Newton's Law of Gravitation

1 Two uniform spheres of masses 5.0 kg and 12 kg have their centres 0.40 m apart. Find the gravitational force between them. Take $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}. **[2]**

2 Calculate the gravitational field strength at the surface of the Earth, given $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}, the mass of the Earth is 5.97×10^{24} kg and its radius is 6.37×10^6 m. **[2]**

3 The gravitational force between two particles is F. Write down the new force, in terms of F, when

(a) the distance between the particles is doubled, **[1]**

(b) one mass is tripled and the distance is halved. **[2]**

4 The gravitational field strength at the surface of the Moon is 1.62 N kg^{-1} and the radius of the Moon is 1.74×10^6 m. Find the mass of the Moon. Take $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}. **[2]**

5 A planet has the same mass as the Earth and three times the radius of the Earth. The gravitational field strength at the surface of the Earth is 9.8 N kg^{-1}. Find the gravitational field strength at the surface of the planet. **[2]**

6 A satellite has weight 600 N on the surface of the Earth. Find its weight when it is at a height above the surface equal to twice the radius of the Earth. **[2]**

7 Find the gravitational field strength at a height of 1.0×10^6 m above the surface of the Earth. Take $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}, $M_E = 5.97 \times 10^{24}$ kg and $R_E = 6.37 \times 10^6$ m. **[3]**

8 The gravitational field strength at the surface of Mars is 3.71 N kg^{-1} and the radius of Mars is 3.39×10^6 m. Take $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}.

(a) Find the mass of Mars. **[2]**

(b) Find the mean density of Mars. **[2]**

9 An astronaut has mass 85 kg. Take $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}, $M_M = 7.35 \times 10^{22}$ kg and $R_M = 1.74 \times 10^6$ m for the Moon, and g = 9.8 N kg^{-1} at the surface of the Earth.

(a) Find the weight of the astronaut on the surface of the Moon. **[2]**

(b) Express this as a fraction of the weight of the astronaut on the surface of the Earth. **[2]**

10 The centres of the Earth and the Moon are 3.84×10^8 m apart. The masses of the Earth and Moon are 5.97×10^{24} kg and 7.35×10^{22} kg. Take $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}.

(a) Find the gravitational force between the Earth and the Moon. **[2]**

(b) Find the acceleration of the Moon towards the Earth. **[2]**

11 At the surface of the Earth the gravitational field strength is 9.8 N kg^{-1}. Find the gravitational field strength at the surface of

(a) planet A, which has 8 times the mass of the Earth and twice its radius, **[2]**

(b) planet B, which has 0.1 times the mass of the Earth and 0.4 times its radius. **[2]**

12 The Earth has mass M and radius R, and g is the gravitational field strength at its surface.

(a) Show that $GM = gR^2$. **[2]**

(b) Using this result, find the weight of a 20 kg object at a height of 3R above the surface of the Earth. Take g = 9.8 N kg^{-1}. **[2]**

13 At the surface of the Earth g = 9.8 N kg^{-1}. Find the height above the surface at which the gravitational field strength is 9.0 N kg^{-1}. Take the radius of the Earth to be 6.37×10^6 m. **[3]**

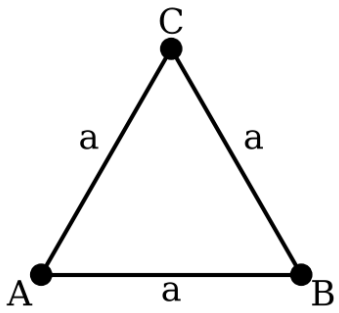
14 The centres of the Earth and the Moon are a distance 3.84×10^8 m apart. Find the distance from the centre of the Earth of the point on the line joining their centres at which the resultant gravitational field strength is zero. Take the mass of the Earth to be 5.97×10^{24} kg and the mass of the Moon to be 7.35×10^{22} kg. **[5]**

15 Particles A and B have masses 9M and M respectively, and AB = 8d.

(a) Find the distance from A of the point C on the line AB at which the resultant gravitational field strength is zero. **[3]**

(b) A particle P of mass m is placed at the midpoint of AB. Find the magnitude and direction of the resultant gravitational force on P. **[3]**

16 Three particles A, B and C, each of mass m, are at the vertices of an equilateral triangle of side a.



(a) Show that the magnitude of the resultant gravitational force on A due to B and C is $\sqrt{3}Gm^2/a^2$. **[3]**

(b) The particle at B is replaced by one of mass 2m. Find the magnitude of the resultant force on A. **[3]**

17 A uniform spherical planet has density ρ and radius R.

(a) Show that the gravitational field strength at its surface is $\frac{4}{3}\pi G\rho R$. **[3]**

(b) A planet has density 5500 kg m^{-3} and radius 3.4×10^6 m. Find the gravitational field strength at its surface. Take $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}. **[2]**

18 The Earth rotates once in 24 hours. Model the Earth as a sphere of radius 6.37×10^6 m, at the surface of which the gravitational field strength is 9.83 N kg^{-1}. A person of mass 70 kg stands on weighing scales.

(a) Find the acceleration of a point on the equator towards the axis of rotation. **[2]**

(b) Find the reading of the scales, in N, when the person is at the equator. **[2]**

(c) Find the percentage by which this is less than the reading at a pole. **[2]**

19 Take the radius of the Earth to be 6.37×10^6 m.

(a) Find the height above the surface of the Earth at which the weight of an object is 10% less than its weight on the surface. **[3]**

(b) Use a binomial approximation to estimate the percentage decrease in the weight of an object carried to a height of 10 km. **[3]**

20 The International Space Station orbits at a height of 4.0×10^5 m above the surface of the Earth. Take $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}, $M_E = 5.97 \times 10^{24}$ kg and $R_E = 6.37 \times 10^6$ m.

(a) Calculate the gravitational field strength at this height. **[3]**

(b) A student says that the astronauts are weightless because the Earth's gravity is zero at this height. Explain why the student is wrong and why the astronauts nevertheless feel weightless. **[2]**

21 Fixed particles A and B have masses M and 9M respectively, with AB = 4a. A particle P of mass m moves on the line AB.

(a) Find the distance from A of the point E on AB at which the resultant gravitational force on P is zero. **[3]**

(b) P is displaced from E a small distance δ towards B. By using a binomial approximation, show that the resultant force on P is then directed towards B, and state what this shows about the equilibrium at E. **[4]**

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22 Assume that the Earth is a uniform sphere of mass M and radius R, and that the matter outside a sphere centred at O (and concentric with the Earth) exerts no force on a particle inside it. The gravitational field strength at the surface is g.

(a) Show that the gravitational field strength at a distance r from the centre, where $r < R$, is $\frac{gr}{R}$. **[4]**

(b) Find the depth below the surface at which the weight of an object is 80% of its weight at the surface. **[2]**

(c) Find the height above the surface at which the weight is 80% of its surface value, and compare it with your answer to part (b). **[3]**

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23 Three particles A, B and C of masses m, 2m and 3m lie in that order on a straight line, with AB = 2a and BC = 3a. Find, in terms of G, m and a, the magnitude and direction of the resultant gravitational force on each particle, and show that the three resultants sum to zero. **[8]**

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24 A ball of water on the near side of the Earth is attracted by the Moon more strongly than one on the far side, which causes tides. The distance between the centres of the Earth and Moon is $d = 3.84 \times 10^8$ m. Take $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}, $M_M = 7.35 \times 10^{22}$ kg and $R = 6.37 \times 10^6$ m for the radius of the Earth.

(a) Find the Moon's gravitational field strength at the point on the Earth's surface nearest the Moon, and at the point farthest from it. **[4]**

(b) Find the difference between these two field strengths. **[1]**

(c) Show that this difference is approximately $\frac{4GM_MR}{d^3}$, and verify that the approximation agrees with your answer to part (b). **[3]**

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25 Planet X has mean density 1.5 times that of the Earth and radius 0.6 times that of the Earth. Both are uniform spheres.

(a) Show that the gravitational field strength at the surface of X is 0.9 times that at the surface of the Earth. **[3]**

(b) Find the mass of X as a multiple of the mass of the Earth. **[2]**

(c) The speed needed to escape from a planet is $\sqrt{2gR}$, where g is the surface gravitational field strength and R the radius. Find the ratio of the escape speed from X to that from the Earth. **[2]**

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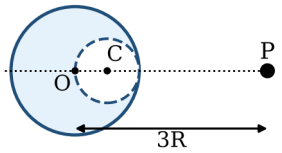
26 The Sun, Earth and Moon have masses 1.99×10^{30} kg, 5.97×10^{24} kg and 7.35×10^{22} kg. The Moon is 3.84×10^8 m from the Earth and 1.50×10^{11} m from the Sun. Take $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}.

(a) Calculate the gravitational force on the Moon due to the Sun, and the gravitational force on the Moon due to the Earth, and find the ratio of the first to the second. **[5]**

(b) The Sun pulls the Moon more strongly than the Earth does, yet the Moon stays in orbit about the Earth. Explain why. **[2]**

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27 A uniform solid sphere has centre O, radius R and mass M. A spherical cavity of radius $\frac{R}{2}$, with centre C where OC = $\frac{R}{2}$, is hollowed out of the sphere. A particle P of mass m lies on the line OC produced, with OP = 3R.



(a) Write down the mass of the material removed to form the cavity. **[1]**

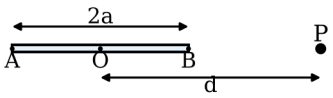
(b) Find the gravitational force exerted on P by the remaining part of the sphere. **[4]**

(c) P is now placed on the opposite side of O at OP = 3R, so that O lies between P and C. Find the gravitational force on P. **[3]**

(d) You may use the result that inside a uniform sphere of mass M and radius R the gravitational field strength at displacement r from the centre is $-\frac{GM}{R^3}r$. Show that the gravitational field in the cavity is uniform and find its magnitude. **[4]**

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28 A uniform thin rod AB has mass M and length 2a, with midpoint O. A particle P of mass m lies on AB produced, with OP = d, where d > a.



(a) An element of the rod of length dx is at a distance x from O, where $-a \leq x \leq a$. Write down the force dF exerted by this element on P. **[2]**

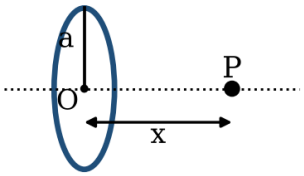
(b) Hence show that the gravitational force on P due to the rod is $\frac{GMm}{(d^2 - a^2)}$. **[5]**

(c) Show that for d much larger than a, $F \approx \frac{GMm}{d^2}(1 + \frac{a^2}{d^2})$. **[2]**

(d) When d = 3a, find the percentage by which the exact force exceeds the force that would be exerted if the whole mass of the rod were concentrated at O. **[2]**

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29 A thin uniform ring has mass M and radius a, with centre O. A particle P of mass m lies on the axis of the ring, at a distance x from O.

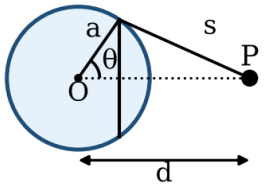


(a) Show that the gravitational force on P has magnitude $\frac{GMmx}{(x^2 + a^2)^{3/2}}$. **[4]**

(b) Find, in terms of a, the value of x for which the force is greatest. **[5]**

(c) Find the greatest value of the force. **[2]**

30 A thin uniform spherical shell has mass M, radius a and centre O. A particle P of mass m is at a distance d from O, where d > a. The point Q on the shell has OQ at angle θ to OP. The thin ring on the shell between angles θ and $\theta + d\theta$ has all its points at the same distance s from P.



(a) Show that the mass of this ring is $\frac{M}{2}\sin\theta\, d\theta$. **[2]**

(b) Show that the component, along PO, of the force exerted by the ring on P is $\frac{GmM\sin\theta\,(d - a\cos\theta)d\theta}{2s^3}$, where $s^2 = a^2 + d^2 - 2ad\cos\theta$. **[3]**

(c) By substituting $u = s^2$, hence prove that the force exerted by the whole shell on P is $\frac{GMm}{d^2}$. **[6]**

Gravitational Potential Energy and Escape Speed

1 Find the gravitational potential energy of a satellite of mass 1000 kg at a distance of 7.0×10^6 m from the centre of the Earth. Take $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2} and $M_E = 5.97 \times 10^{24}$ kg. **[2]**

2 The escape speed from the surface of a planet of mass M and radius R is $\sqrt{\frac{2GM}{R}}$. Find the escape speed from the surface of the Moon, which has mass 7.35×10^{22} kg and radius 1.74×10^6 m. Take $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}. **[2]**

3 A probe of mass m has gravitational potential energy -6.2×10^8 J when it is on the surface of a planet of radius R.

(a) Find its gravitational potential energy when it is at a distance 2R from the centre of the planet. **[1]**

(b) Find the work done in moving the probe from the surface to a distance 2R from the centre. **[1]**

4 The escape speed from the surface of a planet is $\sqrt{2gR}$, where g is the gravitational field strength at the surface and R is the radius. Find the escape speed from the Earth, taking g = 9.8 N kg^{-1} and $R = 6.37 \times 10^6$ m. **[2]**

5 State the value of the gravitational potential energy of a particle at infinity, and explain why the gravitational potential energy of a particle near a planet is negative. **[2]**

6 The escape speed from the surface of the Earth is 11.2 km s^{-1}. Find the escape speed from the surface of a planet which has the same mass as the Earth and one quarter of its radius. **[2]**

7 A satellite of mass 500 kg is raised from the surface of the Earth to a height equal to the radius R of the Earth. Show that the gain in gravitational potential energy is $\frac{1}{2}mgR$ and calculate it, taking g = 9.8 N kg^{-1} and $R = 6.37 \times 10^6$ m. **[3]**

8 A particle is projected vertically upwards from the surface of the Earth with speed 6.0×10^3 m s^{-1}. Ignoring air resistance and the rotation of the Earth, find the greatest height above the surface that it reaches. Take $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}, $M_E = 5.97 \times 10^{24}$ kg and $R_E = 6.37 \times 10^6$ m. **[4]**

9 The gravitational field strength at the surface of Mars is 3.71 N kg^{-1} and its radius is 3.39×10^6 m. Show that the escape speed from the surface of Mars is $\sqrt{2gR}$ and calculate its value. **[3]**

10 A meteorite is at rest at a distance 3R from the centre of the Earth, where R is the radius of the Earth. It falls directly towards the Earth. Find its speed when it reaches the surface, taking g = 9.8 N kg^{-1} and $R = 6.37 \times 10^6$ m, and ignoring air resistance. **[4]**

11 Planet A has twice the mass and half the radius of planet B.

(a) Find the ratio of the escape speed from A to the escape speed from B. **[2]**

(b) Find the ratio of the gravitational potential energy of a given particle on the surface of A to that on the surface of B. **[2]**

12 A satellite of mass 1200 kg is moved from a circular orbit of radius 7.0×10^6 m to a circular orbit of radius 4.22×10^7 m about the centre of the Earth. Find the increase in the gravitational potential energy of the satellite. Take $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2} and $M_E = 5.97 \times 10^{24}$ kg. **[3]**

13 A probe is launched from the surface of the Earth with speed 1.5×10^4 m s^{-1}, which is greater than the escape speed. Find its speed when it is very far from the Earth, ignoring the Sun and other planets. Take $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}, $M_E = 5.97 \times 10^{24}$ kg and $R_E = 6.37 \times 10^6$ m. **[4]**

14 A probe is fired radially from the surface of a planet of radius R at three quarters of the escape speed. Ignoring air resistance and the planet's rotation, find the greatest distance from the centre of the planet that the probe reaches, in terms of R. **[5]**

15 A student models a meteorite falling from rest at a height of 1.0×10^6 m above the surface of the Moon. The student uses $v = \sqrt{2gh}$ with g = 1.62 N kg^{-1} for the speed on impact. The radius of the Moon is 1.74×10^6 m.

(a) Calculate the student's value. **[2]**

(b) Explain what is wrong with the student's method and calculate the correct impact speed. **[3]**

16 The Earth has mass M and radius R. A particle of mass m is at a distance x from the centre of the Earth.

(a) Write down the gravitational force on the particle and, by integration, show that the work done in taking it from the surface of the Earth to infinity is $\frac{GMm}{R}$. **[3]**

(b) Find the work done in raising a 400 kg satellite from the surface of the Earth to a distance 3R from the centre of the Earth. Take $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}, $M = 5.97 \times 10^{24}$ kg and $R = 6.37 \times 10^6$ m. **[3]**

17 A black hole is a body for which the escape speed from its surface is equal to the speed of light, $c = 3.00 \times 10^8$ m s^{-1}. Take $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}.

(a) Show that a body of mass M has this property when its radius is $\frac{2GM}{c^2}$. **[2]**

(b) Calculate this radius for a body of mass 1.99×10^{30} kg. **[2]**

(c) A black hole has radius 1.2×10^{10} m. Find its mass. **[2]**

18 A small asteroid is a uniform sphere of density ρ and radius R.

(a) Show that the escape speed from its surface is $R\sqrt{\frac{8\pi G\rho}{3}}$. **[3]**

(b) An astronaut can leave the ground at a vertical speed of 3.0 m s^{-1}. The asteroid has density 3000 kg m^{-3}. Find the greatest radius of the asteroid for which the astronaut can jump free of the asteroid. Take $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}. **[3]**

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19 A rocket of mass 1500 kg, which may be assumed constant, rises vertically from rest at the surface of the Earth to a height of 3.0×10^6 m above the surface, where its speed is 2000 m s^{-1}. Ignore air resistance. Take $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}, $M_E = 5.97 \times 10^{24}$ kg and $R_E = 6.37 \times 10^6$ m.

(a) Find the gain in the kinetic energy of the rocket. **[1]**

(b) Find the gain in the gravitational potential energy of the rocket. **[3]**

(c) Find the work done by the engines of the rocket. **[1]**

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20 A meteorite of mass 40 kg falls from rest at a very large distance from the Earth towards the centre of the Earth. Ignore air resistance and the motion of the Earth. Take $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}, $M_E = 5.97 \times 10^{24}$ kg and $R_E = 6.37 \times 10^6$ m.

(a) Find its kinetic energy when it is at a height of 4.0×10^5 m above the surface of the Earth. **[2]**

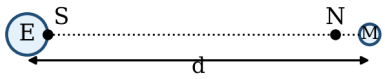
(b) Find its speed at this height. **[1]**

(c) Find its kinetic energy and speed when it reaches the surface of the Earth. **[2]**

(d) Explain why the speed at the surface is equal to the escape speed. **[1]**

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21 A spacecraft of mass m is launched from the surface of the Earth directly towards the Moon along the line of centres. The distance between the centres of the Earth and Moon is 3.84×10^8 m. The resultant gravitational force on the spacecraft is zero at the point N, which is 3.457×10^8 m from the centre of the Earth. Ignore the rotation of the Earth and the motion of the Moon, and take $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}, $M_E = 5.97 \times 10^{24}$ kg, $R_E = 6.37 \times 10^6$ m, $M_M = 7.35 \times 10^{22}$ kg.



(a) Find the gravitational potential energy per unit mass of the spacecraft at the Earth's surface. **[2]**

(b) Find the gravitational potential energy per unit mass at N. **[2]**

(c) Find the minimum launch speed for the spacecraft to reach N. **[3]**

(d) Explain why the spacecraft needs no further energy once it has passed N. **[1]**

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22 A rocket is launched from the equator of the Earth, to escape from the gravitational field of the Earth. Model the Earth as a sphere of radius 6.37×10^6 m, mass 5.97×10^{24} kg, rotating once in 24 hours. Take $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}.

(a) Calculate the escape speed from a stationary Earth. **[2]**

(b) Find the speed of a point on the equator. **[2]**

(c) Find the least launch speed, relative to the ground, for the rocket to escape when it is launched (i) towards the east, (ii) towards the west. Assume that the launches are horizontal initially and that the speed relative to the Earth's centre determines the energy. **[4]**

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23 A particle is projected vertically upwards from the surface of a planet of radius R with speed v, where v is less than the escape speed v_e. Ignore air resistance and rotation. The greatest height reached is h. Let $k = \frac{v}{v_e}$.

(a) Show that $h = \frac{k^2R}{(1-k^2)}$. **[4]**

(b) Find h, in terms of R, when $v = 0.8v_e$. **[2]**

(c) Find, in terms of g and R, the speed needed to reach a height h = R. **[2]**

24 Model the Earth as a uniform sphere of mass M and radius R. The gravitational field strength at a distance r from the centre, where $r \leq R$, is $\frac{GMr}{R^3}$, directed towards the centre. A narrow smooth shaft is drilled through the Earth along a diameter.

(a) Show that the gravitational potential energy of a particle of mass m at the centre of the Earth is $-\frac{3GMm}{2R}$. **[5]**

(b) Find the least speed with which a particle must be projected from the centre of the Earth to escape. Take $g = 9.8$ N kg^{-1} and $R = 6.37 \times 10^6$ m. **[2]**

(c) A particle is released from rest at the surface and falls down the shaft. Find its speed as it passes the centre. **[2]**

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25 A planet can retain a gas in its atmosphere for billions of years only if the escape speed from the planet is at least six times the root mean square speed of the molecules of the gas, $v_{rms} = \sqrt{\frac{3kT}{m}}$, where m is the mass of a molecule and T is the temperature. Take $k = 1.38 \times 10^{-23}$ J K^{-1} and $T = 300$ K. The mass of a hydrogen molecule is 3.3×10^{-27} kg and the mass of an oxygen molecule is 5.3×10^{-26} kg. The escape speeds are 1.12×10^4 m s^{-1} for the Earth and 2.37×10^3 m s^{-1} for the Moon.

(a) Calculate the root mean square speeds of hydrogen and oxygen molecules at this temperature. **[4]**

(b) Show that the Earth can retain oxygen but not hydrogen. **[2]**

(c) Show that the Moon cannot retain oxygen. **[2]**

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26 A particle is released from rest at a height h above the surface of the Earth, of radius R, and falls to the surface. Ignore air resistance. Let g be the gravitational field strength at the surface.

(a) Show that the speed v on reaching the surface satisfies $v^2 = \frac{2gh}{(1 + \frac{h}{R})}$. **[3]**

(b) Find the height h, in terms of R, for which the value of v^2 given by the constant-gravity formula $v^2 = 2gh$ is 10% greater than the true value. **[2]**

(c) Find the true and constant-gravity values of the speed on impact when $h = R$. Take $g = 9.8$ N kg^{-1} and $R = 6.37 \times 10^6$ m. **[2]**

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27 Two uniform spherical asteroids, of masses M and 2M and each of radius a, are released from rest with their centres a distance 10a apart. They move towards each other under their mutual gravitational attraction only, and no other forces act. Let the speeds of the asteroids just before they touch be v_1 (mass M) and v_2 (mass 2M).

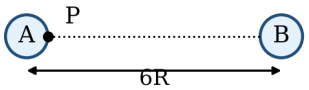
(a) Use conservation of momentum to find v_1 in terms of v_2. **[2]**

(b) Use conservation of energy to show that $v_2^2 = \frac{4GM}{15a}$. **[4]**

(c) Given that $M = 3.0 \times 10^{20}$ kg and $a = 8.0 \times 10^3$ m, find v_1 and v_2. Take $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}. **[3]**

(d) Verify that the speed of approach $v_1 + v_2$ is consistent with $\frac{1}{2}\mu v_{rel}^2$ = loss in GPE, where μ is the reduced mass $\frac{m_1 m_2}{m_1 + m_2}$. **[2]**

28 Two identical uniform spherical planets A and B, each of mass M and radius R, are fixed with their centres a distance 6R apart. A probe of mass m is launched from the surface of A directly towards B along the line of centres. Ignore the rotation of the planets and any other bodies. Let x be the distance of the probe from the centre of A.



(a) Write down an expression for the gravitational potential energy of the probe when $R \leq x \leq 5R$. **[1]**

(b) Show that U has a stationary value at x = 3R, and that this is a maximum. **[4]**

(c) Find the least launch speed for the probe to reach B. Give your answer in terms of G, M and R and then, taking $M = 5.97 \times 10^{24}$ kg, $R = 6.37 \times 10^{6}$ m and $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}, as a number. **[4]**

(d) Explain what happens to the probe after it has passed x = 3R. **[1]**

29 A uniform thick spherical shell has mass M, inner radius a and outer radius b. You may use the results that the gravitational field outside or inside a spherical distribution of matter at distance r from the centre is $\frac{GM'}{r^2}$, where M' is the mass inside radius r, and directed towards the centre.

(a) Show that the gravitational field strength at a point inside the shell at distance r from the centre ($a < r < b$) is $\frac{GM(r^3 - a^3)}{(b^3 - a^3)r^2}$. **[3]**

(b) Hence show that the gravitational potential at the centre of the shell is $-\frac{3GM(b^2 - a^2)}{2(b^3 - a^3)}$. **[6]**

(c) Check this result in the limits $a \to 0$ and $a \to b$. **[2]**

30 The gravitational binding energy of a body is the work needed to separate all of its matter to infinity. A uniform sphere of mass M, radius R and density ρ is built up from infinity by adding thin spherical shells, one at a time.

(a) Show that when the sphere has radius r, the mass of a shell of thickness dr added to its surface is $4\pi\rho r^2$ dr. **[1]**

(b) Show that the work done by gravity when this shell is brought from infinity is $dU = -\frac{16\pi^2 G\rho^2}{3} r^4$ dr in magnitude. **[2]**

(c) Hence show that the binding energy of the sphere is $\frac{3GM^2}{5R}$. **[4]**

(d) Calculate the gravitational binding energy of the Earth, taking $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}, $M = 5.97 \times 10^{24}$ kg and $R = 6.37 \times 10^{6}$ m. **[2]**

Circular Orbits and Satellites

1 A satellite orbits the Earth in a circular orbit of radius 7.0×10^{6} m. Find its orbital speed. Take $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2} and $M_E = 5.97 \times 10^{24}$ kg. **[2]**

2 A satellite moves in a circular orbit of radius 7.0×10^{6} m at a constant speed of 7.5×10^{3} m s^{-1}. Find its period. **[2]**

3 Planet P orbits a star in a circular orbit of radius r with a period of 1.0 year. Planet Q orbits the same star in a circular orbit of radius 4r. Use $T^2 \propto r^3$ to find the period of Q. **[2]**

4 A satellite of mass m moves in a circular orbit of radius r about a planet of mass M at speed v.

State the force that provides the centripetal force, and show that $v = \sqrt{\frac{GM}{r}}$. **[2]**

5 A satellite of mass 500 kg orbits the Earth in a circular orbit of radius 7.0×10^6 m. Find its kinetic energy. Take $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2} and $M_E = 5.97 \times 10^{24}$ kg. **[2]**

6 Satellite A orbits the Earth in a circular orbit of radius r and satellite B in a circular orbit of radius 4r. Find the ratio of the orbital speed of A to the orbital speed of B. **[2]**

7 A geostationary satellite moves in a circular orbit in the equatorial plane with a period equal to the rotation period of the Earth, 8.62×10^4 s. Find the radius of its orbit and its height above the surface of the Earth. Take $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}, $M_E = 5.97 \times 10^{24}$ kg and $R_E = 6.37 \times 10^6$ m. **[4]**

8 The Earth moves in a circular orbit of radius 1.50×10^{11} m about the Sun with a period of 3.16×10^7 s. Find the mass of the Sun. Take $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}. **[3]**

9 A satellite orbits the Earth in a circular orbit at a height of 3.0×10^5 m above the surface. Find its period, giving your answer in minutes. Take $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}, $M_E = 5.97 \times 10^{24}$ kg and $R_E = 6.37 \times 10^6$ m. **[3]**

10 A satellite of mass 800 kg is in a circular orbit of radius 7.2×10^6 m about the Earth. Take $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2} and $M_E = 5.97 \times 10^{24}$ kg.

(a) Find its kinetic energy. **[2]**

(b) Find its gravitational potential energy and its total energy. **[2]**

11 The Jovian moon Io orbits Jupiter in a circular orbit of radius 4.22×10^8 m with a period of 1.77 days. The moon Europa orbits Jupiter in a circular orbit with a period of 3.55 days. Find the radius of the orbit of Europa. **[3]**

12 Satellite A moves in a circular orbit about the Earth with speed 7.6×10^3 m s^{-1}. Satellite B moves in a circular orbit of radius three times that of A.

(a) Find the speed of B. **[2]**

(b) Find the ratio of the period of B to the period of A. **[2]**

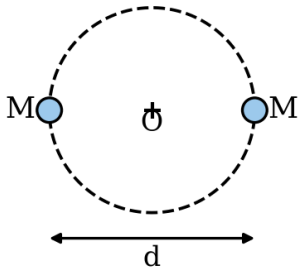
13 A satellite in a low circular orbit loses mechanical energy to air resistance. Explain why its orbital radius decreases and why its speed increases. **[3]**

14 A satellite of mass m moves in a circular orbit of radius r about a planet of mass M with period T.

(a) Show that $T^2 = \frac{4\pi^2 r^3}{GM}$. **[3]**

(b) For the Earth, find the value of the constant $\frac{4\pi^2}{GM}$ in s^2 m^{-3} and use it to find the period of a satellite in a circular orbit of radius 1.0×10^7 m. Take $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2} and $M_E = 5.97 \times 10^{24}$ kg. **[3]**

15 Two stars, each of mass M, move in circular orbits about their common centre of mass O. The stars are always a distance d apart.



(a) Show that the period of the motion is $T = 2\pi\sqrt{\frac{d^3}{2GM}}$. **[4]**

(b) Each star has mass 2.0×10^{30} kg and $d = 1.0 \times 10^{11}$ m. Find T, in days. Take $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}. **[2]**

16 A satellite moves in a circular orbit at a height of 3.0×10^5 m above the equator, in the same direction as the rotation of the Earth. The Earth rotates once in 8.62×10^4 s relative to the stars. Take $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}, $M_E = 5.97 \times 10^{24}$ kg and $R_E = 6.37 \times 10^6$ m.

(a) Find the period of the satellite. **[3]**

(b) Find the time between successive occasions on which the satellite is directly above a given point on the equator. **[3]**

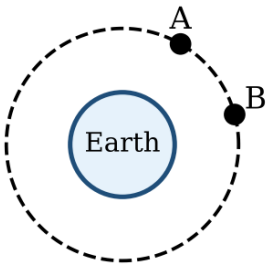
17 A moon orbits the planet X in a circular orbit of radius 2.0×10^8 m with a period of 5.0×10^5 s. The radius of X is 3.0×10^7 m. Take $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}.

(a) Find the mass of X. **[3]**

(b) Find the gravitational field strength at the surface of X. **[2]**

(c) Find the escape speed from the surface of X. **[2]**

18 Two satellites A and B are in the same circular orbit about the Earth, with A directly ahead of B. The pilot of B wishes to catch up with A and fires the engine in the direction of motion to increase the speed. Explain why this will not bring B closer to A, and describe a manoeuvre that would.



[5]

19 A satellite of mass 900 kg is moved from a circular orbit of radius 6.8×10^6 m to a circular orbit of radius 2.6×10^7 m about the Earth. Take $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2} and $M_E = 5.97 \times 10^{24}$ kg.

(a) Find the change in the kinetic energy of the satellite. **[2]**

(b) Find the change in the total mechanical energy of the satellite. **[2]**

(c) Explain why the satellite's speed decreases although its energy increases. **[1]**

20 A satellite of mass m moves in a circular orbit of radius r about a planet of mass M.

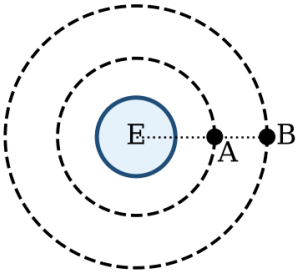
(a) Show that its total energy is $-\frac{GMm}{2r}$. **[3]**

(b) Show that the speed needed to escape from the planet while at this radius is $\sqrt{2}$ times the orbital speed. **[2]**

(c) For a satellite orbiting the Earth at $r = 7.0 \times 10^6$ m, find the extra speed which must be given to it, in the direction of motion, to escape. Take $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2} and $M_E = 5.97 \times 10^{24}$ kg. **[2]**

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21 Satellites A and B move in coplanar circular orbits about the Earth, in the same sense, with radii 7.0×10^6 m and 1.0×10^7 m. At time t = 0 the centre of the Earth, A and B are in a straight line, with A and B on the same side of the Earth. Take $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2} and $M_E = 5.97 \times 10^{24}$ kg.



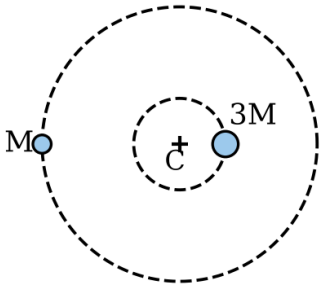
(a) Find the angular speeds of A and B. **[3]**

(b) Find the time at which A, B and the centre of the Earth are next in a straight line with A and B on the same side of the Earth. **[2]**

(c) Find the first time at which A and B are on opposite sides of the Earth in a straight line through its centre. **[2]**

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22 Two stars of masses 3M and M move in circular orbits about their common centre of mass C. The stars are always a distance d apart.



(a) Find the distances of each star from C, in terms of d. **[2]**

(b) Show that the two stars have the same angular speed ω, where $\omega^2 = \frac{4GM}{d^3}$. **[3]**

(c) Find the ratio of the speed of the heavier star to the speed of the lighter star. **[1]**

(d) Given that $M = 2.0 \times 10^{30}$ kg and $d = 3.0 \times 10^{11}$ m, find the period of the motion in years. Take $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2} and one year $= 3.16 \times 10^7$ s. **[3]**

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23 A satellite orbits a uniform spherical planet of density ρ in a circular orbit just above its surface.

(a) Show that the period of the satellite is $\sqrt{\frac{3\pi}{G\rho}}$. **[3]**

(b) Find the period of such an orbit for a planet of density 5500 kg m^{-3}, in minutes. Take $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}. **[2]**

(c) Explain why the period of such an orbit depends on the density of the planet but not on its radius. **[1]**

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24 A satellite of mass 2000 kg is moved from a circular orbit of radius 7.0×10^6 m to a circular orbit of radius 1.4×10^7 m about the Earth. Take $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2} and $M_E = 5.97 \times 10^{24}$ kg.

(a) Show that the kinetic energy of the satellite is halved, and find the change in kinetic energy. **[3]**

(b) Find the change in the gravitational potential energy. **[2]**

(c) Find the net change in the mechanical energy of the satellite. **[1]**

(d) The satellite's engine does positive work on the satellite during the manoeuvre. Explain why the speed of the satellite is nevertheless lower in the new orbit. **[2]**

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25 A satellite of mass 800 kg is launched from rest relative to the ground at a point on the equator, into a circular orbit of radius 7.2×10^6 m about the centre of the Earth. The Earth rotates once in 24 hours. Take $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}, $M_E = 5.97 \times 10^{24}$ kg and $R_E = 6.37 \times 10^6$ m.

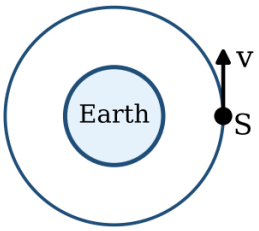
(a) Find the energy that must be supplied to the satellite, taking into account the rotation of the Earth. **[4]**

(b) Find the percentage saving in energy compared with a launch from a non-rotating Earth. **[2]**

(c) Find the speed relative to the ground with which the satellite must be launched horizontally for the orbit to be circular, if launched (i) eastwards (ii) westwards, immediately it enters the orbit. **[2]**

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26 A satellite S is in a circular orbit of radius 7.0×10^6 m about the Earth. A short burst of its engine increases its speed, in the direction of motion, from v to v' = $\sqrt{\frac{3}{2}}$ v, where v is the original speed. The satellite then moves in an elliptical orbit with the burst point as its closest point to the Earth. Take $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2} and $M_E = 5.97 \times 10^{24}$ kg.



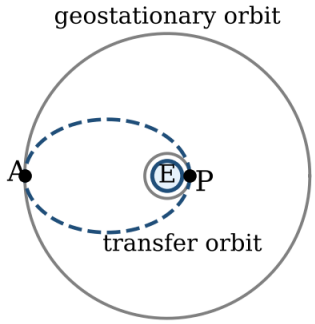
(a) Show that the greatest distance of the satellite from the centre of the Earth in the new orbit is 2.1×10^7 m. You may assume that angular momentum and energy are conserved, and that at the nearest and furthest points the velocity is perpendicular to the radius. **[5]**

(b) Find the speed of the satellite at its furthest point. **[2]**

(c) Given that the period of the original circular orbit was 5830 s, use Kepler's third law with the semi-major axis of the ellipse to find the time taken by the satellite to travel from its nearest point to its furthest point. **[2]**

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27 A satellite is to be moved from a circular parking orbit of radius 6.67×10^6 m about the Earth to a geostationary orbit of radius 4.22×10^7 m. It does this by a transfer ellipse, with the parking orbit touching it at its nearest point P to the Earth and the geostationary orbit touching it at its furthest point A. Short engine burns at P and A change the speed of the satellite. Take $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2} and $M_E = 5.97 \times 10^{24}$ kg.



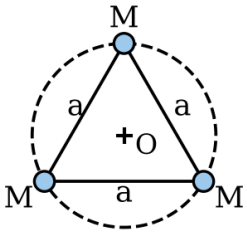
(a) Find the speeds of the satellite in the parking orbit and in the geostationary orbit. **[2]**

(b) By using conservation of angular momentum and of energy, show that the speed at P on the transfer ellipse is $\sqrt{\frac{2GMr_a}{r_p(r_a + r_p)}}$, and find its value and the speed at A. **[6]**

(c) Find the increase in speed needed at P and at A, and the total. **[2]**

(d) Using $T = 2\pi\sqrt{\frac{a^3}{GM}}$ for an orbit of semi-major axis a, find the time taken to travel from P to A. **[2]**

28 Three stars, each of mass M, move in the same circular orbit with their centres at the vertices of an equilateral triangle of side a, which rotates about its centre O.



(a) Show that the gravitational force on one star due to the other two has magnitude $\sqrt{3}\frac{GM^2}{a^2}$, directed towards O. **[2]**

(b) Write down the radius of the orbit of each star in terms of a. **[1]**

(c) Hence show that the angular speed ω satisfies $\omega^2 = \frac{3GM}{a^3}$. **[3]**

(d) Given that $M = 1.0 \times 10^{30}$ kg and $a = 2.0 \times 10^{11}$ m, find the period of the motion in years. Take $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2} and one year $= 3.16 \times 10^7$ s. **[2]**

(e) Show that the total mechanical energy of the system is $-\frac{3GM^2}{2a}$. **[4]**

29 A satellite of mass m moves in a circular orbit of radius r about the Earth, of mass M. The only resistive force is air resistance of magnitude kmv^2, where v is the speed and k is a constant. Assume that the orbit remains circular, with slowly decreasing radius r(t). The total energy of the satellite is $E = -\frac{GMm}{2r}$.

(a) Show that $\frac{dr}{dt} = -2k\sqrt{GMr}$. **[5]**

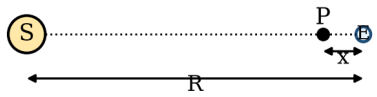
(b) Find an expression for the time taken for the radius to decrease from r_0 to r_1. **[3]**

(c) Find this time, in days, when $k = 1.0 \times 10^{-12}$ m^{-1}, $r_0 = 6.80 \times 10^6$ m and $r_1 = 6.60 \times 10^6$ m. Take $GM = 3.98 \times 10^{14}$ m^3 s^{-2}. **[2]**

(d) Show that the speed of the satellite increases as its orbit decays, and explain how this is consistent with the satellite losing energy. **[2]**

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30 The Earth, of mass M_E, moves in a circular orbit of radius R about the Sun, of mass M_S. A small satellite P moves in a circular orbit about the Sun with the same period as the Earth, and is on the line joining the Sun and the Earth, a distance x from the Earth, where x is much smaller than R. Treat the Sun as fixed, and assume M_E is much smaller than M_S.



(a) Write down the angular speed ω of the Earth in terms of G, M_S and R. **[2]**

(b) Show that $GM_S/(R - x)^2 - GM_E/x^2 = \frac{GM_S(R - x)}{R^3}$. **[3]**

(c) Use binomial approximations to show that $x^3 \approx \frac{R^3M_E}{3M_S}$. **[4]**

(d) Given that $R = 1.50 \times 10^{11}$ m, $M_E = 5.97 \times 10^{24}$ kg and $M_S = 1.99 \times 10^{30}$ kg, find x, and verify that the assumption that x is much smaller than R is reasonable. **[3]**

Motion Under Variable Gravity

1 The Earth is modelled as a fixed sphere of radius R, and the gravitational field strength at its surface is g. The magnitude of the gravitational force on a particle of mass m at a distance x from the centre of the Earth is proportional to $\frac{1}{x^2}$. Show that the acceleration of the particle towards the centre of the Earth is $\frac{gR^2}{x^2}$ when $x \geq R$. **[2]**

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2 A particle moves along the x-axis with $v\frac{dv}{dx} = -\frac{8}{x^2}$, where v is its velocity. Given that $v = 0$ when $x = 2$, find v^2 when $x = 1$. **[3]**

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3 A particle moves along a straight line with displacement x from a fixed point and velocity $v = \frac{dx}{dt}$. Show that its acceleration can be written as $v\frac{dv}{dx}$. **[2]**

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4 A meteorite falls from rest at a distance 2R from the centre of the Earth, directly towards the centre. Taking $g = 9.8$ m s^{-2} and $R = 6.37 \times 10^6$ m, and using $v^2 = 2gR^2(\frac{1}{x} - \frac{1}{x_0})$ for the speed v at distance x when released from rest at x_0, find its speed when it reaches the surface. **[2]**

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5 A particle moves in a straight line so that $v^2 = 2k(\frac{1}{x} - \frac{1}{a})$, where k and a are positive constants. Find the speed of the particle when $x = \frac{a}{2}$. **[2]**

---

6 A particle in a smooth tunnel through the centre of a uniform planet has displacement x from the centre and moves with $\frac{d^2x}{dt^2} = -\frac{g}{R}x$. Find, for g = 9.8 m s^{-2} and R = 6.37 × 10^6 m, the time for one complete oscillation. **[2]**

7 A particle is projected vertically upwards from the surface of the Earth with speed u, where $u^2 = \frac{gR}{2}$. Ignore air resistance, and assume the acceleration at distance x from the centre of the Earth is $\frac{gR^2}{x^2}$ towards the centre. Find the greatest height of the particle above the surface of the Earth, in terms of R, and its value when R = 6.37 × 10^6 m. **[4]**

8 A particle falls from rest at a distance 4R from the centre of the Earth, directly towards the centre. The acceleration at distance x from the centre is $\frac{gR^2}{x^2}$ towards the centre. Find the speed with which the particle reaches the surface, in terms of g and R, and evaluate it for g = 9.8 m s^{-2}, R = 6.37 × 10^6 m. **[4]**

9 A particle is projected vertically upwards from the surface of the Earth. When it is at a height $\frac{R}{2}$ above the surface its speed is $\sqrt{gR}$. Ignore air resistance, and take the acceleration at distance x from the centre to be $\frac{gR^2}{x^2}$ towards the centre. Find the greatest distance of the particle from the centre of the Earth. **[4]**

10 A particle moves along a straight line away from a fixed point O, with acceleration $-\frac{k}{x^2}$, where x is its distance from O and k is a positive constant. It is at rest when x = a. Find v^2 in terms of k, x and a, and hence find its speed when $x = \frac{a}{4}$. **[4]**

11 The acceleration of a particle at a height h above the surface of the Earth is $\frac{gR^2}{(R + h)^2}$ towards the centre. Take g = 9.8 m s^{-2}. Find the magnitude of the acceleration at a height of

(a) 0, **[1]**

(b) R, **[2]**

(c) 3R. **[1]**

12 A rocket is projected vertically upwards from the surface of the Earth with speed 8000 m s^{-1}. Ignore air resistance and take the acceleration at distance x from the centre to be $\frac{gR^2}{x^2}$ towards the centre, with g = 9.8 m s^{-2} and R = 6.37 × 10^6 m. Find the greatest height of the rocket above the surface. **[4]**

13 A particle moves in a straight line away from a fixed point O, and its speed v at distance x from O (x > 0) is given by $v^2 = 36 + \frac{24}{x}$, with x in metres and v in m s^{-1}.

(a) Show that the acceleration of the particle is $-\frac{12}{x^2}$. **[2]**

(b) Find the speed and the acceleration of the particle when x = 4. **[3]**

14 A particle P moves in a straight line towards a fixed point O under an attraction to O of acceleration $\frac{k}{x^3}$, where x is the distance OP and k is a positive constant. P is released from rest at x = a.

(a) Show that $v^2 = k(\frac{1}{x^2} - \frac{1}{a^2})$. **[3]**

(b) Show that the time taken for P to reach O is $\frac{a^2}{\sqrt{k}}$. **[3]**

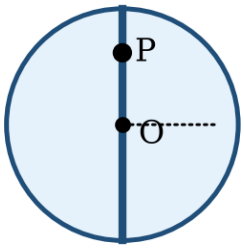
15 A particle is projected vertically upwards from the surface of a planet of radius R with speed u. At a distance x from the centre the downward acceleration is $\frac{gR}{x}$, where g is the gravitational field strength at the surface (this is a model in which the force is inversely proportional to the distance from the centre).

(a) Show that $v^2 = u^2 - 2gR \ln(\frac{x}{R})$. **[3]**

(b) Given that $u^2 = gR$, find the greatest distance from the centre reached. **[2]**

(c) Explain why, in this model, there is no speed of projection for which the particle escapes. **[1]**

16 A smooth narrow tunnel passes through the centre of a uniform planet of radius R. At a distance x from the centre the gravitational acceleration is $\frac{gx}{R}$ directed towards the centre. A particle is released from rest at the surface. Take g = 9.8 m s^{-2} and R = 6.37 × 10^6 m.



(a) Show that the motion is simple harmonic, and find its period. **[3]**

(b) Find the speed of the particle when it is at a distance $\frac{R}{2}$ from the centre. **[2]**

(c) Find the time taken for the particle to reach the centre. **[2]**

17 A rocket moves vertically upwards, away from the Earth, with the engines off. A student writes the following working, where h is the height above the surface, R the radius of the Earth, and g the surface gravitational field strength.

(a) Line 1: $v\frac{dv}{dh} = -\frac{gR^2}{(R+h)^2}$. Line 2: Integrating, $\frac{1}{2}v^2 = \frac{gR^2}{(R+h)^2} + C$. Identify the error in Line 2. **[2]**

(b) The rocket has speed $\sqrt{gR}$ at h = 0. Using the corrected equation, find the greatest height it reaches. **[3]**

18 A projectile is launched vertically upwards from the surface of the Moon with speed 2.0 × 10^3 m s^{-1}. The Moon has radius 1.74 × 10^6 m and gravitational field strength 1.62 N kg^{-1} at its surface. Ignore any atmosphere.

(a) Find the greatest height above the surface reached by the projectile, allowing for the variation of gravity with height. **[4]**

(b) Find the greatest height predicted by assuming the acceleration is constant at 1.62 m s^{-2}, and explain why it is less than your answer to part (a). **[2]**

19 The motion of a particle in a smooth tunnel through the centre of a uniform planet of radius R satisfies $\frac{d^2x}{dt^2} = -\omega^2 x$, where $\omega^2 = \frac{g}{R}$.

(a) Verify that $x = A\cos\omega t + B\sin\omega t$ is a solution of this equation for all constants A and B. **[2]**

(b) The particle is released from rest at $x = R$ when $t = 0$. Find A and B. **[2]**

(c) Find the time taken for the particle first to reach the point $x = \frac{R}{2}$. Take $g = 9.8$ m s^{-2} and $R = 6.37 \times 10^6$ m. **[2]**

20 A projectile is to be fired vertically upwards from the surface of the Earth so that it just reaches a height 2R above the surface. The acceleration at distance x from the centre is $\frac{gR^2}{x^2}$ towards the centre, and air resistance is ignored.

(a) Show that the required speed of projection is $\sqrt{\frac{4gR}{3}}$. **[4]**

(b) Find this speed for $g = 9.8$ m s^{-2}, $R = 6.37 \times 10^6$ m, and express it as a fraction of the escape speed $\sqrt{2gR}$. **[2]**

21 A particle falls from rest at a distance a from a fixed point O, directly towards O, with acceleration $\frac{k}{x^2}$ towards O when it is at a distance x.

(a) Show that $v^2 = 2k(\frac{1}{x} - \frac{1}{a})$. **[2]**

(b) Write down an integral for the time taken to reach O. **[2]**

(c) By using the substitution $x = a\cos^2\theta$, show that the time is $\frac{\pi}{2}\sqrt{\frac{a^3}{2k}}$. **[3]**

(d) If the Earth were suddenly stopped in its orbit, it would fall directly towards the Sun from a distance 1.50×10^{11} m. Find the time it takes to reach the Sun, in days, treating both as points and taking $k = GM_S$ with $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2} and $M_S = 1.99 \times 10^{30}$ kg. **[2]**

22 A particle falls from rest at a distance a from a fixed point mass M, directly towards it. For an elliptical orbit about M with semi-major axis s the period is $T = 2\pi\sqrt{\frac{s^3}{GM}}$.

(a) Imagine the fall as half of a very thin elliptical orbit, with M at one focus and with the particle starting at the far end. Deduce that the time to fall to M is $\frac{\pi}{2}\sqrt{\frac{a^3}{2GM}}$. **[3]**

(b) The Moon is 3.84×10^8 m from the centre of the Earth. If it were stopped in its orbit, find the time it would take to fall to the Earth, in days, treating both as points. Take $GM_E = 3.98 \times 10^{14}$ m^3 s^{-2}. **[2]**

(c) Show that the ratio of this time to the period of the Moon's actual circular orbit of the same radius is $1 : 4\sqrt{2}$. **[2]**

23 A planet is a uniform sphere of radius R, with gravitational field strength g at its surface. Outside the planet the acceleration towards the centre at distance x is $\frac{gR^2}{x^2}$; inside the planet it is $\frac{gx}{R}$. A smooth narrow tunnel passes through the centre. A particle is released from rest outside the planet at a distance 3R from the centre, on the line of the tunnel, and falls into the tunnel.

(a) Show that the speed of the particle on entering the tunnel is $\sqrt{\frac{4gR}{3}}$. **[4]**

(b) Find the speed of the particle when it passes through the centre of the planet. **[4]**

(c) Evaluate this speed for $g = 9.8$ m s^{-2} and $R = 6.37 \times 10^6$ m. **[1]**

24 A rocket of constant mass 2000 kg is fired vertically upwards from rest at the surface of the Earth. The engine produces a constant upward thrust of 4.0×10^4 N, and air resistance is ignored. At a distance x from the centre of the Earth the gravitational acceleration is $\frac{gR^2}{x^2}$. Take g = 9.8 m s^{-2} and R = 6.37×10^6 m.

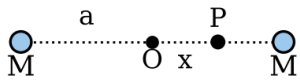
(a) Show that, when the rocket is at a distance x from the centre of the Earth, $v\frac{dv}{dx} = 20 - \frac{gR^2}{x^2}$. **[2]**

(b) Find the speed of the rocket when it is at a height of 1.0×10^5 m above the surface. **[4]**

(c) Find the speed at the same height if the gravitational acceleration were taken as constant at 9.8 m s^{-2}, and the percentage error this introduces. **[2]**

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25 Two particles each of mass M are fixed at the points A and B, a distance 2a apart. A particle P of mass m moves along the line AB, and its displacement from the midpoint O of AB is x, where $-a < x < a$. P is released from rest at the point where $x = \frac{a}{2}$, and moves towards B.



(a) Show that the acceleration of P, in the direction of increasing x, is $GM(\frac{1}{(a-x)^2} - \frac{1}{(a+x)^2})$. **[2]**

(b) Show that $v^2 = 4GMa(\frac{1}{a^2 - x^2} - \frac{4}{3a^2})$. **[4]**

(c) Find the speed of P when $x = \frac{3a}{4}$. **[2]**

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26 A lunar lander of constant mass 2500 kg falls vertically towards the Moon, which has radius 1.74×10^6 m and gravitational field strength 1.62 N kg^{-1} at its surface. The acceleration due to the Moon's gravity at distance x from the centre of the Moon is $\frac{gR^2}{x^2}$, where g = 1.62 m s^{-2} and R is the radius of the Moon. The lander is released from rest at a distance 3R from the centre of the Moon and falls freely. When it is at a distance $\frac{3R}{2}$ from the centre, its engine produces a constant upward thrust F, so that the lander reaches the surface with zero speed.

(a) Find the speed of the lander when its engine starts. **[3]**

(b) Find F. **[4]**

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27 A particle P moves in a straight line towards a fixed point O under an acceleration $\frac{k}{x^2}$ directed towards O, where x is the distance OP and k is a positive constant. At time t = 0, P is at rest at x = a. It is claimed that, for $0 \leq \eta \leq \pi$, the motion is described parametrically by $x = \frac{a}{2}(1 + \cos\eta)$, $t = \sqrt{\frac{a^3}{8k}}(\eta + \sin\eta)$.

(a) Show that $\frac{dx}{dt} = -\sqrt{\frac{2k}{a}}\tan(\frac{\eta}{2})$. **[4]**

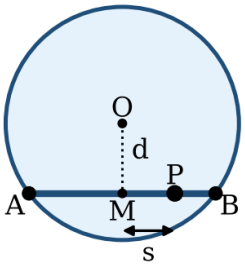
(b) Hence show that $\frac{d^2x}{dt^2} = -\frac{k}{x^2}$. **[4]**

(c) Find, in terms of a and k, the time taken for P to reach O. **[2]**

(d) Find the time taken for P to travel from x = a to $x = \frac{a}{2}$. **[2]**

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28 A uniform planet has radius R, with gravitational field strength g at its surface. At a distance r from the centre ($r \leq R$) the gravitational force on a particle of mass m is $\frac{mgr}{R}$ towards the centre. A smooth straight narrow tunnel AB is drilled along a chord of the planet; the midpoint M of AB is at a distance d from the centre O. A particle P of mass m is released from rest at A and moves along the tunnel. Let s be the displacement of P from M along the tunnel.



(a) Show that the component of the gravitational force on P along the tunnel is $\frac{mgs}{R}$, directed towards M. **[3]**

(b) Hence show that the motion of P is simple harmonic with a period $2\pi\sqrt{\frac{R}{g}}$, independent of the length of the tunnel. **[2]**

(c) Find the normal reaction between P and the tunnel when P is at M, in terms of m, g, d and R. **[2]**

(d) The tunnel has length 3.4×10^5 m. Take g = 9.8 m s^{-2} and R = 6.37×10^6 m. Find the maximum speed of P and the time taken for P to travel from A to B. **[4]**

29 A particle is projected radially outwards from the surface (radius R) of a fixed spherical body with speed u. At a distance $x \geq R$ from the centre the acceleration towards the centre is $\frac{gR^n}{x^n}$, where n is a positive constant and g is the acceleration at the surface.

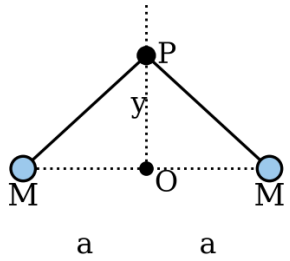
(a) For $n \neq 1$, show that $v^2 = u^2 + \frac{2gR}{n-1}[(\frac{R}{x})^{n-1} - 1]$. **[4]**

(b) For $n > 1$, show that the particle escapes to infinity if and only if $u^2 \geq \frac{2gR}{n-1}$. **[2]**

(c) Find the escape speed for n = 2 and for n = 3, in terms of g and R, and explain why for n = 1 there is no escape speed. **[3]**

(d) Explain why the escape speed for n = 3 is smaller than that for n = 2. **[1]**

30 Two particles each of mass M are fixed at points A and B, a distance 2a apart, with midpoint O. A particle P of mass m moves on the perpendicular bisector of AB, and OP = y. Only the gravitational attraction of A and B acts on P.



(a) Show that the acceleration of P towards O has magnitude $\frac{2GMy}{(a^2 + y^2)^{3/2}}$. **[3]**

(b) Show that, for small y, the motion of P is approximately simple harmonic, and find its period. **[3]**

(c) P is released from rest at $y = \frac{a}{2}$. Show that its speed on passing through O is given exactly by $v^2 = \frac{4GM}{a}(1 - \frac{2}{\sqrt{5}})$. **[4]**

(d) Compare this with the speed at O predicted by the small-oscillation (simple harmonic) model, and find the percentage error of that model. **[2]**