Newton's Law of Gravitation

1 (2 marks)

$F = \frac{Gm_1m_2}{r^2} = \frac{6.67 \times 10^{-11} \times 5.0 \times 12}{0.40^2}$ **M1**

$= 2.50 \times 10^{-8}$ N **A1**

Answer: 2.50×10^{-8} N **2 marks**

2 (2 marks)

$g = \frac{GM}{R^2} = \frac{6.67 \times 10^{-11} \times 5.97 \times 10^{24}}{(6.37 \times 10^6)^2}$ **M1**

$= 9.81$ N kg^{-1} **A1**

Answer: 9.81 N kg^{-1} **2 marks**

3 (3 marks)

(a) Force $\propto \frac{1}{r^2}$, so the force becomes $\frac{F}{4}$ **B1**

Answer: $\frac{F}{4}$ **1 mark**

(b) Force becomes $3 \times 2^2 \times F$ **M1**

$= 12F$ **A1**

Answer: 12F **2 marks**

4 (2 marks)

$g = \frac{GM}{R^2}$ so $M = \frac{gR^2}{G}$ **M1**

$= \frac{1.62 \times (1.74 \times 10^6)^2}{6.67 \times 10^{-11}} = 7.35 \times 10^{22}$ kg **A1**

Answer: 7.35×10^{22} kg **2 marks**

5 (2 marks)

$g \propto \frac{1}{R^2}$, so $g = \frac{9.8}{3^2}$ **M1**

$= 1.09$ N kg^{-1} **A1**

Answer: 1.09 N kg^{-1} **2 marks**

6 (2 marks)

Distance from the centre = 3R, so weight $= \frac{600}{3^2}$ **M1**

$= 66.7$ N **A1**

Answer: 66.7 N **2 marks**

7 (3 marks)

$r = 6.37 \times 10^6 + 1.0 \times 10^6 = 7.37 \times 10^6$ m **B1**

$g = \frac{GM}{r^2} = \frac{6.67 \times 10^{-11} \times 5.97 \times 10^{24}}{(7.37 \times 10^6)^2}$ **M1**

$= 7.33$ N kg^{-1} **A1**

Answer: 7.33 N kg^{-1} **3 marks**

8 (4 marks)

(a) $M = \frac{gR^2}{G} = \frac{3.71 \times (3.39 \times 10^6)^2}{6.67 \times 10^{-11}}$ **M1**

$= 6.39 \times 10^{23}$ kg **A1**

Answer: 6.39×10^{23} kg **2 marks**

(b) $\rho = \dfrac{M}{\frac{4}{3}\pi R^3}$ **M1**

$= 3920 \text{ kg m}^{-3}$ **A1**

Answer: 3920 kg m^{-3} **2 marks**

9 (4 marks)

(a) $W = \dfrac{GM_M m}{R_M^2} = \dfrac{6.67 \times 10^{-11} \times 7.35 \times 10^{22} \times 85}{(1.74 \times 10^6)^2}$ **M1**

$= 138 \text{ N}$ **A1**

Answer: 138 N **2 marks**

(b) Earth weight = $85 \times 9.8 = 833$ N, and ratio = $\dfrac{138}{833}$ **M1**

$= 0.165$ (about $\dfrac{1}{6}$) **A1**

Answer: 0.165 (about one sixth) **2 marks**

10 (4 marks)

(a) $F = \dfrac{GM_E M_M}{d^2} = \dfrac{6.67 \times 10^{-11} \times 5.97 \times 10^{24} \times 7.35 \times 10^{22}}{(3.84 \times 10^8)^2}$ **M1**

$= 1.98 \times 10^{20} \text{ N}$ **A1**

Answer: 1.98 × 10^{20} N **2 marks**

(b) $a = \dfrac{F}{M_M} = \dfrac{1.98 \times 10^{20}}{7.35 \times 10^{22}}$ **M1**

$= 2.70 \times 10^{-3} \text{ m s}^{-2}$ **A1**

Answer: 2.70 × 10^{-3} m s^{-2} **2 marks**

11 (4 marks)

(a) $g \propto \dfrac{M}{R^2}$, so $g_A = 9.8 \times \dfrac{8}{2^2}$ **M1**

$= 19.6 \text{ N kg}^{-1}$ **A1**

Answer: 19.6 N kg^{-1} **2 marks**

(b) $g_B = 9.8 \times \dfrac{0.1}{0.4^2}$ **M1**

$= 6.13 \text{ N kg}^{-1}$ **A1**

Answer: 6.13 N kg^{-1} **2 marks**

12 (4 marks)

(a) At the surface, $mg = \dfrac{GMm}{R^2}$ **M1**

Cancelling m, $gR^2 = GM$ **A1**

Answer: GM = gR^2 **2 marks**

(b) Distance from the centre = 4R, so $g' = \dfrac{GM}{(4R)^2} = \dfrac{g}{16}$ **M1**

$W = 20 \times \dfrac{9.8}{16} = 12.25 \text{ N}$ **A1**

Answer: 12.25 N **2 marks**

13 (3 marks)

$\dfrac{g_h}{g_0} = \dfrac{R^2}{(R+h)^2}$, so $\dfrac{9.0}{9.8} = \dfrac{R^2}{(R+h)^2}$ **M1**

$R + h = R\sqrt{\dfrac{9.8}{9.0}} = 1.0435R$ **M1**

$h = 0.0435 \times 6.37 \times 10^6 = 2.77 \times 10^5 \text{ m}$ **A1**

Answer: 2.77 × 10^5 m **3 marks**

14 (5 marks)

Let the point be x from the Earth: $\frac{GM_E}{x^2} = \frac{GM_M}{(d-x)^2}$ **M1**

$\frac{x}{(d-x)} = \sqrt{\frac{M_E}{M_M}} = 9.012$ **M1 A1**

$x = \frac{9.012}{1+9.012} \times 3.84 \times 10^8$ **M1**

$x = 3.46 \times 10^8$ m **A1**

Answer: 3.46×10^8 m from the centre of the Earth **5 marks**

15 (6 marks)

(a) $\frac{9GM}{x^2} = \frac{GM}{(8d-x)^2}$ **M1**

$\frac{3}{x} = \frac{1}{(8d-x)}$, so $24d - 3x = x$ **M1**

$x = 6d$ **A1**

Answer: AC = 6d **3 marks**

(b) Force towards A $= \frac{G \times 9M \times m}{(4d)^2} = \frac{9GMm}{16d^2}$, towards B $= \frac{GMm}{16d^2}$ **M1**

Resultant $= \frac{8GMm}{16d^2}$ **M1**

$= \frac{GMm}{2d^2}$ directed towards A **A1**

Answer: $\frac{GMm}{2d^2}$ towards A **3 marks**

16 (6 marks)

(a) Each force has magnitude $\frac{Gm^2}{a^2}$, along AB and AC, at 60° to each other **B1**

The resultant lies along the bisector and has magnitude $2 \times \frac{Gm^2}{a^2} \times \cos 30°$ **M1**

$= \frac{\sqrt{3}Gm^2}{a^2}$ **A1**

Answer: Resultant = $\frac{\sqrt{3}Gm^2}{a^2}$ **3 marks**

(b) Forces on A: $\frac{2Gm^2}{a^2}$ along AB and $\frac{Gm^2}{a^2}$ along AC **B1**

Components along and perpendicular to AB: $\frac{Gm^2}{a^2}(2 + \frac{1}{2})$ and $\frac{Gm^2}{a^2}(\frac{\sqrt{3}}{2})$ **M1**

Magnitude $= \frac{Gm^2}{a^2}\sqrt{\frac{25}{4} + \frac{3}{4}} = \sqrt{7}\frac{Gm^2}{a^2}$ **A1**

Answer: $\sqrt{7}\frac{Gm^2}{a^2}$ **3 marks**

17 (5 marks)

(a) $M = \rho \times \frac{4}{3}\pi R^3$ **M1**

$g = \frac{GM}{R^2}$ **M1**

$= \frac{4}{3}\pi G\rho R$ **A1**

Answer: $g = \frac{4}{3}\pi G\rho R$ **3 marks**

(b) $g = \frac{4}{3}\pi \times 6.67 \times 10^{-11} \times 5500 \times 3.4 \times 10^6$ **M1**

$= 5.22$ N kg^{-1} **A1**

Answer: 5.22 N kg^{-1} **2 marks**

18 (6 marks)

(a) $\omega = \frac{2\pi}{86400} = 7.27 \times 10^{-5}$ rad s^{-1} **M1**

$a = \omega^2 R = 0.0337$ m s^{-2} **A1**

Answer: 0.0337 m s^{-2} **2 marks**

(b) $70g - R_{eq} = 70a$ **M1**

$R_{eq} = 70(9.83 - 0.0337) = 686$ N **A1**

Answer: 686 N **2 marks**

(c) Reading at a pole $= 70 \times 9.83 = 688$ N **M1**

Percentage $= \frac{688 - 686}{688} \times 100 = 0.343\%$ **A1**

Answer: 0.343% **2 marks**

19 (6 marks)

(a) $\frac{R^2}{(R+h)^2} = 0.9$ **M1**

$h = R(\frac{1}{\sqrt{0.9}} - 1)$ **M1**

$= 3.45 \times 10^5$ m **A1**

Answer: 3.45×10^5 m **3 marks**

(b) Weight $\propto (1 + \frac{h}{R})^{-2} \approx 1 - \frac{2h}{R}$ **M1**

$\frac{2 \times 1.0 \times 10^4}{6.37 \times 10^6} = 3.14 \times 10^{-3}$ **M1**

Decrease $\approx 0.314\%$ **A1**

Answer: 0.314% **3 marks**

20 (5 marks)

(a) $r = 6.77 \times 10^6$ m **B1**

$g = \frac{GM_E}{r^2} = 8.69$ N kg^{-1} **M1 A1**

Answer: 8.69 N kg^{-1} **3 marks**

(b) Gravity at this height is still about 88% of its surface value, so the statement is wrong **E1**

Station and astronauts are in free fall with the same acceleration, so there is no contact force between them **E1**

Answer: Gravity is still about 88% of its surface value; both are in free fall together **2 marks**

21 (7 marks)

(a) $\frac{GMm}{x^2} = \frac{9GMm}{(4a - x)^2}$ **M1**

$4a - x = 3x$ **M1**

$x = a$ **A1**

Answer: AE = a **3 marks**

(b) Force towards B $= GMm(\frac{9}{(3a - \delta)^2} - \frac{1}{(a + \delta)^2})$ **M1**

$= \frac{GMm}{a^2}[(1 - \frac{\delta}{3a})^{-2} - (1 + \frac{\delta}{a})^{-2}]$ **M1**

$\approx \frac{GMm}{a^2}[(1 + \frac{2\delta}{3a}) - (1 - \frac{2\delta}{a})] = \frac{8GMm\delta}{3a^3}$ **A1**

This is positive, so P is pushed further from E: the equilibrium is unstable **E1**

Answer: Force $\approx \frac{8GMm\delta}{3a^3}$ towards B; E is an unstable equilibrium **4 marks**

22 (9 marks)

(a) Mass within radius r = $M \times \frac{r^3}{R^3}$ **M1**

Field strength = $\frac{G \times \frac{Mr^3}{R^3}}{r^2} = \frac{GMr}{R^3}$ **M1 A1**

Using $GM = gR^2$, field strength = $\frac{gr}{R}$ **A1**

Answer: $\frac{gr}{R}$ **4 marks**

(b) $\frac{r}{R} = 0.8$ **M1**

Depth = $0.2R = 0.2 \times 6.37 \times 10^6 = 1.27 \times 10^6$ m **A1**

Answer: 1.27×10^6 m **2 marks**

(c) $\frac{R^2}{(R+h)^2} = 0.8$, so $h = R(\frac{1}{\sqrt{0.8}} - 1)$ **M1**

$h = 0.118R = 7.52 \times 10^5$ m **A1**

$0.118R < 0.2R$, so the weight falls more quickly with height above the surface than with depth below it **E1**

Answer: $h = 7.52 \times 10^5$ m, which is smaller than the depth in (b) **3 marks**

23 (8 marks)

Take the direction from A towards C as positive, and units of $\frac{Gm^2}{a^2}$ **B1**

On A: from B $\frac{2}{4} = \frac{1}{2}$; from C $\frac{3}{25}$; total $\frac{1}{2} + \frac{3}{25} = \frac{31}{50}$, towards C **M1 A1**

On B: from A $-\frac{2}{4} = -\frac{1}{2}$; from C $\frac{6}{9} = \frac{2}{3}$; total $\frac{1}{6}$ towards C **M1 A1**

On C: from A $-\frac{3}{25}$; from B $-\frac{6}{9} = -\frac{2}{3}$; total $-\frac{59}{75}$, i.e. $\frac{59}{75}$ towards A **M1 A1**

Sum = $\frac{31}{50} + \frac{1}{6} - \frac{59}{75} = \frac{93 + 25 - 118}{150} = 0$ **B1**

Answer: A: $\frac{31Gm^2}{50a^2}$ towards C; B: $\frac{Gm^2}{6a^2}$ towards C; C: $\frac{59Gm^2}{75a^2}$ towards A; sum is zero **8 marks**

24 (8 marks)

(a) Nearest: $\frac{GM_M}{(d-R)^2} = 3.438 \times 10^{-5}$ N kg^{-1} **M1 A1**

Farthest: $\frac{GM_M}{(d+R)^2} = 3.217 \times 10^{-5}$ N kg^{-1} **M1 A1**

Answer: Nearest: 3.438×10^{-5} N kg^{-1}; farthest: 3.217×10^{-5} N kg^{-1} **4 marks**

(b) $3.438 \times 10^{-5} - 3.217 \times 10^{-5} = 2.21 \times 10^{-6}$ N kg^{-1} **A1**

Answer: 2.21×10^{-6} N kg^{-1} **1 mark**

(c) $\frac{GM_M}{d^2}[(1 - \frac{R}{d})^{-2} - (1 + \frac{R}{d})^{-2}]$ **M1**

$\approx \frac{GM_M}{d^2}[(1 + \frac{2R}{d}) - (1 - \frac{2R}{d})] = \frac{4GM_MR}{d^3}$ **M1**

$= 2.21 \times 10^{-6}$ N kg^{-1}, in agreement **A1**

Answer: $\frac{4GM_MR}{d^3} = 2.21 \times 10^{-6}$ N kg^{-1} **3 marks**

25 (7 marks)

(a) $M = \frac{4}{3}\pi\rho R^3$, so $g = \frac{GM}{R^2} = \frac{4}{3}\pi G\rho R$, giving $g \propto \rho R$ **M1 M1**

$\frac{g_X}{g_E} = 1.5 \times 0.6 = 0.9$ **A1**

Answer: $g_X = 0.9g_E$ **3 marks**

(b) $M \propto \rho R^3$ **M1**

$\frac{M_X}{M_E} = 1.5 \times 0.6^3 = 0.324$ **A1**

Answer: $M_X = 0.324\ M_E$ **2 marks**

(c) Ratio $= \sqrt{\frac{g_X R_X}{g_E R_E}}$ **M1**

$= \sqrt{0.9 \times 0.6} = 0.735$ **A1**

Answer: 0.735 **2 marks**

26 (7 marks)

(a) $F_S = \frac{GM_S M_M}{(1.50 \times 10^{11})^2} = 4.34 \times 10^{20}$ N **M1 A1**

$F_E = \frac{GM_E M_M}{(3.84 \times 10^8)^2} = 1.98 \times 10^{20}$ N **M1 A1**

Ratio = 2.18 **A1**

Answer: $F_S = 4.34 \times 10^{20}$ N, $F_E = 1.98 \times 10^{20}$ N, ratio 2.18 **5 marks**

(b) The Sun gives the Earth almost the same acceleration as it gives the Moon **E1**

So relative to the Earth the Moon's motion is determined by the Earth's pull, and it orbits the Earth while both orbit the Sun **E1**

Answer: The Sun accelerates Earth and Moon almost equally, so only the Earth's pull affects the Moon's motion relati **2 marks**

27 (12 marks)

(a) Mass ∝ volume ∝ radius cubed, so the mass removed is $\frac{M}{8}$ **B1**

Answer: $\frac{M}{8}$ **1 mark**

(b) Treat the remaining part as the full sphere minus the removed sphere **M1**

Full sphere: $\frac{GMm}{(3R)^2} = \frac{GMm}{9R^2}$; removed sphere, with PC $= \frac{5R}{2}$: $\frac{G \times \frac{M}{8} \times m}{(\frac{5R}{2})^2} = \frac{GMm}{50R^2}$ **M1 A1**

Force $= \frac{GMm}{R^2}(\frac{1}{9} - \frac{1}{50}) = \frac{41GMm}{450R^2}$ **A1**

Answer: $\frac{41GMm}{450R^2}$ towards the sphere **4 marks**

(c) PC $= \frac{7R}{2}$ **B1**

Force $= \frac{GMm}{R^2}(\frac{1}{9} - \frac{1}{8 \times (\frac{7}{2})^2})$ **M1**

$= \frac{GMm}{R^2}(\frac{1}{9} - \frac{1}{98}) = \frac{89GMm}{882R^2}$ **A1**

Answer: $\frac{89GMm}{882R^2}$ **3 marks**

(d) Field of full sphere at displacement r from O: $-\frac{GM}{R^3}r$ **M1**

Removed sphere has density the same, mass $\frac{M}{8}$, radius $\frac{R}{2}$, centre C: its field is $-\frac{G \times \frac{M}{8}}{(\frac{R}{2})^3}(r - c) =$ **M1**

$-\frac{GM}{R^3}(r - c)$

Field in cavity $= -\frac{GM}{R^3}r + \frac{GM}{R^3}(r - c)$ **M1**

$= -\frac{GM}{R^3}c$, which is constant, of magnitude $\frac{GM}{R^3} \times \frac{R}{2} = \frac{GM}{2R^2}$ **A1**

Answer: Uniform field of magnitude $\frac{GM}{2R^2}$, directed from C towards O **4 marks**

28 (11 marks)

(a) Mass of element = $\frac{M}{2a}dx$ **B1**

$dF = \frac{Gm \times \frac{M}{2a}\,dx}{(d-x)^2}$ **B1**

Answer: dF = $\frac{\mathbf{GMm\,dx}}{\mathbf{2a(d-x)^2}}$ **2 marks**

(b) $F = \frac{GMm}{2a}\int_{-a}^{a} \frac{dx}{(d-x)^2}$ **M1**

$= \frac{GMm}{2a}[\frac{1}{(d-x)}]_{-a}^{a}$ **M1 A1**

$= \frac{GMm}{2a}(\frac{1}{(d-a)} - \frac{1}{(d+a)}) = \frac{GMm}{2a} \times \frac{2a}{(d^2-a^2)}$ **M1**

$= \frac{GMm}{(d^2-a^2)}$ **A1**

Answer: $\frac{\mathbf{GMm}}{\mathbf{(d^2-a^2)}}$ **5 marks**

(c) $F = \frac{GMm}{d^2}(1 - \frac{a^2}{d^2})^{-1}$ **M1**

$\approx \frac{GMm}{d^2}(1 + \frac{a^2}{d^2})$ **A1**

Answer: $F \approx \frac{GMm}{d^2}(1 + \frac{a^2}{d^2})$ **2 marks**

(d) Exact force = $\frac{GMm}{8a^2}$; point mass: $\frac{GMm}{9a^2}$ **M1**

Excess = $\frac{9}{8} - 1 = 12.5\%$ **A1**

Answer: 12.5% **2 marks**

29 (11 marks)

(a) Each element dM of the ring is at distance $s = \sqrt{x^2 + a^2}$ from P, exerting force $\frac{Gm\,dM}{s^2}$ **M1**

The components perpendicular to the axis cancel in pairs by symmetry **E1**

The component along the axis has factor $\cos\varphi = \frac{x}{s}$, giving $\frac{Gm\,dM\,x}{s^3}$ **M1**

Summing over the ring, total dM = M, so $F = \frac{GMmx}{(x^2+a^2)^{3/2}}$ **A1**

Answer: $\frac{\mathbf{GMmx}}{\mathbf{(x^2+a^2)^{3/2}}}$ **4 marks**

(b) $\frac{dF}{dx} = GMm[(x^2+a^2)^{-3/2} - 3x^2(x^2+a^2)^{-5/2}]$ **M1 A1**

$= GMm(x^2+a^2)^{-5/2}(a^2 - 2x^2) = 0$ **M1**

$x = \frac{a}{\sqrt{2}}$ **A1**

$F \to 0$ as $x \to 0$ and as $x \to \infty$, so this is the maximum **E1**

Answer: $x = \frac{\mathbf{a}}{\sqrt{\mathbf{2}}}$ **5 marks**

(c) $F = \frac{GMm \times \frac{a}{\sqrt{2}}}{(\frac{3a^2}{2})^{3/2}}$ **M1**

$= \frac{2GMm}{3\sqrt{3}a^2}$ **A1**

Answer: $\frac{\mathbf{2GMm}}{\mathbf{3\sqrt{3}a^2}}$ **2 marks**

30 (11 marks)

(a) Area of ring = $2\pi a \sin\theta \times a\, d\theta$ **M1**

Mass = $M \times \frac{2\pi a^2 \sin\theta\, d\theta}{4\pi a^2} = \frac{M}{2}\sin\theta\, d\theta$ **A1**

Answer: $\frac{M}{2}\sin\theta\, d\theta$ **2 marks**

(b) Cosine rule in triangle OQP gives s^2 **M1**

Force of ring has magnitude $\frac{Gm\, dM}{s^2}$, with $dM = \frac{M}{2}\sin\theta\, d\theta$ **M1**

The cosine of the angle between PQ and PO is $\frac{d - a\cos\theta}{s}$, giving the stated component **A1**

Answer: $dF = \frac{GmM\sin\theta\,(d - a\cos\theta)d\theta}{2s^3}$ **3 marks**

(c) $du = 2ad\sin\theta\, d\theta$ and $d - a\cos\theta = \frac{u + d^2 - a^2}{2d}$ **M1 M1**

Limits: u from $(d - a)^2$ to $(d + a)^2$ **M1**

$F = \frac{GMm}{8ad^2}\int_{(d-a)^2}^{(d+a)^2} (u^{-1/2} + (d^2 - a^2)u^{-3/2})\, du$ **A1**

$= \frac{GMm}{8ad^2}[2u^{1/2} - 2(d^2 - a^2)u^{-1/2}]$ **M1**

$= \frac{GMm}{4ad^2}[2a + 2a] = \frac{GMm}{d^2}$ **A1**

Answer: $F = \frac{GMm}{d^2}$, as if all the mass were at O **6 marks**

Gravitational Potential Energy and Escape Speed

1 (2 marks)

$GPE = -\frac{GMm}{r} = -\frac{6.67 \times 10^{-11} \times 5.97 \times 10^{24} \times 1000}{7.0 \times 10^6}$ **M1**

$= -5.69 \times 10^{10}$ J **A1**

Answer: -5.69×10^{10} J **2 marks**

2 (2 marks)

$v = \sqrt{\frac{2 \times 6.67 \times 10^{-11} \times 7.35 \times 10^{22}}{1.74 \times 10^6}}$ **M1**

$= 2370$ m s^{-1} **A1**

Answer: 2370 m s^{-1} **2 marks**

3 (2 marks)

(a) $GPE \propto \frac{1}{r}$, so $GPE = \frac{-6.2 \times 10^8}{2} = -3.1 \times 10^8$ J **B1**

Answer: -3.1×10^8 J **1 mark**

(b) Work done = $-3.1 \times 10^8 - (-6.2 \times 10^8) = 3.1 \times 10^8$ J **B1**

Answer: 3.1×10^8 J **1 mark**

4 (2 marks)

$v = \sqrt{2 \times 9.8 \times 6.37 \times 10^6}$ **M1**

$= 1.12 \times 10^4$ m s^{-1} **A1**

Answer: 1.12×10^4 m s^{-1} **2 marks**

5 (2 marks)

The gravitational potential energy at infinity is zero — B1

Gravity is attractive, so work must be done against it to take the particle to infinity, so its energy near the planet is below zero — E1

Answer: Zero at infinity; negative near a planet because work must be done to remove the particle — 2 marks

6 (2 marks)

$v \propto \sqrt{\frac{1}{R}}$, so the speed is multiplied by $\sqrt{4} = 2$ — M1

$v = 22.4 \text{ km s}^{-1}$ — A1

Answer: 22.4 km s^{-1} — 2 marks

7 (3 marks)

$\text{Gain} = GMm\left(\frac{1}{R} - \frac{1}{2R}\right) = \frac{GMm}{2R}$ — M1

Using $GM = gR^2$, gain $= \frac{1}{2}mgR$ — A1

$= \frac{1}{2} \times 500 \times 9.8 \times 6.37 \times 10^6 = 1.56 \times 10^{10}$ J — A1

Answer: 1.56 × 10^{10} J — 3 marks

8 (4 marks)

$\frac{1}{2}v^2 - \frac{GM}{R} = -\frac{GM}{r}$ — M1

$\frac{GM}{r} = \frac{6.67 \times 10^{-11} \times 5.97 \times 10^{24}}{6.37 \times 10^6} - \frac{1}{2}(6.0 \times 10^3)^2 = 4.45 \times 10^7$ — M1 A1

$r = 8.95 \times 10^6$ m, so $h = r - R = 2.58 \times 10^6$ m — A1

Answer: 2.58 × 10^6 m — 4 marks

9 (3 marks)

$\frac{1}{2}mv^2 = \frac{GMm}{R}$ — M1

Using $GM = gR^2$: $v = \sqrt{2gR}$ — A1

$v = \sqrt{2 \times 3.71 \times 3.39 \times 10^6} = 5020 \text{ m s}^{-1}$ — A1

Answer: 5020 m s^{-1} — 3 marks

10 (4 marks)

$\frac{1}{2}v^2 = GM\left(\frac{1}{R} - \frac{1}{3R}\right) = \frac{2GM}{3R}$ — M1 A1

$v^2 = \frac{4gR}{3}$ (using $GM = gR^2$) — M1

$v = 9120 \text{ m s}^{-1}$ — A1

Answer: 9120 m s^{-1} — 4 marks

11 (4 marks)

(a) $v \propto \sqrt{\frac{M}{R}}$ — M1

Ratio $= \sqrt{\frac{2}{0.5}} = 2$ — A1

Answer: 2 : 1 — 2 marks

(b) $\text{GPE} \propto \frac{M}{R}$ — M1

Ratio $= \frac{2}{0.5} = 4$ — A1

Answer: 4 : 1 — 2 marks

12 (3 marks)

$\Delta U = GMm(\frac{1}{r_1} - \frac{1}{r_2})$ **M1**

$= 6.67 \times 10^{-11} \times 5.97 \times 10^{24} \times 1200 \times (\frac{1}{7.0 \times 10^6} - \frac{1}{4.22 \times 10^7})$ **M1**

$= 5.69 \times 10^{10}$ J **A1**

Answer: 5.69×10^{10} J **3 marks**

13 (4 marks)

$\frac{1}{2}v_\infty^2 = \frac{1}{2}v^2 - \frac{GM}{R}$ **M1**

$v_\infty^2 = (1.5 \times 10^4)^2 - \frac{2 \times 6.67 \times 10^{-11} \times 5.97 \times 10^{24}}{6.37 \times 10^6} = 1.00 \times 10^8$ **M1 A1**

$v_\infty = 10000$ m s^{-1} **A1**

Answer: 10000 m s^{-1} **4 marks**

14 (5 marks)

$v_e^2 = \frac{2GM}{R}$, so $v^2 = \frac{9}{16} \times \frac{2GM}{R} = \frac{9GM}{8R}$ **M1**

$\frac{1}{2}v^2 = GM(\frac{1}{R} - \frac{1}{r})$ gives $\frac{9GM}{16R} = \frac{GM}{R} - \frac{GM}{r}$ **M1 A1**

$\frac{1}{r} = \frac{7}{16R}$ **M1**

$r = \frac{16R}{7}$ **A1**

Answer: $\frac{16R}{7}$ **5 marks**

15 (5 marks)

(a) $v = \sqrt{2 \times 1.62 \times 1.0 \times 10^6} = 1800$ m s^{-1} **M1 A1**

Answer: 1800 m s^{-1} **2 marks**

(b) g is not constant over a height comparable with the radius of the Moon **E1**

$\frac{1}{2}v^2 = GM(\frac{1}{R} - \frac{1}{(R+h)}) = \frac{gR\,h}{(R+h)}$ **M1**

$v = \sqrt{\frac{2 \times 1.62 \times 1.74 \times 10^6 \times 1.0 \times 10^6}{2.74 \times 10^6}} = 1430$ m s^{-1} **A1**

Answer: g decreases with height so the student overestimates; correct speed 1430 m s^{-1} **3 marks**

16 (6 marks)

(a) Force $= \frac{GMm}{x^2}$ **B1**

Work $= \int_R^\infty \frac{GMm}{x^2}\,dx = GMm[-\frac{1}{x}]_R^\infty$ **M1**

$= \frac{GMm}{R}$ **A1**

Answer: $\frac{GMm}{R}$ **3 marks**

(b) Work $= GMm(\frac{1}{R} - \frac{1}{3R}) = \frac{2GMm}{3R}$ **M1 A1**

$= 1.67 \times 10^{10}$ J **A1**

Answer: 1.67×10^{10} J **3 marks**

17 (6 marks)

(a) $\frac{1}{2}mc^2 = \frac{GMm}{r}$ **M1**

$r = \frac{2GM}{c^2}$ **A1**

Answer: $r = \frac{2GM}{c^2}$ **2 marks**

(b) $r = \frac{2 \times 6.67 \times 10^{-11} \times 1.99 \times 10^{30}}{(3.00 \times 10^8)^2}$ **M1**

$= 2950$ m **A1**

Answer: 2950 m **2 marks**

(c) $M = \frac{c^2 r}{2G} = \frac{(3.00 \times 10^8)^2 \times 1.2 \times 10^{10}}{2 \times 6.67 \times 10^{-11}}$ **M1**

$= 8.10 \times 10^{36}$ kg **A1**

Answer: 8.10×10^{36} kg **2 marks**

18 (6 marks)

(a) $M = \frac{4}{3}\pi R^3 \rho$ **M1**

$\frac{1}{2}v^2 = \frac{GM}{R} = \frac{4\pi G\rho R^2}{3}$ **M1**

$v = R\sqrt{\frac{8\pi G\rho}{3}}$ **A1**

Answer: $v = R\sqrt{\frac{8\pi G\rho}{3}}$ **3 marks**

(b) $R = \frac{v}{\sqrt{\frac{8\pi G\rho}{3}}}$ **M1**

$\sqrt{\frac{8\pi \times 6.67 \times 10^{-11} \times 3000}{3}} = 1.29 \times 10^{-3}$ **M1**

$R = 2320$ m **A1**

Answer: 2320 m **3 marks**

19 (5 marks)

(a) $\frac{1}{2} \times 1500 \times 2000^2 = 3.00 \times 10^9$ J **B1**

Answer: 3.00×10^9 J **1 mark**

(b) $\Delta U = GMm\left(\frac{1}{R} - \frac{1}{R+h}\right)$ **M1**

$= 6.67 \times 10^{-11} \times 5.97 \times 10^{24} \times 1500 \times \left(\frac{1}{6.37 \times 10^6} - \frac{1}{9.37 \times 10^6}\right)$ **M1**

$= 3.00 \times 10^{10}$ J **A1**

Answer: 3.00×10^{10} J **3 marks**

(c) Work done $= \Delta KE + \Delta U = 3.30 \times 10^{10}$ J **A1**

Answer: 3.30×10^{10} J **1 mark**

20 (6 marks)

(a) KE + GPE = 0 at infinity, so $KE = \frac{GMm}{r}$ **M1**

$= \frac{6.67 \times 10^{-11} \times 5.97 \times 10^{24} \times 40}{6.77 \times 10^6} = 2.35 \times 10^9$ J **A1**

Answer: 2.35×10^9 J **2 marks**

(b) $v = \sqrt{\frac{2 \times KE}{m}} = 1.08 \times 10^4$ m s^{-1} **A1**

Answer: 1.08×10^4 m s^{-1} **1 mark**

(c) $KE = \frac{GMm}{R} = 2.50 \times 10^9$ J **M1**

$v = 1.12 \times 10^4$ m s^{-1} **A1**

Answer: KE = 2.50×10^9 J, $v = 1.12 \times 10^4$ m s^{-1} **2 marks**

(d) The energy needed to escape from the surface is the same as the energy gained falling in from infinity, because gravity is conservative **E1**

Answer: Gravity is conservative: the energy gained falling from infinity equals the energy needed to escape **1 mark**

21 (8 marks)

(a) $V = -\frac{GM_E}{R_E} - \frac{GM_M}{(d - R_E)}$ **M1**

$= -6.251 \times 10^7 - 1.298 \times 10^4 = -6.252 \times 10^7$ J kg^{-1} **A1**

Answer: -6.252×10^7 J kg^{-1} **2 marks**

(b) $V = -\frac{GM_E}{x} - \frac{GM_M}{(d - x)}$ **M1**

$= -1.152 \times 10^6 - 1.280 \times 10^5 = -1.280 \times 10^6$ J kg^{-1} **A1**

Answer: -1.280×10^6 J kg^{-1} **2 marks**

(c) $\frac{1}{2}v^2 = V_N - V_S$ **M1**

$v^2 = 2(-1.280 \times 10^6 - -6.252 \times 10^7)$ **M1**

$v = 1.11 \times 10^4$ m s^{-1} **A1**

Answer: 1.11×10^4 m s^{-1} **3 marks**

(d) Beyond N the Moon's pull is greater than the Earth's, so the resultant force is towards the Moon and the spacecraft is drawn on **E1**

Answer: Beyond N the resultant force is towards the Moon **1 mark**

22 (8 marks)

(a) $v = \sqrt{\frac{2GM}{R}} = \sqrt{\frac{2 \times 6.67 \times 10^{-11} \times 5.97 \times 10^{24}}{6.37 \times 10^6}}$ **M1**

$= 1.12 \times 10^4$ m s^{-1} **A1**

Answer: 1.12×10^4 m s^{-1} **2 marks**

(b) $v = \frac{2\pi R}{T} = \frac{2\pi \times 6.37 \times 10^6}{86400}$ **M1**

$= 463$ m s^{-1} **A1**

Answer: 463 m s^{-1} **2 marks**

(c) (i) The ground moves east, so the rocket already has speed 463: required speed relative to the ground $= 1.12 \times 10^4 - 463$ **M1 A1**

$= 1.07 \times 10^4$ m s^{-1} **A1**

(ii) Required speed $= 1.12 \times 10^4 + 463 = 1.16 \times 10^4$ m s^{-1} **A1**

Answer: (i) 1.07×10^4 m s^{-1}; (ii) 1.16×10^4 m s^{-1} **4 marks**

23 (8 marks)

(a) $\frac{1}{2}v^2 = GM(\frac{1}{R} - \frac{1}{R + h}) = \frac{GMh}{R(R + h)}$ **M1**

$v^2 = \frac{2GMh}{R(R + h)}$ and $v_e{}^2 = \frac{2GM}{R}$, so $k^2 = \frac{h}{R + h}$ **M1 A1**

$k^2R + k^2h = h$, so $h = \frac{k^2R}{(1 - k^2)}$ **A1**

Answer: $h = \frac{k^2R}{(1 - k^2)}$ **4 marks**

(b) $h = \frac{0.64R}{0.36}$ **M1**

$= \frac{16R}{9}(1.78R)$ **A1**

Answer: $\frac{16R}{9}$ **2 marks**

(c) $k^2 = \frac{1}{2}$, so $v^2 = \frac{1}{2}v_e{}^2 = \frac{1}{2} \times 2gR = gR$ **M1**

$v = \sqrt{gR}$ **A1**

Answer: $\sqrt{gR}$ **2 marks**

24 (9 marks)

(a) Work done in moving from infinity to the surface $= \frac{GMm}{R}$ **M1 A1**

Work done in moving from the surface to the centre $= \int_0^R \frac{GMmr}{R^3}\,dr$ **M1**

$= \frac{GMm}{2R}$ **A1**

Total work done by gravity $= \frac{3GMm}{2R}$, so GPE $= -\frac{3GMm}{2R}$ **A1**

Answer: GPE at the centre $= -\frac{3GMm}{2R}$ **5 marks**

(b) $\frac{1}{2}v^2 = \frac{3GM}{2R}$, so $v = \sqrt{\frac{3GM}{R}} = \sqrt{3gR}$ **M1**

$= 1.37 \times 10^4$ m s^{-1} **A1**

Answer: 1.37×10^4 m s^{-1} **2 marks**

(c) $\frac{1}{2}v^2 = U_{surface} - U_{centre}$ per unit mass $= \frac{3GM}{2R} - \frac{GM}{R} = \frac{GM}{2R}$ **M1**

$v = \sqrt{\frac{GM}{R}} = \sqrt{gR} = 7900$ m s^{-1} **A1**

Answer: 7900 m s^{-1} **2 marks**

25 (8 marks)

(a) $v_H = \sqrt{\frac{3 \times 1.38 \times 10^{-23} \times 300}{3.3 \times 10^{-27}}}$ **M1 A1**

$= 1940$ m s^{-1} **A1**

$v_O = \sqrt{\frac{3 \times 1.38 \times 10^{-23} \times 300}{5.3 \times 10^{-26}}} = 484$ m s^{-1} **A1**

Answer: Hydrogen: 1940 m s^{-1}; oxygen: 484 m s^{-1} **4 marks**

(b) Minimum escape speeds: $6 \times 1940 = 1.16 \times 10^4$ (hydrogen), $6 \times 484 = 2900$ (oxygen) **M1**

$1.12 \times 10^4 < 1.16 \times 10^4$ but $1.12 \times 10^4 > 2900$ **A1**

Answer: Hydrogen needs 1.16×10^4 m s^{-1} ($> 1.12 \times 10^4$); oxygen needs only 2900 m s^{-1} **2 marks**

(c) Required escape speed 2900 m s^{-1} $> 2.37 \times 10^3$ m s^{-1} **M1 A1**

Answer: The Moon's escape speed (2.37×10^3) is below the 2900 needed **2 marks**

26 (7 marks)

(a) $\frac{1}{2}v^2 = GM\left(\frac{1}{R} - \frac{1}{R+h}\right) = \frac{GMh}{R(R+h)}$ **M1**

Using $GM = gR^2$: $v^2 = \frac{2gRh}{(R+h)}$ **M1**

$= \frac{2gh}{\left(1 + \frac{h}{R}\right)}$ **A1**

Answer: $v^2 = \frac{2gh}{\left(1 + \frac{h}{R}\right)}$ **3 marks**

(b) $2gh = 1.1 \times \frac{2gh}{\left(1 + \frac{h}{R}\right)}$ **M1**

$1 + \frac{h}{R} = 1.1$, so $h = 0.1R = 6.37 \times 10^5$ m **A1**

Answer: $h = 0.1R = 6.37 \times 10^5$ m **2 marks**

(c) True: $v^2 = gR$, so $v = 7900$ m s^{-1} **A1**

Constant gravity: $v = \sqrt{2gR} = 1.12 \times 10^4$ m s^{-1} **A1**

Answer: True: 7900 m s^{-1}; constant gravity: 1.12×10^4 m s^{-1} **2 marks**

27 (11 marks)

(a) $Mv_1 = 2Mv_2$ **M1**

$v_1 = 2v_2$ **A1**

Answer: $v_1 = 2v_2$ **2 marks**

(b) Initial GPE $= -\frac{G \times M \times 2M}{10a}$; GPE when touching (centres 2a apart) $= -\frac{G \times M \times 2M}{2a}$ **M1 M1**

$\frac{1}{2}Mv_1^2 + \frac{1}{2}(2M)v_2^2 = 2GM^2(\frac{1}{2a} - \frac{1}{10a}) = \frac{4GM^2}{5a}$ **M1**

$3Mv_2^2 = \frac{4GM^2}{5a}$ (using $v_1 = 2v_2$), so $v_2^2 = \frac{4GM}{15a}$ **A1**

Answer: $v_2^2 = \frac{4GM}{15a}$ **4 marks**

(c) $v_2 = \sqrt{\frac{4 \times 6.67 \times 10^{-11} \times 3.0 \times 10^{20}}{15 \times 8.0 \times 10^3}}$ **M1**

$v_2 = 817$ m s^{-1} **A1**

$v_1 = 1630$ m s^{-1} **A1**

Answer: $v_1 = 1630$ m s^{-1}, $v_2 = 817$ m s^{-1} **3 marks**

(d) $\mu = \frac{M \times 2M}{3M} = \frac{2M}{3}$; $\frac{1}{2} \cdot \frac{2M}{3} \cdot v_{rel}^2 = \frac{4GM^2}{5a}$ **M1**

$v_{rel}^2 = \frac{12GM}{5a} = (3v_2)^2 = 9 \times \frac{4GM}{15a}$, as required **A1**

Answer: Both give $v_{rel}^2 = \frac{12GM}{5a}$ **2 marks**

28 (10 marks)

(a) $U = -\frac{GMm}{x} - \frac{GMm}{6R - x}$ **B1**

Answer: $U = -GMm(\frac{1}{x} + \frac{1}{(6R - x)})$ **1 mark**

(b) $\frac{dU}{dx} = GMm(\frac{1}{x^2} - \frac{1}{(6R - x)^2})$ **M1**

$= 0$ when $x = 6R - x$, i.e. $x = 3R$ **A1**

$\frac{d^2U}{dx^2} = -GMm(\frac{2}{x^3} + \frac{2}{(6R - x)^3})$ **M1**

which is negative, so U has a maximum at $x = 3R$ **A1**

Answer: Stationary at x = 3R; second derivative negative so it is a maximum **4 marks**

(c) The probe reaches B if it reaches $x = 3R$, where U is greatest **M1**

$\frac{1}{2}mv^2 = U(3R) - U(R) = -\frac{2GMm}{3R} + \frac{GMm}{R} + \frac{GMm}{5R}$ **M1 A1**

$v^2 = \frac{16GM}{15R}$, so $v = 8170$ m s^{-1} **A1**

Answer: $v = \sqrt{\frac{16GM}{15R}} = 8170$ m s^{-1} **4 marks**

(d) The net force on the probe is then towards B, so it is accelerated into B **E1**

Answer: It is accelerated towards B and falls onto it **1 mark**

29 (11 marks)

(a) Volume $\propto (r^3 - a^3)$ for the matter within radius r **M1**

$M' = M \times \frac{r^3 - a^3}{b^3 - a^3}$ **M1**

Field $= \frac{GM'}{r^2}$ as stated **A1**

Answer: $\frac{GM(r^3 - a^3)}{(b^3 - a^3)r^2}$ **3 marks**

(b) There is no field inside the cavity, so $V(0) = V(a)$ **B1**

$V(a) = -\frac{GM}{b} - \int_a^b \frac{GM(r^3 - a^3)}{(b^3 - a^3)r^2}\,dr$ **M1**

$\int_a^b (r - \frac{a^3}{r^2})\,dr = \frac{b^2 - a^2}{2} + a^3(\frac{1}{b} - \frac{1}{a})$ **M1 A1**

$V(0) = -\frac{GM}{(b^3 - a^3)}[b^2 + \frac{b^2}{2} - \frac{3a^2}{2}]$ **M1**

$= -\frac{3GM(b^2 - a^2)}{2(b^3 - a^3)}$ **A1**

Answer: $V(0) = -\frac{3GM(b^2 - a^2)}{2(b^3 - a^3)}$ **6 marks**

(c) $a \to 0$ gives $-\frac{3GM}{2b}$, the value at the centre of a uniform sphere **B1**

$a \to b$ gives $-\frac{GM}{b}$, the potential inside a thin shell **B1**

Answer: $-3GM/2b$ (solid sphere) and $-GM/b$ (thin shell) **2 marks**

30 (9 marks)

(a) Surface area $4\pi r^2$ times thickness dr times density **B1**

Answer: $4\pi\rho r^2\,dr$ **1 mark**

(b) Mass of the sphere already formed $= \frac{4}{3}\pi\rho r^3$ **M1**

Work done by gravity $= \frac{G \times \frac{4}{3}\pi\rho r^3 \times 4\pi\rho r^2\,dr}{r} = \frac{16\pi^2 G\rho^2}{3} r^4\,dr$ **A1**

Answer: Work done $= \frac{16\pi^2 G\rho^2}{3} r^4\,dr$ **2 marks**

(c) Total work $= \int_0^R \frac{16\pi^2 G\rho^2}{3} r^4\,dr$ **M1**

$= \frac{16\pi^2 G\rho^2 R^5}{15}$ **A1**

$M = \frac{4}{3}\pi\rho R^3$, so $\rho^2 = \frac{9M^2}{16\pi^2 R^6}$ **M1**

Binding energy $= \frac{16\pi^2 G\,R^5}{15} \times \frac{9M^2}{16\pi^2 R^6} = \frac{3GM^2}{5R}$ **A1**

Answer: $\frac{3GM^2}{5R}$ **4 marks**

(d) $U = \frac{3 \times 6.67 \times 10^{-11} \times (5.97 \times 10^{24})^2}{5 \times 6.37 \times 10^6}$ **M1**

$= 2.24 \times 10^{32}$ J **A1**

Answer: 2.24×10^{32} J **2 marks**

Circular Orbits and Satellites

1 (2 marks)

$\frac{GMm}{r^2} = \frac{mv^2}{r}$, so $v = \sqrt{\frac{GM}{r}}$ **M1**

$= \sqrt{\frac{6.67 \times 10^{-11} \times 5.97 \times 10^{24}}{7.0 \times 10^6}} = 7540$ m s^{-1} **A1**

Answer: 7540 m s^{-1} **2 marks**

2 (2 marks)

$T = \frac{2\pi r}{v} = \frac{2\pi \times 7.0 \times 10^6}{7.5 \times 10^3}$ **M1**

$= 5860$ s **A1**

Answer: 5860 s **2 marks**

3 (2 marks)

$T^2 \propto r^3$, so $T_Q{}^2 = 4^3 \times 1.0^2 = 64$ **M1**

$T_Q = 8.0$ years **A1**

Answer: 8.0 years **2 marks**

4 (2 marks)

Gravitational attraction of the planet: $\frac{GMm}{r^2} = \frac{mv^2}{r}$ **M1**

$v^2 = \frac{GM}{r}$, so $v = \sqrt{\frac{GM}{r}}$ **A1**

Answer: $v = \sqrt{\frac{GM}{r}}$ **2 marks**

5 (2 marks)

$KE = \frac{1}{2}mv^2 = \frac{GMm}{2r}$ **M1**

$= \frac{6.67 \times 10^{-11} \times 5.97 \times 10^{24} \times 500}{2 \times 7.0 \times 10^6} = 1.42 \times 10^{10}$ J **A1**

Answer: 1.42×10^{10} J **2 marks**

6 (2 marks)

$v \propto \frac{1}{\sqrt{r}}$ **M1**

$v_A : v_B = \sqrt{4} : 1 = 2 : 1$ **A1**

Answer: 2 : 1 **2 marks**

7 (4 marks)

$\frac{GMm}{r^2} = m\omega^2 r$ with $\omega = \frac{2\pi}{T}$ **M1**

$r^3 = \frac{GMT^2}{4\pi^2} = \frac{6.67 \times 10^{-11} \times 5.97 \times 10^{24} \times (8.62 \times 10^4)^2}{4\pi^2}$ **M1**

$r = 4.22 \times 10^7$ m **A1**

$h = r - R_E = 3.58 \times 10^7$ m **A1**

Answer: $r = 4.22 \times 10^7$ m, $h = 3.58 \times 10^7$ m **4 marks**

8 (3 marks)

$\frac{GM_Sm}{r^2} = m\omega^2 r$ with $\omega = \frac{2\pi}{T}$, so $M_S = \frac{4\pi^2 r^3}{GT^2}$ **M1 M1**

$= \frac{4\pi^2 \times (1.50 \times 10^{11})^3}{6.67 \times 10^{-11} \times (3.16 \times 10^7)^2} = 2.00 \times 10^{30}$ kg **A1**

Answer: 2.00×10^{30} kg **3 marks**

9 (3 marks)

$r = 6.37 \times 10^6 + 3.0 \times 10^5 = 6.67 \times 10^6$ m **B1**

$T^2 = \frac{4\pi^2 r^3}{GM}$ **M1**

$T = 5420$ s $= 90.4$ minutes **A1**

Answer: 90.4 minutes **3 marks**

10 (4 marks)

(a) $v^2 = \frac{GM}{r}$, $KE = \frac{1}{2}mv^2 = \frac{GMm}{2r}$ **M1**

$= 2.21 \times 10^{10}$ J **A1**

Answer: 2.21×10^{10} J **2 marks**

(b) $GPE = -\frac{GMm}{r} = -4.42 \times 10^{10}$ J **B1**

$Total = KE + GPE = -2.21 \times 10^{10}$ J **A1**

Answer: GPE = -4.42×10^{10} J, total = -2.21×10^{10} J **2 marks**

11 (3 marks)

$T^2 \propto r^3$, so $\frac{r_E}{r_I} = (\frac{T_E}{T_I})^{2/3}$ **M1 M1**

$r_E = 4.22 \times 10^8 \times (\frac{3.55}{1.77})^{2/3} = 6.71 \times 10^8$ m **A1**

Answer: 6.71×10^8 m **3 marks**

12 (4 marks)

(a) $v \propto \frac{1}{\sqrt{r}}$, so $v_B = \frac{7.6 \times 10^3}{\sqrt{3}}$ **M1**

$= 4390$ m s^{-1} **A1**

Answer: 4390 m s^{-1} **2 marks**

(b) $T \propto r^{3/2}$ **M1**

$\frac{T_B}{T_A} = 3^{3/2} = 3\sqrt{3} = 5.20$ **A1**

Answer: $3\sqrt{3} = 5.20$ **2 marks**

13 (3 marks)

Total energy $E = -\frac{GMm}{2r}$; as E decreases (becomes more negative) the orbital radius r must decrease **E1**

$v = \sqrt{\frac{GM}{r}}$, so a smaller radius gives a larger speed **E1**

The gravitational potential energy falls by twice as much as the kinetic energy rises, the difference being lost to the air **E1**

Answer: Energy loss makes E more negative, so r decreases; $v = \sqrt{\frac{GM}{r}}$ then increases **3 marks**

14 (6 marks)

(a) $\frac{GMm}{r^2} = m\omega^2 r$ **M1**

$\omega = \frac{2\pi}{T}$, so $\frac{GM}{r^2} = \frac{4\pi^2 r}{T^2}$ **M1**

$T^2 = \frac{4\pi^2 r^3}{GM}$ **A1**

Answer: $T^2 = \frac{4\pi^2 r^3}{GM}$ **3 marks**

(b) $\frac{4\pi^2}{GM} = 9.91 \times 10^{-14}$ s^2 m^{-3} **B1**

$T = \sqrt{9.91 \times 10^{-14} \times (1.0 \times 10^7)^3}$ **M1**

$= 9960$ s **A1**

Answer: 9.91×10^{-14} s^2 m^{-3}; T = 9960 s **3 marks**

15 (6 marks)

(a) Each star moves on a circle of radius $\frac{d}{2}$ **B1**

$\frac{GM^2}{d^2} = M\omega^2 \frac{d}{2}$ **M1 A1**

$\omega^2 = \frac{2GM}{d^3}$, so $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{d^3}{2GM}}$ **A1**

Answer: $T = 2\pi\sqrt{\frac{d^3}{2GM}}$ **4 marks**

(b) $T = 2\pi\sqrt{\frac{(1.0 \times 10^{11})^3}{2 \times 6.67 \times 10^{-11} \times 2.0 \times 10^{30}}}$ **M1**

$= 1.22 \times 10^7$ s = 141 days **A1**

Answer: 141 days **2 marks**

16 (6 marks)

(a) $r = 6.67 \times 10^6$ m, $T = 2\pi\sqrt{\frac{r^3}{GM}}$ **M1 A1**

= 5420 s **A1**

Answer: 5420 s **3 marks**

(b) The satellite gains one revolution on the point: $\frac{2\pi}{T_s} - \frac{2\pi}{T_E} = \frac{2\pi}{T}$ **M1**

$\frac{1}{T} = \frac{1}{5424} - \frac{1}{8.62 \times 10^4}$ **M1**

T = 5790 s **A1**

Answer: 5790 s **3 marks**

17 (7 marks)

(a) $M = \frac{4\pi^2 r^3}{GT^2}$ **M1**

$= \frac{4\pi^2 \times (2.0 \times 10^8)^3}{6.67 \times 10^{-11} \times (5.0 \times 10^5)^2} = 1.89 \times 10^{25}$ kg **M1 A1**

Answer: 1.89×10^{25} kg **3 marks**

(b) $g = \frac{GM}{R^2}$ **M1**

= 1.40 N kg^{-1} **A1**

Answer: 1.40 N kg^{-1} **2 marks**

(c) $v = \sqrt{\frac{2GM}{R}}$ **M1**

= 9180 m s^{-1} **A1**

Answer: 9180 m s^{-1} **2 marks**

18 (5 marks)

Increasing the speed increases the total energy of B, so its orbit radius increases **E1**

At a larger radius the orbital period is longer ($T^2 \propto r^3$) and the speed is lower than before **E1**

So B falls behind A rather than catching it **E1**

B should first slow down, moving to a lower orbit with a shorter period, so that it gains on A **E1**

When B is close to A it should speed up again to return to the original orbit **E1**

Answer: Speeding up raises B into a larger orbit with a longer period, so it falls behind; B should drop into a lower orbi **5 marks**

19 (5 marks)

(a) $KE = \frac{GMm}{2r}$, so $\Delta KE = \frac{GMm}{2}\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$ **M1**

$= -1.95 \times 10^{10}$ J **A1**

Answer: -1.95×10^{10} J (a decrease) **2 marks**

(b) $E = -\frac{GMm}{2r}$, so $\Delta E = \frac{GMm}{2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$ **M1**

$= 1.95 \times 10^{10}$ J **A1**

Answer: 1.95×10^{10} J (an increase) **2 marks**

(c) The increase in the potential energy (twice the increase in total energy) is greater than the energy supplied, so kinetic energy falls **E1**

Answer: Potential energy increases by 3.89×10^{10} J, more than the total energy gain, so kinetic energy falls **1 mark**

20 (7 marks)

(a) $v^2 = \frac{GM}{r}$, so $KE = \frac{GMm}{2r}$ **M1 A1**

Total $= \frac{GMm}{2r} - \frac{GMm}{r} = -\frac{GMm}{2r}$ **A1**

Answer: $E = -\frac{GMm}{2r}$ **3 marks**

(b) Escape requires $\frac{1}{2}mv^2 = \frac{GMm}{r}$, so $v = \sqrt{\frac{2GM}{r}}$ **M1**

$= \sqrt{2} \times \sqrt{\frac{GM}{r}}$ **A1**

Answer: $v_{esc} = \sqrt{2}v_{orbit}$ **2 marks**

(c) $\Delta v = (\sqrt{2} - 1)v = (\sqrt{2} - 1) \times 7540$ **M1**

$= 3120 \text{ m s}^{-1}$ **A1**

Answer: 3120 m s^{-1} **2 marks**

21 (7 marks)

(a) $\omega = \sqrt{\frac{GM}{r^3}}$ **M1**

$\omega_A = 1.08 \times 10^{-3}$ rad s^{-1}, $\omega_B = 6.31 \times 10^{-4}$ rad s^{-1} **A1 A1**

Answer: $\omega_A = 1.08 \times 10^{-3}$ rad s^{-1}, $\omega_B = 6.31 \times 10^{-4}$ rad s^{-1} **3 marks**

(b) A gains 2π radians on B: $(\omega_A - \omega_B)t = 2\pi$ **M1**

$t = \frac{2\pi}{1.077 \times 10^{-3} - 6.310 \times 10^{-4}} = 1.41 \times 10^4$ s **A1**

Answer: 1.41×10^4 s **2 marks**

(c) A gains π radians on B: $(\omega_A - \omega_B)t = \pi$ **M1**

$t = 7040$ s **A1**

Answer: 7040 s **2 marks**

22 (9 marks)

(a) $3M \times r_1 = M \times r_2$ with $r_1 + r_2 = d$ **M1**

$r_1 = \frac{d}{4}$, $r_2 = \frac{3d}{4}$ **A1**

Answer: Heavier star: $\frac{d}{4}$; lighter star: $\frac{3d}{4}$ **2 marks**

(b) The line joining them always passes through C, so they have equal angular speeds **E1**

For the lighter star, $\frac{G \cdot 3M \cdot M}{d^2} = M\omega^2 \times \frac{3d}{4}$ **M1**

$\omega^2 = \frac{4GM}{d^3}$ **A1**

Answer: $\omega^2 = \frac{4GM}{d^3}$ **3 marks**

(c) $v = r\omega$, so $\frac{v_1}{v_2} = \frac{r_1}{r_2} = \frac{1}{3}$ **B1**

Answer: 1 : 3 **1 mark**

(d) $\omega^2 = \frac{4GM}{d^3} = \frac{4 \times 6.67 \times 10^{-11} \times 2.0 \times 10^{30}}{(3.0 \times 10^{11})^3}$ **M1**

$T = \frac{2\pi}{\omega} = 4.47 \times 10^7$ s **A1**

$= 1.41$ years **A1**

Answer: 1.41 years **3 marks**

23 (6 marks)

(a) $\frac{GMm}{R^2} = m\omega^2 R$ with $M = \frac{4}{3}\pi R^3 \rho$ **M1**

$\omega^2 = \frac{GM}{R^3} = \frac{4\pi G\rho}{3}$ **M1**

$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{3}{4\pi G\rho}} = \sqrt{\frac{3\pi}{G\rho}}$ **A1**

Answer: $T = \sqrt{\frac{3\pi}{G\rho}}$ **3 marks**

(b) $T = \sqrt{\frac{3\pi}{6.67 \times 10^{-11} \times 5500}}$ **M1**

$= 5070$ s $= 84.5$ minutes **A1**

Answer: 84.5 minutes **2 marks**

(c) $M \propto R^3$ while the gravitational field strength at the surface $\propto \frac{M}{R^2} \propto R$, and the centripetal acceleration $\omega^2 R \propto R$, so R cancels **E1**

Answer: Because $\omega^2 = \frac{4\pi G\rho}{3}$: the radius cancels **1 mark**

24 (8 marks)

(a) $KE = \frac{GMm}{2r} \propto \frac{1}{r}$, so doubling r halves the KE **E1**

$\Delta KE = -\frac{GMm}{4r} = -2.84 \times 10^{10}$ J **M1 A1**

Answer: -2.84×10^{10} J **3 marks**

(b) $GPE = -\frac{GMm}{r}$, so $\Delta GPE = \frac{GMm}{r} - \frac{GMm}{2r} = \frac{GMm}{2r}$ **M1**

$= 5.69 \times 10^{10}$ J **A1**

Answer: 5.69×10^{10} J **2 marks**

(c) $\Delta E = \Delta KE + \Delta GPE = \frac{GMm}{4r} = 2.84 \times 10^{10}$ J **A1**

Answer: 2.84×10^{10} J **1 mark**

(d) The work done by the engine increases the total energy by $\frac{GMm}{4r}$ **B1**

Gravity does negative work, raising the potential energy by $\frac{GMm}{2r}$, which is more than the energy supplied, so the kinetic energy falls **E1**

Answer: The potential energy rises by twice the energy supplied, so kinetic energy falls **2 marks**

25 (8 marks)

(a) Initial energy $= -\frac{GMm}{R} + \frac{1}{2}m(\omega R)^2$, final energy $= -\frac{GMm}{2r}$ **M1 M1**

$\omega R = 463$ m s^{-1} **A1**

Energy supplied $= GMm(\frac{1}{R} - \frac{1}{2r}) - \frac{1}{2}m(\omega R)^2 = 2.78 \times 10^{10}$ J **A1**

Answer: 2.78×10^{10} J **4 marks**

(b) Non-rotating: 2.79×10^{10} J **M1**

Saving = 0.308% **A1**

Answer: 0.308% **2 marks**

(c) Orbital speed $= \sqrt{\frac{GM}{r}} = 7440$ m s^{-1} **M1**

(i) $7440 - 463 = 6970$ m s^{-1}; (ii) $7440 + 463 = 7900$ m s^{-1} **A1**

Answer: (i) 6970 m s^{-1}; (ii) 7900 m s^{-1} **2 marks**

26 (9 marks)

(a) Angular momentum: $v'r = v_a r_a$, so $v_a = \frac{v'r}{r_a}$ **M1**

Energy: $\frac{1}{2}v'^2 - \frac{GM}{r} = \frac{1}{2}v_a^2 - \frac{GM}{r_a}$, with $v'^2 = \frac{3GM}{2r}$ **M1**

Put $r_a = \lambda r$: $\frac{3}{4} - 1 = \frac{3}{4\lambda^2} - \frac{1}{\lambda}$ **M1**

$-\frac{1}{4} = \frac{3}{4\lambda^2} - \frac{1}{\lambda}$ gives $\lambda^2 - 4\lambda + 3 = 0$, so $(\lambda - 1)(\lambda - 3) = 0$ **A1**

$\lambda = 1$ is the burst point, so $r_a = 3r = 2.1 \times 10^7$ m **A1**

Answer: $r_a = 2.1 \times 10^7$ m **5 marks**

(b) $v_a = \frac{v'r}{3r} = \frac{\sqrt{\frac{3}{2}}v}{3}$ **M1**

$= 3080$ m s^{-1} **A1**

Answer: 3080 m s^{-1} **2 marks**

(c) Semi-major axis $a = \frac{r + 3r}{2} = 2r$, so $T \propto a^{3/2}$ gives $T = 5830 \times 2^{3/2}$ **M1**

Half of this: 8250 s **A1**

Answer: 8250 s **2 marks**

27 (12 marks)

(a) $v = \sqrt{\frac{GM}{r}}$ **M1**

$v_1 = 7730$ m s^{-1}, $v_2 = 3070$ m s^{-1} **A1**

Answer: Parking: 7730 m s^{-1}; geostationary: 3070 m s^{-1} **2 marks**

(b) $v_P r_p = v_A r_a$ **B1**

$\frac{1}{2}v_P^2 - \frac{GM}{r_p} = \frac{1}{2}v_P^2\frac{r_p^2}{r_a^2} - \frac{GM}{r_a}$ **M1**

$\frac{1}{2}v_P^2\left(\frac{r_a^2 - r_p^2}{r_a^2}\right) = GM\left(\frac{r_a - r_p}{r_p r_a}\right)$ **M1**

$v_P^2 = \frac{2GMr_a}{r_p(r_a + r_p)}$ **A1**

$v_P = 1.02 \times 10^4$ m s^{-1}, $v_A = 1600$ m s^{-1} **A1 A1**

Answer: $v_P = 1.02 \times 10^4$ m s^{-1}, $v_A = 1600$ m s^{-1} **6 marks**

(c) $\Delta v_P = 1.02 \times 10^4 - 7730 = 2430$ m s^{-1} **M1**

$\Delta v_A = 3070 - 1600 = 1470$ m s^{-1}; total $= 3890$ m s^{-1} **A1**

Answer: $\Delta v_P = 2430$ m s^{-1}, $\Delta v_A = 1470$ m s^{-1}, total 3890 m s^{-1} **2 marks**

(d) $a = \frac{r_p + r_a}{2} = 2.44 \times 10^7$ m **M1**

Time $= \frac{1}{2}T = \pi\sqrt{\frac{a^3}{GM}} = 1.90 \times 10^4$ s $= 5.28$ hours **A1**

Answer: 1.90×10^4 s (5.28 hours) **2 marks**

28 (12 marks)

(a) Each force has magnitude $\frac{GM^2}{a^2}$, at 30° to the line to O **M1**

Resultant $= 2 \times \frac{GM^2}{a^2} \times \cos 30° = \sqrt{3}\frac{GM^2}{a^2}$ **A1**

Answer: $\sqrt{3}\frac{GM^2}{a^2}$ **2 marks**

(b) Distance from a vertex to the centre of an equilateral triangle $= \frac{a}{\sqrt{3}}$ **B1**

Answer: $\mathbf{\frac{a}{\sqrt{3}}}$ **1 mark**

(c) $\sqrt{3}\frac{GM^2}{a^2} = M\omega^2\frac{a}{\sqrt{3}}$ **M1 M1**

$\omega^2 = \frac{3GM}{a^3}$ **A1**

Answer: $\mathbf{\omega^2 = \frac{3GM}{a^3}}$ **3 marks**

(d) $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{a^3}{3GM}}$ **M1**

$= 3.97 \times 10^7$ s $= 1.26$ years **A1**

Answer: 1.26 years **2 marks**

(e) Speed of each star $v^2 = \omega^2\frac{a^2}{3} = \frac{GM}{a}$, so total KE $= 3 \times \frac{1}{2}M \times \frac{GM}{a} = \frac{3GM^2}{2a}$ **M1 A1**

Total GPE $= -3\frac{GM^2}{a}$ **M1**

$E = \frac{3GM^2}{2a} - \frac{3GM^2}{a} = -\frac{3GM^2}{2a}$ **A1**

Answer: $\mathbf{E = -\frac{3GM^2}{2a}}$ **4 marks**

29 (12 marks)

(a) Rate of loss of energy = force × speed $= kmv^3$, so $\frac{dE}{dt} = -kmv^3$ **M1**

$\frac{dE}{dt} = \frac{GMm}{2r^2}\frac{dr}{dt}$ **M1**

$v^3 = (\frac{GM}{r})^{3/2}$ **B1**

$\frac{dr}{dt} = -\frac{2r^2}{GMm} \times km(\frac{GM}{r})^{3/2}$ **M1**

$= -2k\sqrt{GMr}$ **A1**

Answer: $\mathbf{\frac{dr}{dt} = -2k\sqrt{GMr}}$ **5 marks**

(b) $\int_{r_0}^{r_1} r^{-1/2}\,dr = -2k\sqrt{GM}\int_0^t dt$ **M1**

$2(\sqrt{r_1} - \sqrt{r_0}) = -2k\sqrt{GM}\ t$ **M1**

$t = \frac{\sqrt{r_0} - \sqrt{r_1}}{k\sqrt{GM}}$ **A1**

Answer: $\mathbf{t = \frac{\sqrt{r_0} - \sqrt{r_1}}{k\sqrt{GM}}}$ **3 marks**

(c) $t = \frac{\sqrt{6.80 \times 10^6} - \sqrt{6.60 \times 10^6}}{1.0 \times 10^{-12} \times \sqrt{3.98 \times 10^{14}}}$ **M1**

$= 1.94 \times 10^6$ s $= 22.4$ days **A1**

Answer: 22.4 days **2 marks**

(d) $v = \sqrt{\frac{GM}{r}}$ increases as r decreases **B1**

The gravitational potential energy decreases by twice the amount that the kinetic energy increases, so the total energy decreases **E1**

Answer: $\mathbf{v = \sqrt{\frac{GM}{r}}}$ **increases as r decreases; the fall in GPE exceeds the rise in KE** **2 marks**

30 (12 marks)

(a) $\frac{GM_SM_E}{R^2} = M_E\omega^2R$ **M1**

$\omega^2 = \frac{GM_S}{R^3}$ **A1**

Answer: $\boldsymbol{\omega^2 = \frac{GM_S}{R^3}}$ **2 marks**

(b) The Sun attracts P towards the Sun with force per unit mass $\frac{GM_S}{(R-x)^2}$, and the Earth attracts P away from the Sun with force per unit mass $\frac{GM_E}{x^2}$ **B1**

The resultant provides the centripetal acceleration $\omega^2(R-x)$ of P, which has the Earth's angular speed **M1**

Substituting ω^2 from (a) gives the required equation **A1**

Answer: $\boldsymbol{\frac{GM_S}{(R-x)^2} - \frac{GM_E}{x^2} = \frac{GM_S(R-x)}{R^3}}$ **3 marks**

(c) $\frac{1}{(R-x)^2} = \frac{1}{R^2}(1-\frac{x}{R})^{-2} \approx \frac{1}{R^2}(1+\frac{2x}{R})$ **M1**

$\frac{GM_S}{R^2}(1+\frac{2x}{R}) - \frac{GM_E}{x^2} = \frac{GM_S}{R^2}(1-\frac{x}{R})$ **M1**

$\frac{3GM_Sx}{R^3} = \frac{GM_E}{x^2}$ **M1**

$x^3 = \frac{R^3M_E}{3M_S}$ **A1**

Answer: $\boldsymbol{x^3 \approx \frac{R^3M_E}{3M_S}}$ **4 marks**

(d) $x = R(\frac{M_E}{3M_S})^{1/3} = 1.50 \times 10^{11} \times (1.00 \times 10^{-6})^{1/3}$ **M1**

$= 1.50 \times 10^9$ m **A1**

$\frac{x}{R} = 0.01$, which is small **E1**

Answer: $\boldsymbol{x = 1.50 \times 10^9}$ m; $\boldsymbol{x/R = 0.01}$ **3 marks**

Motion Under Variable Gravity

1 (2 marks)

Force $= \frac{k\,m}{x^2}$, and at $x = R$ the acceleration is g, so $k = gR^2$ **M1**

Acceleration $= \frac{gR^2}{x^2}$ **A1**

Answer: $\boldsymbol{\frac{gR^2}{x^2}}$ **2 marks**

2 (3 marks)

$\frac{1}{2}v^2 = \frac{8}{x} + C$ **M1**

$v = 0$, $x = 2$: $C = -4$, so at $x = 1$, $\frac{1}{2}v^2 = 4$ **M1**

$v^2 = 8$ **A1**

Answer: $\boldsymbol{v^2 = 8}$ **3 marks**

3 (2 marks)

$\frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{dv}{dx} \times \frac{dx}{dt}$ **M1**

$= v\frac{dv}{dx}$ **A1**

Answer: $a = v\frac{dv}{dx}$ **2 marks**

4 (2 marks)

$v^2 = 2gR^2(\frac{1}{R} - \frac{1}{2R}) = gR$ **M1**

$v = \sqrt{9.8 \times 6.37 \times 10^6} = 7900 \text{ m s}^{-1}$ **A1**

Answer: 7900 m s^{-1} **2 marks**

5 (2 marks)

$v^2 = 2k(\frac{2}{a} - \frac{1}{a}) = \frac{2k}{a}$ **M1**

$v = \sqrt{\frac{2k}{a}}$ **A1**

Answer: $\sqrt{\frac{2k}{a}}$ **2 marks**

6 (2 marks)

$\omega^2 = \frac{g}{R}$, so $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{R}{g}}$ **M1**

$= 5070$ s **A1**

Answer: 5070 s **2 marks**

7 (4 marks)

$v\frac{dv}{dx} = -\frac{gR^2}{x^2}$, so $\frac{1}{2}v^2 = \frac{gR^2}{x} + C$ **M1**

$x = R$, $v^2 = \frac{gR}{2}$: $C = \frac{gR}{4} - gR = -\frac{3gR}{4}$ **M1 A1**

$v = 0$: $\frac{gR^2}{x} = \frac{3gR}{4}$, $x = \frac{4R}{3}$, so height $= \frac{R}{3} = 2.12 \times 10^6$ m **A1**

Answer: $\frac{R}{3} = 2.12 \times 10^6$ m **4 marks**

8 (4 marks)

$v\frac{dv}{dx} = -\frac{gR^2}{x^2}$, so $\frac{1}{2}v^2 = \frac{gR^2}{x} + C$; $v = 0$ at $x = 4R$ gives $C = -\frac{gR}{4}$ **M1 A1**

At $x = R$: $\frac{1}{2}v^2 = gR - \frac{gR}{4} = \frac{3gR}{4}$ **M1**

$v = \sqrt{\frac{3gR}{2}} = 9680 \text{ m s}^{-1}$ **A1**

Answer: $\sqrt{\frac{3gR}{2}} = 9680$ m s^{-1} **4 marks**

9 (4 marks)

$\frac{1}{2}v^2 = \frac{gR^2}{x} + C$ **M1**

At $x = \frac{3R}{2}$, $v^2 = gR$: $\frac{1}{2}gR = \frac{2gR}{3} + C$, so $C = -\frac{gR}{6}$ **M1 A1**

$v = 0$: $\frac{gR^2}{x} = \frac{gR}{6}$, so $x = 6R$ **A1**

Answer: 6R **4 marks**

10 (4 marks)

$v\frac{dv}{dx} = -\frac{k}{x^2}$ gives $\frac{1}{2}v^2 = \frac{k}{x} + C$ **M1**

$v = 0$, $x = a$: $C = -\frac{k}{a}$ **A1**

$v^2 = 2k(\frac{1}{x} - \frac{1}{a})$ **A1**

At $x = \frac{a}{4}$: $v^2 = 2k(\frac{4}{a} - \frac{1}{a}) = \frac{6k}{a}$, so $v = \sqrt{\frac{6k}{a}}$ **A1**

Answer: $v^2 = 2k(\frac{1}{x} - \frac{1}{a})$; $v = \sqrt{\frac{6k}{a}}$ **4 marks**

11 (4 marks)

(a) $a = 9.8$ m s^{-2} **B1**

Answer: 9.8 m s^{-2} **1 mark**

(b) $a = \frac{9.8}{4}$ **M1**

$= 2.45$ m s^{-2} **A1**

Answer: 2.45 m s^{-2} **2 marks**

(c) $a = \frac{9.8}{16} = 0.612$ m s^{-2} **A1**

Answer: 0.612 m s^{-2} **1 mark**

12 (4 marks)

$\frac{1}{2}v^2 = \frac{gR^2}{x} + C$ **M1**

$x = R$, $v = 8000$: $C = \frac{1}{2}(8000)^2 - gR$ **M1**

$v = 0$: $\frac{gR^2}{x} = gR - 3.2 \times 10^7 = 3.04 \times 10^7$, so $x = 1.31 \times 10^7$ m **A1**

Height $= x - R = 6.70 \times 10^6$ m **A1**

Answer: 6.70×10^6 m **4 marks**

13 (5 marks)

(a) $a = v\frac{dv}{dx} = \frac{d}{dx}(\frac{1}{2}v^2)$ **M1**

$= \frac{d}{dx}(18 + 12x^{-1}) = -\frac{12}{x^2}$ **A1**

Answer: $a = -\frac{12}{x^2}$ **2 marks**

(b) $v^2 = 36 + 6 = 42$, so $v = 6.48$ m s^{-1} **M1 A1**

$a = -\frac{12}{16} = -0.75$ m s^{-2} **A1**

Answer: $v = 6.48$ m s^{-1}; $a = -0.75$ m s^{-2} **3 marks**

14 (6 marks)

(a) $v\frac{dv}{dx} = -\frac{k}{x^3}$ gives $\frac{1}{2}v^2 = \frac{k}{2x^2} + C$ **M1**

$v = 0$, $x = a$: $C = -\frac{k}{2a^2}$ **M1**

$v^2 = k(\frac{1}{x^2} - \frac{1}{a^2})$ **A1**

Answer: $v^2 = k(\frac{1}{x^2} - \frac{1}{a^2})$ **3 marks**

(b) Speed $= \frac{\sqrt{k}\sqrt{a^2 - x^2}}{ax}$, so $t = \int_0^a \frac{ax}{\sqrt{k}\sqrt{a^2 - x^2}}\,dx$ **M1**

$= \frac{a}{\sqrt{k}}[-\sqrt{a^2 - x^2}]_0^a$ **M1**

$= \frac{a^2}{\sqrt{k}}$ **A1**

Answer: $\frac{\mathbf{a^2}}{\sqrt{\mathbf{k}}}$ **3 marks**

15 (6 marks)

(a) $v\frac{dv}{dx} = -\frac{gR}{x}$ gives $\frac{1}{2}v^2 = -gR \ln x + C$ **M1**

$x = R$, $v = u$: $C = \frac{1}{2}u^2 + gR \ln R$ **M1**

$v^2 = u^2 - 2gR \ln(\frac{x}{R})$ **A1**

Answer: $v^2 = u^2 - 2gR \ln(\frac{x}{R})$ **3 marks**

(b) $v = 0$: $\ln(\frac{x}{R}) = \frac{1}{2}$ **M1**

$x = Re^{1/2} = 1.65\ R$ **A1**

Answer: $Re^{1/2} = 1.65\ R$ **2 marks**

(c) For any u, v^2 becomes zero at $\ln(\frac{x}{R}) = \frac{u^2}{2gR}$, a finite distance **E1**

Answer: v^2 reaches zero at a finite x for every u **1 mark**

16 (7 marks)

(a) $\frac{d^2x}{dt^2} = -\frac{g}{R}x$ **M1**

This is of the form $\ddot{x} = -\omega^2 x$ with $\omega^2 = \frac{g}{R}$, so SHM **A1**

Period $= 2\pi\sqrt{\frac{R}{g}} = 5070$ s **A1**

Answer: SHM with period 5070 s **3 marks**

(b) $v\frac{dv}{dx} = -\frac{g}{R}x$ gives $v^2 = \frac{g}{R}(R^2 - x^2)$ **M1**

At $x = \frac{R}{2}$: $v = \frac{1}{2}\sqrt{3gR} = 6840$ m s^{-1} **A1**

Answer: 6840 m s^{-1} **2 marks**

(c) $x = R \cos \omega t$; $x = 0$ when $\omega t = \frac{\pi}{2}$ **M1**

$t = \frac{\pi}{2}\sqrt{\frac{R}{g}} = 1270$ s **A1**

Answer: 1270 s **2 marks**

17 (5 marks)

(a) $\int -gR^2(R + h)^{-2}\, dh = gR^2(R + h)^{-1} + C$ **M1**

The student has left the power as 2 instead of integrating to power -1, so Line 2 should read $\frac{1}{2}v^2$ **A1**

$= \frac{gR^2}{(R + h)} + C$

Answer: $\frac{1}{2}\mathbf{v^2} = \frac{\mathbf{gR^2}}{\mathbf{(R + h)}} + \mathbf{C}$ **2 marks**

(b) $h = 0$: $\frac{1}{2}gR = gR + C$, so $C = -\frac{gR}{2}$ **M1**

$v = 0$: $\frac{gR^2}{(R + h)} = \frac{gR}{2}$, so $R + h = 2R$ **M1**

$h = R$ **A1**

Answer: h = R **3 marks**

18 (6 marks)

(a) $\frac{1}{2}v^2 = \frac{gR^2}{x} + C$, with x = R, v = u — M1

$\frac{1}{2}u^2 - gR = -\frac{gR^2}{x_{max}}$ — M1

$\frac{1}{x_{max}} = \frac{1}{R} - \frac{u^2}{2gR^2} = 1.67 \times 10^{-7}$ — A1

Height = $x_{max} - R = 4.25 \times 10^6$ m — A1

Answer: 4.25×10^6 m — 4 marks

(b) $\frac{u^2}{2g} = 1.23 \times 10^6$ m — B1

Gravity weakens with height, so the true deceleration is less than 1.62 and the projectile goes higher — E1

Answer: 1.23×10^6 m; real gravity weakens with height — 2 marks

19 (6 marks)

(a) $\frac{dx}{dt} = -A\omega \sin \omega t + B\omega \cos \omega t$ — M1

$\frac{d^2x}{dt^2} = -A\omega^2 \cos \omega t - B\omega^2 \sin \omega t = -\omega^2 x$ — A1

Answer: x = A cos ωt + B sin ωt is a solution — 2 marks

(b) x(0) = R gives A = R — B1

$\frac{dx}{dt}(0) = 0$ gives Bω = 0, so B = 0 — B1

Answer: A = R, B = 0 — 2 marks

(c) cos ωt = $\frac{1}{2}$, so ωt = $\frac{\pi}{3}$ — M1

$t = \frac{\pi}{3}\sqrt{\frac{R}{g}} = 844$ s — A1

Answer: 844 s — 2 marks

20 (6 marks)

(a) $\frac{1}{2}v^2 = \frac{gR^2}{x} + C$ — M1

v = 0 at x = 3R: $C = -\frac{gR}{3}$; v = u at x = R: $\frac{1}{2}u^2 = gR - \frac{gR}{3}$ — M1 A1

$u = \sqrt{\frac{4gR}{3}}$ — A1

Answer: $u = \sqrt{\frac{4gR}{3}}$ — 4 marks

(b) u = 9120 m s^{-1} — B1

Fraction = $\sqrt{\frac{4}{3}} \div \sqrt{2} = \sqrt{\frac{2}{3}} = 0.816$ — B1

Answer: 9120 m s^{-1}, which is 0.816 of the escape speed — 2 marks

21 (9 marks)

(a) $\frac{1}{2}v^2 = \frac{k}{x} + C$ — M1

$v^2 = 2k(\frac{1}{x} - \frac{1}{a})$ — A1

Answer: $v^2 = 2k(\frac{1}{x} - \frac{1}{a})$ — 2 marks

(b) Speed = $\sqrt{\frac{2k(a-x)}{ax}}$ — M1

$t = \int_0^a \sqrt{\frac{ax}{2k(a-x)}}\,dx = \sqrt{\frac{a}{2k}} \int_0^a \sqrt{\frac{x}{(a-x)}}\,dx$ — A1

Answer: $t = \sqrt{\frac{a}{2k}} \int_0^a \sqrt{\frac{x}{(a-x)}}\,dx$ — 2 marks

(c) $\sqrt{\frac{x}{(a-x)}} = \cot\theta$ and $dx = -2a\cos\theta\sin\theta\, d\theta$ **M1**

$\int_0^a \sqrt{\frac{x}{(a-x)}}\, dx = \int_0^{\pi/2} 2a\cos^2\theta\, d\theta$ **M1**

$= a\int_0^{\pi/2}(1+\cos 2\theta)\, d\theta = \frac{\pi a}{2}$, so $t = \sqrt{\frac{a}{2k}} \times \frac{\pi a}{2} = \frac{\pi}{2}\sqrt{\frac{a^3}{2k}}$ **A1**

Answer: $t = \frac{\pi}{2}\sqrt{\frac{a^3}{2k}}$ **3 marks**

(d) $k = 1.33 \times 10^{20}\ m^3\ s^{-2}$ **B1**

$t = \frac{\pi}{2}\sqrt{\frac{(1.50\times10^{11})^3}{2k}} = 5.60 \times 10^6\ s = 64.8$ days **A1**

Answer: 64.8 days **2 marks**

22 (7 marks)

(a) The thin ellipse has length $2s = a$ at its longest, so $s = \frac{a}{2}$ **M1**

$T = 2\pi\sqrt{\frac{(a/2)^3}{GM}} = 2\pi\sqrt{\frac{a^3}{8GM}}$ **M1**

Time of fall $= \frac{T}{2} = \pi\sqrt{\frac{a^3}{8GM}} = \frac{\pi}{2}\sqrt{\frac{a^3}{2GM}}$ **A1**

Answer: $\frac{\pi}{2}\sqrt{\frac{a^3}{2GM}}$ **3 marks**

(b) $t = \frac{\pi}{2}\sqrt{\frac{(3.84\times10^8)^3}{2\times3.98\times10^{14}}}$ **M1**

$= 4.19 \times 10^5\ s = 4.85$ days **A1**

Answer: 4.85 days **2 marks**

(c) Circular period $= 2\pi\sqrt{\frac{a^3}{GM}}$ **M1**

Ratio $= \frac{\frac{\pi}{2}\times\frac{1}{\sqrt{2}}}{2\pi} = \frac{1}{4\sqrt{2}}$ **A1**

Answer: $1 : 4\sqrt{2}$ **2 marks**

23 (9 marks)

(a) $\frac{1}{2}v^2 = \int_R^{3R} \frac{gR^2}{x^2}\, dx$ **M1**

$= gR^2[-\frac{1}{x}]_R^{3R} = gR^2(\frac{1}{R} - \frac{1}{3R}) = \frac{2gR}{3}$ **M1 A1**

$v = \sqrt{\frac{4gR}{3}}$ **A1**

Answer: $\sqrt{\frac{4gR}{3}}$ **4 marks**

(b) Inside: $\frac{1}{2}v_c^2 = \frac{1}{2}v_s^2 + \int_0^R \frac{gx}{R}\, dx$ **M1**

$= \frac{2gR}{3} + \frac{gR}{2}$ **M1 A1**

$v_c = \sqrt{\frac{7gR}{3}}$ **A1**

Answer: $\sqrt{\frac{7gR}{3}}$ **4 marks**

(c) $v_c = 1.21 \times 10^4\ m\ s^{-1}$ **B1**

Answer: $1.21 \times 10^4\ m\ s^{-1}$ **1 mark**

24 (8 marks)

(a) Newton's second law: $2000a = 4.0 \times 10^4 - \frac{2000gR^2}{x^2}$ **M1**

$a = v\frac{dv}{dx} = 20 - \frac{gR^2}{x^2}$ **A1**

Answer: $v\frac{dv}{dx} = 20 - \frac{gR^2}{x^2}$ **2 marks**

(b) $\frac{1}{2}v^2 = 20h - gR^2(\frac{1}{R} - \frac{1}{R+h})$ **M1 M1**

$= 2.0 \times 10^6 - 9.8 \times 6.37 \times 10^6 \times \frac{10^5}{6.47 \times 10^6} = 1.04 \times 10^6$ **A1**

$v = 1440$ m s^{-1} **A1**

Answer: 1440 m s^{-1} **4 marks**

(c) $\frac{1}{2}v^2 = (20 - 9.8) \times 10^5$, $v = 1430$ m s^{-1} **M1**

Error $= -0.734\%$ **A1**

Answer: 1430 m s^{-1}, an error of -0.734% **2 marks**

25 (8 marks)

(a) Attraction to B (towards +x): $\frac{GM}{(a-x)^2}$; attraction to A (towards −x): $\frac{GM}{(a+x)^2}$ **M1**

Net acceleration $= GM(\frac{1}{(a-x)^2} - \frac{1}{(a+x)^2})$ **A1**

Answer: $a = GM(\frac{1}{(a-x)^2} - \frac{1}{(a+x)^2})$ **2 marks**

(b) $v\frac{dv}{dx} = GM(\frac{1}{(a-x)^2} - \frac{1}{(a+x)^2})$, so $\frac{1}{2}v^2 = GM(\frac{1}{a-x} + \frac{1}{a+x}) + C = \frac{2GMa}{a^2-x^2} + C$ **M1 A1**

$x = \frac{a}{2}$, $v = 0$: $C = -\frac{2GMa \times 4}{3a^2}$ **M1**

$v^2 = 4GMa(\frac{1}{a^2-x^2} - \frac{4}{3a^2})$ **A1**

Answer: $v^2 = 4GMa(\frac{1}{a^2-x^2} - \frac{4}{3a^2})$ **4 marks**

(c) $v^2 = 4GMa(\frac{16}{7a^2} - \frac{4}{3a^2}) = \frac{4GM}{a} \times \frac{20}{21}$ **M1**

$v = \sqrt{\frac{80GM}{21a}}$ **A1**

Answer: $\sqrt{\frac{80GM}{21a}}$ **2 marks**

26 (7 marks)

(a) $\frac{1}{2}v^2 = gR^2(\frac{2}{3R} - \frac{1}{3R})$ **M1 A1**

$v^2 = \frac{2gR}{3}$, so $v = \sqrt{\frac{2 \times 1.62 \times 1.74 \times 10^6}{3}} = 1370$ m s^{-1} **A1**

Answer: 1370 m s^{-1} **3 marks**

(b) Work-energy from $\frac{3R}{2}$ to R: $0 - \frac{1}{2}mv^2 = mgR^2(\frac{1}{R} - \frac{2}{3R}) - F \times \frac{R}{2}$ **M1 M1**

$\frac{FR}{2} = \frac{mgR}{3} + \frac{mgR}{3}$, so $F = \frac{4mg}{3}$ **A1**

$F = \frac{4 \times 2500 \times 1.62}{3} = 5400$ N **A1**

Answer: 5400 N **4 marks**

27 (12 marks)

(a) $\frac{dx}{d\eta} = -\frac{a}{2}\sin\eta$ and $\frac{dt}{d\eta} = \sqrt{\frac{a^3}{8k}}(1 + \cos\eta)$ **M1 M1**

$\frac{dx}{dt} = \frac{dx}{d\eta} \div \frac{dt}{d\eta}$; $\sin\eta = 2\sin(\frac{\eta}{2})\cos(\frac{\eta}{2})$ and $1 + \cos\eta = 2\cos^2(\frac{\eta}{2})$ **M1**

$\frac{dx}{dt} = -\frac{a}{2}\sqrt{\frac{8k}{a^3}}\tan(\frac{\eta}{2}) = -\sqrt{\frac{2k}{a}}\tan(\frac{\eta}{2})$ **A1**

Answer: $\mathbf{\frac{dx}{dt} = -\sqrt{\frac{2k}{a}}\tan(\frac{\eta}{2})}$ **4 marks**

(b) $\frac{d^2x}{dt^2} = \frac{d}{d\eta}(\frac{dx}{dt}) \div \frac{dt}{d\eta}$ **M1**

$= -\sqrt{\frac{2k}{a}} \times \frac{1}{2}\sec^2(\frac{\eta}{2}) \div (2\sqrt{\frac{a^3}{8k}}\cos^2(\frac{\eta}{2}))$ **M1**

$= -\frac{k}{a^2}\sec^4(\frac{\eta}{2})$ **A1**

$x = a\cos^2(\frac{\eta}{2})$, so $\frac{k}{x^2} = \frac{k}{a^2}\sec^4(\frac{\eta}{2})$, as required **A1**

Answer: $\mathbf{\frac{d^2x}{dt^2} = -\frac{k}{x^2}}$ **4 marks**

(c) $x = 0$ when $\eta = \pi$ **M1**

$t = \pi\sqrt{\frac{a^3}{8k}} = \frac{\pi}{2}\sqrt{\frac{a^3}{2k}}$ **A1**

Answer: $\mathbf{\frac{\pi}{2}\sqrt{\frac{a^3}{2k}}}$ **2 marks**

(d) $x = \frac{a}{2}$ when $\cos\eta = 0$, $\eta = \frac{\pi}{2}$ **M1**

$t = \sqrt{\frac{a^3}{8k}}(\frac{\pi}{2} + 1)$ **A1**

Answer: $\mathbf{\sqrt{\frac{a^3}{8k}}(\frac{\pi}{2} + 1)}$ **2 marks**

28 (11 marks)

(a) The force has magnitude $\frac{mgr}{R}$ directed along PO, where $r = OP$ **B1**

The angle between PO and PM has sine $\frac{s}{r}$, so the component along the tunnel is $\frac{mgr}{R} \times \frac{s}{r}$ **M1 A1**

Answer: $\mathbf{\frac{mgs}{R}}$ **towards M** **3 marks**

(b) $\frac{d^2s}{dt^2} = -\frac{g}{R}s$ **M1**

$\omega = \sqrt{\frac{g}{R}}$, period $2\pi\sqrt{\frac{R}{g}}$, which does not involve d **A1**

Answer: Period $\mathbf{2\pi\sqrt{\frac{R}{g}}}$ **2 marks**

(c) At M the gravitational force is $\frac{mgd}{R}$ towards O, perpendicular to the tunnel **M1**

Normal reaction = $\frac{mgd}{R}$, directed away from O **A1**

Answer: $\mathbf{\frac{mgd}{R}}$ **2 marks**

(d) Amplitude = 1.7×10^5 m, so $v_{max} = A\omega = 1.7 \times 10^5 \times \sqrt{\frac{9.8}{6.37 \times 10^6}} = 211$ m s^{-1} **M1 A1**

Time = half a period = $\pi\sqrt{\frac{R}{g}} = 2530$ s (42.2 minutes) **M1 A1**

Answer: Maximum speed 211 m s^{-1}; time 2530 s (42.2 minutes) **4 marks**

29 (10 marks)

(a) $v\frac{dv}{dx} = -\frac{gR^n}{x^n}$, so $\frac{1}{2}v^2 = \frac{gR^n x^{1-n}}{n-1} + C$ **M1 A1**

$x = R$, $v = u$: $C = \frac{1}{2}u^2 - \frac{gR}{n-1}$ **M1**

$v^2 = u^2 + \frac{2gR}{n-1}[(\frac{R}{x})^{n-1} - 1]$ **A1**

Answer: $v^2 = u^2 + \frac{2gR}{n-1}[(\frac{R}{x})^{n-1} - 1]$ **4 marks**

(b) As $x \to \infty$, $(\frac{R}{x})^{n-1} \to 0$ for $n > 1$ **M1**

So $v^2 \to u^2 - \frac{2gR}{n-1}$, and the particle never stops if this is not negative **A1**

Answer: Escape speed squared = $\frac{2gR}{n-1}$ **2 marks**

(c) $n = 2$: $\sqrt{2gR}$; $n = 3$: $\sqrt{gR}$ **B1 B1**

For $n = 1$, $v\frac{dv}{dx} = -\frac{gR}{x}$ gives $v^2 = u^2 - 2gR\ln(\frac{x}{R})$, which becomes zero at a finite x for any u **E1**

Answer: n = 2: $\sqrt{2gR}$; n = 3: $\sqrt{gR}$; for n = 1, v^2 reaches zero at a finite distance for every u **3 marks**

(d) The force falls off more quickly with distance when $n = 3$, so less work needs to be done against it to reach infinity **E1**

Answer: Gravity weakens faster for n = 3, so less energy is needed **1 mark**

30 (12 marks)

(a) Each of A and B attracts P with acceleration $\frac{GM}{a^2 + y^2}$; the components perpendicular to OP **M1 A1**

cancel; each has component along PO of $\frac{GM}{a^2 + y^2} \times \frac{y}{\sqrt{a^2 + y^2}}$

Total = $\frac{2GMy}{(a^2 + y^2)^{3/2}}$ **A1**

Answer: $\frac{2GMy}{(a^2 + y^2)^{3/2}}$ **3 marks**

(b) For y small compared with a: $\frac{d^2y}{dt^2} \approx -\frac{2GM}{a^3}y$ **M1 A1**

$\omega^2 = \frac{2GM}{a^3}$, period $= 2\pi\sqrt{\frac{a^3}{2GM}}$ **A1**

Answer: Period $\approx 2\pi\sqrt{\frac{a^3}{2GM}}$ **3 marks**

(c) $v\frac{dv}{dy} = -\frac{2GMy}{(a^2 + y^2)^{3/2}}$, so $\frac{1}{2}v^2 = \frac{2GM}{\sqrt{a^2 + y^2}} + C$ **M1 A1**

At $y = \frac{a}{2}$, $v = 0$, $\sqrt{a^2 + y^2} = \frac{a\sqrt{5}}{2}$ **M1**

At $y = 0$: $\frac{1}{2}v^2 = 2GM(\frac{1}{a} - \frac{2}{a\sqrt{5}})$, giving the result **A1**

Answer: $v^2 = \frac{4GM}{a}(1 - \frac{2}{\sqrt{5}})$ **4 marks**

(d) SHM: $v^2 = \omega^2(y_0)^2 = \frac{2GM}{a^3} \times \frac{a^2}{4} = \frac{GM}{2a}$ **M1**

Exact $v^2 = 0.4223\frac{GM}{a}$, approximate $0.5\frac{GM}{a}$: error = 18.4% **A1**

Answer: Exact 0.4223 GM/a, SHM 0.5 GM/a: the SHM model overestimates v^2 by 18.4% **2 marks**