

Two-Person Zero-Sum Games

1 (1 mark)

The total of the payoffs to the two players is always zero: whatever one player gains the other loses

B1

Answer: One player's gain equals the other's loss

1 mark

2 (1 mark)

Row A_2 , column B_3 : 0

B1

Answer: 0

1 mark

3 (1 mark)

Zero-sum, so B's payoff is -4

B1

Answer: -4

1 mark

4 (1 mark)

A loses that amount (and B gains it)

B1

Answer: A loses; B gains

1 mark

5 (2 marks)

A has 3 strategies (rows) and B has 4 (columns)

B1 B1

Answer: A: 3, B: 4

2 marks

6 (2 marks)

Column B_2 : A_1 gives -2 , A_2 gives 5

M1

A should play A_2 and gains 5

A1

Answer: A_2 , payoff 5

2 marks

7 (3 marks)

Row minima: 1, -1 , -2

M1 A1

Maximin = 1, so A's play-safe strategy is A_1

A1

Answer: Maximin 1; play A_1

3 marks

8 (3 marks)

Column maxima: 4, 5, 6

M1 A1

Minimax = 4, so B's play-safe strategy is B_1

A1

Answer: Minimax 4; play B_1

3 marks

9 (4 marks)

Row minima 2, -1 , -3 : A plays A_1

M1 A1

Column maxima 5, 3: B plays B_2

A1

A's play-safe strategy maximises the minimum payoff A can get (guarantees at least 2)

E1

Answer: A_1 and B_2

4 marks

10 (3 marks)

Row minima 3, 1: maximin = 3; column maxima 6, 3, 5: minimax = 3

M1 A1

They are equal, so the game is stable (has a saddle point at A_1, B_2) and the value is 3

A1

Answer: Stable; value 3

3 marks

11 (3 marks)

Row minima -1 , -2 : maximin = -1

M1

Column maxima 3, 4: minimax = 3

M1

$-1 \neq 3$, so there is no saddle point (not stable)

A1

Answer: maximin -1 , minimax 3: not stable

3 marks

12 (3 marks)

Matrix $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ (rows: A shows 1 or 2; columns: B shows 1 or 2) B1

Row minima $-2, -2$ give maximin -2 ; column maxima $2, 2$ give minimax 2 ; these differ, so there is no saddle point M1

Expected payoff $= \frac{1}{4}(2 - 2 - 2 + 2) = 0$, so the game is fair A1

Answer: Matrix $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$; no saddle point; expected payoff 0, fair 3 marks

13 (3 marks)

X's row minima: R: 3, I: 1; maximin 3, so X plays R M1 A1

Y's column maxima: R: 6, I: 3; minimax 3, so Y plays I M1

Answer: X plays R; Y plays I 3 marks

14 (5 marks)

Whatever one wins, the other loses the same amount, so the game is zero-sum E1

(1, 1): total 2, even: $+2$; (1, 2): total 3, odd: -3 ; (2, 1): -3 ; (2, 2): total 4: $+4$ M1 A1

Matrix $\begin{pmatrix} 2 & -3 \\ -3 & 4 \end{pmatrix}$; row minima $-3, -3$ so maximin $= -3$ M1

Column maxima $2, 4$, so minimax $= 2 \neq -3$; not stable A1

Answer: Matrix $\begin{pmatrix} 2 & -3 \\ -3 & 4 \end{pmatrix}$; maximin -3 , minimax 2 5 marks

15 (5 marks)

$E(A_1) = 1 + \frac{3}{4} + \frac{1}{4} = 2$ M1 A1

$E(A_2) = 2 - \frac{1}{4} + \frac{1}{2} = \frac{9}{4}$ A1

$E(A_3) = 0 + \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$ A1

A's best reply is A_2 (expected payoff $\frac{9}{4}$) A1

Answer: $2, \frac{9}{4}, \frac{1}{4}$; A_2 5 marks

16 (3 marks)

B's payoffs are the negatives of A's M1

Rows are now B's strategies, so transpose M1

B's matrix: $\begin{pmatrix} -4 & -3 \\ 1 & 0 \\ -2 & 2 \end{pmatrix}$ A1

Answer: B's matrix: $\begin{pmatrix} -4 & -3 \\ 1 & 0 \\ -2 & 2 \end{pmatrix}$ 3 marks

17 (4 marks)

Maximin is the largest of the row minima, not the largest entry E1

Row minima 1 and 2, so maximin $= 2$ M1 A1

The largest entry is $5 \neq 2$ A1

Answer: Maximin 2 but the largest entry is 5 4 marks

18 (4 marks)

Row minima: 2, 0, 2; maximin $= 2$ (A_1 or A_3) M1 A1

Column maxima: 6, 4, 5; minimax $= 4$ (B_2) M1

$2 \neq 4$, so the game is not stable A1

Answer: A_1 or A_3 (maximin 2), B_2 (minimax 4); not stable 4 marks

19 (4 marks)

A's play-safe strategy is A_1 (row minima $-1, -2$)

B1

If B realises that A always plays A_1 , B always plays B_2 , giving A only -1 each time

M1 A1

A should vary the strategy randomly (a mixed strategy) to avoid being exploited

E1

Answer: B can exploit a predictable A

4 marks

20 (6 marks)

Row minima: 4, 2, 3; maximin = 4 (A_1)

M1 A1

Column maxima: 8, 4, 9; minimax = 4 (B_2)

M1 A1

Maximin = minimax, so the game is stable with the saddle point (A_1, B_2)

A1

Value 4: if both play optimally A gains 4 (and B loses 4) each time

E1

Answer: Stable; A_1, B_2 ; value 4

6 marks

21 (7 marks)

Row minima: 3, 2, 3; maximin = 3

M1 A1

Column maxima: 3, 7, 6; minimax = 3

M1 A1

Maximin = minimax = 3, so the value is 3

A1

Saddle points: row A_1 , column B_1 and row A_3 , column B_1 , since each entry is 3, the minimum of its row and the maximum of its column

M1 A1

Answer: Value 3; saddle points (A_1, B_1) and (A_3, B_1)

7 marks

22 (7 marks)

Let r be a row giving the maximin, so maximin = min over j of a_{rj}

M1

Let c be a column giving the minimax, so minimax = max over i of a_{ic}

M1

Then $a_{rc} \geq \min \text{ over } j \text{ of } a_{rj} = \text{maximin}$ (it is one of the entries in row r)

M1 A1

and $a_{rc} \leq \max \text{ over } i \text{ of } a_{ic} = \text{minimax}$ (it is one of the entries in column c)

M1 A1

So maximin $\leq a_{rc} \leq \text{minimax}$

A1

Answer: maximin $\leq a_{rc} \leq \text{minimax}$

7 marks

23 (7 marks)

Row minima: 1, 5, 2; maximin = 5

M1 A1

Column maxima: 8, 9, 5; minimax = 5

M1 A1

Saddle point at row A_2 , column B_3 ; value 5

A1

If B moves away from B_3 while A plays A_2 , B concedes 6 or 7 (more than 5); if A moves away while B plays B_3 , A gets 1 or 2 (less than 5)

E1 A1

Answer: Saddle point (A_2, B_3); value 5

7 marks

24 (7 marks)

$$\text{vs } B_1: \frac{1}{3}(3) + \frac{2}{3}(-2) = -\frac{1}{3}$$

M1 A1

$$\text{vs } B_2: -\frac{1}{3} + \frac{8}{3} = \frac{7}{3}$$

A1

$$\text{vs } B_3: \frac{2}{3} + \frac{2}{3} = \frac{4}{3}$$

A1

B's best reply is B_1 , giving A $-\frac{1}{3}$

A1

A's maximin is -1 (row minima $-1, -2$); the mixed strategy guarantees $-\frac{1}{3} > -1$, so mixing improves A's guaranteed payoff

E1 A1

Answer: $-\frac{1}{3}, \frac{7}{3}, \frac{4}{3}$; B plays B_1 ; $-\frac{1}{3}$ beats the maximin -1

7 marks

25 (6 marks)

The striker's chance + the goalkeeper's chance = 100, so subtracting 50 gives payoffs of equal size and opposite sign

E1

$$\text{Matrix} \begin{pmatrix} -10 & 40 \\ 30 & -20 \end{pmatrix}$$

M1 A1

Row minima $-10, -20$: maximin = -10 (striker aims L)

M1

Column maxima $30, 40$: minimax = 30 (goalkeeper dives L)

M1

$-10 \neq 30$, so not stable

A1

Answer: Matrix $\begin{pmatrix} -10 & 40 \\ 30 & -20 \end{pmatrix}$; maximin -10 , minimax 30

6 marks

26 (6 marks)

The student has used row maxima and column minima

E1

Maximin uses row minima: 2 and 3, so maximin = 3

M1 A1

Minimax uses column maxima: 5 and 4, so minimax = 4

M1 A1

Always maximin $\leq$ minimax, so the student's claim is impossible; the game is not stable

E1

Answer: Maximin 3, minimax 4

6 marks

27 (6 marks)

If A changes to row i while B keeps column c , A gets a_{ic}

M1

a_{rc} is the maximum of column c , so $a_{ic} \leq a_{rc}$: A cannot do better

A1

If B changes to column j while A keeps row r , B pays a_{rj}

M1

a_{rc} is the minimum of row r , so $a_{rj} \geq a_{rc}$: B pays at least as much, so B cannot do better

A1

Hence neither player gains by a unilateral change (a stable pair of strategies)

E1 A1

Answer: Neither player can improve by changing alone

6 marks

28 (10 marks)

Row minima: $\min(k, 3)$ and 2, so maximin = $\max(\min(k, 3), 2)$

M1 A1

Column maxima: $\max(k, 2)$ and 4, so minimax = $\min(\max(k, 2), 4)$

M1 A1

$k \leq 2$: maximin = 2, minimax = 2, stable with value 2

M1 A1

$2 < k \leq 3$: maximin = k , minimax = k , stable with value k

A1

$k > 3$: maximin = 3 but minimax = $\min(k, 4) > 3$, so not stable

M1 A1

Stable for $k \leq 3$, with value $\max(k, 2)$

A1

Answer: Stable iff $k \leq 3$; value 2 for $k \leq 2$, value k for $2 \leq k \leq 3$

10 marks

29 (9 marks)

Row minima: 2, 0, 1, 1; maximin = 2 (X plays level 1)

M1 A1

Column maxima: 4, 5, 2, 6; minimax = 2 (Y plays level 3)

M1 A1

Maximin = minimax = 2, so there is a saddle point at (X1, Y3)

A1

Optimal strategies: X plays level 1 and Y plays level 3; value 2

A1

X gains 200 sales per week and Y loses 200 sales per week if both play optimally

E1

If Y changes alone, X gets 3, 5 or 4 (at least 2); if X changes alone, X gets 0, 2 or 1 (at most 2):

M1 A1

neither gains by changing

Answer: Saddle point (X1, Y3), value 2

9 marks

30 (8 marks)

$a_{rc} \leq a_{rd}$ since a_{rc} is the minimum of row r

M1 A1

$a_{rd} \leq a_{sd}$ since a_{sd} is the maximum of column d

M1 A1

Similarly $a_{sd} \leq a_{sc}$ (minimum of row s) and $a_{sc} \leq a_{rc}$ (maximum of column c)

M1

So $a_{rc} \leq a_{rd} \leq a_{sd} \leq a_{sc} \leq a_{rc}$: all four are equal

A1

a_{rd} : equal to a_{rc} , the minimum of row r ; and equal to a_{sd} , the maximum of column d : so a saddle point

M1

Likewise $a_{sc} = a_{sd}$ is the minimum of row s and a_{rc} is the maximum of column c : a saddle point

A1

Answer: All four entries are equal; (r, d), (s, c) are saddle points

8 marks

Payoff Matrices

1 (1 mark)

Matrix $\begin{pmatrix} 3 & -2 \\ 0 & 4 \end{pmatrix}$

B1

Answer: $\begin{pmatrix} 3 & -2 \\ 0 & 4 \end{pmatrix}$

1 mark

2 (1 mark)

Row 3, column 2: 5

B1

Answer: 5

1 mark

3 (1 mark)

4×3 (4 rows, 3 columns)

B1

Answer: 4×3

1 mark

4 (1 mark)

Row minima: $-1, 0, -2$

B1

Answer: $-1, 0, -2$

1 mark

5 (1 mark)

Column maxima: 4, 5, 3

B1

Answer: 4, 5, 3

1 mark

6 (2 marks)

Zero-sum: negate every entry

M1

Matrix $\begin{pmatrix} -3 & 2 \\ 0 & -4 \end{pmatrix}$

A1

Answer: $\begin{pmatrix} -3 & 2 \\ 0 & -4 \end{pmatrix}$

2 marks

7 (2 marks)

(H, H): +3; (T, T): +1

B1

(H, T) and (T, H): -2

B1

Answer: Matrix $\begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix}$

2 marks

8 (3 marks)

B's payoffs are the negatives of A's

M1

Rows become B's strategies, so transpose: $\begin{pmatrix} -3 & 0 \\ 1 & -4 \\ -2 & 2 \end{pmatrix}$

M1 A1

Answer: $\begin{pmatrix} -3 & 0 \\ 1 & -4 \\ -2 & 2 \end{pmatrix}$

3 marks

9 (4 marks)

Row minima 2, 1: maximin 2; column maxima 2, 5: minimax 2, so a saddle point at (A_1, B_1) , value 2

M1 A1

New matrix $\begin{pmatrix} 5 & 8 \\ 4 & 6 \end{pmatrix}$

B1

Same saddle point, with value 5 (the value increases by 3)

A1

Answer: Value 2; new matrix $\begin{pmatrix} 5 & 8 \\ 4 & 6 \end{pmatrix}$, value 5

4 marks

10 (3 marks)

$$\text{New matrix } \begin{pmatrix} 4 & 10 \\ 2 & 6 \end{pmatrix}$$

B1

Every payoff is doubled, so the value is doubled: 4; the saddle point is unchanged

B1 B1**Answer: Value 4****3 marks****11 (4 marks)**

$$P(A_1, B_1) = \frac{1}{8}, P(A_1, B_2) = \frac{3}{8}, P(A_2, B_1) = \frac{1}{8}, P(A_2, B_2) = \frac{3}{8}$$

M1

$$E = \frac{4}{8} - \frac{9}{8} - \frac{2}{8} + \frac{15}{8}$$

M1 A1

$$E = 1$$

A1**Answer: 1****4 marks****12 (3 marks)**The smallest entry is -4 , so $k = 5$ (giving smallest entry 1)**M1 A1**

$$\text{New matrix } \begin{pmatrix} 2 & 7 & 5 \\ 6 & 1 & 10 \end{pmatrix}$$

A1

$$\text{Answer: } k = 5; \begin{pmatrix} 2 & 7 & 5 \\ 6 & 1 & 10 \end{pmatrix}$$

3 marks**13 (2 marks)**When A plays A_2 and B plays B_3 , A's sales fall by 400 per week**E1**

and B's sales rise by 400 per week (zero-sum)

E1**Answer: A loses 400 sales and B gains 400****2 marks****14 (5 marks)**(R, S): +3 and (S, R): -3 **M1**(P, R): +1 and (R, P): -1 ; (S, P): +2 and (P, S): -2 **M1**

$$\text{Matrix } \begin{pmatrix} 0 & -1 & 3 \\ 1 & 0 & -2 \\ -3 & 2 & 0 \end{pmatrix}$$

A1

$$\text{B's matrix is the negative transpose, } \begin{pmatrix} 0 & -1 & 3 \\ 1 & 0 & -2 \\ -3 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 3 \\ 1 & 0 & -2 \\ -3 & 2 & 0 \end{pmatrix}, \text{ the same}$$

M1 A1

$$\text{Answer: } \begin{pmatrix} 0 & -1 & 3 \\ 1 & 0 & -2 \\ -3 & 2 & 0 \end{pmatrix}$$

5 marks**15 (5 marks)**

X's share + Y's share = 100, so after subtracting 50 the payoffs to X and Y are equal and opposite

E1

$$\text{Matrix } \begin{pmatrix} 5 & -10 & 10 \\ -5 & 15 & 0 \end{pmatrix}$$

M1 A1Row minima $-10, -5$; maximin = -5 , so X plays X_2 **M1**Column maxima 5, 15, 10; minimax = 5, so Y plays Y_1 **A1**

$$\text{Answer: Matrix } \begin{pmatrix} 5 & -10 & 10 \\ -5 & 15 & 0 \end{pmatrix}; \text{ X plays } X_2, \text{ Y plays } Y_1$$

5 marks

16 (6 marks)

B's matrix = negative transpose: $\begin{pmatrix} -3 & -1 & 1 \\ 2 & -4 & 0 \end{pmatrix}$

M1 A1

B's row minima: $-3, -4$, so B's maximin = -3

M1 A1

A's column maxima: $3, 4$, so A's minimax = 3

A1

B's maximin = $-(A's \text{ minimax})$: B's guaranteed payoff is the negative of the most A can be made to win

E1

Answer: $\begin{pmatrix} -3 & -1 & 1 \\ 2 & -4 & 0 \end{pmatrix}; -3 \text{ and } 3$

6 marks

17 (6 marks)

vs B_1 : $2p - (1 - p) = 3p - 1$

M1 A1

vs B_2 : $-3p + 4(1 - p) = 4 - 7p$

A1

vs B_3 : $p + 0 = p$

A1

When $p = \frac{1}{3}$: $0, \frac{5}{3}, \frac{1}{3}$

M1

B's best choice is B_1 (A's payoff 0)

A1

Answer: $3p - 1, 4 - 7p, p$; B_1

6 marks

18 (4 marks)

Adding a constant 5 to every payoff adds 5 to the value

M1

Value of the original game = $\frac{9}{2} - 5 = -\frac{1}{2}$

A1

The expected payoffs for every pair of strategies are all increased by the same amount, so the optimal strategies are unchanged: $(\frac{1}{2}, \frac{1}{2})$

E1 A1

Answer: Value $-\frac{1}{2}$; $(\frac{1}{2}, \frac{1}{2})$

4 marks

19 (6 marks)

$2(3) + 1 = 7, 2(-1) + 1 = -1, 2(-2) + 1 = -3, 2(4) + 1 = 9$

M1 A1

Every expected payoff E in the first game becomes $2E + 1$ in the second

M1

So the optimal strategy is unchanged: $(\frac{3}{5}, \frac{2}{5})$

A1

Value = $2(1) + 1 = 3$

A1

Check: $\frac{3}{5}(7) + \frac{2}{5}(-3) = \frac{21}{5} - \frac{6}{5} = 3$

A1

Answer: Value 3; strategy $(\frac{3}{5}, \frac{2}{5})$

6 marks

20 (4 marks)

The student has negated the entries but has not transposed the matrix

E1

B's strategies (the rows of B's matrix) are the 2 columns of A's matrix, so B's matrix must be 2×3

E1

$\begin{pmatrix} -3 & -1 & 1 \\ 2 & -4 & 0 \end{pmatrix}$

B1 B1

Answer: $\begin{pmatrix} -3 & -1 & 1 \\ 2 & -4 & 0 \end{pmatrix}$

4 marks

21 (7 marks)

Subtract 50: the server's payoff is 'points won $- 50$ ' and the receiver's is the negative

E1

$$\text{Matrix } \begin{pmatrix} -10 & 30 \\ 40 & 0 \end{pmatrix}$$

M1 A1

Row minima $-10, 0$; maximin $= 0$

M1

Column maxima $40, 30$; minimax $= 30$

M1

$0 \neq 30$, so there is no saddle point

A1

The entry 30 means that when the server serves forehand and the receiver covers backhand, the server wins 80% of points, which is 30 more than 50%

E1

Answer: Matrix $\begin{pmatrix} -10 & 30 \\ 40 & 0 \end{pmatrix}$; maximin 0, minimax 30

7 marks

22 (6 marks)

$$A_1: \frac{3}{2} - \frac{1}{4} + \frac{1}{2} = \frac{7}{4}$$

M1 A1

$$A_2: 0 + 1 - \frac{3}{4} = \frac{1}{4}$$

A1

$$A_3: 1 + \frac{1}{4} + \frac{1}{4} = \frac{3}{2}$$

A1

If B's strategy were optimal, A could not obtain more than the value 1 against it

E1

But A_1 gives A $\frac{7}{4} > 1$, so B's strategy is not optimal

A1

Answer: $\frac{7}{4}, \frac{1}{4}, \frac{3}{2}; \frac{7}{4} > 1$

6 marks

23 (7 marks)

Row minima: $\min(3, a)$ and 2, so maximin $= \max(\min(3, a), 2)$

M1 A1

Column maxima: 3 and $\max(a, 5) \geq 5$, so minimax $= 3$

M1 A1

Stable when maximin $= 3$, which needs $\min(3, a) = 3$, that is $a \geq 3$

M1 A1

Value 3

A1

Answer: $a \geq 3$; value 3

7 marks

24 (7 marks)

B's matrix is $-S^T$

M1

Each entry of S^T is the negative of S, so $-S^T = S$

A1

$$\text{vs } B_1: 0 + \frac{1}{6}(-2) + \frac{1}{3}(1) = 0$$

M1 A1

$$\text{vs } B_2: \frac{1}{2}(2) + 0 + \frac{1}{3}(-3) = 0$$

A1

$$\text{vs } B_3: \frac{1}{2}(-1) + \frac{1}{6}(3) + 0 = 0$$

A1

The game is symmetric, so the value is 0 and the strategy guarantees A at least 0

E1

Answer: Same matrix; expected payoff 0 against each column

7 marks

25 (6 marks)

A's payoffs are the negatives of B's

M1

$$\text{A's strategies are the columns of B's table, so transpose: } \begin{pmatrix} -2 & 0 & 1 \\ 3 & -4 & -1 \\ -1 & 2 & -3 \end{pmatrix}$$

M1 A1

A's row minima: $-2, -4, -3$; maximin $= -2$ (A_1)

M1

Column maxima: 3, 2, 1; minimax $= 1$ (B_3)

M1

$-2 \neq 1$, so the game is not stable

A1

Answer: $\begin{pmatrix} -2 & 0 & 1 \\ 3 & -4 & -1 \\ -1 & 2 & -3 \end{pmatrix}$; A_1 (maximin -2), B_3 (minimax 1); not stable

6 marks

26 (6 marks)

B's matrix has entries $b_{ji} = -a_{ij}$ (row j for B's strategy j , column i for A's strategy i) M1 A1
 a_{rc} is the minimum of row r of A's matrix, so $-a_{rc}$ is the maximum of column r of B's matrix M1 A1
 a_{rc} is the maximum of column c of A's matrix, so $-a_{rc}$ is the minimum of row c of B's matrix M1
 So $b_{cr} = -v$ is the minimum of its row and the maximum of its column: a saddle point with value A1
 $-v$

Answer: Saddle at (c, r), value $-v$

6 marks

27 (8 marks)

$E' = \sum_i \sum_j p_i q_j (\lambda a_{ij} + \mu)$ M1
 $= \lambda \sum_i \sum_j p_i q_j a_{ij} + \mu \sum_i \sum_j p_i q_j$ M1
 Since $\sum p_i = \sum q_j = 1$, $\sum \sum p_i q_j = 1$, so $E' = \lambda E + \mu$ A1
 A's strategy p guarantees at least v against every column in G if and only if $\lambda(\dots) + \mu \geq \lambda v + \mu$ M1 A1
 in G' ($\lambda > 0$ preserves inequalities)
 Likewise for B: $E \leq v$ for every row if and only if $E' \leq \lambda v + \mu$ M1
 So the same strategies are optimal for G' and the value is $\lambda v + \mu$ A1 E1
Answer: $E' = \lambda E + \mu$; value $\lambda v + \mu$; same strategies 8 marks

28 (8 marks)

G : no saddle point (maximin 1, minimax 3) M1
 A plays A_1 with probability p : $3p + (1 - p) = p + 3(1 - p)$, so $p = \frac{1}{2}$ M1 A1
 Value = 2, and by symmetry B plays B_1 with probability $\frac{1}{2}$ A1
 New matrix $\begin{pmatrix} 7 & 5 \\ 1 & 3 \end{pmatrix}$: row minima 5, 1, so maximin = 5; column maxima 7, 5, so minimax = 5 M1 A1
 Saddle point at (A_1, B_2) , value 5 A1
 Now A always plays A_1 (not a 50:50 mix) and B plays B_2 : the optimal strategies have changed, so the student is wrong E1
Answer: Original: $(\frac{1}{2}, \frac{1}{2})$, value 2; new: pure A_1, B_2 , value 5 8 marks

29 (9 marks)

Subtract 50: X's payoff is the percentage above 50 and Y's is the negative, so the game is zero-sum E1
 Matrix $\begin{pmatrix} 2 & 5 & 10 \\ -2 & 3 & 0 \\ 8 & 6 & 12 \end{pmatrix}$ M1 A1
 Row minima: 2, -2, 6; maximin = 6 (X_3) M1 A1
 Column maxima: 8, 6, 12; minimax = 6 (Y_2) M1 A1
 Saddle point at (X_3, Y_2) ; value 6, so X attracts 56% of customers A1
 If X changes alone (Y keeps Y_2), X's share would be 55% or 53%, less than 56%; if Y changes alone (X keeps X_3), X's share would be 58% or 62%, more than 56%, which is worse for Y E1
Answer: Saddle point (X_3, Y_2) ; X gets 56% 9 marks

30 (8 marks)

Entries: row 1: -5, -4, -3, -2; row 2: -4, -2, 0, 2; row 3: -3, 0, 3, 6; row 4: -2, 2, 6, 10 M1 A1
 $ij - 6$ increases with i for every j , so A prefers $i = 4$ whatever B does E1
 $ij - 6$ increases with j for every i , so B (who pays A) prefers $j = 1$ whatever A does E1
 Hence the entry for (4, 1) is the maximum of column 1 and the minimum of row 4: a saddle point M1 A1
 Value = $4 \times 1 - 6 = -2$ A1
 The value is negative, so the game favours B (A loses 2 each time with optimal play) E1
Answer: Saddle at (4, 1); value -2; favours B 8 marks

Dominance

1 (1 mark)

Every entry of row i is at least as large as the corresponding entry of row k , and at least one is strictly larger

B1

Answer: Each entry of row $i \geq$ the entry of row k , and one is larger

1 mark

2 (1 mark)

Every entry of column j is at most as large as the corresponding entry of column k , and at least one is strictly smaller (B prefers smaller entries)

B1

Answer: Each entry of column $j \leq$ the entry of column k , and one is smaller

1 mark

3 (1 mark)

$3 > 2$ and $5 > 4$, so row 1 dominates row 2

B1

Answer: Row 1 dominates row 2

1 mark

4 (2 marks)

B wants small values: column 1 (3, 2) has smaller entries than column 2 (5, 4)

M1

Column 1 dominates column 2

A1

Answer: Column 1 dominates column 2

2 marks

5 (1 mark)

Whatever B does, the dominating row gives A at least as much, so the dominated row is never better

E1

Answer: The dominating row is always at least as good

1 mark

6 (1 mark)

$4 \geq 2$, $1 \geq 1$, $3 \geq 3$ with $4 > 2$, so row 2 dominates row 1

B1

Answer: Row 2 dominates row 1

1 mark

7 (3 marks)

Row 1 (3, 1, 4) $\geq$ row 2 (2, 0, 3) in every column

M1 A1

Row 2 is dominated; the new matrix is $\begin{pmatrix} 3 & 1 & 4 \\ 1 & 5 & 2 \end{pmatrix}$

A1

Answer: Row 2 dominated; $\begin{pmatrix} 3 & 1 & 4 \\ 1 & 5 & 2 \end{pmatrix}$

3 marks

8 (3 marks)

Column 1 (3, 1) $\leq$ column 3 (4, 2) in each row, so for B, column 3 is dominated by column 1

M1 A1

New matrix $\begin{pmatrix} 3 & 1 \\ 1 & 5 \end{pmatrix}$

A1

Answer: $\begin{pmatrix} 3 & 1 \\ 1 & 5 \end{pmatrix}$

3 marks

9 (4 marks)

Row 1 (4, 2, 6) $\geq$ row 2 (3, 1, 5): delete row 2

M1 A1

In the remaining matrix column 1 (4, 2) $\leq$ column 3 (6, 4): B never plays column 3; delete it

M1

Reduced matrix $\begin{pmatrix} 4 & 2 \\ 2 & 3 \end{pmatrix}$

A1

Answer: $\begin{pmatrix} 4 & 2 \\ 2 & 3 \end{pmatrix}$

4 marks

10 (4 marks)

Row 1 (2, 5, 3) beats row 2 (1, 4, 2) in every column: delete row 2

M1 A1

Then column 1 (2, 3) is below column 3 (3, 6) for B: delete column 3

M1

Reduced matrix $\begin{pmatrix} 2 & 5 \\ 3 & 2 \end{pmatrix}$

A1

Answer: $\begin{pmatrix} 2 & 5 \\ 3 & 2 \end{pmatrix}$

4 marks

11 (2 marks)

Row 1 is better in column 1 ($3 > 2$) but worse in column 2 ($1 < 4$), so neither row dominates

E1

Column 1 (3, 2) and column 2 (1, 4): $3 > 1$ but $2 < 4$, so neither column dominates

E1

Answer: Each pair differs in direction

2 marks

12 (2 marks)

The rows are identical, so neither strictly dominates the other

E1

They are equivalent, so one can be deleted without changing the value of the game

E1

Answer: Identical rows: delete one

2 marks

13 (2 marks)

Compare S_1 (4, 6) with S_3 (4, 2)

M1

$4 \geq 4$ and $6 > 2$, so S_1 dominates S_3 : S_3 is never better

A1

Answer: S_1 dominates S_3

2 marks

14 (6 marks)

Row 1 (3, 2, 4) $\geq$ row 4 (0, 2, 1): delete row 4

M1 A1

Column 1 (3, 1, 2) $\leq$ column 3 (4, 2, 5) for B: delete column 3

M1 A1

Row 1 (3, 2) $\geq$ row 3 (2, 1): delete row 3

M1

Reduced matrix $\begin{pmatrix} 3 & 2 \\ 1 & 3 \end{pmatrix}$

A1

Answer: $\begin{pmatrix} 3 & 2 \\ 1 & 3 \end{pmatrix}$

6 marks

15 (6 marks)

Row 1 (5, 2, 3) $\geq$ row 3 (2, 0, 2) in every column: delete row 3

M1 A1

Column 2 (2, 1) is smaller than column 1 (5, 4) and column 3 (3, 6): delete columns 1 and 3

M1 A1

The matrix left is column B_2 with entries 2 and 1: row 1 dominates row 2

M1

A plays A_1 , B plays B_2 , and the value is 2

A1

Answer: A_1 , B_2 ; value 2

6 marks

16 (6 marks)

Columns (7, 1, 2), (0, 3, 4), (1, 1, 7): no column has entries all $\leq$ another's (for example $7 > 0$ but $1 < 3$)

E1

Row 3 (2, 4, 7) $\geq$ row 2 (1, 3, 1): delete row 2

M1 A1

Remaining rows (7, 0, 1) and (2, 4, 7): column 2 (0, 4) $\leq$ column 3 (1, 7), so delete column 3

M1 A1

Reduced matrix $\begin{pmatrix} 7 & 0 \\ 2 & 4 \end{pmatrix}$

A1

Answer: $\begin{pmatrix} 7 & 0 \\ 2 & 4 \end{pmatrix}$

6 marks

17 (7 marks)

Row 1 $\geq$ row 2: delete row 2; column 1 $\leq$ column 3 for B: delete column 3

M1 A1

Reduced matrix $\begin{pmatrix} 4 & 2 \\ 2 & 3 \end{pmatrix}$, no saddle (maximin 2, minimax 3)

A1

A plays A_1 with probability p : $4p + 2(1 - p) = 2p + 3(1 - p)$, so $2 + 2p = 3 - p$, $p = \frac{1}{3}$

M1 A1

Value = $\frac{8}{3}$

A1

A plays (A_1, A_2, A_3) with probabilities $(\frac{1}{3}, 0, \frac{2}{3})$

A1

Answer: $(\frac{1}{3}, 0, \frac{2}{3})$, value $\frac{8}{3}$

7 marks

18 (6 marks)

Row 2 (1, 4, 2) is below row 1 (2, 5, 3): delete it; then column 3 (3, 6) is above column 1 (2, 3) for B: delete it

M1 A1

Reduced matrix $\begin{pmatrix} 2 & 5 \\ 3 & 2 \end{pmatrix}$; A plays row 1 with probability p : $2p + 3(1 - p) = 5p + 2(1 - p)$

M1

$3 - p = 2 + 3p$, so $p = \frac{1}{4}$

A1

Value = $3 - \frac{1}{4} = \frac{11}{4}$

A1

A attracts on average $\frac{11}{4}$ hundred = 275 extra customers per day with optimal play

E1

Answer: A: $(\frac{1}{4}, 0, \frac{3}{4})$; value $\frac{11}{4}$

6 marks

19 (4 marks)

Dominance needs one row to be at least as good in every column

E1

Row 1 is better than row 2 against B_1 ($3 > 2$)

M1 A1

So row 1 is not dominated

A1

Answer: Row 1 is better in column 1: no domination

4 marks

20 (5 marks)

For B, larger payoffs are better

E1

Row 2 $(-2, 0) \geq$ row 1 $(-3, -1)$ and $\geq$ row 3 $(-5, -4)$

M1 A1

B plays row 2, so B's payoffs are -2 and 0 against A_1 and A_2

M1

A prefers A_1 , which gives B -2 , that is A gets $+2$; the value to A is 2

A1

Answer: B plays row 2; A plays A_1 ; value 2 to A

5 marks

21 (8 marks)

Row 2 is dominated by row 1; then column 3 is dominated by column 1 (for B)

M1 A1

Reduced matrix $\begin{pmatrix} 3 & 1 \\ 1 & 5 \end{pmatrix}$, no saddle point (maximin 1, minimax 3)

A1

A plays row 1 with probability p : $3p + 1 - p = p + 5 - 5p$, so $p = \frac{2}{3}$

M1 A1

Value = $1 + 2(\frac{2}{3}) = \frac{7}{3}$

A1

B plays column 1 with probability q : $3q + (1 - q) = 5 - 4q$, so $q = \frac{2}{3}$

M1 A1

Answer: A: $(\frac{2}{3}, 0, \frac{1}{3})$; B: $(\frac{2}{3}, \frac{1}{3}, 0)$; value $\frac{7}{3}$

8 marks

22 (9 marks)

Row 2 is dominated by row 1; column 1 is dominated by column 2 (for B)

M1 A1

Reduced matrix $\begin{pmatrix} 2 & 5 & 3 \\ 4 & 3 & 6 \\ 3 & 6 & 1 \end{pmatrix}$

A1

A's expected payoff against B's columns 2, 3, 4: $\frac{3}{4}(4) + \frac{1}{4}(3) = \frac{15}{4}$, $\frac{3}{4}(3) + \frac{1}{4}(6) = \frac{15}{4}$, $\frac{3}{4}(6) + \frac{1}{4}(1) = \frac{19}{4}$

M1 A1

A guarantees at least $\frac{15}{4}$

A1

B's expected loss against A's rows: $\frac{3}{4}(2) + \frac{1}{4}(5) = \frac{11}{4}$, $\frac{3}{4}(4) + \frac{1}{4}(3) = \frac{15}{4}$, $\frac{3}{4}(3) + \frac{1}{4}(6) = \frac{15}{4}$

M1 A1

A's guaranteed minimum equals B's guaranteed maximum, so the value is $\frac{15}{4}$

E1

Answer: Value $\frac{15}{4}$ verified

9 marks

23 (7 marks)

Row 1 (0, 1, -1) against row 2 (-1, 0, 1): $0 > -1$ but $1 > 0$ and $-1 < 1$, so neither dominates; similarly for every other pair of rows

M1 A1

Columns: the same argument applies (each pair differs in sign pattern), so no column is dominated

A1

Matrix is skew-symmetric, so the game is fair and the value is 0

E1

By symmetry, A plays each row with probability $\frac{1}{3}$: the expected payoff is $\frac{1}{3}(0 + (-1) + 1) = 0$ against each column

M1 A1

So A guarantees 0

A1

Answer: No dominance; value 0; A: $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$

7 marks

24 (7 marks)

Row 1 $\geq$ row 2 in every column, so delete row 2

M1 A1

Expected payoffs: $B_1: 3 + 2p$; $B_2: 4 - p$; $B_3: 2 + 4p$; $B_4: 6 - 4p$

M1 A1

Lower envelope is formed by B_2 and B_3 : $4 - p = 2 + 4p$

M1

$p = \frac{2}{5}$, value $\frac{18}{5}$

A1

Check: B_1 gives $\frac{19}{5}$ and B_4 gives $\frac{22}{5}$, both above $\frac{18}{5}$

A1

Answer: $p = \frac{2}{5}$; value $\frac{18}{5}$

7 marks

25 (7 marks)

Row 2 is dominated by row 1; then column 3 is dominated by column 1 (for B)

M1 A1

Reduced matrix $\begin{pmatrix} 4 & 2 \\ 2 & 3 \end{pmatrix}$

A1

B plays column 1 with probability q : $4q + 2(1 - q) = 2q + 3(1 - q)$

M1

$2 + 2q = 3 - q$, so $q = \frac{1}{3}$

A1

Value $= 2 + \frac{2}{3} = \frac{8}{3}$

A1

B plays promotions 1, 2, 3 with probabilities $\frac{1}{3}, \frac{2}{3}, 0$; A gains $\pounds \frac{800}{3}$ per week on average

E1

Answer: B: $(\frac{1}{3}, \frac{2}{3}, 0)$; value $\frac{8}{3}$

7 marks

26 (7 marks)

Dominance: row 3 is dominated by row 1

M1

Column 2 (2, 1) is smaller than column 1 and column 3 in both rows: delete columns 1 and 3, then row 2 is dominated by row 1

M1 A1

Value 2 at (A_1, B_2)

A1

Row minima 2, 1, 0 so maximin = 2; column maxima 5, 2, 6 so minimax = 2

M1 A1

The methods agree: saddle point, value 2

E1

Answer: Value 2 both ways

7 marks

27 (9 marks)

Order 1: row 4 is dominated by row 1; column 3 is then dominated by column 1; row 3 is then dominated by row 1

M1 A1 A1

Order 2: column 3 $(4, 2, 5, 1) \geq$ column 1 $(3, 1, 2, 0)$ entry by entry, so delete column 3; row 4 is dominated by row 1, then row 3 by row 1

M1 A1

Both give $\begin{pmatrix} 3 & 2 \\ 1 & 3 \end{pmatrix}$

A1

$3p + 1 - p = 2p + 3 - 3p$, so $p = \frac{2}{3}$

M1 A1

Value = $\frac{7}{3}$

A1

Answer: Both orders give $\begin{pmatrix} 3 & 2 \\ 1 & 3 \end{pmatrix}$; value $\frac{7}{3}$

9 marks

28 (9 marks)

Row 3 is not dominated by row 1 ($1 > 0$) or by row 2 ($1 > 0$)

E1

Mixing rows 1 and 2 with probabilities $\frac{1}{2}$: payoffs are (2, 2), and $(2, 2) \geq (1, 1)$, so row 3 is dominated by the mixture

M1 A1

Delete row 3: matrix $\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$ has no saddle point

A1

$4p = 4(1 - p)$, so $p = \frac{1}{2}$

M1 A1

Value = 2

A1

A plays $(A_1, A_2, A_3) = (\frac{1}{2}, \frac{1}{2}, 0)$, B plays $(\frac{1}{2}, \frac{1}{2})$ by symmetry

M1 A1

Answer: Value 2; A: $(\frac{1}{2}, \frac{1}{2}, 0)$; B: $(\frac{1}{2}, \frac{1}{2})$

9 marks

29 (8 marks)

Row 1 dominates row 2 and row 4: delete rows 2 and 4

M1 A1

In rows 1, 3, 5: column 3 $(6, 4, 2) \leq$ column 1 $(7, 6, 5)$, so delete column 1; column 2 $(4, 5, 6) \leq$ column 5 $(8, 7, 6)$, so delete column 5

M1 A1

Reduced matrix $\begin{pmatrix} 4 & 6 & 5 \\ 5 & 4 & 3 \\ 6 & 2 & 4 \end{pmatrix}$ (rows 1, 3, 5; columns 2, 3, 4)

A1

Against column 2: $\frac{2}{3}(4) + \frac{1}{3}(6) = \frac{14}{3}$

M1

Against column 3: $\frac{2}{3}(6) + \frac{1}{3}(2) = \frac{14}{3}$; against column 4: $\frac{2}{3}(5) + \frac{1}{3}(4) = \frac{14}{3}$

A1

So A guarantees $\frac{14}{3}$, which is the value of the game

E1

Answer: Value $\frac{14}{3}$

8 marks

30 (9 marks)

Row 1 $\geq$ row 2 with $3 > 1$ in column 1 and equality in column 2: row 2 is weakly dominated

M1 A1

Row minima 3, 1 so maximin = 3

M1

Column maxima 3, 3 so minimax = 3, so the game is stable with value 3

M1 A1

Saddle points are (A_1, B_1) and (A_1, B_2) ; row 2 contains none

M1 A1

A plays A_1

B1

B may play either column, or any mixture, since A_1 pays 3 whichever is played and A_2 pays at most 3

E1

Answer: Value 3; saddle points (A_1, B_1) , (A_1, B_2) ; A plays A_1

9 marks

Optimal Strategies

1 (1 mark)

An entry that is the minimum of its row and the maximum of its column

B1

Answer: Smallest in its row and largest in its column

1 mark

2 (1 mark)

The payoff at the saddle point, which equals both the maximin value and the minimax value

B1

Answer: The payoff at the saddle point

1 mark

3 (2 marks)

Row minima 3, 2 so maximin = 3; column maxima 3, 5 so minimax = 3

M1

Saddle point (A_1, B_1) ; value 3

A1

Answer: (A_1, B_1) ; value 3

2 marks

4 (2 marks)

Maximin = 2 and minimax = 3

M1

They are not equal, so there is no saddle point

A1

Answer: 2 and 3; no saddle point

2 marks

5 (2 marks)

The value is 4

B1

The game is stable since maximin = minimax

B1

Answer: Value 4; stable

2 marks

6 (2 marks)

The entry is the largest in its column, so if B stays put, A cannot gain by changing row

E1

The entry is the smallest in its row, so if A stays put, B cannot gain by changing column

E1

Answer: Column maximum and row minimum

2 marks

7 (5 marks)

Row minima: 4, 2; maximin = 4 (A plays A_1)

M1 A1

Column maxima: 6, 4, 7; minimax = 4 (B plays B_2)

M1 A1

Maximin = minimax = 4, so there is a saddle point at (A_1, B_2) ; the value is 4

A1

Answer: Saddle point at (A_1, B_2) ; value 4

5 marks

8 (5 marks)

Row minima: 2, 4, 1; maximin = 4 (A plays A_2)

M1 A1

Column maxima: 4, 7; minimax = 4 (B plays B_1)

M1 A1

Maximin = minimax = 4, so there is a saddle point at (A_2, B_1) ; the value is 4

A1

Answer: Saddle point at (A_2, B_1) ; value 4

5 marks

9 (5 marks)Row minima: $-1, -2, -3$; maximin = -1 (A plays A1)

M1 A1

Column maxima: $-1, 4, 5$; minimax = -1 (B plays B1)

M1 A1

Maximin = minimax = -1 , so there is a saddle point at (A1, B1); the value is -1

A1

Answer: Saddle point at (A1, B1); value -1

5 marks

10 (5 marks)Row minima: $2, 1$; maximin = 2 (A plays A1)

M1 A1

Column maxima: $5, 4$; minimax = 4 (B plays B2)

M1 A1

Maximin $2 \neq$ minimax 4 , so there is no saddle point and the game is not stable

A1

Answer: Maximin 2, minimax 4: no saddle point

5 marks

11 (5 marks)Row minima: $3, 2, 5$; maximin = 5 (A plays A3)

M1 A1

Column maxima: $5, 6, 5$; minimax = 5 (B plays B1)

M1 A1

Saddle points at (A_3, B_1) and (A_3, B_3) , each with value 5

A1

Answer: (A_3, B_1) and (A_3, B_3) ; value 5

5 marks

12 (3 marks)Row 3 is $(7, 9, 8)$, whose minimum is 7 , the entry at (A_3, B_1)

M1

Column 1 is $(4, 2, 7)$, whose maximum is 7 , the same entry

M1

So (A_3, B_1) is a saddle point; the value is 7

A1

Answer: Value 7

3 marks

13 (4 marks)Matrix $\begin{pmatrix} 0 & -2 & -4 \\ 1 & 0 & -2 \\ 2 & 1 & 0 \end{pmatrix}$ (rows: A chooses 1, 2, 3; columns: B chooses 1, 2, 3)

M1 A1

Row minima $-4, -2, 0$ give maximin = 0 (A_3); column maxima $2, 1, 0$ give minimax = 0 (B_3)

M1

Maximin = minimax, so there is a saddle point at (A_3, B_3) and the value is 0 , so the game is fair

A1

Answer: Matrix $\begin{pmatrix} 0 & -2 & -4 \\ 1 & 0 & -2 \\ 2 & 1 & 0 \end{pmatrix}$; saddle point (A_3, B_3) ; value 0

4 marks

14 (5 marks)Row minima: $2, 4, 1$; maximin = 4 (A plays A2)

M1 A1

Column maxima: $4, 7, 6$; minimax = 4 (B plays B1)

M1 A1

Maximin = minimax = 4 , so there is a saddle point at $(A2, B1)$; the value is 4

A1

Answer: Saddle point at $(A2, B1)$; value 4

5 marks

15 (7 marks)Row minima: $-2, 0, -3$; maximin = 0 (A plays A2)

M1 A1

Column maxima: $0, 2, 3$; minimax = 0 (B plays B1)

M1 A1

Saddle point (A_2, B_1) ; value 0

A1

Each plays the opening corresponding to the saddle point: A plays opening 2 and B plays opening 1

B1

Value 0 , so the game is fair (neither player has an advantage with best play)

E1

Answer: A: opening 2; B: opening 1; value 0 , fair

7 marks

16 (6 marks)

B wants the largest payoff, so B chooses the row with the largest minimum: row minima $-4, -7, -5$, so B plays B_1 M1 A1

A wants B's payoff small, so A chooses the column with the smallest maximum: column maxima $-4, -2$, so A plays A_1 M1 A1

The entry -4 is the minimum of its row and the maximum of its column, so it is a saddle point M1

The value to B is -4 (and 4 to A) A1

Answer: B plays B_1 ; A plays A_1 ; value -4 to B 6 marks

17 (6 marks)

Row minima: 2, 1, 3, 2; maximin = 3 (A plays A_3) M1 A1

Column maxima: 5, 7, 7, 8; minimax = 5 (B plays B_1) M1 A1

Maximin 3 $\neq$ minimax 5, so there is no saddle point A1

A can guarantee at least 3 and B can hold A to at most 5, so the value lies between 3 and 5 E1

Answer: No saddle point; $3 \leq \text{value} \leq 5$ 6 marks

18 (5 marks)

The saddle value is 4, which must be the minimum of row 2 (4, x, 6) M1

So $x \geq 4$ A1

Column 1 is (2, 4, 1) with maximum 4, so the entry is a column maximum for every x M1 A1

The value of the game is 4 A1

Answer: $x \geq 4$; value 4 5 marks

19 (4 marks)

A saddle point is the minimum of its row and the maximum of its column; 1 is the minimum of row 1, but column 2 is (1, 5, 4), whose maximum is 5 M1 A1

Row minima 1, 2, 3 so maximin = 3; column maxima 7, 5, 6 so minimax = 5 M1

These are not equal, so there is no saddle point at all A1

Answer: 1 is not a column maximum; no saddle point 4 marks

20 (6 marks)

Row minima: 2, 3; maximin = 3 (A plays A_2) M1 A1

Column maxima: 6, 5; minimax = 5 (B plays B_2) M1 A1

Maximin 3 $\neq$ minimax 5, so there is no saddle point A1

$3 < 4 < 5$, as A can guarantee at least maximin and B can hold A to at most minimax E1

Answer: $3 < 4 < 5$ 6 marks

21 (8 marks)

Row minima: $\min(k, 4)$ and 3; column maxima: $\max(k, 3)$ and 5 M1 A1

If $k \leq 3$: maximin = 3 and minimax = 3, saddle point (A_2, B_1), value 3 M1 A1

If $3 \leq k \leq 4$: maximin = k and minimax = k, saddle point (A_1, B_1), value k M1 A1

If $k > 4$: maximin = 4 but minimax = $\min(k, 5) > 4$, no saddle point M1

So a saddle point exists exactly when $k \leq 4$ A1

Answer: $k \leq 4$; value 3 for $k \leq 3$ and k for $3 \leq k \leq 4$ 8 marks

22 (6 marks)

Row minima: $\min(2, k)$ and 3, so maximin = 3 for every k M1 A1

Column maxima: 5 and $\max(k, 3)$ M1

Minimax = $\min(5, \max(k, 3)) = 3$ exactly when $k \leq 3$; if $k > 3$ then minimax is greater than 3 M1 A1

Saddle point (A_2, B_2), value 3 A1

Answer: $k \leq 3$; saddle point (A_2, B_2); value 3 6 marks

23 (6 marks)

A plays A_1 : B's best reply is B_2 (payoff 0 instead of 3) M1 A1
 Against B_2 , A's best reply is A_2 (2 instead of 0) M1
 Against A_2 , B's best reply is B_1 (1 instead of 2); against B_1 , A's best reply is A_1 (3 instead of 1) M1 A1
 The sequence returns to the start and never settles, so there is no pair from which neither player wants to move E1
Answer: Cycle A_1, B_2, A_2, B_1, A_1 : no saddle point 6 marks

24 (6 marks)

a_{ij} is the minimum of row i , so $a_{ij} \leq a_{il}$ M1
 a_{kl} is the maximum of column l , so $a_{il} \leq a_{kl}$; hence $a_{ij} \leq a_{kl}$ M1 A1
 Similarly $a_{kl} \leq a_{kj} \leq a_{ij}$, so $a_{ij} = a_{kl}$ M1 A1
 Then $a_{ij} = a_{il} = a_{kl}$, so a_{il} is the minimum of row i and the maximum of column l : a saddle point E1
Answer: Equal values; (A_i, B_l) is a saddle point 6 marks

25 (7 marks)

Row minima: 3, 4, 1; maximin = 4 (A plays A_2) M1 A1
 Column maxima: 8, 4, 9, 9; minimax = 4 (B plays B_2) M1 A1
 Maximin = minimax = 4, so there is a saddle point at (A_2, B_2) A1
 A plays policy 2 and B plays policy 2 B1
 The value is 4, so A gains 4% market share with optimal play E1
Answer: (A_2, B_2) ; value 4; A gains 4% 7 marks

26 (6 marks)

Game 1: row minima 3, 2; column maxima 3, 6, 4; saddle point (A_1, B_1) , value 3 M1 A1
 Game 2 matrix $\begin{pmatrix} 9 & 13 & 11 \\ 7 & 15 & 9 \end{pmatrix}$ A1
 Row minima 9, 7; column maxima 9, 15, 11; saddle point (A_1, B_1) , value 9 M1 A1
 The saddle point is unchanged and the value follows the same rule: $2 \times 3 + 3 = 9$ E1
Answer: Game 2: $\begin{pmatrix} 9 & 13 & 11 \\ 7 & 15 & 9 \end{pmatrix}$; saddle (A_1, B_1) , value 9 6 marks

27 (8 marks)

Let row r give the maximin and column c give the minimax M1
 Maximin = min over j of $a_{rj} \leq a_{rc}$, as a_{rc} is one of the entries in row r M1 A1
 Minimax = max over i of $a_{ic} \geq a_{rc}$, as a_{rc} is one of the entries in column c M1 A1
 So maximin $\leq a_{rc} \leq$ minimax A1
 A saddle point exists exactly when equality holds, and then the common value is the value of the game E1
 A can always guarantee the maximin and B can always hold A to the minimax, so a gap between them leaves room for mixed strategies E1
Answer: maximin $\leq$ minimax 8 marks

28 (8 marks)

Row minima: $\min(2, k), \min(3, k), 2$; column maxima: 6, $\max(5, k), \max(3, k)$ M1 A1
 If $k < 3$: row 3 and row 1 give at most 2, and maximin = $\max(2, \min(3, k)) < 3$; minimax = $\min(6, \max(5, k), 3) = 3$. They differ M1 A1
 If $k > 3$: maximin = 3 (from row 2); minimax = $\min(6, \max(5, k), k) = \min(6, k) > 3$. They differ M1 A1
 If $k = 3$: maximin = 3 and minimax = 3, equal M1
 With $k = 3$ the matrix is $\begin{pmatrix} 4 & 3 & 2 \\ 3 & 5 & 3 \\ 6 & 2 & 3 \end{pmatrix}$, so the saddle point is (A_2, B_3) and the value is 3 A1
Answer: $k = 3$; saddle point (A_2, B_3) ; value 3 8 marks

29 (8 marks)

B's payoffs are the negatives of A's, transposed: $\begin{pmatrix} -4 & -2 \\ -6 & -7 \\ -5 & -3 \end{pmatrix}$ M1 A1

A's table: row minima 4, 2 so maximin = 4; column maxima 4, 7, 5 so minimax = 4; saddle point (A_1, B_1) , value 4 M1 A1

B's table: row minima $-4, -7, -5$, so B's maximin = -4 M1 A1

Column maxima $-4, -2$, so A's minimax in B's table is -4 M1

They are equal: a saddle point at (B_1, A_1) and the value to B is $-4 = -(\text{value to A})$ A1

Answer: Value to B is -4 8 marks

30 (10 marks)

Row minima 2, 3, 6; column maxima 6, 7, 6; saddle points (A_3, B_1) and (A_3, B_3) , value 6 M1 A1

B plays B_1 with probability q and B_3 with probability $1 - q$; A's expected payoffs for rows 1, 2, 3 are $6, 3q + 5(1 - q) = 5 - 2q, 6$ M1 A1

None exceeds 6 for any q with $0 \leq q \leq 1$, so every such q is optimal for B E1

A plays row 1 with probability x and row 3 with probability $1 - x$: expected payoffs against B_1, B_2, B_3 are $6, 2x + 7(1 - x) = 7 - 5x, 6$ M1 A1

A needs at least 6 in every column: $7 - 5x \geq 6$, so $x \leq \frac{1}{5}$ M1 A1

A plays row 1 with probability x , row 3 with probability $1 - x$, for $0 \leq x \leq \frac{1}{5}$ A1

Answer: B: any q in $[0, 1]$; A: x in $[0, \frac{1}{5}]$ 10 marks

Mixed Strategies**1 (1 mark)**

A strategy in which A chooses each of the available pure strategies with a stated probability, the probabilities summing to 1 B1

Answer: Choosing among pure strategies with fixed probabilities 1 mark

2 (1 mark)

If A always plays the same pure strategy, B can find out and respond to it, so A should keep B guessing by randomising E1

Answer: Otherwise B can exploit a predictable choice 1 mark

3 (2 marks)

$0.3 \times 4 + 0.7 \times 2$ M1

$= 2.6$ A1

Answer: 2.6 2 marks

4 (2 marks)

$3p - 2(1 - p)$ M1

$= 5p - 2$ A1

Answer: $5p - 2$ 2 marks

5 (2 marks)

$\frac{1}{2} \times 2 + \frac{1}{2} \times 4$ M1

$= 3$ A1

Answer: 3 2 marks

6 (3 marks)

Against B_1 : $\frac{1}{2}(1) + \frac{1}{2}(4) = \frac{5}{2}$

M1

Against B_2 : $\frac{1}{2}(3) + \frac{1}{2}(2) = \frac{5}{2}$

A1

Equal, so B gains nothing by choosing one over the other

E1

Answer: $\frac{5}{2}$ against each; no gain for B

3 marks

7 (4 marks)

A plays A_1 with probability p and A_2 with probability $1 - p$. Expected payoff against B_1 : $2 + 3p$;
against B_2 : $4 - 3p$

M1

Equating: $2 + 3p = 4 - 3p$, so $p = \frac{1}{3}$

M1 A1

Value = 3

A1

Answer: A: $(\frac{1}{3}, \frac{2}{3})$; value 3

4 marks

8 (4 marks)

A plays A_1 with probability p and A_2 with probability $1 - p$. Expected payoff against B_1 : $-2 + 5p$;
against B_2 : $4 - 5p$

M1

Equating: $-2 + 5p = 4 - 5p$, so $p = \frac{3}{5}$

M1 A1

Value = 1

A1

Answer: A: $(\frac{3}{5}, \frac{2}{5})$; value 1

4 marks

9 (3 marks)

B plays B_1 with probability q and B_2 with probability $1 - q$. Expected loss against A_1 : $6 - 4q$;
against A_2 : $3 + 2q$

M1

Equating: $6 - 4q = 3 + 2q$, so $q = \frac{1}{2}$

M1 A1

Answer: B: $(\frac{1}{2}, \frac{1}{2})$

3 marks

10 (4 marks)

$p = (6 - 3) / (7 - 2 - 3 + 6) = \frac{3}{8}$

M1 A1

Expected payoff against B_1 : $3 + 4p = 3 + \frac{3}{2} = \frac{9}{2}$

M1

Value = $\frac{9}{2}$

A1

Answer: $p = \frac{3}{8}$; value $\frac{9}{2}$

4 marks

11 (3 marks)

Against B_1 : $\frac{1}{4}(3) + \frac{3}{4}(1) = \frac{3}{2}$

M1

Against B_2 : $\frac{1}{4}(0) + \frac{3}{4}(2) = \frac{3}{2}$

M1

Both equal $\frac{3}{2}$

A1

Answer: $\frac{3}{2}$ in each case

3 marks

12 (4 marks)

A plays A_1 with probability p and A_2 with probability $1 - p$. Expected payoff against B_1 : $-1 + 4p$; against B_2 : $1 - 3p$

M1

Equating: $-1 + 4p = 1 - 3p$, so $p = \frac{2}{7}$

M1 A1

Value = $\frac{1}{7}$

A1

Answer: Value $\frac{1}{7}$; favours A

4 marks

13 (4 marks)

If A plays A_1 : $\frac{1}{3}(4) + \frac{2}{3}(2) = \frac{8}{3}$; if A plays A_2 : $\frac{1}{3}(1) + \frac{2}{3}(5) = \frac{11}{3}$

M1 A1

Expected payoff = $\frac{1}{4}(\frac{8}{3}) + \frac{3}{4}(\frac{11}{3}) = \frac{41}{12}$

M1 A1

Answer: $\frac{41}{12}$

4 marks

14 (6 marks)

Against B_1 : $1 + 2p$; against B_2 : $4p$; against B_3 : $5 - 4p$

M1 A1

A is guaranteed the lowest of the three lines; the highest point of this lower boundary is where the lines for B_1 and B_3 cross

M1

$1 + 2p = 5 - 4p$, so $p = \frac{2}{3}$

A1

Value = $1 + \frac{4}{3} = \frac{7}{3}$

A1

At $p = \frac{2}{3}$ the line for B_2 gives $\frac{8}{3}$, above $\frac{7}{3}$, so B never uses B_2

E1

Answer: A: $(\frac{2}{3}, \frac{1}{3})$; value $\frac{7}{3}$

6 marks

15 (6 marks)

Row minima 60, 30, so maximin = 60; column maxima 80, 90, so minimax = 80; no saddle point

M1 A1

A plays A_1 with probability p and A_2 with probability $1 - p$. Expected payoff against B_1 : $80 - 20p$; against B_2 : $30 + 60p$

M1

Equating: $80 - 20p = 30 + 60p$, so $p = \frac{5}{8}$

M1 A1

Value = $\frac{135}{2}$

A1

Answer: Shoot left with probability $\frac{5}{8}$; scores $\frac{135}{2}\%$

6 marks

16 (6 marks)

Against A_1 : $3q + (1 - q) = 1 + 2q$; against A_2 : $4q$; against A_3 : $q + 5(1 - q) = 5 - 4q$

M1 A1

B wants the lowest highest line; the highest line is smallest where the lines for A_2 and A_3 cross

M1

$4q = 5 - 4q$, so $q = \frac{5}{8}$

M1 A1

Value = $4(\frac{5}{8}) = \frac{5}{2}$; A_1 gives $\frac{9}{4} < \frac{5}{2}$ at this q , so it is not used

A1

Answer: B: $(\frac{5}{8}, \frac{3}{8})$; value $\frac{5}{2}$

6 marks

17 (7 marks)

Row minima 2, 1, so maximin = 2; column maxima 6, 4, so minimax = 4; no saddle point

M1 A1

A plays A_1 with probability p and A_2 with probability $1 - p$. Expected payoff against B_1 : $1 + 5p$;
against B_2 : $4 - 2p$

M1

Equating: $1 + 5p = 4 - 2p$, so $p = \frac{3}{7}$

M1 A1

Value = $\frac{22}{7}$

A1

A's mix makes the two expected payoffs equal, so whichever column B plays (or any mixture of them) the result is the same

E1

Answer: A: $(\frac{3}{7}, \frac{4}{7})$; value $\frac{22}{7}$

7 marks

18 (7 marks)

A plays A_1 with probability p and A_2 with probability $1 - p$. Expected payoff against B_1 : $4 - 6p$;
against B_2 : $-1 + 4p$

M1

Equating: $4 - 6p = -1 + 4p$, so $p = \frac{1}{2}$

M1 A1

Value = 1

A1

B plays B_1 with probability q and B_2 with probability $1 - q$. Expected loss against A_1 : $3 - 5q$;
against A_2 : $-1 + 5q$

M1

Equating: $3 - 5q = -1 + 5q$, so $q = \frac{2}{5}$

M1 A1

Answer: A: $(\frac{1}{2}, \frac{1}{2})$; B: $(\frac{2}{5}, \frac{3}{5})$; value 1

7 marks

19 (6 marks)

With $(\frac{1}{2}, \frac{1}{2})$: against B_1 : $\frac{1}{2}(1) + \frac{1}{2}(4) = \frac{5}{2}$; against B_2 : $\frac{1}{2}(5) + \frac{1}{2}(2) = \frac{7}{2}$

M1 A1

B plays B_1 , holding A to $\frac{5}{2}$

A1

Correct strategy: $4 - 3p = 2 + 3p$, so $p = \frac{1}{3}$

M1 A1

Value = $4 - 1 = 3$, which is more than $\frac{5}{2}$

A1

Answer: $\frac{5}{2} < 3$; $p = \frac{1}{3}$, value 3

6 marks

20 (5 marks)

A plays A_1 with probability p and A_2 with probability $1 - p$. Expected payoff against B_1 : $7 - 4p$;
against B_2 : $2 + 3p$

M1

Equating: $7 - 4p = 2 + 3p$, so $p = \frac{5}{7}$

M1 A1

Value = $\frac{29}{7}$

A1

Against both B_1 and B_2 A's expected payoff is $\frac{29}{7}$, so against any mixture it is $0.9(\frac{29}{7}) + 0.1(\frac{29}{7})$

E1

= $\frac{29}{7}$, equal to the value

Answer: $\frac{29}{7}$

5 marks

21 (8 marks)

Against B_1 : $4 - 2p$; against B_2 : $7 - 6p$; against B_3 : $3 + 2p$

M1 A1

The lowest boundary is highest where B_1 and B_3 cross: $4 - 2p = 3 + 2p$, so $p = \frac{1}{4}$

M1 A1

At $p = \frac{1}{4}$, B_2 gives $\frac{11}{2}$, which is higher, so B does not use B_2

M1

$$\text{Value} = 4 - \frac{1}{2} = \frac{7}{2}$$

A1

B plays B_1 with probability q and B_3 with $1 - q$: against A_1 : $5 - 3q$, against A_2 : $3 + q$

M1

$$5 - 3q = 3 + q, \text{ so } q = \frac{1}{2}$$

A1

Answer: A: $(\frac{1}{4}, \frac{3}{4})$; B: $(\frac{1}{2}, 0, \frac{1}{2})$; value $\frac{7}{2}$

8 marks

22 (8 marks)

The matrix is skew-symmetric ($a_{ji} = -a_{ij}$), so the game is symmetric and neither player has an advantage: value 0

E1

Let A play (p_1, p_2, p_3) . Against B_1 : $-2p_2 + p_3 = 0$; against B_2 : $2p_1 - 3p_3 = 0$; against B_3 : $-p_1 + 3p_2 = 0$

M1 A1

From the first and third: $p_3 = 2p_2$ and $p_1 = 3p_2$

M1 A1

$$p_1 + p_2 + p_3 = 6p_2 = 1, \text{ so } p_2 = \frac{1}{6}$$

M1

$$\text{A: } (\frac{1}{2}, \frac{1}{6}, \frac{1}{3})$$

A1

B's optimal strategy is the same, by symmetry

E1

Answer: A and B both play $(\frac{1}{2}, \frac{1}{6}, \frac{1}{3})$; value 0

8 marks

23 (8 marks)

$$\text{A's expected payoff against } B_1: \frac{1}{2}(2) + \frac{1}{3}(8) + \frac{1}{6}(3) = \frac{25}{6}$$

M1 A1

$$\text{Against } B_2: \frac{1}{2}(5) + \frac{1}{3}(3) + \frac{1}{6}(4) = \frac{25}{6}; \text{ against } B_3: \frac{1}{2}(3) + \frac{1}{3}(5) + \frac{1}{6}(6) = \frac{25}{6}$$

M1 A1

So A guarantees at least $\frac{25}{6}$

A1

$$\text{B's expected loss against } A_1: \frac{1}{6}(2) + \frac{2}{3}(5) + \frac{1}{6}(3) = \frac{25}{6}; \text{ against } A_2: \frac{1}{6}(8) + \frac{2}{3}(3) + \frac{1}{6}(5) = \frac{25}{6};$$

M1 A1

$$\text{against } A_3: \frac{1}{6}(3) + \frac{2}{3}(4) + \frac{1}{6}(6) = \frac{25}{6}$$

B holds A to at most $\frac{25}{6}$, so the strategies are optimal and the value is $\frac{25}{6}$

E1

Answer: Value $\frac{25}{6}$

8 marks

24 (9 marks)

Row minima $-2, 0$, so maximin = 0; column maxima 1, 1, so minimax = 1; no saddle point

M1

A plays A_1 with probability p and A_2 with probability $1 - p$. Expected payoff against B_1 : $1 - 3p$; against B_2 : p

M1

$$\text{Equating: } 1 - 3p = p, \text{ so } p = \frac{1}{4}$$

M1 A1

$$\text{Value} = \frac{1}{4}$$

A1

B plays B_1 with probability q and B_2 with probability $1 - q$. Expected loss against A_1 : $1 - 3q$; against A_2 : q

M1

$$\text{Equating: } 1 - 3q = q, \text{ so } q = \frac{1}{4}$$

M1 A1

$$\text{Expected profit over 100 games: } 100 \times \frac{1}{4} = \text{£}25$$

A1

Answer: A bluffs with probability $\frac{1}{4}$; B calls with probability $\frac{1}{4}$; value $\frac{1}{4}$; profit £25

9 marks

25 (8 marks)

Row minima: 2 and 1, so maximin = 2; column maxima: k and 3, so minimax = $\min(k, 3) > 2$ M1 A1

Against B_1 : $1 + (k - 1)p$; against B_2 : $3 - p$ M1

$1 + (k - 1)p = 3 - p$, so $kp = 2$, $p = 2/k$ M1 A1

Value = $3 - 2/k$ A1

$3 - 2/k = \frac{5}{2}$, so $2/k = \frac{1}{2}$ and $k = 4$ M1 A1

Answer: $p = 2/k$; value $3 - 2/k$; $k = 4$ 8 marks

26 (9 marks)

Row 3 (2, 1, 3) is dominated by row 2 (3, 5, 4); delete it M1 A1

Column 1 (6, 3) $\leq$ column 3 (7, 4) for B; delete column 3 M1 A1

Reduced matrix $\begin{pmatrix} 6 & 2 \\ 3 & 5 \end{pmatrix}$, no saddle point A1

A: $3 + 3p = 5 - 3p$, so $p = \frac{1}{3}$; value $3 + 1 = 4$ M1 A1

B: $2 + 4q = 5 - 2q$, so $q = \frac{1}{2}$ M1 A1

Answer: A: $(\frac{1}{3}, \frac{2}{3}, 0)$; B: $(\frac{1}{2}, \frac{1}{2}, 0)$; value 4 9 marks

27 (11 marks)

A's expected payoffs against B_1, B_2, B_3 are all equal to the value v: $6p_1 + 4p_2 + 5p_3 = 5p_1 + 6p_2 + 4p_3 = p_1 + 5p_2 + 6p_3$ M1 A1

First two: $p_1 - 2p_2 + p_3 = 0$; with $p_1 + p_2 + p_3 = 1$ this gives $p_2 = \frac{1}{3}$ M1 A1

Last two: $4p_1 + p_2 - 2p_3 = 0$, so $4p_1 - 2(\frac{2}{3} - p_1) = -\frac{1}{3}$, giving $p_1 = \frac{1}{6}$ and $p_3 = \frac{1}{2}$ M1 A1

Value = $6(\frac{1}{6}) + 4(\frac{1}{3}) + 5(\frac{1}{2}) = \frac{29}{6}$ A1

For B (q_1, q_2, q_3): $6q_1 + 5q_2 + q_3 = 4q_1 + 6q_2 + 5q_3 = 5q_1 + 4q_2 + 6q_3$ M1

Differences: $2q_1 - q_2 - 4q_3 = 0$ and $-q_1 + 2q_2 - q_3 = 0$, so $q_2 = \frac{1}{3}$, $q_1 + q_3 = \frac{2}{3}$ M1 A1

$q_1 = \frac{1}{2}$, $q_3 = \frac{1}{6}$ A1

Answer: A: $(\frac{1}{6}, \frac{1}{3}, \frac{1}{2})$; B: $(\frac{1}{2}, \frac{1}{3}, \frac{1}{6})$; value $\frac{29}{6}$ 11 marks

28 (10 marks)

Expected payoff against B_1 : $ap + c(1 - p)$; against B_2 : $bp + d(1 - p)$ M1

Equal at the optimum: $ap + c - cp = bp + d - dp$ M1

$p(a - c - b + d) = d - c$, so $p = (d - c) / (a - b - c + d)$ M1 A1

Value = $c + (a - c)p = [c(a - b - c + d) + (a - c)(d - c)] / (a - b - c + d)$ M1

Numerator = $ca - cb - c^2 + cd + ad - ac - cd + c^2 = ad - bc$ M1 A1

For $\begin{pmatrix} 8 & 3 \\ 2 & 6 \end{pmatrix}$: $a - b - c + d = 8 - 3 - 2 + 6 = 9$ M1

$p = (6 - 2)/9 = \frac{4}{9}$ A1

Value = $(48 - 6)/9 = \frac{14}{3}$ A1

Answer: $p = \frac{4}{9}$; value $\frac{14}{3}$ 10 marks

29 (11 marks)Against B_1 : $-2 + (k + 2)p$; against B_2 : $4 - 7p$

M1 A1

Equating: $(k + 9)p = 6$, so $p = 6/(k + 9)$

M1 A1

Value = $4 - 42/(k + 9) = (4k - 6)/(k + 9)$

M1 A1

Fair when $4k - 6 = 0$, so $k = \frac{3}{2}$

A1

Row minima $-3, -2$, so maximin = -2 ; column maxima $\frac{3}{2}, 4$, so minimax = $\frac{3}{2}$; these differ, so no saddle point

M1

 $p = 6/\frac{21}{2} = \frac{4}{7}$, so A plays $(\frac{4}{7}, \frac{3}{7})$

A1

B plays B_1 with probability q : against A_1 : $\frac{9}{2}q - 3$, against A_2 : $4 - 6q$; equating gives $\frac{21}{2}q = 7$, so $q = \frac{2}{3}$

M1 A1

Answer: $k = \frac{3}{2}$; **A:** $(\frac{4}{7}, \frac{3}{7})$; **B:** $(\frac{2}{3}, \frac{1}{3})$

11 marks

30 (12 marks)The 2×2 game is $\begin{pmatrix} 3 & 6 \\ 4 & 1 \end{pmatrix}$, with no saddle point (maximin 3, minimax 4)

M1 A1

A plays A_1 with probability p : $3p + 4(1 - p) = 6p + (1 - p)$, so $4 - p = 1 + 5p$ and $p = \frac{1}{2}$

M1 A1

Value = $\frac{7}{2}$

A1

B plays B_1 with probability q : $3q + 6(1 - q) = 4q + (1 - q)$, so $6 - 3q = 1 + 3q$ and $q = \frac{5}{6}$

M1 A1

A's expected payoff against B_3 is $\frac{1}{2}(2) + \frac{1}{2}(7) = \frac{9}{2}$, which is at least $\frac{7}{2}$

M1 A1

B's expected loss against A_2 is $0 \times \frac{5}{6} + 5 \times \frac{1}{6} = \frac{5}{6}$, which is at most $\frac{7}{2}$

M1 A1

So both strategies are optimal in the full game and the value is $\frac{7}{2}$

E1

Answer: **A:** $(\frac{1}{2}, 0, \frac{1}{2})$; **B:** $(\frac{5}{6}, \frac{1}{6}, 0)$; **value** $\frac{7}{2}$

12 marks

Game Theory and Linear Programming**1 (2 marks)**It makes the value of the game positive, so that quantities such as q_j/V are defined and the variables are non-negative

E1

The value of the original game is recovered by subtracting the constant at the end

E1

Answer: So that the value is positive; subtract the constant afterwards

2 marks

2 (2 marks)The smallest entry is -2 , so add 3 (giving 1)

M1

New matrix $\begin{pmatrix} 1 & 6 \\ 7 & 2 \end{pmatrix}$

A1

Answer: $\begin{pmatrix} 1 & 6 \\ 7 & 2 \end{pmatrix}$ (add 3)

2 marks

3 (1 mark) $5 - 3 = 2$

B1

Answer: 2

1 mark

4 (2 marks)

$$6q_1 + 2q_2 \leq V$$

B1

$$q_1 + 7q_2 \leq V$$

B1

Answer: $6q_1 + 2q_2 \leq V$ and $q_1 + 7q_2 \leq V$

2 marks

5 (2 marks)

$$y_1 + y_2 = (q_1 + q_2)/V$$

M1

$$q_1 + q_2 = 1, \text{ so } y_1 + y_2 = 1/V$$

A1

Answer: $y_1 + y_2 = 1/V$

2 marks

6 (1 mark)

$$6y_1 + 2y_2 + r = 1, \text{ with } r \geq 0$$

B1

Answer: $6y_1 + 2y_2 + r = 1$

1 mark

7 (4 marks)

New matrix $\begin{pmatrix} 6 & 3 \\ 1 & 8 \end{pmatrix}$

B1

$$\text{Maximise } Z = y_1 + y_2$$

B1

$$\text{Subject to } 6y_1 + 3y_2 \leq 1, y_1 + 8y_2 \leq 1, y_1, y_2 \geq 0$$

B1 B1

Answer: Maximise $Z = y_1 + y_2$ subject to $6y_1 + 3y_2 \leq 1, y_1 + 8y_2 \leq 1, y_1, y_2 \geq 0$

4 marks

8 (3 marks)

$$\text{Maximise } Z = y_1 + y_2 + y_3$$

B1

$$\text{Subject to } 3y_1 + 5y_2 + 2y_3 \leq 1$$

B1

$$4y_1 + y_2 + 6y_3 \leq 1 \text{ and } y_1, y_2, y_3 \geq 0$$

B1

Answer: Maximise $Z = y_1 + y_2 + y_3$ subject to $3y_1 + 5y_2 + 2y_3 \leq 1, 4y_1 + y_2 + 6y_3 \leq 1$

3 marks

9 (4 marks)

$$\text{Maximise } Z = y_1 + y_2$$

B1

$$\text{Subject to } 2y_1 + 4y_2 \leq 1, 5y_1 + y_2 \leq 1$$

B1 B1

$$3y_1 + 3y_2 \leq 1 \text{ and } y_1, y_2 \geq 0$$

B1

Answer: Maximise $Z = y_1 + y_2$ subject to $2y_1 + 4y_2 \leq 1, 5y_1 + y_2 \leq 1, 3y_1 + 3y_2 \leq 1, y_1, y_2 \geq 0$

4 marks

10 (3 marks)

$$6y_1 + 2y_2 + r = 1, y_1 + 7y_2 + s = 1$$

B1

$$\text{Constraint rows } (6, 2, 1, 0 \mid 1) \text{ and } (1, 7, 0, 1 \mid 1)$$

B1

$$\text{Z row } (-1, -1, 0, 0 \mid 0)$$

B1

Answer: Constraint rows $(6, 2, 1, 0 \mid 1)$ and $(1, 7, 0, 1 \mid 1)$; Z row $(-1, -1, 0, 0 \mid 0)$

3 marks

b.v.	y_1	y_2	r	s	Value
r	6	2	1	0	1
s	1	7	0	1	1
Z	-1	-1	0	0	0

11 (4 marks)

$$Z = \frac{1}{4}, y_1 = \frac{1}{8}, y_2 = \frac{1}{8}$$

B1

$$V = 1/Z = 4$$

M1

$$q_1 = y_1/Z = \frac{1}{2}, q_2 = \frac{1}{2}$$

M1 A1

Answer: $V = 4$; B: $(\frac{1}{2}, \frac{1}{2})$

4 marks

12 (4 marks)

The Z row entries under r and s are $\frac{3}{20}$ and $\frac{1}{10}$

B1

$$p_1 = \frac{3}{20} \div \frac{1}{4} = \frac{3}{5}$$

M1 A1

$$p_2 = \frac{1}{10} \div \frac{1}{4} = \frac{2}{5}$$

A1

Answer: A: $(\frac{3}{5}, \frac{2}{5})$

4 marks**13 (2 marks)**

Value of the new game = $1/Z = 4$

M1

Value of the original game = $4 - 3 = 1$

A1

Answer: 1

2 marks**14 (5 marks)**

Iteration 1: most negative entry in the Z row is -1 , so y_1 enters; ratios $\frac{1}{6}$, 1, least is $\frac{1}{6}$, so r leaves and the pivot is 6

M1 A1

Row operations: $R_1 \div 6 \rightarrow R_1$; $R_2 - R_1 \rightarrow R_2$; $Z + R_1 \rightarrow Z$ (new tableau in the diagram)

M1 A1

After the first iteration $Z = \frac{1}{6}$

A1

Answer: $Z = \frac{1}{6}$ after one iteration

5 marks

b.v.	y_1	y_2	r	s	Value
y_1	1	$\frac{1}{3}$	$\frac{1}{6}$	0	$\frac{1}{6}$
s	0	$\frac{20}{3}$	$-\frac{1}{6}$	1	$\frac{5}{6}$
Z	0	$-\frac{2}{3}$	$\frac{1}{6}$	0	$\frac{1}{6}$

15 (7 marks)

Iteration 2: most negative entry in the Z row is $-\frac{2}{3}$, so y_2 enters; ratios $\frac{1}{2}$, $\frac{1}{8}$, least is $\frac{1}{8}$, so s leaves and the pivot is $\frac{20}{3}$

M1 A1

Row operations: $R_2 \div \frac{20}{3} \rightarrow R_2$; $R_1 - \frac{1}{3} R_2 \rightarrow R_1$; $Z + \frac{2}{3} R_2 \rightarrow Z$ (new tableau in the diagram)

M1 A1

Optimal tableau reached: $Z = \frac{1}{4}$, $y_1 = y_2 = \frac{1}{8}$

B1

$V = 1/Z = 4$ and $q_j = y_j/Z = \frac{1}{2}, \frac{1}{2}$

M1 A1

Answer: $V = 4$; B: $(\frac{1}{2}, \frac{1}{2})$

7 marks

b.v.	y_1	y_2	r	s	Value
y_1	1	0	$\frac{7}{40}$	$-\frac{1}{20}$	$\frac{1}{8}$
y_2	0	1	$-\frac{1}{40}$	$\frac{3}{20}$	$\frac{1}{8}$
Z	0	0	$\frac{3}{20}$	$\frac{1}{10}$	$\frac{1}{4}$

16 (7 marks)

The smallest payoff is -2 , so add 3: $\begin{pmatrix} 6 & 2 \\ 1 & 7 \end{pmatrix}$

M1 A1

Maximise $Z = y_1 + y_2$ subject to $6y_1 + 2y_2 \leq 1$, $y_1 + 7y_2 \leq 1$, $y_1, y_2 \geq 0$

M1 A1

$1/Z = 4$, so the value of the modified game is 4 and the value of the original game is $4 - 3 = 1$

M1 A1

With best play company A gains 1% market share on average

E1

Answer: Value 1: A gains 1% share

7 marks

17 (4 marks)

$1/Z = 9$ is the value of the game after adding 5

M1

The constant must be subtracted: $9 - 5 = 4$

M1 A1

The student gave the value of the modified game, not the original

E1

Answer: Value 4

4 marks

18 (5 marks)

Add 3 (any larger integer is also acceptable): $\begin{pmatrix} 7 & 1 & 4 \\ 3 & 6 & 2 \\ 5 & 8 & 9 \end{pmatrix}$

M1 A1

Maximise $Z = y_1 + y_2 + y_3$

B1

Subject to $7y_1 + y_2 + 4y_3 \leq 1$, $3y_1 + 6y_2 + 2y_3 \leq 1$

B1

$5y_1 + 8y_2 + 9y_3 \leq 1$ and $y_1, y_2, y_3 \geq 0$

B1

Answer: Add 3; maximise $Z = y_1 + y_2 + y_3$ with constraint rows $\begin{pmatrix} 7 & 1 & 4 \\ 3 & 6 & 2 \\ 5 & 8 & 9 \end{pmatrix} \leq 1$

5 marks

19 (6 marks)

$Z = \frac{5}{17}$, so $V = 1/Z = \frac{17}{5}$

M1 A1

$q = y/Z$: $y_1 = \frac{4}{17}$, $y_2 = \frac{1}{17}$, $y_3 = 0$ give $q = (\frac{4}{5}, \frac{1}{5}, 0)$

M1 A1

Z row under r and s: $\frac{3}{17}, \frac{2}{17}$; $p = (\frac{3}{17}, \frac{2}{17}) \div \frac{5}{17} = (\frac{3}{5}, \frac{2}{5})$

M1 A1

Answer: $V = \frac{17}{5}$; A: $(\frac{3}{5}, \frac{2}{5})$; B: $(\frac{4}{5}, \frac{1}{5}, 0)$

6 marks

20 (4 marks)

From the tableau $p_1 = \frac{3}{20} \div \frac{1}{4} = \frac{3}{5}$

M1 A1

Formula: $p = (4 - (-2)) / (3 - (-1) - (-2) + 4) = 6/10 = \frac{3}{5}$

M1

The two results agree

A1

Answer: $\frac{3}{5}$ by both methods

4 marks

21 (9 marks)

Iteration 1: most negative entry in the Z row is -1 , so y_1 enters; ratios $\frac{1}{3}, \frac{1}{4}$, least is $\frac{1}{4}$, so s

M1 A1

leaves and the pivot is 4

Row operations: $R_2 \div 4 \rightarrow R_2$; $R_1 - 3R_2 \rightarrow R_1$; $Z + R_2 \rightarrow Z$ (new tableau in the diagram)

M1

Iteration 2: most negative entry in the Z row is $-\frac{3}{4}$, so y_2 enters; ratios $\frac{1}{17}, 1$, least is $\frac{1}{17}$, so r

M1 A1

leaves and the pivot is $\frac{17}{4}$

Row operations: $R_1 \div \frac{17}{4} \rightarrow R_1$; $R_2 - \frac{1}{4}R_1 \rightarrow R_2$; $Z + \frac{3}{4}R_1 \rightarrow Z$ (new tableau in the diagram)

M1

$Z = \frac{5}{17}$, $y = (\frac{4}{17}, \frac{1}{17}, 0)$, so $V = \frac{17}{5}$ and $q = (\frac{4}{5}, \frac{1}{5}, 0)$

M1 A1 A1

Answer: $V = \frac{17}{5}$; B: $(\frac{4}{5}, \frac{1}{5}, 0)$

9 marks

b.v.	y_1	y_2	y_3	r	s	Value
r	3	5	2	1	0	1
s	4	1	6	0	1	1
Z	-1	-1	-1	0	0	0

b.v.	y_1	y_2	y_3	r	s	Value
r	0	$\frac{17}{4}$	$-\frac{5}{2}$	1	$-\frac{3}{4}$	$\frac{1}{4}$
y_1	1	$\frac{1}{4}$	$\frac{3}{2}$	0	$\frac{1}{4}$	$\frac{1}{4}$
Z	0	$-\frac{3}{4}$	$\frac{1}{2}$	0	$\frac{1}{4}$	$\frac{1}{4}$

b.v.	y_1	y_2	y_3	r	s	Value
y_2	0	1	$-\frac{10}{17}$	$\frac{4}{17}$	$-\frac{3}{17}$	$\frac{1}{17}$
y_1	1	0	$\frac{28}{17}$	$-\frac{1}{17}$	$\frac{5}{17}$	$\frac{4}{17}$
Z	0	0	$\frac{1}{17}$	$\frac{3}{17}$	$\frac{2}{17}$	$\frac{5}{17}$

22 (8 marks)

Iteration 1: most negative entry in the Z row is -1 , so y_1 enters; ratios $\frac{1}{3}$, $\frac{1}{2}$, least is $\frac{1}{3}$, so r

M1 A1

leaves and the pivot is 3

Row operations: $R_1 \div 3 \rightarrow R_1$; $R_2 - 2R_1 \rightarrow R_2$; $Z + R_1 \rightarrow Z$ (new tableau in the diagram)

M1 A1

$Z = \frac{1}{3}$, $y_1 = \frac{1}{3}$, $y_2 = 0$, so $V = 3$ and $q = (1, 0)$: B plays B_1 only

M1 A1

Z row under r and s : $\frac{1}{3}$ and 0, so $p = (1, 0)$: A plays A_1 only

A1

Row minima 3, 2 and column maxima 3, 5 give a saddle point (A_1, B_1) with value 3, the same

E1

Answer: $V = 3$; A_1 and B_1 ; saddle point (A_1, B_1)

8 marks

b.v.	y_1	y_2	r	s	Value
r	3	5	1	0	1
s	2	4	0	1	1
Z	-1	-1	0	0	0

b.v.	y_1	y_2	r	s	Value
y_1	1	$\frac{5}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$
s	0	$\frac{2}{3}$	$-\frac{2}{3}$	1	$\frac{1}{3}$
Z	0	$\frac{2}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$

23 (7 marks)

$Z = \frac{2}{7}$, so the value of the modified game is $\frac{7}{2}$

M1

Value of the original game = $\frac{7}{2} - 4 = -\frac{1}{2}$

A1

$y = (\frac{1}{7}, \frac{1}{7})$, so $q = y/Z = (\frac{1}{2}, \frac{1}{2})$

M1 A1

Z row under r and s : $\frac{5}{28}$, $\frac{3}{28}$, so $p = (\frac{5}{8}, \frac{3}{8})$

M1 A1

The value is negative, so the game favours B

E1

Answer: $V = -\frac{1}{2}$; A: $(\frac{5}{8}, \frac{3}{8})$; B: $(\frac{1}{2}, \frac{1}{2})$; favours B

7 marks

24 (6 marks)

A's expected payoff against B_1 is $6p_1 + p_2$ and against B_2 is $2p_1 + 7p_2$

M1

Each must be at least V , so $V \leq 6p_1 + p_2$ and $V \leq 2p_1 + 7p_2$, and A wants V as large as possible

M1 A1

The probabilities satisfy $p_1 + p_2 = 1$; with all entries positive A gains nothing from $p_1 + p_2 < 1$, so ≤ 1 can be used

E1

Slack form: $V - 6p_1 - p_2 + r = 0$, $V - 2p_1 - 7p_2 + s = 0$, $p_1 + p_2 + t = 1$

M1

Rows $(1, -6, -1, 1, 0, 0 | 0)$, $(1, -2, -7, 0, 1, 0 | 0)$, $(0, 1, 1, 0, 0, 1 | 1)$ and P row $(-1, 0, 0, 0, 0, 0 | 0)$

A1

Answer: $V - 6p_1 - p_2 + r = 0$, $V - 2p_1 - 7p_2 + s = 0$, $p_1 + p_2 + t = 1$

6 marks

b.v.	V	p_1	p_2	r	s	t	Value
r	1	-6	-1	1	0	0	0
s	1	-2	-7	0	1	0	0
t	0	1	1	0	0	1	1
P	-1	0	0	0	0	0	0

25 (7 marks)

$Z = \frac{1}{4}$, so the value of the modified game is 4 and of the original game is $4 - 4 = 0$

M1 A1

$y = (\frac{1}{8}, \frac{1}{24}, \frac{1}{12})$, so $q = y/Z = (\frac{1}{2}, \frac{1}{6}, \frac{1}{3})$

M1 A1

Z row under r, s, t: $\frac{1}{8}, \frac{1}{24}, \frac{1}{12}$, so $p = (\frac{1}{2}, \frac{1}{6}, \frac{1}{3})$

M1 A1

The value is 0, so the game is fair

E1

Answer: $V = 0$; both players: $(\frac{1}{2}, \frac{1}{6}, \frac{1}{3})$; fair

7 marks

26 (7 marks)

$Z = \frac{3}{17}$, so the modified value is $\frac{17}{3}$ and the original value is $\frac{17}{3} - 3 = \frac{8}{3}$

M1 A1

$y = (\frac{7}{51}, \frac{2}{51}, 0)$, so $q = y/Z = (\frac{7}{9}, \frac{2}{9}, 0)$

M1 A1

Z row under r, s, t: $\frac{1}{17}, 0, \frac{2}{17}$, so $p = (\frac{1}{3}, 0, \frac{2}{3})$

M1 A1

A never plays A_2

B1

Answer: $V = \frac{8}{3}$; A: $(\frac{1}{3}, 0, \frac{2}{3})$; B: $(\frac{7}{9}, \frac{2}{9}, 0)$; A_2 never used

7 marks

27 (11 marks)

Iteration 1: most negative entry in the Z row is -1 , so y_1 enters; ratios $\frac{1}{6}$, $\frac{1}{4}$, $\frac{1}{5}$, least is $\frac{1}{6}$, so r leaves and the pivot is 6

M1 A1

Row operations: $R_1 \div 6 \rightarrow R_1$; $R_2 - 4 R_1 \rightarrow R_2$; $R_3 - 5 R_1 \rightarrow R_3$; $Z + R_1 \rightarrow Z$ (new tableau in the diagram)

A1

Iteration 2: most negative entry in the Z row is $-\frac{5}{6}$, so y_3 enters; ratios 1 , $\frac{1}{13}$, $\frac{1}{31}$, least is $\frac{1}{31}$, so t

M1 A1

leaves and the pivot is $\frac{31}{6}$

Row operations: $R_3 \div \frac{31}{6} \rightarrow R_3$; $R_1 - \frac{1}{6} R_3 \rightarrow R_1$; $R_2 - \frac{13}{3} R_3 \rightarrow R_2$; $Z + \frac{5}{6} R_3 \rightarrow Z$ (new tableau in the diagram)

A1

Iteration 3: most negative entry in the Z row is $-\frac{6}{31}$, so y_2 enters; ratios $\frac{5}{26}$, $\frac{2}{29}$, $-\frac{1}{31}$, least is $\frac{2}{29}$, so

M1 A1

s leaves and the pivot is $\frac{87}{31}$

Row operations: $R_2 \div \frac{87}{31} \rightarrow R_2$; $R_1 - \frac{26}{31} R_2 \rightarrow R_1$; $R_3 + \frac{1}{31} R_2 \rightarrow R_3$; $Z + \frac{6}{31} R_2 \rightarrow Z$ (new tableau in the diagram)

A1

$Z = \frac{6}{29}$, $y = (\frac{3}{29}, \frac{2}{29}, \frac{1}{29})$; $V = 1/Z = \frac{29}{6}$

A1

$q = y/Z = (\frac{1}{2}, \frac{1}{3}, \frac{1}{6})$

A1

Answer: $V = \frac{29}{6}$; $B: (\frac{1}{2}, \frac{1}{3}, \frac{1}{6})$

11 marks

b.v.	y_1	y_2	y_3	r	s	t	Value
r	6	5	1	1	0	0	1
s	4	6	5	0	1	0	1
t	5	4	6	0	0	1	1
Z	-1	-1	-1	0	0	0	0

b.v.	y_1	y_2	y_3	r	s	t	Value
y_1	1	$\frac{5}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	0	0	$\frac{1}{6}$
s	0	$\frac{8}{3}$	$\frac{13}{3}$	$-\frac{2}{3}$	1	0	$\frac{1}{3}$
t	0	$-\frac{1}{6}$	$\frac{31}{6}$	$-\frac{5}{6}$	0	1	$\frac{1}{6}$
Z	0	$-\frac{1}{6}$	$-\frac{5}{6}$	$\frac{1}{6}$	0	0	$\frac{1}{6}$

b.v.	y_1	y_2	y_3	r	s	t	Value
y_1	1	$\frac{26}{31}$	0	$\frac{6}{31}$	0	$-\frac{1}{31}$	$\frac{5}{31}$
s	0	$\frac{87}{31}$	0	$\frac{1}{31}$	1	$-\frac{26}{31}$	$\frac{6}{31}$
y_3	0	$-\frac{1}{31}$	1	$-\frac{5}{31}$	0	$\frac{6}{31}$	$\frac{1}{31}$
Z	0	$-\frac{6}{31}$	0	$\frac{1}{31}$	0	$\frac{5}{31}$	$\frac{6}{31}$

b.v.	y_1	y_2	y_3	r	s	t	Value
y_1	1	0	0	$\frac{16}{87}$	$-\frac{26}{87}$	$\frac{19}{87}$	$\frac{3}{29}$
y_2	0	1	0	$\frac{1}{87}$	$\frac{31}{87}$	$-\frac{26}{87}$	$\frac{2}{29}$
y_3	0	0	1	$-\frac{14}{87}$	$\frac{1}{87}$	$\frac{16}{87}$	$\frac{1}{29}$
Z	0	0	0	$\frac{1}{29}$	$\frac{2}{29}$	$\frac{3}{29}$	$\frac{6}{29}$

28 (9 marks)

B holds A's expected payoff to at most V whichever row A plays: $b_{i1}q_1 + \dots + b_{in}q_n \leq V$ for every i

M1 A1

Divide by $V > 0$: $b_{i1}(q_1/V) + \dots + b_{in}(q_n/V) \leq 1$, that is $b_{i1}y_1 + \dots + b_{in}y_n \leq 1$

M1 A1

B wants V as small as possible

B1

$q_1 + \dots + q_n = 1$ gives $y_1 + \dots + y_n = 1/V$

M1 A1

Minimising V is the same as maximising $1/V = Z$

A1

So $V = 1/Z$ and $q_j = y_j V = y_j/Z$, and $y_j \geq 0$ because $q_j \geq 0$

E1

Answer: $V = 1/Z$; $q_j = y_j/Z$

9 marks

29 (12 marks)

The constraint coefficients in the tableau are 7, 1, 4 / 3, 6, 2 / 5, 8, 9, each equal to the original entry plus 3, so the constant added was 3

M1 A1

Iteration 1: most negative entry in the Z row is -1 , so y_1 enters; ratios $\frac{1}{7}, \frac{1}{3}, \frac{1}{5}$, least is $\frac{1}{7}$, so r leaves and the pivot is 7

M1 A1

Row operations: $R_1 \div 7 \rightarrow R_1$; $R_2 - 3 R_1 \rightarrow R_2$; $R_3 - 5 R_1 \rightarrow R_3$; $Z + R_1 \rightarrow Z$ (new tableau in the diagram)

A1

Iteration 2: most negative entry in the Z row is $-\frac{6}{7}$, so y_2 enters; ratios $1, \frac{4}{39}, \frac{2}{51}$, least is $\frac{2}{51}$, so t leaves and the pivot is $\frac{51}{7}$

M1 A1

Row operations: $R_3 \div \frac{51}{7} \rightarrow R_3$; $R_1 - \frac{1}{7} R_3 \rightarrow R_1$; $R_2 - \frac{39}{7} R_3 \rightarrow R_2$; $Z + \frac{6}{7} R_3 \rightarrow Z$ (new tableau in the diagram)

A1

$Z = \frac{3}{17}$, so $V = \frac{17}{3} - 3 = \frac{8}{3}$

M1 A1

$q = (\frac{7}{9}, \frac{2}{9}, 0)$ and $p = (\frac{1}{3}, 0, \frac{2}{3})$

A1 A1

Answer: $V = \frac{8}{3}$; A: $(\frac{1}{3}, 0, \frac{2}{3})$; B: $(\frac{7}{9}, \frac{2}{9}, 0)$

12 marks

b.v.	y_1	y_2	y_3	r	s	t	Value
r	7	1	4	1	0	0	1
s	3	6	2	0	1	0	1
t	5	8	9	0	0	1	1
Z	-1	-1	-1	0	0	0	0

b.v.	y_1	y_2	y_3	r	s	t	Value
y_1	1	$\frac{1}{7}$	$\frac{4}{7}$	$\frac{1}{7}$	0	0	$\frac{1}{7}$
s	0	$\frac{39}{7}$	$\frac{2}{7}$	$-\frac{3}{7}$	1	0	$\frac{4}{7}$
t	0	$\frac{51}{7}$	$\frac{43}{7}$	$-\frac{5}{7}$	0	1	$\frac{2}{7}$
Z	0	$-\frac{6}{7}$	$-\frac{3}{7}$	$\frac{1}{7}$	0	0	$\frac{1}{7}$

b.v.	y_1	y_2	y_3	r	s	t	Value
y_1	1	0	$\frac{23}{51}$	$\frac{8}{51}$	0	$-\frac{1}{51}$	$\frac{7}{51}$
s	0	0	$-\frac{75}{17}$	$\frac{2}{17}$	1	$-\frac{13}{17}$	$\frac{6}{17}$
y_2	0	1	$\frac{43}{51}$	$-\frac{5}{51}$	0	$\frac{7}{51}$	$\frac{2}{51}$
Z	0	0	$\frac{5}{17}$	$\frac{1}{17}$	0	$\frac{2}{17}$	$\frac{3}{17}$

30 (8 marks)

$$\text{Sum of } y \text{ values} = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

M1

$$\text{Sum of the slack entries} = \frac{3}{20} + \frac{1}{10} = \frac{1}{4}, \text{ the same as } Z$$

M1 A1

At the optimum the value of the problem for B equals that of the dual problem for A, so both sums equal Z

E1

$$q_j = y_j/Z, \text{ so the } q \text{ values sum to } (\text{sum of } y)/Z = 1$$

M1 A1

$$p_i = x_i/Z, \text{ so the } p \text{ values sum to } (\text{sum of } x)/Z = 1$$

M1

$$q = \left(\frac{1}{2}, \frac{1}{2}\right) \text{ and } p = \left(\frac{3}{5}, \frac{2}{5}\right)$$

A1

$$\text{Answer: Both sums are } \frac{1}{4}; q = \left(\frac{1}{2}, \frac{1}{2}\right); p = \left(\frac{3}{5}, \frac{2}{5}\right)$$

8 marks