text{ (or } \cosh x \geq 1, \text{ minimum value 1 at } x = 0)$$

E1

$$\frac{1}{2} < 1, \text{ so there is no real solution}$$

E1

Answer: $\cosh x \geq 1$ for all real x

2 marks

5 (2 marks)

$$\sinh x + \cosh x = e^x$$

M1

$$e^x = 4, \text{ so } x = \ln 4$$

A1

Answer: $x = \ln 4$

2 marks

6 (3 marks)

$$\operatorname{sech} x = \frac{1}{2} \text{ gives } \cosh x = 2$$

B1

$$e^{2x} - 4e^x + 1 = 0, \text{ so } e^x = 2 \pm \sqrt{3}$$

M1

$$x = \pm \ln(2 + \sqrt{3}) = \pm 1.32$$

A1

Answer: $x = \pm \ln(2 + \sqrt{3}) = \pm 1.32$

3 marks

7 (4 marks)

$$(2 \sinh x + 1)(\sinh x - 1) = 0, \text{ so } \sinh x = 1 \text{ or } \sinh x = -\frac{1}{2}$$

M1 A1

$$\sinh x = 1: x = \ln(1 + \sqrt{2})$$

A1

$$\sinh x = -\frac{1}{2}: x = \ln\left(-\frac{1}{2} + \frac{\sqrt{5}}{2}\right) = \ln\left(\frac{\sqrt{5} - 1}{2}\right)$$

A1

Answer: $x = \ln(1 + \sqrt{2}), x = \ln\left(\frac{\sqrt{5} - 1}{2}\right)$

4 marks

8 (4 marks)

$$(3 \tanh x - 1)(\tanh x + 1) = 0$$

M1 A1

$$\tanh x = -1 \text{ is impossible as } -1 < \tanh x < 1$$

E1

$$\tanh x = \frac{1}{3}: x = \frac{1}{2} \ln\left(\frac{1 + \frac{1}{3}}{1 - \frac{1}{3}}\right) = \frac{1}{2} \ln 2$$

A1

Answer: $x = \frac{1}{2} \ln 2$

4 marks

9 (3 marks)

$$\text{Divide by } \cosh x: 1 = 2 \tanh x, \text{ so } \tanh x = \frac{1}{2}$$

M1

$$x = \frac{1}{2} \ln\left(\frac{1 + \frac{1}{2}}{1 - \frac{1}{2}}\right)$$

M1

$$= \frac{1}{2} \ln 3$$

A1

Answer: $x = \frac{1}{2} \ln 3$

3 marks

10 (4 marks)

In exponentials: $\frac{3(e^x + e^{-x}) - 5(e^x - e^{-x})}{2} = 3$, that is $-2e^x + 8e^{-x} = 6$ M1 A1

Multiply by e^x : $-2u^2 - 6u + 8 = 0$, so $u^2 + 3u - 4 = 0$ with $u = e^x$ M1

$(u + 4)(u - 1) = 0$; $u > 0$ so $u = 1$ and $x = 0$ A1

Answer: $x = 0$ 4 marks

11 (4 marks)

$\cosh 2x = 2\cosh^2 x - 1$, so $2\cosh^2 x - 3\cosh x + 1 = 0$ M1 A1

$(2\cosh x - 1)(\cosh x - 1) = 0$ M1

$\cosh x = \frac{1}{2}$ is impossible, so $\cosh x = 1$ and $x = 0$ A1

Answer: $x = 0$ 4 marks

12 (4 marks)

$2\sinh x \cosh x - 3\sinh x = 0$, so $\sinh x (2\cosh x - 3) = 0$ M1 A1

$\sinh x = 0$ gives $x = 0$ A1

$\cosh x = \frac{3}{2}$ gives $e^x = \frac{3 \pm \sqrt{5}}{2}$, so $x = \pm \ln\left(\frac{3 + \sqrt{5}}{2}\right)$ A1

Answer: $x = 0, \pm \ln\left(\frac{3 + \sqrt{5}}{2}\right)$ 4 marks

13 (3 marks)

(a) Two solutions ($x = \pm \operatorname{arcosh} 3$) B1

Answer: 2 1 mark

(b) One solution ($x = 0$) B1

Answer: 1 1 mark

(c) No solutions since $\cosh x \geq 1$ B1

Answer: 0 1 mark

14 (5 marks)

$\tanh^2 x = 1 - \operatorname{sech}^2 x$, so $3 - 3\operatorname{sech}^2 x + 2\operatorname{sech} x = 2$ M1

$3\operatorname{sech}^2 x - 2\operatorname{sech} x - 1 = 0$ A1

$(3\operatorname{sech} x + 1)(\operatorname{sech} x - 1) = 0$ M1

$\operatorname{sech} x = -\frac{1}{3}$ is impossible since $\operatorname{sech} x > 0$ E1

$\operatorname{sech} x = 1$ gives $\cosh x = 1$, so $x = 0$ A1

Answer: $x = 0$ 5 marks

15 (6 marks)

$\coth^2 x = 1 + \operatorname{cosech}^2 x$, so $2 + 2\operatorname{cosech}^2 x - 5\operatorname{cosech} x + 1 = 0$ M1

$2\operatorname{cosech}^2 x - 5\operatorname{cosech} x + 3 = 0$, so $(2\operatorname{cosech} x - 3)(\operatorname{cosech} x - 1) = 0$ M1 A1

$\operatorname{cosech} x = 1$: $\sinh x = 1$, $x = \ln(1 + \sqrt{2})$ A1

$\operatorname{cosech} x = \frac{3}{2}$: $\sinh x = \frac{2}{3}$, $x = \ln\left(\frac{2}{3} + \sqrt{\frac{4}{9} + 1}\right) = \ln\left(\frac{2 + \sqrt{13}}{3}\right)$ M1 A1

Answer: $x = \ln(1 + \sqrt{2})$, $x = \ln\left(\frac{2 + \sqrt{13}}{3}\right)$ 6 marks

16 (7 marks)

(a) $R \cosh(x + \alpha) = R \cosh \alpha \cosh x + R \sinh \alpha \sinh x$, so $R \cosh \alpha = 13$ and $R \sinh \alpha = 5$ M1

$R^2(\cosh^2 \alpha - \sinh^2 \alpha) = 169 - 25 = 144$, so $R = 12$ M1 A1

$\tanh \alpha = \frac{5}{13}$, so $\alpha = \frac{1}{2} \ln\left(\frac{1+5}{13/1 - 5/13}\right) = \frac{1}{2} \ln\left(\frac{9}{4}\right) = \ln\left(\frac{3}{2}\right)$ A1

Answer: $R = 12$, $\alpha = \ln\left(\frac{3}{2}\right)$ 4 marks

(b) $12 \cosh\left(x + \ln\left(\frac{3}{2}\right)\right) = 15$, so $\cosh\left(x + \ln\left(\frac{3}{2}\right)\right) = \frac{5}{4} = \cosh(\ln 2)$ M1

$$x + \ln \frac{3}{2} = \pm \ln 2 \quad \text{A1}$$

$$x = \ln 2 - \ln \frac{3}{2} = \ln \frac{4}{3} \text{ or } x = -\ln 2 - \ln \frac{3}{2} = -\ln 3 \quad \text{A1}$$

Answer: $x = \ln \left(\frac{4}{3}\right)$ or $x = -\ln 3$ 3 marks

17 (4 marks)

(a) $\cosh x = 2$ at $x = \pm \ln(2 + \sqrt{3})$ M1

$\cosh x > 2$ for $x < -\ln(2 + \sqrt{3})$ or $x > \ln(2 + \sqrt{3})$ A1

Answer: $x < -\ln(2 + \sqrt{3})$ or $x > \ln(2 + \sqrt{3})$ 2 marks

(b) $\sinh x$ is increasing and $\sinh(\ln 2) = \frac{3}{4}$ M1

So $x \leq \ln 2$ A1

Answer: $x \leq \ln 2$ 2 marks

18 (5 marks)

$\tanh 2x = \frac{2 \tanh x}{1 + \tanh^2 x}$, so $\frac{2t}{1+t^2} = 3t$ where $t = \tanh x$ M1 A1

$t(2 - 3 - 3t^2) = 0$, so $t(1 + 3t^2) = 0$ M1

$1 + 3t^2 > 0$ for real t , so $t = 0$ A1

$\tanh x = 0$ gives $x = 0$ A1

Answer: $x = 0$ 5 marks

19 (5 marks)

$\cosh 5x + \cosh x = 2 \cosh 3x \cosh 2x$ M1 A1

$2 \cosh 3x \cosh 2x = 6 \cosh 3x$, so $\cosh 3x (\cosh 2x - 3) = 0$ M1

$\cosh 3x \geq 1$ so $\cosh 2x = 3$ A1

$2x = \pm \ln(3 + 2\sqrt{2})$, so $x = \pm \frac{1}{2} \ln(3 + 2\sqrt{2}) = \pm \ln(1 + \sqrt{2})$ A1

Answer: $x = \pm \ln(1 + \sqrt{2})$ 5 marks

20 (6 marks)

$2 \cosh x + \sinh x = \frac{3e^x + e^{-x}}{2}$, so $3u^2 - 2ku + 1 = 0$ with $u = e^x > 0$ M1 A1

Real roots need $4k^2 - 12 \geq 0$, so $k \geq \sqrt{3}$ or $k \leq -\sqrt{3}$ M1

Positive roots need $k > 0$ (sum of roots $\frac{2k}{3} > 0$), so $k \geq \sqrt{3}$ A1

$k = 2$: $3u^2 - 4u + 1 = 0$, so $(3u - 1)(u - 1) = 0$ M1

$u = 1$ or $\frac{1}{3}$, so $x = 0$ or $x = -\ln 3$ A1

Answer: $k \geq \sqrt{3}$; $x = 0$ or $-\ln 3$ 6 marks

21 (6 marks)

In exponentials: $\frac{3(e^x + e^{-x})}{2} - 3 = e^x - e^{-x}$ M1

$3u + \frac{3}{u} - 6 = 2u - \frac{2}{u}$, so $u^2 - 6u + 5 = 0$ after multiplying by u and simplifying M1 A1

$(u - 1)(u - 5) = 0$ A1

$u = 1$ gives $x = 0$ A1

$u = 5$ gives $x = \ln 5$ A1

Answer: $x = 0$ or $x = \ln 5$ 6 marks

22 (6 marks)

$$\cosh 2x = 1 + 2 \sinh^2 x, \text{ so } 1 + 2 \sinh^2 x + 3 \sinh x = 3$$

M1

$$2 \sinh^2 x + 3 \sinh x - 2 = 0, \text{ so } (2 \sinh x - 1)(\sinh x + 2) = 0$$

M1 A1

$$\sinh x = \frac{1}{2}: x = \ln \left(\frac{1}{2} + \frac{\sqrt{5}}{2} \right) = \ln \left(\frac{1 + \sqrt{5}}{2} \right)$$

M1 A1

$$\sinh x = -2: x = \ln (-2 + \sqrt{5})$$

A1

$$\text{Answer: } x = \ln \left(\frac{1 + \sqrt{5}}{2} \right), x = \ln (\sqrt{5} - 2)$$

6 marks

23 (7 marks)

$$\text{Multiply the second equation by 2: } 6 \sinh x - 4 \sinh y = 5$$

M1

$$\text{Add to the first equation: } 7 \sinh x = \frac{13}{3} + 5 = \frac{28}{3}$$

M1 A1

$$\sinh x = \frac{4}{3}$$

A1

$$\text{Then } 4 \sinh y = \frac{13}{3} - \frac{4}{3} = 3, \text{ so } \sinh y = \frac{3}{4}$$

A1

$$x = \operatorname{arsinh} \left(\frac{4}{3} \right) = \ln \left(\frac{4}{3} + \frac{5}{3} \right) = \ln 3$$

A1

$$y = \operatorname{arsinh} \left(\frac{3}{4} \right) = \ln \left(\frac{3}{4} + \frac{5}{4} \right) = \ln 2$$

A1

$$\text{Answer: } x = \ln 3, y = \ln 2$$

7 marks

24 (6 marks)

Dividing by $\sinh x$ is not allowed unless $\sinh x \neq 0$: it loses the solutions with $\sinh x = 0$

E1

Check: $x = 0$ gives $\sinh 0 = 0 = 4 \sinh 0$, so $x = 0$ satisfies the equation but is missing from the student's answer

B1

$$\text{Correct: } 2 \sinh x \cosh x - 4 \sinh x = 0, \text{ so } 2 \sinh x (\cosh x - 2) = 0$$

M1 A1

$$\sinh x = 0 \text{ gives } x = 0$$

A1

$$\cosh x = 2 \text{ gives } x = \pm \ln (2 + \sqrt{3}), \text{ so the solutions are } x = 0, \pm \ln (2 + \sqrt{3})$$

A1

$$\text{Answer: } x = 0, \pm \ln (2 + \sqrt{3})$$

6 marks

25 (7 marks)

$$\text{(a) } 1 + 2 \sinh^2 x = 4 \sinh x + k, \text{ so } 2s^2 - 4s + 1 - k = 0 \text{ with } s = \sinh x$$

M1 A1

$$s \text{ is real, so the discriminant } 16 - 8(1 - k) \geq 0$$

M1

$$8 + 8k \geq 0, \text{ so } k \geq -1$$

A1

$$\text{Answer: if the equation has a solution, } k \geq -1$$

4 marks

$$\text{(b) } 2s^2 - 4s = 0, \text{ so } s = 0 \text{ or } s = 2$$

M1

$$\sinh x = 0 \text{ gives } x = 0$$

A1

$$\sinh x = 2 \text{ gives } x = \ln (2 + \sqrt{5})$$

A1

$$\text{Answer: } x = 0 \text{ or } x = \ln (2 + \sqrt{5})$$

3 marks

26 (6 marks)

$$\text{(a) } x = 0: y = 12 \cosh 0 = 12 \text{ m}$$

B1

$$\text{Answer: } 12 \text{ m}$$

1 mark

$$\text{(b) } 12 \cosh \left(\frac{x}{12} \right) = 13, \text{ so } \cosh \left(\frac{x}{12} \right) = \frac{13}{12}$$

M1

$$\sinh^2 \left(\frac{x}{12} \right) = \frac{169}{144} - 1 = \frac{25}{144}, \text{ so } \sinh \left(\frac{x}{12} \right) = \pm \frac{5}{12}$$

M1 A1

$$e^{x/12} = \cosh + \sinh = \frac{13}{12} \pm \frac{5}{12}, \text{ that is } \frac{3}{2} \text{ or } \frac{2}{3}$$

M1

$$x = \pm 12 \ln \left(\frac{3}{2} \right); \text{ distance} = 24 \ln \left(\frac{3}{2} \right) = 9.73 \text{ m}$$

A1

$$\text{Answer: } 24 \ln \left(\frac{3}{2} \right) = 9.73 \text{ m}$$

5 marks

27 (9 marks)

(a) With $c = \cosh x$, $\cosh 2x = 2c^2 - 1$, so $2c^2 - 10c - 1 = -1$, that is $2c(c - 5) = 0$ M1

$\cosh x \geq 1$, so $c = 5$ ($c = 0$ is rejected) A1

$$x = \pm \operatorname{arcosh} 5 = \pm \ln(5 + \sqrt{24}) = \pm \ln(5 + 2\sqrt{6})$$
 A1

Answer: $x = \pm \ln(5 + 2\sqrt{6})$ 3 marks

(b) $f(x) = k$ becomes $2c^2 - 10c - (1 + k) = 0$, with $c = \cosh x \geq 1$; each root $c > 1$ gives two values of x , and $c = 1$ gives one ($x = 0$) M1 A1

(i) $k = -14$: discriminant $100 + 8(1 + k) = -4 < 0$, so no solutions B1

(ii) $k = -\frac{27}{2}$: discriminant 0 , $c = \frac{5}{2} > 1$, so two solutions B1

(iii) $k = -10$: $c = \frac{5 \pm \sqrt{7}}{2}$, both greater than 1 (about 3.82 and 1.18), so four solutions B1

(iv) $k = -9$: $c = 1$ or $c = 4$, giving $x = 0$ and two more, so three solutions; (v) $k = 0$: $c = \frac{5 \pm 3\sqrt{3}}{2}$, B1

only $c \approx 5.10 > 1$ is valid, so two solutions

Answer: (i) 0, (ii) 2, (iii) 4, (iv) 3, (v) 2 6 marks

28 (9 marks)

(a) $\sinh(x + \ln 2) = \sinh x \cosh(\ln 2) + \cosh x \sinh(\ln 2)$ M1

$$\cosh(\ln 2) = \frac{5}{4} \text{ and } \sinh(\ln 2) = \frac{3}{4}$$
 A1

$$\text{Therefore } 4 \sinh(x + \ln 2) = 4 \times \frac{3}{4} \cosh x + 4 \times \frac{5}{4} \sinh x = 5 \sinh x + 3 \cosh x$$
 A1

Answer: $5 \sinh x + 3 \cosh x \equiv 4 \sinh(x + \ln 2)$ 3 marks

(b) $\sinh(x + \ln 2) = 3$, so $x + \ln 2 = \operatorname{arsinh} 3 = \ln(3 + \sqrt{10})$ M1 A1

$$x = \ln(3 + \sqrt{10}) - \ln 2 = \ln\left(\frac{3 + \sqrt{10}}{2}\right)$$
 A1

Answer: $x = \ln\left(\frac{3 + \sqrt{10}}{2}\right)$ 3 marks

(c) $\sinh(x + \ln 2) = -2$ B1

$$x + \ln 2 = \operatorname{arsinh}(-2) = \ln(-2 + \sqrt{5})$$
 M1

$$x = \ln(\sqrt{5} - 2) - \ln 2 = \ln\left(\frac{\sqrt{5} - 2}{2}\right)$$
 A1

Answer: $x = \ln\left(\frac{\sqrt{5} - 2}{2}\right)$ 3 marks

29 (9 marks)

(a) $7 \cosh x = 5 + 5 \sinh x$, so $\frac{7(e^x + e^{-x})}{2} = 5 + \frac{5(e^x - e^{-x})}{2}$ M1

$$7u + \frac{7}{u} = 10 + 5u - \frac{5}{u}, \text{ so } 2u^2 - 10u + 12 = 0 \text{ with } u = e^x$$
 M1 A1

$$u^2 - 5u + 6 = 0, \text{ so } u = 2 \text{ or } 3$$
 A1

$$x = \ln 2: y = 7 \times \frac{5}{4} = \frac{35}{4}; x = \ln 3: y = 7 \times \frac{5}{3} = \frac{35}{3}. \text{ Points } (\ln 2, \frac{35}{4}) \text{ and } (\ln 3, \frac{35}{3})$$
 A1

Answer: $(\ln 2, \frac{35}{4})$ and $(\ln 3, \frac{35}{3})$ 5 marks

(b) Area = $\int_{\ln 2}^{\ln 3} (5 + 5 \sinh x - 7 \cosh x) dx$ M1

$$= [5x + 5 \cosh x - 7 \sinh x]_{\ln 2}^{\ln 3}$$
 A1

$$= 5 \ln 3 + 5 \times \frac{5}{3} - 7 \times \frac{4}{3} - 5 \ln 2 - 5 \times \frac{5}{4} + 7 \times \frac{3}{4}$$
 M1

$$= 5 \ln\left(\frac{3}{2}\right) + \frac{25 - 28}{3} + \frac{-25 + 21}{4} = 5 \ln\left(\frac{3}{2}\right) - 1 - 1 = 5 \ln\left(\frac{3}{2}\right) - 2$$
 A1

Answer: $5 \ln\left(\frac{3}{2}\right) - 2$ 4 marks

30 (11 marks)

$$(a) v = \frac{ds}{dt} = 10 \sinh 2t - 6 \cosh 2t$$

M1 A1

$$v = 0: \tanh 2t = \frac{3}{5}$$

M1

$$2t = \operatorname{artanh}\left(\frac{3}{5}\right) = \ln 2, \text{ so } t = \frac{1}{2} \ln 2$$

A1

$$s = 5 \times \frac{5}{4} - 3 \times \frac{3}{4} = 4 \text{ m}$$

A1

$$\text{Answer: } t = \frac{1}{2} \ln 2; s = 4 \text{ m}$$

5 marks

$$(b) \frac{d^2s}{dt^2} = 20 \cosh 2t - 12 \sinh 2t = 4s > 0 \text{ when } s = 4$$

M1

So the stationary point is a minimum, $s = 4$

A1

Answer: Minimum

2 marks

$$(c) \text{ Let } u = e^{2t}: 5 \cosh 2t - 3 \sinh 2t = u + \frac{4}{u}$$

M1

$$u + \frac{4}{u} = 7, \text{ so } u^2 - 7u + 4 = 0$$

M1

$$u = \frac{7 \pm \sqrt{33}}{2}; u < 1 \text{ for the smaller root } (u = 0.63), \text{ which gives } t < 0 \text{ and is rejected}$$

A1

$$t = \frac{1}{2} \ln \left(\frac{7 + \sqrt{33}}{2} \right) \approx 0.926 \text{ s}$$

A1

$$\text{Answer: } t = \frac{1}{2} \ln \left(\frac{7 + \sqrt{33}}{2} \right) = 0.926 \text{ s}$$

4 marks