Graph Colouring

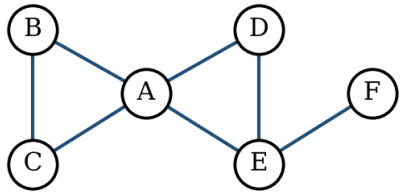
1 Write down the chromatic number of the complete graph K_5, giving a reason. **[2]**

2 Write down the chromatic number of each of the following graphs.

(a) The cycle C_6 (a single cycle on six vertices). **[1]**

(b) The cycle C_7. **[1]**

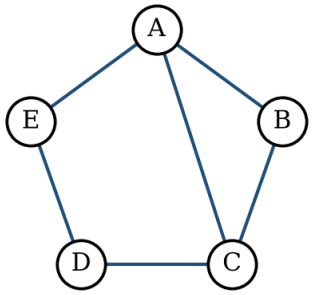
3 The graph G is shown in the diagram.



(a) Write down the maximum degree of G. **[1]**

(b) Hence state an upper bound for the chromatic number of G. **[1]**

4 The graph shown is coloured as follows: A red, B blue, C red, D blue, E green.



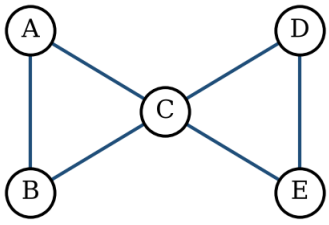
(a) Explain why this is not a proper colouring. **[1]**

(b) Change the colour of one vertex to obtain a proper colouring. **[1]**

5 State the four colour theorem. **[2]**

6 Find the number of ways of properly colouring K_4 using 5 available colours. **[2]**

7 The graph shown consists of two triangles ABC and CDE which share the vertex C.

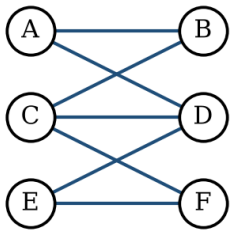


(a) Explain why the chromatic number of the graph is at least 3. **[1]**

(b) Find a proper colouring using three colours and hence state the chromatic number. **[2]**

8 A graph has vertices A, B, C, D, E, F and edges AB, BC, CD, DE, EF, FA and AD. Use the greedy algorithm, taking the vertices in the order A, B, C, D, E, F and giving each vertex the lowest numbered colour not used by an adjacent coloured vertex, to colour the graph. State the number of colours used. **[3]**

9 The graph G is shown in the diagram. Show that G is bipartite, stating the two sets of vertices, and write down its chromatic number.



[3]

10 Write down the chromatic number of each graph, giving a brief reason.

(a) The complete bipartite graph $K_{3,4}$. **[1]**

(b) A path on 9 vertices. **[1]**

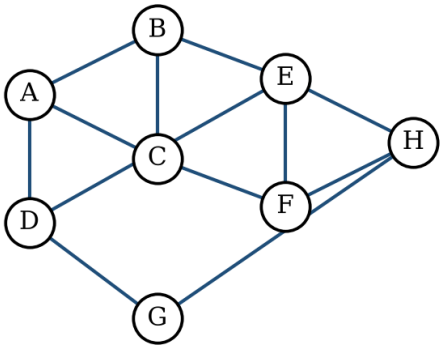
(c) The cycle C_9. **[1]**

11 A path has 6 vertices in a row. Find the number of ways of properly colouring it with 4 colours. **[3]**

12 Six examinations A, B, C, D, E, F are to be timetabled. Two examinations cannot be held at the same time if some student sits both. The pairs with students in common are AB, AC, BC, BD, CE, DE, DF and EF.
Find the least number of time slots needed, justifying your answer, and give a suitable timetable. **[4]**

13 The graph G is obtained from the complete graph K_6 by deleting one edge. Find the chromatic number of G, justifying your answer. **[3]**

14 The graph G is shown in the diagram. In this question, when vertices have equal degree they are taken in alphabetical order.

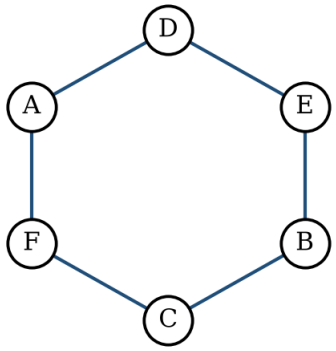


(a) List the vertices in order of decreasing degree, showing the degrees. **[2]**

(b) Use the Welsh-Powell algorithm to colour G, and state the number of colours used. **[3]**

(c) Explain why the chromatic number of G is 3. **[2]**

15 The graph shown has six vertices and edges AD, AF, CB, CF, EB and ED.



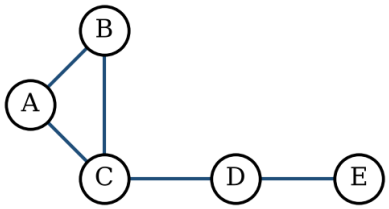
(a) Use the greedy algorithm with the vertices taken in the order A, B, C, D, E, F to colour the graph. State the number of colours used. **[3]**

(b) Show that the graph is bipartite, and hence comment on the result of part (a). **[3]**

16 The chromatic polynomial P(G, k) of a graph G gives the number of proper colourings of G with k colours. For an edge e, P(G, k) = P(G − e, k) − P(G / e, k), where G − e is G with e deleted and G / e is G with e contracted.

Given that a path with 4 vertices has chromatic polynomial $k(k-1)^3$ and K_3 has chromatic polynomial $k(k-1)(k-2)$, use deletion-contraction to show that the cycle C_4 has chromatic polynomial $k(k-1)(k^2-3k+3)$. Hence find the number of proper colourings of C_4 using 3 colours. **[5]**

17 The graph shown consists of a triangle ABC with a path C, D, E attached at C.
Find, in factorised form, the chromatic polynomial of the graph, and hence the number of proper colourings using 5 colours.



[4]

18 You may use the result that a connected simple planar graph with $v \geq 3$ vertices has at most $3v - 6$ edges, and Euler's formula $v - e + f = 2$.

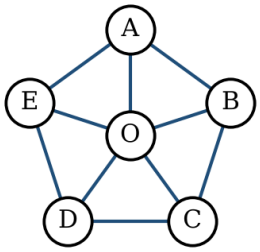
(a) A connected simple graph has 8 vertices and 19 edges. Show that it is not planar. **[2]**

(b) The graph $K_{3,3}$ has 6 vertices and 9 edges and contains no triangles. Assuming it is planar, show that this leads to a contradiction. **[4]**

19 The complete bipartite graph $K_{2,3}$ has two vertices U and V in one set and three vertices X, Y and Z in the other set, and each of U and V is joined to each of X, Y and Z (so there are no other edges).

By considering separately the cases where U and V have the same colour and where they have different colours, find the chromatic polynomial of $K_{2,3}$, and hence the number of proper colourings of $K_{2,3}$ using 3 colours. **[5]**

20 The wheel graph has a hub joined to every vertex of a rim cycle. The diagram shows the wheel with a rim of 5 vertices. Let W_n be the wheel with a rim of n vertices.



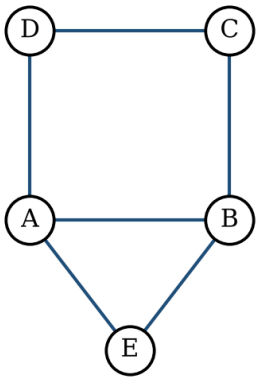
(a) Find the chromatic number of W_6, explaining your answer. **[2]**

(b) Find the chromatic number of W_7, explaining your answer. **[2]**

(c) State the chromatic number of W_n in terms of whether n is even or odd. **[1]**

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21 The graph H shown has vertices A, B, C, D, E and edges AB, BC, CD, DA, AE, BE. The chromatic polynomial of the cycle C_4 is $k(k-1)(k^2-3k+3)$.

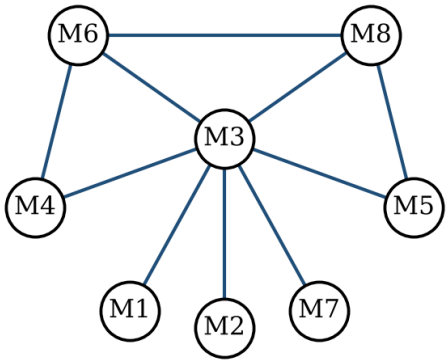


(a) By applying deletion-contraction to the edge AE, show that the chromatic polynomial of H is $k(k-1)(k-2)(k^2-3k+3)$. **[5]**

(b) Find the number of proper colourings of H using 4 colours, and deduce the chromatic number of H. **[3]**

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22 A conference has eight meetings M1 to M8. The graph shows which pairs of meetings cannot be held at the same time (each edge joins a pair that share a delegate).

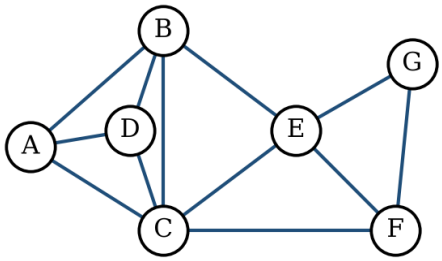


(a) Find the least number of time slots needed if any number of meetings may run at the same time in a slot. **[3]**

(b) Only three rooms are available, so at most three meetings can run in a slot. Find the least number of slots now, justifying your answer. **[4]**

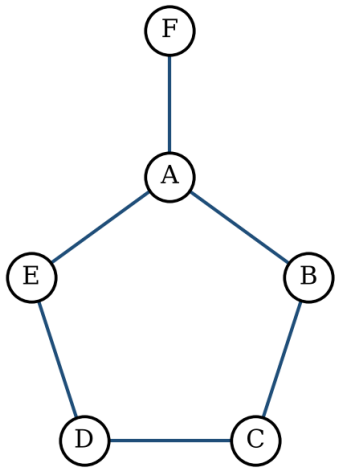
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23 Seven radio transmitters A to G are to be assigned frequencies. Two transmitters whose ranges overlap, shown by an edge in the graph, must use different frequencies.
Find the least number of frequencies needed, and give an assignment which uses this number.
Explain why this number is consistent with the four colour theorem.



[5]

24 The graph G has no triangles. A student writes: "G has no triangle, so there is no clique of size 3, so G can be coloured with 2 colours."
Explain the error in the student's reasoning, and find the chromatic number of G.



[4]

25 A connected simple graph G has chromatic polynomial $P(k) = k^4 - 4k^3 + 5k^2 - 2k$.

(a) Write down the number of vertices of G and the number of edges of G. **[2]**

(b) Factorise P(k) and hence find the chromatic number of G. **[3]**

(c) Find the number of proper colourings of G with 6 colours. **[1]**

26 Nine lectures must be given in rooms, and a room cannot host two lectures whose times overlap. The lectures run at these times (24 hour clock; a lecture ending at 11 does not overlap one starting at 11):
L1 9 to 11, L2 10 to 12, L3 11 to 13, L4 9 to 10, L5 12 to 14, L6 13 to 15, L7 10 to 11, L8 14 to 16, L9 12 to 13.
Model the allocation as a graph colouring problem, and find the least number of rooms needed. Justify that this number cannot be reduced, and give a room allocation. **[6]**

27 Let $P(C_n, k)$ denote the chromatic polynomial of the cycle C_n ($n \geq 3$), and recall that a path on n vertices has chromatic polynomial $k(k-1)^{n-1}$.
Use deletion-contraction and induction on n to prove that $P(C_n, k) = (k-1)^n + (-1)^n(k-1)$ for all $n \geq 3$. **[8]**

28 G is a simple graph with chromatic number c.

(a) By considering a proper colouring of G with c colours and two of its colour classes, prove that G has at least $c(c-1)/2$ edges. **[4]**

(b) Hence find the greatest possible chromatic number of a simple graph with 12 edges, and give an example of a graph that achieves it. **[4]**

29 In this question you may use the result that a simple connected planar graph with $v \geq 3$ vertices has at most $3v - 6$ edges.
Prove by induction on the number of vertices that every simple planar graph can be properly coloured using at most six colours. **[9]**

30 The chromatic polynomial of the cycle C_n is $(k-1)^n + (-1)^n(k-1)$. The wheel W_n is formed from C_n by adding a hub vertex joined to every vertex of the cycle.

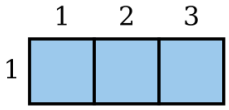
(a) Explain why the number of proper colourings of W_n with k colours is k times the number of proper colourings of C_n with $k-1$ colours, and hence find the chromatic polynomial of W_n. **[3]**

(b) Show that $P(W_4, k) = k(k-1)(k-2)(k^2 - 5k + 7)$, and find the number of proper colourings of W_4 using 4 colours. **[3]**

(c) Show that W_5 cannot be properly coloured with 3 colours but can with 4, and find the number of proper colourings using 4 colours. **[3]**

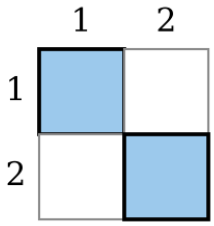
Rook Polynomials and Restricted Permutations

1 The rook polynomial of a board is $R(x) = 1 + r_1x + r_2x^2 + ...$, where r_k is the number of ways of placing k rooks on squares of the board so that no two rooks are in the same row or column.
Write down the rook polynomial of the board shown, which has three squares in a single row.



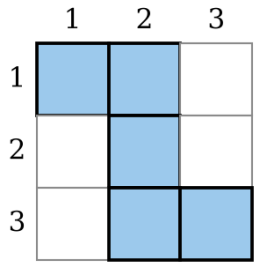
[2]

2 Write down the rook polynomial of the board shown, whose two squares are in different rows and different columns.



[2]

3 For the board shown, write down the value of r_0 and the value of r_1.



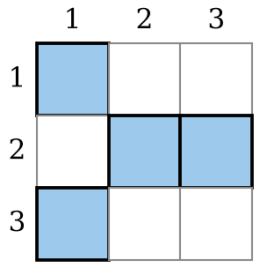
[2]

4 The rook polynomial of a board of forbidden squares in a 3 × 3 grid is $1 + 2x + x^2$. Use the formula for the number of permutations of 1, 2, 3 with no object in a forbidden position, $N = 3! - r_1 \times 2! + r_2 \times 1!$, to find N. **[2]**

5 All the squares of a board lie in one column, and there are n of them. Write down its rook polynomial, and explain your answer. **[2]**

6 A derangement of 1, 2, 3 is a permutation in which no number is in its own position. List all the derangements of 1, 2, 3 and state how many there are. **[2]**

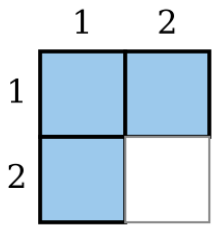
7 In the 3 × 3 grid shown, the shaded squares are the forbidden positions for a permutation of 1, 2, 3 (row i, column j means that i is placed in position j).



(a) Find the rook polynomial of the shaded board. **[2]**

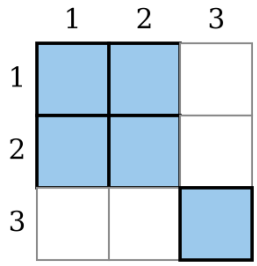
(b) Hence find the number of permutations of 1, 2, 3 with no number in a forbidden position. **[2]**

8 Find the rook polynomial of the board shown.



[3]

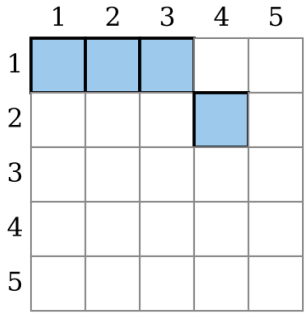
9 The board B shown is made up of a 2 × 2 block B_1 and a single square B_2 which share no row or column.



(a) Find the rook polynomial of B_1. **[2]**

(b) Explain why $R(B) = R(B_1)\,R(B_2)$, and hence find R(B). **[3]**

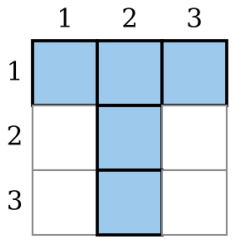
10 The board shown is the set of forbidden squares in a 5 × 5 grid, so that the permutation $p_1p_2p_3p_4p_5$ of 1 to 5 is not allowed to have $p_1 \in 1, 2, 3$ or $p_2 = 4$.
Find the rook polynomial of the board and hence the number of permutations of 1 to 5 that satisfy the restrictions.



[3]

11 A full board has m rows and n columns. By choosing the rows, then the columns, and then pairing them, explain why the number of ways of placing k non-attacking rooks on it is C(m, k) × C(n, k) × k!. Hence find the number of ways of placing 3 non-attacking rooks on a full 4 × 6 board. **[3]**

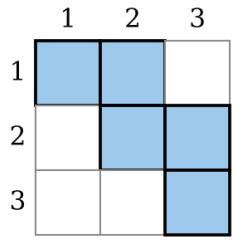
12 Find the number of ways of placing two rooks on the board shown so that neither attacks the other.



[3]

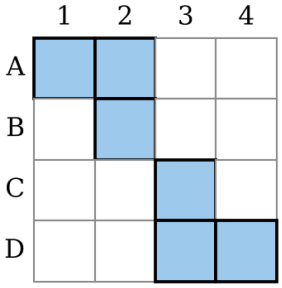
13 A derangement of 1, 2, ..., n is a permutation with no number in its own position. The board of forbidden squares for n = 5 consists of the diagonal squares (i, i).
Show that its rook polynomial is $(1 + x)^5$, and hence find the number of derangements of 1, 2, 3, 4, 5. **[4]**

14 For a board B and a chosen square s, let B_i be the board obtained by deleting the row and column containing s, and B_e the board obtained by deleting just the square s. Then $R(B, x) = x\,R(B_i, x) + R(B_e, x)$.
Use this result with s = (row 2, column 2) to find the rook polynomial of the board shown.



[5]

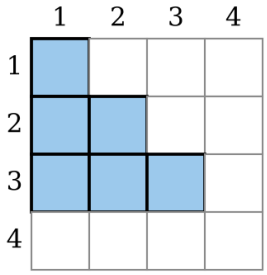
15 Four workers A, B, C, D are each to be given a different one of four jobs 1, 2, 3, 4. The diagram shows the pairings that are not allowed (shaded): A cannot do jobs 1 or 2, B cannot do job 2, C cannot do job 3, and D cannot do jobs 3 or 4.
By splitting the board into two parts that share no row or column, find its rook polynomial, and hence find the number of ways of allocating the jobs.



[6]

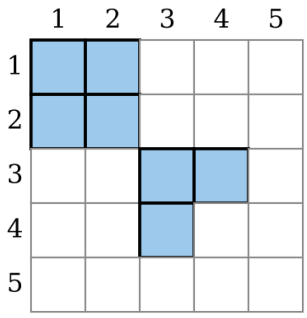
16 Five letters are put at random into five addressed envelopes, one letter in each envelope. Find the number of ways in which neither letter 1 nor letter 2 goes into its own envelope, using a rook polynomial. **[3]**

17 The staircase board shown has rows of 1, 2 and 3 squares, and lies in the top three rows of a 4 × 4 grid. It is the set of forbidden squares for permutations $p_1p_2p_3p_4$ of 1, 2, 3, 4.
Find the rook polynomial of the staircase board, and hence the number of permutations with no rook on a shaded square. Check your answer by a direct argument.



[5]

18 The diagram shows the forbidden squares for a permutation $p_1p_2p_3p_4p_5$ of 1 to 5 (row i, column j means $p_i = j$ is forbidden).
Find the rook polynomial of the shaded board and hence the number of permutations with no number in a forbidden position.



[5]

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19 In a permutation $p_1p_2...p_6$ of 1 to 6, the numbers p_1 and p_2 must both be from 4, 5, 6. The forbidden squares form a 2×3 block in a 6×6 grid.
Find the rook polynomial of the block, use it to find the number of permutations satisfying the condition, and check your answer by a direct argument. **[4]**

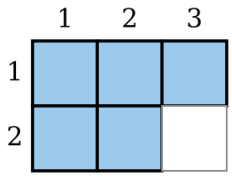
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20 A student finds the number of derangements of 1, 2, 3, 4 as $4! - 4 \times 3! + 6 \times 2! - 4 \times 1! = 8$, but the correct number is 9. Find and correct the error in the student's working. **[3]**

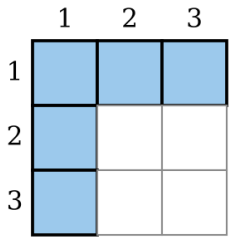
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21 Two boards with the same rook polynomial are called rook-equivalent. The boards A and B are shown.

(a) Find the rook polynomial of A. **[2]**



(b) Find the rook polynomial of B, and hence show that A and B are rook-equivalent. **[3]**



(c) Deduce that the number of permutations of 1 to 5 avoiding A equals the number avoiding B (A and B sitting in the top-left corner of a 5×5 grid), and find this number. **[2]**

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22 The number of permutations of 1, ..., n that have exactly m elements in forbidden positions is $e_m = \Sigma_{k=m}^{n} (-1)^{k-m} C(k, m) r_k (n - k)!$, where r_k are the coefficients of the rook polynomial of the forbidden board. Take n = 4 and the forbidden board to be the four diagonal squares (i, i), so that e_m counts permutations with exactly m fixed points.

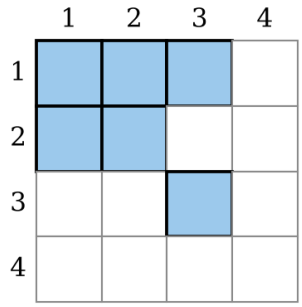
(a) Write down $r_0, ..., r_4$. **[1]**

(b) Find e_1, the number of permutations with exactly one fixed point. **[3]**

(c) Find e_2. **[2]**

(d) Given that $e_0 = 9$, $e_3 = 0$ and $e_4 = 1$, show that your values are consistent with the total number of permutations of 1, 2, 3, 4. **[1]**

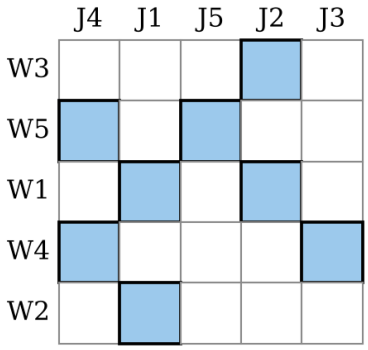
23 For a board B and a square s, $R(B, x) = x\,R(B_i, x) + R(B_e, x)$, where B_i is B with the row and column of s deleted and B_e is B with the square s deleted.
The board shown is the set of forbidden squares in a 4 × 4 grid. Use s = (row 1, column 3) to find its rook polynomial, and hence find the number of permutations of 1, 2, 3, 4 that avoid it.



[8]

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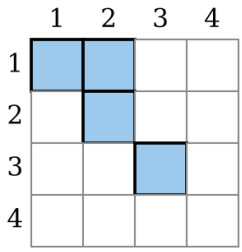
24 Five workers W1 to W5 are each to be assigned a different one of five jobs J1 to J5. The shaded squares show the pairings that are not allowed.
By reordering the rows and columns of the board, split it into two parts that share no row or column, find the rook polynomial of the whole board, and hence find the number of possible assignments.



[8]

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25 The board B shown is the set of forbidden squares in a 4 × 4 grid. A student splits B into (1, 1), (1, 2) and (2, 2), (3, 3), and writes $R(B) = (1 + 2x)(1 + x)^2 = 1 + 4x + 5x^2 + 2x^3$.



(a) Explain the error in the student's method. **[2]**
(b) Find the correct rook polynomial of B, using the square (2, 2). **[3]**
(c) Hence find the number of permutations of 1, 2, 3, 4 that avoid B. **[1]**

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26 The number of derangements D_n of 1, ..., n is the number of permutations avoiding the diagonal board.

(a) Show that $D_n = \Sigma_{k=0}^{n} (-1)^k C(n, k)(n - k)!$ and deduce that $D_n = n!\,\Sigma_{k=0}^{n} \frac{(-1)^k}{k!}$. **[3]**
(b) Calculate D_6. **[2]**
(c) Find $D_6/6!$ to 4 decimal places and compare with 1/e = 0.36788 to 5 decimal places. **[1]**

27 A full block of m rows and m columns is placed in the top-left corner of an n × n grid (n ≥ 2m). The block is the set of forbidden squares for a permutation of 1, ..., n.

(a) Show that the rook polynomial of the block has $r_k = C(m, k)^2\, k!$. **[3]**

(b) By a direct argument show that the number of allowed permutations is $((n - m)!)^2/(n - 2m)!$. **[3]**

(c) Verify the agreement of the two methods for n = 6 and m = 3 by evaluating the inclusion-exclusion sum $\Sigma(-1)^k\, r_k(6 - k)!$ and the formula in (b). **[3]**

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28 The number of derangements of 1, ..., n is $D_n = n!\, \Sigma_{k=0}^{n}\, (-1)^k/k!$.

(a) Prove that $D_n = nD_{n-1} + (-1)^n$ for n ≥ 2. **[4]**

(b) Given that $D_6 = 265$, find D_7. **[1]**

(c) Prove by induction that D_n is even if n is odd and odd if n is even. **[4]**

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29 For n ≥ 1, let P_n be the board with the squares (i, i) for 1 ≤ i ≤ n and (i, i + 1) for 1 ≤ i ≤ n − 1, in an n × n grid. It is the set of forbidden squares for permutations $p_1...p_n$ of 1, ..., n in which $p_i \neq i$ for all i and $p_i \neq i + 1$ for i < n.

(a) List the squares of P_n in the order (1, 1), (1, 2), (2, 2), (2, 3), ..., (n − 1, n), (n, n). Show that two squares of P_n are in the same row or column if and only if they are consecutive in this list, and deduce that $r_k = C(2n - k, k)$. **[5]**

(b) Find the number of permutations of 1, ..., 6 with $p_i \neq i$ for all i and $p_i \neq i + 1$ for i < 6. **[4]**

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30 For i = 1, ..., 5 the forbidden squares in a 5 × 5 grid are (i, i) and (i, i + 1), where column 6 means column 1. So the board has ten squares, two in every row and two in every column.

(a) Find r_1 and r_2 for this board. **[4]**

(b) Given that $r_3 = 50$, $r_4 = 25$ and $r_5 = 2$, find the number of permutations $p_1...p_5$ of 1, ..., 5 with $p_i \notin i, i + 1$ (taking 6 as 1). **[3]**

(c) Explain why $r_5 = 2$ by describing the two placements of five rooks. **[2]**

Generating Functions for Counting

1 The generating function of a sequence $a_0, a_1, a_2, ...$ is $\Sigma\, a_n x^n$. Write the generating function of the sequence 1, 1, 1, 1, ... as a single fraction, for |x| < 1. **[2]**

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2 Find the coefficient of x^3 in the expansion of $(1 + x)^5$. **[2]**

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3 Expand $(1 + 2x)^4$ in ascending powers of x. **[3]**

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4 A bag is to hold some identical apples and some identical bananas. Write down the generating function (a polynomial in x, where the power of x counts the pieces of fruit) for the number of ways of choosing:

(a) from 0 to 3 apples. **[1]**

(b) exactly 2 or exactly 4 bananas. **[1]**

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5 Find the generating function of the sequence $a_n = 3^n$, n ≥ 0, as a single fraction, stating the values of x for which it is valid. **[2]**

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6 Use the binomial expansion to write down the first four terms of $(1 - x)^{-2}$. **[3]**

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7 Expand $(1 - x)^{-3}$ as a series in ascending powers of x up to and including the term in x^3, and state the coefficient of x^n. **[3]**

8 A shop sells three flavours of ice cream. Find the number of ways of choosing 4 scoops (scoops of the same flavour are identical, and any number of each flavour may be chosen), by finding a coefficient in a generating function. **[3]**

9 Find the coefficient of x^5 in $(1 + x + x^2)^3$. **[3]**

10 Find the number of solutions of $a + b + c = 8$ in which a, b, c are integers with $0 \leq a, b, c \leq 4$, using a generating function. **[4]**

11 The sequence $a_n = n$ has generating function $A(x) = \Sigma_{n=0}^{\infty} n x^n$. Starting from $\frac{1}{1 - x} = \Sigma x^n$, show that $A(x) = \frac{x}{(1 - x)^2}$. **[3]**

12 A sequence has generating function $\frac{1}{(1 - 2x)^2}$. Find a formula for a_n, the coefficient of x^n. **[3]**

13 A partition of n is a way of writing n as a sum of positive integers where order does not matter. The generating function for the number of partitions of n into parts of size at most 3 is $\frac{1}{(1 - x)(1 - x^2)(1 - x^3)}$. Explain why, and list the partitions of 5 that use parts of size at most 3 to find the coefficient of x^5. **[4]**

14 Ten identical sweets are shared between three children so that each child gets at least 1 and at most 5 sweets. Use a generating function to find the number of ways of doing this. **[5]**

15 Three fair six-sided dice are thrown. By finding a coefficient in a suitable generating function, find the number of ways of getting a total of 12, and hence the probability of a total of 12. **[5]**

16 Find the number of ways of making exactly 10p using any number of 1p, 2p and 5p coins, by finding a coefficient in a generating function. (The order of the coins does not matter.) **[4]**

17 Use the binomial expansion to find the first four terms of $(1 - 2x)^{-1/2}$ in ascending powers of x, and state the range of values of x for which the expansion is valid. **[4]**

18 By putting $x = 1$ in $(1 + x)^n = \Sigma_{k=0}^{n} C(n, k)x^k$, and in the derivative of both sides, prove that

(a) $\Sigma_{k=0}^{n} C(n, k) = 2^n$ **[1]**

(b) $\Sigma_{k=0}^{n} k\, C(n, k) = n\, 2^{n-1}$ **[3]**

19 A fruit bowl is to contain 6 pieces of fruit chosen from apples, bananas and cherries, with an even number of apples, an odd number of bananas and any number of cherries (pieces of the same fruit are identical). Write down the generating function for the number of selections, and find the number of ways of choosing the 6 pieces. **[5]**

20 A partition of n into distinct parts uses each part at most once. Explain why the generating function for the number of partitions into distinct parts is $(1 + x)(1 + x^2)(1 + x^3)$ ..., and list the partitions of 8 into distinct parts to find the coefficient of x^8. **[4]**

21 A fair coin is tossed repeatedly until the first head appears. The random variable N is the number of tosses, so that $P(N = n) = (1/2)^n$ for $n = 1, 2, 3, \ldots$ You may use $(1 - x)^{-3} = \Sigma_{n=0}^{\infty} C(n + 2, 2)\, x^n$ for $|x| < 1$.

(a) Show that $\Sigma_{n=1}^{\infty} \frac{n(n + 1)}{2} x^n = \frac{x}{(1 - x)^3}$. **[2]**

(b) Hence find the exact value of $\Sigma_{n=1}^{\infty} \frac{n(n + 1)}{2^n}$. **[2]**

(c) Find $E[N(N + 1)]$, and hence find $E[N^2]$ given that $E[N] = 2$. **[2]**

22 Let $p_3(n)$ be the number of partitions of n into at most 3 parts.

(a) List the partitions of 8 into at most 3 parts, and so find $p_3(8)$. **[3]**

(b) By reading a partition with at most 3 parts down its columns instead of along its rows, explain why $p_3(n)$ is also the number of partitions of n into parts of size at most 3. **[2]**

(c) Expand $\frac{1}{(1-x^2)(1-x^3)}$ as a series up to x^8, and then multiply by $\frac{1}{1-x}$ to confirm the value of $p_3(8)$. **[3]**

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23 Find the number of ways of making exactly 15p using any number of 1p coins, at most four 2p coins and at most three 5p coins (the order of coins does not matter). Write down the generating function and find the required coefficient. **[6]**

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24 The sequence $a_n = n^2$ has generating function $A(x) = \Sigma_{n=0}^{\infty} n^2x^n$. Starting from $\Sigma\, n\, x^n = \frac{x}{(1-x)^2}$, show that $A(x) = \frac{x(1+x)}{(1-x)^3}$. Hence, by considering $A(x)/(1-x)$, prove that $\Sigma_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$. **[8]**

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25 A student is asked to find the number of ways of sharing 12 identical sweets between 3 children so that every child gets at least one sweet. The student writes: "The generating function is $(1-x)^{-3}$, and the answer is the coefficient of x^{12}, which is C(14, 2) = 91." Explain the error and find the correct answer. **[4]**

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26 Ten identical chocolates are chosen from four types A, B, C and D. At least one and at most four of type A must be chosen, and at most four of each other type. Find the number of different selections, using a generating function. **[6]**

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27 By considering the coefficient of x^n in $(1+x)^{2n} = (1+x)^n(1+x)^n$, answer the following.

(a) Prove that $\Sigma_{k=0}^{n} C(n,k)^2 = C(2n,n)$. **[4]**

(b) Check this for n = 5 by evaluating the sum directly. **[2]**

(c) Use $(1+x)^{m+n} = (1+x)^m(1+x)^n$ to prove $\Sigma_{k=0}^{r} C(m,k)C(n,r-k) = C(m+n,r)$. Hence find the number of committees of 4 that can be chosen from 6 men and 5 women, and state how the sum splits the committees. **[3]**

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28 The generating function for partitions into distinct parts is $F(x) = \prod_{k=1}^{\infty}(1+x^k)$, and for partitions into odd parts is $H(x) = \prod_{k=1}^{\infty}\frac{1}{1-x^{2k-1}}$.

(a) Prove that F(x) = H(x), so that the number of partitions of n into distinct parts equals the number into odd parts. **[5]**

(b) List the partitions of 9 into distinct parts and the partitions of 9 into odd parts, and verify the result of part (a) for n = 9. **[4]**

29 The exponential generating function of a sequence a_n is $\Sigma_{n=0}^{\infty} a_n \frac{x^n}{n!}$. You may use the fact that if A(x) and B(x) are the exponential generating functions of a_n and b_n, then the coefficient of $\frac{x^n}{n!}$ in A(x)B(x) is $\Sigma_{k=0}^{n} C(n, k)\, a_k b_{n-k}$.

Let w_n be the number of strings of length n made from the letters A, B and C in which the letter A appears an even number of times (zero counts as even).

(a) Write down the exponential generating function of the sequence 1, 0, 1, 0, 1, ... in terms of e^x and e^{-x}, and the exponential generating function of the sequence 1, 1, 1, ... **[2]**

(b) Show that the exponential generating function of w_n is $\frac{e^{3x} + e^x}{2}$, and hence find a formula for w_n. **[3]**

(c) Check the formula for n = 2 by listing the strings, and find w_6. **[3]**

30 Find the number of four-digit positive integers (1000 to 9999) whose digits sum to 18, using a generating function. **[9]**

Recurrence Relations and Generating Functions

1 A sequence is defined by $a_0 = 0$ and $a_n = 2a_{n-1} + 1$ for $n \geq 1$. Write down a_1, a_2, a_3, a_4. **[2]**

2 A sequence has generating function $A(x) = \frac{3}{1 - 2x}$. Find a formula for a_n. **[2]**

3 The Fibonacci numbers satisfy $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Write down F_2, F_3, ..., F_8. **[2]**

4 Find the constants A and B such that $\frac{1}{(1 - x)(1 - 2x)} = \frac{A}{1 - x} + \frac{B}{1 - 2x}$. **[3]**

5 Write each of the following as a single fraction in x.

(a) The generating function of the sequence 1, 0, 1, 0, 1, 0, ... **[1]**

(b) The generating function of the sequence $a_n = 2^n + 3^n$, $n \geq 0$. **[2]**

6 A sequence has generating function $A(x) = \frac{1}{1 - x} - \frac{1}{1 - 4x}$. Find a formula for a_n. **[2]**

7 Use generating functions to solve $a_n = 3a_{n-1}$ for $n \geq 1$, with $a_0 = 5$. **[3]**

8 Use generating functions to solve $a_n = a_{n-1} + 2$ for $n \geq 1$, with $a_0 = 1$. **[4]**

9 A sequence satisfies $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2$, with $a_0 = 1$ and $a_1 = 0$. Use generating functions to find a formula for a_n. **[6]**

10 A staircase has n steps. A climber goes up one or two steps at a time, and s_n is the number of different ways of climbing n steps, with $s_0 = 1$. Explain why $s_n = s_{n-1} + s_{n-2}$ for $n \geq 2$, and find s_6. **[4]**

11 A sequence has generating function $A(x) = \frac{x}{1 - 5x + 6x^2}$. Use partial fractions to find a formula for a_n. **[4]**

12 A sequence has generating function $A(x) = \frac{2 - x}{1 - x - 2x^2}$.

(a) Find a recurrence relation satisfied by a_n, together with a_0 and a_1. **[3]**

(b) Use partial fractions to find a formula for a_n. **[3]**

13 Use generating functions to solve $a_n = a_{n-1} + 2^n$ for $n \geq 1$, with $a_0 = 1$. **[5]**

14 The Fibonacci numbers satisfy $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ $(n \geq 2)$. Let $F(x) = \Sigma F_n x^n$.

(a) Show that $F(x) = \frac{x}{1 - x - x^2}$. **[3]**

(b) Let $\alpha = \frac{1 + \sqrt{5}}{2}$ and $\beta = \frac{1 - \sqrt{5}}{2}$, the roots of $t^2 = t + 1$. Show that $1 - x - x^2 = (1 - \alpha x)(1 - \beta x)$, and hence find Binet's formula $F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}$. **[4]**

15 Use generating functions to solve $a_n = 4a_{n-1} - 4a_{n-2}$ for $n \geq 2$, with $a_0 = 1$ and $a_1 = 3$. **[6]**

16 A frog hops along a line of stones numbered 0, 1, 2, ..., starting on stone 0. Each hop is 1 or 3 stones forward. Let t_n be the number of different sequences of hops that take the frog from stone 0 to stone n, with $t_0 = 1$.

(a) Explain why $t_n = t_{n-1} + t_{n-3}$ for $n \geq 3$, and write down t_1 and t_2. **[3]**

(b) Show that $\Sigma t_n x^n = \frac{1}{1 - x - x^3}$ and use it to find t_9. **[3]**

17 Let a_n be the number of strings of length n over the alphabet 0, 1, 2 that do not contain two consecutive 2s, with $a_0 = 1$ and $a_1 = 3$.

(a) Show that $a_n = 2a_{n-1} + 2a_{n-2}$ for $n \geq 2$. **[3]**

(b) Show that $\Sigma a_n x^n = \frac{1 + x}{1 - 2x - 2x^2}$ and hence find a_5. **[3]**

18 n straight lines are drawn in a plane, no two parallel and no three through a point. Let r_n be the number of regions into which they divide the plane, with $r_0 = 1$. By considering the effect of adding the n-th line, which crosses the other $n - 1$ lines at $n - 1$ distinct points, explain why $r_n = r_{n-1} + n$. Use generating functions to find a formula for r_n. **[6]**

19 In the Tower of Hanoi puzzle with n discs, the least number of moves h_n satisfies $h_n = 2h_{n-1} + 1$ for $n \geq 1$, with $h_0 = 0$. Use generating functions to show that $h_n = 2^n - 1$. **[5]**

20 A $2 \times n$ board is to be covered exactly with 1×2 dominoes (placed either way) and 2×2 squares. Let a_n be the number of coverings, with $a_0 = 1$ and $a_1 = 1$. Explain why $a_n = a_{n-1} + 2a_{n-2}$ for $n \geq 2$, and solve this using generating functions. **[6]**

21 Sequences a_n and b_n satisfy $a_n = 2a_{n-1} + b_{n-1}$ and $b_n = a_{n-1} + 2b_{n-1}$ for $n \geq 1$, with $a_0 = 1$ and $b_0 = 0$. Let $A(x) = \Sigma a_n x^n$ and $B(x) = \Sigma b_n x^n$. Find formulae for a_n and b_n. **[8]**

22 Let a_n be the number of strings of length n over the alphabet 0, 1, 2, 3 that contain an even number of 0s (including none), with $a_0 = 1$. By considering the last symbol, explain why $a_n = 2a_{n-1} + 4^{n-1}$ for $n \geq 1$, and solve this using generating functions. **[6]**

23 Use generating functions to solve $a_n = 2a_{n-1} + n$ for $n \geq 1$, with $a_0 = 1$. **[6]**

24 A student solves $a_n = 3a_{n-1} - 2$ for $n \geq 1$, with $a_0 = 1$, by writing $A(x) = 3xA(x) - \frac{2}{1-x}$, and then obtains an answer that does not give $a_0 = 1$.

(a) Explain the error in the student's equation. **[2]**

(b) Solve the recurrence correctly. **[3]**

25 Using $F(x) = \frac{x}{1 - x - x^2}$ for the Fibonacci numbers, prove that $\Sigma_{k=0}^{n} F_k = F_{n+2} - 1$. **[6]**

26 A composition of n is an ordered list of positive integers that sum to n. Let c_n be the number of compositions of n in which every part is odd, with $c_0 = 1$.

(a) Show that the generating function of c_n is $C(x) = \frac{1 - x^2}{1 - x - x^2}$. **[3]**

(b) Hence show that $c_n = F_n$ for $n \geq 1$, where F_n is the n-th Fibonacci number, and find c_7. **[3]**

27 The numbers C_n satisfy $C_0 = 1$ and $C_{n+1} = \Sigma_{k=0}^{n} C_k C_{n-k}$ for $n \geq 0$ (they count, for example, the ways of triangulating a convex (n + 2)-sided polygon). Let $C(x) = \Sigma\, C_n x^n$.

(a) Show that $C(x) = 1 + x(C(x))^2$. **[3]**

(b) Solve for C(x), explaining your choice of root, and show that $C(x) = \frac{1 - \sqrt{1 - 4x}}{2x}$. **[3]**

(c) Given that the coefficient of x^m in $(1 - 4x)^{1/2}$ is $-\frac{2}{m}C(2m - 2, m - 1)$ for $m \geq 1$, show that $C_n = \frac{1}{n+1}C(2n, n)$, and find C_5. **[4]**

28 A fair coin is tossed n times. Let b_n be the number of sequences of n results (H or T) that do not contain three consecutive Hs, with $b_0 = 1$.

(a) By considering how a valid sequence ends (T, TH or THH, or is short), show that $b_n = b_{n-1} + b_{n-2} + b_{n-3}$ for $n \geq 3$, and find b_0, b_1, b_2. **[4]**

(b) Show that the generating function is $B(x) = \frac{1 + x + x^2}{1 - x - x^2 - x^3}$. **[3]**

(c) Find b_7, and the probability that 7 tosses contain no three consecutive heads. **[2]**

29 Use generating functions to solve $a_{n+2} - 4a_{n+1} + 4a_n = 2^n$ for $n \geq 0$, with $a_0 = 0$ and $a_1 = 1$. **[9]**

30 The exponential generating function of a sequence a_n is $\Sigma_{n=0}^{\infty} a_n \frac{x^n}{n!}$. You may use the fact that if A(x) and B(x) are the exponential generating functions of a_n and b_n, then the coefficient of $\frac{x^n}{n!}$ in A(x)B(x) is $\Sigma_{k=0}^{n} C(n, k)\, a_k b_{n-k}$.

Let D_n be the number of derangements of 1, 2, ..., n (permutations with no number in its own position), with $D_0 = 1$, and let $D(x) = \Sigma_{n=0}^{\infty} D_n \frac{x^n}{n!}$.

(a) By classifying the n! permutations of 1, 2, ..., n according to which numbers are in their own positions, show that $\Sigma_{k=0}^{n} C(n, k)\, D_{n-k} = n!$. **[3]**

(b) Hence show that $e^x D(x) = \frac{1}{1 - x}$. **[3]**

(c) Deduce that $D_n = n!\, \Sigma_{k=0}^{n} \frac{(-1)^k}{k!}$, and find D_5. **[3]**