

Graph Theory

1 (2 marks)

(a) Edges at D: BD, CD, DA, so degree 3

B1

Answer: 3

1 mark

(b) AB, BC, CD, DA, BD, CE: 6 edges

B1

Answer: 6

1 mark

2 (2 marks)

A path is a walk in which no vertex is repeated

B1

A cycle is a closed path: it starts and ends at the same vertex, and no other vertex is repeated

B1

Answer: A cycle is a closed path (starts and ends at the same vertex)

2 marks

3 (2 marks)

Each edge contributes 2 to the sum of the degrees

M1

Sum of degrees = $2 \times 9 = 18$

A1

Answer: 18

2 marks

4 (2 marks)

Each of the 6 vertices is joined to the other 5, so the sum of degrees is $6 \times 5 = 30$

M1

Edges = $30 \div 2 = 15$

A1

Answer: 15

2 marks

5 (1 mark)

A tree with n vertices has $n - 1$ edges, so 11

B1

Answer: 11

1 mark

6 (2 marks)

Entry is 1 if the two vertices are joined by an edge, otherwise 0; the matrix is symmetric

M1

$$\text{Matrix} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

A1

$$\text{Answer: } \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

2 marks

7 (3 marks)

(a) For example A, C, D, E

B1

Answer: A, C, D, E

1 mark

(b) A, B, C, A

B1

Answer: A, B, C, A

1 mark

(c) There is a path between every pair of vertices, for example A, B, C, D, E joins all five

E1

Answer: Every pair of vertices is joined by a path

1 mark

8 (3 marks)

(a) The row sums give the degrees

M1

2, 3, 3, 2, 2

A1

Answer: 2, 3, 3, 2, 2

2 marks

(b) Sum of degrees = 12, so edges = 6

B1

Answer: 6

1 mark

9 (2 marks)

(a) Vertices A and B are joined by two edges (a multiple edge)

E1

Answer: A and B are joined by two edges

1 mark

(b) Two edges to B and one to D: degree 3

B1

Answer: 3**1 mark****10 (3 marks)**

Count the arcs entering and leaving each vertex

M1

In-degrees: P 1, Q 1, R 2, S 1

A1

Out-degrees: P 2, Q 1, R 1, S 1

A1**Answer: In: 1, 1, 2, 1; out: 2, 1, 1, 1****3 marks****11 (2 marks)****(a)** $3 + 4 = 7$ **B1****Answer: 7****1 mark****(b)** Each of the 3 vertices is joined to 4 vertices, so $3 \times 4 = 12$ **B1****Answer: 12****1 mark****12 (3 marks)**

Every vertex of H is a vertex of G and every edge of H is an edge of G

M1

So H is a subgraph of G

A1

Deleting E (and with it the edges DE and EA) leaves exactly H

B1**Answer: Yes; delete E****3 marks****13 (3 marks)****(a)** $7 + 5 + 3 = 15$ **B1****Answer: 15 km****1 mark****(b)** Routes: ABDE $7 + 5 + 3 = 15$, ACDE $4 + 6 + 3 = 13$, ACBDE $4 + 2 + 5 + 3 = 14$, ACE $4 + 10 = 14$, ABCDE $7 + 2 + 6 + 3 = 18$ **M1**

Shortest: A, C, D, E of length 13 km

A1**Answer: A, C, D, E; 13 km****2 marks****14 (4 marks)****(a)** Sum of degrees = $2 \times 13 = 26$ **M1** $x = 26 - 24 = 2$ **A1****Answer: 2****2 marks****(b)** If $x = 3$ the sum of the degrees would be 27**M1**

The sum of the degrees is twice the number of edges, so must be even

E1**Answer: Sum of degrees would be odd****2 marks****15 (4 marks)**

$$\mathbf{M}^2 = \begin{pmatrix} 3 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 3 & 1 \\ 1 & 2 & 1 & 2 \end{pmatrix}$$

M1 A1

Entry in row B, column D is 2

M1

Walks: B, A, D and B, C, D

A1**Answer: 2 walks: B, A, D and B, C, D****4 marks****16 (4 marks)****(a)** If there are n vertices the sum of the degrees is $3n$, which equals twice the number of edges, so $3n$ is even**M1** $3n$ is even only when n is even**E1****Answer: n must be even****2 marks****(b)** Sum of degrees = 30**M1**

Edges = 15

A1**Answer: 15****2 marks**

17 (5 marks)

(a) Each vertex has degree $n - 1$, so the sum of the degrees is $n(n - 1)$

M1

Number of edges = half of this = $\frac{n(n - 1)}{2}$

A1

Answer: K_n has $\frac{n(n - 1)}{2}$ edges

2 marks

(b) $\frac{n(n - 1)}{2} = 105$ so $n^2 - n - 210 = 0$

M1

$(n - 15)(n + 14) = 0$

M1

$n = 15$ ($n > 0$)

A1

Answer: 15

3 marks

18 (5 marks)

(a) Rows give arcs leaving each vertex

M1

$$D = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

A1

Answer: $\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$

2 marks

(b) $D^2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$

M1 A1

Row S, column R is 1: the route S, Q, R

A1

Answer: 1 route: S, Q, R

3 marks

19 (7 marks)

(a) Degrees: A 3, B 2, C 3, D 3, E 2, F 3, G 2

M1

Sum of degrees = 18, so $18 \div 2 = 9$ friendships

A1

Degree sequence 3, 3, 3, 3, 2, 2, 2

A1

Answer: 9 friendships; 3, 3, 3, 3, 2, 2, 2

3 marks

(b) B is friends with A and E, G is friends with D and F; A and D are friends, and E and F are friends, but no student is a friend of both B and G, so there is no chain with 2 friendships

M1

B, A, D, G uses 3 friendships, so the smallest number is 3

A1

Answer: 3; B, A, D, G

2 marks

(c) The students with most friends (3) are A, C, D, F, so 4 new edges: $9 + 4 = 13$ friendships

M1

A, C, D, F now have degree 4, H has degree 4 and B, E, G still have degree 2: 4, 4, 4, 4, 4, 2, 2, 2

A1

Answer: 13 friendships; 4, 4, 4, 4, 4, 2, 2, 2

2 marks

20 (5 marks)

(a) Difference 1: 12, 23, 34, 45, 56

M1

Difference 4: 15, 26

A1

Answer: 12, 23, 34, 45, 56, 15, 26

2 marks

(b) Degrees: 2, 3, 2, 2, 3, 2

B1

Sum $14 = 2 \times 7$

B1

Answer: 2, 3, 2, 2, 3, 2; sum $14 = 2 \times 7$

2 marks

(c) For example 1, 2, 6, 5, 1

B1

Answer: 1, 2, 6, 5, 1

1 mark

21 (6 marks)

(a) Each edge is incident with exactly two vertices (its ends), so it adds 2 to the total of the degrees

M1

Hence the sum of all degrees is $2e$

A1

Answer: The sum of the degrees is $2e$

2 marks

(b) Split the vertices into those of even degree and those of odd degree

M1

Sum over even-degree vertices is even

B1

Total $2e$ is even, so the sum over the odd-degree vertices is even

M1

A sum of k odd numbers is even only if k is even, so the number of odd vertices is even

E1

Answer: G has an even number of vertices of odd degree

4 marks

22 (7 marks)

(a) Number of games = number of arcs = $\frac{6 \times 5}{2}$

M1

= 15, and each arc leaves exactly one vertex, so the out-degrees sum to 15

A1

Answer: The out-degrees sum to 15

2 marks

(b) Two teams with 5 wins would each have beaten every other team

M1

Including each other

M1

But only one of them can win that game

E1

Answer: Two teams cannot both beat each other

3 marks

(c) Wins lie between 0 and 5, giving six possible values and six teams, so each value occurs once

M1

5, 4, 3, 2, 1, 0 (sum 15, as required)

A1

Answer: 5, 4, 3, 2, 1, 0

2 marks

23 (7 marks)

(a) Symmetric 0-1 matrix

M1

$$\mathbf{M} = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

A1

$$\text{Answer: } \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

2 marks

(b) Row A, column D of $\mathbf{M}^3$ is 2

B1

Walks: ABCD; ACED

M1 A1

Answer: 2 walks: ABCD, ACED

3 marks

(c) Each triangle gives 6 closed walks of length 3 (3 starting points, 2 directions), so triangles = $12 \div 6 = 2$

M1

Triangles ABC and CDE

A1

Answer: 2 triangles: ABC, CDE

2 marks

24 (6 marks)

(a) The sum of the degrees counts each edge twice, once at each end

M1

So the number of edges is half the sum of the degrees

E1

Answer: Each edge is counted twice

2 marks

(b) $16 \div 2 = 8$

B1

Answer: 8

1 mark

(c) There are 7 vertices, so a tree has 6 edges

M1

The graph has 8 edges, which is more than 6

M1

So it must contain a cycle (and cannot be a tree)

E1

Answer: A tree on 7 vertices has 6 edges, not 8

3 marks

25 (8 marks)

- (a) Each edge appears twice in the table; the weights above the diagonal are 5, 3, 2, 6, 4, 5, 3, 6
 8 edges M1
 Total weight 34 A1
Answer: 8 edges; total weight 34 3 marks
- (b) Degrees: A 2, B 3, C 3, D 3, E 3, F 2 M1
 Four vertices of odd degree: B, C, D, E A1
Answer: 4: B, C, D, E 2 marks
- (c) Routes: ACEF $3 + 4 + 6 = 13$, ABDF $5 + 6 + 3 = 14$, ACEDF $3 + 4 + 5 + 3 = 15$ M1 A1
 Shortest A, C, E, F of length 13 A1
Answer: A, C, E, F; 13 3 marks

26 (6 marks)

- (a) Number of vertices $n = 5 + L$, so number of edges $= 4 + L$ M1
 Sum of degrees $= 3 \times 4 + 2 \times 3 + L = 18 + L$ M1
 $18 + L = 2(4 + L)$ M1
 $L = 10$ A1
Answer: 10 4 marks
- (b) Vertices $5 + 10 = 15$ B1
 Edges 14 B1
Answer: 15 vertices, 14 edges 2 marks

27 (8 marks)

- Let the graph have $n \geq 2$ vertices. In a simple graph a vertex can be joined to at most the other $n - 1$ vertices, so each degree is one of $0, 1, \dots, n - 1$ B1
- Case 1: some vertex has degree $n - 1$. It is joined to every other vertex, so no vertex has degree 0 M1
 The degrees then lie in $1, 2, \dots, n - 1$, which has $n - 1$ possible values A1
- Case 2: no vertex has degree $n - 1$. The degrees then lie in $0, 1, \dots, n - 2$, which again has only $n - 1$ possible values M1
- In both cases the n vertices have degrees taken from at most $n - 1$ different values A1
 By the pigeonhole principle at least two vertices have the same degree M1 A1
 Every simple graph falls into one of the two cases, so the statement is proved E1
Answer: Two vertices have the same degree 8 marks

28 (9 marks)

- (a) $2(n - 1) = 2n - 2$ B1
Answer: $2n - 2$ 1 mark
- (b) T is connected with $n \geq 2$ vertices, so every vertex has degree at least 1 B1
 Suppose T has at most one leaf; then at least $n - 1$ vertices have degree at least 2 M1
 Sum of degrees $\geq 1 + 2(n - 1) = 2n - 1$ (or $\geq 2n$ if there is no leaf) M1 A1
 This exceeds $2n - 2$, a contradiction, so T has at least two leaves E1
Answer: T has at least two leaves 5 marks
- (c) With exactly two leaves the other $n - 2$ vertices have degree at least 2 (T is connected, so none has degree 0) M1
 The degrees sum to $2n - 2 = 2 + 2(n - 2)$, so every other vertex has degree exactly 2 A1
 Start at one leaf and walk along the only unused edge at each vertex: each vertex of degree 2 is entered and left, and T has no cycle so no vertex is revisited, so the walk can only end at the other leaf; it passes through all n vertices because T is connected, so T is a path E1
Answer: A tree with exactly two leaves is a path 3 marks

29 (9 marks)

- (a) Diagonal entry of A^2 : a row of **A** has three 1s and a 0, so row $\cdot$ row $= 1 + 1 + 1 = 3$ M1
 Off-diagonal entry (rows i and j , $i \neq j$): both rows have a 1 in the two columns other than i and j , and 0 in column i for row i , so the dot product is $1 + 1 = 2$ M1
 So A^2 has 3 on the diagonal and 2 elsewhere, which equals $3I + 2A$ ($2A$ has 2 off the diagonal and 0 on it) A1
Answer: $A^2 = 2A + 3I$ 3 marks

$$(b) A^3 = 2A^2 + 3A = 2(2A + 3I) + 3A \\ = 7A + 6I$$

M1

A1

6 walks back to the same vertex; 7 walks to a different vertex

A1

Answer: $7A + 6I$; 6 and 7

3 marks

$$(c) A^4 = 7A^2 + 6A$$

M1

$$= 7(2A + 3I) + 6A = 20A + 21I$$

M1

The diagonal entry is 21

A1

Answer: 21

3 marks

30 (9 marks)

(a) From A: 3 edges (to D, E, F), each returns to A, so 3 walks A to A

M1

A to B: via D, E or F, so 3 walks

A1

A to D: a walk of length 2 would need an edge joining two vertices of different sets after leaving A, impossible, so 0

A1

Answer: 3, 3, 0

3 marks

(b) Each edge changes set, so after an odd number of steps the walk is in the other set

M1

Hence it cannot end in the starting set

E1

Answer: Each step changes set

2 marks

(c) A walk A, X, Y, D has X in D, E, F (3 choices)

M1

Y in A, B, C (3 choices) and Y is joined to D

M1

$$3 \times 3 = 9$$

A1

Every choice of X and Y gives a valid walk, as X is joined to A and Y, and Y is joined to X and D, so there are no other walks

A1

Answer: 9

4 marks

Graph Properties

1 (2 marks)

Degrees: A 3, B 2, C 3, D 3, E 1

M1

Degree sequence 3, 3, 3, 2, 1

A1

Answer: 3, 3, 3, 2, 1

2 marks

2 (2 marks)

A simple graph has no loops (no edge joins a vertex to itself)

B1

and no multiple edges (at most one edge joins any pair of vertices)

B1

Answer: No loops and no multiple edges

2 marks

3 (2 marks)

The sum of the degrees is 11

M1

The sum of the degrees is twice the number of edges, so must be even

E1

Answer: Sum of degrees (11) is odd

2 marks

4 (2 marks)

$$K_5 \text{ has } \frac{5 \times 4}{2} = 10 \text{ edges}$$

M1

Complement has $10 - 7 = 3$ edges

A1

Answer: 3

2 marks

5 (2 marks)

The vertices can be split into two sets

B1

so that every edge joins a vertex in one set to a vertex in the other set (no edge joins two vertices of the same set)

B1

Answer: Vertices split into two sets with every edge between the sets

2 marks

6 (2 marks)

Each of the 3 vertices is joined to all 5 vertices of the other set: degree 5

B1

Each of the 5 vertices is joined to the 3 vertices of the other set: degree 3

B1

Answer: Three vertices of degree 5, five of degree 3

2 marks

7 (3 marks)

A needs 3 edges: AB, AC, AD

M1

Add BC (B and C now have degree 2) and DE (D degree 2, E degree 1)

M1

Degrees A 3, B 2, C 2, D 2, E 1 as required

A1

Answer: Edges AB, AC, AD, BC, DE

3 marks

8 (3 marks)

K_5 has 10 edges and G has 6, so the complement has 4 edges

M1

Pairs not joined in G: AD, BD, BE, CE

M1 A1

Answer: AD, BD, BE, CE

3 marks

9 (3 marks)

Degrees: A and C have degree 3, B and D degree 2 in G; Q and S have degree 3, P and R degree 2 in H

M1

$A \rightarrow Q, C \rightarrow S, B \rightarrow P, D \rightarrow R$

A1

Edges map as $AB \rightarrow QP, BC \rightarrow PS, CD \rightarrow SR, DA \rightarrow RQ, AC \rightarrow QS$: all are edges of H

A1

Answer: $A \rightarrow Q, B \rightarrow P, C \rightarrow S, D \rightarrow R$

3 marks

10 (3 marks)

Degrees of G: 1, 2, 2, 2, 2, 1

B1

Degrees of H: 5, 1, 1, 1, 1, 1

B1

Isomorphic graphs have the same degree sequence, so these graphs are not isomorphic

E1

Answer: Different degree sequences

3 marks

11 (4 marks)

(a) Take the sets A, B, C and D, E, F

M1

Every edge listed joins one vertex from each set

A1

Answer: Sets A, B, C and D, E, F

2 marks

(b) A, B and D would form a triangle A, B, D, A (AB, BD, DA)

M1

Bipartite graphs contain no odd cycles, so the new graph is not bipartite

E1

Answer: Triangle ABD is an odd cycle

2 marks

12 (3 marks)

(a) A, B, C, D, E, F, G: 3 components

B1

Answer: 3

1 mark

(b) Each new edge can join at most two components

M1

So $3 - 1 = 2$ edges

A1

Answer: 2

2 marks

13 (7 marks)

(a) Smallest: $8 - 1 = 7$ edges, a tree

M1 A1

Largest: $\frac{8 \times 7}{2} = 28$ edges, the complete graph K_8

A1

Answer: 7 (a tree) and 28 (K_8)

3 marks

(b) The sum of the degrees is $2 \times 5 = 10$ and each degree is at most 3 (there are only 3 other vertices)

M1

Only 3, 3, 3, 1 and 3, 3, 2, 2 are possible

B1

3, 3, 3, 1 is impossible: each vertex of degree 3 is joined to every other vertex, so the fourth vertex would have degree at least 3, not 1

M1

With 3, 3, 2, 2 the two vertices of degree 3 are joined to all the others, so the two vertices of degree 2 are not joined to each other: the graph is K_4 with one edge deleted

A1

Answer: K_4 with one edge deleted (degrees 3, 3, 2, 2)

4 marks

14 (6 marks)

(a) Remove 6 and subtract 1 from the other six numbers: 4, 2, 2, 1, 1, 0 M1

Remove 4 and subtract 1 from the next four: 1, 1, 0, 0, 0 M1

Remove 1 and reduce the next: 0, 0, 0, 0, so the sequence is graphical A1

Answer: Graphical 3 marks

(b) The sum 22 is even, so use the algorithm: remove 6 to give 5, 3, 1, 1, 0, 0 M1

Next remove 5: it requires five positive entries but only three entries (3, 1, 1) are positive M1

So the sequence is not graphical A1

Answer: Not graphical 3 marks

15 (4 marks)

(a) K_4 B1

Answer: K_4 1 mark

(b) A vertex of degree 4 is joined to all of the other four vertices M1

So each of the three vertices of degree 4 is joined to both vertices of degree 1, giving those vertices degree at least 3 M1

This contradicts degree 1 E1

Answer: Impossible 3 marks

16 (5 marks)

(a) G and its complement have the same number of edges, and together they have $\frac{n(n-1)}{2}$ edges M1

So G has half of this, $\frac{n(n-1)}{4}$ A1

Answer: G has $\frac{n(n-1)}{4}$ edges 2 marks

(b) $\frac{6 \times 5}{4} = 7.5$ is not a whole number E1

Answer: 7.5 edges is impossible 1 mark

(c) $\frac{5 \times 4}{4} = 5$ edges B1

The cycle C_5 , whose complement is another 5-cycle B1

Answer: 5 edges; C_5 2 marks

17 (5 marks)

(a) G: A 3, D 3, others 2 B1

H: Q 3, T 3, others 2, so both have degree sequence 3, 3, 2, 2, 2, 2 B1

Answer: 3, 3, 2, 2, 2, 2 for both 2 marks

(b) The vertices of degree 3 are adjacent in both graphs: $A \rightarrow Q, D \rightarrow T$ M1

$B \rightarrow P, C \rightarrow S, E \rightarrow U, F \rightarrow R$ A1

All 7 edges of G map to edges of H, e.g. $AB \rightarrow QP, BC \rightarrow PS, CD \rightarrow ST, DE \rightarrow TU, EF \rightarrow UR,$ A1

$FA \rightarrow RQ, AD \rightarrow QT$

Answer: $A \rightarrow Q, B \rightarrow P, C \rightarrow S, D \rightarrow T, E \rightarrow U, F \rightarrow R$ 3 marks

18 (5 marks)

(a) Each of the a vertices can be joined to at most $10 - a$ vertices of the other set E1

Answer: At most $a(10 - a)$ edges 1 mark

(b) $10a - a^2 = 25 - (a^2 - 10a + 25)$ M1 A1

Largest when $a = 5$, giving 25 edges ($K_{5,5}$) A1

Answer: 25 3 marks

(c) A connected graph on 10 vertices needs at least 9 edges (a tree is bipartite) B1

Answer: 9 1 mark

19 (5 marks)

(a) 7 vertices B1

6 spokes + 6 sides = 12 edges B1

Hub degree 6, every other vertex degree 3 (check: $6 + 18 = 24 = 2 \times 12$) B1

Answer: 7, 12; 6 and 3 3 marks

(b) O, A, B form a triangle, an odd cycle

M1

A bipartite graph has no odd cycle

E1

Answer: Contains a triangle

2 marks

20 (5 marks)

$$\text{Reordered matrix} \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

M1 A1 A1

The top-left 3×3 block and the bottom-right 3×3 block are all zeros

M1

So no two of V_1, V_3, V_5 are adjacent and no two of V_2, V_4, V_6 are adjacent: every edge joins the two sets, so the graph is bipartite

E1

Answer: Zero diagonal blocks, so bipartite with sets V_1, V_3, V_5 and V_2, V_4, V_6

5 marks

21 (6 marks)

(a) K_6 has 15 edges, $K_{3,3}$ has 9

M1

Complement has 6 edges

A1

Answer: 6

2 marks

(b) Each vertex has degree 5 in K_6 and 3 in G

M1

So every vertex of the complement has degree 2

A1

Answer: 2

2 marks

(c) Vertices in the same set are joined in the complement but vertices in different sets are not: two disjoint triangles

M1

It is not connected (two components)

A1

Answer: Two disjoint triangles; not connected

2 marks

22 (7 marks)

Let u and v be any two vertices; we show there is a path between them

B1

If u and v are adjacent there is nothing to prove, so suppose not

M1

Each of u and v has at least $\frac{n-1}{2}$ neighbours, all in the other $n-2$ vertices

M1

If they had no common neighbour, the number of vertices other than u and v would be at least

M1

$$\frac{n-1}{2} + \frac{n-1}{2}$$

$$= n-1$$

A1

This is greater than $n-2$, a contradiction

M1

So u and v have a common neighbour and are joined by a path of length 2, so G is connected

E1

Answer: G is connected

7 marks

23 (7 marks)

(a) 4 edges, so the sum is 8

B1

Answer: 8

1 mark

(b) The degrees are at least 1 and sum to 8

M1

Degrees 2, 2, 2, 1, 1: the path of length 4

A1

Degrees 3, 2, 1, 1, 1: a vertex of degree 3 with one of its branches extended by one edge

A1

Degrees 4, 1, 1, 1, 1: the star $K_{1,4}$

A1

Answer: Three trees

4 marks

(c) The degrees would sum to at least $3 + 3 + 1 + 1 + 1 = 9$

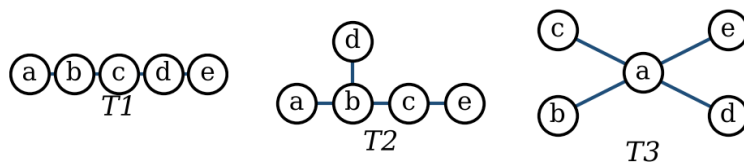
M1

which exceeds 8

E1

Answer: Degree sum would be at least 9

2 marks

**24 (7 marks)**

Call the vertices a, b, c, d, e, f, g with degrees $6, 5, 4, 3, 2, 1, x$. The vertex a is joined to all the others, so f is joined only to a M1

Remove a : the remaining degrees are $b\ 4, c\ 3, d\ 2, e\ 1, f\ 0, g\ x - 1$ M1

b needs 4 neighbours among c, d, e, f, g ; f has degree 0, so b is joined to c, d, e and g M1

Remove b : $c\ 2, d\ 1, e\ 0, f\ 0, g\ x - 2$ M1

c needs 2 neighbours and only d and g have positive degree, so c is joined to d and g M1

Remove c : $d\ 0$ and $g\ x - 3$, and all remaining degrees must be 0, so $x - 3 = 0$ M1

$x = 3$ (a graph with degrees $6, 5, 4, 3, 3, 2, 1$ does exist, by the Havel-Hakimi algorithm) A1

Answer: $x = 3$ 7 marks

25 (6 marks)

(a) Both have 6 vertices and 6 edges B1

Every vertex of both graphs has degree 2 B1

Answer: 6 vertices, 6 edges, all degrees 2 2 marks

(b) G is connected but H has two components M1

Isomorphic graphs have the same number of connected components, so G and H are not isomorphic A1

Answer: G is connected, H is not 2 marks

(c) The claim is false: having the same degree sequence is necessary but not sufficient for isomorphism E1

G has no triangle (its only cycle has length 6) whereas H has two triangles E1

Answer: Claim false; H has triangles and G has none 2 marks

26 (6 marks)

(a) The entry in row i , column i counts walks of length 2 from i back to i , one for each edge at i M1 E1

Degrees $A\ 3, B\ 2, C\ 2, D\ 2, E\ 1$ A1

Answer: 3, 2, 2, 2, 1 3 marks

(b) Sum of the degrees is 10, so 5 edges B1

Answer: 5 1 mark

(c) Entry in row B , column C is 1 M1

So B and C have exactly 1 common neighbour A1

Answer: 1 2 marks

27 (8 marks)

Model the people as the vertices of a graph, with an edge when two people know each other. We must show that the graph or its complement contains a triangle B1

Choose one person P . Each of the other five people either knows P or is a stranger to P M1

By the pigeonhole principle at least 3 of them are in the same category (since $5 > 2 \times 2$) M1 A1

Case 1: three people X, Y, Z all know P . If any two of X, Y, Z know each other, they form a group of three mutual acquaintances with P M1

Otherwise X, Y, Z are three mutual strangers A1

Case 2: three people X, Y, Z are all strangers to P . If any two of them are strangers to each other, they form a group of three mutual strangers with P ; otherwise X, Y, Z all know each other M1

In every case the required group of three exists E1

Answer: Three mutual acquaintances or three mutual strangers always exist 8 marks

28 (9 marks)

- (a) Going round a cycle, each edge moves from one set to the other M1
 so the vertices of the cycle alternate between the two sets M1
 To return to the start the number of edges must be even E1
Answer: Every cycle in a bipartite graph has even length 3 marks
 (b) Suppose u and w are both in X (or both in Y) and are joined by an edge M1
 Take shortest paths from v to u and from v to w , of lengths a and b ; a and b have the same parity M1
 so $a + b$ is even
 The two paths together with the edge uw form a closed walk of length $a + b + 1$, which is odd M1 A1
 A closed walk of odd length contains an odd cycle: if no vertex is repeated it is a cycle; otherwise M1
 split it at a repeated vertex into two shorter closed walks whose lengths add to an odd number, so
 one of them is odd; repeat until an odd cycle is found
 This contradicts the assumption, so every edge joins X to Y and G is bipartite E1
Answer: G is bipartite, with parts X and Y 6 marks

29 (9 marks)

- (a) u and v have no common neighbour, otherwise a triangle would exist M1
 so their neighbours are all different vertices among the n vertices, giving $d(u) + d(v) \leq n$ E1
Answer: $d(u) + d(v) \leq n$ 2 marks
 (b) Each vertex v lies in $d(v)$ of the edges, so the sum over edges of $d(u) + d(v)$ equals the sum of M1
 $d(v)^2$
 The sum of e terms each at most n is at most en A1
Answer: The sum of $d(v)^2$ is at most en 2 marks
 (c) The sum of the degrees is $2e$, so $4e^2 \leq n \times (\text{sum of } d(v)^2)$ M1
 $4e^2 \leq n \times en$ M1
 $e \leq \frac{n^2}{4}$ A1
Answer: $e \leq \frac{n^2}{4}$ 3 marks
 (d) $10^2 \div 4 = 25$ B1
 $K_{5,5}$ B1
Answer: 25 edges, $K_{5,5}$ 2 marks

30 (9 marks)

- (a) Each of the 6 possible edges is either present or absent M1
 $2^6 = 64$ A1
Answer: 64 2 marks
 (b) Choose 3 of the 6 edges: ${}^6C_3 = 20$ B1
Answer: 20 1 mark
 (c) If the 3 edges contain a cycle, it must be a triangle, leaving one vertex isolated; otherwise the M1
 graph has no cycle and 3 edges, so it is a tree on 4 vertices
 Triangle plus an isolated vertex: choose the isolated vertex in 4 ways M1 A1
 The remaining $20 - 4 = 16$ graphs are trees A1
Answer: 4 triangle graphs, 16 trees 4 marks
 (d) Paths: $24 \div 2 = 12$ M1
 $12 + 4 = 16$, as in (c) A1
Answer: $12 + 4 = 16$ 2 marks

Eulerian and Hamiltonian Graphs

1 (2 marks)

- An Eulerian trail uses every edge exactly once B1
 An Eulerian circuit is an Eulerian trail that starts and ends at the same vertex B1
Answer: A circuit starts and ends at the same vertex 2 marks

2 (2 marks)

The graph must be connected

B1

and every vertex must have even degree

B1

Answer: All vertices have even degree

2 marks

3 (2 marks)

Degrees: A 2, B 2, C 4, D 2, E 2

M1

The graph is connected and every vertex has even degree

A1

Answer: Connected and all degrees even

2 marks

4 (2 marks)

A cycle (closed path) in the graph

B1

that passes through every vertex exactly once

B1

Answer: A cycle visiting every vertex exactly once

2 marks

5 (2 marks)

Every vertex of K_5 has degree 4

M1

All degrees are even and K_5 is connected, so yes

A1

Answer: Yes

2 marks

6 (2 marks)

Exactly two vertices have odd degree

B1

and all other vertices have even degree; the trail starts at one odd vertex and ends at the other

B1

Answer: Exactly two odd vertices

2 marks

7 (3 marks)

(a) Degrees: A 3, B 2, C 3, D 2

M1

The trail starts and finishes at the two odd vertices A and C

A1

Answer: A and C

2 marks

(b) A, C, D, A, B, C

B1

Answer: A, C, D, A, B, C

1 mark

8 (3 marks)

(a) Every vertex has degree 3 (odd), so no

B1

Answer: No

1 mark

(b) Every vertex has degree 6 (even), so yes

B1

Answer: Yes

1 mark

(c) Degrees are 4 and 2 (all even), so yes

B1

Answer: Yes

1 mark

9 (2 marks)

A cycle through all six vertices: A, B, D, F, E, C, A

M1

All the edges AB, BD, DF, FE, EC, CA are in the graph

A1

Answer: A, B, D, F, E, C, A

2 marks

10 (3 marks)

Number of vertices of odd degree = 4

B1

An Eulerian trail needs 0 or 2 odd vertices

M1

So no trail uses every edge exactly once

A1

Answer: Four odd vertices

3 marks

11 (3 marks)

(a) Every degree is even, so the graph is Eulerian

B1

Answer: Eulerian

1 mark

(b) Exactly two vertices (the two of degree 3) have odd degree, so the graph is semi-Eulerian

B1

Answer: Semi-Eulerian

1 mark

(c) Four vertices have odd degree, which is neither 0 nor 2, so the graph is neither

B1

Answer: Neither

1 mark

12 (3 marks)

Vertex E has degree 1

B1

A cycle must enter and leave every vertex it visits, so uses two edges at each vertex

M1

E has only one edge, so no cycle can pass through E

E1

Answer: E has degree 1

3 marks

13 (3 marks)

Odd vertices: A, C, D, F

M1

Adjacent pairs among them: AC, CD, DF, FA, so join the non-adjacent pairs A with D and C with F

M1

Two extra edges: AD and CF

A1

Answer: 2 edges: AD and CF

3 marks

14 (4 marks)

(a) Degrees: A 3, B 4, C 3, D 2, E 2, F 2: exactly two odd vertices

M1

So there is an Eulerian trail from A to C but no Eulerian circuit

A1

Answer: Exactly two odd vertices

2 marks

(b) Trail: A, E, B, F, C, D, A, B, C

M1 A1

Answer: A, E, B, F, C, D, A, B, C

2 marks

15 (4 marks)

(a) $n = 6$, so every degree must be at least 3

M1

Every vertex has degree 3

A1

Answer: All degrees are 3

2 marks

(b) By Dirac's theorem it is Hamiltonian

B1

A, B, C, D, E, F, A

B1

Answer: A, B, C, D, E, F, A

2 marks

16 (4 marks)

(a) Each vertex has degree 3

M1

which is odd, so there is no Eulerian circuit

E1

Answer: All degrees are 3

2 marks

(b) Gray code order: 000, 001, 011, 010, 110, 111, 101, 100

M1

Consecutive strings (and the last and first) differ in one digit, so it is a Hamiltonian cycle

A1

Answer: 000, 001, 011, 010, 110, 111, 101, 100, 000

2 marks

17 (4 marks)

(a) Every edge joins a vertex of one set to a vertex of the other

E1

Answer: Edges only join different sets

1 mark

(b) A cycle alternates between the sets so visits equally many vertices of each set

M1

A Hamiltonian cycle would need to visit all $3 + 4 = 7$ vertices, with 3 from one set and 4 from the other

M1

These numbers are not equal, so no Hamiltonian cycle exists

E1

Answer: Unequal sets

3 marks

18 (4 marks)

Each vertex of K_n has degree $n - 1$

B1

An Eulerian circuit exists if and only if all degrees are even, so $n - 1$ is even, so n is odd

M1 A1

K_n has a Hamiltonian cycle for every $n \geq 3$, as the vertices can be visited in any order

A1

Answer: n odd; Hamiltonian for all $n \geq 3$

4 marks

19 (6 marks)

(a) A 3, B 3, C 3, D 3, E 2, F 2

M1 A1

Answer: A, B, C, D: 3; E, F: 2

2 marks

(b) Four rooms have odd degree

M1

A walk through every doorway once needs at most two odd vertices

E1

Answer: Four odd vertices

2 marks

(c) Now A and B have degree 2 and only C and D are odd	M1
The walk must start at one of C, D and finish at the other	A1
Answer: Yes; start at C (or D), end at D (or C)	2 marks
20 (4 marks)	
(a) Fix a starting vertex: the other $n - 1$ vertices can be arranged in $(n - 1)!$ orders	M1
Each cycle is counted twice, once in each direction	M1
Number of cycles = $\frac{(n - 1)!}{2}$	A1
Answer: K_n has $\frac{(n - 1)!}{2}$ Hamiltonian cycles	3 marks
(b) $\frac{6!}{2} = 360$	B1
Answer: 360	1 mark
21 (8 marks)	
(a) Odd vertices: A, B, D, E, which is four of them	M1
An Eulerian trail needs exactly 0 or 2 odd vertices	E1
Answer: Four odd vertices	2 marks
(b) Now only B and E have odd degree (A and D become even)	M1
So an Eulerian trail exists	A1
It starts at B and finishes at E (or vice versa)	A1
Answer: B to E	3 marks
(c) All degrees become 2	M1
The graph is the hexagon, a single cycle	A1
Circuit A, B, C, D, E, F, A	A1
Answer: A, B, C, D, E, F, A	3 marks
22 (6 marks)	
Consider an Eulerian circuit; it uses every edge exactly once	B1
Each time the circuit passes through a vertex it uses one edge to arrive and a different edge to leave	M1
So each pass through a vertex uses 2 edges at that vertex	A1
For the starting vertex, the first edge (leaving) and the last edge (arriving) together form one more such pair	M1
Every edge at a vertex is used exactly once, so the degree is a sum of pairs	M1
Hence the degree of every vertex is even	E1
Answer: Every vertex has even degree	6 marks
23 (7 marks)	
(a) The four edge-midpoint vertices b, d, f, h have degree 3 (odd)	M1
Four odd vertices, so no Eulerian trail	A1
Answer: Four odd vertices	2 marks
(b) Adjacent vertices have different colours, so a cycle alternates between colours	M1
There are 5 vertices of one colour and 4 of the other	M1
A Hamiltonian cycle would need equal numbers, so none exists	E1
Answer: 5 black and 4 white vertices	3 marks
(c) Snake path: a, b, c, f, e, d, g, h, i	M1
Every consecutive pair is joined by an edge	A1
Answer: a, b, c, f, e, d, g, h, i	2 marks
24 (8 marks)	
(a) Degrees: C has degree 6 and all other vertices have degree 2	M1
All degrees are even and the graph is connected, so an Eulerian circuit exists	A1
Answer: Degrees 6, 2, 2, 2, 2, 2	2 marks
(b) A and B each have degree 2, so a Hamiltonian cycle must use both edges at A (AB, AC) and both edges at B (BA, BC)	M1
Then the edges AB, BC, CA already form a closed cycle of length 3	M1

A cycle that has closed after 3 vertices cannot include the other four vertices	M1
So no Hamiltonian cycle exists: degree at least 2 is not sufficient	E1
Answer: No Hamiltonian cycle, so the student is wrong	4 marks
(c) C, A, B, C, D, E, C, F, G, C	M1 A1
Answer: C, A, B, C, D, E, C, F, G, C	2 marks
25 (6 marks)	
(a) Degrees: A 3, B 3, C 4, D 3, E 3, F 2	M1
Non-adjacent pairs include A and E ($3 + 3 = 6$), A and F ($3 + 2 = 5$), B and D (6), B and F (5), D and F (5), C and E (7)	M1
Pairs with sum less than 6: AF, BF, DF	A1
Answer: AF, BF, DF	3 marks
(b) The condition fails, so Ore's theorem tells us nothing about whether G is Hamiltonian	E1
Answer: Nothing	1 mark
(c) Cycle: A, B, C, F, E, D, A	M1
The condition is sufficient but not necessary	E1
Answer: A, B, C, F, E, D, A	2 marks
26 (7 marks)	
(a) Each of the 4 vertices in one set has degree 4: $4 \times 4 = 16$ edges	M1 A1
All degrees are 4, which is even, and the graph is connected, so an Eulerian circuit exists	E1
Answer: 16 edges; all degrees even	3 marks
(b) The cycle alternates between the sets, so every edge used (AE, EB, BF, FC, CG, GD, DH, HA) joins a vertex of A, B, C, D to a vertex of E, F, G, H and is an edge of $K_{4,4}$	M1
All 8 vertices appear exactly once before returning to A	A1
Answer: A, E, B, F, C, G, D, H, A is a Hamiltonian cycle	2 marks
(c) The cycle uses 8 edges and 2 at each vertex	M1
Each vertex now has degree $4 - 2 = 2$	A1
Answer: Every vertex has degree 2	2 marks
27 (10 marks)	
Start at any vertex v and build a trail, never repeating an edge, until it cannot be extended	B1
Whenever the trail enters a vertex other than v, it has used an odd number of edges there; as the degree is even, an unused edge remains, so the trail can only get stuck at v	M1 A1
So the trail T is closed	A1
Suppose T does not use every edge. As the graph is connected, some vertex w on T has an unused edge	M1
Remove the edges of T: every vertex still has even degree (T uses an even number of edges at each vertex)	M1
Starting at w, build another closed trail T' from unused edges in the same way	M1
Insert T' into T at w to get a longer closed trail	A1
Repeat; as the number of edges is finite the process ends with a closed trail using all the edges	M1
This is an Eulerian circuit	E1
Answer: The graph has an Eulerian circuit	10 marks
28 (8 marks)	
Let the sets be X (size m) and Y (size n)	B1
Every edge joins X to Y, so a cycle alternates between X and Y	M1
A cycle therefore contains equal numbers of vertices from X and Y	M1
A Hamiltonian cycle contains all $m + n$ vertices, so $m = n$	A1
Conversely suppose $m = n$: label $X = x_1, \dots, x_n$ and $Y = y_1, \dots, y_n$	M1
The cycle $x_1, y_1, x_2, y_2, \dots, x_n, y_n, x_1$ uses only edges between X and Y	M1
All edges exist because $K_{n,n}$ is complete bipartite	A1
It visits every vertex once, so it is a Hamiltonian cycle	E1
Answer: $K_{m,n}$ is Hamiltonian if and only if $m = n$	8 marks

29 (9 marks)

Pair the $2k$ odd vertices into k pairs	B1
Add k new edges, one joining each pair, to form a graph G'	M1
Every vertex of G' has even degree and G' is connected	A1
So G' has an Eulerian circuit	M1
Remove the k new edges from the circuit: it breaks into k trails (one gap for each removed edge on the circle)	M1 A1
These k trails together use every edge of G exactly once	A1
Lower bound: each trail has two ends, and a vertex of odd degree must be the end of at least one trail (a trail passing through a vertex uses edges in pairs)	M1
There are $2k$ odd vertices, each needing a trail end, and each trail supplies two ends, so at least k trails	E1
Answer: k trails are enough, and fewer than k is impossible	9 marks

30 (10 marks)

(a) Vertices 2^n	B1
Each vertex has n neighbours, so degree n	B1
Edges = $\frac{n \times 2^n}{2} = n 2^{n-1}$	B1
Answer: 2^n; $n 2^{n-1}$; n	3 marks
(b) All degrees equal n and the graph is connected	M1
So n must be even	A1
Answer: n even	2 marks
(c) Base case $n = 2$: 00, 01, 11, 10, 00 is a Hamiltonian cycle of Q_2	B1
Assume Q_k ($k \geq 2$) has a Hamiltonian cycle C and let uv be an edge of C	M1
Q_{k+1} consists of a copy of Q_k with strings $0u, 0v, \dots$ and a copy with strings $1u, 1v, \dots$, where each vertex $0x$ is joined to $1x$. Both copies contain the cycle C	M1
Delete edge uv from C in each copy to leave a path from $0u$ to $0v$ through all vertices of the first copy, and a path from $1v$ to $1u$ through all vertices of the second	M1
Follow the first path from $0u$ to $0v$, cross the edge $0v-1v$, follow the second path from $1v$ to $1u$, then cross $1u-0u$ to close the cycle. Every vertex of Q_{k+1} is visited once, so Q_{k+1} is Hamiltonian, and by induction Q_n is Hamiltonian for all $n \geq 2$	E1
Answer: Q_n has a Hamiltonian cycle for every $n \geq 2$	5 marks

Planar Graphs**1 (2 marks)**

A graph that can be drawn in the plane	B1
so that no two edges cross (edges meet only at vertices)	B1
Answer: Can be drawn with no edges crossing	2 marks

2 (1 mark)

$v - e + f = 2$	B1
Answer: $v - e + f = 2$	1 mark

3 (2 marks)

$6 - 9 + f = 2$	M1
$f = 5$	A1
Answer: 5	2 marks

4 (2 marks)

Faces: the outer face, the inner triangle DEF, and three quadrilaterals	M1
$f = 5$ and $6 - 9 + 5 = 2$	A1
Answer: 5 faces; $6 - 9 + 5 = 2$	2 marks

5 (2 marks) K_5 : 5 vertices and 10 edges

B1

 $K_{3,3}$: 6 vertices and 9 edges

B1

Answer: K_5 : 5, 10; $K_{3,3}$: 6, 9

2 marks

6 (2 marks)

A graph is non-planar if and only if it contains a subgraph

B1

that is a subdivision of K_5 or of $K_{3,3}$

B1

Answer: Contains a subdivision of K_5 or $K_{3,3}$

2 marks

7 (2 marks)

$$v - 15 + 8 = 2$$

M1

$$v = 9$$

A1

Answer: 9

2 marks

8 (3 marks) K_5 has $v = 5$ and $e = 10$

B1

$$3v - 6 = 9$$

M1

 $10 > 9$, so K_5 cannot be planar

A1

Answer: $10 > 9$

3 marks

9 (3 marks) $K_{3,3}$ has $v = 6$ and $e = 9$

B1

$$2v - 4 = 8$$

M1

 $9 > 8$, so $K_{3,3}$ is not planar

A1

Answer: $9 > 8$

3 marks

10 (3 marks)

Edges AD and BE cross

B1

Redraw BE as a curve outside the hexagon ABCDEF, leaving AD inside

M1

No other edges cross, so the graph is planar

A1

Answer: Draw BE outside the hexagon

3 marks

11 (3 marks)

(a) Sum of face degrees = $3 \times 12 = 36 = 2e$

M1

$$e = 18$$

A1

Answer: 18

2 marks

(b) $v = 2 + e - f = 2 + 18 - 12 = 8$

B1

Answer: 8

1 mark

12 (3 marks)

$$v - e + f = 2 \text{ so } 5 - 6 + f = 2, f = 3$$

M1

Put the two vertices of the 2-set at the top and bottom and the three vertices of the 3-set in a row between them

M1

Each of the three middle vertices is joined to both outer vertices without crossings, giving 3 faces

A1

Answer: $f = 3$; drawable without crossings

3 marks

13 (3 marks)

(a) Vertices $5 + 1 = 6$

B1

Edges $10 + 1 = 11$

B1

Answer: 6 vertices, 11 edges

2 marks

(b) It is a subdivision of K_5 , which is non-planar (Kuratowski)

E1

Answer: Subdivision of K_5

1 mark

14 (6 marks)**(a)** Non-cycle edges: AC, AD, BF, EG, drawn inside or outside the cycle **B1**Two such edges conflict (cannot be on the same side) if their ends interlace around the cycle **M1**AC and BF interlace, AD and BF interlace, BF and EG interlace; no other pair interlaces **M1 A1**Put BF inside; then AC, AD and EG all go outside and do not interlace with each other **M1****Answer: Planar** **5 marks****(b)** Inside: BF; outside: AC, AD, EG. The graph is planar **A1****Answer: Planar** **1 mark****15 (6 marks)****(a)** Non-cycle edges: AD, BE, CF **B1**AD and BE interlace, AD and CF interlace, BE and CF interlace **M1 A1**So all three must be on different sides of the cycle, but there are only two sides (inside and outside) **M1**Hence the graph is not planar **A1****Answer: Not planar** **5 marks****(b)** Hexagon with three long diagonals is $K_{3,3}$ (sets A, C, E and B, D, F) **B1****Answer: $K_{3,3}$** **1 mark****16 (5 marks)****(a)** $14 - 12 + f = 1 + 3$ **M1** $f = 2$ **A1****Answer: 2** **2 marks****(b)** A forest of c trees on v vertices has $v - c$ edges, so $18 = 20 - c$, $c = 2$ **M1 A1**One face: $20 - 18 + f = 1 + 2$ gives $f = 1$ **A1****Answer: $c = 2$, $f = 1$** **3 marks****17 (5 marks)**Each face is bounded by at least 3 edges **B1**Each edge borders at most 2 faces, so the sum of the face degrees is $2e$, hence $3f \leq 2e$ **M1 A1**Euler: $f = 2 - v + e$, so $3(2 - v + e) \leq 2e$ **M1** $6 - 3v + 3e \leq 2e$ so $e \leq 3v - 6$ **A1****Answer: $e \leq 3v - 6$** **5 marks****18 (4 marks)**Sum of degrees ≥ 50 , so $e \geq 25$ **M1 A1**For a simple planar graph $e \leq 3v - 6 = 24$ **M1** $25 > 24$, a contradiction **E1****Answer: Contradiction** **4 marks****19 (5 marks)****(a)** $v = 5$ and $e = 6$, so $f = 2 - 5 + 6 = 3$ **M1**Faces: two triangles ABC, ACD and the outer face **A1****Answer: 3** **2 marks****(b)** Triangular faces: degree 3 each **B1**Outer face boundary A, B, C, D, E, D, A: degree 6 (DE counted twice) **B1** $3 + 3 + 6 = 12 = 2 \times 6$ **B1****Answer: 3, 3, 6; sum 12** **3 marks****20 (5 marks)****(a)** $v = 6$, $e = 8$ **B1****Answer: 6, 8** **1 mark****(b)** $f = 2 - 6 + 8 = 4$ **B1****Answer: 4** **1 mark****(c)** H is bipartite so has no odd cycles; each face has degree at least 4 **M1**Sum of face degrees $= 2e = 16$ **M1**4 faces each of degree at least 4 with sum 16 means each has degree 4 **E1**

Answer: Each face has degree 4**3 marks****21 (6 marks)**Suppose K_5 is planar with f faces. $v = 5$, $e = 10$ **B1**Euler: $5 - 10 + f = 2$ so $f = 7$ **M1 A1**

Every face has degree at least 3 (simple graph), so the sum of the face degrees is at least 21

M1The sum of the face degrees is $2e = 20$ **A1** $21 \leq 20$ is false, so K_5 is not planar**E1****Answer: K_5 is not planar****6 marks****22 (6 marks)**Suppose $K_{3,3}$ is planar with f faces. $v = 6$, $e = 9$, so $f = 2 - 6 + 9 = 5$ **M1 A1** $K_{3,3}$ is bipartite so has no odd cycles; every face has degree at least 4**M1 A1**Sum of face degrees $\geq 4 \times 5 = 20$ **M1**But the sum is $2e = 18$, a contradiction**E1****Answer: $K_{3,3}$ is not planar****6 marks****23 (6 marks)**

Suppose every vertex has degree at least 6

B1Then $2e \geq 6v$ so $e \geq 3v$ **M1 A1**A simple planar graph has $e \leq 3v - 6$ **M1**So $3v \leq 3v - 6$, which is false**A1**

Hence some vertex has degree at most 5

E1**Answer: Some vertex has degree at most 5****6 marks****24 (6 marks)**The complement of G has the same 11 vertices; the edges of G and its complement together make K_{11} , so $e + e' = 55$ **M1 A1**If both were planar, each would have at most $3 \times 11 - 6 = 27$ edges**M1**So $e + e' \leq 54$ **M1 A1** $55 > 54$, a contradiction**E1****Answer: G and its complement cannot both be planar****6 marks****25 (6 marks)****(a)** $v = 6$ and $e = 9$ **B1** $3v - 6 = 12$ and $9 \leq 12$ **B1****Answer: $9 \leq 12$** **2 marks****(b)** $K_{3,3}$ is not planar**B1**

so the inequality is necessary for planarity but not sufficient

E1**Answer: $K_{3,3}$ is non-planar****2 marks****(c)** $2v - 4 = 8$ **M1** $9 > 8$ **A1****Answer: $9 > 8$** **2 marks****26 (6 marks)****(a)** After deleting GH , G has degree 2 (edges AG , GD) and H has degree 2 (edges BH , HE)**M1**Replace A, G, D by a single edge AD and B, H, E by a single edge BE **M1**The remaining vertices A, B, C and D, E, F have every vertex of A, B, C joined to every vertex of D, E, F **M1**This is $K_{3,3}$, so the graph is a subdivision of $K_{3,3}$ **A1****Answer: Subdivision of $K_{3,3}$ with sets A, B, C, D, E, F** **4 marks****(b)** It contains a subdivision of $K_{3,3}$ as a subgraph**M1**

By Kuratowski's theorem it is non-planar

E1**Answer: Contains a subdivision of $K_{3,3}$** **2 marks**

27 (10 marks)

Base case $e = 0$: a single vertex, so $v = 1$, $f = 1$ and $1 - 0 + 1 = 2$

B1

Inductive hypothesis: the formula holds for every connected planar graph with k edges

M1

Let G be a connected planar graph with $k + 1$ edges

B1

Case 1: G is a tree. A tree has $e = v - 1$ and one face, so $v - (v - 1) + 1 = 2$

M1 A1

Case 2: G has a cycle. Remove one edge from a cycle: G stays connected and has k edges

M1

An edge on a cycle has the inside of the cycle on one side and the outside on the other, so it borders two different faces; removing it merges these two faces into one, so f decreases by 1

M1

The new graph has v vertices, k edges, $f - 1$ faces and $v - k + (f - 1) = 2$ by the hypothesis

M1

So $v - (k + 1) + f = 2$, as required

A1

By induction the formula holds for all connected planar graphs

E1

Answer: $v - e + f = 2$

10 marks

28 (8 marks)

(a) G is connected and has a cycle, so every face boundary contains a cycle, of length at least 5; each edge lies on at most two faces (or twice on one), so the face degrees add up to $2e$ and $2e \geq 5f$

M1 A1

Euler: $f = 2 - v + e$

M1

$2e \geq 5(2 - v + e)$ so $3e \leq 5v - 10$

M1

$$e \leq \frac{5(v - 2)}{3}$$

A1

Answer: $e \leq \frac{5(v - 2)}{3}$

5 marks

(b) Bound for $v = 10$: $e \leq \frac{5 \times 8}{3} = 13.33\dots$

M1

The Petersen graph has $15 > 13$

M1

So it is not planar

A1

Answer: $15 > 13.3$

3 marks

29 (9 marks)

(a) Each face has n edges and each edge lies on 2 faces: $nf = 2e$

M1

Each vertex has m edges and each edge has 2 ends: $mv = 2e$

A1

Answer: $2e = nf = mv$

2 marks

$$(b) v = \frac{2e}{m}, f = \frac{2e}{n}$$

M1

$$\frac{2e}{m} - e + \frac{2e}{n} = 2$$

M1

$$\text{Dividing by } 2e: \frac{1}{m} + \frac{1}{n} = \frac{1}{2} + \frac{1}{e}$$

A1

$$\text{Answer: } \frac{1}{m} + \frac{1}{n} = \frac{1}{2} + \frac{1}{e}$$

3 marks

$$(c) \text{ Since } e > 0, \frac{1}{m} + \frac{1}{n} > \frac{1}{2}$$

B1

$$\text{If } m = 3: \frac{1}{n} > \frac{1}{6} \text{ so } n = 3, 4, 5, \text{ giving } e = 6, 12, 30$$

M1

$$\text{If } m = 4: \frac{1}{n} > \frac{1}{4} \text{ so } n = 3, \text{ and } m = 5: n = 3; (m \geq 6 \text{ impossible})$$

M1

$$\text{Faces } f = \frac{2e}{n}: (3,3): 4; (3,4): 6; (3,5): 12; (4,3): 8; (5,3): 20$$

A1

Answer: (3,3), (3,4), (3,5), (4,3), (5,3): 4, 6, 12, 8, 20 faces

4 marks

30 (10 marks)

(a) Each edge borders two faces, so $2e = 5p + 6h$

M1

$$\text{Each vertex has degree 3, so } 3v = 2e, v = \frac{5p + 6h}{3}$$

A1

$$\text{Answer: } e = \frac{5p + 6h}{2} \text{ and } v = \frac{5p + 6h}{3}$$

2 marks

$$(b) f = p + h$$

B1

$$\frac{5p + 6h}{3} - \frac{5p + 6h}{2} + p + h = 2$$

M1

Common denominator 6: $\frac{2(5p + 6h) - 3(5p + 6h)}{6} + p + h = 2$

M1

$$-\frac{5p + 6h}{6} + p + h = 2$$

M1

$$\frac{p}{6} = 2 \text{ so } p = 12$$

A1

Answer: p = 12

5 marks

(c) $5p + 6h = 60 + 120 = 180$

M1

$$e = 90$$

A1

$$v = 60$$

A1

Answer: 60 vertices, 90 edges

3 marks

Minimum Spanning Trees

1 (2 marks)

A subgraph that is a tree (connected, no cycles)
that includes every vertex of the graph

B1

B1

Answer: A tree containing all the vertices

2 marks

2 (1 mark)

$$9 - 1 = 8$$

B1

Answer: 8

1 mark

3 (2 marks)

A spanning tree of the network
whose total weight is as small as possible

B1

B1

Answer: Spanning tree of least total weight

2 marks

4 (2 marks)

Tree edges AC 4, CD 3, BC 5, DE 5

M1

Total = 17

A1

Answer: 17

2 marks

5 (2 marks)

A spanning tree has $6 - 1 = 5$ edges

M1

$11 - 5 = 6$ edges removed

A1

Answer: 6

2 marks

6 (2 marks)

If a spanning tree T does not contain e, adding e to T creates a cycle, and removing another (heavier) edge of that cycle gives a spanning tree of smaller weight

M1

So T was not a minimum spanning tree: every minimum spanning tree contains e

E1

Answer: Otherwise a swap gives a lighter tree

2 marks

7 (3 marks)

Choose shortest links that do not form a cycle: DE 6, BE 7, BC 8

M1

Next AC 9 (AB 12 or CD 10 would make cycles)

M1

Total = $6 + 7 + 8 + 9 = 30$ km: links DE, BE, BC, AC

A1

Answer: 30 km

3 marks

8 (2 marks)

A tree has no cycles

B1

so if all the edges of a cycle were included the subgraph would contain a cycle and not be a tree

E1

Answer: A tree contains no cycles

2 marks

9 (3 marks)

A spanning tree has 4 of the 5 edges

M1

Removing any one edge leaves a path through all 5 vertices (a tree)

M1

So there are 5 spanning trees

A1

Answer: 5

3 marks

10 (3 marks)

Weight 2 edge CD must be included, then the weight 3 edges AB and BC

M1

Then one of DE or EA (both weight 4) completes the tree: tree 1 is CD, AB, BC, DE

A1

Tree 2 is CD, AB, BC, EA; both have weight 12, so the minimum spanning tree is not unique

A1

Answer: CD, AB, BC with DE or EA; weight 12

3 marks

11 (4 marks)

Shortest edges in order: DE 3, BC 4, AB 5, CE 6

M1 A1

Edge AC 7 would form a cycle with AB, BC; BD 8 would form a cycle; so next DF 9

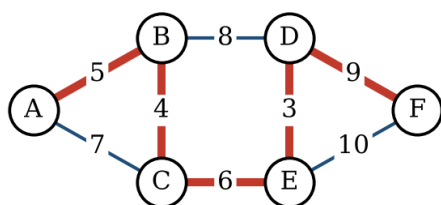
M1

Total = $3 + 4 + 5 + 6 + 9 = 27$ km

A1

Answer: 27 km: DE, BC, AB, CE, DF

4 marks

**12 (4 marks)**

Choose edges in descending order of weight, rejecting any that form a cycle: DE 10, AC 9, BD 8

M1 A1

BC 7 joins A, C to B, D, E without a cycle

M1

Total = $10 + 9 + 8 + 7 = 34$

A1

Answer: 34: DE, AC, BD, BC

4 marks

13 (4 marks)

Before: Kruskal selects CD 4, AB 5, DE 6; then BC 7 joins A, B to C, D, E; AC 8 and EA 9 would form cycles

M1 A1

Before total = $4 + 5 + 6 + 7 = 22$

A1

After: XB 3 is the lightest edge and is selected; CD 4, AB 5, DE 6 and BC 7 are selected as before, while XA 6 and XC 8 would form cycles, so the weight is $3 + 22 = 25$

A1

Answer: Before 22; after 25

4 marks

14 (5 marks)

Order the edges by weight: FG 3, EF 4, AC 5, DE 6, AB 7, ...

M1

Select FG, EF, AC, DE, AB in order (none forms a cycle)

M1 A1

BD 8 would join A, B, C to D, E, F, G so is accepted: the next edges BC 9, DF 10 would make cycles

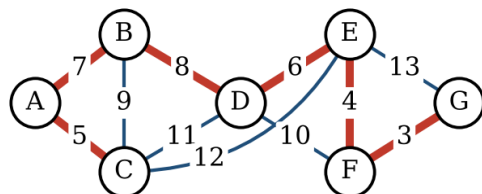
M1

Total = $3 + 4 + 5 + 6 + 7 + 8 = 33$

A1

Answer: 33: FG, EF, AC, DE, AB, BD

5 marks

**15 (6 marks)**

(a) Edges leaving A, B: AC 8, BC 6, BD 9

M1

Lightest is BC 6

A1

Answer: BC 6

2 marks

(b) Let e be the lightest edge leaving a set S and suppose a spanning tree T does not contain e . T contains a path joining the two ends of e ; the path starts inside S and ends outside S , so it uses some other edge f leaving S	M1
f is heavier than e , so $T - f + e$ is a spanning tree of smaller weight; hence T is not a minimum spanning tree, so e is in every minimum spanning tree	E1
MST by checking: EF 2, CD 3, AB 4, CE 5, BC 6 (joins A, B to C, D, E, F)	M1
BC 6 is in the tree; total 20	A1
Answer: BC 6 is in the MST; weight 20	4 marks
16 (5 marks)	
Minimum spanning tree: DE 2, DF 3, AC 4, BC 5, then CE 7 (CE joins A, B, C to D, E, F; AB 6 would make a cycle)	M1 A1 A1
The route from A to F in the tree is A, C, E, D, F	M1
Largest weight on it is 7	A1
Answer: A, C, E, D, F; largest weight 7	5 marks
17 (5 marks)	
Minimum spanning tree: AB 4, BD 5, AC 6, CE 7	M1 A1
Removing AB splits the tree into A, C, E and B, D	M1
The cheapest other edge joining the two parts is BC 8	M1
The tree remains a minimum spanning tree for $x \leq 8$	A1
Answer: $x \leq 8$ (for $x < 8$ it is the unique MST)	5 marks
18 (5 marks)	
A spanning tree uses 3 of the 5 edges	B1
Number of ways to choose 3 edges = ${}^5C_3 = 10$	M1 A1
Selections that are not trees contain a cycle: the triangles ABC and ACD	M1
Each triangle is exactly one such selection, so $10 - 2 = 8$	A1
Answer: 8	5 marks
19 (5 marks)	
Unconstrained MST: EF 4, AB 5, BC 6, CD 7, CE 9: total 31	M1 A1
Include DE 12 first, then add the lightest edges that avoid cycles: EF 4, AB 5, BC 6, CD 7	M1
Check: DE joins D and E; CD 7 joins A, B, C to D, E, F	M1
Total with DE = $12 + 4 + 5 + 6 + 7 = 34$, which is 3 more than the unconstrained minimum	A1
Answer: 34 (increase of 3)	5 marks
20 (5 marks)	
Edges by weight: DE 4, BC 5, AB 6, EF 6, BD 7, ...	M1
Accept DE 4, BC 5, AB 6, EF 6	M1 A1
BD 7 joins A, B, C to D, E, F	M1
Length = $4 + 5 + 6 + 6 + 7 = 28$ km, so the cost is $28 \times 1200 = \text{£}33,600$	A1
Answer: £33,600	5 marks
21 (8 marks)	
(a) Sort the costs: EF 7, GH 8, CD 9, FG 10, BC 11, DE 12, FH 13, ...	M1
Select EF 7, GH 8, CD 9, FG 10, BC 11, DE 12	M1 A1
FH 13 forms a cycle with F, G, H; AB 14 joins A	M1
Total = $7 + 8 + 9 + 10 + 11 + 12 + 14 = 71$	A1
Answer: 71: EF, GH, CD, FG, BC, DE, AB	5 marks
(b) Remove DE 12, so the sets A, B, C, D and E, F, G, H must be joined by another link	M1
Cheapest crossing links: BD 16 is inside; CE 18, DF 20, EG... the cheapest is CE 18	M1
New total = $71 - 12 + 18 = 77$	A1
Answer: 77	3 marks

22 (6 marks)

Suppose T and T' are two different minimum spanning trees. Both have $n - 1$ edges, so each contains an edge that is not in the other **B1**

Among all edges that are in exactly one of T , T' let e be the lightest (it is unique as the weights are distinct); say e is in T but not in T' **M1**

Adding e to T' creates a cycle C **M1**

C is not wholly inside T (T has no cycle), so C contains an edge f not in T . Every edge of C other than e is in T' , so f is in T' but not in T , and $f \neq e$ **M1**

f is in exactly one of the trees and e is the lightest such edge, so $w(f) > w(e)$ (distinct weights) **M1**

$T' + e - f$ is a spanning tree of weight $w(T') - w(f) + w(e) < w(T')$, contradicting that T' is a minimum spanning tree. So the minimum spanning tree is unique **A1**

Answer: The minimum spanning tree is unique **6 marks**

23 (6 marks)

A spanning tree is a tree on all n vertices. A tree is connected with no cycles, and we prove by induction on n that a tree on n vertices has $n - 1$ edges **B1**

$n = 1$: one vertex and no edges, and $1 - 1 = 0$ **B1**

Assume every tree on k vertices has $k - 1$ edges, and let T be a tree on $k + 1$ vertices ($k \geq 1$) **M1**

T has a leaf: the last vertex of a longest path in T has degree 1, since another neighbour would either extend the path or form a cycle **M1**

Removing this leaf and its edge leaves a connected graph with no cycle, a tree on k vertices, which has $k - 1$ edges **M1**

So T has $(k - 1) + 1 = k = (k + 1) - 1$ edges, which completes the induction **A1**

Answer: A spanning tree has $n - 1$ edges **6 marks**

24 (7 marks)

(a) Edges in order: FG 3, DE 4, AC 5, BD 6, AB 7, CE 8 **M1**

Accept FG, DE, AC, BD, AB **A1**

BD joins B to D, E; AB joins A, C to B, D, E; CE 8 would form a cycle; DF 10 joins F, G to the rest **M1**

Tree: FG 3, DE 4, AC 5, BD 6, AB 7, DF 10, total 35 **A1**

Answer: 35 **4 marks**

(b) Removing BD splits the tree into A, B, C and D, E, F, G **M1**

Cheapest edge joining them: CE 8 **M1**

New weight = $35 - 6 + 8 = 37$ **A1**

Answer: CE 8; weight 37 **3 marks**

25 (6 marks)

(a) AB 3 and BC 3 are the two lightest edges and do not form a cycle **M1**

Weight 6 **A1**

Answer: AB, BC; weight 6 **2 marks**

(b) In the tree: A, B, C = 6 **B1**

In the network: AC = 5 **B1**

Answer: 6 and 5 **2 marks**

(c) The tree path (6) is longer than the shortest path (5) **M1**

So the claim is false: an MST minimises total weight, not distances between pairs **E1**

Answer: Claim false **2 marks**

26 (7 marks)

(a) Edges in order: DE 3, AB 4, BC 5 **M1**

AC 6 forms a cycle (reject); CD 7 joins A, B, C to D, E; BD 8, CE 9 would form cycles; EF 10 joins F **M1**

Weight 29 **A1**

Answer: 29 **3 marks**

(b) Add each edge not in the minimum spanning tree (AC 6, BD 8, CE 9, DF 11) to the tree in turn: it forms a cycle with the tree path between its ends, and removing the heaviest edge of that path gives a new spanning tree **M1**

AC 6 (path A, B, C: remove BC 5) increases the weight by 1; BD 8 (path B, C, D: remove CD 7) by 1; CE 9 (path C, D, E: remove CD 7) by 2; DF 11 (path D, E, F: remove EF 10) by 1
The smallest increase is 1, so the second best weight is 30, for example by replacing BC 5 with AC 6 (replacing CD 7 with BD 8, or EF 10 with DF 11, also gives 30)

Answer: 30, e.g. replace BC 5 by AC 6

M1 A1

A1

4 marks

27 (9 marks)

Let T be a minimum spanning tree and suppose $e = uv$ is not in T , with u in S and v not in S

B1

T is connected, so it contains a path P from u to v

M1

P starts in S and ends outside S , so it contains an edge $f = xy$ with x in S and y not in S

M1 A1

f is not e , and e is the lightest edge crossing from S , so $w(e) < w(f)$ (weights distinct)

M1

Form $T' = T - f + e$

M1

T' is connected (the path P is replaced via e) and has $n - 1$ edges, so is a spanning tree

M1

$w(T') = w(T) - w(f) + w(e) < w(T)$

A1

This contradicts the minimality of T , so e is in every minimum spanning tree

E1

Answer: e is in every minimum spanning tree

9 marks

28 (8 marks)

Suppose a minimum spanning tree T contains $e = uv$

B1

Remove e from T : this splits T into two components, one containing u and the other containing v

M1 A1

The rest of cycle C is a path from u to v , so it must contain an edge f joining the two components

M1 A1

f is in C and $f \neq e$, so $w(f) < w(e)$

M1

$T - e + f$ is a spanning tree of smaller weight

M1

Contradiction, so e is not in any minimum spanning tree

E1

Answer: e is in no minimum spanning tree

8 marks

29 (8 marks)

(a) Each degree is 3 and each off-diagonal entry of A is 1, so L has 3 on the diagonal and -1 elsewhere

M1

$\det = 3(9 - 1) + 1(-3 - 1) - 1(1 + 3) = 24 - 4 - 4$

M1

$= 16$

A1

Answer: 16

3 marks

(b) Rows for A, B, C : diagonal entries 3, 2, 3; A, B, C are mutually adjacent so all other entries are -1

M1

$\det = 3(6 - 1) + 1(-3 - 1) - 1(1 + 2)$

M1

$= 15 - 4 - 3 = 8$, agreeing with the direct count of 8

A1

Answer: 8

3 marks

(c) Let D_m be the determinant of the $m \times m$ matrix with 2 on the leading diagonal and -1 on the two neighbouring diagonals. Expanding along the first row gives $D_m = 2D_{m-1} - D_{m-2}$

M1

$D_1 = 2$ and $D_2 = 3$, so $D_3 = 2 \times 3 - 2 = 4$ and $D_4 = 2 \times 4 - 3 = 5$, so there are 5 spanning trees

A1

Answer: 5

2 marks

30 (8 marks)

Each spanning tree of K_4 has 3 edges

B1

Number of pairs $(T, e) = 16 \times 3 = 48$

M1 A1

All 6 edges of K_4 are equivalent by symmetry, so each lies in the same number x of spanning trees

M1

$6x = 48$

M1

$x = 8$

A1

Spanning trees not containing a given edge: $16 - 8 = 8$

M1 A1

Answer: 8 and 8

8 marks

Kruskal's Algorithm

1 (2 marks)

Sort all the edges into ascending order of weight

B1

Select the edge of least weight, then keep adding the next shortest edge, rejecting any edge that would form a cycle

B1

Answer: Sort edges; add the shortest edge that does not form a cycle

2 marks

2 (2 marks)

Arrange by increasing weight

M1

DA 3, BC 4, AC 6, AB 7, BD 8, CD 9

A1

Answer: DA, BC, AC, AB, BD, CD

2 marks

3 (1 mark)

When adding it would create a cycle (its two ends are already connected)

B1

Answer: If it forms a cycle

1 mark

4 (3 marks)

Order: BC 3, CD 4, AB 5, AC 6, BD 7, AD 8

M1

Accept BC, CD, AB (AC would form a cycle ABC)

M1

Weight $3 + 4 + 5 = 12$

A1

Answer: BC, CD, AB; 12

3 marks

5 (1 mark)

A spanning tree has $8 - 1 = 7$ edges

B1

Answer: 7

1 mark

6 (2 marks)

BD joins the two separate components A, B and C, D, so BD 5 is accepted

B1

AC would now join two vertices already connected (cycle A, B, D, C), so AC 6 is rejected

B1

Answer: BD accepted; AC rejected

2 marks

7 (4 marks)

Order: BC 3 accept; DE 4 accept; CE 5 accept; AB 6 accept

M1 A1

Stop after 4 edges: BC, DE, CE, AB

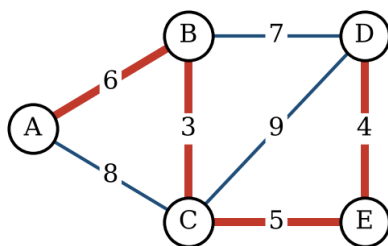
A1

Weight $3 + 4 + 5 + 6 = 18$

A1

Answer: BC, DE, CE, AB; 18

4 marks



8 (4 marks)

Order of edges: BC 7, DE 8, AC 9, CD 10, EF 11, AB 12, DF 13, BE 14, CF 16, AD 20

M1

Process: BC 7 accept; DE 8 accept; AC 9 accept; CD 10 accept; EF 11 accept

M1 A1

Total = 45 km

A1

Answer: 45 km

4 marks

9 (4 marks)

AB 4 and BC 4 are accepted (in either order)

M1

CD 5 and DA 5 have equal weight: accepting CD (then rejecting DA, which would form a cycle) gives tree AB, BC, CD

M1 A1

Accepting DA instead (rejecting CD) gives AB, BC, DA; both have weight 13

A1

Answer: AB, BC, CD or AB, BC, DA; 13

4 marks

10 (3 marks)

A and C are already connected by the path A, B, C

M1

Adding AC would create the cycle A, B, C, A

A1

An accepted edge must join two vertices that are not yet connected (in different components)

E1

Answer: AC would form a cycle

3 marks

11 (4 marks)

Order: EF 4, AB 5, DE 6, AD 7, BC 8, CE 9, DF 10, BE 11

M1

Process: EF 4 accept; AB 5 accept; DE 6 accept; AD 7 accept; BC 8 accept

M1 A1

Cost = $30 \times 6500 = \text{£}195,000$

A1

Answer: £195,000

4 marks

12 (3 marks)

Order: EF 2, AB 3, CD 4, BC 5, DE 6, GA 7, FG 8, BE 9, AD 10, CF 11, CG 12

M1

Process: EF 2 accept; AB 3 accept; CD 4 accept; BC 5 accept; DE 6 accept; GA 7 accept

M1

The sixth accepted edge, GA 7, completes the tree, so stop there; 6 edges examined

A1

Answer: GA; 6 edges examined

3 marks

13 (4 marks)

AB, CD, AC are accepted (AC joins A, B to C, D)

M1 A1

BD is rejected (B and D already connected via A, C)

A1

DE and EF are accepted, which gives 5 edges; total = $2 + 3 + 4 + 6 + 7 = 22$

A1

Answer: 22

4 marks

14 (6 marks)

Sorted edges: HF 3, HB 4, AB 5, DE 6, CD 7, FG 8, BC 9, GA 10, EF 11, HC 12, HG 13, HD 14

M1

Process: HF 3 accept; HB 4 accept; AB 5 accept; DE 6 accept; CD 7 accept; FG 8 accept; BC 9 accept

M1 A1 A1

The tree has 7 edges: HF, HB, AB, DE, CD, FG, BC

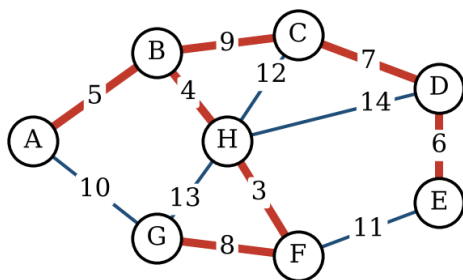
A1

Total weight 42

A1

Answer: 42

6 marks

**15 (6 marks)**

Sorted: EF 6, DE 7, BC 8, AB 9, CE 10, FG 11, BD 12, CD 13, AC 14, EG 15, DF 16

M1

Process: EF 6 accept; DE 7 accept; BC 8 accept; AB 9 accept; CE 10 accept; FG 11 accept

M1 A1 A1

Edges accepted: EF, DE, BC, AB, CE, FG

A1

Cost £5100

A1

Answer: £5100

6 marks

16 (5 marks)

The student accepted AC 5, which forms a cycle A, B, C

M1 A1

Correct selection: DE 2, AB 3, BC 4, then AC rejected

M1

CD 6 accepted (joins A, B, C to D, E); CE 7 rejected

M1

Correct weight $2 + 3 + 4 + 6 = 15$

A1

Answer: DE, AB, BC, CD; 15

5 marks

17 (5 marks)

Unconstrained: EF 3 accept; CE 4 accept; AC 5 accept; BC 6 accept; AB 7 reject (cycle); CD 8 accept, total 26

M1 A1

Start with BD 9 selected, then add the remaining edges in order of weight, rejecting cycles: EF 3, CE 4, AC 5

M1

BC 6 joins B, D to A, C, E, F; no further edges are needed

M1

Total = $9 + 3 + 4 + 5 + 6 = 27$, which is 1 more than the unconstrained minimum

A1

Answer: 27 (increase of 1)

5 marks

18 (5 marks)

Full network: EF 3 accept; AC 4 accept; BD 5 accept; AB 6 accept; BC 7 reject (cycle); CE 8 accept; DE 9 reject (cycle); DF 10 reject (cycle); FG 11 accept

M1 A1

Weight 37

A1

Without A, the edges AB and AC are removed; apply Kruskal's algorithm again to the remaining edges: EF 3 accept; BD 5 accept; BC 7 accept; CE 8 accept; DE 9 reject (cycle); DF 10 reject (cycle); FG 11 accept

M1

Weight 34

A1

Answer: 37 and 34

5 marks

19 (5 marks)

(a) Process: AB 4 accept; CD 5 accept; DE 6 accept; BC 7 accept

M1 A1

Weight 22

A1

Answer: 22

3 marks

(b) The order of the edges by weight is unchanged, so Kruskal's algorithm makes the same choices

M1

New weight = $2 \times 22 + 5 \times 4 = 64$

A1

Answer: 64

2 marks

20 (5 marks)

(a) Process: AB 2 accept; CD 3 accept; BC 4 accept; AD 5 reject (cycle); DE 6 accept; EF 7 accept

M1 A1

Weight 22

A1

Answer: 22

3 marks

(b) 6 edges are examined before the fifth edge is accepted

M1

Rejected: AD 5

A1

Answer: 6 examined; rejected AD

2 marks

21 (7 marks)

Sorted: GH 3, EF 4, BE 5, AB 6, DE 7, AD 8, BC 9, FI 10, DG 11, CF 12, EH 13, HI 14

M1

Accept: GH, EF, BE, AB, DE, BC, FI, DG

M1 A1

Rejected before the tree is complete: AD, each joining two vertices already connected

M1 A1

The tree has 8 edges, one fewer than the number of vertices

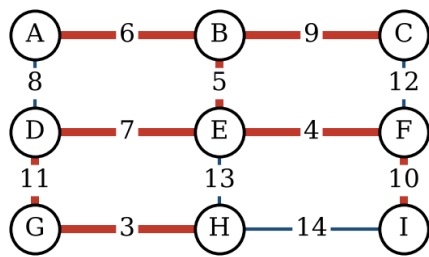
B1

Total weight 55

A1

Answer: 55

7 marks

**22 (7 marks)**

(a) Weight 3 edges: accept two of AB, BC, CA (the third forms a cycle)

M1

Weight 5 edges: accept three of DE, EF, FG, GD (the fourth forms a cycle); then CD 8 joins the two components

M1

Weight $3 + 3 + 5 + 5 + 5 + 8 = 29$

A1

Answer: 29

3 marks

(b) Triangle ABC: choose which edge to reject, 3 ways

M1 A1

Square DEFG: choose which edge to reject, 4 ways

M1

The other edges accepted are forced, so $3 \times 4 = 12$

A1

Answer: 12

4 marks

23 (6 marks)

Edges are only accepted if they do not form a cycle, so the selected subgraph T has no cycle

B1 M1

Suppose T is not connected

M1

Since the network is connected, some edge e of the network joins two different components of T

M1

When e was considered, its ends were in different components of the (smaller) subgraph then

M1

selected, so e would have been accepted

This is a contradiction, so T is connected and acyclic, hence a spanning tree

E1

Answer: Kruskal's edges form a spanning tree

6 marks

24 (6 marks)

(a) EA 8 has been placed before CD 7

M1

The list is not in ascending order of weight

E1

Answer: EA 8 listed before CD 7

2 marks

(b) Correct order: DE 4, AB 5, BC 6, CD 7, EA 8, AC 9

M1

Accept DE, AB, BC, CD; EA then forms a cycle and is rejected

M1

Weight $4 + 5 + 6 + 7 = 22$

A1

Answer: DE, AB, BC, CD; 22

3 marks

(c) A heavier edge could otherwise be accepted before a lighter edge that would have been a better choice, so the tree would not be minimal

E1

Answer: Otherwise a heavier edge may be chosen first

1 mark

25 (7 marks)

The existing cables AB, CD, EF form three components A, B, C, D, E, F

B1

A spanning tree needs 5 edges, so 2 more are needed

M1

Remaining useful edges in order: DE 5, BC 8, AC 9, CE 10, BD 11, DF 12

M1

DE 5 joins C, D to E, F: accept

M1 A1

The shortest edge joining A, B to the other villages is BC 8: accept

M1

Extra length $5 + 8 = 13$ km

A1

Answer: 13 km: DE and BC

7 marks

26 (6 marks)

(a) Sorted order AB 5, CD 6, EF 7, GH 8, BC 9, DE 10, FG 11, ...

M1

Accepted: AB, CD, EF, GH, BC, DE, FG

A1

Answer: AB, CD, EF, GH, BC, DE, FG

2 marks

(b) After AB, CD, EF, GH: components A, B, C, D, E, F, G, H

B1

Each accepted edge joins two components, so each reduces the number by 1

M1

After k edges there are $8 - k$ components

A1

Answer: A, B, C, D, E, F, G, H; 8 – k**3 marks**

(c) Total $5 + 6 + 7 + 8 + 9 + 10 + 11 = 56$; the algorithm stops after the seventh accepted edge (FG 11), so AH 12 and BE 13 are not examined

B1**Answer: 56; AH and BE are not examined****1 mark****27 (10 marks)**

Kruskal's algorithm gives a spanning tree K

B1Let T be a minimum spanning tree, and suppose $K \neq T$ **M1**

Let e be the first edge selected by Kruskal's algorithm that is not in T

M1

Adding e to T creates a cycle C

M1

C contains an edge f not in K (otherwise K would contain a cycle)

M1If $w(f) < w(e)$, then f was considered before e and not selected, so it formed a cycle with earlier selected edges**M1**Those earlier edges all lie in T, as does f, which gives a cycle in T: impossible; so $w(f) > w(e)$ (weights distinct)**A1** $T' = T + e - f$ is a spanning tree**M1** $w(T') < w(T)$, contradicting the minimality of T**A1**So $K = T$, and Kruskal's algorithm gives the minimum spanning tree**E1****Answer: Kruskal's tree is a minimum spanning tree****10 marks****28 (8 marks)**

Let K be Kruskal's tree and let its heaviest edge be e of weight b

B1

When e was accepted, it joined two components of the subgraph formed by the edges selected so far, so the edges considered before e do not connect the network

M1 A1

These edges have weight at most b

M1

Let T be any spanning tree with all edges of weight less than b

M1

Then T would connect the network using only edges that were considered before e, contradicting the previous step

M1 A1

So every spanning tree has an edge of weight at least b, and K has the smallest bottleneck

E1**Answer: Kruskal's tree has the smallest possible bottleneck****8 marks****29 (10 marks)**

(a) Process: EF 4 accept; AB 5 accept; CD 6 accept; FA 7 accept; BC 8 accept

M1 A1Weight $4 + 5 + 6 + 7 + 8 = 30$ **A1****Answer: 30****3 marks**

(b) The tree path from A to D is A, B, C, D with weights 5, 8, 6

M1

AD forms a cycle with this path, and is accepted when it is lighter than the heaviest edge BC 8 (or equal in a tie)

M1 A1 $x \leq 8$ **A1****Answer: $x \leq 8$** **4 marks**

(c) AD replaces BC 8 in the tree

M1Weight $= 30 - 8 + x$ **M1** $= 22 + x$ **A1****Answer: $22 + x$** **3 marks****30 (9 marks)**

(a) The three weight 3 edges are considered first and any two connect A, B, C; the third would form a cycle

M1

So exactly two are accepted

A1

3 choices (choose the edge to omit)

A1**Answer: 3 choices****3 marks**

(b) BD 5 and CD 5 both join D to A, B, C

M1

Once one is accepted the other forms a cycle and is rejected

M1

2 choices

A1**Answer: 2 choices****3 marks**

(c) DE 6 is the only edge reaching E before AE 9, which then forms a cycle

M1Number of trees $3 \times 2 = 6$ **A1**

Weight $3 + 3 + 5 + 6 = 17$

Answer: 6 trees of weight 17

A1

3 marks

Prim's Algorithm

1 (2 marks)

Choose any vertex to start the tree

B1

Add the shortest edge joining a vertex in the tree to a vertex not yet in the tree

B1

Answer: Start anywhere; add the shortest edge from the tree to a new vertex

2 marks

2 (3 marks)

From A the shortest edge is AC 3

M1

Then the shortest edge from A, C to a new vertex is BC 4, then CD 6

M1

Weight $3 + 4 + 6 = 13$

A1

Answer: AC, BC, CD; 13

3 marks

3 (1 mark)

Edges that join a vertex already in the tree to a vertex not in the tree

B1

Answer: Edges from the tree to a vertex outside it

1 mark

4 (2 marks)

Circle the smallest entry in the column of the chosen vertex; the row containing it gives the next vertex, and that row is then deleted

B1

Continue by circling the smallest entry in the undeleted rows of the columns of all the vertices chosen so far, deleting the row of the new vertex each time, until all the rows have been deleted

B1

Answer: Circle the smallest entry in its column, delete the new vertex's row, and repeat using the columns of all chosen

5 (1 mark)

$10 - 1 = 9$

B1

Answer: 9

1 mark

6 (2 marks)

True

B1

The minimum spanning tree is unique when all edge weights are distinct

E1

Answer: True

2 marks

7 (4 marks)

Start at A: shortest edge from A is AC 4

M1

Tree A, C: candidates AB 9, BC 7, CD 5, CE 8; add CD 5

M1

Tree A, C, D: add DE 3, then BD 6 (BC 7, BD 6, AB 9 for B)

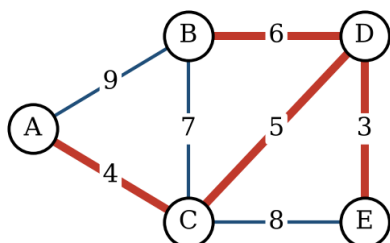
A1

Weight $4 + 5 + 3 + 6 = 18$

A1

Answer: AC, CD, DE, BD; 18

4 marks



8 (4 marks)

Starting at C: CB 5, BA 7, AF 4, FE 8, ED 6

M1 A1

Starting at E: ED 6, EF 8, FA 4, AB 7, BC 5

M1

Both trees consist of the same edges, with weight 30: the minimum spanning tree is unique because the weights are distinct

A1

Answer: Same tree of weight 30

4 marks

9 (4 marks)

Start at A: smallest entry in column A is AC 5

M1

Then CD 6 (candidates AB 8, AD 12, BC 7, CD 6, CE 10)

M1

Then DE 4, then BC 7

A1

Vertices in order A, C, D, E, B; cost £22 thousand

A1

Answer: A, C, D, E, B; £22 thousand

4 marks

10 (4 marks)

Edges added in order: AB 6, BC 4, BD 7, DE 5, EF 3

M1 A1

Length 25 km

A1

Cost $25 \times 2400 = £60,000$

A1

Answer: £60,000

4 marks

11 (3 marks)

(a) Edges at A: AB 7, AC 4, AD 9: add AC 4

B1

Answer: AC

1 mark

(b) Tree A, C: candidate edges CD 5, BC 6, AB 7, CE 8, AD 9

M1

The shortest is CD 5

A1

Answer: CD 5, BC 6, AB 7, CE 8, AD 9; add CD

2 marks

12 (3 marks)

AB 4 and AC 4 are equal: choose either, then the other one can be added

M1

Next BD 5 and CD 5 are equal: choose either

M1

Total weight $4 + 4 + 5 = 13$ in every case

A1

Answer: 13

3 marks

13 (4 marks)

Start at E: the smallest entry in column E is DE 6

M1

Order of arcs: ED 6, DB 7, EF 8, BC 11, CA 9

M1 A1

Order of vertices E, D, B, F, C, A; total weight 41

A1

Answer: 41

4 marks

14 (6 marks)

Edges at A: AB 6, AG 7; add AB 6

M1

Order of edges: AB 6, BH 3, HF 4, AG 7, BC 10, CD 8, DE 5

M1 A1 A1

Order of vertices: A, B, H, F, G, C, D, E

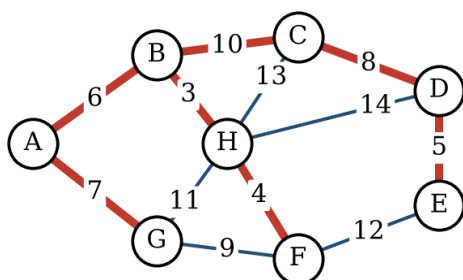
A1

Total weight 43

A1

Answer: 43

6 marks



15 (6 marks)

Start at D: the smallest entry in column D is DE 12

M1

Order of arcs: DE 12, EF 6, EC 7, CA 8, CB 9, FG 10

M1 A1 A1

Order of vertices: D, E, F, C, A, B, G

A1

Cost £5200

A1

Answer: £5200

6 marks

16 (5 marks)**(a)** Order: AC 7, CE 6, EF 8, CB 9, CD 10

M1 A1

Weight 40

A1

Answer: 40

3 marks

(b) Order: FE 8, EC 6, CA 7, CB 9, CD 10

M1

The same set of edges and the same weight 40, since the minimum spanning tree is unique

A1

Answer: Same tree

2 marks

17 (5 marks)**(a)** AB 5, then BC 3, then CD 6

M1

AD 8 forms a cycle so is not a candidate (D already in the tree)

A1

Answer: AB, BC, CD

2 marks

(b) No edge joins A, B, C, D to E or F

M1

The network is not connected, so it has no spanning tree

E1

Answer: Network is not connected

2 marks

(c) One extra edge, joining E, F to A, B, C, D

B1

Answer: 1

1 mark

18 (5 marks)

Order of edges: XB 3, BA 5, BC 7, CD 4, DE 6

M1 A1 A1

New weight 25

A1

The original tree had weight 22, so the change is 3

B1

Answer: 25; change 3

5 marks

19 (5 marks)

Stage 1: tree A: candidates AC 5, AB 7

M1

Edges added in order: AC 5, CD 6, DE 4, AB 7

M1 A1

Numbers of candidate edges at the four stages: 2, 4, 5, 3

M1 A1

Answer: Candidates: 2, 4, 5, 3

5 marks

20 (5 marks)**(a)** From A: AB 7 and AC 4, so AC is added; from A, C: AB 7, BC 5, CD 9, CE 10, so BC is added

M1

From A, B, C the candidates are BD x , CD 9 and CE 10. BD is added, so $x < 9$ (strictly, as there are no ties)

M1

 x is a positive integer, so $x = 1, 2, 3, 4, 5, 6, 7$ or 8

A1

Answer: $x = 1, 2, \dots, 8$

3 marks

(b) The edges are AC 4, BC 5, BD x , DE 6, EF 3

M1

Total weight = $18 + x$

A1

Answer: $18 + x$

2 marks

21 (7 marks)

Edges at E: BE 5, DE 7, EF 4, EH 13; add EF 4

M1

Order of edges: EF 5, EB 6, BA 7, ED 8, BC 10, FI 11, IH 12, HG 4

M1 A1 A1

Order of vertices: E, F, B, A, D, C, I, H, G

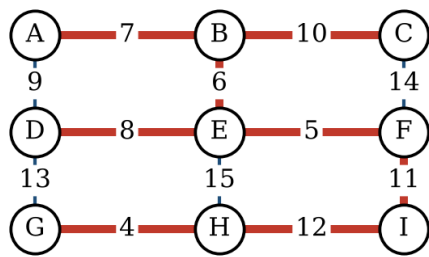
A1

Tree has 8 edges, total weight 63

M1 A1

Answer: 63

7 marks

**22 (6 marks)**

(a) AB 4 and AC 4 are tied: add both (either order)

M1

BD 6 and CD 6 are tied: add one of them, the other would form a cycle

M1

Then DE 8 and EF 3 (EF 3 joins F to E after E is added)

M1

Tree of weight $4 + 4 + 6 + 8 + 3 = 25$

A1

Answer: 25

4 marks

(b) The ties are AB 4 and AC 4 (both are added, so the order does not matter) and BD 6 and CD 6 (once one of them is added the other would form a cycle with A, B, C, D, so only one can be used)

M1

Every other edge is forced: DE 8 is added before CE 9 (and CE 9 would then form a cycle), then EF 3. So the only choice is BD or CD, giving exactly 2 minimum spanning trees

A1

Answer: 2

2 marks

23 (6 marks)

The algorithm starts with one vertex and, at each step, adds an edge joining a vertex in the tree to a new vertex

B1

The tree built so far is connected, since each new vertex is joined to it

M1

It has no cycle, since each new edge has one end outside the current tree

M1

If the algorithm stopped with a vertex v outside the tree, then no edge would join the tree to the other vertices

M1

But the network is connected, so there is a path from the start vertex to v , and some edge of this path joins the tree to a vertex outside it

M1

So the algorithm only stops when all the vertices are in the tree, giving a spanning tree

E1

Answer: Prim's edges form a spanning tree

6 marks

24 (6 marks)

(a) DE 2 was added although neither D nor E was in the tree

M1

At each stage Prim's algorithm must add an edge joining the tree to a vertex outside it

E1

Answer: DE 2 is not connected to the tree

2 marks

(b) AB 4, then AC 5 (the shortest edge from A, B)

M1

CD 6, then DE 2

M1

EF 7

A1

Answer: AB, AC, CD, DE, EF

3 marks

(c) Weight $4 + 5 + 6 + 2 + 7 = 24$; the student's edges are the same set

B1

Answer: 24

1 mark

25 (7 marks)

Original: AB 5, BC 6, CD 7, DE 9, EF 4

M1 A1

Original weight 31

A1

BC removed: AB 5, AC 8, CD 7, DE 9, EF 4

M1 A1

New weight 33

A1

Increase 2

B1

Answer: 33; increase 2

7 marks

26 (6 marks)

(a) AB 4 first (shorter than AC 6)

M1

Then BC 5 (shorter than AC 6, which would form a cycle) so the order is A, B, C

A1

Answer: A, B, C; edges AB 4, BC 5

2 marks

(b) Edges across: CE 7, CD 8, BD 9

M1

Next is CE 7

A1

Answer: CE 7, CD 8, BD 9; add CE

2 marks

(c) Order of edges: AB 4, BC 5, CE 7, ED 3, EF 10

M1

Total weight 29

A1

Answer: 29

2 marks

27 (10 marks)

Let P be the tree produced by Prim's algorithm with edges $e_1, e_2, \dots, e_{n-1}$ in the order added

B1

We show that each e_k belongs to every minimum spanning tree T

M1

At the stage when e_k is added, the current tree S has vertex set V_k , and e_k is the lightest edge joining V_k to the other vertices

M1

Suppose $e_k = uv$ is not in T , with u in V_k and v not in V_k

M1

T contains a path from u to v , which must contain an edge f joining V_k to a vertex outside V_k

M1 A1

$f \neq e_k$ and e_k is the lightest such edge, so $w(e_k) < w(f)$

M1

$T' = T - f + e_k$ is a spanning tree with smaller weight than T

M1

This contradicts the minimality of T , so e_k is in T

A1

Hence P is contained in T , and since both have $n - 1$ edges $P = T$, a minimum spanning tree

E1

Answer: Prim's tree is a minimum spanning tree

10 marks

28 (9 marks)

(a) There are k vertices in the tree and $n - k$ outside

M1

Each pair of one of each is joined by an edge, giving $k(n - k)$

A1

Answer: $k(n - k)$ edges

2 marks

(b) Total = sum from $k = 1$ to $n - 1$ of $k(n - k)$

M1

$$= n \times \frac{(n-1)n}{2} - \frac{(n-1)n(2n-1)}{6}$$

M1 A1

$$= \frac{n(n-1)}{6} \times (3n - (2n-1))$$

M1

$$= \frac{n(n-1)(n+1)}{6} = \frac{n(n^2-1)}{6}$$

A1

$$\text{Answer: Total} = \frac{n(n^2-1)}{6}$$

5 marks

$$\text{(c)} \frac{10 \times 99}{6} = 165$$

M1

K_{10} has 45 edges, so the same edges are examined several times (165 examinations)

A1

Answer: 165 compared with 45

2 marks

29 (9 marks)

$$\text{(a)} AB = 4, AD = \sqrt{2^2 + 3^2} = \sqrt{13}, BD = \sqrt{2^2 + 3^2} = \sqrt{13}, BE = \sqrt{13}, DE = 4, DF = \sqrt{2^2 + 3^2} = \sqrt{13}$$

M1 A1

Answer: 4, $\sqrt{13}$, $\sqrt{13}$, $\sqrt{13}$, 4, $\sqrt{13}$

2 marks

(b) The distances $\sqrt{13} \approx 3.606$ occur for AD, BD, BE, CE, DF and EF; AB, BC and DE are 4; AC = 8 and the rest are longer

M1

From A the nearest sensor is D ($AD = \sqrt{13}$, $AB = 4$)

M1

Add BD (tied with DF), then BE (tied with DF), then CE and DF (tied with EF): the links are AD, BD, BE, CE, DF, five links each of length $\sqrt{13}$

M1 A1

Total length $5\sqrt{13} = 18.0$ m (3 s.f.)

A1

Answer: $5\sqrt{13} = 18.0$ m

5 marks

(c) At several stages two or more candidate edges have the same length $\sqrt{13}$, for example BD and DF once A and D are in the tree

M1

Choosing a different one of the tied edges gives a different tree of the same length: the links AD and CE are always used together with three of the four edges BD, BE, EF, DF, so there are 4 minimum spanning trees

A1

Answer: Equal distances give ties; there are 4 minimum spanning trees

2 marks

30 (8 marks)

Let P be the spanning tree found by Prim's algorithm (from any start) and K the one found by Kruskal's algorithm. Both are spanning trees of the network, so each has $n - 1$ edges	B1
Let $e = uv$ be any edge added by Prim's algorithm, when the tree contains the set V of vertices, with u in V and v not in V . Then e is the lightest edge joining V to the other vertices	M1
Suppose Kruskal's algorithm rejects e . Then when e is considered, u and v are already joined by a path of selected edges, all of which were considered earlier and so are lighter than e (the weights are distinct)	M1
This path starts at u in V and ends at v outside V , so it contains an edge joining V to a vertex outside V , which is lighter than e	M1
That contradicts the fact that e is the lightest such edge	A1
So Kruskal's algorithm accepts every edge of P , that is, P is contained in K	M1
P and K both have $n - 1$ edges, so $P = K$	A1
Hence the two algorithms give the same tree	E1
Answer: Prim's and Kruskal's trees are the same	8 marks

Shortest Path Algorithms

1 (2 marks)

The shortest (least weight) path	B1
from a given start vertex to every other vertex (or to a chosen end vertex)	B1
Answer: Shortest paths from a start vertex	2 marks

2 (2 marks)

The final label is the length of the shortest path from the start vertex	B1
to that vertex, fixed when the vertex is permanently labelled	B1
Answer: Shortest distance from the start	2 marks

3 (2 marks)

Routes: ABD $5 + 6 = 11$, ACD $3 + 4 = 7$, $ABCD$ $5 + 1 + 4 = 10$	M1
Shortest A , C , D of length 7	A1
Answer: A, C, D; 7	2 marks

4 (2 marks)

Floyd's algorithm finds the shortest distance between every pair of vertices	B1
whereas Dijkstra's algorithm finds distances from one start vertex only	B1
Answer: All pairs of vertices	2 marks

5 (1 mark)

One iteration for each vertex, so 6	B1
Answer: 6	1 mark

6 (2 marks)

Arcs can only be travelled in one direction	B1
so the routes available from A to B can differ from the routes from B to A	E1
Answer: Arcs are one-way	2 marks

7 (5 marks)

Labels (working values, order): A : 0 (1st); C : 3 (2nd); B : 4 (3rd); D : 9 (4th); F : 11 (5th); E : 12 (6th)	M1 A1 A1
Shortest path: A , C , D , F	A1
Length 11	A1
Answer: A, C, D, F; 11	5 marks

8 (5 marks)

Labels (working values, order): A: 0 (1st); B: 7 (2nd); C: 9 (3rd); F: 14, 11 (4th); D: 22, 20 (5th); E: 20 (6th)

M1 A1 A1

Shortest path A, C, F, E

A1

Length 20

A1

Answer: A, C, F, E; 20

5 marks

9 (4 marks)

Shortest A to C is the direct road AC, 4 (A, B, C would be $6 + 3 = 9$)

B1

Shortest C to F: C, D, F = $9 + 2 = 11$; C, E, F = $7 + 6 = 13$; C, B, D, F = $3 + 5 + 2 = 10$

M1 A1

Shortest route through C is A, C, B, D, F of length $4 + 10 = 14$ km; the shortest route from A to F overall is A, B, D, F of length $6 + 5 + 2 = 13$ km, so it is 1 km longer

A1

Answer: A, C, B, D, F (14 km); 1 km longer

4 marks

10 (5 marks)

Labels (working values, order): A: 0 (1st); C: 2 (2nd); B: 4, 3 (3rd); D: 10, 8 (4th); E: 12, 10 (5th); F: 14, 13 (6th)

M1 A1 A1

Shortest path A, C, B, D, E, F

A1

Length 13

A1

Answer: A, C, B, D, E, F; 13

5 marks

11 (4 marks)

For each entry $D(i, j)$ compare it with $D(i, A) + D(A, j)$, using column A (0, 5, 8, ∞) and row A (0, 5, 8, ∞)

M1

B to C: $5 + 8 = 13$ is less than ∞ , so $D(B, C) = D(C, B) = 13$; B to D: $5 + \infty$ and C to D: $8 + \infty$ give no improvement, and all other entries are unchanged

M1

New matrix $\begin{pmatrix} 0 & 5 & 8 & \infty \\ 5 & 0 & 13 & 7 \\ 8 & 13 & 0 & 3 \\ \infty & 7 & 3 & 0 \end{pmatrix}$

A1 A1

Answer: $\begin{pmatrix} 0 & 5 & 8 & \infty \\ 5 & 0 & 13 & 7 \\ 8 & 13 & 0 & 3 \\ \infty & 7 & 3 & 0 \end{pmatrix}$

4 marks

12 (4 marks)

(a) Time is proportional to n^3 , so time = $k \times n^3$ with $k = \frac{0.8}{50^3}$

M1

200 is 4 times 50, so the time is multiplied by $4^3 = 64$

M1

$0.8 \times 64 = 51.2$ seconds

A1

Answer: 51.2 seconds

3 marks

(b) Order n^3 means that the time is only approximately proportional to n^3 (lower order terms are ignored, and the speed of the computer may vary)

E1

Answer: Order n^3 is only approximately proportional to n^3

1 mark

13 (5 marks)

Labels: A: 0 (1st); C: 5 (2nd); B: 8 (3rd); D: 14, 11 (4th); E: 16, 15 (5th); F: 21 (6th)

M1 A1 A1

Shortest route A, C, D, F

A1

Length 21 km

A1

Answer: A, C, D, F; 21 km

5 marks

14 (7 marks)

Labels (working values, order): A: 0 (1st); B: 6 (2nd); C: 9 (3rd); D: 11, 9 (4th); E: 16, 14 (5th); G: 22, 18 (6th); F: 21 (7th); H: 27 (8th)

M1 A1 A1 A1

Final labels give H = 27

A1

Trace back from H through the vertices whose labels differ by the arc weights: A, B, D, E, G, H

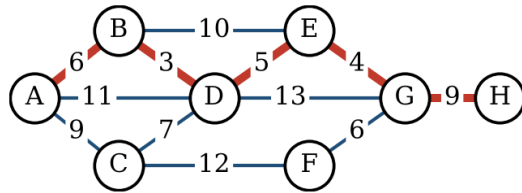
M1

Length 27

A1

Answer: A, B, D, E, G, H; 27

7 marks

**15 (9 marks)**

(a) Initial tables: distance $\begin{pmatrix} 0 & 5 & 2 & \infty \\ 5 & 0 & 10 & 6 \\ 2 & 10 & 0 & 3 \\ \infty & 6 & 3 & 0 \end{pmatrix}$, route $\begin{pmatrix} A & B & C & D \\ A & B & C & D \\ A & B & C & D \\ A & B & C & D \end{pmatrix}$

B1

After iteration 1 (A): distance $\begin{pmatrix} 0 & 5 & 2 & \infty \\ 5 & 0 & 7 & 6 \\ 2 & 7 & 0 & 3 \\ \infty & 6 & 3 & 0 \end{pmatrix}$, route $\begin{pmatrix} A & B & C & D \\ A & B & A & D \\ A & A & C & D \\ A & B & C & D \end{pmatrix}$

M1

After iteration 2 (B): distance $\begin{pmatrix} 0 & 5 & 2 & 11 \\ 5 & 0 & 7 & 6 \\ 2 & 7 & 0 & 3 \\ 11 & 6 & 3 & 0 \end{pmatrix}$, route $\begin{pmatrix} A & B & C & B \\ A & B & A & D \\ A & A & C & D \\ B & B & C & D \end{pmatrix}$

M1 A1

After iteration 3 (C): distance $\begin{pmatrix} 0 & 5 & 2 & 5 \\ 5 & 0 & 7 & 6 \\ 2 & 7 & 0 & 3 \\ 5 & 6 & 3 & 0 \end{pmatrix}$, route $\begin{pmatrix} A & B & C & C \\ A & B & A & D \\ A & A & C & D \\ C & B & C & D \end{pmatrix}$

M1 A1

After iteration 4 (D): distance $\begin{pmatrix} 0 & 5 & 2 & 5 \\ 5 & 0 & 7 & 6 \\ 2 & 7 & 0 & 3 \\ 5 & 6 & 3 & 0 \end{pmatrix}$, route $\begin{pmatrix} A & B & C & C \\ A & B & A & D \\ A & A & C & D \\ C & B & C & D \end{pmatrix}$; no entry changes, so

A1

these are the final tables

Answer: $\begin{pmatrix} 0 & 5 & 2 & 5 \\ 5 & 0 & 7 & 6 \\ 2 & 7 & 0 & 3 \\ 5 & 6 & 3 & 0 \end{pmatrix}$

7 marks

(b) Row D, column A of the route table is C, so the route starts D to C; row C, column A is A, so the route is D, C, A

M1

Length: row D, column A of the distance table, 5 (3 + 2)

A1

Answer: D, C, A; 5

2 marks

16 (6 marks)

Labels: A: 0 (1st); B: 5 (2nd); C: 8, 7 (3rd); D: 11 (4th); E: 11 (5th); F: 18 (6th)

M1 A1 A1

Shortest distances: B 5, C 7, D 11, E 11, F 18

A1

Furthest vertex F at distance 18

A1

Route A, B, D, F

A1

Answer: Furthest is F (18)

6 marks

17 (6 marks)

Labels: A: 0 (1st); B: 5 (2nd); C: 8, 7 (3rd); D: 14, 11 (4th); E: 17, 14 (5th); F: 18 (6th)

M1 A1

Quickest route A, B, C, D, F, time 18 minutes

A1

With CD removed: A: 0 (1st); B: 5 (2nd); C: 8, 7 (3rd); D: 14 (4th); E: 17 (5th); F: 21 (6th)

M1 A1

New route A, B, D, F, time 21 minutes, an increase of 3 minutes

A1

Answer: A, B, C, D, F (18 min); A, B, D, F (21 min)

6 marks

18 (5 marks)

Labels: A: 0 (1st); B: 4 (2nd); C: 6 (3rd); D: 9 (4th); E: 15 (5th)

M1 A1

Shortest length 15

A1

All shortest paths: A, B, C, D, E; A, B, C, E; A, B, D, E; A, C, D, E; A, C, E

M1 A1

Answer: 15; A, B, C, D, E; A, B, C, E; A, B, D, E; A, C, D, E; A, C, E

5 marks

19 (6 marks)

Labels: A: 0 (1st); C: 3 (2nd); B: 5, 4 (3rd); D: 12, 10 (4th); E: 10 (5th); F: 14, 12 (6th)

M1 A1 A1

Final label of F is 12

A1

Route A, C, E, F

M1

Time 12 minutes

A1

Answer: A, C, E, F; 12 minutes

6 marks

20 (6 marks)

Shortest distances between all pairs (by Dijkstra or Floyd): A: B 5, C 10, D 13, E 9; B: A 5, C 7,

M1 A1

D 8, E 12; C: A 10, B 7, D 6, E 10; D: A 13, B 8, C 6, E 4; E: A 9, B 12, C 10, D 4

Greatest distance from each town: A 13, B 12, C 10, D 13, E 12

M1 A1

Smallest is C with 10 km

A1 B1

Answer: Town C, 10 km

6 marks

21 (8 marks)Initial:
$$\begin{pmatrix} 0 & 6 & 9 & \infty & \infty \\ 6 & 0 & 4 & 8 & \infty \\ 9 & 4 & 0 & 5 & 11 \\ \infty & 8 & 5 & 0 & 3 \\ \infty & \infty & 11 & 3 & 0 \end{pmatrix}$$

B1

After A:
$$\begin{pmatrix} 0 & 6 & 9 & \infty & \infty \\ 6 & 0 & 4 & 8 & \infty \\ 9 & 4 & 0 & 5 & 11 \\ \infty & 8 & 5 & 0 & 3 \\ \infty & \infty & 11 & 3 & 0 \end{pmatrix}$$

M1

After B:
$$\begin{pmatrix} 0 & 6 & 9 & 14 & \infty \\ 6 & 0 & 4 & 8 & \infty \\ 9 & 4 & 0 & 5 & 11 \\ 14 & 8 & 5 & 0 & 3 \\ \infty & \infty & 11 & 3 & 0 \end{pmatrix}$$

M1 A1

After C:
$$\begin{pmatrix} 0 & 6 & 9 & 14 & 20 \\ 6 & 0 & 4 & 8 & 15 \\ 9 & 4 & 0 & 5 & 11 \\ 14 & 8 & 5 & 0 & 3 \\ 20 & 15 & 11 & 3 & 0 \end{pmatrix}$$

M1 A1

Final matrix after D and E:
$$\begin{pmatrix} 0 & 6 & 9 & 14 & 17 \\ 6 & 0 & 4 & 8 & 11 \\ 9 & 4 & 0 & 5 & 8 \\ 14 & 8 & 5 & 0 & 3 \\ 17 & 11 & 8 & 3 & 0 \end{pmatrix}$$

A1

Shortest distance from A to E is 17

A1

Answer: 17

8 marks

22 (8 marks)

(a) The routes from A to D are A, B, D = $6 + 8 = 14$; A, C, D = $9 + 4 = 13$; A, B, C, D = $6 + x + 4 = 10 + x$ (routes through E are much longer)

M1

The shortest distance is 13, so $10 + x$ must not be less than 13

M1

$x \geq 3$ (for $x = 3$ the routes A, B, C, D and A, C, D tie at 13)

A1

Answer: $x \geq 3$

3 marks

(b) Labels: A: 0 (1st); B: 6 (2nd); C: 9, 8 (3rd); D: 14, 12 (4th); E: 18, 17 (5th)

M1 A1 A1

Shortest path A, B, C, D, E

A1

Length 17

A1

Answer: A, B, C, D, E; 17

5 marks

23 (7 marks)

(a)
$$\begin{pmatrix} 0 & 2 & \infty & 5 \\ \infty & 0 & -3 & \infty \\ 0 & \infty & 0 & 4 \\ \infty & \infty & \infty & 0 \end{pmatrix}$$

B1

Answer:
$$\begin{pmatrix} 0 & 2 & \infty & 5 \\ \infty & 0 & -3 & \infty \\ 0 & \infty & 0 & 4 \\ \infty & \infty & \infty & 0 \end{pmatrix}$$

1 mark

(b) After the iteration using A:
$$\begin{pmatrix} 0 & 2 & \infty & 5 \\ \infty & 0 & -3 & \infty \\ 0 & 2 & 0 & 4 \\ \infty & \infty & \infty & 0 \end{pmatrix}$$

M1

After B:
$$\begin{pmatrix} 0 & 2 & -1 & 5 \\ \infty & 0 & -3 & \infty \\ 0 & 2 & -1 & 4 \\ \infty & \infty & \infty & 0 \end{pmatrix}$$

M1

After C:
$$\begin{pmatrix} -1 & 1 & -2 & 2 \\ -3 & -1 & -4 & 0 \\ -1 & 1 & -2 & 2 \\ \infty & \infty & \infty & 0 \end{pmatrix}$$

M1 A1

Final matrix:
$$\begin{pmatrix} -1 & 1 & -2 & 2 \\ -3 & -1 & -4 & 0 \\ -1 & 1 & -2 & 2 \\ \infty & \infty & \infty & 0 \end{pmatrix}$$

A1

Negative entries on the diagonal show that a vertex can return to itself with total weight below 0: the cycle A, B, C, A has weight $2 - 3 + 0 = -1$

E1

Answer: Negative cycle A, B, C, A (weight -1)

6 marks

24 (6 marks)

(a) The student has taken the shortest edge at each step (a greedy choice)

M1

A short first edge need not lie on the shortest path: the whole path length matters, and Dijkstra's algorithm keeps all working values

E1

Answer: A greedy choice is not Dijkstra's algorithm

2 marks

(b) Labels: A: 0 (1st); B: 2 (2nd); C: 3 (3rd); D: 4 (4th); E: 11, 5 (5th)

M1 A1

Shortest path A, C, D, E

A1

Length 5

A1

Answer: A, C, D, E; 5

4 marks

25 (7 marks)

(a) Cost = $0.4 \times \text{length} + \text{toll}$

M1

AB 13, AC 14, BC 6, BD 12, CD 10, CE 10, DE 7.4

A1

Answer: AB 13, AC 14, BC 6, BD 12, CD 10, CE 10, DE 7.4

2 marks

(b) Labels: A: 0 (1st); B: 13 (2nd); C: 14 (3rd); D: 25, 24 (4th); E: 24 (5th)

M1 A1

Cheapest route A, C, E costing £24	A1
Answer: A, C, E; £24	3 marks
(c) Shortest by distance: A, B, C, E, 55 km	M1
Its cost is £29, so it is not the cheapest route	A1
Answer: A, B, C, E; not the cheapest	2 marks
26 (6 marks)	
(a) A is labelled 0; B gets 2 and C gets 3; the smallest working value is B 2, which becomes permanent	M1
From B, D gets 3; then C 3 becomes permanent, and the arc C to B ($3 - 2 = 1$) is ignored since B is permanent	M1
Student's final labels: B 2, C 3, D 3	A1
Answer: B 2, C 3, D 3	3 marks
(b) A to C to B has length $3 - 2 = 1$, shorter than 2	M1
True distances: B 1, D $1 + 1 = 2$	A1
Answer: B 1, D 2	2 marks
(c) Dijkstra's algorithm assumes a longer route can never become shorter by adding more arcs, which is false when there are negative weights	E1
Answer: It requires non-negative weights	1 mark
27 (10 marks)	
Use induction on the number of vertices already permanent. The start vertex A is labelled 0, which is correct	B1
Assume that all the vertices already permanent have correct labels, and let v be the next vertex made permanent, with label $d(v)$ (the smallest working value)	M1
$d(v)$ is the length of an actual path P: the shortest path to a permanent vertex u followed by an arc uv	M1
Suppose there is a shorter path Q from A to v	M1
Q starts at A (permanent) and ends at v (not permanent), so has a first arc xy with x permanent and y not	M1 A1
The label of y is at most $d(x) + w(xy)$, which is at most the length of the part of Q up to y	M1
Since weights are non-negative, the part of Q up to y has length at most the length of Q, which is less than $d(v)$	M1
Then y has working value less than $d(v)$, contradicting that v had the smallest working value	A1
So no shorter path exists and $d(v)$ is correct	E1
Answer: $d(v)$ is the length of the shortest path from A to v	10 marks
28 (10 marks)	
(a) $k = 0$: D contains only direct arcs, so the property holds	B1
Assume it holds after iteration $k - 1$	M1
A shortest path from i to j using intermediate vertices from 1, ..., k either does not use vertex k (then its length is the old $D(i, j)$ by the hypothesis) or uses vertex k exactly once, splitting into a path from i to k and from k to j using only 1, ..., $k - 1$	M1 A1
So the shortest length is the smaller of the two options, which is the update rule	E1
Answer: After iteration k, $D(i, j)$ is the shortest distance using only vertices 1 to k as intermediates	6 marks
(b) After iteration n all vertices are allowed as intermediate vertices	M1
so $D(i, j)$ is the length of the shortest path overall	E1
Answer: All vertices allowed	2 marks
(c) Each of the n iterations considers every one of the n^2 entries of the matrix once, making one comparison for each	M1
so the total number of comparisons is $n \times n^2 = n^3$	A1
Answer: n^3 comparisons	2 marks
29 (9 marks)	
(a) Labels: A: 0 (1st); B: 1 (2nd); C: 2 (3rd); D: 3 (4th); E: 3 (5th); F: 8, 4 (6th)	M1 A1
Shortest A, B, C, D, F of length 4 using 4 roads	A1
Answer: 4 km, 4 roads	3 marks

(b) $k = 1$: only AF 8	B1
$k = 2$: AE then EF gives 6, better than 8	M1
$k = 3$: nothing better than 6 (the path A, B, C, D, F needs 4 roads)	A1
$k = 4$: 4	A1
Answer: 8, 6, 6, 4	4 marks
(c) The best path with at most k roads need not be made of shortest paths to intermediate vertices (the number of roads matters as well as the length)	M1
Build up $d_k(v)$ from d_{k-1} : $d_k(v) = \min \text{ of } d_{k-1}(v) \text{ and } d_{k-1}(u) + w(uv) \text{ over neighbours } u$	A1
Answer: Use a recurrence on the number of roads	2 marks

30 (10 marks)

Labels: A: 0 (1st); B: 2 (2nd); C: 2 (3rd); D: 5 (4th); E: 7 (5th); F: 7 (6th); G: 10 (7th)	M1 A1 A1 A1
Shortest length 10	A1
Number of shortest paths to a vertex = sum of the numbers for the vertices that give its final label	M1
A 1, B 1, C 1	B1
D: reached from B ($2 + 3 = 5$) and from C ($2 + 3 = 5$): $1 + 1 = 2$ paths	M1
E: from B ($2 + 5 = 7$) and D ($5 + 2 = 7$): $1 + 2 = 3$ paths; F similarly 3 paths	A1
G: from E and F, both giving 10: $3 + 3 = 6$ paths	A1
Answer: 10; 6 paths	10 marks

Route Inspection

1 (2 marks)

Find a route that traverses every edge of a network at least once	B1
and starts and finishes at the same vertex with the least possible total length	B1
Answer: Shortest closed route using every edge at least once	2 marks

2 (2 marks)

The network has an Eulerian circuit, so no edge needs to be repeated	M1
Length 85	A1
Answer: 85	2 marks

3 (2 marks)

Degrees: A 2, B 3, C 3, D 3, E 1	M1
Odd vertices: B, C, D, E	A1
Answer: B, C, D, E	2 marks

4 (1 mark)

The sum of the degrees is 17 (odd), but must equal twice the number of edges	E1
Answer: Sum of degrees must be even	1 mark

5 (2 marks)

Find the shortest path between the two odd vertices	B1
Repeat the edges of that path; the length of the route is the total weight plus the length of the path	B1
Answer: Repeat the shortest path between the two odd vertices	2 marks

6 (2 marks)

AB and CD	B1
AC and BD, and AD and BC	B1
Answer: AB + CD; AC + BD; AD + BC	2 marks

7 (5 marks)

All four vertices have degree 3, so all are odd	B1
Pairings: $AB + CD = 7$; $AC + BD = 9$; $AD + BC = 9$	M1 A1
Repeat the edges of the shortest pairing: $AB + CD$; total weight is 27	M1
Route length $27 + 7 = 34$	A1
Answer: 34	5 marks

8 (4 marks)

(a) Degrees: A 3, B 2, C 3, D 2

B1

Answer: A and C

1 mark

(b) There is an Eulerian trail from A to C since there are exactly two odd vertices

M1

No repeats needed

A1

Length = total weight = $6 + 5 + 7 + 4 + 8 = 30$

A1

Answer: 30

3 marks

9 (2 marks)

The first vertex can be paired with 5 others, the next unpaired vertex with 3 others, the last two are paired

M1

$5 \times 3 \times 1 = 15$

A1

Answer: 15

2 marks

10 (5 marks)

Odd vertices: B, E

B1

Pairings: BE = 8

M1 A1

Total weight 37 plus 8

M1

Route length 45 km

A1

Answer: 45 km

5 marks

11 (5 marks)

Degrees: A 2, B 3, C 4, D 3, E 2: odd vertices B and D

M1 A1

Shortest path B to D: B, D of length 7

M1

Total weight 42

A1

Route length $42 + 7 = 49$

A1

Answer: 49

5 marks

12 (5 marks)

Degrees: A 2, B 3, C 3, D 4, E 2: odd vertices B and C

M1 A1

Shortest path from B to C is B, C of length 6 (direct BC is 6)

M1

Total weight 42

A1

Length 48

A1

Answer: 48

5 marks

13 (4 marks)

Degrees: A 2, B 2, C 4, D 2, E 2: all even, so an Eulerian circuit exists

M1 A1

Length = $3 + 4 + 5 + 2 + 6 + 7 = 27$

A1

Route: A, C, E, D, C, B, A

A1

Answer: 27; A, C, E, D, C, B, A

4 marks

14 (6 marks)

Odd vertices: B, C, D, E

M1 A1

Pairings and sums: BC + DE = 7; BD + CE = 15; BE + CD = 21

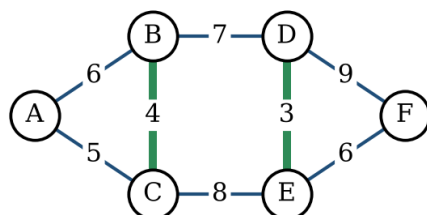
M1 A1

Repeat BC and DE; total weight 48, route length $48 + 7 = 55$

M1 A1

Answer: 55

6 marks



15 (7 marks)**(a)** Smallest pairing sum is 7 (CD 2 with AB 5)

M1

Length $27 + 7 = 34$

A1

Answer: 34

2 marks

(b) A and the finishing vertex are left unpaired; the other two odd vertices must be paired

M1

Options: pair B, C (4), pair B, D (6) or pair C, D (2)

M1 A1

Smallest is C, D = 2, so the route finishes at B

A1

Length $27 + 2 = 29$

A1

Answer: 29, finishing at B

5 marks

16 (6 marks)

Degrees: A 2, B 3, C 4, D 4, E 3, F 2; odd vertices B, E

M1 A1

Pairings: BE = 11

M1 A1

Total weight 72

A1

Route length $72 + 11 = 83$ km

A1

Answer: 83 km

6 marks

17 (5 marks)

Repeat the edges of the best pairing (AB + CD)

M1

The resulting network has all degrees even

A1

A route: A, B, D, C, D, A, C, B, A

M1 A1

Length 34, which equals $27 + 7$

A1

Answer: A, B, D, C, D, A, C, B, A

5 marks

18 (5 marks)

Before: length 45

B1

With BF added the odd vertices change: odd vertices are now E, F

M1

Shortest path joining them: E, F of length 4

M1 A1

New total weight 40, route length 44, which is shorter than before

A1

Answer: 44

5 marks

19 (7 marks)

Every vertex has degree 3, so all six vertices are odd

B1

There are 15 ways of pairing six odd vertices

M1

Shortest distances used (the shortest path between two vertices may not be the direct path)

M1

Smallest sums: AD + BE + CF = 8; AD + BF + CE = 14; AE + BD + CF = 14

M1 A1

Total weight 47, so route length $47 + 8 = 55$ (hundred metres)

M1 A1

Answer: 5500 m

7 marks

20 (5 marks)

Odd vertices B and D (degrees A 2, B 3, C 4, D 3, E 2), shortest path B to D is the direct edge

M1 A1

BD, length 7

Repeating BD gives degrees A 2, B 4, C 4, D 4, E 2

M1

A vertex of degree d in the closed route is passed through d/2 times, but A is passed through d/2

M1

– 1 times since the route starts and ends there

A 0, B 2, C 2, D 2, E 1 (total 7, which is one fewer than the 8 edges used)

A1

Answer: A 0, B 2, C 2, D 2, E 1

5 marks

21 (8 marks)

Degrees: A 2, B 3, C 3, D 2, E 2, F 3, G 3, H 2; odd vertices B, C, F, G

M1 A1

Pairings and sums of shortest distances: BC + FG = 11; BG + CF = 21; BF + CG = 31

M1 A1 A1

Smallest is BC + FG = 11: repeat BC and FG

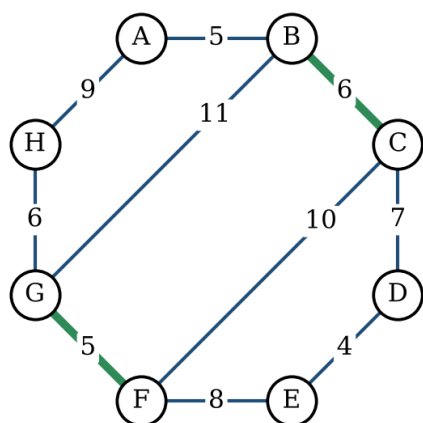
A1

Total weight 71, length $71 + 11 = 82$

M1 A1

Answer: 82

8 marks

**22 (6 marks)**

(a) The route uses every edge at least once
so its length is at least the sum of all the edge weights

M1

E1

Answer: Each edge used at least once

2 marks

(b) If every degree is even there is an Eulerian circuit, a closed route using each edge once, of length W

B1 M1

Conversely, a route of length W uses each edge exactly once (weights positive), so it is an Eulerian circuit

M1

In an Eulerian circuit, each visit to a vertex uses two edges, so all degrees are even

E1

Answer: The shortest route has length W if and only if every vertex has even degree

4 marks

23 (7 marks)

(a) Corner vertices have degree 2, edge vertices b, d, f, h have degree 3, centre e has degree 4

M1

So the odd vertices are b, d, f, h

A1

Answer: b, d, f, h

2 marks

(b) $bd = 2$, $bf = 2$, $bh = 2$, $df = 2$, $dh = 2$, $fh = 2$

M1 A1

$bd + fh = 4$, $bf + dh = 4$, $bh + df = 4$

A1

Answer: All three pairings total 4

3 marks

(c) 12 streets, total weight 12, so route length $12 + 4 = 16$ km

M1

All three pairings give the same extra length 4, so any of them can be used

A1

Answer: 16 km

2 marks

24 (6 marks)

(a) Odd vertices: B, E

M1

Pairings: $BE = 8$

M1 A1

Answer: $BE = 8$

3 marks

(b) The repeated edges add to the length of the route, so the pairing with the smallest total must be chosen

E1

Smallest pairing $BE = 8$

M1

Total weight 37, so shortest length 45

A1

Answer: 45

3 marks

25 (7 marks)

Degrees: A 3, B 3, C 2, D 3, E 3, F 2; odd vertices A, B, D, E

M1 A1

Pairings: $AB + DE = 6$; $AD + BE = 17$; $AE + BD = 17$

M1 A1

Best pairing $AB + DE$; repeated paths A, B; D, E

M1

A route: A, B, E, D, A, F, E, D, C, B, A

A1

Length $40 + 6 = 46$

A1

Answer: 46; A, B, E, D, A, F, E, D, C, B, A

7 marks

26 (7 marks)

Every vertex has degree 3, so all four vertices are odd M1 A1
 $AB + CD = 4 + 4 = 8$, $AD + BC = 4 + 4 = 8$, $AC + BD = 7 + 6 = 13$ (no shorter paths exist M1 A1
 between A and C or between B and D)
 Two pairings give the smallest total 8, so there are 2 optimal choices: repeat AB and CD, or M1
 repeat AD and BC
 Total weight $4 + 4 + 4 + 4 + 7 + 6 = 29$, route length $29 + 8 = 37$ A1
 Both optimal choices add exactly 8 to the total weight, so the routes have equal length E1
Answer: 37, with 2 optimal choices 7 marks

27 (10 marks)

R is a closed trail that uses each edge of M exactly once, so every vertex of M has even degree B1 M1
 Let X be the multigraph formed by the extra copies only, so that M is G together with X and the M1
 length of R is $W + \text{length}(X)$
 At each vertex, degree in M = degree in G + degree in X M1
 The degree in M is even, so degree in X and degree in G have the same parity A1
 So the odd vertices of X are exactly the odd vertices of G, say $2k$ of them A1
 Take a vertex v of odd degree in X and walk along edges of X, never reusing an edge. Each M1
 vertex of even degree can be left whenever it is entered, so the walk can only stop at a different
 odd vertex w . Remove this path from X: v and w become even (or isolated) and every other
 degree changes by 0 or 2, so the parities are kept. Repeat k times
 Therefore X splits into k edge-disjoint paths joining the $2k$ odd vertices in pairs, plus (possibly) M1
 some cycles of positive length
 Each path has length at least the shortest distance between its two ends, so $\text{length}(X)$ is at least the M1
 sum of the shortest distances for this pairing, which is at least the minimum over all pairings
 Hence $\text{length}(R) = W + \text{length}(X)$ is at least $W + \text{the minimum pairing sum}$ E1
Answer: Length of R is at least W + the minimum pairing sum 10 marks

28 (8 marks)

(a) Every vertex has degree $n - 1$, which is even M1
 So $\text{length} = \frac{n(n-1)}{2}$ A1
Answer: $\frac{n(n-1)}{2}$ 2 marks
 (b) All n vertices have odd degree $n - 1$ B1
 Every pair of vertices is adjacent, so each pairing repeats one edge of weight 1 per pair M1
 There are $\frac{n}{2}$ pairs, so the repeated length is $\frac{n}{2}$ A1
 Total $\frac{n(n-1)}{2} + \frac{n}{2} = \frac{n^2}{2}$ A1
Answer: $\frac{n^2}{2}$ 4 marks
 (c) $n = 10$: $\frac{100}{2} = 50$ B1
 $n = 9$: $\frac{9 \times 8}{2} = 36$ B1
Answer: 50 and 36 2 marks

29 (8 marks)

(a) The first vertex can be paired with any of the other $2k - 1$ M1
 The next unpaired vertex can be paired with any of the remaining $2k - 3$ M1
 and so on, giving $(2k - 1)(2k - 3) \dots \times 3 \times 1$ A1
Answer: The number of pairings is $1 \times 3 \times 5 \times \dots \times (2k - 1)$ 3 marks
 (b) $k = 3$: 15 B1
 $k = 4$: 105 B1
 $k = 10$: 654 729 075 B1
Answer: 15, 105, 654 729 075 3 marks
 (c) The number of pairings grows very quickly (faster than exponentially) E1

Use a matching algorithm such as Edmonds' algorithm, which runs in polynomial time

E1

Answer: Use a matching algorithm

2 marks

30 (10 marks)

(a) Each vertex lies on exactly three of the six edges

M1

Total weight = $x + 22$

A1

Answer: $x + 22$

2 marks

(b) Shortest AB is the smaller of x and A , C , $B = 7$, shortest $CD = 2$: $AB + CD = \min(x, 7) + 2$

M1 A1

$AC + BD = 3 + 6 = 9$

B1

$AD + BC$: shortest AD is A , C , $D = 5$ (direct 7), so $AD + BC = 5 + 4 = 9$

B1

Answer: $\min(x, 7) + 2, 9, 9$

4 marks

(c) For $x < 7$ the smallest pairing sum is $x + 2$ (less than 9)

M1

Length = $x + 22 + x + 2 = 2x + 24$

A1

For $x \geq 7$ the smallest sum is 9, length = $x + 31$

A1

Answer: $2x + 24$ for $x < 7$; $x + 31$ for $x \geq 7$

3 marks

(d) $2x + 24 = 30$ so $x = 3$

B1

Answer: 3

1 mark

Travelling Salesperson Problem

1 (2 marks)

Find a route that visits every vertex of a complete network exactly once and returns to the start, with minimum total weight

B1

B1

Answer: A minimum weight Hamiltonian cycle in a complete network

2 marks

2 (2 marks)

In the practical problem the network need not be complete

B1

and vertices may be visited more than once (the triangle inequality may not hold)

B1

Answer: Practical: not complete and vertices can be repeated

2 marks

3 (1 mark)

8 (it is a cycle through 8 vertices)

B1

Answer: 8

1 mark

4 (2 marks)

Start at a vertex and move to the nearest unvisited vertex, repeating until all vertices are visited

B1

Then return to the start

B1

Answer: Always go to the nearest unvisited vertex

2 marks

5 (2 marks)

Upper bound

B1

The optimal tour is the shortest tour, so it is no longer than any tour found

E1

Answer: Upper bound

2 marks

6 (2 marks)

Fix a start; the other 4 vertices can be ordered in $4! = 24$ ways

M1

Each tour is counted twice (two directions): 12

A1

Answer: 12

2 marks

7 (4 marks)

From A the nearest is C (15)

M1

Order: A, C, E, D, B, A

M1 A1

Length = $15 + 7 + 14 + 28 + 17 = 81$

A1

Answer: A, C, E, D, B, A, 81

4 marks

8 (3 marks)

Tour 1: $6 + 11 + 16 + 22 + 28 = 83$

M1 A1

Tour 2: $12 + 26 + 15 + 27 + 30 = 110$, so tour 2 is longer

A1

Answer: 83 and 110

3 marks

9 (5 marks)

Delete A and find the minimum spanning tree of B, C, D, E: CE (7), BC (11), DE (14)

M1 A1

Weight 32

A1

Add the two shortest edges at A: AC (15), AE (16), total 31

M1

Lower bound 63

A1

Answer: 63

5 marks

10 (4 marks)

Pairs joined directly: AB 4, BC 6, CD 5, DE 3, EA 7, BD 9 but check for shorter routes

M1

BD: B, C, D = 11, direct is 9, so BD stays 9

B1

Indirect pairs: AC = 10; AD = 10; BE = 11; CE = 8

M1 A1

Answer: AB 4; AC 10; AD 10; AE 7; BC 6; BD 9; BE 11; CD 5; CE 8; DE 3

4 marks

11 (3 marks)

The optimal tour is no longer than the shortest tour found, 45

M1

and no shorter than the lower bound, 41

M1

$41 \leq \text{optimal} \leq 45$

A1

Answer: 41 to 45

3 marks

12 (5 marks)

Starting at A: A, B, C, F, E, D, A

M1 A1

Length 48

A1

Starting at B: B, C, F, A, D, E, B, length 45

M1

The better upper bound is 45

A1

Answer: 45 from B

5 marks

13 (3 marks)

The optimal tour is at least 47 and at most 52

M1

The greatest excess is when the optimum is 47: $\frac{(52 - 47)}{47} \times 100$

M1

= 10.6%

A1

Answer: 10.6%

3 marks

14 (7 marks)

Nearest neighbour from A: A, B, C, F, E, D, A

M1 A1

Upper bound 48

A1

Delete A: MST of B to F = DE (4), CF (5), BC (6), CD (8), weight 23

M1

Add the two shortest edges at A: AB (7), AD (9)

A1

Lower bound 39, so $39 \leq \text{optimal} \leq 48$

A1

Greatest error $\frac{(48 - 39)}{39} \times 100 = 23.1\%$

A1

Answer: 39 to 48; 23.1%

7 marks

15 (7 marks)

Least distances found by inspection / Dijkstra: AB 6, AC 8, AD 20, AE 5, AF 15, BC 2, BD 14, BE 9, BF 19, CD 12, CE 7, CF 17, DE 19, DF 14, EF 10

M1 A1

Nearest neighbour from A: A, E, C, B, D, F, A

M1 A1

Length 57

A1

Route in the original network: A, E, C, B, C, D, F, E, A

M1 A1

Answer: A, E, C, B, C, D, F, E, A, 57 km

7 marks

16 (5 marks)

Removing any one edge from a tour leaves a spanning tree **M1**
 So the tour is at least as long as the minimum spanning tree **E1**
 Minimum spanning tree: AB (6), BC (11), AD (12), BE (15), weight 44 **A1**
 Deleting A: MST of the rest BC (11), BE (15), CD (16) = 42; two shortest edges at A: AB (6), **M1**
 AD (12) = 18; bound 60, bound 60
 The deleted vertex bound 60 is better (larger) than 44 **A1**
Answer: MST bound 44; deleted vertex bound 60 **5 marks**

17 (6 marks)

Method: for each vertex, MST of the remaining five vertices plus the two shortest edges at the **M1**
 deleted vertex
 A: 39 **A1**
 B: 39, C: 42 **A1**
 D: 40, E: 40, F: 41 **A1**
 The best (largest) lower bound is 42, from deleting C **A1**
 So the optimal tour has length at least 42 **B1**
Answer: 42 (delete C) **6 marks**

18 (5 marks)

Original tour A, B, C, F, E, D, A **B1**
 New tour: A, B, E, F, C, D, A **M1**
 Edges AB 7, BE 10, EF 17, FC 5, CD 8, DA 9 **M1**
 Length 56, which is longer than 48 **A1 A1**
Answer: 56 **5 marks**

19 (5 marks)

Number of tours = $\frac{(n-1)!}{2}$ **M1**
 $n = 7: \frac{6!}{2} = 360$; $n = 10: \frac{9!}{2} = 181\,440$ **A1 A1**
 $n = 15: \frac{14!}{2} = 43,589,145,600$ **M1**
 Time = 43 600 s = 12.1 hours (3 s.f.) **A1**
Answer: 360; 181 440; 43 600 s **5 marks**

20 (6 marks)

There are $4!/2 = 12$ tours **B1**
 Lengths: ABCDEA 49; ABCEDA 43; ABDCEA 53; ABDECA 59; ABECDA 46; ABEDCA 58; **M1 A1 A1**
 ACBDEA 59; ACBEDA 52; ACDBEA 62; ACEBDA 56; ADBCEA 47; ADCBEA 46
 The shortest is A, B, C, E, D, A with length 43 **A1**
 This is the optimal tour **B1**
Answer: A, B, C, E, D, A, 43 **6 marks**

21 (8 marks)

Nearest neighbour from B: B, C, D, G, F, E, A, B **M1 A1**
 Upper bound 56 **A1**
 Delete B: MST of the other six = FG (4), DG (5), CD (7), AE (8), AC (10), weight 34 **M1 A1**
 Two shortest edges at B: BC (3), BF (6); lower bound 43 **M1 A1**
 Interval $43 \leq \text{optimal} \leq 56$, greatest excess $\frac{(56-43)}{43} \times 100 = 30.2\%$ **A1**
Answer: 43 to 56; 30.2% **8 marks**

22 (6 marks)

Minimum spanning tree: BC (2), AE (5), AB (6), EF (10), CD (12), weight 35	M1 A1
Double every edge: the graph has all degrees even, and an Eulerian circuit of length 70	M1
Follow this circuit, but skip any vertex already visited, going directly to the next new vertex	M1
By the triangle inequality a direct edge is no longer than the path it replaces, so the tour has length at most 70	A1
Upper bound 70, compared with 57 from nearest neighbour	A1
Answer: 70 (nearest neighbour 57)	6 marks

23 (6 marks)

(a) A tour must leave A and return to A, so it uses two edges at A	M1
The student should add the two shortest edges at A, not one	E1
Answer: Two edges at A are needed	2 marks
(b) Two shortest edges at A: AC 15 and AE 16	M1
Lower bound = $32 + 15 + 16$	M1
= 63	A1
Answer: 63	3 marks
(c) $63 \leq \text{optimal} \leq 81$	B1
Answer: $63 \leq \text{optimal} \leq 81$	1 mark

24 (7 marks)

(a) The network has no Hamiltonian cycle: A and B have degree 1, so no cycle can pass through them	E1
Answer: A and B each have only one road	1 mark
(b) Direct roads as given; AB 7, AD 8, AE 9, BD 9, BE 10	M1 A1
Answer: AB 7, AC 3, AD 8, AE 9, BC 4, BD 9, BE 10, CD 5, CE 6, DE 7	2 marks
(c) Nearest to A is C (3); then from C: B (4); then D (9); then E (7); then back to A (9)	M1 A1
Tour A, C, B, D, E, A, length 32	A1
Route A, C, B, C, D, E, C, A, which passes through C three times	A1
Answer: A, C, B, C, D, E, C, A, 32 km	4 marks

25 (8 marks)

(a) Edges in order: DE 8, BC 9, CD 10 (BD 16 and BE 12 would make cycles)	M1
Weight $8 + 9 + 10 = 27$	A1
Answer: 27	2 marks
(b) The edges at A are x, 14, 20, 11, so with $x < 14$ the two shortest are x and AE 11	M1
Lower bound $27 + x + 11 = 47$	M1
$x = 9$	A1
Answer: $x = 9$	3 marks
(c) Nearest to A is B (9), then C (9), then D (10), then E (8), then back to A (11)	M1 A1
Tour A, B, C, D, E, A has length 47, equal to the lower bound, so it is optimal	A1
Answer: 47; the tour is optimal	3 marks

26 (6 marks)

(a) Remove any one edge e from the optimal tour: the remaining $n - 1$ edges form a path through all vertices, a spanning tree	M1 A1
So $L - w(e) \geq W$	M1
Since $w(e) \geq m$, $L \geq W + w(e) \geq W + m$	E1
Answer: $L \geq W + m$	4 marks
(b) Minimum spanning tree: BC (6), AD (7), CE (8), AB (9), $W = 30$; lightest edge $m = 6$	M1
Lower bound $30 + 6 = 36$	A1
Answer: 36	2 marks

27 (9 marks)

Let T be any tour, a cycle through all the vertices	B1
T uses exactly two edges at v , say e_1 and e_2	M1
Deleting v from T leaves a path through all the other $n - 1$ vertices	M1 A1
This path is a spanning tree of $G - v$, so its weight is at least the weight W of a minimum spanning tree of $G - v$	M1
So the length of T is at least $W + w(e_1) + w(e_2)$	M1
The two edges e_1, e_2 are distinct edges at v , so their total weight is at least the sum of the two lightest edges at v	M1 A1
This holds for every tour, so it holds for the optimal tour: the stated lower bound is valid	E1
Answer: Every tour is at least as long as $MST(G - v)$ plus the two lightest edges at v	9 marks

28 (8 marks)

(a) Removing an edge from the optimal tour leaves a spanning tree whose weight is at least W , so $L \geq W$	M1 A1
Answer: $L \geq W$	2 marks
(b) The doubled tree has all degrees even, so has an Eulerian circuit of length $2W$	M1 A1
Traverse this circuit, but whenever the next vertex has already been visited, go directly to the next unvisited vertex	M1
By the triangle inequality each shortcut does not increase the length, so the tour has length at most $2W$	A1
Answer: A tour of length at most $2W$ exists	4 marks
(c) The tour has length at most $2W$	M1
and $W \leq L$, so the tour has length at most $2L$	A1
Answer: The tour has length at most $2L$	2 marks

29 (8 marks)

(a) Fix the starting vertex, the other $n - 1$ can be ordered in $(n - 1)!$ ways	M1
A tour and its reverse are the same cycle, so divide by 2	A1
Answer: There are $\frac{(n - 1)!}{2}$ tours	2 marks
(b) Number of tours $\frac{24!}{2} = 3.102 \times 10^{23}$	B1
Time = $\frac{3.102 \times 10^{23}}{10^9} = 3.10 \times 10^{14}$ s	M1 A1
= 9.83×10^6 years (3 s.f.)	A1
Answer: 9.83×10^6 years	4 marks
(c) Number of tours is multiplied by $(25 - 1) = 24$ (from $\frac{(n - 2)!}{2}$ to $\frac{(n - 1)!}{2}$)	M1
Factorial growth: exact methods become impractical, so bounds and heuristics are used	E1
Answer: $\times 24$	2 marks

30 (10 marks)

(a) Nearest to A is E (9), then from E: D (15), then C (10), then B (14), then back to A (12)	M1 A1
Tour A, E, D, C, B, A, length 60	A1
Answer: A, E, D, C, B, A, 60	3 marks
(b) MST of B, C, D, E: CD (10), BC (14), DE (15), weight 39	M1 A1
Two shortest edges at A: AE (9), AB (12)	M1
Lower bound 60	A1
Answer: 60	4 marks
(c) The upper and lower bounds are equal	M1
so the optimal tour has length 60	A1
Optimal tour A, E, D, C, B, A	A1
Answer: 60; A, E, D, C, B, A	3 marks