

Hyperbolic Sine, Cosine and Tangent

- 1 Write down the definitions of $\sinh x$ and $\cosh x$ in terms of exponentials. Hence evaluate $\sinh 0$, $\cosh 0$ and $\tanh 0$. [3]
- 2 Use your calculator to find $\sinh 2$ and $\cosh 2$, giving each answer to 3 significant figures. [2]
- 3 Show that $\sinh (\ln 3) = \frac{4}{3}$. [2]
- 4 State the range of the function $f(x) = \cosh x$, $x \in \mathbb{R}$, and the range of $g(x) = \tanh x$, $x \in \mathbb{R}$. [2]
- 5 By writing $\tanh x$ in terms of exponentials, show that $\tanh x = \frac{e^{2x} - 1}{e^{2x} + 1}$. [2]
- 6 The functions $\operatorname{sech} x$, $\operatorname{cosech} x$ and $\operatorname{coth} x$ are defined as $\frac{1}{\cosh x}$, $\frac{1}{\sinh x}$ and $\frac{1}{\tanh x}$. Find the value of each at $x = 1$, to 3 significant figures. [3]
- 7 Show that $\cosh (\ln 2) = \frac{5}{4}$, and find the exact value of $\tanh (\ln 2)$. [3]
- 8 Using the definitions in terms of exponentials, show that $\sinh (-x) = -\sinh x$ and $\cosh (-x) = \cosh x$ for all real x . [3]
- 9 Solve the equation $\sinh x = 0$, and the equation $\cosh x = 1$, using the definitions in terms of exponentials. [3]
- 10 As $x \rightarrow \infty$, find the limits of $\tanh x$ and $\coth x$. Hence state the equations of the horizontal asymptotes of $y = \tanh x$, and write down the equation of the vertical asymptote of $y = \coth x$. [3]
- 11 Given that $\sinh x = \frac{3}{4}$, use the definition of $\sinh x$ to find the exact value of x . [4]
- 12 Sketch the graph of $y = 2 + \cosh x$, stating the coordinates of the point where it crosses the y -axis and of its minimum point. [3]
- 13 A cable hangs between two supports so that its height above the ground, in metres, is modelled by $h = 5 \cosh \left(\frac{x}{5}\right)$, $-10 \leq x \leq 10$, where x is the horizontal distance in metres from the midpoint between the supports.
Find the height of the cable above the ground at its lowest point and at each support, giving answers to 3 significant figures. [4]
- 14 Show that, for $k > 0$, $\sinh (\ln k) = \frac{k^2 - 1}{2k}$. Hence find the exact value of $\sinh (\ln 5)$ and of $\cosh (\ln 5)$. [4]
- 15 Solve the equation $\cosh x = 2$, giving the solutions in exact logarithmic form. [4]
- 16 Given that $\operatorname{sech} w = \frac{5}{13}$ and $w > 0$, find the exact values of $\cosh w$, $\sinh w$ and $\tanh w$, and show that $w = \ln 5$. [5]
- 17 Show that $\cosh x - 1 = \frac{(e^{x/2} - e^{-x/2})^2}{2}$. Hence prove that $\cosh x \geq 1$ for all real x , and state when equality holds. [4]

- 18** Sketch the graph of $y = 2 \tanh x - 1$, showing the coordinates of any axis intercepts and the equations of the asymptotes. [4]
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- 19** Show that $\cosh x + \sinh x = e^x$. Hence show that $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$ for all positive integers n . [5]
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- 20** The arch of a footbridge has height, in metres, $y = 25 - 5 \cosh\left(\frac{x}{10}\right)$, where x is the horizontal distance in metres from the centre of the arch and the ground level is $y = 0$. Find the width of the arch at ground level, giving the exact answer in terms of natural logarithms and also to 3 significant figures. [5]
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- 21** Given that $\cosh x = \frac{5}{4}$, use the definition of $\cosh x$ to find the two possible values of e^x . Hence find the exact values of $\sinh x$ and $\tanh x$ in each case. [6]
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- 22** Let $f(x) = \tanh x$ for $x \in \mathbb{R}$.
 (a) Show that $\tanh x = 1 - \frac{2}{e^{2x} + 1}$. [2]
 (b) Hence prove that $-1 < \tanh x < 1$ for all x , and that $\tanh x$ increases as x increases. [4]
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- 23** A uniform chain hangs between two vertical poles of equal height. In a model, the shape of the chain is $y = 15 \cosh\left(\frac{x}{15}\right)$, where x and y are measured in metres, the origin O is on the ground directly below the lowest point of the chain, and each pole has its top at the point where $y = 25$.
 (a) Write down the height above the ground of the lowest point of the chain. [1]
 (b) Show that the poles are at $x = \pm 15 \ln 3$. [3]
 (c) Find the distance between the poles to 3 significant figures. [2]
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- 24** A student attempts to solve $\tanh x = 2$ and writes: ' $e^{2x} - 1 = 2(e^{2x} + 1)$, so $-e^{2x} = 3$, so $e^{2x} = -3$ and $x = \frac{1}{2} \ln(-3)$.'
 Explain what is wrong with the last step and what this tells you about the equation $\tanh x = 2$.
 Then show that $\tanh x = k$ has a solution $x = \frac{1}{2} \ln\left(\frac{1+k}{1-k}\right)$ if and only if $-1 < k < 1$. [6]
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- 25** The function f is defined by $f(x) = 3 \cosh x - 4$, $x \in \mathbb{R}$.
 (a) State the range of f . [2]
 (b) Solve $f(x) = 5$, giving the solutions in exact logarithmic form. [4]
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- 26** Let $k > 1$.
 (a) Show that $\coth(\ln k) = \frac{k^2 + 1}{k^2 - 1}$. [3]
 (b) Hence find the exact value of $\coth(\ln 3)$, and find k such that $\coth(\ln k) = \frac{13}{12}$. [4]
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- 27** The angle θ , where $0 < \theta < \frac{\pi}{2}$, is defined for a real number $x > 0$ by $\tan \theta = \sinh x$.
 Show that $\cos \theta = \frac{1}{\cosh x}$, and that $\sin \theta = \tanh x$. [8]
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- 28** Consider the equation $5 \cosh x + 3 \sinh x = k$, where k is a constant.
 (a) Show that $5 \cosh x + 3 \sinh x = 4e^x + e^{-x}$. [2]
 (b) Show that the equation has real solutions if and only if $k \geq 4$. [5]
 (c) Solve the equation when $k = 5$, giving the solutions in exact form. [3]
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- 29** Show that $\sinh(\ln(x + \sqrt{x^2 + 1})) = x$ for all real x .
Show also that $\cosh(\ln(x + \sqrt{x^2 - 1})) = x$ for $x \geq 1$. [9]
- 30** The real numbers a and b satisfy $\tanh a = \frac{1}{2}$ and $\tanh b = \frac{1}{3}$.
(a) Show that $e^{2a} = 3$ and $e^{2b} = 2$. [3]
(b) Hence find the exact value of $\tanh(a + b)$. [3]
(c) Hence show that $a + b = \frac{1}{2} \ln 6$. [2]

Hyperbolic Identities

- 1** Given that $\sinh x = \frac{3}{4}$ and $x > 0$, use the identity $\cosh^2 x - \sinh^2 x = 1$ to find the exact value of $\cosh x$. [2]
- 2** Divide both sides of $\cosh^2 x - \sinh^2 x = 1$ by $\cosh^2 x$ to show that $1 - \tanh^2 x = \operatorname{sech}^2 x$. [2]
- 3** Write each of the following as a single hyperbolic function.
(a) $2 \sinh 3x \cosh 3x$ [1]
(b) $\cosh^2 4x + \sinh^2 4x$ [1]
- 4** Given that $\tanh x = \frac{3}{5}$, find $\operatorname{sech}^2 x$ and hence the exact value of $\cosh x$ (x real). [2]
- 5** Given that $\cosh x = 3$, use the identity $\cosh 2x = 2\cosh^2 x - 1$ to find the value of $\cosh 2x$. [2]
- 6** Verify that $\sinh(A + B) = \sinh A \cosh B + \cosh A \sinh B$ when $A = \ln 2$ and $B = \ln 3$, by finding the exact value of each side. [3]
- 7** Using the definitions of $\cosh x$ and $\sinh x$ in terms of exponentials, prove that $\cosh^2 x - \sinh^2 x = 1$. [3]
- 8** Using the definitions in terms of exponentials, prove that $\sinh 2x = 2 \sinh x \cosh x$. [3]
- 9** Show that $-\ln(1 - \tanh x) = x + \ln(\cosh x)$ for all real x . [4]
- 10** Given that $5 \sinh x - \cosh x = 1$, find the exact value of $\tanh x$. [5]
- 11** Use the identities for $\sinh 2x$ and $\cosh 2x$ to prove that $\tanh 2x = \frac{2 \tanh x}{1 + \tanh^2 x}$. [4]
- 12** Simplify $(\cosh x + \sinh x)(\cosh x - \sinh x)$. Hence show that, if $\cosh x + \sinh x = 3$, then $\cosh x - \sinh x = \frac{1}{3}$, and find the exact values of $\cosh x$ and $\sinh x$. [4]
- 13** Solve the equation $\cosh 2x = 3$, giving the answers in exact logarithmic form. [4]
- 14** Using the definitions in terms of exponentials, prove that $\sinh(A + B) = \sinh A \cosh B + \cosh A \sinh B$. [4]
- 15** Throughout this question you may use $\cosh(\ln 2) = \frac{5}{4}$ and $\sinh(\ln 2) = \frac{3}{4}$.
(a) Prove that $\cosh(A + B) = \cosh A \cosh B + \sinh A \sinh B$ using the definitions in terms of exponentials. [3]
(b) Hence solve $\cosh(x + \ln 2) = 3 \sinh x$, giving the answer as an exact logarithm. [4]

- 16** Solve the equation $\cosh 2x + 3 = 5 \cosh x$, giving the solutions as exact logarithms. [6]
- 17** Solve the equation $2 \operatorname{sech}^2 x + 3 \tanh x = 3$, giving the answer in exact logarithmic form. [5]
- 18** Prove that $\sinh 3x = 3 \sinh x + 4 \sinh^3 x$. [5]
- 19** Show that $\cosh X + \cosh Y = 2 \cosh\left(\frac{X+Y}{2}\right) \cosh\left(\frac{X-Y}{2}\right)$.
Hence write $\cosh 5x + \cosh x$ as a product. [4]
- 20** Find a Cartesian equation for each of the following curves, given parametrically.
(a) $x = 2 \sinh t, y = \cosh 2t$ [2]
(b) $x = 3 \cosh t, y = 2 \sinh t$ [3]
- 21** Throughout this question x is real.
(a) Prove that $\cosh 3x = 4 \cosh^3 x - 3 \cosh x$. [4]
(b) Hence solve $\cosh 3x = 13 \cosh x$, giving the solutions in exact logarithmic form. [4]
- 22** Express $\cosh^4 x$ in the form $a \cosh 4x + b \cosh 2x + c$, where a, b and c are constants to be found. [6]
- 23** Let $t = \tanh\left(\frac{x}{2}\right)$.
(a) Show that $\sinh x = \frac{2t}{1-t^2}$ and $\cosh x = \frac{1+t^2}{1-t^2}$. [4]
(b) Hence solve $5 \cosh x - 4 \sinh x = 3$, giving the answer as an exact logarithm. [5]
- 24** A student solves $\cosh 2x = 7$ as follows: ' $\cosh 2x = 2 \cosh^2 x + 1$, so $2 \cosh^2 x + 1 = 7$, so $\cosh^2 x = 3$ and $\cosh x = \sqrt{3}$, giving $x = \pm \ln(\sqrt{3} + \sqrt{2})$.'
Show by substituting $x = 0$ that the identity used is wrong, give the correct identity, and find the correct solutions. [6]
- 25** Show that $\sinh 3x + \sinh x = 2 \sinh 2x \cosh x$, and hence solve $\sinh 3x + \sinh x = 4 \sinh 2x$, giving the solutions in exact form. [7]
- 26** Prove that $\tanh 3x = \frac{3 \tanh x + \tanh^3 x}{1 + 3 \tanh^2 x}$. [7]
- 27** Given the identities $\cosh 3x = 4 \cosh^3 x - 3 \cosh x$ and $\cosh 2x = 2 \cosh^2 x - 1$, prove that $\cosh 5x = 16 \cosh^5 x - 20 \cosh^3 x + 5 \cosh x$. [8]
- 28** Let $x \neq 0$ be a real number and let n be a positive integer.
(a) Show that $2 \sinh\left(\frac{x}{2}\right) \sinh(rx) = \cosh\left((r + \frac{1}{2})x\right) - \cosh\left((r - \frac{1}{2})x\right)$. [3]
(b) Hence prove that $\sum_{r=1}^n \sinh(rx) = \frac{\sinh\left(\frac{nx}{2}\right) \sinh\left(\frac{(n+1)x}{2}\right)}{\sinh\left(\frac{x}{2}\right)}$. [6]
- 29** A curve is given parametrically by $x = 2 \cosh t + 3 \sinh t, y = \cosh t + 2 \sinh t, t \in \mathbb{R}$.
Show that the Cartesian equation of the curve is $3x^2 - 8xy + 5y^2 = 1$, and show that this can be written in the form $(3x - 5y)(x - y) = 1$. [8]
- 30** Show that $\sinh x + \sinh 3x + \sinh 5x = \sinh 3x (1 + 2 \cosh 2x)$. Hence solve $\sinh x + \sinh 3x + \sinh 5x = 4 \sinh 3x$, giving the exact solutions in terms of natural logarithms and simplifying fully. [9]

Inverse Hyperbolic Functions

- 1 Use your calculator to find $\operatorname{arsinh} 2$, $\operatorname{arcosh} 3$ and $\operatorname{artanh} \frac{1}{2}$, giving each answer to 3 significant figures. [3]
- 2 State the largest possible domain of each of the functions $\operatorname{arsinh} x$, $\operatorname{arcosh} x$ and $\operatorname{artanh} x$. [3]
- 3 Write down the exact values of $\operatorname{arsinh} 0$, $\operatorname{arcosh} 1$ and $\operatorname{artanh} 0$. [3]
- 4 Given that $y = \operatorname{arsinh} \left(\frac{4}{3}\right)$, find the exact values of $\cosh y$ and $\tanh y$. [3]
- 5 Use the logarithmic form $\operatorname{artanh} x = \frac{1}{2} \ln \left(\frac{1+x}{1-x}\right)$ to show that $\operatorname{artanh} \left(\frac{3}{5}\right) = \ln 2$. [2]
- 6 Use the logarithmic form $\operatorname{arsinh} x = \ln (x + \sqrt{x^2 + 1})$ to show that $\operatorname{arsinh} \left(\frac{3}{4}\right) = \ln 2$. [2]
- 7 Let $y = \operatorname{arsinh} x$. By writing $x = \sinh y$ in terms of exponentials, prove that $\operatorname{arsinh} x = \ln (x + \sqrt{x^2 + 1})$. [4]
- 8 Let $y = \operatorname{arcosh} x$, where $x \geq 1$ and $y \geq 0$. By writing $x = \cosh y$ in terms of exponentials, prove that $\operatorname{arcosh} x = \ln (x + \sqrt{x^2 - 1})$. [4]
- 9 Let $y = \operatorname{artanh} x$, where $-1 < x < 1$. By writing $x = \tanh y$ in terms of exponentials, prove that $\operatorname{artanh} x = \frac{1}{2} \ln \left(\frac{1+x}{1-x}\right)$. [4]
- 10 Solve $\operatorname{arsinh} x = \ln 3$, giving x as an exact fraction. [3]
- 11 Find the exact value of $\sinh (2 \operatorname{arsinh} 3)$. [3]
- 12 Solve the equation $\operatorname{artanh} x = \ln 5$ without using a calculator, giving x as an exact fraction. [3]
- 13 Show that $\operatorname{arcosh} \left(\frac{13}{5}\right) = \ln 5$. [3]
- 14 Solve $\operatorname{arsinh} x = 2 \operatorname{arsinh} \left(\frac{3}{4}\right)$, giving x as an exact fraction. [4]
- 15 Sketch the graph of $y = \operatorname{arcosh} x$, stating its domain and range and the coordinates of any intercepts with the axes. [4]
- 16 State the domain and range of each of the following functions.
 - (a) $f(x) = 3 \operatorname{arsinh} (x - 1) + 2$ [1]
 - (b) $g(x) = \operatorname{arcosh} x + 1$ [2]
 - (c) $h(x) = \operatorname{artanh} \left(\frac{x}{2}\right)$ [2]
- 17 Show that, for $x > 0$, $\operatorname{artanh} \left(\frac{x^2 - 1}{x^2 + 1}\right) = \ln x$. [4]
- 18 The function f is defined by $f(x) = \operatorname{arcosh} (2x - 3)$. State the largest possible domain of f , and solve $f(x) = \ln (2 + \sqrt{3})$, giving x as an exact fraction. [4]
- 19 Prove that, for $-1 < x < 1$ and $-1 < y < 1$, $\operatorname{artanh} x + \operatorname{artanh} y = \operatorname{artanh} \left(\frac{x + y}{1 + xy}\right)$. [5]

- 20** Throughout this question $k > 0$.
- (a) Show that $\tanh\left(\frac{1}{2} \ln k\right) = \frac{k-1}{k+1}$ for $k > 0$. [2]
- (b) Hence solve the equation $\operatorname{artanh} x = \frac{1}{2} \ln 7$. [2]
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- 21** The function f is defined by $f(x) = 2 \sinh 3x - 1$, $x \in \mathbb{R}$. Find an expression for $f^{-1}(x)$, giving your answer in terms of natural logarithms and simplifying as far as possible, and state the domain and range of f^{-1} . [6]
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- 22** Throughout this question θ lies in the interval $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.
- (a) Show that $\operatorname{arsinh}(\tan \theta) = \ln(\sec \theta + \tan \theta)$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. [3]
- (b) Hence solve $\operatorname{arsinh}(\tan \theta) = \ln 3$, giving θ to 3 significant figures. [4]
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- 23** Show that, for $-1 < x < 1$, $\operatorname{artanh}\left(\frac{2x}{1+x^2}\right) = 2 \operatorname{artanh} x$. [6]
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- 24** Let $y = \operatorname{arcosh} x$, where $x \geq 1$. Show that $y = 2 \operatorname{arsinh} \sqrt{\frac{x-1}{2}}$. [6]
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- 25** A student claims that the equation $\operatorname{arcosh} x = -1$ has the solution $x = \cosh(-1) = \cosh 1 \approx 1.54$.
- (a) Show that the student's value of x does not satisfy the equation, and explain the error. [3]
- (b) Find the range of values of k for which $\operatorname{arcosh} x = k$ has a solution, and give the solution in terms of k . [3]
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- 26** Solve $\operatorname{arsinh} x + \operatorname{arsinh}\left(\frac{4}{3}\right) = \operatorname{arsinh}\left(\frac{15}{8}\right)$, giving x as an exact fraction. [7]
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- 27** Consider the series $\operatorname{artanh}\left(\frac{1}{2}\right) + \operatorname{artanh}\left(\frac{1}{4}\right) + \operatorname{artanh}\left(\frac{1}{6}\right) + \dots + \operatorname{artanh}\left(\frac{1}{2n}\right)$.
- (a) Show that $\operatorname{artanh}\left(\frac{1}{2r}\right) = \frac{1}{2} \ln\left(\frac{2r+1}{2r-1}\right)$ for every integer $r \geq 1$. [2]
- (b) Hence show that $\sum_{r=1}^n \operatorname{artanh}\left(\frac{1}{2r}\right) = \frac{1}{2} \ln(2n+1)$. [4]
- (c) Find the least value of n for which the sum exceeds 2. [3]
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- 28** Let $x > 0$ and let $y = \operatorname{arsinh} x$. Prove that
- (a) $y = \operatorname{arcosh} \sqrt{1+x^2}$ [3]
- (b) $y = \operatorname{artanh}\left(\frac{x}{\sqrt{1+x^2}}\right)$ [3]
- (c) Hence show that $\operatorname{arsinh} 2 = \operatorname{arcosh} \sqrt{5} = \operatorname{artanh}\left(\frac{2}{\sqrt{5}}\right)$, and give this common value as a natural logarithm. [3]
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- 29** Solve the simultaneous equations
- $$\operatorname{arcosh} x + \operatorname{arcosh} y = \ln 6,$$
- $$\operatorname{arcosh} x - \operatorname{arcosh} y = \ln\left(\frac{3}{2}\right),$$
- where $x \geq 1$ and $y \geq 1$. Give x and y as exact fractions. [8]
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- 30** Solve the equation $\operatorname{arcosh} x - \operatorname{arsinh} x = -\ln 2$, where $x \geq 1$, giving the exact solution. [9]
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Hyperbolic Differentiation

- 1 Differentiate each of the following with respect to x .
 - (a) $\sinh 5x$ [1]
 - (b) $\cosh\left(\frac{x}{2}\right)$ [1]
- 2 Find $\frac{dy}{dx}$ when $y = \tanh 2x$. [2]
- 3 Differentiate $y = x^2 \cosh x$ with respect to x . [2]
- 4 Given that $y = \operatorname{arsinh} 3x$, find $\frac{dy}{dx}$. [2]
- 5 Use the definition $\sinh x = \frac{e^x - e^{-x}}{2}$ to prove that $\frac{d}{dx}(\sinh x) = \cosh x$. [2]
- 6 Find the gradient of the curve $y = \cosh x$ at the point where $x = \ln 3$. [2]
- 7 Using the quotient rule and the derivatives of $\sinh x$ and $\cosh x$, prove that $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$. [3]
- 8 Prove that $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$. [3]
- 9 Differentiate $y = e^x \sinh 2x$ with respect to x . [3]
- 10 Differentiate each of the following with respect to x , simplifying your answers.
 - (a) $\ln(\cosh x)$ [1]
 - (b) $\ln(\sinh 3x)$ [2]
- 11 Given that $y = \cosh^2 x$, show that $\frac{dy}{dx} = \sinh 2x$. [3]
- 12 Differentiate each of the following with respect to x , simplifying your answers.
 - (a) $\operatorname{arcosh} 2x$ [2]
 - (b) $\operatorname{artanh}\left(\frac{x}{2}\right)$ [2]
- 13 Find an equation of the tangent to the curve $y = \sinh x$ at the point where $x = \ln 2$. [4]
- 14 Let $y = \operatorname{arsinh} x$. Prove that $\frac{dy}{dx} = \frac{1}{\sqrt{x^2 + 1}}$. [4]
- 15 Let $y = \operatorname{artanh} x$, where $-1 < x < 1$. Prove that $\frac{dy}{dx} = \frac{1}{1 - x^2}$. [4]
- 16 A curve has equation $y = 5 \cosh x - 4 \sinh x$. Find the exact coordinates of the stationary point of the curve and determine its nature. [6]
- 17 Given that $y = x \operatorname{arsinh} x - \sqrt{x^2 + 1}$, show that $\frac{dy}{dx} = \operatorname{arsinh} x$. [5]
- 18 Given that $y = \tanh(\ln x)$, where $x > 0$, show that $\frac{dy}{dx} = \frac{4x}{(x^2 + 1)^2}$. [5]

- 19** A curve has parametric equations $x = t - \tanh t$, $y = \operatorname{sech} t$, $t > 0$. Show that $\frac{dy}{dx} = -\operatorname{cosech} t$. [5]
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- 20** The curve C has equation $y = \operatorname{sech} x$.
Show that $\frac{d^2y}{dx^2} = \operatorname{sech} x (\tanh^2 x - \operatorname{sech}^2 x)$, and hence find the exact coordinates of the points of inflection of C. [7]
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- 21** The curve C has equation $y = 7x - \cosh x - 5 \sinh x$.
Find the exact coordinates of the stationary points of C and determine their nature. [9]
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- 22** A cable hangs between two supports. In a model, its height above the ground is $y = 20 \cosh\left(\frac{x}{20}\right)$ metres, where x metres is the horizontal distance from the lowest point.
(a) Find the gradient of the cable at the point where $x = 15$, and the angle the cable makes with the horizontal there, to the nearest 0.1° . [3]
(b) Find the exact value of x at the point where the cable has gradient 1, and the height of the cable there. [4]
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- 23** The following parts are about the functions $\arctan(\sinh x)$ and $2 \arctan(e^x)$.
(a) Show that $\frac{d}{dx}[\arctan(\sinh x)] = \operatorname{sech} x$. [3]
(b) Show that $\frac{d}{dx}[2 \arctan(e^x)] = \operatorname{sech} x$. [3]
(c) Hence find the constant c such that $\arctan(\sinh x) = 2 \arctan(e^x) + c$. [3]
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- 24** Given that $y = (\operatorname{arsinh} x)^2$, show that $(1 + x^2) \frac{d^2y}{dx^2} + x \frac{dy}{dx} = 2$. [7]
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- 25** A student writes the following derivatives.
(i) $\frac{d}{dx}(\cosh 3x) = \sinh 3x$ (ii) $\frac{d}{dx}(\tanh x) = \operatorname{sech} x$ (iii) $\frac{d}{dx}(x \sinh x) = \cosh x$
Identify the error in each statement and write down the correct derivative. [6]
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- 26** Let $f(x) = \ln(\cosh x)$. Find $f'(x)$, $f''(x)$, $f'''(x)$ and $f^{(4)}(x)$, and hence show that the Maclaurin series of $f(x)$ begins $\frac{x^2}{2} - \frac{x^4}{12} + \dots$. [7]
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- 27** The curve C has equation $y = 4x \operatorname{sech} x$, $x \geq 0$.
(a) Show that C has a stationary point where $x \tanh x = 1$. [4]
(b) Show that the equation $x \tanh x = 1$ has a root between 1 and 1.5. [2]
(c) Using the Newton-Raphson method with $x_0 = 1.2$, find the root to 3 decimal places, and hence the maximum value of y to 3 significant figures. [5]
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- 28** The curve H has parametric equations $x = 2 \cosh t$, $y = 3 \sinh t$, $t \in \mathbb{R}$. The point P on H has parameter p , where $p \neq 0$.
(a) Show that the tangent to H at P has equation $3x \cosh p - 2y \sinh p = 6$. [4]
(b) The tangent meets the line $y = \frac{3}{2}x$ at A and the line $y = -\frac{3}{2}x$ at B. Find the coordinates of A and B in terms of p . [4]
(c) Hence show that P is the midpoint of AB. [2]
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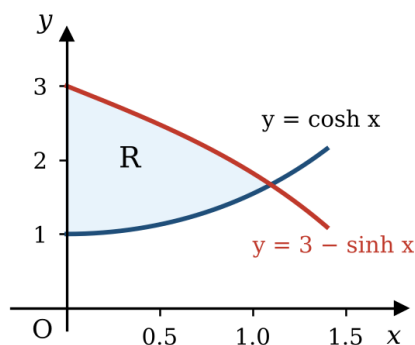
- 29** A chain hangs in the shape of the curve $y = a \cosh\left(\frac{x}{a}\right)$, where a is a positive constant.
- (a) Show that $a \frac{d^2y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$. [3]
- (b) For $a = 5$, find the gradient of the chain at each point where the height is 13, giving exact values. [4]
- (c) Find the acute angle between the chain and the horizontal at these points, to the nearest 0.1° . [1]
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- 30** Let $g(x) = \cosh x \sin x + \sinh x \cos x$.
- (a) Show that $g'(x) = 2 \cosh x \cos x$. [3]
- (b) Show that $g'''(x) = -4g(x)$. [5]
- (c) Hence find the Maclaurin series of $g(x)$ up to and including the term in x^5 . [3]

Hyperbolic Integration

- 1** Find each of the following integrals.
- (a) $\int \sinh 3x \, dx$ [1]
- (b) $\int \cosh(2x + 1) \, dx$ [1]
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- 2** Find each of the following integrals.
- (a) $\int \operatorname{sech}^2 x \, dx$ [1]
- (b) $\int 4 \operatorname{sech}^2 2x \, dx$ [1]
-
- 3** Evaluate $\int_0^1 \cosh x \, dx$, giving your answer to 3 significant figures. [2]
-
- 4** Show that $\int \tanh x \, dx = \ln(\cosh x) + c$. [2]
-
- 5** Write down each of the following integrals.
- (a) $\int \frac{1}{\sqrt{x^2 + 1}} \, dx$ [1]
- (b) $\int \frac{1}{\sqrt{x^2 + 4}} \, dx$ [1]
-
- 6** Show that $\int_0^{\ln 2} \sinh x \, dx = \frac{1}{4}$. [2]
-
- 7** Show that $\int_0^{\ln 3} \cosh 2x \, dx = \frac{20}{9}$. [3]
-
- 8** Find $\int \cosh^2 x \, dx$. [3]
-
- 9** Find the exact value of $\int_0^1 \sinh^2 x \, dx$. [3]
-
- 10** Show that $\int_0^3 \frac{1}{\sqrt{x^2 + 16}} \, dx = \ln 2$. [3]
-
- 11** Find the exact value of $\int_5^{15} \frac{1}{\sqrt{x^2 - 9}} \, dx$, giving your answer as a single natural logarithm. [4]
-
- 12** Show that $\int_0^{1/2} \frac{1}{1 - x^2} \, dx = \frac{1}{2} \ln 3$. [3]

- 13** Find the exact area of the region bounded by the curve $y = \sinh 2x$, the x -axis and the line $x = \ln 2$. [4]
- 14** Use integration by parts to find $\int x \sinh x \, dx$. [4]
- 15** Use the substitution $x = \sinh u$ to show that $\int \frac{1}{\sqrt{x^2 + 1}} \, dx = \operatorname{arsinh} x + c$. [4]
- 16** Given that $\frac{dy}{dx} = \cosh x \sqrt{1 + y^2}$ and that $y = 0$ when $x = 0$, find y in terms of x . [5]
- 17** Find the exact value of $\int_0^{\ln 2} x \cosh x \, dx$. [5]
- 18** Show that $\int_2^3 \frac{1}{x^2 - 1} \, dx = \frac{1}{2} \ln\left(\frac{3}{2}\right)$. [5]
- 19** Show that $\operatorname{sech} x = \frac{2e^x}{e^{2x} + 1}$. Hence find $\int \operatorname{sech} x \, dx$. [5]
- 20** Find the exact value of $\int_0^1 x \operatorname{sech}^2 x \, dx$. [5]
- 21** Show that $\int_{-2}^1 \frac{1}{\sqrt{x^2 + 4x + 13}} \, dx = \ln(1 + \sqrt{2})$. [6]
- 22** Use the substitution $x = \sinh u$ to show that $\int \sqrt{x^2 + 1} \, dx = \frac{1}{2} x \sqrt{x^2 + 1} + \frac{1}{2} \operatorname{arsinh} x + c$. [7]
- 23** The region R is bounded by the curve $y = \cosh x$, the x -axis, the y -axis and the line $x = 1$. Find the exact volume of the solid formed when R is rotated through 2π radians about the x -axis, and give the value to 3 significant figures. [6]
- 24** The arc of the curve $y = \cosh x$ between $x = 0$ and $x = \ln 2$ has length s . Use the formula $s = \int_0^{\ln 2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$ to show that $s = \frac{3}{4}$. [6]
- 25** Use integration by parts to find $\int \operatorname{arsinh} x \, dx$, and hence show that $\int_0^{3/4} \operatorname{arsinh} x \, dx = \frac{3}{4} \ln 2 - \frac{1}{4}$. [6]
- 26** Show that $\int \operatorname{sech} x \, dx = \arctan(\sinh x) + c$. Hence show that $\int_0^{\ln(1+\sqrt{2})} \operatorname{sech} x \, dx = \frac{\pi}{4}$. [6]
- 27** Let $I_n = \int_0^{\ln 2} \cosh^n x \, dx$ for integers $n \geq 0$.
 (a) Show that $n I_n = [\cosh^{n-1} x \sinh x]_0^{\ln 2} + (n - 1) I_{n-2}$ for $n \geq 2$. [5]
 (b) Hence find I_4 exactly. [5]
- 28** Let $J = \int_0^{\ln 3} \frac{1}{\cosh x + 3} \, dx$.
 (a) Use the substitution $t = \tanh\left(\frac{x}{2}\right)$ to show that $J = \int_0^{1/2} \frac{1}{2 - t^2} \, dt$. You may use $\cosh x = \frac{1 + t^2}{1 - t^2}$ and $\frac{dx}{dt} = \frac{1}{1 - t^2}$. [5]
 (b) Hence show that $J = \frac{1}{2\sqrt{2}} \ln\left(\frac{2\sqrt{2} + 1}{2\sqrt{2} - 1}\right)$. [4]

- 29** The diagram shows the curves $y = \cosh x$ and $y = 3 - \sinh x$, which meet at the point A. The shaded region R lies between the two curves and the y-axis.



- (a) Show that the curves meet where $e^x = 3$, and find the exact coordinates of A. [3]
 (b) Find the exact area of R. [3]
 (c) Find the exact volume generated when R is rotated through 2π radians about the x-axis. [5]
-
- 30** Show that $\int_0^{\ln 2} \frac{1}{\sinh x + 2 \cosh x} dx = \frac{2}{\sqrt{3}} \arctan\left(\frac{\sqrt{3}}{7}\right)$. You may use the result $\arctan a - \arctan b = \arctan\left(\frac{a - b}{1 + ab}\right)$ for $a, b > 0$ with $ab > -1$. [8]

Solving Hyperbolic Equations

- 1** Solve $\tanh x = \frac{3}{5}$, giving the answer as an exact natural logarithm. [2]
- 2** Solve $\cosh x = \frac{5}{4}$, giving the answers as exact natural logarithms. [2]
- 3** Solve $2 \sinh x = 3$, giving your answer to 3 significant figures. [2]
- 4** Explain why the equation $\cosh x = \frac{1}{2}$ has no real solutions. [2]
- 5** Given that $\sinh x + \cosh x = 4$, find the exact value of x . [2]
- 6** Solve $\operatorname{sech} x = \frac{1}{2}$, giving the answers as exact logarithms and to 3 significant figures. [3]
- 7** Solve $2 \sinh^2 x - \sinh x - 1 = 0$, giving the answers as exact natural logarithms. [4]
- 8** Solve $3 \tanh^2 x + 2 \tanh x - 1 = 0$, giving the answer as an exact natural logarithm. [4]
- 9** Solve $\cosh x = 2 \sinh x$, giving the answer as an exact natural logarithm. [3]
- 10** Solve $3 \cosh x - 5 \sinh x = 3$, giving the exact solution. [4]
- 11** Solve $\cosh 2x - 3 \cosh x + 2 = 0$. [4]
- 12** Solve $\sinh 2x = 3 \sinh x$, giving the exact solutions. [4]
- 13** State the number of real solutions of the equation $\cosh x = k$ for each value of k .
 (a) $k = 3$ [1]
 (b) $k = 1$ [1]
 (c) $k = \frac{1}{2}$ [1]

- 14** Solve $3 \tanh^2 x + 2 \operatorname{sech} x = 2$. [5]
-
- 15** Solve $2 \coth^2 x - 5 \operatorname{cosech} x + 1 = 0$, giving the answers as exact natural logarithms. [6]
-
- 16** It is given that $13 \cosh x + 5 \sinh x = R \cosh (x + \alpha)$, where $R > 0$ and $\alpha > 0$.
 (a) Find the exact values of R and α . [4]
 (b) Hence solve the equation $13 \cosh x + 5 \sinh x = 15$, giving the answers as exact natural logarithms. [3]
-
- 17** Solve each of the following inequalities, giving exact answers.
 (a) $\cosh x > 2$ [2]
 (b) $\sinh x \leq \frac{3}{4}$ [2]
-
- 18** Solve $\tanh 2x = 3 \tanh x$. [5]
-
- 19** Solve $\cosh 5x + \cosh x = 6 \cosh 3x$, giving the exact solutions. [5]
-
- 20** Find the range of values of k for which the equation $2 \cosh x + \sinh x = k$ has real solutions. Solve the equation when $k = 2$. [6]
-
- 21** Solve $3(\cosh x - 1) = 2 \sinh x$, giving the exact solutions. [6]
-
- 22** Solve $\cosh 2x + 3 \sinh x = 3$, giving the answers as exact natural logarithms. [6]
-
- 23** Solve the simultaneous equations
 $\sinh x + 4 \sinh y = \frac{13}{3}$,
 $3 \sinh x - 2 \sinh y = \frac{5}{2}$,
 giving x and y as exact natural logarithms. [7]
-
- 24** A student solves $\sinh 2x = 4 \sinh x$ as follows.
 'sinh $2x = 2 \sinh x \cosh x$, so $2 \sinh x \cosh x = 4 \sinh x$. Divide both sides by $\sinh x$ to get $2 \cosh x = 4$, so $\cosh x = 2$ and $x = \pm \ln(2 + \sqrt{3})$.'
 The student has missed a solution. Identify the mistake, and find all the solutions of the equation. [6]
-
- 25** The equation $\cosh 2x = 4 \sinh x + k$, where k is a constant, may or may not have real solutions.
 (a) Show that, if the equation has a solution, $k \geq -1$. [4]
 (b) Solve the equation when $k = 1$, giving exact answers. [3]
-
- 26** A cable between two towers hangs in the shape $y = 12 \cosh\left(\frac{x}{12}\right)$, where x and y are in metres and the origin is at ground level below the lowest point of the cable.
 (a) Find the height of the cable at its lowest point. [1]
 (b) At two points on the cable the height is 13 m. Find the exact horizontal distance between these points, and give it to 3 significant figures. [5]
-
- 27** Let $f(x) = \cosh 2x - 10 \cosh x$, for $x \in \mathbb{R}$.
 (a) Solve $f(x) = -1$, giving the answers as exact natural logarithms. [3]
 (b) Find the number of real solutions of $f(x) = k$ for each of the following values of k .
 (i) $k = -14$ (ii) $k = -\frac{27}{2}$ (iii) $k = -10$ (iv) $k = -9$ (v) $k = 0$ [6]
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- 28** The following parts use the values $\cosh(\ln 2) = \frac{5}{4}$ and $\sinh(\ln 2) = \frac{3}{4}$.
- (a) Show that $5 \sinh x + 3 \cosh x \equiv 4 \sinh(x + \ln 2)$. [3]
- (b) Hence solve $5 \sinh x + 3 \cosh x = 12$, giving the exact answer in the form $\ln\left(\frac{a + \sqrt{b}}{c}\right)$. [3]
- (c) Solve $5 \sinh x + 3 \cosh x = -8$, giving the exact answer. [3]
-
- 29** The curves C_1 and C_2 have equations $y = 7 \cosh x$ and $y = 5 + 5 \sinh x$ respectively.
- (a) Show that the x -coordinates of the points where C_1 and C_2 meet satisfy $e^{2x} - 5e^x + 6 = 0$, and hence find the exact coordinates of the points of intersection. [5]
- (b) Between these points, C_2 lies above C_1 . Find the exact area of the region enclosed by the two curves. [4]
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- 30** A particle moves in a straight line so that its displacement from a fixed point O at time t seconds ($t \geq 0$) is $s = 5 \cosh 2t - 3 \sinh 2t$ metres.
- (a) Show that the particle is instantaneously at rest when $t = \frac{1}{2} \ln 2$, and find its displacement at that instant. [5]
- (b) Show that this is the minimum displacement. [2]
- (c) Find the exact time at which the displacement is 7 m, giving your answer in terms of a natural logarithm and ignoring any root that corresponds to a negative time. [4]
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