Graph Colouring

1 (2 marks)

Every vertex of K_5 is adjacent to every other vertex, so all five vertices need different colours — **E1**

Chromatic number = 5 — **B1**

Answer: 5 — **2 marks**

2 (2 marks)

(a) Colour alternately round the cycle with two colours: 2 — **B1**

Answer: 2 — **1 mark**

(b) An odd cycle cannot be coloured with two colours, but three suffice: 3 — **B1**

Answer: 3 — **1 mark**

3 (2 marks)

(a) Vertex A is joined to B, C, D and E, so the maximum degree is 4 — **B1**

Answer: 4 — **1 mark**

(b) The chromatic number is at most the maximum degree plus 1, so at most 5 — **B1**

Answer: At most 5 — **1 mark**

4 (2 marks)

(a) A and C are joined by an edge but are both red — **E1**

Answer: A and C are adjacent and both red — **1 mark**

(b) Colour C green: its neighbours A, B, D are red, blue, blue, so there is no clash and no other edge is affected — **B1**

Answer: e.g. colour C green — **1 mark**

5 (2 marks)

Any planar graph (equivalently any map) can be properly coloured — **B1**

using at most four colours — **B1**

Answer: Every planar graph has chromatic number at most 4 — **2 marks**

6 (2 marks)

The four vertices must all have different colours: $5 \times 4 \times 3 \times 2$ — **M1**

$= 120$ — **A1**

Answer: 120 — **2 marks**

7 (3 marks)

(a) A, B, C are mutually adjacent (a triangle), so three colours are needed — **E1**

Answer: ABC is a triangle — **1 mark**

(b) For example A and D red, B and E blue, C green: no edge joins two vertices of the same colour — **M1**

Chromatic number = 3 — **A1**

Answer: A, D red; B, E blue; C green; chromatic number 3 — **2 marks**

8 (3 marks)

A: 1. B (adjacent to A): 2. C (adjacent to B): 1. D (adjacent to C and A): 2 — **M1 A1**

E (adjacent to D): 1. F (adjacent to E and A): 2. Two colours are used — **A1**

Answer: A, C, E colour 1; B, D, F colour 2; 2 colours — **3 marks**

9 (3 marks)

Take the sets {A, C, E} and {B, D, F} — **M1**

Every edge (AB, AD, CB, CF, ED, EF, CD) joins a vertex of the first set to a vertex of the second set, so G is bipartite — **E1**

G has edges, so one colour is not enough; chromatic number = 2 — **B1**

Answer: Bipartite with sets A, C, E and B, D, F; chromatic number 2 — **3 marks**

10 (3 marks)

(a) It is bipartite and has edges: 2 — **B1**

Answer: 2 — **1 mark**

(b) A path is a tree, which is bipartite: 2 — **B1**

Answer: 2 — **1 mark**

(c) An odd cycle needs three colours: 3 — **B1**

Answer: 3 — **1 mark**

11 (3 marks)

The first vertex can be coloured in 4 ways — **B1**

Each later vertex must differ from the previous one: 3 choices each — **M1**

$4 \times 3^5 = 972$ — **A1**

Answer: 972 — **3 marks**

12 (4 marks)

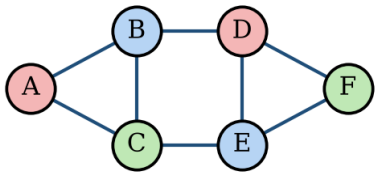
Draw the conflict graph with an edge for each pair above — **M1**

A, B, C are mutually adjacent (a triangle), so at least 3 slots are needed — **B1**

Slot 1: A, D. Slot 2: B, E. Slot 3: C, F: no pair in a slot is in the list — **M1**

Least number of slots = 3 — **A1**

Answer: 3 slots: (A, D), (B, E), (C, F) — **4 marks**



13 (3 marks)

Let the deleted edge join P and Q. The other four vertices are adjacent to everything, so need 4 colours different from each other and from the colours on P and Q — **M1**

P and Q are not adjacent so can share a fifth colour: 5 colours suffice — **M1**

At least 5 are needed since G contains K_5 (the four other vertices and P) — **A1**

Answer: 5 — **3 marks**

14 (7 marks)

(a) Degrees: A 3, B 3, C 3, D 3, E 4, F 3, G 2, H 3 — **M1**

Order: E, A, B, C, D, F, H, G — **A1**

Answer: E, A, B, C, D, F, H, G — **2 marks**

(b) Colour 1: take E, then A (not adjacent to E); B, C, D are adjacent to A or E, and F and H are adjacent to E; G (adjacent to D and H only) is also free — **M1**

Colour 2: B, then D and F (neither is adjacent to B or to each other); C is adjacent to B and H is adjacent to F — **M1**

Colour 3: C and H (not adjacent). Three colours are used — **A1**

Answer: 3 colours — **3 marks**

(c) Welsh-Powell gives a proper 3-colouring, so the chromatic number is at most 3 — **M1**

A, B, C form a triangle, so at least 3 colours are needed — **E1**

Answer: Triangle ABC forces at least 3; the colouring shows 3 suffice — **2 marks**

15 (6 marks)

(a) A: 1. B (no coloured neighbour): 1. C (adjacent to B): 2 — **M1**

D (adjacent to A): 2. E (adjacent to B and D): 3 — **A1**

F (adjacent to A and C): 3. Three colours — **A1**

Answer: 3 colours — **3 marks**

(b) Sets A, C, E and B, D, F: every edge joins the two sets (AD, AF, CB, CF, EB, ED) — **M1**

So the chromatic number is 2 — **A1**

The greedy algorithm does not always give the minimum: the order of the vertices matters — **E1**

Answer: Bipartite; greedy gave 3 but chromatic number is 2 — **3 marks**

16 (5 marks)

Delete one edge of C_4: the result is a path on 4 vertices — M1

Contract that edge: the result is K_3 — M1

$P(C_4, k) = k(k-1)^3 - k(k-1)(k-2)$ — A1

$= k(k-1)[(k-1)^2 - (k-2)] = k(k-1)(k^2 - 3k + 3)$ — A1

$k = 3$: $3 \times 2 \times 3 = 18$ — A1

Answer: 18 — 5 marks

17 (4 marks)

Colour the triangle first: $k(k-1)(k-2)$ ways — M1 A1

D must differ from C: $(k-1)$ ways; E must differ from D: $(k-1)$ ways — M1

$P(k) = k(k-1)^3(k-2)$; at $k = 5$: $5 \times 64 \times 3 = 960$ — A1

Answer: $k(k-1)^3(k-2)$; 960 — 4 marks

18 (6 marks)

(a) A planar graph with 8 vertices has at most $3 \times 8 - 6 = 18$ edges — M1

$19 > 18$, so the graph is not planar — E1

Answer: Not planar since $19 > 18$ — 2 marks

(b) If planar, Euler gives $f = 2 - 6 + 9 = 5$ faces — M1

No triangles and bipartite means every face is bounded by at least 4 edges, so the sum of the face degrees is at least $4 \times 5 = 20$ — M1 E1

But each edge borders two faces so the sum of face degrees is $2 \times 9 = 18 < 20$, a contradiction — A1

Answer: Contradiction, so $K_{3,3}$ is not planar — 4 marks

19 (5 marks)

U and V the same colour: k choices of colour, then each of X, Y, Z can use any of the other $k - 1$ colours, giving $k(k-1)^3$ colourings — M1 A1

U and V different colours: $k(k-1)$ choices, then each of X, Y, Z must avoid both, giving $k(k-1)(k-2)^3$ colourings — M1 A1

$P(k) = k(k-1)^3 + k(k-1)(k-2)^3$, so with $k = 3$: $3 \times 8 + 3 \times 2 \times 1 = 24 + 6 = 30$ — A1

Answer: $P(k) = k(k-1)^3 + k(k-1)(k-2)^3$; 30 — 5 marks

20 (5 marks)

(a) The hub is adjacent to every rim vertex so needs its own colour — E1

The rim is an even cycle, 2 colours: total 3 — A1

Answer: 3 — 2 marks

(b) The rim is an odd cycle so needs 3 colours — M1

The hub needs a fourth colour: 4 — A1

Answer: 4 — 2 marks

(c) 3 if n is even, 4 if n is odd — B1

Answer: 3 (n even), 4 (n odd) — 1 mark

21 (8 marks)

(a) Deleting AE leaves cycle ABCD with E hanging from B by the single edge BE — M1

E can be coloured in $(k-1)$ ways once ABCD is coloured: $P(H - AE) = (k-1)\,P(C_4)$ — M1 A1

Contracting AE merges E into A; the edge BE becomes the existing edge AB, so the result is C_4 — M1

$P(H) = (k-1)P(C_4) - P(C_4) = (k-2)\,k(k-1)(k^2 - 3k + 3)$ — A1

Answer: $P(H, k) = k(k-1)(k-2)(k^2 - 3k + 3)$ — 5 marks

(b) $k = 4$: $4 \times 3 \times 2 \times 7 = 168$ — M1 A1

$P(2) = 0$ and $P(3) = 3 \times 2 \times 1 \times 3 = 18 > 0$, so the chromatic number is 3 — A1

Answer: 168; chromatic number 3 — 3 marks

22 (7 marks)

(a) M3, M6 and M8 are mutually adjacent, so at least 3 slots — **B1**

Slots (M3), (M1, M2, M5, M6, M7), (M4, M8): no edge lies inside any slot — **M1**

So 3 slots suffice and the least number is 3 — **A1**

Answer: 3 — **3 marks**

(b) M3 is adjacent to all seven other meetings, so it must be in a slot on its own — **M1**

The other 7 meetings need slots of at most 3 meetings, so at least 3 more slots — **M1 A1**

Four slots work: (M3), (M1, M2, M6), (M4, M5, M7), (M8), so the least number is 4 — **A1**

Answer: 4 — **4 marks**

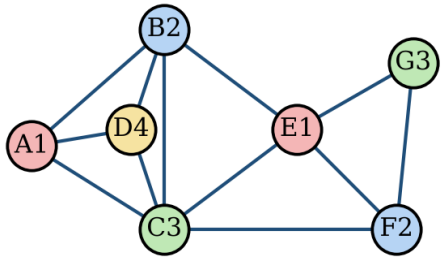
23 (5 marks)

A, B, C, D are mutually adjacent (AB, AC, AD, BC, BD, CD all edges), so at least 4 frequencies — **M1 A1**

Assignment: A 1, B 2, C 3, D 4, E 1, F 2, G 3 (E is adjacent to B, C, F, G; F to C, E, G; G to E, F) — **M1 A1**

The graph is planar, so by the four colour theorem 4 frequencies always suffice; this agrees with the minimum 4 — **E1**

Answer: 4 frequencies, e.g. A1 B2 C3 D4 E1 F2 G3 — **5 marks**



24 (4 marks)

Having no triangle only removes one reason for needing 3 colours; it does not make the graph bipartite — **E1**

ABCDE is a cycle with 5 vertices, an odd cycle, which cannot be coloured with 2 colours — **M1 E1**

Three colours suffice: A 1, B 2, C 1, D 2, E 3, F 2. Chromatic number = 3 — **A1**

Answer: Odd cycle ABCDE needs 3 colours; chromatic number 3 — **4 marks**

25 (6 marks)

(a) The degree of P is the number of vertices: 4 — **B1**

The coefficient of k^3 is minus the number of edges: 4 edges — **B1**

Answer: 4 vertices, 4 edges — **2 marks**

(b) $P(k) = k(k^3 - 4k^2 + 5k - 2)$ — **M1**

$= k(k - 1)^2(k - 2)$ — **A1**

$P(1) = P(2) = 0$ and $P(3) > 0$, so the chromatic number is 3 — **A1**

Answer: $k(k - 1)^2(k - 2)$; 3 — **3 marks**

(c) $6 \times 5^2 \times 4 = 600$ — **B1**

Answer: 600 — **1 mark**

26 (6 marks)

Vertices are lectures, with an edge when two lectures overlap in time; a proper colouring is a room allocation — **M1**

Overlaps: L1L2, L1L4, L1L7, L2L3, L2L7, L3L5, L3L9, L5L6, L5L9, L6L8 — **M1 A1**

L1, L2 and L7 are mutually adjacent (all running from 10 to 11), so at least 3 rooms are needed — **M1 E1**

Room 1: L1, L3, L6. Room 2: L2, L4, L5, L8. Room 3: L7, L9. No two in a room overlap, so 3 rooms suffice — **A1**

Answer: 3 rooms: (L1, L3, L6), (L2, L4, L5, L8), (L7, L9) — **6 marks**

27 (8 marks)

Base case n = 3: $(k-1)^3-(k-1)=(k-1)(k^2-2k)=k(k-1)(k-2)$, the polynomial of K_3 **M1 A1**

Assume true for $n=m-1$ $(m\geq 4)$ **B1**

Delete an edge of C_m: a path on m vertices. Contract it: C_{m-1} **M1**

$P(C_m,k)=k(k-1)^{m-1}-P(C_{m-1},k)$ **M1**

$=k(k-1)^{m-1}-(k-1)^{m-1}-(-1)^{m-1}(k-1)$ **A1**

$=(k-1)^{m-1}(k-1)+(-1)^m(k-1)=(k-1)^m+(-1)^m(k-1)$ **A1**

True for n = 3 and true for m whenever true for $m-1$, so true for all $n\geq 3$ by induction **E1**

Answer: $P(C_n,k)=(k-1)^n+(-1)^n(k-1)$ for all $n\geq 3$ **8 marks**

28 (8 marks)

(a) Take a colouring with exactly c colours (colour classes are non-empty) **M1**

If no edge joined two classes the classes could be merged to give a proper colouring with $c-1$ colours, contradicting minimality **E1 E1**

So each of the $c(c-1)/2$ pairs of classes has an edge, giving at least $c(c-1)/2$ edges **A1**

Answer: A graph with chromatic number c has at least $c(c-1)/2$ edges **4 marks**

(b) $c(c-1)/2\leq 12$ **M1**

$c=5$ gives $10\leq 12$ but $c=6$ gives $15>12$, so $c\leq 5$ **M1 A1**

K_5 with two extra pendant edges has 12 edges and chromatic number 5 **B1**

Answer: 5; e.g. K_5 with two pendant edges **4 marks**

29 (9 marks)

Base case: a graph with at most 6 vertices can be given a different colour at every vertex **B1**

Assume every planar graph with $m-1$ vertices $(m\geq 7)$ is 6-colourable, and let G be planar with m vertices **B1**

It suffices to consider each connected component, so assume G is connected, with $e\leq 3m-6$ **M1**

Sum of degrees $=2e\leq 6m-12<6m$ **M1 A1**

So the average degree is below 6, hence some vertex v has degree at most 5 **E1**

$G-v$ is planar with $m-1$ vertices, so can be properly coloured with 6 colours **M1**

The at most 5 neighbours of v use at most 5 colours, so a sixth colour is free for v **M1**

Colouring v with it gives a proper 6-colouring of G, completing the induction **A1**

Answer: Every simple planar graph can be 6-coloured **9 marks**

30 (9 marks)

(a) The hub can be coloured in k ways and then every rim vertex must avoid the hub colour **M1**

So the rim is properly coloured using the remaining $k-1$ colours **E1**

$P(W_n,k)=k[(k-2)^n+(-1)^n(k-2)]$ **A1**

Answer: $k[(k-2)^n+(-1)^n(k-2)]$ **3 marks**

(b) $P(W_4,k)=k(k-2)[(k-2)^3+1]$ **M1**

$(k-2)^3+1=(k-1)[(k-2)^2-(k-2)+1]=(k-1)(k^2-5k+7)$ **M1**

$k=4$: $4\times 2\times 3\times 3=72$ **A1**

Answer: 72 **3 marks**

(c) $P(W_5,3)=3[1-1]=0$, so 3 colours are not enough **M1 A1**

$P(W_5,4)=4[32-2]=120>0$, so 4 colours work in 120 ways **A1**

Answer: 120 **3 marks**

Rook Polynomials and Restricted Permutations

1 (2 marks)

Any two squares lie in the same row, so no two rooks can be placed: $r_1=3$, $r_2=0$ **M1**

$R(x)=1+3x$ **A1**

Answer: 1 + 3x **2 marks**

2 (2 marks)

$r_1 = 2$ (one rook on either square) and $r_2 = 1$ (rooks on both squares) **M1**

$R(x) = 1 + 2x + x^2$ **A1**

Answer: $1 + 2x + x^2$ **2 marks**

3 (2 marks)

$r_0 = 1$ (the empty placement) **B1**

r_1 is the number of squares: 5 **B1**

Answer: $r_0 = 1$, $r_1 = 5$ **2 marks**

4 (2 marks)

$N = 6 - 2 \times 2 + 1 \times 1$ **M1**

$= 3$ **A1**

Answer: 3 **2 marks**

5 (2 marks)

Any two squares are in the same column, so at most one rook can be placed **E1**

$r_1 = n$, so $R(x) = 1 + nx$ **B1**

Answer: $1 + nx$ **2 marks**

6 (2 marks)

The permutations 2 3 1 and 3 1 2 have no fixed point **M1**

The other four permutations all fix at least one number, so there are 2 derangements **A1**

Answer: 231 and 312; 2 **2 marks**

7 (4 marks)

(a) $r_1 = 4$ squares. Non-attacking pairs: (1, 1)&(2, 2), (1, 1)&(2, 3), (2, 2)&(3, 1), (2, 3)&(3, 1), so $r_2 = 4$, and $r_3 = 0$ **M1**

$R(x) = 1 + 4x + 4x^2$ **A1**

Answer: $1 + 4x + 4x^2$ **2 marks**

(b) $N = 3! - 4 \times 2! + 4 \times 1! = 6 - 8 + 4$ **M1**

$= 2$ **A1**

Answer: 2 **2 marks**

8 (3 marks)

$r_1 = 3$ squares **B1**

Two rooks: only the squares (row 1, column 2) and (row 2, column 1) share no row or column, so $r_2 = 1$ **M1**

$R(x) = 1 + 3x + x^2$ **A1**

Answer: $1 + 3x + x^2$ **3 marks**

9 (5 marks)

(a) $r_1 = 4$. Two rooks must be on a diagonal of the block: $r_2 = 2$ **M1**

$R(B_1) = 1 + 4x + 2x^2$ **A1**

Answer: $1 + 4x + 2x^2$ **2 marks**

(b) B_1 and B_2 share no row or column, so rooks on one never attack rooks on the other, and any placement on B is a placement on B_1 together with one on B_2 **E1**

$R(B) = (1 + 4x + 2x^2)(1 + x)$ **M1**

$= 1 + 5x + 6x^2 + 2x^3$ **A1**

Answer: $1 + 5x + 6x^2 + 2x^3$ **3 marks**

10 (3 marks)

$r_1 = 4$. For two rooks, one must be at (2, 4) and the other in row 1: $r_2 = 3$ **M1 A1**

$R(x) = 1 + 4x + 3x^2$, so $N = 5! - 4 \times 4! + 3 \times 3! = 120 - 96 + 18 = 42$ **A1**

Answer: 42 **3 marks**

11 (3 marks)

Choose k of the m rows and k of the n columns, then match the rows to the columns in k! ways (each match is a square, and no two share a row or column) **E1**

4×6 board: $C(4, 3) \times C(6, 3) \times 3! = 4 \times 20 \times 6$ **M1**

$= 480$ **A1**

Answer: 480 **3 marks**

12 (3 marks)

Two rooks cannot both be on the vertical arm (column 2), and the square (row 1, column 2) shares a row or column with every other square **M1**

Valid pairs: (1, 1) with (2, 2) or (3, 2); (1, 3) with (2, 2) or (3, 2) **M1**

$r_2 = 4$ **A1**

Answer: 4 **3 marks**

13 (4 marks)

The five squares are in different rows and columns, so any k of them can hold rooks: $r_k = C(5, k)$ **M1**

$R(x) = (1 + x)^5 = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$ **A1**

$N = 120 - 5 \times 24 + 10 \times 6 - 10 \times 2 + 5 \times 1 - 1$ **M1**

$= 44$ **A1**

Answer: 44 **4 marks**

14 (5 marks)

B_i: delete row 2 and column 2, leaving the squares (1, 1) and (3, 3), so $R(B_i) = (1 + x)^2$ **M1 A1**

B_e: squares (1, 1), (1, 2), (2, 3), (3, 3); the pair in row 1 and the pair in column 3 share no row or column, so $R(B_e) = (1 + 2x)(1 + 2x)$ **M1 A1**

$R(B) = x(1 + x)^2 + (1 + 2x)^2 = x + 2x^2 + x^3 + 1 + 4x + 4x^2 = 1 + 5x + 6x^2 + x^3$ **A1**

Answer: $1 + 5x + 6x^2 + x^3$ **5 marks**

15 (6 marks)

Part 1: (A, 1), (A, 2), (B, 2); part 2: (C, 3), (D, 3), (D, 4). These share no row or column **M1 A1**

Each part is an L-shape with $r_1 = 3$, $r_2 = 1$, so has polynomial $1 + 3x + x^2$ **M1**

$R(x) = (1 + 3x + x^2)^2 = 1 + 6x + 11x^2 + 6x^3 + x^4$ **A1**

$N = 4! - 6 \times 3! + 11 \times 2! - 6 \times 1! + 1 \times 0! = 24 - 36 + 22 - 6 + 1$ **M1**

$= 5$ **A1**

Answer: 5 ways **6 marks**

16 (3 marks)

The forbidden squares are (1, 1) and (2, 2), which share no row or column, so $R(x) = (1 + x)^2 = 1 + 2x + x^2$ **M1**

$N = 5! - 2 \times 4! + 1 \times 3!$ **M1**

$= 120 - 48 + 6 = 78$ **A1**

Answer: 78 **3 marks**

17 (5 marks)

$r_1 = 6$. Non-attacking pairs: (1, 1)&(2, 2), (1, 1)&(3, 2), (1, 1)&(3, 3), (2, 1)&(3, 2), (2, 1)&(3, 3), (2, 2)&(3, 1), (2, 2)&(3, 3), so $r_2 = 7$ **M1 A1**

Only (1, 1), (2, 2), (3, 3) form three mutually non-attacking squares, so $r_3 = 1$ and $R(x) = 1 + 6x + 7x^2 + x^3$ **A1**

$N = 24 - 6 \times 3! + 7 \times 2! - 1 \times 1! = 24 - 36 + 14 - 1$ **M1**

$= 1$. Direct: p_3 must be 4, then $p_2 = 3$, $p_1 = 2$, $p_4 = 1$, so exactly one **A1**

Answer: $R(x) = 1 + 6x + 7x^2 + x^3$; 1 permutation **5 marks**

18 (5 marks)

The 2 × 2 block (rows 1 and 2, columns 1 and 2) and the L-shape (rows 3 and 4, columns 3 and 4) share no row or column — **M1**

$R = (1 + 4x + 2x^2)(1 + 3x + x^2)$ — **M1**

$= 1 + 7x + 15x^2 + 10x^3 + 2x^4$ — **A1**

$N = 120 - 7 \times 24 + 15 \times 6 - 10 \times 2 + 2 \times 1$ — **M1**

$= 24$ — **A1**

Answer: 24 — **5 marks**

19 (4 marks)

For a full 2 × 3 block $r_1 = 6$ and $r_2 = C(2, 2) \times C(3, 2) \times 2! = 6$, so $R(x) = 1 + 6x + 6x^2$ — **M1 A1**

$N = 720 - 6 \times 120 + 6 \times 24 = 144$ — **M1**

Direct: p_1, p_2 are chosen in $3 \times 2 = 6$ ways and the other four numbers in $4! = 24$ ways: 144 — **A1**

Answer: 144 — **4 marks**

20 (3 marks)

The board is the diagonal, with rook polynomial $(1 + x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$, so there is a fifth term — **M1 E1**

The student has omitted $+ r_4 \times 0! = + 1$: $24 - 24 + 12 - 4 + 1 = 9$ — **A1**

Answer: The term +1 × 0! was omitted; $D_4 = 9$ — **3 marks**

21 (7 marks)

(a) $r_1 = 5$. Two rooks: one in row 1 and one in row 2 in different columns: (1, 1)&(2, 2), (1, 2)&(2, 1), (1, 3)&(2, 1), (1, 3)&(2, 2), so $r_2 = 4$; $r_3 = 0$ since there are only 2 rows — **M1**

$R(A) = 1 + 5x + 4x^2$ — **A1**

Answer: $1 + 5x + 4x^2$ — **2 marks**

(b) $r_1 = 5$. Pairs: (1, 2)&(2, 1), (1, 2)&(3, 1), (1, 3)&(2, 1), (1, 3)&(3, 1): $r_2 = 4$ — **M1 A1**

Three rooks would need squares from rows 1, 2, 3, but (2, 1) and (3, 1) share a column, so $r_3 = 0$ and $R(B) = 1 + 5x + 4x^2 = R(A)$ — **A1**

Answer: $1 + 5x + 4x^2$; equal to R(A) — **3 marks**

(c) The count depends only on r_k: $N = \Sigma(-1)^k r_k(5 - k)!$, the same for both boards — **M1**

$N = 120 - 5 \times 24 + 4 \times 6 = 24$ — **A1**

Answer: 24 — **2 marks**

22 (7 marks)

(a) The rook polynomial is $(1 + x)^4$: 1, 4, 6, 4, 1 — **B1**

Answer: 1, 4, 6, 4, 1 — **1 mark**

(b) $e_1 = C(1, 1) \times 4 \times 3! - C(2, 1) \times 6 \times 2! + C(3, 1) \times 4 \times 1! - C(4, 1) \times 1 \times 0!$ — **M1 A1**

$= 24 - 24 + 12 - 4 = 8$ — **A1**

Answer: 8 — **3 marks**

(c) $e_2 = C(2, 2) \times 6 \times 2! - C(3, 2) \times 4 \times 1! + C(4, 2) \times 1 \times 0! = 12 - 12 + 6$ — **M1**

$= 6$ — **A1**

Answer: 6 — **2 marks**

(d) $9 + 8 + 6 + 0 + 1 = 24 = 4!$ — **B1**

Answer: Total 24 — **1 mark**

23 (8 marks)

B_i: delete row 1 and column 3, leaving (2, 1) and (2, 2) in one row, so $R(B_i) = 1 + 2x$ **M1 A1**

B_e: a 2 × 2 block (rows 1 and 2, columns 1 and 2) together with the square (3, 3), which share no row or column **M1**

$R(B_e) = (1 + 4x + 2x^2)(1 + x) = 1 + 5x + 6x^2 + 2x^3$ **M1 A1**

$R(B) = x(1 + 2x) + 1 + 5x + 6x^2 + 2x^3 = 1 + 6x + 8x^2 + 2x^3$ **A1**

$N = 4! - 6 \times 3! + 8 \times 2! - 2 \times 1! = 24 - 36 + 16 - 2$ **M1**

$= 2$ **A1**

Answer: $R(x) = 1 + 6x + 8x^2 + 2x^3$; 2 **8 marks**

24 (8 marks)

Rows W1, W2, W3 only meet columns J1, J2; rows W4, W5 only meet columns J3, J4, J5. The board is two disjoint parts **M1 A1**

Part 1: (W1, J1), (W1, J2), (W2, J1), (W3, J2): $r_1 = 4$, $r_2 = 3$ (W1J1&W3J2, W1J2&W2J1, W2J1&W3J2), $r_3 = 0$ **M1 A1**

Part 2: (W4, J3), (W4, J4), (W5, J4), (W5, J5): $r_1 = 4$, $r_2 = 3$ (W4J3&W5J4, W4J3&W5J5, W4J4&W5J5) **M1**

$R = (1 + 4x + 3x^2)^2 = 1 + 8x + 22x^2 + 24x^3 + 9x^4$ **A1**

$N = 120 - 8 \times 24 + 22 \times 6 - 24 \times 2 + 9 \times 1 = 120 - 192 + 132 - 48 + 9 = 21$ **M1 A1**

Answer: $R(x) = 1 + 8x + 22x^2 + 24x^3 + 9x^4$; 21 **8 marks**

25 (6 marks)

(a) The squares (1, 2) and (2, 2) are in the same column, so the two parts are not independent **E1**

Rooks on the two parts can attack each other, so the product rule does not apply (for example the student counts (1, 2) and (2, 2) together as allowed) **E1**

Answer: (1, 2) and (2, 2) share a column **2 marks**

(b) B_i (delete row 2, column 2) has (1, 1) and (3, 3): $(1 + x)^2$ **M1**

B_e has (1, 1), (1, 2), (3, 3): $(1 + 2x)(1 + x) = 1 + 3x + 2x^2$, so $R = x(1 + x)^2 + 1 + 3x + 2x^2$ **M1**

$R(B) = 1 + 4x + 4x^2 + x^3$ **A1**

Answer: $1 + 4x + 4x^2 + x^3$ **3 marks**

(c) $N = 24 - 4 \times 6 + 4 \times 2 - 1 \times 1 = 7$ **B1**

Answer: 7 **1 mark**

26 (6 marks)

(a) The diagonal squares share no row or column so $r_k = C(n, k)$: $R(x) = (1 + x)^n$ **M1**

$D_n = \Sigma(-1)^k r_k(n - k)! = \Sigma(-1)^k C(n, k)(n - k)!$ **A1**

$C(n, k)(n - k)! = n!/k!$, giving $D_n = n! \Sigma (-1)^k/k!$ **A1**

Answer: $D_n = n! \Sigma_{k=0}^{n} (-1)^k/k!$ **3 marks**

(b) $720(1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720})$ **M1**

$= 720 - 720 + 360 - 120 + 30 - 6 + 1 = 265$ **A1**

Answer: 265 **2 marks**

(c) 265/720 = 0.3681, close to 1/e **B1**

Answer: 0.3681 **1 mark**

27 (9 marks)

(a) Choose the k rows in C(m, k) ways and the k columns in C(m, k) ways **M1**

Match the chosen rows to the chosen columns in k! ways; each pairing is a square of the block and no two rooks share a line **E1 A1**

Answer: $r_k = C(m, k)^2 k!$ **3 marks**

(b) The first m positions must take values from the n − m values outside the block columns: $(n - m)!/(n - 2m)!$ ways **M1 A1**

The remaining n − m positions take the remaining n − m values in $(n - m)!$ ways, giving the product **A1**

Answer: $((n - m)!)^2/(n - 2m)!$ **3 marks**

(c) $r_0, ..., r_3 = 1, 9, 18, 6$ **B1**

$720 - 9 \times 120 + 18 \times 24 - 6 \times 6 = 720 - 1080 + 432 - 36 = 36$ **M1**

$(3!)^2/0! = 36$, agreeing **A1**

Answer: Both give 36 **3 marks**

28 (9 marks)

(a) Split off the last term: $D_n = n! \sum_{k=0}^{n-1} (-1)^k/k! + n! (-1)^n/n!$ **M1 A1**

$n! = n \times (n - 1)!$, so the first part is $n \times (n - 1)! \sum_{k=0}^{n-1} (-1)^k/k! = nD_{n-1}$ **M1**

The last term is $(-1)^n$, so $D_n = nD_{n-1} + (-1)^n$ **A1**

Answer: $D_n = nD_{n-1} + (-1)^n$ **4 marks**

(b) $D_7 = 7 \times 265 - 1 = 1854$ **B1**

Answer: 1854 **1 mark**

(c) $D_1 = 0$ is even, and $D_2 = 1$ is odd, so true for the first cases **B1**

Suppose it is true for $n - 1$. If n is even then $n D_{n-1}$ is even and $(-1)^n = 1$, so D_n is odd **M1**

If n is odd then $n - 1$ is even so D_{n-1} is odd by hypothesis; $n D_{n-1}$ is odd, and subtracting 1 gives an even number **M1**

So true for n whenever true for $n - 1$, and hence for all n **A1**

Answer: D_n is even when n is odd and odd when n is even **4 marks**

29 (9 marks)

(a) The squares are (1, 1), (1, 2), (2, 2), (2, 3), ...; (i, i) and (i, i + 1) share row i, and (i, i + 1) and (i + 1, i + 1) share column i + 1 **M1 A1**

No other pairs share a line: rows and columns of non-consecutive squares differ **E1**

So a non-attacking placement of k rooks is a choice of k non-consecutive squares from the $2n - 1$ in the list **M1**

The number of ways to choose k non-consecutive items from N is $C(N - k + 1, k) = C(2n - k, k)$ **A1**

Answer: $r_k = C(2n - k, k)$ **5 marks**

(b) $r_k = C(12 - k, k)$: 1, 11, 45, 84, 70, 21, 1 **M1 A1**

$N = 720 - 11 \times 120 + 45 \times 24 - 84 \times 6 + 70 \times 2 - 21 \times 1 + 1 \times 1$ **M1**

$= 720 - 1320 + 1080 - 504 + 140 - 21 + 1 = 96$ **A1**

Answer: 96 **4 marks**

30 (9 marks)

(a) $r_1 = 10$ **B1**

Pairs of squares: $C(10, 2) = 45$ **M1**

Pairs in the same row: 5; pairs in the same column: 5 (and none is both) **M1**

$r_2 = 45 - 10 = 35$ **A1**

Answer: $r_1 = 10$, $r_2 = 35$ **4 marks**

(b) $N = 5! - 10 \times 4! + 35 \times 3! - 50 \times 2! + 25 \times 1! - 2 \times 0!$ **M1**

$= 120 - 240 + 210 - 100 + 25 - 2$ **M1**

$= 13$ **A1**

Answer: 13 **3 marks**

(c) Five rooks need one square in each row and column: either all diagonal squares (i, i), or all squares (i, i + 1) **M1**

Mixing the two types forces a repeated column, so there are exactly 2 **E1**

Answer: All (i, i) or all (i, i + 1) **2 marks**

Generating Functions for Counting

1 (2 marks)

$1 + x + x^2 + ...$ is a geometric series with first term 1 and common ratio x — **M1**

Sum to infinity = $\frac{1}{1 - x}$ — **A1**

Answer: $\frac{1}{1 - x}$ — **2 marks**

2 (2 marks)

Coefficient = C(5, 3) — **M1**

= 10 — **A1**

Answer: 10 — **2 marks**

3 (3 marks)

$(1 + 2x)^4 = 1 + 4(2x) + 6(2x)^2 + 4(2x)^3 + (2x)^4$ — **M1 A1**

$= 1 + 8x + 24x^2 + 32x^3 + 16x^4$ — **A1**

Answer: $1 + 8x + 24x^2 + 32x^3 + 16x^4$ — **3 marks**

4 (2 marks)

(a) $1 + x + x^2 + x^3$ — **B1**

Answer: $1 + x + x^2 + x^3$ — **1 mark**

(b) $x^2 + x^4$ — **B1**

Answer: $x^2 + x^4$ — **1 mark**

5 (2 marks)

$\Sigma\, 3^n x^n = \Sigma\, (3x)^n$ is geometric with ratio 3x — **M1**

$= \frac{1}{1 - 3x}$, valid for $|x| < \frac{1}{3}$ — **A1**

Answer: $\frac{1}{1 - 3x}$, $|x| < \frac{1}{3}$ — **2 marks**

6 (3 marks)

$(1 - x)^{-2} = 1 + (-2)(-x) + \frac{(-2)(-3)}{2!}(-x)^2 + \frac{(-2)(-3)(-4)}{3!}(-x)^3$ — **M1 A1**

$= 1 + 2x + 3x^2 + 4x^3$ — **A1**

Answer: $1 + 2x + 3x^2 + 4x^3$ — **3 marks**

7 (3 marks)

$(1 - x)^{-3} = 1 + (-3)(-x) + \frac{(-3)(-4)}{2!}(-x)^2 + \frac{(-3)(-4)(-5)}{3!}(-x)^3 + ... = 1 + 3x + 6x^2 + 10x^3$ — **M1 A1**

Coefficient of x^n is $C(n + 2, 2) = \frac{(n + 1)(n + 2)}{2}$ — **A1**

Answer: $1 + 3x + 6x^2 + 10x^3$; $C(n + 2, 2)$ — **3 marks**

8 (3 marks)

Each flavour contributes $1 + x + x^2 + ... = \frac{1}{1 - x}$, so the generating function is $(1 - x)^{-3}$ — **M1**

The number of ways is the coefficient of x^4, which is C(6, 2) — **M1**

= 15 — **A1**

Answer: 15 — **3 marks**

9 (3 marks)

Three terms with exponents from 0, 1, 2 summing to 5: arrangements of (2, 2, 1) — **M1**

There are 3 such arrangements (the 1 can be in any of the three positions) — **M1**

Coefficient = 3 — **A1**

Answer: 3 — **3 marks**

10 (4 marks)

Each variable has generating function $1 + x + x^2 + x^3 + x^4 = \frac{1 - x^5}{1 - x}$ **M1**

Need the coefficient of x^8 in $(1 - x^5)^3(1 - x)^{-3}$ **M1**

= coefficient in $(1 - 3x^5 + ...)(1 + 3x + 6x^2 + ...)$: $C(10, 2) - 3 \times C(5, 2)$ **M1**

$= 45 - 30 = 15$ **A1**

Answer: 15 **4 marks**

11 (3 marks)

Differentiate both sides: $\frac{1}{(1 - x)^2} = \Sigma\, n\, x^{n-1}$ **M1 A1**

Multiply by x: $\Sigma\, n\, x^n = \frac{x}{(1 - x)^2}$ **A1**

Answer: A(x) = $\frac{x}{(1 - x)^2}$ **3 marks**

12 (3 marks)

$(1 - 2x)^{-2} = \Sigma\, (n + 1)(2x)^n$, using $(1 - y)^{-2} = \Sigma\, (n + 1)y^n$ with $y = 2x$ **M1 A1**

$a_n = (n + 1)2^n$ **A1**

Answer: $a_n = (n + 1)2^n$ **3 marks**

13 (4 marks)

The factor $\frac{1}{(1 - x^j)} = 1 + x^j + x^{2j} + ...$ records using the part j any number of times **E1**

Partitions of 5 with parts at most 3: 3 + 2, 3 + 1 + 1, 2 + 2 + 1, 2 + 1 + 1 + 1, 1 + 1 + 1 + 1 + 1 **M1 A1**

Coefficient of x^5 is 5 **A1**

Answer: 5 **4 marks**

14 (5 marks)

Each child has generating function $x + x^2 + ... + x^5 = x\frac{1 - x^5}{1 - x}$ **M1**

Need the coefficient of x^{10} in $x^3(1 - x^5)^3(1 - x)^{-3}$, that is of x^7 in $(1 - x^5)^3(1 - x)^{-3}$ **M1 A1**

$(1 - x^5)^3 = 1 - 3x^5 + ...$ **B1**

$C(9, 2) - 3 \times C(4, 2) = 36 - 18 = 18$ **A1**

Answer: 18 **5 marks**

15 (5 marks)

One die: $x + x^2 + ... + x^6 = x\frac{1 - x^6}{1 - x}$; three dice: $x^3(1 - x^6)^3(1 - x)^{-3}$ **M1 A1**

Need the coefficient of x^9 in $(1 - 3x^6 + ...)(1 - x)^{-3}$ **M1**

$C(11, 2) - 3 \times C(5, 2) = 55 - 30 = 25$ **A1**

Probability = $\frac{25}{216}$ **A1**

Answer: 25 ways; $\frac{25}{216}$ **5 marks**

16 (4 marks)

Generating function $\frac{1}{(1 - x)(1 - x^2)(1 - x^5)}$ **M1**

Number of 5p coins 0, 1 or 2 leaves 10p, 5p or 0p to make with 1p and 2p coins **M1**

With 1p and 2p: the number of 2p coins can be 0 to $\lfloor n/2 \rfloor$, giving 6, 3 and 1 ways **M1**

Total = 6 + 3 + 1 = 10 **A1**

Answer: 10 **4 marks**

17 (4 marks)

$(1 - 2x)^{-1/2} = 1 + (-\frac{1}{2})(-2x) + \frac{(-1}{2)(-3/2)/2!}(-2x)^2 + \frac{(-1}{2)(-3/2)(-5/2)/3!}(-2x)^3$ **M1 A1**

$= 1 + x + \frac{3}{2}x^2 + \frac{5}{2}x^3$ **A1**

Valid for $|2x| < 1$, that is $|x| < \frac{1}{2}$ **B1**

Answer: $1 + x + \frac{3}{2}x^2 + \frac{5}{2}x^3$; $|x| < \frac{1}{2}$ **4 marks**

18 (4 marks)

(a) Put $x = 1$: $2^n = \Sigma\, C(n, k)$ **B1**

Answer: $\Sigma\, C(n, k) = 2^n$ **1 mark**

(b) Differentiate: $n(1 + x)^{n-1} = \Sigma\, k\, C(n, k)x^{k-1}$ **M1 A1**

Put $x = 1$: $n\, 2^{n-1} = \Sigma\, k\, C(n, k)$ **A1**

Answer: $\Sigma\, k\, C(n, k) = n\, 2^{n-1}$ **3 marks**

19 (5 marks)

Apples: $1 + x^2 + x^4 + \ldots = \frac{1}{1 - x^2}$; bananas: $x + x^3 + \ldots = \frac{x}{1 - x^2}$; cherries: $\frac{1}{1 - x}$ **M1 A1**

Generating function $\frac{x}{(1 - x^2)^2(1 - x)}$ **A1**

Coefficient of x^6: apples a $\in$ 0, 2, 4, bananas b odd with $a + b \leq 6$, cherries make up the rest **M1**

a = 0: b = 1, 3, 5 (3 ways); a = 2: b = 1, 3 (2 ways); a = 4: b = 1 (1 way); total 6 **A1**

Answer: 6 **5 marks**

20 (4 marks)

Each part size j is either not used (1) or used once (x^j), so each factor is $1 + x^j$ **E1**

8, 7 + 1, 6 + 2, 5 + 3, 5 + 2 + 1, 4 + 3 + 1 **M1 A1**

Coefficient of x^8 is 6 **A1**

Answer: 6 **4 marks**

21 (6 marks)

(a) The coefficient of x^{n-1} in $(1 - x)^{-3}$ is $C(n + 1, 2) = \frac{n(n + 1)}{2}$, so $(1 - x)^{-3} = \Sigma_{n=1}^{\infty} \frac{n(n + 1)}{2}$ **M1**

x^{n-1}

Multiplying by x gives the result **A1**

Answer: $\Sigma \frac{n(n + 1)}{2} x^n = \frac{x}{(1 - x)^3}$ **2 marks**

(b) Put $x = \frac{1}{2}$: $\Sigma \frac{n(n + 1)}{2}(\frac{1}{2})^n = \frac{1}{2} \div (\frac{1}{2})^3 = 4$ **M1**

Doubling: $\Sigma \frac{n(n + 1)}{2^n} = 8$ **A1**

Answer: 8 **2 marks**

(c) $E[N(N + 1)] = \Sigma\, n(n + 1)(\frac{1}{2})^n = 8$ **M1**

$E[N^2] = E[N(N + 1)] - E[N] = 8 - 2 = 6$ **A1**

Answer: $E[N(N + 1)] = 8$; $E[N^2] = 6$ **2 marks**

22 (8 marks)

(a) One part: 8. Two parts: 7 + 1, 6 + 2, 5 + 3, 4 + 4 **M1**

Three parts: 6 + 1 + 1, 5 + 2 + 1, 4 + 3 + 1, 4 + 2 + 2, 3 + 3 + 2 **A1**

$p_3(8) = 1 + 4 + 5 = 10$ **A1**

Answer: 10 **3 marks**

(b) A partition with at most 3 parts, drawn as rows of dots, has at most 3 columns; the column lengths form a partition with each part at most 3 **E1**

This is reversible, so the two counts are equal **E1**

Answer: Conjugation of the partition (rows to columns) **2 marks**

(c) Coefficient of x^n counts solutions of 2a + 3b = n: 1, 0, 1, 1, 1, 1, 2, 1, 2 for n = 0 to 8 **M1 A1**

Multiplying by $\frac{1}{1-x}$ forms running totals, so the coefficient of x^8 is 1 + 0 + 1 + 1 + 1 + 1 + 2 + 1 + 2 = 10 **A1**

Answer: 10 **3 marks**

23 (6 marks)

1p coins: $\frac{1}{1-x}$; 2p coins: $1 + x^2 + x^4 + x^6 + x^8$; 5p coins: $1 + x^5 + x^{10} + x^{15}$ **M1 A1**

Each choice of numbers of 2p and 5p coins totalling at most 15p is completed by exactly one number of 1p coins **M1**

No 5p: 0 to 4 twos give 5 ways. One 5p: $2t \leq 10$, t = 0 to 4: 5 ways **A1**

Two 5p: $2t \leq 5$, t = 0, 1, 2: 3 ways. Three 5p: t = 0: 1 way **A1**

Total = 5 + 5 + 3 + 1 = 14 **A1**

Answer: 14 **6 marks**

24 (8 marks)

Differentiate $\Sigma\, n\, x^n = x(1-x)^{-2}$: $\Sigma\, n^2 x^{n-1} = (1-x)^{-2} + 2x(1-x)^{-3} = \frac{1+x}{(1-x)^3}$ **M1 A1**

Multiply by x: $A(x) = \frac{x(1+x)}{(1-x)^3}$ **A1**

Multiplying a series by $\frac{1}{1-x}$ replaces each coefficient by the running total, so the coefficient of x^n in $A(x)/(1-x)$ is $\Sigma_{r=1}^{n}\, r^2$ **E1**

$A(x)/(1-x) = (x + x^2)(1-x)^{-4}$; the coefficient of x^m in $(1-x)^{-4}$ is C(m + 3, 3), so the coefficient of x^n is C(n + 2, 3) + C(n + 1, 3) **M1 A1**

$= \frac{(n+2)(n+1)n}{6} + \frac{(n+1)n(n-1)}{6} = \frac{n(n+1)}{6}[(n+2) + (n-1)]$ **M1**

$= \frac{n(n+1)(2n+1)}{6}$ **A1**

Answer: $\Sigma\, r^2 = \frac{n(n+1)(2n+1)}{6}$ **8 marks**

25 (4 marks)

$(1-x)^{-3}$ allows a child to receive 0 sweets, so it counts selections that are not allowed **E1**

Each child needs $x + x^2 + \ldots = \frac{x}{1-x}$, so the generating function is $x^3(1-x)^{-3}$ **M1**

Need the coefficient of x^9 in $(1-x)^{-3}$: C(11, 2) **M1**

= 55 **A1**

Answer: 55 **4 marks**

26 (6 marks)

Type A: $x + x^2 + x^3 + x^4 = x\frac{1-x^4}{1-x}$; types B, C, D: $\frac{1-x^5}{1-x}$ each **M1 A1**

Need the coefficient of x^9 in $(1-x^4)(1-x^5)^3(1-x)^{-4}$ **M1**

$(1-x^4)(1-x^5)^3 = 1 - x^4 - 3x^5 + 3x^9 +$ (higher powers) **A1**

C(12, 3) − C(8, 3) − 3 × C(7, 3) + 3 × C(3, 3) **M1**

= 220 − 56 − 105 + 3 = 62 **A1**

Answer: 62 **6 marks**

27 (9 marks)

(a) Coefficient of x^n on the left is C(2n, n) **B1**

On the right the coefficient of x^n is $\Sigma_{k=0}^{n}$ C(n, k) C(n, n − k) **M1 A1**

C(n, n − k) = C(n, k), so Σ C(n, k)^2 = C(2n, n) **A1**

Answer: Σ C(n, k)^2 = C(2n, n) **4 marks**

(b) 1 + 25 + 100 + 100 + 25 + 1 **M1**

= 252 = C(10, 5) **A1**

Answer: 252 **2 marks**

(c) Equate the coefficients of x^r on both sides of $(1 + x)^{m+n} = (1 + x)^m(1 + x)^n$: $C(m + n, r) = \Sigma C(m, k)\, C(n, r - k)$ **M1**

For m = 6, n = 5, r = 4: C(11, 4) = 330 **A1**

The term k counts committees with k men and 4 − k women **E1**

Answer: 330 **3 marks**

28 (9 marks)

(a) $1 + x^k = \dfrac{1 - x^{2k}}{1 - x^k}$ **M1**

$F(x) = \prod (1 - x^{2k}) / \prod (1 - x^k)$ **M1**

The numerator contains exactly the factors $1 - x^j$ with j even **E1**

These cancel with the even-j factors in the denominator, leaving the factors with j odd in the denominator **M1**

$F(x) = \prod \dfrac{1}{1 - x^{2k-1}} = H(x)$ **A1**

Answer: F(x) = H(x) **5 marks**

(b) Distinct: 9, 8 + 1, 7 + 2, 6 + 3, 5 + 4, 6 + 2 + 1, 5 + 3 + 1, 4 + 3 + 2 **M1 A1**

Odd: 9, 7 + 1 + 1, 5 + 3 + 1, 5 + 1 + 1 + 1 + 1, 3 + 3 + 3, 3 + 3 + 1 + 1 + 1, 3 + 1 + 1 + 1 + 1 + 1 + 1 + 1, nine 1s **A1**

Both lists have 8 partitions **A1**

Answer: 8 in each case **4 marks**

29 (8 marks)

(a) $1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \ldots = \dfrac{e^x + e^{-x}}{2}$ **B1**

$1 + x + \dfrac{x^2}{2!} + \ldots = e^x$ **B1**

Answer: $\mathbf{\dfrac{e^x + e^{-x}}{2}}$**;** $\mathbf{e^x}$ **2 marks**

(b) The strings are built from the letters A (an even number of times), B and C, so by the product fact $W(x) = \dfrac{e^x + e^{-x}}{2} \times e^x \times e^x$ **M1**

$= \dfrac{e^{3x} + e^x}{2}$ **A1**

The coefficient of $\dfrac{x^n}{n!}$ in e^{3x} is 3^n and in e^x is 1, so $w_n = \dfrac{3^n + 1}{2}$ **A1**

Answer: $\mathbf{w_n = \dfrac{3^n + 1}{2}}$ **3 marks**

(c) n = 2: AA, BB, BC, CB, CC, which is 5 strings, and $\dfrac{9 + 1}{2} = 5$ **M1 A1**

$w_6 = \dfrac{729 + 1}{2} = 365$ **A1**

Answer: 5 (agrees); $\mathbf{w_6 = 365}$ **3 marks**

30 (9 marks)

The first digit has generating function $x + x^2 + ... + x^9 = x\frac{1 - x^9}{1 - x}$ **M1**

Each other digit has $1 + x + ... + x^9 = \frac{1 - x^{10}}{1 - x}$ **A1**

Product = $x(1 - x^9)(1 - x^{10})^3(1 - x)^{-4}$; need the coefficient of x^{17} in $(1 - x^9)(1 - x^{10})^3(1 - x)^{-4}$ **M1 A1**

$(1 - x^9)(1 - x^{10})^3 = 1 - x^9 - 3x^{10}$ + (terms of degree 19 or more) **B1**

Terms of degree above 17 do not contribute **E1**

Coefficient = $C(20, 3) - C(11, 3) - 3 \times C(10, 3)$ **M1**

$= 1140 - 165 - 360$ **A1**

$= 615$ **A1**

Answer: 615 **9 marks**

Recurrence Relations and Generating Functions

1 (2 marks)

$a_1 = 2 \times 0 + 1 = 1$, $a_2 = 2 \times 1 + 1 = 3$ **M1**

$a_3 = 7$, $a_4 = 15$ **A1**

Answer: 1, 3, 7, 15 **2 marks**

2 (2 marks)

$\frac{3}{1 - 2x} = 3\ \Sigma\ (2x)^n$ **M1**

$a_n = 3 \times 2^n$ **A1**

Answer: $a_n = 3 \times 2^n$ **2 marks**

3 (2 marks)

$F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$ **M1**

$F_6 = 8$, $F_7 = 13$, $F_8 = 21$ **A1**

Answer: 1, 2, 3, 5, 8, 13, 21 **2 marks**

4 (3 marks)

$1 = A(1 - 2x) + B(1 - x)$ **M1**

Put $x = 1$: $1 = -A$, so $A = -1$. Put $x = \frac{1}{2}$: $1 = \frac{B}{2}$, so $B = 2$ **M1 A1**

Answer: A = −1, B = 2 **3 marks**

5 (3 marks)

(a) The series is $1 + x^2 + x^4 + ...$, geometric with ratio x^2: $\frac{1}{1 - x^2}$ **B1**

Answer: $\frac{1}{1 - x^2}$ **1 mark**

(b) $\Sigma\ 2^n x^n = \frac{1}{1 - 2x}$ and $\Sigma\ 3^n x^n = \frac{1}{1 - 3x}$ **M1**

Sum = $\frac{(1 - 3x) + (1 - 2x)}{(1 - 2x)(1 - 3x)} = \frac{2 - 5x}{(1 - 2x)(1 - 3x)}$ **A1**

Answer: $\frac{2 - 5x}{(1 - 2x)(1 - 3x)}$ **2 marks**

6 (2 marks)

Coefficients of $\frac{1}{1 - x}$ are 1 and of $\frac{1}{1 - 4x}$ are 4^n **M1**

$a_n = 1 - 4^n$ **A1**

Answer: $a_n = 1 - 4^n$ **2 marks**

7 (3 marks)

$A(x) - 5 = 3xA(x)$ **M1**

$A(x) = \frac{5}{1-3x}$ **A1**

$a_n = 5 \times 3^n$ **A1**

Answer: $a_n = 5 \times 3^n$ **3 marks**

8 (4 marks)

$A(x) - 1 = xA(x) + 2\sum_{n\geq 1} x^n = xA(x) + \frac{2x}{1-x}$ **M1 A1**

$A(x) = \frac{1}{1-x} + \frac{2x}{(1-x)^2}$ **A1**

Coefficient of x^n: $1 + 2n$ **A1**

Answer: $a_n = 2n + 1$ **4 marks**

9 (6 marks)

$(1 - 5x + 6x^2)A(x) = a_0 + (a_1 - 5a_0)x = 1 - 5x$ **M1 A1**

$A(x) = \frac{1-5x}{(1-2x)(1-3x)} = \frac{P}{1-2x} + \frac{Q}{1-3x}$ **M1**

$1 - 5x = P(1 - 3x) + Q(1 - 2x)$: $x = \frac{1}{2}$ gives $P = 3$, $x = \frac{1}{3}$ gives $Q = -2$ **M1 A1**

$a_n = 3 \times 2^n - 2 \times 3^n$ **A1**

Answer: $a_n = 3 \times 2^n - 2 \times 3^n$ **6 marks**

10 (4 marks)

The last move is one step (leaving a climb of $n - 1$ steps: s_{n-1} ways) **E1**

or two steps (leaving $n - 2$ steps: s_{n-2} ways); these cases are different and cover everything **E1**

$s_1 = 1$, $s_2 = 2$, $s_3 = 3$, $s_4 = 5$, $s_5 = 8$ **M1**

$s_6 = 13$ **A1**

Answer: 13 **4 marks**

11 (4 marks)

$1 - 5x + 6x^2 = (1 - 2x)(1 - 3x)$ **B1**

$\frac{x}{(1-2x)(1-3x)} = \frac{P}{1-3x} + \frac{Q}{1-2x}$: $x = P(1 - 2x) + Q(1 - 3x)$, giving $P = 1$, $Q = -1$ **M1 A1**

$a_n = 3^n - 2^n$ **A1**

Answer: $a_n = 3^n - 2^n$ **4 marks**

12 (6 marks)

(a) $(1 - x - 2x^2)A(x) = 2 - x$ **M1**

Comparing coefficients: $a_0 = 2$, $a_1 - a_0 = -1$ so $a_1 = 1$ **A1**

$a_n = a_{n-1} + 2a_{n-2}$ for $n \geq 2$ **A1**

Answer: $a_n = a_{n-1} + 2a_{n-2}$; $a_0 = 2$, $a_1 = 1$ **3 marks**

(b) $1 - x - 2x^2 = (1 - 2x)(1 + x)$; $\frac{2-x}{(1-2x)(1+x)} = \frac{P}{1-2x} + \frac{Q}{1+x}$ **M1**

$2 - x = P(1 + x) + Q(1 - 2x)$: $P = 1$, $Q = 1$ **A1**

$a_n = 2^n + (-1)^n$ **A1**

Answer: $a_n = 2^n + (-1)^n$ **3 marks**

13 (5 marks)

$A(x) - 1 = xA(x) + \Sigma_{n\geq 1}(2x)^n = xA(x) + \frac{2x}{1-2x}$ **M1 A1**

$A(x)(1-x) = 1 + \frac{2x}{1-2x} = \frac{1}{1-2x}$ **A1**

$A(x) = \frac{1}{(1-x)(1-2x)} = \frac{2}{1-2x} - \frac{1}{1-x}$ **M1**

$a_n = 2^{n+1} - 1$ **A1**

Answer: $a_n = 2^{n+1} - 1$ **5 marks**

14 (7 marks)

(a) $F(x) - F_0 - F_1x = x(F(x) - F_0) + x^2F(x)$ **M1 A1**

$F(x)(1 - x - x^2) = x$, so $F(x) = \frac{x}{1-x-x^2}$ **A1**

Answer: $F(x) = \frac{x}{1-x-x^2}$ **3 marks**

(b) $\alpha + \beta = 1$ and $\alpha\beta = -1$, so $(1-\alpha x)(1-\beta x) = 1 - x - x^2$ **B1**

$\frac{x}{(1-\alpha x)(1-\beta x)} = \frac{1}{\alpha-\beta}[\frac{1}{1-\alpha x} - \frac{1}{1-\beta x}]$ **M1**

$\alpha - \beta = \sqrt{5}$ **A1**

$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}$ **A1**

Answer: $F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}$ **4 marks**

15 (6 marks)

$(1 - 4x + 4x^2)A(x) = a_0 + (a_1 - 4a_0)x = 1 - x$ **M1 A1**

$A(x) = \frac{1-x}{(1-2x)^2}$ **A1**

$1 - x = \frac{1}{2}(1-2x) + \frac{1}{2}$, so $A(x) = \frac{1}{2}\frac{1}{1-2x} + \frac{1}{2}\frac{1}{(1-2x)^2}$ **M1 A1**

$a_n = \frac{1}{2}2^n + \frac{1}{2}(n+1)2^n = (n+2)2^{n-1}$ **A1**

Answer: $a_n = (n+2)2^{n-1}$ **6 marks**

16 (6 marks)

(a) The last hop is 1 (from stone $n-1$: t_{n-1} ways) or 3 (from stone $n-3$: t_{n-3} ways) **E1 E1**

$t_1 = 1$ and $t_2 = 1$ (only 1 + 1) **B1**

Answer: $t_1 = t_2 = 1$ **3 marks**

(b) $(1 - x - x^3)T(x) = t_0 + (t_1 - t_0)x + (t_2 - t_1)x^2 = 1$ **M1 A1**

t: 1, 1, 1, 2, 3, 4, 6, 9, 13, 19, so $t_9 = 19$ **A1**

Answer: $t_9 = 19$ **3 marks**

17 (6 marks)

(a) A valid string ends in 0 or 1 (2 choices) after a valid string of length $n-1$: $2a_{n-1}$ **M1 E1**

or ends in 2 preceded by 0 or 1 (2 choices) after a valid string of length $n-2$: $2a_{n-2}$ **E1**

Answer: $a_n = 2a_{n-1} + 2a_{n-2}$ **3 marks**

(b) $(1 - 2x - 2x^2)A(x) = 1 + (3-2)x = 1 + x$ **M1 A1**

$a_2 = 8$, $a_3 = 22$, $a_4 = 60$, $a_5 = 164$ **A1**

Answer: $a_5 = 164$ **3 marks**

18 (6 marks)

The new line is cut into n pieces by the $n - 1$ crossings and each piece splits one region in two, so adds n regions **E1**

$R(x) - 1 = xR(x) + \Sigma_{n\geq 1}\, n\, x^n = xR(x) + \frac{x}{(1-x)^2}$ **M1 A1**

$R(x) = \frac{1}{1-x} + \frac{x}{(1-x)^3}$ **A1**

The coefficient of x^n in $x(1-x)^{-3}$ is $C(n+1, 2)$ **M1**

$r_n = 1 + \frac{n(n+1)}{2}$ **A1**

Answer: $r_n = 1 + \frac{n(n+1)}{2}$ **6 marks**

19 (5 marks)

$H(x) = 2xH(x) + \Sigma_{n\geq 1}\, x^n = 2xH(x) + \frac{x}{1-x}$ **M1 A1**

$H(x) = \frac{x}{(1-x)(1-2x)}$ **A1**

$= \frac{1}{1-2x} - \frac{1}{1-x}$ **M1**

$h_n = 2^n - 1$ **A1**

Answer: $h_n = 2^n - 1$ **5 marks**

20 (6 marks)

The last column is covered by a vertical domino (a_{n-1} ways) **E1**

or the last two columns by two horizontal dominoes or one square (2 ways each leaving a_{n-2}) **E1**

$(1 - x - 2x^2)A(x) = 1 + (1-1)x = 1$ **M1**

$A(x) = \frac{1}{(1-2x)(1+x)} = \frac{2}{3}\frac{1}{1-2x} + \frac{1}{3}\frac{1}{1+x}$ **M1 A1**

$a_n = \frac{2^{n+1} + (-1)^n}{3}$ **A1**

Answer: $a_n = \frac{2^{n+1} + (-1)^n}{3}$ **6 marks**

21 (8 marks)

$A(x) - 1 = x(2A + B)$ and $B(x) = x(A + 2B)$ **M1 A1**

From the second: $B = \frac{xA}{1-2x}$ **A1**

Substituting: $A(1-2x) - \frac{x^2 A}{1-2x} = 1$, so $A\frac{(1-2x)^2 - x^2}{1-2x} = 1$, giving $A = \frac{1-2x}{(1-x)(1-3x)}$ **M1 A1**

$\frac{1-2x}{(1-x)(1-3x)} = \frac{1}{2}\frac{1}{1-x} + \frac{1}{2}\frac{1}{1-3x}$, so $a_n = \frac{3^n + 1}{2}$ **M1 A1**

$B = \frac{x}{(1-x)(1-3x)} = \frac{1}{2}[\frac{1}{1-3x} - \frac{1}{1-x}]$, so $b_n = \frac{3^n - 1}{2}$ **A1**

Answer: $a_n = \frac{3^n + 1}{2}$, $b_n = \frac{3^n - 1}{2}$ **8 marks**

22 (6 marks)

If the last symbol is not 0 (3 choices) the first $n-1$ symbols have an even number of 0s: $3a_{n-1}$. If it is 0, the first $n-1$ symbols have an odd number of 0s: $4^{n-1} - a_{n-1}$ **M1 E1**

Total $a_n = 3a_{n-1} + 4^{n-1} - a_{n-1} = 2a_{n-1} + 4^{n-1}$ **A1**

$A(x) - 1 = 2xA(x) + \frac{x}{1-4x}$, so $A(x) = \frac{1-3x}{(1-2x)(1-4x)}$ **M1**

$\frac{1-3x}{(1-2x)(1-4x)} = \frac{1}{2}\frac{1}{1-2x} + \frac{1}{2}\frac{1}{1-4x}$ **M1**

$a_n = \frac{4^n + 2^n}{2}$ **A1**

Answer: $a_n = \frac{4^n + 2^n}{2}$ **6 marks**

23 (6 marks)

$A(x) - 1 = 2xA(x) + \Sigma_{n\geq 1}\, n\, x^n = 2xA(x) + \frac{x}{(1-x)^2}$ **M1 A1**

$A(x)(1-2x) = 1 + \frac{x}{(1-x)^2} = \frac{1-x+x^2}{(1-x)^2}$ **A1**

$A(x) = \frac{1-x+x^2}{(1-2x)(1-x)^2} = \frac{3}{1-2x} - \frac{1}{1-x} - \frac{1}{(1-x)^2}$ **M1 A1**

$a_n = 3 \times 2^n - 1 - (n+1) = 3 \times 2^n - n - 2$ **A1**

Answer: $a_n = 3 \times 2^n - n - 2$ **6 marks**

24 (5 marks)

(a) The recurrence only holds for $n \geq 1$, so $A(x)$ contains the term $a_0 = 1$ which the right-hand side does not **E1**

The constant -2 appears only for $n \geq 1$, so its series is $-2x - 2x^2 - \ldots = -\frac{2x}{1-x}$, not $-\frac{2}{1-x}$ **E1**

Answer: a_0 omitted; the series for -2 starts at x^1 **2 marks**

(b) $A(x) - 1 = 3xA(x) - \frac{2x}{1-x}$ **M1**

$A(x)(1-3x) = \frac{1-x-2x}{1-x} = \frac{1-3x}{1-x}$, so $A(x) = \frac{1}{1-x}$ **M1 A1**

Answer: $a_n = 1$ for all n **3 marks**

25 (6 marks)

Multiplying $F(x)$ by $\frac{1}{1-x}$ gives running totals, so $\Sigma_{k=0}^{n} F_k$ is the coefficient of x^n in **M1 E1**

$\frac{x}{(1-x)(1-x-x^2)}$

The coefficient of x^n in F_{n+2} is that of $\frac{F(x)-x}{x^2} = \frac{1+x}{1-x-x^2}$ **M1 A1**

The coefficient of x^n in -1 is that of $-\frac{1}{1-x}$, so $F_{n+2} - 1$ has generating function $\frac{1+x}{1-x-x^2} - \frac{1}{1-x} = \frac{(1+x)(1-x) - (1-x-x^2)}{(1-x)(1-x-x^2)}$ **M1**

The numerator is x, so this equals $\frac{x}{(1-x)(1-x-x^2)}$, and the coefficients agree **A1**

Answer: $\Sigma_{k=0}^{n} F_k = F_{n+2} - 1$ **6 marks**

26 (6 marks)

(a) One odd part has generating function $x + x^3 + x^5 + \ldots = \frac{x}{1 - x^2}$ **M1**

A composition is a sequence of any number of parts: $C(x) = \frac{1}{1 - \frac{x}{1 - x^2}}$ **M1**

$= \frac{1 - x^2}{1 - x^2 - x}$ **A1**

Answer: $C(x) = \frac{1 - x^2}{1 - x - x^2}$ **3 marks**

(b) $\frac{1 - x^2}{1 - x - x^2} = \frac{1}{1 - x - x^2} - \frac{x^2}{1 - x - x^2}$, and the coefficient of x^n in $\frac{1}{1 - x - x^2}$ is F_{n+1} **M1**

$c_n = F_{n+1} - F_{n-1} = F_n$ **A1**

$c_7 = F_7 = 13$ **A1**

Answer: $c_7 = 13$ **3 marks**

27 (10 marks)

(a) $\Sigma_{n\geq 0}\, C_{n+1}x^{n+1} = C(x) - 1$ **M1**

The right-hand side summed is $x\, \Sigma_{n\geq 0}\, (\Sigma_k\, C_k C_{n-k})\, x^n = x(C(x))^2$, a convolution product **M1 A1**

Answer: $C(x) = 1 + x(C(x))^2$ **3 marks**

(b) $xC^2 - C + 1 = 0$, so $C = \frac{1 \pm \sqrt{1 - 4x}}{2x}$ **M1**

As $x \to 0$, $C(x) \to 1$ but with + the expression is unbounded, so take the minus sign **E1**

$C(x) = \frac{1 - \sqrt{1 - 4x}}{2x}$ **A1**

Answer: $C(x) = \frac{1 - \sqrt{1 - 4x}}{2x}$ **3 marks**

(c) The coefficient of x^{n+1} in $\sqrt{1 - 4x}$ is $-\frac{2}{n + 1}C(2n, n)$ **M1**

C_n is the coefficient of x^n in $\frac{1 - \sqrt{1 - 4x}}{2x}$, which is $-\frac{1}{2}$ times that **M1**

$C_n = \frac{1}{n + 1}C(2n, n)$ **A1**

$C_5 = \frac{1}{6} \times 252 = 42$ **A1**

Answer: $C_n = \frac{1}{n + 1}C(2n, n)$; $C_5 = 42$ **4 marks**

28 (9 marks)

(a) A valid sequence ends in T (preceded by a valid sequence of length $n - 1$), in TH (valid sequence of length $n - 2$), or in THH (valid of length $n - 3$); these are the only cases when $n \geq 3$ **M1 E1 E1**

$b_0 = 1$, $b_1 = 2$, $b_2 = 4$ **B1**

Answer: $b_0 = 1$, $b_1 = 2$, $b_2 = 4$ **4 marks**

(b) $(1 - x - x^2 - x^3)B(x) = b_0 + (b_1 - b_0)x + (b_2 - b_1 - b_0)x^2$ **M1 A1**

$= 1 + x + x^2$ **A1**

Answer: $B(x) = \frac{1 + x + x^2}{1 - x - x^2 - x^3}$ **3 marks**

(c) $b_3 = 7$, $b_4 = 13$, $b_5 = 24$, $b_6 = 44$, $b_7 = 81$ **M1**

Probability $= \frac{81}{128}$ **A1**

Answer: $b_7 = 81$; $\frac{81}{128}$ **2 marks**

29 (9 marks)

Multiply by x^n and sum: $\frac{A - a_0 - a_1x}{x^2} - 4\frac{A - a_0}{x} + 4A = \frac{1}{1 - 2x}$ **M1 A1**

Multiply by x^2: $A - x - 4xA + 4x^2A = \frac{x^2}{1 - 2x}$ **A1**

$A(1 - 2x)^2 = x + \frac{x^2}{1 - 2x} = \frac{x - x^2}{1 - 2x}$, so $A = \frac{x(1 - x)}{(1 - 2x)^3}$ **M1 A1**

Let $u = 1 - 2x$: $x = \frac{1 - u}{2}$ and $1 - x = \frac{1 + u}{2}$, so $x(1 - x) = \frac{1 - u^2}{4}$ **M1**

$A = \frac{1}{4}[\frac{1}{u^3} - \frac{1}{u}]$ **A1**

Coefficient of x^n in $(1 - 2x)^{-3}$ is $C(n + 2, 2)2^n$ **M1**

$a_n = \frac{1}{4}2^n[\frac{(n + 1)(n + 2)}{2} - 1] = 2^{n-3}n(n + 3)$ **A1**

Answer: $a_n = 2^{n-3}\, n(n + 3)$ **9 marks**

30 (9 marks)

(a) Choose the k numbers that are fixed: $C(n, k)$ ways **M1**

The other $n - k$ numbers must be deranged among themselves: D_{n-k} ways **M1**

Each permutation has exactly one set of fixed numbers, so summing over k counts every permutation once, giving n! **E1**

Answer: $\Sigma\, C(n, k)\, D_{n-k} = n!$ **3 marks**

(b) $e^x = \Sigma\, \frac{x^n}{n!}$ has every coefficient equal to 1, so the coefficient of $\frac{x^n}{n!}$ in $e^xD(x)$ is $\Sigma\, C(n, k)\, D_{n-k}$ **M1 A1**

$= n!$

$\Sigma\, n!\, \frac{x^n}{n!} = \Sigma\, x^n = \frac{1}{1 - x}$ **A1**

Answer: $e^xD(x) = \frac{1}{1 - x}$ **3 marks**

(c) $D(x) = \frac{e^{-x}}{1 - x} = (\Sigma\, \frac{(-1)^kx^k}{k!})(\Sigma\, x^j)$ **M1**

The coefficient of x^n is $\Sigma_{k=0}^{\,n}\, \frac{(-1)^k}{k!}$, so $D_n = n!\, \Sigma_{k=0}^{\,n}\, \frac{(-1)^k}{k!}$ **A1**

$D_5 = 120 - 120 + 60 - 20 + 5 - 1 = 44$ **A1**

Answer: $D_n = n!\, \Sigma\, \frac{(-1)^k}{k!}$; $D_5 = 44$ **3 marks**