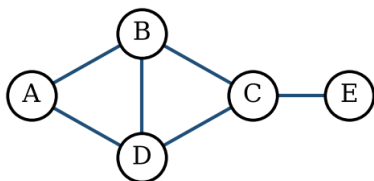


Graph Theory

- 1 The graph G is shown in the diagram.



- (a) Write down the degree of vertex D. [1]
 (b) Write down the number of edges of G. [1]

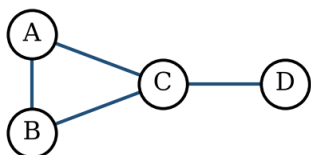
- 2 State the difference between a path and a cycle in a graph. [2]

- 3 A graph has 9 edges. Find the sum of the degrees of all its vertices. [2]

- 4 Find the number of edges in the complete graph K_6 . [2]

- 5 A tree has 12 vertices. Write down the number of edges in the tree. [1]

- 6 Write down the adjacency matrix for the graph shown, using the vertices in the order A, B, C, D.



[2]

- 7 A graph has vertices A, B, C, D, E and edges AB, AC, BC, CD, DE.

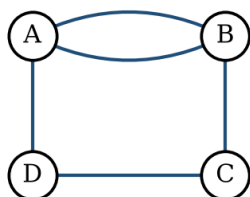
- (a) Write down a path from A to E that passes through exactly four vertices. [1]
 (b) Write down a cycle in the graph. [1]
 (c) Explain why the graph is connected. [1]

- 8 The adjacency matrix of a graph with vertices V_1, V_2, V_3, V_4, V_5 is

$$\mathbf{M} = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

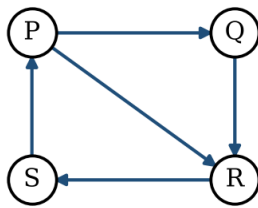
- (a) Find the degree of each vertex. [2]
 (b) Hence find the number of edges. [1]

- 9 The diagram shows a graph H.



- (a) Explain why H is not a simple graph. [1]
 (b) Write down the degree of vertex A. [1]

- 10** The directed graph (digraph) shown has four vertices P, Q, R and S. Write down the in-degree and the out-degree of each vertex.



[3]

- 11** The complete bipartite graph $K_{3,4}$ has one set of 3 vertices and another set of 4 vertices, and every vertex in one set is joined to every vertex in the other set.

- (a) Write down the number of vertices of $K_{3,4}$.
 (b) Find the number of edges of $K_{3,4}$.

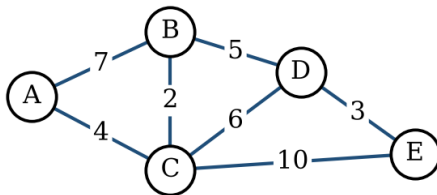
[1]

[1]

- 12** A graph G has vertices A, B, C, D, E and edges AB, BC, CD, DE, EA, AC. A graph H has vertices A, B, C, D and edges AB, BC, CD, AC. State whether H is a subgraph of G, giving a reason, and state which vertex of G has to be deleted to produce H.

[3]

- 13** The weighted graph shown models a network of roads between five towns, with weights giving distances in km.



- (a) Find the length of the path A, B, D, E.
 (b) By considering all sensible routes, find the shortest path from A to E and state its length.

[1]

[2]

- 14** A graph has 8 vertices. Seven of the vertices have degrees 3, 3, 4, 4, 2, 5 and 3, and the eighth vertex has degree x.

- (a) Given that the graph has 13 edges, find x.
 (b) Explain why x cannot be 3.

[2]

[2]

- 15** The adjacency matrix of a graph with vertices A, B, C, D (in that order) is

$$\mathbf{M} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

Find $\mathbf{M}^2$, and use it to find the number of different walks of length 2 from B to D. List these walks.

[4]

- 16** Each vertex of a simple graph has degree 3.

- (a) Show that the graph must have an even number of vertices.
 (b) Find the number of edges when the graph has 10 vertices.

[2]

[2]

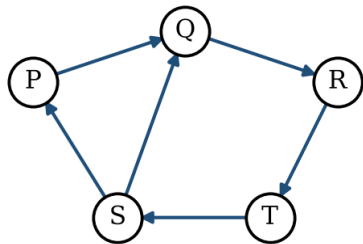
- 17** The complete graph K_n has n vertices, and every pair of vertices is joined by exactly one edge.

- (a) Show that K_n has $\frac{n(n-1)}{2}$ edges.
 (b) A complete graph has 105 edges. Find the number of vertices.

[2]

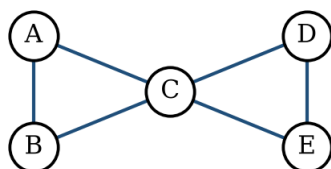
[3]

- 18** The digraph shown has five vertices P, Q, R, S, T.



- (a) Write down the adjacency matrix **D**, where the entry in row *i*, column *j* is the number of arcs from vertex *i* to vertex *j*, using the order P, Q, R, S, T. [2]
- (b) Find **D**² and hence the number of routes of exactly two arcs from S to R. [3]
-
- 19** A graph models friendships between seven students A to G. A is friends with B, C and D; B with A and E; C with A, D and F; D with A, C and G; E with B and F; F with C, E and G; G with D and F.
- (a) Find the number of friendships, and write down the degree sequence of the graph in descending order. [3]
- (b) Find the smallest number of friendships a message must pass along to get from B to G, and write down one such chain. [2]
- (c) A new student H becomes friends with exactly the four students who have the most friends. Find the new number of friendships, and write down the new degree sequence in descending order. [2]
-
- 20** A graph has vertices numbered 1, 2, 3, 4, 5, 6. Two distinct vertices *i* and *j* are joined by an edge when the difference between *i* and *j* is 1 or 4.
- (a) List the edges of the graph. [2]
- (b) Write down the degree of each vertex and verify that the sum of the degrees is twice the number of edges. [2]
- (c) Find a cycle of length 4 in the graph. [1]
-
- 21** Let *G* be a graph with *e* edges.
- (a) Show that the sum of the degrees of the vertices of *G* is $2e$. [2]
- (b) Hence prove that *G* has an even number of vertices of odd degree. [4]
-
- 22** In a league of 6 teams every team plays every other team once and there are no draws. A digraph models the results, with an arc from the winner to the loser of each game.
- (a) Show that the sum of the out-degrees of the vertices is 15. [2]
- (b) Explain why the numbers of wins 5, 5, 3, 1, 1, 0 are impossible, even though they sum to 15. [3]
- (c) Given that all six teams have different numbers of wins, find the numbers of wins. [2]

- 23** The graph shown has five vertices A, B, C, D, E.



- (a) Write down the adjacency matrix $\mathbf{M}$ of the graph, using the order A, B, C, D, E. [2]
- (b) Given that $\mathbf{M}^3 = \begin{pmatrix} 2 & 3 & 5 & 2 & 2 \\ 3 & 2 & 5 & 2 & 2 \\ 5 & 5 & 4 & 5 & 5 \\ 2 & 2 & 5 & 2 & 3 \\ 2 & 2 & 5 & 3 & 2 \end{pmatrix}$, state the number of walks of length 3 from A to D, and list them. [3]
- (c) The sum of the diagonal entries of $\mathbf{M}^3$ is 12. Explain how this relates to the number of triangles in the graph and verify it. [2]

- 24** A student writes: "The degrees of the vertices of a connected graph are 4, 3, 3, 2, 2, 1, 1. The sum of the degrees is 16, so the graph has 16 edges."

- (a) Identify the student's error. [2]
- (b) Write down the correct number of edges. [1]
- (c) Use the number of vertices and edges to explain why the graph is not a tree. [3]

- 25** The table shows the weights on the edges of a connected graph with vertices A to F. A dash means that there is no edge.

	A	B	C	D	E	F
A	-	5	3	-	-	-
B	5	-	2	6	-	-
C	3	2	-	-	4	-
D	-	6	-	-	5	3
E	-	-	4	5	-	6
F	-	-	-	3	6	-

- (a) Find the number of edges of the graph and the sum of all the edge weights. [3]
- (b) State the number of vertices of odd degree, and name them. [2]
- (c) Find the shortest path from A to F and its length. [3]

- 26** A tree has three vertices of degree 4, two vertices of degree 3, and all of its other vertices are leaves (vertices of degree 1). Let L be the number of leaves.

- (a) Find L . [4]
- (b) Write down the total number of vertices and the number of edges of the tree. [2]

- 27** Prove that every simple graph with at least two vertices has two vertices of the same degree. [8]

- 28** A tree is a connected graph with no cycles, and a tree with n vertices has $n - 1$ edges. A leaf is a vertex of degree 1. Let T be a tree with $n \geq 2$ vertices.

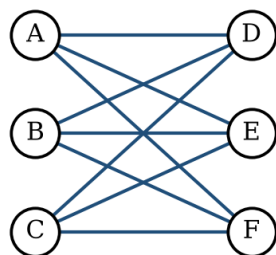
- (a) Write down the sum of the degrees of the vertices of T . [1]
- (b) Prove that T has at least two leaves. [5]
- (c) Deduce that a tree with exactly two leaves is a path. [3]

29

The complete graph K_4 has adjacency matrix $\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$.

- Show that $\mathbf{A}^2 = 2\mathbf{A} + 3\mathbf{I}$, where $\mathbf{I}$ is the 4×4 identity matrix. [3]
- Hence find $\mathbf{A}^3$ in the form $p\mathbf{A} + q\mathbf{I}$, and state the number of walks of length 3 from a vertex to itself, and from a vertex to a different vertex. [3]
- Find the number of walks of length 4 from a vertex to itself. [3]

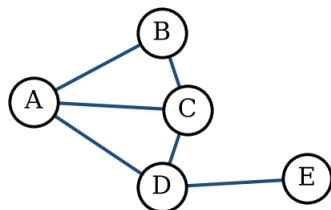
- 30 The graph $K_{3,3}$ shown has vertices A, B, C in one set and D, E, F in the other, with every vertex of one set joined to every vertex of the other. Its adjacency matrix is $\mathbf{A}$.



- Find the number of walks of length 2 from A to A, from A to B, and from A to D. [3]
- Explain why the number of walks of any odd length between two vertices in the same set is zero. [2]
- Find the number of walks of length 3 from A to D. [4]

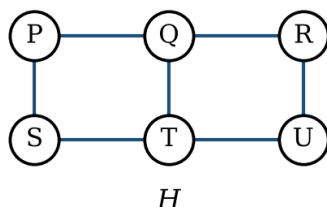
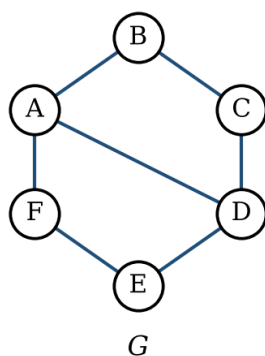
Graph Properties

- 1 The graph shown has five vertices. Write down its degree sequence, listing the degrees in descending order.



- [2]
- State what is meant by a simple graph. [2]
- Explain why there is no graph whose vertices have degrees 3, 3, 3, 2. [2]
- A simple graph G has 5 vertices and 7 edges. Find the number of edges in the complement of G . [2]
- State what is meant by a bipartite graph. [2]
- Write down the degree of each vertex of $K_{3,5}$. [2]
- Show that there is a simple graph with five vertices A, B, C, D, E whose degrees are 3, 2, 2, 1 respectively, by listing its edges. [3]
- A graph G has vertices A, B, C, D, E and edges AB, BC, CD, DE, EA, AC. List the edges of the complement of G . [3]

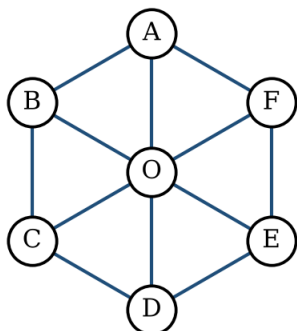
- 9** G is the graph with edges AB, BC, CD, DA, AC and H is the graph with edges PQ, QR, RS, SP, QS.
Show that G and H are isomorphic by giving a correspondence between their vertices. [3]
- 10** G has vertices A to F and edges AB, BC, CD, DE, EF. H has vertices P to U and edges PQ, PR, PS, PT, PU.
Show that G and H are not isomorphic. [3]
- 11** A graph has vertices A, B, C, D, E, F and edges AD, AE, BD, BF, CE, CF, AF.
(a) Show that the graph is bipartite. [2]
(b) Explain why adding the edge AB would give a graph that is not bipartite. [2]
- 12** A graph has vertices A, B, C, D, E, F, G and edges AB, BC, CA, DE, FG.
(a) Write down the number of connected components. [1]
(b) Find the smallest number of extra edges needed to make the graph connected. [2]
- 13** A simple connected graph has 8 vertices.
(a) Write down the smallest and the largest possible number of edges, and describe the graph in each case. [3]
(b) Show that, apart from the labelling of the vertices, there is only one simple connected graph with 4 vertices and 5 edges, and describe it. [4]
- 14** The Havel-Hakimi algorithm tests whether a list of numbers is the degree sequence of a simple graph: remove the largest number d , subtract 1 from each of the next d largest numbers, and repeat.
(a) Use the algorithm to show that 6, 5, 3, 3, 2, 2, 1 is the degree sequence of a simple graph. [3]
(b) Show that 6, 6, 4, 2, 2, 1, 1 is not the degree sequence of a simple graph. [3]
- 15** A simple graph has 5 vertices.
(a) Give an example of a graph whose vertices have degrees 3, 3, 3, 3 (on four vertices). [1]
(b) Show that the degrees of the five vertices cannot be 4, 4, 4, 1, 1. [3]
- 16** A simple graph is self-complementary if it is isomorphic to its complement. Let G be a self-complementary graph with n vertices.
(a) Show that G has $\frac{n(n-1)}{4}$ edges. [2]
(b) Explain why no self-complementary graph has 6 vertices. [1]
(c) Find the number of edges for $n = 5$ and give an example of a self-complementary graph with 5 vertices. [2]
- 17** The graphs G and H are shown.



- (a) Write down the degree sequence of each graph. [2]
(b) Show that G and H are isomorphic. [3]

- 18** A simple bipartite graph has 10 vertices, with a vertices in one set and $10 - a$ vertices in the other set.
- (a) Show that the number of edges is at most $a(10 - a)$. [1]
- (b) Show that $a(10 - a) = 25 - (a - 5)^2$, and hence find the largest possible number of edges. [3]
- (c) Find the smallest possible number of edges if the graph is also connected. [1]

- 19** The wheel graph W_6 has a hub O joined to each of the six vertices A, B, C, D, E, F of a hexagon $ABCDEF$, as shown.



- (a) Write down the number of vertices, the number of edges and the degree of each vertex. [3]
- (b) Explain why W_6 is not bipartite. [2]

- 20** A graph has vertices $V_1, \dots, V_6$ and adjacency matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

Reorder the rows and columns of $\mathbf{A}$ into the order $V_1, V_3, V_5, V_2, V_4, V_6$ and explain how the new matrix shows that the graph is bipartite.

[5]

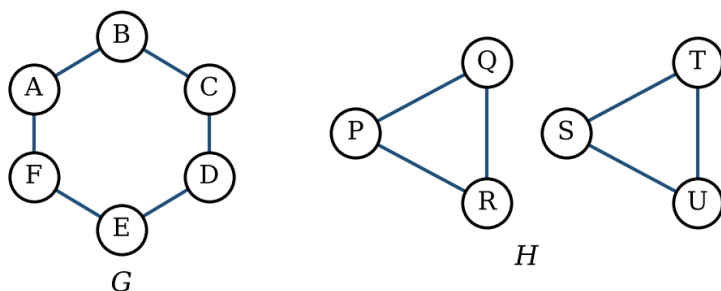
- 21** The graph G is $K_{3,3}$.
- (a) Find the number of edges of the complement of G . [2]
- (b) Find the degree of each vertex of the complement. [2]
- (c) Describe the complement of G , and state whether it is connected. [2]

- 22** G is a simple graph with n vertices in which every vertex has degree at least $\frac{n-1}{2}$. Prove that G is connected. [7]

- 23** A tree with 5 vertices is a connected graph with 5 vertices and no cycles. Two trees are the same if they are isomorphic.
- (a) Write down the sum of the degrees of the vertices of a tree with 5 vertices. [1]
- (b) Find all non-isomorphic trees with 5 vertices, by giving the degree sequence of each and describing its shape. [4]
- (c) Explain why a tree with 5 vertices cannot have two vertices of degree 3. [2]

- 24** A simple graph has 7 vertices with degrees 6, 5, 4, 3, 2, 1 and x . Find x . [7]

- 25** A student claims: "Two graphs with the same number of vertices, the same number of edges and the same degree sequence must be isomorphic."
The graphs G and H are shown.



- Show that G and H have the same number of vertices, the same number of edges and the same degree sequence. [2]
- Explain why G and H are not isomorphic. [2]
- State what this shows about the student's claim, and give a cycle property that also distinguishes G from H. [2]

- 26** A simple graph has vertices A, B, C, D, E and adjacency matrix $\mathbf{M}$, where

$$\mathbf{M}^2 = \begin{pmatrix} 3 & 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 & 0 \\ 1 & 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

- Explain why the leading diagonal of $\mathbf{M}^2$ gives the degrees of the vertices, and write down the degree of each vertex. [3]
- Find the number of edges of the graph. [1]
- Use the matrix to find the number of common neighbours of B and C. [2]

- 27** Six people attend a meeting. Any two of them either know each other or are strangers. Prove that there are three people who all know each other, or three people who are all strangers to each other. [8]

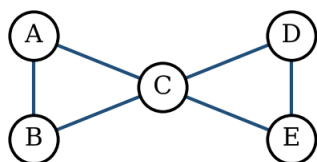
- 28** A graph is bipartite if its vertices can be divided into two sets so that every edge joins a vertex in one set to a vertex in the other.
- Prove that every cycle in a bipartite graph has even length. [3]
 - Let G be a connected graph with no cycle of odd length. Fix a vertex v and let X be the set of vertices whose shortest-path distance from v is even, and Y the set of vertices whose distance from v is odd. Prove that no edge joins two vertices of X, and no edge joins two vertices of Y, so that G is bipartite. [6]

- 29** Let G be a simple graph with n vertices and e edges that contains no triangle. Let $d(v)$ denote the degree of vertex v .
- Show that if u and v are adjacent then $d(u) + d(v) \leq n$. [2]
 - By summing the inequality in (a) over all the edges, show that the sum of the values $d(v)^2$, taken over all vertices v , is at most en . [2]
 - Use the Cauchy-Schwarz inequality, in the form $(\text{sum of } d(v))^2 \leq n \times (\text{sum of } d(v)^2)$, to prove that $e \leq \frac{n^2}{4}$. [3]
 - State the largest number of edges a triangle-free graph on 10 vertices can have, and give a graph that achieves it. [2]

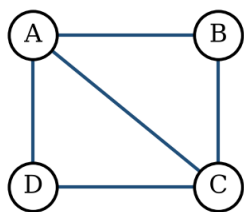
- 30** Four vertices are labelled A, B, C, D. A labelled simple graph is determined by which of the six possible edges are present.
- (a) Find the total number of labelled simple graphs on these four vertices. [2]
 - (b) Show that exactly 20 of these graphs have 3 edges. [1]
 - (c) Show that a graph with 3 edges on these vertices is either a tree or a triangle with an isolated vertex, and find how many graphs there are of each type. [4]
 - (d) Verify the number of trees in (c) by counting by shape: a path on 4 vertices can be labelled in $\frac{4!}{2}$ ways and a star $K_{1,3}$ in 4 ways. [2]

Eulerian and Hamiltonian Graphs

- 1** State the difference between an Eulerian trail and an Eulerian circuit. [2]
- 2** State the necessary and sufficient condition for a connected graph to have an Eulerian circuit. [2]
- 3** Explain why the graph shown has an Eulerian circuit.

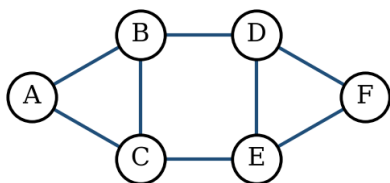


- [2]
- 4** State what is meant by a Hamiltonian cycle. [2]
- 5** Does the complete graph K_5 have an Eulerian circuit? Give a reason. [2]
- 6** State the condition on the vertex degrees for a connected graph to be semi-Eulerian, that is, to have an Eulerian trail that is not a circuit. [2]
- 7** The graph shown has an Eulerian trail.



- (a) Identify the vertices at which the trail must start and finish. [2]
 - (b) Write down an Eulerian trail. [1]
- 8** Determine whether each of the following has an Eulerian circuit, giving reasons.
- (a) K_4 [1]
 - (b) K_7 [1]
 - (c) $K_{2,4}$ [1]

- 9** Find a Hamiltonian cycle in the graph shown, starting and finishing at A.



[2]

- 10** The famous bridges of Königsberg problem can be modelled by a graph with four vertices (land areas) of degrees 5, 3, 3 and 3. Explain why it is impossible to cross each bridge exactly once.

[3]

- 11** Each list gives the degrees of the vertices of a connected graph. In each case state, with a reason, whether the graph is Eulerian (it has an Eulerian circuit), semi-Eulerian (it has an Eulerian trail but no Eulerian circuit) or neither.

(a) 4, 4, 2, 2, 2

[1]

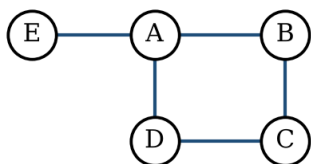
(b) 3, 3, 2, 2, 2

[1]

(c) 3, 3, 3, 3, 2, 2

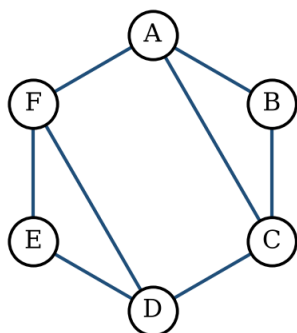
[1]

- 12** Explain why the graph shown has no Hamiltonian cycle.



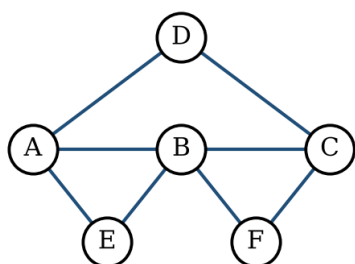
[3]

- 13** The graph shown has four vertices of odd degree. Find the smallest number of extra edges that must be added (duplicating an existing pair is not allowed) to give a graph with an Eulerian circuit, and say which pairs of vertices they join.



[3]

- 14** The graph shown has an Eulerian trail.



(a) Show that the graph has an Eulerian trail but not an Eulerian circuit.

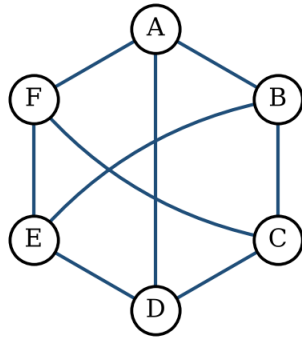
[2]

(b) Find an Eulerian trail starting at A.

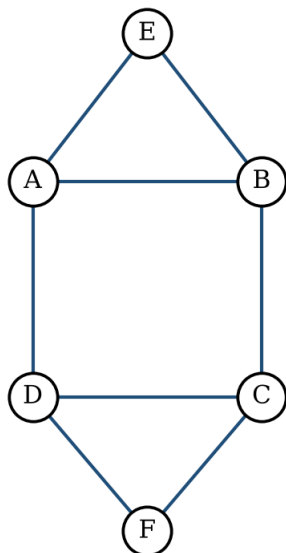
[2]

- 15** Dirac's theorem states that a simple graph with $n \geq 3$ vertices is Hamiltonian if every vertex has degree at least $\frac{n}{2}$.

The graph shown has six vertices.



- (a) Show that the graph satisfies the condition of Dirac's theorem. [2]
 (b) Hence state that the graph is Hamiltonian and write down a Hamiltonian cycle. [2]
-
- 16** The cube graph Q_3 has 8 vertices, the corners of a cube, and 12 edges, the edges of the cube. [2]
 (a) Explain why Q_3 does not have an Eulerian circuit. [2]
 (b) Describe a Hamiltonian cycle of Q_3 by labelling the corners with binary strings that differ in one digit for adjacent corners, such as 000 and 001. [2]
-
- 17** The complete bipartite graph $K_{m,n}$ has m vertices in one set and n in the other, with $m < n$. [1]
 (a) Explain why any cycle in $K_{m,n}$ must alternate between the two sets. [1]
 (b) Hence show that $K_{3,4}$ has no Hamiltonian cycle. [3]
-
- 18** Show that the complete graph K_n has an Eulerian circuit if and only if n is odd ($n \geq 3$), and state, with a reason, which complete graphs have a Hamiltonian cycle. [4]
-
- 19** The diagram shows the plan of a gallery. Each vertex is a room and each edge is a doorway between two rooms.

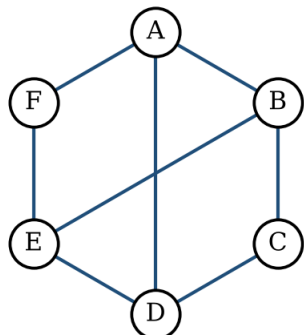


- (a) Find the degree of each room. [2]
 (b) Explain why a visitor cannot walk through every doorway exactly once. [2]
 (c) The gallery closes one doorway, AB. Show that the visitor can now walk through each remaining doorway exactly once, and say where the walk must start and finish. [2]

- 20** In the complete graph K_n a Hamiltonian cycle is a cyclic ordering of the vertices, where a cycle and its reverse are the same.

- (a) Show that the number of different Hamiltonian cycles in K_n is $\frac{(n-1)!}{2}$. [3]
 (b) Find the number of Hamiltonian cycles in K_7 . [1]

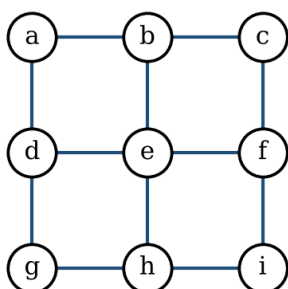
- 21** The graph G is shown.



- (a) Show that G has neither an Eulerian circuit nor an Eulerian trail. [2]
 (b) Show that removing the single edge AD gives a graph with an Eulerian trail, and state where the trail starts and finishes. [3]
 (c) Show that removing both AD and BE gives a graph with an Eulerian circuit, and write one down starting at A . [3]

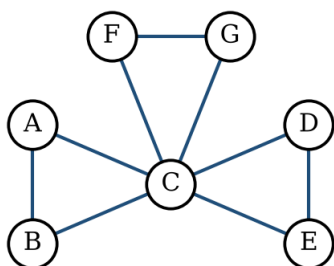
- 22** Prove that if a connected graph has an Eulerian circuit then every vertex has even degree. [6]

- 23** The graph shown is a 3×3 grid with vertices a to i .



- (a) Explain why the grid has no Eulerian trail. [2]
 (b) Colour the vertices alternately black and white like a chessboard. Use this to explain why the grid has no Hamiltonian cycle. [3]
 (c) Show that the grid does have a path through all nine vertices. [2]

- 24** A student writes: "Every vertex of this graph has degree at least 2, so the graph has a Hamiltonian cycle." The graph shown has seven vertices.

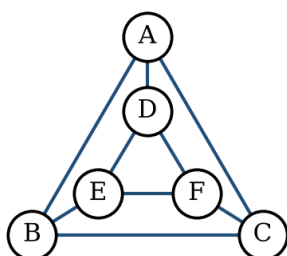


- (a) Show that every vertex has degree at least 2 and that the graph has an Eulerian circuit. [2]
 (b) Show that the graph has no Hamiltonian cycle, and hence that the student is wrong. [4]
 (c) Write down an Eulerian circuit starting at C . [2]

- 25** Ore's theorem states that a simple graph with $n \geq 3$ vertices is Hamiltonian if, for every pair of non-adjacent vertices u and v , $\deg(u) + \deg(v) \geq n$.
Let G be the graph with vertices A, B, C, D, E, F and edges $AB, AC, AD, BC, BE, CD, CF, DE, EF$.
- Find all the pairs of non-adjacent vertices whose degrees add to less than 6. [3]
 - State what can be deduced about G from Ore's theorem. [1]
 - Find a Hamiltonian cycle in G , and explain why this does not contradict (b). [2]
-
- 26** The complete bipartite graph $K_{4,4}$ has sets A, B, C, D and E, F, G, H .
- Find the number of edges, and explain why $K_{4,4}$ has an Eulerian circuit. [3]
 - Show that $A, E, B, F, C, G, D, H, A$ is a Hamiltonian cycle. [2]
 - Show that, after deleting the edges of this Hamiltonian cycle from $K_{4,4}$, every vertex has degree 2. [2]
-
- 27** Prove that a connected graph in which every vertex has even degree has an Eulerian circuit. [10]
-
- 28** Let $m \geq 2$. Prove that the complete bipartite graph $K_{m,n}$ (with $m \leq n$) has a Hamiltonian cycle if and only if $m = n$. [8]
-
- 29** A connected graph G has exactly $2k$ odd vertices, where $k \geq 1$. Prove that the edges of G can be split into k trails (each edge used exactly once overall), and that fewer than k trails is not possible. [9]
-
- 30** The hypercube graph Q_n has vertices the binary strings of length n , with two vertices joined by an edge when they differ in exactly one digit.
- Find the number of vertices, the number of edges, and the degree of each vertex of Q_n . [3]
 - For which values of n does Q_n have an Eulerian circuit? [2]
 - Prove by induction that Q_n has a Hamiltonian cycle for every $n \geq 2$. [5]

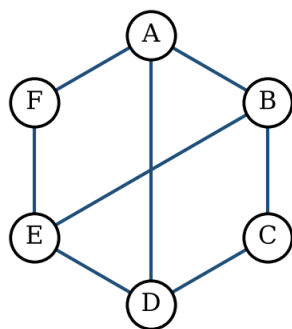
Planar Graphs

- State what is meant by a planar graph. [2]
- State Euler's formula for a connected planar graph with v vertices, e edges and f faces. [1]
- A connected planar graph has 6 vertices and 9 edges. Find the number of faces, including the outer face. [2]
- Count the faces, including the outer face, of the planar graph shown, and verify Euler's formula.

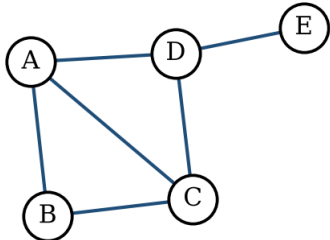
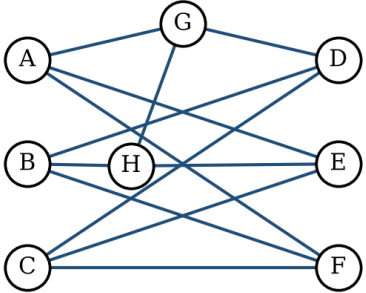


- State the number of vertices and edges of K_5 and of $K_{3,3}$. [2]
- State Kuratowski's theorem for non-planar graphs. [2]

- 7 A connected planar graph has 15 edges and 8 faces, including the outer face. Find the number of vertices. [2]
- 8 For a simple connected planar graph with $v \geq 3$ vertices, $e \leq 3v - 6$. Use this to show that K_5 is not planar. [3]
- 9 For a simple connected planar bipartite graph with $v \geq 3$ vertices, $e \leq 2v - 4$. Use this to show that $K_{3,3}$ is not planar. [3]
- 10 The graph shown is drawn with two edges crossing. Show that it is planar by describing how to redraw it with no crossings. [3]



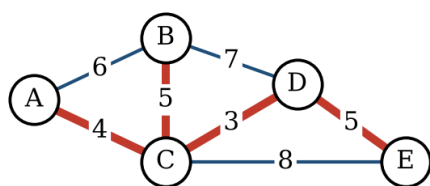
- 11 A connected planar graph has 12 faces, including the outer face, and every face is bounded by exactly 3 edges.
(a) Find the number of edges. [2]
(b) Find the number of vertices. [1]
- 12 The graph $K_{2,3}$ has 5 vertices and 6 edges.
Show that if it is planar then it has 3 faces, and draw it with no crossings by describing a planar drawing. [3]
- 13 A graph is obtained from K_5 by placing one extra vertex in the middle of an edge, so that the edge is replaced by two edges.
(a) Write down the number of vertices and edges of the new graph. [2]
(b) Explain why the new graph is not planar. [1]
- 14 A graph has vertices A, B, C, D, E, F, G. It has the Hamiltonian cycle A, B, C, D, E, F, G, A and also the edges AC, AD, BF and EG.
(a) Use the planarity algorithm to determine whether the graph is planar. You must make your working clear. [5]
(b) Hence write down which edges are drawn inside the cycle and which outside, and state your conclusion. [1]
- 15 A graph has vertices A, B, C, D, E, F and edges AB, BC, CD, DE, EF, FA, AD, BE, CF.
(a) Use the planarity algorithm with the Hamiltonian cycle ABCDEFA to show that the graph is not planar. [5]
(b) Identify the graph by name. [1]
- 16 A planar graph has c connected components, v vertices, e edges and f faces (including the outer face). The formula $v - e + f = 1 + c$ holds.
(a) A planar graph with 3 components has 14 vertices and 12 edges. Find the number of faces. [2]
(b) A planar graph has 20 vertices and 18 edges and is a forest (no cycles) of c trees. Find c , and check using the formula that it has only one face. [3]

- 17** A simple connected planar graph has $v \geq 3$ vertices, e edges and f faces. Show that $e \leq 3v - 6$. [5]
- 18** A planar graph has 10 vertices and every vertex has degree at least 5. Show that this is impossible. [4]
- 19** The planar graph shown has a pendant edge DE.
The degree of a face is the number of edges on its boundary, with an edge counted twice if the face lies on both sides of it.
- 
- (a) Find the number of faces and use Euler's formula to check your answer. [2]
(b) Find the degree of each face and verify that the face degrees sum to $2e$. [3]
- 20** $K_{3,3}$ has one edge removed to give a graph H. [1]
(a) Write down the number of vertices and edges of H. [1]
(b) Given that H is planar, find its number of faces. [1]
(c) Explain why every face of H has degree exactly 4. [3]
- 21** Prove that K_5 is not planar, using Euler's formula. [6]
- 22** Prove that $K_{3,3}$ is not planar, using Euler's formula and the fact that $K_{3,3}$ is bipartite. [6]
- 23** Prove that every simple connected planar graph has a vertex of degree at most 5. [6]
- 24** G is a simple graph with 11 vertices. Prove that G and its complement cannot both be planar. [6]
- 25** A student writes: "A simple connected graph with v vertices and e edges satisfying $e \leq 3v - 6$ must be planar."
(a) Show that $K_{3,3}$ satisfies $e \leq 3v - 6$. [2]
(b) Explain why this shows that the student is wrong. [2]
(c) Show that $K_{3,3}$ fails the sharper inequality for bipartite planar graphs, $e \leq 2v - 4$. [2]
- 26** The graph shown has 8 vertices and 12 edges.
- 
- (a) Delete the edge GH. Show that the resulting graph is a subdivision of $K_{3,3}$, by stating the two sets of vertices. [4]
(b) Hence explain why the original graph is not planar. [2]

- 27** Prove Euler's formula $v - e + f = 2$ for every connected planar graph, by induction on the number of edges e . [10]
- 28** The girth of a graph is the length of its shortest cycle. Let G be a simple connected planar graph with v vertices, e edges and girth at least 5 (and at least one cycle).
- (a) Prove that $e \leq \frac{5(v-2)}{3}$. [5]
- (b) The Petersen graph has 10 vertices, 15 edges and girth 5. Show that it is not planar. [3]
- 29** A convex polyhedron has f faces, each an n -gon, and every vertex has degree m ($m \geq 3$, $n \geq 3$), so that v , e and f satisfy Euler's formula $v - e + f = 2$.
- (a) Show that $2e = nf = mv$. [2]
- (b) Hence show that $\frac{1}{m} + \frac{1}{n} = \frac{1}{2} + \frac{1}{e}$. [3]
- (c) Deduce that $\frac{1}{m} + \frac{1}{n} > \frac{1}{2}$, and find all possible pairs (m, n) , giving the number of faces for each. [4]
- 30** A connected planar graph has every vertex of degree 3 and every face either a pentagon or a hexagon. Let it have p pentagonal faces and h hexagonal faces.
- (a) Show that $e = \frac{5p+6h}{2}$ and $v = \frac{5p+6h}{3}$. [2]
- (b) Use Euler's formula to prove that $p = 12$, whatever the value of h . [5]
- (c) A football has this structure with $h = 20$. Find the number of vertices and edges. [3]

Minimum Spanning Trees

- 1** State what is meant by a spanning tree of a connected graph. [2]
- 2** A connected network has 9 vertices. Write down the number of edges in any spanning tree of the network. [1]
- 3** State what is meant by a minimum spanning tree of a weighted network. [2]
- 4** The weighted network shown has a spanning tree highlighted in bold. Find its total weight. [2]



- 5** A connected network has 11 edges and 6 vertices. Find the number of edges that must be removed to form a spanning tree. [2]
- 6** A network has a single edge e whose weight is smaller than that of every other edge. Explain why e must belong to every minimum spanning tree. [2]

- 7 The table shows the distances in km between five villages A to E. Find the minimum total length of cable that connects all five villages, and state which links are used.

	A	B	C	D	E
A	-	12	9	15	11
B	12	-	8	14	7
C	9	8	-	10	13
D	15	14	10	-	6
E	11	7	13	6	-

[3]

- 8 Explain why a spanning tree can never contain all the edges of a cycle.

[2]

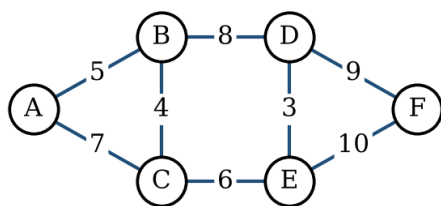
- 9 How many different spanning trees does a cycle on 5 vertices have? Explain your answer.

[3]

- 10 A network has edges AB 3, BC 3, CD 2, DE 4, EA 4 and AC 5. Show that the network has more than one minimum spanning tree by finding two of them.

[3]

- 11 The weighted network shows possible cable routes (lengths in km) between six offices A to F. Find the minimum total length of cable needed to connect all the offices, giving the routes used.



[4]

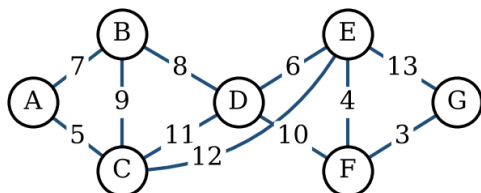
- 12 A spanning tree of maximum total weight is a maximum spanning tree. Find the maximum spanning tree of the network with edges AB 6, AC 9, AD 4, BC 7, BD 8, CD 5, CE 3, DE 10, and state its weight.

[4]

- 13 A network on vertices A to E has edges AB 5, BC 7, CD 4, DE 6, EA 9 and AC 8. A new vertex X is added with edges XA 6, XB 3 and XC 8. Find the weight of the minimum spanning tree before and after X is added.

[4]

- 14 The network shown has seven vertices. Find a minimum spanning tree and its total weight, listing the edges in the order in which you select them.



[5]

- 15 A network has edges AB 4, AC 8, BC 6, BD 9, CD 3, CE 5, DE 7, EF 2, DF 10 (all weights distinct).
- List the edges that join the set A, B to the rest of the network, and identify the lightest.
 - Explain why the lightest edge leaving any set of vertices must belong to the minimum spanning tree, and verify this for BC by finding the minimum spanning tree.

[2]

[4]

- 16** A network has edges AB 6, AC 4, BC 5, BD 9, CD 8, CE 7, DF 3, EF 10, DE 2. A route from A to F is to be chosen so that the largest edge weight on the route is as small as possible. Use a minimum spanning tree to find the route and state this largest weight. [5]

- 17** A network has edges AB 4, AC 6, BC 8, BD 5, CD 9, CE 7, DE 10. Its minimum spanning tree has weight 22 and includes the edge AB. The weight of AB is increased to x . Find the range of values of x for which the original minimum spanning tree is still a minimum spanning tree. [5]

- 18** The graph K_4 with one edge removed has 4 vertices and 5 edges (a square ABCD with one diagonal AC). Show that it has exactly 8 spanning trees. [5]

- 19** A network has edges AB 5, AC 8, BC 6, BD 11, CD 7, CE 9, DE 12, EF 4, DF 10. The edge DE must be included in the cable network for safety reasons. Find the minimum total weight of a spanning tree that contains DE, and compare it with the unconstrained minimum. [5]

- 20** Six villages A to F are to be connected by new water pipes along the routes shown, with lengths in km: AB 6, AC 9, BC 5, BD 7, CE 8, DE 4, DF 9, EF 6. Pipes cost £1200 per km to lay. Find the minimum cost of connecting all six villages. [5]

- 21** The table shows the cost in £ thousand of linking pairs of eight sites A to H (a dash means no direct link is possible).

	A	B	C	D	E	F	G	H
A	-	14	21	-	-	-	-	-
B	14	-	11	16	-	-	-	-
C	21	11	-	9	18	-	-	25
D	-	16	9	-	12	20	-	-
E	-	-	18	12	-	7	15	-
F	-	-	-	20	7	-	10	13
G	-	-	-	-	15	10	-	8
H	-	-	25	-	-	13	8	-

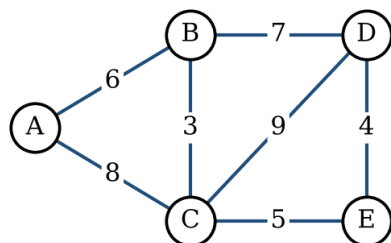
- (a) Find the minimum spanning tree and its total cost, listing the links in the order selected. [5]
 (b) The link between D and E is unavailable. Find the new minimum total cost. [3]
- 22** Prove that if all the edge weights of a connected network are distinct, then the minimum spanning tree is unique. [6]
- 23** Prove that every spanning tree of a connected graph with n vertices has exactly $n - 1$ edges. [6]
- 24** The network has edges AB 7, AC 5, BC 9, BD 6, CE 8, DE 4, DF 10, EF 11, EG 12, FG 3.
 (a) Find the minimum spanning tree and its weight. [4]
 (b) The link BD fails. Find the cheapest way to reconnect the network and the new minimum total weight. [3]
- 25** A student claims: "The path between two vertices in a minimum spanning tree is always the shortest path between them in the network."
 The network has edges AB 3, BC 3 and AC 5.
 (a) Find the minimum spanning tree and its weight. [2]
 (b) Find the length of the path from A to C in the tree, and the length of the shortest path from A to C in the network. [2]
 (c) Explain what this shows about the student's claim. [2]

- 26** The network has edges AB 4, AC 6, BC 5, BD 8, CD 7, CE 9, DE 3, DF 11, EF 10. The second best spanning tree is a spanning tree of the smallest weight that is greater than the weight of the minimum spanning tree.
- (a) Find the minimum spanning tree and its weight. [3]
- (b) Find the weight of the second best spanning tree, and state the change made to the minimum spanning tree to obtain it. [4]
-
- 27** Let G be a connected network with distinct edge weights, and let S be any non-empty proper subset of the vertices. Prove that the lightest edge e joining a vertex in S to a vertex not in S belongs to every minimum spanning tree of G . [9]
-
- 28** Let C be a cycle in a connected network with distinct edge weights, and let e be the heaviest edge of C . Prove that e belongs to no minimum spanning tree. [8]
-
- 29** The matrix-tree theorem states that the number of spanning trees of a connected graph equals the determinant of its Laplacian matrix $L = D - A$ (D is the diagonal matrix of vertex degrees and A the adjacency matrix) after deleting any one row and the corresponding column.
- (a) Show that the reduced matrix for K_4 is $\begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}$, and find its determinant, so that K_4 has 16 spanning trees. [3]
- (b) A square ABCD with the diagonal AC has A and C of degree 3 and B and D of degree 2. Show that deleting the row and column of D gives the matrix $\begin{pmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 3 \end{pmatrix}$, and find the number of spanning trees. [3]
- (c) Use the matrix $\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$ to find the number of spanning trees of the cycle on 5 vertices. [2]
-
- 30** In the complete graph K_4 there are 16 spanning trees and 6 edges. By counting pairs (T, e) , where T is a spanning tree and e is an edge of T , show that each edge of K_4 belongs to exactly 8 spanning trees, and deduce the number of spanning trees that do not contain a given edge. [8]

Kruskal's Algorithm

- 1** Kruskal's algorithm is used to find a minimum spanning tree. State the first two steps of the algorithm. [2]
-
- 2** A network has edges AB 7, BC 4, CD 9, DA 3, AC 6 and BD 8. List the edges in the order in which Kruskal's algorithm considers them. [2]
-
- 3** In Kruskal's algorithm, state when an edge is rejected. [1]
-
- 4** Use Kruskal's algorithm on the network with edges AB 5, BC 3, CD 4, AC 6, BD 7 and AD 8 to find a minimum spanning tree and its weight. [3]
-
- 5** A network has 8 vertices. How many edges does Kruskal's algorithm accept before it stops? [1]
-
- 6** Kruskal's algorithm has accepted AB 2 and CD 3. The next edges in order are BD 5 and AC 6. State, with reasons, whether each is accepted. [2]

- 7 Use Kruskal's algorithm to find the minimum spanning tree of the network shown. State the order in which edges are accepted or rejected, and give the weight of the tree.



[4]

- 8 The table gives the distances in km between six towns A to F. Use Kruskal's algorithm to find the shortest total length of road that links all six towns.

	A	B	C	D	E	F
A	-	12	9	20	-	-
B	12	-	7	-	14	-
C	9	7	-	10	-	16
D	20	-	10	-	8	13
E	-	14	-	8	-	11
F	-	-	16	13	11	-

[4]

- 9 A network has edges AB 4, BC 4, CD 5, DA 5 and AC 6. Use Kruskal's algorithm to find two different minimum spanning trees, and give their common weight.

[4]

- 10 Using Kruskal's algorithm on a network, the edges AB, DE and BC have been accepted. The next edge in the sorted list is AC. Explain why AC is rejected and state what has to be true of the vertices of an accepted edge.

[3]

- 11 Six farms A to F are to be linked by underground cables along the routes AB 5, BC 8, AD 7, CE 9, DE 6, EF 4, DF 10, BE 11 (lengths in km). Cable costs £6500 per km. Use Kruskal's algorithm to find the minimum cost of linking all six farms.

[4]

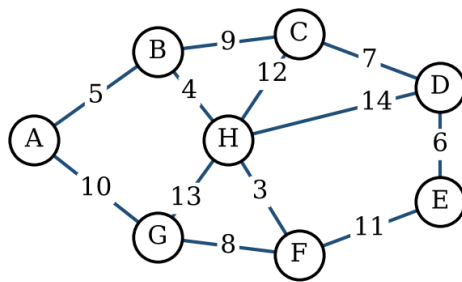
- 12 A network with 7 vertices has edges AB 3, BC 5, CD 4, DE 6, EF 2, FG 8, GA 7, BE 9, AD 10, CF 11 and CG 12. Apply Kruskal's algorithm. State the edge after which the algorithm can stop, and how many edges were examined in total, including those rejected.

[3]

- 13 The edges of a network are listed in order of weight: AB 2, CD 3, AC 4, BD 5, DE 6, EF 7, CE 8, AF 9, BF 10, DF 11. State which of the first seven edges Kruskal's algorithm accepts, giving a brief reason for each rejection, and find the total weight of the tree.

[4]

- 14** Use Kruskal's algorithm to find the minimum spanning tree of the network shown. List the edges in the order considered, stating whether each is accepted or rejected, and give the total weight.



[6]

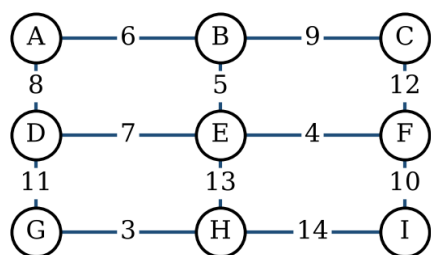
- 15** The table shows the cost, in £ hundreds, of connecting seven sites A to G by direct links (a dash means no link is possible). Use Kruskal's algorithm to find a minimum cost network and state its cost.

	A	B	C	D	E	F	G
A	-	9	14	-	-	-	-
B	9	-	8	12	-	-	-
C	14	8	-	13	10	-	-
D	-	12	13	-	7	16	-
E	-	-	10	7	-	6	15
F	-	-	-	16	6	-	11
G	-	-	-	-	15	11	-

[6]

- 16** A student uses Kruskal's algorithm on the network with edges AB 3, BC 4, AC 5, CD 6, DE 2, CE 7 and selects DE, AB, BC, AC, CD, saying the total weight is 20. Identify the student's error, and find the correct minimum spanning tree and its weight. [5]
- 17** A network has edges AB 7, AC 5, BC 6, BD 9, CD 8, CE 4, DE 10, EF 3, DF 11. The edge BD must be included in the network. Use Kruskal's algorithm, starting with BD already selected, to find the minimum total weight of such a spanning tree, and compare with the unconstrained minimum. [5]
- 18** A network has edges AB 6, AC 4, BC 7, BD 5, CE 8, DE 9, DF 10, EF 3, EG 12, FG 11. Find the weight of the minimum spanning tree. Find also the weight of the minimum spanning tree of the network formed by deleting vertex A and its edges. [5]
- 19** A network has edges AB 4, BC 7, CD 5, DE 6, EA 9, AC 8 and BD 10. The minimum spanning tree is found using Kruskal's algorithm. Every edge weight w is then replaced by $2w + 5$.
 (a) Find the minimum spanning tree and its weight for the original network. [3]
 (b) Explain why the minimum spanning tree has the same edges after the change, and find the new weight. [2]
- 20** Kruskal's algorithm is applied to the network with edges AB 2, CD 3, BC 4, AD 5, DE 6, EF 7, CF 8, BE 9 and AF 10.
 (a) Find the minimum spanning tree and its weight. [3]
 (b) State how many edges are examined, and list the edges that are rejected before the algorithm stops. [2]

- 21** Use Kruskal's algorithm to find the minimum spanning tree of the network shown, and its weight. Show the order of consideration, with a reason for each rejection.



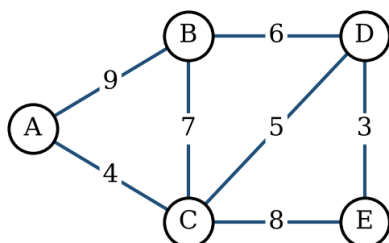
[7]

- 22** A network has edges AB 3, BC 3, CA 3, DE 5, EF 5, FG 5, GD 5, CD 8, AD 9 and BG 10.
- (a) Use Kruskal's algorithm to find the weight of a minimum spanning tree. [3]
- (b) Show that the network has exactly 12 minimum spanning trees. [4]
- 23** Prove that the edges selected by Kruskal's algorithm on a connected network always form a spanning tree. [6]
- 24** A student applies Kruskal's algorithm to the network with edges AB 5, DE 4, BC 6, EA 8, CD 7 and AC 9. The student lists the edges as DE 4, AB 5, BC 6, EA 8, CD 7, AC 9, accepts the first four and says the weight is 23.
- (a) Identify the error in the student's list. [2]
- (b) Find the correct minimum spanning tree and its weight. [3]
- (c) Explain why it is important to consider the edges in ascending order. [1]
- 25** Six villages A to F have possible cable routes AB 6, AC 9, BC 8, BD 11, CD 7, CE 10, DE 5, DF 12, EF 4 (km). Cables already exist along AB, CD and EF. Use Kruskal's algorithm, treating the existing cables as already selected, to find the least extra length of cable needed to connect all six villages, and which routes to use. [7]
- 26** A network on vertices A to H has edges AB 5, CD 6, EF 7, GH 8, BC 9, DE 10, FG 11, AH 12 and BE 13. Kruskal's algorithm is applied.
- (a) List the edges in the order in which they are accepted. [2]
- (b) Write down the connected components after the first four edges have been accepted, and state how many components there are after k edges have been accepted. [3]
- (c) Give the total weight of the tree, and state which edges are not examined. [1]
- 27** A connected network has distinct edge weights. Prove that Kruskal's algorithm produces a minimum spanning tree. [10]
- 28** The bottleneck of a spanning tree is the largest edge weight in the tree. Prove that a spanning tree found by Kruskal's algorithm has the smallest possible bottleneck among all spanning trees of the network. [8]
- 29** A network has edges AB 5, BC 8, CD 6, DE 9, EF 4, FA 7, BE 10, CF 11 and also an edge AD of weight x , where x is a positive integer.
- (a) Find the weight of the minimum spanning tree when the edge AD is not present. [3]
- (b) Find the values of x for which the edge AD is in a minimum spanning tree (including cases of a tie). [4]
- (c) Find, in terms of x , the weight of the minimum spanning tree for $x < 8$. [3]

- 30** A network has edges AB 3, BC 3, CA 3, BD 5, CD 5, DE 6 and AE 9.
- (a) Show that every minimum spanning tree contains exactly two of the three edges AB, BC, CA, and state how many choices there are. [3]
- (b) Show that exactly one of BD and CD is then accepted, and state the number of choices. [3]
- (c) Show that DE 6 is always accepted and AE 9 always rejected, and hence find the number of minimum spanning trees and their common weight. [3]

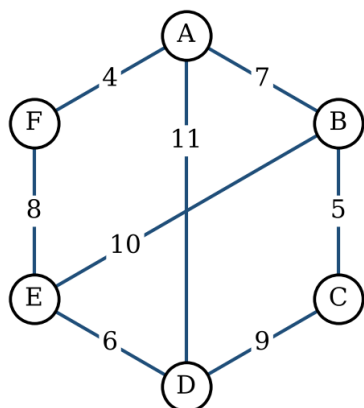
Prim's Algorithm

- 1** State the first two steps of Prim's algorithm for a weighted network. [2]
- 2** A network has edges AB 5, AC 3, BC 4, CD 6 and BD 8. Use Prim's algorithm starting at A to find a minimum spanning tree and its weight. [3]
- 3** At each stage of Prim's algorithm, which edges are candidates to be added to the tree? [1]
- 4** In the matrix form of Prim's algorithm you choose a starting vertex and delete its row. State what you do next, and how the algorithm continues. [2]
- 5** A network has 10 vertices. How many edges are in the minimum spanning tree found by Prim's algorithm? [1]
- 6** True or false: for a network with distinct edge weights, Prim's algorithm and Kruskal's algorithm give the same minimum spanning tree. Give a reason. [2]
- 7** Use Prim's algorithm, starting at A, to find a minimum spanning tree of the network shown. State the order in which the edges are added and the total weight.



[4]

- 8** Use Prim's algorithm on the network shown, first starting at C and then starting at E. State the order in which edges are added in each case, and comment on the two minimum spanning trees.



[4]

- 9** The table gives the cost, in £ thousand, of laying cable between five buildings A to E. Use Prim's algorithm, starting at A, to find the minimum cost of connecting all five buildings, and give the order in which the vertices are added.

	A	B	C	D	E
A	-	8	5	12	-
B	8	-	7	-	9
C	5	7	-	6	10
D	12	-	6	-	4
E	-	9	10	4	-

[4]

- 10** Six villages are to be linked by a new water main along the routes shown by the edges AB 6, AC 9, BC 4, BD 7, CE 8, DE 5, DF 10 and EF 3 (km). Pipes cost £2400 per km. Use Prim's algorithm starting at A to find the minimum cost.

[4]

- 11** Prim's algorithm is applied to a network with edges AB 7, AC 4, AD 9, BC 6, BE 12, CD 5, CE 8, DF 10 and EF 3, starting at A.

(a) Write down the first edge added to the tree.

[1]

(b) List all the candidate edges at the second stage, and state which is added.

[2]

- 12** A network has edges AB 4, AC 4, BC 6, BD 5 and CD 5. Use Prim's algorithm starting at A. Explain what happens when there is a tie, and show that the total weight is the same whichever choice is made.

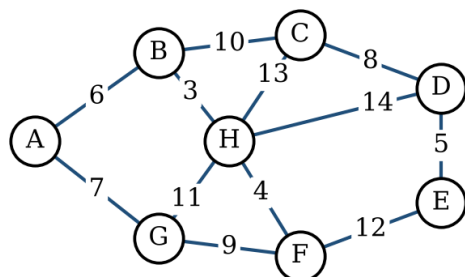
[3]

- 13** The table shows the distances between six sites A to F. Use the matrix form of Prim's algorithm starting at E to find the minimum spanning tree, giving the order in which the vertices are selected, and the total weight.

	A	B	C	D	E	F
A	-	14	9	-	-	-
B	14	-	11	7	-	10
C	9	11	-	-	13	-
D	-	7	-	-	6	12
E	-	-	13	6	-	8
F	-	10	-	12	8	-

[4]

- 14** Use Prim's algorithm starting at A to find the minimum spanning tree of the network shown. State the order in which the vertices are added and the total weight of the tree.



[6]

- 15** The table shows the costs, in £ hundreds, of linking seven towns A to G. Use Prim's algorithm starting at D to find a minimum cost connection, and state the order in which the vertices are selected.

	A	B	C	D	E	F	G
A	-	11	8	-	-	-	-
B	11	-	9	15	-	-	-
C	8	9	-	16	7	-	-
D	-	15	16	-	12	14	-
E	-	-	7	12	-	6	13
F	-	-	-	14	6	-	10
G	-	-	-	-	13	10	-

[6]

- 16** A network has edges AB 12, AC 7, BC 9, BD 15, CD 10, CE 6, DE 11, DF 13, EF 8.

- (a) Use Prim's algorithm starting at A. Give the edges in the order added and the total weight. [3]
 (b) Repeat starting at F, and state what you notice. [2]

- 17** A network has vertices A to F and edges AB 5, BC 3, CD 6, AD 8 and EF 2. Prim's algorithm is started at A.

- (a) List the edges added by the algorithm. [2]
 (b) Explain why the algorithm stops with only four vertices in the tree, and what this tells you about the network. [2]
 (c) State the smallest number of extra edges needed to make the network connected. [1]

- 18** The minimum spanning tree of a network with edges AB 5, BC 7, CD 4, DE 6, AE 9 and AC 8 is found using Prim's algorithm. A new vertex X is added with edges XA 6, XB 3 and XD 9.

Use Prim's algorithm starting at X to find the new minimum spanning tree and its weight, and state the change in weight.

[5]

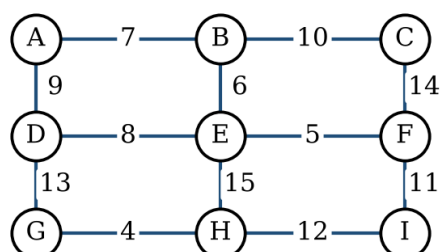
- 19** Prim's algorithm is applied to the network with edges AB 7, AC 5, BC 8, BD 9, CD 6, CE 11 and DE 4, starting at A. At each stage, state how many candidate edges there are (edges from the tree to a vertex not yet in the tree), and the edge added.

[5]

- 20** A network has edges AB 7, AC 4, BC 5, BD x , CD 9, CE 10, DE 6 and EF 3, where x is a positive integer. Prim's algorithm, starting at A, adds the edges in the order AC, BC, BD, DE, EF, and never has to choose between two edges of equal weight.

- (a) Find the possible values of x . [3]
 (b) Find, in terms of x , the total weight of the minimum spanning tree. [2]

- 21** Use Prim's algorithm starting at E to find the minimum spanning tree of the network shown, giving the order in which the vertices are added, and the weight of the tree.



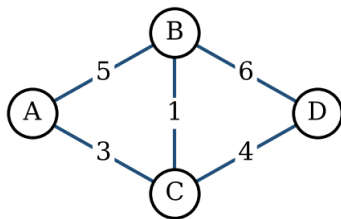
[7]

- 22** A network has edges AB 4, AC 4, BD 6, CD 6, CE 9, DE 8, EF 3 and DF 10.
 (a) Use Prim's algorithm starting at A to find a minimum spanning tree, stating each choice you make where there is a tie. [4]
 (b) Show that the network has exactly two minimum spanning trees. [2]
-
- 23** Prove that the edges selected by Prim's algorithm on a connected network always form a spanning tree. [6]
-
- 24** A student applies Prim's algorithm, starting at A, to the network with edges AB 4, AC 5, BC 9, CD 6, DE 2, EF 7 and DF 8. The student adds the edges in the order AB 4, DE 2, AC 5, CD 6, EF 7.
 (a) Explain the student's error. [2]
 (b) Use Prim's algorithm correctly, starting at A, and give the edges in the order added. [3]
 (c) Find the total weight and explain why in this case the student's final tree has the same weight. [1]
-
- 25** A network has edges AB 5, AC 8, BC 6, BD 11, CD 7, CE 12, DE 9, DF 10 and EF 4. Use Prim's algorithm starting at A to find the minimum spanning tree. The route BC is then closed for repairs. Use Prim's algorithm again to find the new minimum spanning tree and the increase in total weight. [7]
-
- 26** A network has edges AB 4, AC 6, BC 5, BD 9, CD 8, CE 7, DE 3 and EF 10. Prim's algorithm is applied starting at A. After three vertices (A, B and one other) have been added, the tree contains A, B and C.
 (a) Find the order in which A, B and C are added, with the edges used. [2]
 (b) List the edges joining A, B, C to the other vertices, and state which edge is added next. [2]
 (c) Complete the algorithm and give the total weight. [2]
-
- 27** A connected network has distinct edge weights. Prove that Prim's algorithm produces a minimum spanning tree. [10]
-
- 28** Prim's algorithm is applied to the complete network K_n with n vertices. At the stage when the tree contains k vertices, the algorithm examines every edge that joins a vertex in the tree to a vertex outside the tree.
 (a) Show that the number of edges examined at this stage is $k(n - k)$. [2]
 (b) Show that the total number of edges examined over all stages is $\frac{n(n^2 - 1)}{6}$. [5]
 (c) Evaluate this total for $n = 10$, and compare it with the number of edges in K_{10} . [2]
-
- 29** Six sensors A(0, 0), B(4, 0), C(8, 0), D(2, 3), E(6, 3), F(4, 6) are positioned on a grid, with units in metres. A cable is to connect all six sensors, and the cable between two sensors has length equal to the straight line distance between them.
 (a) Find the exact distances AB, AD, BD, BE, DE and DF. [2]
 (b) Use Prim's algorithm starting at A to find the minimum length of cable, giving your answer to 3 significant figures, and list the links used. [5]
 (c) Explain why the minimum spanning tree is not unique here. [2]
-
- 30** Prove that Prim's algorithm and Kruskal's algorithm produce the same spanning tree when all the edge weights of a connected network are distinct. [8]
-

Shortest Path Algorithms

- 1** State what Dijkstra's algorithm finds in a weighted network. [2]
-
- 2** In Dijkstra's algorithm each vertex is given a box containing the order of labelling, the final label and working values. State what is meant by the final label of a vertex. [2]
-

- 3 Find, by inspection, the shortest path from A to D in the network shown and state its length.



[2]

- 4 State what Floyd's algorithm finds that Dijkstra's algorithm (applied once) does not.

[2]

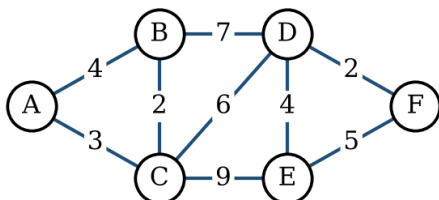
- 5 Floyd's algorithm is applied to a network with 6 vertices. How many iterations (passes through the distance matrix) are needed?

[1]

- 6 Explain why, in a directed network, the shortest distance from A to B may be different from the shortest distance from B to A.

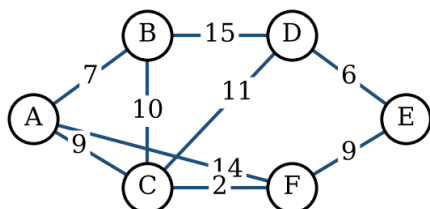
[2]

- 7 Use Dijkstra's algorithm to find the shortest path from A to F in the network shown, and state its length. Show the order of labelling and the working values at each vertex.



[5]

- 8 Use Dijkstra's algorithm to find the shortest path from A to E in the network shown, stating the length of the path and showing your working.

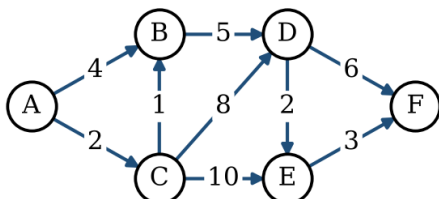


[5]

- 9 A delivery van must travel from A to F, calling at C on the way. The roads are AB 6, AC 4, BC 3, BD 5, CD 9, CE 7, DF 2 and EF 6 (km). Find the shortest route from A to F that passes through C, and say how much longer it is than the shortest route from A to F.

[4]

- 10 The directed network shown has arcs with the weights marked. Use Dijkstra's algorithm to find the shortest path from A to F and its length.



[5]

- 11** The distance matrix for a network with vertices A, B, C, D is shown, where ∞ means no direct edge.

$$D_0 = \begin{pmatrix} 0 & 5 & 8 & \infty \\ 5 & 0 & \infty & 7 \\ 8 & \infty & 0 & 3 \\ \infty & 7 & 3 & 0 \end{pmatrix}$$

Write down the matrix after the first iteration of Floyd's algorithm (using vertex A as the intermediate vertex).

[4]

- 12** Floyd's algorithm is of order n^3 , where n is the number of vertices of the network. A computer takes 0.8 seconds to apply Floyd's algorithm to a network with 50 vertices.

- (a) Estimate the time the computer takes to apply Floyd's algorithm to a network with 200 vertices.
- (b) Explain why your answer to (a) is only an estimate.

[3]

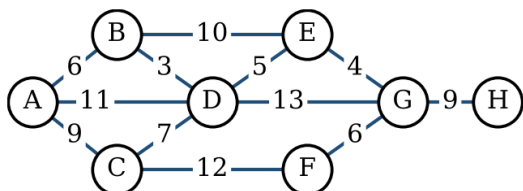
[1]

- 13** The table shows the direct distances in km between six towns (a dash means no direct road). Use Dijkstra's algorithm to find the shortest route from A to F.

	A	B	C	D	E	F
A	-	8	5	14	-	-
B	8	-	4	-	9	-
C	5	4	-	6	11	-
D	14	-	6	-	4	10
E	-	9	11	4	-	7
F	-	-	-	10	7	-

[5]

- 14** Use Dijkstra's algorithm to find the shortest path from A to H in the network shown. Show all the working values, the order of labelling, and state the length of the shortest path.



[7]

- 15** A network has edges AB 5, AC 2, BC 10, BD 6 and CD 3. The route table has entry (i, j) equal to the first vertex after i on the best route found so far from i to j , and at the start every entry in column j is j .

- (a) Use Floyd's algorithm to find the shortest distance between every pair of vertices. Show the distance table and the route table after each iteration.
- (b) Use the final route table to find the shortest route from D to A, and state its length.

[7]

[2]

- 16** Use Dijkstra's algorithm starting at A to find the shortest distance from A to every other vertex of the network with edges AB 5, AC 8, BC 2, BD 6, CE 4, DE 3, DF 7 and EF 9. Hence find the vertex furthest from A and the route to it.

[6]

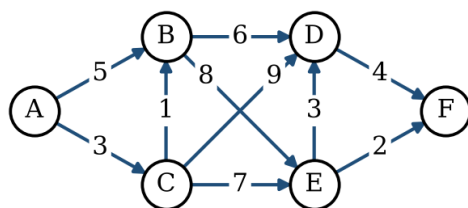
- 17** The edges AB 5, AC 8, BC 2, BD 9, CD 4, CE 10, DE 3, DF 7 and EF 6 give journey times in minutes. Find the quickest route from A to F. The road CD is then closed. Find the new quickest route and the extra time taken.

[6]

- 18** A network has edges AB 4, AC 6, BC 2, BD 5, CD 3, DE 6 and CE 9. Use Dijkstra's algorithm from A to find the length of the shortest path to E, and find all the shortest paths.

[5]

- 19** The directed network shows one-way roads, with journey times in minutes. Use Dijkstra's algorithm to find the quickest route from A to F and its time.



[6]

- 20** A company wishes to place a depot at one of five towns A to E so that the furthest town is as close as possible. The roads are AB 5, BC 7, CD 6, DE 4, EA 9, BD 8 and AC 10 (km). Find, using shortest distances, the best town for the depot, and the greatest distance from it.

[6]

- 21** A network has vertices A to E and edges AB 6, AC 9, BC 4, BD 8, CD 5, CE 11 and DE 3. Use Floyd's algorithm to find the matrix of shortest distances between all pairs of vertices, showing the initial matrix and the matrix after each of the first three iterations. Hence find the shortest distance from A to E.

[8]

- 22** A network has edges AB 6, AC 9, BC x , BD 8, CD 4, CE 10 and DE 5, where x is a positive integer.

(a) Given that the shortest distance from A to D is 13, find the possible values of x .

[3]

(b) For $x = 2$, use Dijkstra's algorithm to find the shortest path from A to E and its length.

[5]

- 23** A directed network has arcs A to B of weight 2, B to C of weight -3 , C to A of weight 0, C to D of weight 4 and A to D of weight 5. Floyd's algorithm is applied, with the vertices in the order A, B, C, D.

(a) Write down the initial distance matrix, with ∞ for no arc and 0 on the leading diagonal.

[1]

(b) Complete the algorithm, and use the leading diagonal of the final matrix to show that the network has a negative cycle. Identify the cycle.

[6]

- 24** A student looks for the shortest path from A to E in the network with edges AB 2, AC 3, BE 9, CD 1 and DE 1. The student moves from A to the nearest vertex B (distance 2), then from B takes the only remaining route to E, and says that the shortest path is A, B, E of length 11.

(a) Explain the error in the student's method.

[2]

(b) Use Dijkstra's algorithm to find the true shortest path and its length.

[4]

- 25** The table gives the length (km) and toll (£) of each road between five towns: AB 20 km £5, AC 35 km £0, BC 10 km £2, BD 30 km £0, CD 15 km £4, CE 25 km £0, DE 11 km £3. A lorry costs £0.40 per km to run. Find the cheapest route from A to E, and compare it with the shortest route by distance.

(a) Work out the total cost, in £, of using each road.

[2]

(b) Use Dijkstra's algorithm with these costs to find the cheapest route from A to E and its cost.

[3]

(c) Find the shortest route by distance and say whether it is also the cheapest.

[2]

- 26** A directed network has arcs A to B of weight 2, A to C of weight 3, C to B of weight -2 and B to D of weight 1. A student applies Dijkstra's algorithm from A, giving each vertex a permanent label in turn, and never changing a permanent label.

(a) Show the labels the student obtains for B, C and D.

[3]

(b) Find the true shortest distances from A to B and from A to D.

[2]

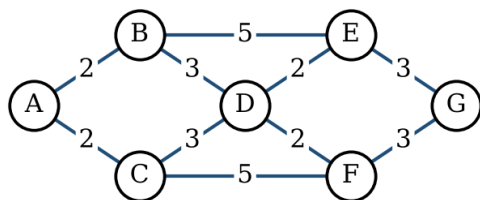
(c) Explain why Dijkstra's algorithm fails here.

[1]

- 27** A network has non-negative arc weights. Dijkstra's algorithm labels vertices permanently one at a time, starting at A. Prove that when a vertex v is made permanent, its label is the length of the shortest path from A to v .

[10]

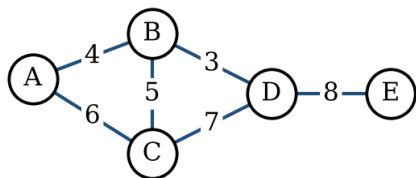
- 28** Floyd's algorithm updates a matrix D so that after iteration k , the entry $D(i, j)$ is the length of the shortest path from i to j that uses only vertices $1, 2, \dots, k$ as intermediate vertices (for $k = 0$, only direct arcs). The update rule is $D(i, j) = \min(D(i, j), D(i, k) + D(k, j))$.
- Prove, by induction on k , that this property holds for every k . [6]
 - Hence explain why, after all n iterations, $D(i, j)$ is the shortest distance between i and j . [2]
 - State, with a reason, the number of comparisons Floyd's algorithm makes in total for a network with n vertices. [2]
-
- 29** A courier must travel from A to F along the roads AB 1, BC 1, CD 1, DF 1, AE 3, EF 3 and AF 8 (km), but is allowed to use at most k roads.
- Use Dijkstra's algorithm to find the shortest distance from A to F with no restriction, and the number of roads used. [3]
 - Let $d_k(v)$ be the shortest distance from A to v using at most k roads. Find $d_k(F)$ for $k = 1, 2, 3$ and 4 . [4]
 - Explain why Dijkstra's algorithm cannot be used directly to answer the question when k is restricted, and describe a method that can. [2]
-
- 30** The network shown has edges with the weights marked. Find the length of the shortest path from A to G , and the number of different shortest paths from A to G , explaining how the number of shortest paths to each vertex can be built up as the vertices are labelled.



[10]

Route Inspection

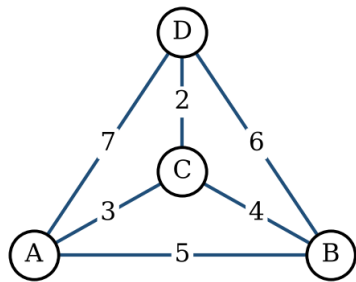
- Describe the route inspection (Chinese postman) problem. [2]
- A connected network has every vertex of even degree and the total weight of its edges is 85. Write down the length of the shortest route that uses every edge and returns to its start, and explain why. [2]
- List the vertices of odd degree in the network shown.



[2]

- A network has 5 vertices of degree 3 and 1 vertex of degree 2, which gives an odd number of odd vertices. Explain why this is impossible. [1]
- A network has exactly two vertices of odd degree. Describe how the shortest closed route that uses every edge can be found. [2]
- A network has four odd vertices A, B, C and D . List the three ways of pairing them. [2]

- 7 Find the length of the shortest route that starts and finishes at A and traverses every edge of the network shown at least once. Show the pairings of the odd vertices.



[5]

- 8 A network has edges AB 6, BC 5, CD 7, DA 4 and AC 8. A route must traverse every edge at least once, starting at A and finishing at C.

(a) Show that A and C are the only vertices of odd degree.

[1]

(b) Find the length of the shortest such route, and explain why no edges need to be repeated.

[3]

- 9 A network has six vertices of odd degree. Find the number of ways of pairing them up.

[2]

- 10 A gritting lorry must travel along every road in the network (distances in km: AB 4, BC 5, CD 3, DE 6, EF 4, FA 7, BE 8), starting and finishing at the depot A. Find the length of the shortest such route.

[5]

- 11 A network has edges AB 5, AC 8, BD 7, CD 4, BC 6, CE 9 and DE 3. A route inspection is needed starting and ending at A. Find the odd vertices, the shortest repeated path between the pairs, and the length of the shortest route.

[5]

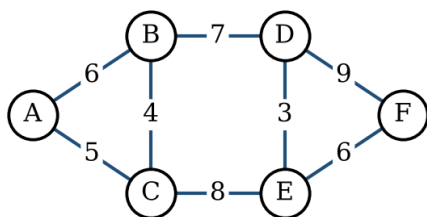
- 12 A network has edges AB 4, BC 6, CD 3, DA 5, BD 9, CE 7 and DE 8. Find the shortest route that traverses every edge at least once and starts and ends at the same vertex.

[5]

- 13 A network has edges AB 3, BC 4, CA 5, CD 2, DE 6 and EC 7. Show that no edges need to be repeated in a route that starts and ends at A and uses every edge, find the length of such a route, and write down one.

[4]

- 14 Find the length of the shortest route that starts and finishes at A and traverses every edge of the network shown at least once. State which edges are repeated.



[6]

- 15 A network has edges AB 5, AC 3, AD 7, BC 4, BD 6 and CD 2 (total weight 27). Find the length of the shortest route that traverses every edge at least once, (a) starting and finishing at A, (b) starting at A and finishing at a different vertex of your choice. State where the route in (b) finishes.

(a) Starting and finishing at A.

[2]

(b) Starting at A and finishing at a different vertex.

[5]

- 16** The table shows the distances in km along the roads between six junctions A to F (a dash means there is no road). A road inspector must travel along every road, starting and finishing at A. Find the length of the shortest route.

	A	B	C	D	E	F
A	-	7	9	-	-	-
B	7	-	6	11	-	-
C	9	6	-	8	5	-
D	-	11	8	-	10	4
E	-	-	5	10	-	12
F	-	-	-	4	12	-

[6]

- 17** A network has edges AB 5, AC 3, AD 7, BC 4, BD 6 and CD 2. Write down a complete route of minimum length, starting and finishing at A, listing the vertices in order, and verify its length.

[5]

- 18** A road network has edges AB 4, BC 5, CD 3, DE 6, EF 4, FA 7 and BE 8, and the shortest route starting and finishing at A that uses every road has length 45. A new road BF of length 3 is built. Find the length of the shortest route starting and finishing at A that uses every road, and say whether it is longer or shorter than before.

[5]

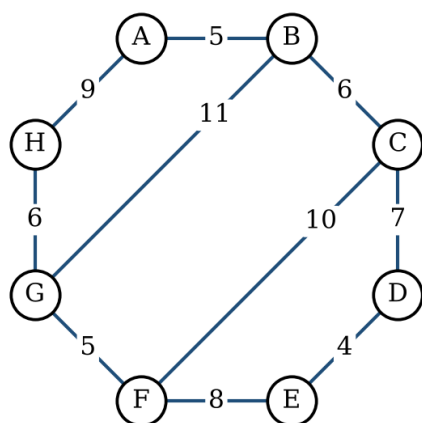
- 19** A network of paths in a park joins six points A, B, C and D, E, F, with the paths AD 3, AE 5, AF 8, BD 6, BE 2, BF 7, CD 9, CE 4 and CF 3 (hundreds of metres). A gardener must walk along every path, starting and finishing at A. Find the length of the shortest route.

[7]

- 20** A network has edges AB 5, AC 8, BD 7, CD 4, BC 6, CE 9 and DE 3. A shortest route that starts and finishes at A and uses every edge is found. Not counting the start and the finish at A, find how many times the route passes through each vertex.

[5]

- 21** Find the length of the shortest route that starts and finishes at A and uses every edge of the network shown at least once, giving the edges that are repeated.

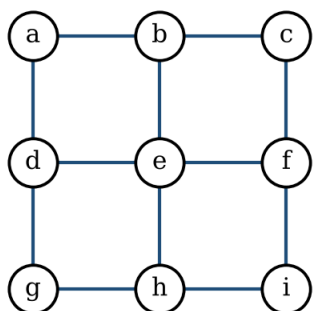


[8]

- 22** The route inspection problem asks for the shortest closed route that uses every edge of a connected network at least once. Let the total weight of the network be W .

- (a) Explain why the length of any such route is at least W . [2]
 (b) Prove that the length of the shortest route equals W if and only if every vertex has even degree. [4]

- 23** The diagram shows a 3×3 grid of streets with vertices a to i. Each street (edge) is 1 km long. A cleaning crew must travel along every street, starting and finishing at e.



- (a) Show that the vertices of odd degree are b, d, f and h. [2]
- (b) Find the shortest distance between each pair of odd vertices and evaluate each of the three pairings. [3]
- (c) Find the length of the shortest route, and explain why the choice of pairing does not matter. [2]
-
- 24** A network has edges AB 4, BC 5, CD 3, DE 6, EF 4, FA 7 and BE 8. A student finds the odd vertices and the three pairings, and then writes: "I will repeat the pairing with the largest total, because it covers more roads." [2]
- (a) Find the odd vertices and the sums for the three pairings. [3]
- (b) Explain the student's error and find the length of the shortest route. [3]
-
- 25** A network has edges AB 4, BC 3, CD 6, DE 2, EF 5, FA 3, BE 9 and AD 8. A route must use every edge, starting and finishing at A. Find the shortest such route, give the route as a list of vertices, and state its length. [7]
-
- 26** A network has edges AB 4, BC 4, CD 4, DA 4, AC 7 and BD 6. Find the length of the shortest route that starts and finishes at A and uses every edge. State the number of different optimal choices of repeated edges and explain why they give routes of the same length. [7]
-
- 27** Let G be a connected network with positive weights and let R be a closed route that uses every edge at least once. Let the multigraph M consist of the edges of G together with an extra copy of each edge for every additional time that R uses it. Prove that the length of R is at least the total weight of G plus the minimum over all pairings of the odd vertices of G of the sum of the shortest distances within the pairs. [10]
-
- 28** K_n is the complete graph on n vertices with every edge of weight 1. [2]
- (a) Find the length of the shortest closed route using every edge when n is odd. [2]
- (b) Show that when n is even the length of the shortest closed route using every edge is $\frac{n^2}{2}$. [4]
- (c) Find the length for $n = 10$ and $n = 9$. [2]
-
- 29** A network has $2k$ vertices of odd degree. [3]
- (a) Show that the number of ways of pairing the odd vertices is $1 \times 3 \times 5 \times \dots \times (2k - 1)$. [3]
- (b) Evaluate this number for $k = 3$, $k = 4$ and $k = 10$. [3]
- (c) Explain why listing all the pairings is not practical for large networks, and name one way round this. [2]

- 30** A network has edges AB x , AC 3, AD 7, BC 4, BD 6 and CD 2, where x is a positive integer.
- (a) Explain why all four vertices have odd degree, and write down the total weight of the network in terms of x . [2]
- (b) Find, in terms of x , the sum of the shortest distances for each of the three pairings. [4]
- (c) Hence find the length of the shortest route starting and finishing at A, in terms of x , for $x < 7$ and for $x \geq 7$. [3]
- (d) Find x when the shortest route has length 30. [1]

Travelling Salesperson Problem

- 1** State what is meant by the classical travelling salesperson problem. [2]
- 2** State two differences between the classical and the practical travelling salesperson problems. [2]
- 3** A tour in the classical problem visits 8 cities. How many edges does the tour contain? [1]
- 4** Describe the nearest neighbour algorithm for finding a tour. [2]
- 5** State whether the length of a tour found by the nearest neighbour algorithm is an upper bound or a lower bound for the optimal tour, and give a reason. [2]
- 6** Find the number of different tours (Hamiltonian cycles) in the complete graph K_5 . [2]
- 7** The table gives the distances between five cities A to E. Use the nearest neighbour algorithm starting at A to find an upper bound for the travelling salesperson problem.

	A	B	C	D	E
A	-	17	15	21	16
B	17	-	11	28	25
C	15	11	-	26	7
D	21	28	26	-	14
E	16	25	7	14	-

[4]

- 8** The table gives the distances between five cities A to E. Calculate the length of each of the tours A, B, C, D, E, A and A, D, B, E, C, A, and state which is shorter.

	A	B	C	D	E
A	-	6	30	12	28
B	6	-	11	26	15
C	30	11	-	16	27
D	12	26	16	-	22
E	28	15	27	22	-

[3]

- 9** Use the deleted vertex method, deleting A, to find a lower bound for the travelling salesperson problem for the five cities whose distances are given in the table.

	A	B	C	D	E
A	-	17	15	21	16
B	17	-	11	28	25
C	15	11	-	26	7
D	21	28	26	-	14
E	16	25	7	14	-

[5]

- 10** A practical travelling salesperson problem has roads AB 4, BC 6, CD 5, DE 3, EA 7 and BD 9 (km). Complete the table of least distances between every pair of vertices.

[4]

- 11** The nearest neighbour algorithm starting at A gives a tour of length 48 for a network. A different tour is found of length 45, and the deleted vertex method gives a lower bound of 41. State the range of values within which the length of the optimal tour lies, explaining your answer.

[3]

- 12** The table gives the distances between six towns A to F. Use the nearest neighbour algorithm starting at A, and then starting at B. Give each tour and its length, and state the better upper bound.

	A	B	C	D	E	F
A	-	7	12	9	15	11
B	7	-	6	14	10	16
C	12	6	-	8	13	5
D	9	14	8	-	4	18
E	15	10	13	4	-	17
F	11	16	5	18	17	-

[5]

- 13** A tour of length 52 has been found for a travelling salesperson problem, and the deleted vertex method gives a lower bound of 47. Find the greatest possible percentage by which the tour of length 52 exceeds the optimal tour, to 3 significant figures.

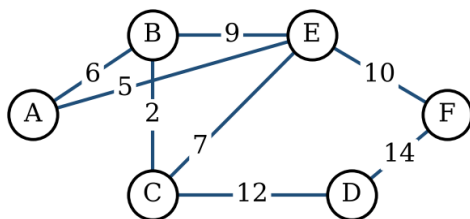
[3]

- 14** Use the distance table for the six towns A to F to find (a) an upper bound using the nearest neighbour algorithm starting at A and (b) a lower bound by deleting A. State the interval that contains the length of the optimal tour, and calculate the greatest possible percentage error if the upper bound is used as the estimate (to 3 s.f.).

	A	B	C	D	E	F
A	-	7	12	9	15	11
B	7	-	6	14	10	16
C	12	6	-	8	13	5
D	9	14	8	-	4	18
E	15	10	13	4	-	17
F	11	16	5	18	17	-

[7]

- 15** A practical travelling salesperson problem is modelled by the network shown (distances in km). Construct the complete table of least distances, use the nearest neighbour algorithm starting at A on this table, and give the route in the original network and its length.



[7]

- 16** Using the distance table for the five cities below, show that the weight of a minimum spanning tree is a lower bound for the optimal tour, and compare it with the bound obtained by deleting vertex A.

	A	B	C	D	E
A	-	6	30	12	28
B	6	-	11	26	15
C	30	11	-	16	27
D	12	26	16	-	22
E	28	15	27	22	-

[5]

- 17** For the six towns A to F whose distances are given in the table, find the lower bound given by deleting each of the vertices A, B, C, D, E, F in turn. State the best lower bound.

	A	B	C	D	E	F
A	-	7	12	9	15	11
B	7	-	6	14	10	16
C	12	6	-	8	13	5
D	9	14	8	-	4	18
E	15	10	13	4	-	17
F	11	16	5	18	17	-

[6]

- 18** A tour of the six towns A to F whose distances are given in the table is A, B, C, F, E, D, A with length 48. Show that reversing the section C, F, E (replacing the edges BC and ED by BE and CD) gives a shorter tour, and state its length.

	A	B	C	D	E	F
A	-	7	12	9	15	11
B	7	-	6	14	10	16
C	12	6	-	8	13	5
D	9	14	8	-	4	18
E	15	10	13	4	-	17
F	11	16	5	18	17	-

[5]

- 19** A complete network has n vertices. Find the number of different tours for $n = 7$ and for $n = 10$. A computer checks one million tours per second. Find the time to check every tour for $n = 15$, giving your answer to 3 significant figures.

[5]

- 20** The distances between five cities A to E are as shown. List all the different tours starting and finishing at A (a tour and its reverse count as the same), find the length of each, and hence find the optimal tour.

	A	B	C	D	E
A	-	9	14	7	11
B	9	-	6	15	12
C	14	6	-	10	8
D	7	15	10	-	13
E	11	12	8	13	-

[6]

- 21** The table shows the distances between seven towns A to G. A courier starts and finishes at B and must visit every town. Find an upper bound using the nearest neighbour algorithm from B, and a lower bound by deleting B. Hence find the interval in which the optimal tour length lies, and the maximum percentage by which the upper bound could exceed the optimal length (3 s.f.).

	A	B	C	D	E	F	G
A	-	12	10	19	8	21	16
B	12	-	3	18	22	6	9
C	10	3	-	7	20	11	14
D	19	18	7	-	15	13	5
E	8	22	20	15	-	17	23
F	21	6	11	13	17	-	4
G	16	9	14	5	23	4	-

[8]

- 22** A practical travelling salesperson problem has roads EF 10, AE 5, CD 12, AB 6, BE 9, BC 2, CE 7 and DF 14 (km). The table of least distances, shown, satisfies the triangle inequality. Find the weight of a minimum spanning tree, and explain how doubling every edge of this tree and taking shortcuts gives a tour of length at most twice this weight. State the upper bound obtained, and compare it with the nearest neighbour bound 57.

	A	B	C	D	E	F
A	-	6	8	20	5	15
B	6	-	2	14	9	19
C	8	2	-	12	7	17
D	20	14	12	-	19	14
E	5	9	7	19	-	10
F	15	19	17	14	10	-

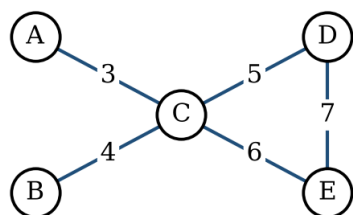
[6]

- 23** A student finds a lower bound for the five cities whose distances are given in the table, by deleting A, as follows: "The minimum spanning tree of B, C, D, E has weight 32. A is joined to the nearest city by an edge of length 15. So the lower bound is $32 + 15 = 47$."

	A	B	C	D	E
A	-	17	15	21	16
B	17	-	11	28	25
C	15	11	-	26	7
D	21	28	26	-	14
E	16	25	7	14	-

- (a) Explain the error in the student's method. [2]
 (b) Find the correct lower bound. [3]
 (c) Given that the nearest neighbour algorithm from A gives a tour of length 81, state what can be said about the optimal tour. [1]

- 24** A practical travelling salesperson problem has roads CA 3, CB 4, CD 5, CE 6 and DE 7 (km), as shown. A salesperson must visit every town and return to A, and is allowed to pass through a town more than once.



- (a) Explain why the classical problem has no solution for this network. [1]
 (b) Find the table of least distances. [2]
 (c) Use the nearest neighbour algorithm from A on the table to find a tour, and give the corresponding route in the network with its length. [4]
-
- 25** The distances between five depots A to E are AB x , AC 14, AD 20, AE 11, BC 9, BD 16, BE 12, CD 10, CE 15 and DE 8, where x is a positive integer less than 14. Deleting vertex A gives a lower bound of 47 for the length of the optimal tour.
- (a) Find the weight of the minimum spanning tree of B, C, D and E. [2]
 (b) Write down the two shortest edges at A in terms of x , and hence find x . [3]
 (c) For this value of x , use the nearest neighbour algorithm from A to find a tour, and state what you can deduce about the optimal tour. [3]
-
- 26** Let L be the length of an optimal tour of a complete network, and let W be the weight of a minimum spanning tree of the network.
- (a) Prove that $L \geq W + m$, where m is the weight of the lightest edge in the network. [4]
 (b) Apply this to the five cities with distances AB 9, AC 14, AD 7, AE 11, BC 6, BD 15, BE 12, CD 10, CE 8, DE 13 to find a lower bound. [2]
-
- 27** Let G be a complete network with $n \geq 3$ vertices and let v be any vertex. Prove that the length of every tour of G is at least the weight of a minimum spanning tree of $G - v$ (the network with v and its edges deleted) plus the sum of the weights of the two lightest edges at v . [9]
-
- 28** A complete network satisfies the triangle inequality: for any three vertices, $w(xz) \leq w(xy) + w(yz)$. Let W be the weight of a minimum spanning tree and L the length of an optimal tour.
- (a) Show that $L \geq W$ (briefly). [2]
 (b) Doubling every edge of the minimum spanning tree gives a closed walk through every vertex. Show that a tour of length at most $2W$ can be obtained from it. [4]
 (c) Deduce that this tour has length at most $2L$. [2]
-
- 29** The travelling salesperson problem for n cities can be solved exactly by checking every tour.
- (a) Show that there are $\frac{(n-1)!}{2}$ different tours in the complete network with n vertices. [2]
 (b) A computer can check 10^9 tours per second. Find the time needed for 25 cities, in years to 3 significant figures (1 year = 3.156×10^7 s, taking $\frac{24!}{2} = 3.10 \times 10^{23}$). [4]
 (c) Find the factor by which the number of tours increases when a 25th city is added to a 24 city problem, and comment. [2]
-
- 30** The distances between five depots A to E are AB 12, AC 18, AD 25, AE 9, BC 14, BD 20, BE 16, CD 10, CE 22 and DE 15.
- (a) Use the nearest neighbour algorithm from A to find an upper bound. [3]
 (b) Use the deleted vertex method, deleting A, to find a lower bound. [4]
 (c) Hence state the length of the optimal tour, giving a reason, and write down an optimal tour. [3]