

Two-Person Zero-Sum Games

1 Explain what is meant by a zero-sum game. [1]

2 The table shows the payoffs to player A in a zero-sum game. Write down the payoff to A when A plays A_2 and B plays B_3 .

	B1	B2	B3
A1	3	-2	4
A2	-1	5	0

[1]

3 In a zero-sum game, player A's payoff is +4 when A plays A_1 and B plays B_3 . Write down B's payoff. [1]

4 In a payoff table for player A, state what a negative entry means. [1]

5 A game is represented by a 3×4 payoff table, with the rows for player A. State the number of strategies available to each player. [2]

6 The table shows A's payoffs. Player B is known to play B_2 . State which strategy A should play, and the payoff.

	B1	B2	B3
A1	3	-2	4
A2	-1	5	0

[2]

7 The table shows the payoffs to A in a zero-sum game. Find the row minimum for each row and hence A's maximin value, stating A's play-safe strategy.

	B1	B2	B3
A1	2	1	4
A2	3	0	-1
A3	1	-2	2

[3]

8 For the game with payoff table shown (payoffs to A), find the column maxima and hence B's minimax value, stating B's play-safe strategy.

	B1	B2	B3
A1	4	-1	3
A2	2	5	-2
A3	1	0	6

[3]

- 9 The game with the payoff table shown (payoffs to A) is played. State the play-safe strategy for each player, and explain what 'play-safe' means for A.

	B1	B2
A1	5	2
A2	-1	3
A3	4	-3

[4]

- 10 Show that the zero-sum game with the payoff table shown (payoffs to A) is stable, and state the value of the game.

	B1	B2	B3
A1	4	3	5
A2	6	2	1

[3]

- 11 The payoffs to A in a zero-sum game are given by the matrix $\begin{pmatrix} 3 & -1 \\ -2 & 4 \end{pmatrix}$. Show that the game is not stable.

[3]

- 12 Two players each show either one or two fingers at the same moment. If the number of fingers is the same, A wins 2 points from B; otherwise B wins 2 points from A. Write down the payoff matrix for A and show that the game has no saddle point. Each player now chooses one or two fingers with probability $\frac{1}{2}$. Find A's expected payoff per game and state whether the game is fair.

[3]

- 13 Two shops, X and Y, each decide whether to advertise on radio (R) or on the internet (I). The table shows the percentage change in X's market share (a gain for X is a loss for Y). Write down the play-safe strategy for each shop.

	Y: R	Y: I
X: R	6	3
X: I	1	2

[3]

- 14 Two players each show either 1 or 2 fingers. If the total number of fingers is even, A wins that number of points from B; if odd, B wins that number of points from A. Show that the game is zero-sum, write down the payoff matrix for A, and show that the game has no stable solution.

[5]

- 15 Player B plays B_1 , B_2 and B_3 with probabilities $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{4}$ respectively. The payoff matrix for A is $\begin{pmatrix} 2 & 3 & 1 \\ 4 & -1 & 2 \\ 0 & 2 & -1 \end{pmatrix}$.

Find A's expected payoff for each of A's three strategies, and state A's best reply.

[5]

- 16 The payoff table for A in a zero-sum game is $\begin{pmatrix} 4 & -1 & 2 \\ 3 & 0 & -2 \end{pmatrix}$.

Write down the payoff matrix for B (the payoffs to B), with B's strategies as the rows.

[3]

- 17** A student says: 'In a zero-sum game, A's maximin is the biggest number in the payoff table.'
Explain the error and give a counterexample using $\begin{pmatrix} 1 & 5 \\ 2 & 3 \end{pmatrix}$. [4]

- 18** Two airlines compete on route pricing in a zero-sum game: the payoffs to A are in the matrix $\begin{pmatrix} 3 & 2 & 5 \\ 1 & 4 & 0 \\ 6 & 2 & 3 \end{pmatrix}$.
Find A's and B's play-safe strategies. State whether the game is stable and, if it is, the value. [4]

- 19** In an unstable zero-sum game player A always plays the same play-safe strategy. Explain why this is a weakness, using $\begin{pmatrix} 3 & -1 \\ -2 & 4 \end{pmatrix}$ as an example. [4]

- 20** Two supermarkets choose a pricing policy in a zero-sum game; the payoffs to A are in the matrix $\begin{pmatrix} 7 & 4 & 6 \\ 3 & 2 & 5 \\ 8 & 3 & 9 \end{pmatrix}$.
Show that the game is stable, state the optimal strategy for each supermarket, and interpret the value of the game. [6]

- 21** The payoff matrix for A in a zero-sum game is $\begin{pmatrix} 3 & 5 & 3 \\ 2 & 4 & 6 \\ 3 & 7 & 3 \end{pmatrix}$.
Find the maximin and minimax values, and show that the game has two saddle points. State where they are and the value of the game. [7]

- 22** Prove that, for any zero-sum game with payoff matrix (a_{ij}) , $\text{maximin} \leq \text{minimax}$; that is, $\max \text{ over } i \text{ of } (\min \text{ over } j \text{ of } a_{ij}) \leq \min \text{ over } j \text{ of } (\max \text{ over } i \text{ of } a_{ij})$. [7]

- 23** The payoff matrix for A in a zero-sum game is $\begin{pmatrix} 8 & 9 & 1 \\ 6 & 7 & 5 \\ 3 & 4 & 2 \end{pmatrix}$.
Show that the game has a saddle point and state its position and the value of the game. Explain, using the saddle point, why neither player gains by changing strategy alone. [7]

- 24** The payoff matrix for A is $\begin{pmatrix} 3 & -1 & 2 \\ -2 & 4 & 1 \end{pmatrix}$. A plays A_1 with probability $\frac{1}{3}$ and A_2 with probability $\frac{2}{3}$.
Find A's expected payoff against each of B's strategies, and state B's best reply. Compare A's guaranteed payoff with A's maximin value and comment. [7]

- 25** In a penalty shoot-out the striker aims left (L) or right (R) and the goalkeeper dives left or right. The table shows the percentage chance that the striker scores.
Show that subtracting 50 from every entry gives a zero-sum game for the striker, write down this matrix, and show that the game is not stable.

	Dive L	Dive R
Aim L	40	90
Aim R	80	30

[6]

- 26** A student calculates the maximin of $\begin{pmatrix} 5 & 2 \\ 3 & 4 \end{pmatrix}$ as 4, and the minimax as 3, and says: 'maximin is bigger than minimax, so A can get more than B will allow.' Explain the error and find the correct values. [6]

- 27** In a zero-sum game with payoff matrix (a_{ij}) , the entry a_{rc} is a saddle point if it is the minimum of row r and the maximum of column c .
Prove that if A plays row r and B plays column c , neither player can improve the result by changing strategy while the other keeps to the same strategy. [6]

- 28** The payoff matrix for A in a zero-sum game is $\begin{pmatrix} k & 3 \\ 2 & 4 \end{pmatrix}$, where k is a constant.
Find, in terms of k , the maximin and minimax values, and hence find the range of values of k for which the game is stable. State the value of the game in each case. [10]

- 29** Two companies, X and Y, each choose one of four advertising levels. The table shows the change in X's weekly sales (in hundreds) for each pair of choices; Y's change is the negative of this.
Find X's and Y's play-safe strategies, show that the game has a saddle point, and state the optimal strategies and the value. Interpret the value for X and Y.

	Y1	Y2	Y3	Y4
X1	3	5	2	4
X2	1	2	0	3
X3	4	1	2	6
X4	2	3	1	5

[9]

- 30** Suppose a zero-sum game has two saddle points at positions (r, c) and (s, d) in the payoff matrix (a_{ij}) .
Prove that $a_{rc} = a_{sd}$, so the value of the game is well defined, and that (r, d) and (s, c) are also saddle points. [8]

Payoff Matrices

- 1** Player A chooses A_1 or A_2 and player B chooses B_1 or B_2 . The payoffs to A are: (A_1, B_1) 3, (A_1, B_2) -2, (A_2, B_1) 0, (A_2, B_2) 4.
Write down the payoff matrix for A, with A's strategies as the rows. [1]

- 2** The table shows the payoffs to A. Write down the payoff to A when A plays A_3 and B plays B_2 .

	B1	B2	B3
A1	4	-2	1
A2	0	3	-1
A3	2	5	6

[1]

- 3** A has four strategies and B has three. State the dimensions of the payoff matrix for A, with A's strategies as the rows. [1]

- 4** Find the row minimum of each row of the payoff matrix for A: $\begin{pmatrix} 3 & -1 & 2 \\ 4 & 0 & 1 \\ -2 & 5 & 3 \end{pmatrix}$ [1]
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- 5** Find the column maximum of each column of the payoff matrix for A: $\begin{pmatrix} 3 & -1 & 2 \\ 4 & 0 & 1 \\ -2 & 5 & 3 \end{pmatrix}$ [1]
-
- 6** The payoff matrix for A is $\begin{pmatrix} 3 & -2 \\ 0 & 4 \end{pmatrix}$. Write down the matrix giving the payoffs to B, with B's strategies as the columns as before. [2]
-
- 7** Players A and B each show a coin, heads (H) or tails (T). If both show heads, A wins 3 points from B. If both show tails, A wins 1 point from B. If the coins differ, B wins 2 points from A. Write down the payoff matrix for A, with rows H, T for A and columns H, T for B. [2]
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- 8** The payoff matrix for A (rows A_1, A_2 ; columns B_1, B_2, B_3) is $\begin{pmatrix} 3 & -1 & 2 \\ 0 & 4 & -2 \end{pmatrix}$. Write down the payoff matrix for B, with B's strategies as the rows and A's as the columns. [3]
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- 9** A zero-sum game has payoff matrix $\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$ for A. Show that the game has a saddle point and find its value. Every entry is then increased by 3; write down the new matrix, and state the new saddle point and value. [4]
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- 10** A game with payoff matrix $\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$ has value 2. Every payoff is multiplied by 2 to give a new game. Write down the new matrix and the value of the new game, giving a reason. [3]
-
- 11** The payoff matrix for A is $\begin{pmatrix} 4 & -3 \\ -2 & 5 \end{pmatrix}$. A plays A_1 and A_2 with probabilities $\frac{1}{2}$ each, and B plays B_1 with probability $\frac{1}{4}$ and B_2 with probability $\frac{3}{4}$. Find A's expected payoff. [4]
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- 12** The payoff matrix for A is $\begin{pmatrix} -3 & 2 & 0 \\ 1 & -4 & 5 \end{pmatrix}$. Find the smallest positive integer k that, when added to every entry, makes all the entries of the matrix positive, and write down the new matrix. [3]
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- 13** In a payoff matrix for A showing the change in A's weekly sales (in hundreds) when A and B choose strategies, the entry in row 2, column 3 is -4 . Interpret this entry. [2]
-
- 14** Players A and B each choose rock (R), paper (P) or scissors (S). Rock beats scissors by 3 points, scissors beats paper by 2 points and paper beats rock by 1 point. A draw scores 0. Write down the payoff matrix for A (rows and columns in the order R, P, S) and show that it is the same as the payoff matrix for B. [5]
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- 15** Two companies share a market. The table shows the percentage of the market gained by company X for each pair of strategies (company Y gets the rest). Explain why subtracting 50 from every entry gives a zero-sum game, write down the zero-sum payoff matrix for X, and find the play-safe strategy for each company.

	Y1	Y2	Y3
X1	55	40	60
X2	45	65	50

[5]

- 16** The payoff matrix for A is $\begin{pmatrix} 3 & -2 \\ 1 & 4 \\ -1 & 0 \end{pmatrix}$.

Write down the payoff matrix for B, with B's strategies as rows. Find B's maximin value from this matrix and A's minimax value from the original matrix, and compare them.

[6]

- 17** The payoff matrix for A is $\begin{pmatrix} 2 & -3 & 1 \\ -1 & 4 & 0 \end{pmatrix}$. A plays A_1 with probability p and A_2 with probability $1 - p$.

Find A's expected payoff, in terms of p , against each of B's strategies. When $p = \frac{1}{3}$, find which of B's strategies is best for B.

[6]

- 18** A zero-sum game has payoff matrix $\begin{pmatrix} -2 & 3 \\ 1 & -4 \end{pmatrix}$ for A. Five is added to every entry to give the matrix $\begin{pmatrix} 3 & 8 \\ 6 & 1 \end{pmatrix}$, and the value of this new game is $\frac{9}{2}$, with A's optimal strategy $(A_1, A_2) = (\frac{1}{2}, \frac{1}{2})$.

Find the value of the original game and A's optimal strategy for it, and explain your answer.

[4]

- 19** Two games have payoff matrices for A $\begin{pmatrix} 3 & -1 \\ -2 & 4 \end{pmatrix}$ and $\begin{pmatrix} 7 & -1 \\ -3 & 9 \end{pmatrix}$.

Show that the second matrix is obtained from the first by the rule entry $\rightarrow 2 \times \text{entry} + 1$.

Given that the optimal strategy for A in the first game is $(\frac{3}{5}, \frac{2}{5})$ with value 1, find the value of the second game and A's optimal strategy for it, and justify your answers.

[6]

- 20** The payoff matrix for A is $\begin{pmatrix} 3 & -2 \\ 1 & 4 \\ -1 & 0 \end{pmatrix}$ (3 rows, 2 columns). A student writes the payoff matrix for B as $\begin{pmatrix} -3 & 2 \\ -1 & -4 \\ 1 & 0 \end{pmatrix}$.

Explain the error in the student's matrix and write down the correct matrix.

[4]

- 21** In a tennis rally the server serves to the forehand (F) or backhand (B) and the receiver moves to the forehand or backhand side. The table shows the percentage of points won by the server. Convert the game to a zero-sum game for the server, write down the matrix, and show that there is no pure-strategy solution. Interpret the entry in the first row, second column of your matrix.

	Cover F	Cover B
Serve F	40	80
Serve B	90	50

[7]

- 22** The payoff matrix for A is $\begin{pmatrix} 3 & -1 & 2 \\ 0 & 4 & -3 \\ 2 & 1 & 1 \end{pmatrix}$. B plays B_1 , B_2 and B_3 with probabilities $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{4}$.

Find A's expected payoff for each of A's strategies. Given that the value of the game is 1, show that B's mixed strategy is not optimal for B.

[6]

- 23** The payoff matrix for A is $\begin{pmatrix} 3 & a \\ 2 & 5 \end{pmatrix}$, where a is a constant.

Find the range of values of a for which the game has a saddle point, and state the value of the game when it does.

[7]

- 24** The payoff matrix for A in a zero-sum game is $\begin{pmatrix} 0 & 2 & -1 \\ -2 & 0 & 3 \\ 1 & -3 & 0 \end{pmatrix}$.

Show that B's payoff matrix is identical to A's. Show that A's strategy $(\frac{1}{2}, \frac{1}{6}, \frac{1}{3})$ gives A an expected payoff of 0 against each of B's strategies.

[7]

- 25** The table shows the payoffs to B in a zero-sum game, with B's strategies as the rows and A's strategies as the columns.

Write down the payoff matrix for A, with A's strategies as the rows. Find the play-safe strategy for each player and show that the game is not stable.

	A1	A2	A3
B1	2	-3	1
B2	0	4	-2
B3	-1	1	3

[6]

- 26** A zero-sum game has a saddle point at row r, column c of the payoff matrix (a_{ij}) for A, with value $v = a_{rc}$.

Write down B's payoff matrix in terms of a_{ij} , and prove that it has a saddle point at row c, column r with value $-v$.

[6]

- 27** A game G has payoff matrix (a_{ij}) for A. A new game G' has payoff matrix $(\lambda a_{ij} + \mu)$, where $\lambda > 0$ and μ are constants.

Prove that if A plays row i with probability p_i and B plays column j with probability q_j , the expected payoff in G' is $\lambda E + \mu$, where E is the expected payoff in G. Hence prove that the optimal strategies for G' are the same as for G, and that the value of G' is $\lambda v + \mu$.

[8]

- 28** A student says: 'Adding the same constant to every entry in one row of a payoff matrix does not change the optimal strategies.' By considering $\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$, and the matrix obtained by adding 4 to every entry of the first row, show that the student is wrong. Find the solution of each game. [8]

- 29** Two supermarkets, X and Y, each choose a pricing strategy. The table shows the percentage of customers who choose X for each pair of strategies (the others choose Y). Show that this is equivalent to a zero-sum game, find the play-safe strategies and show that there is a saddle point. State the optimal strategies and the percentage of customers X attracts if both play optimally. Explain why neither supermarket would change strategy alone. [9]

	Y1	Y2	Y3
X1	52	55	60
X2	48	53	50
X3	58	56	62

- 30** In a game, A chooses an integer i from 1 to 4 and B chooses an integer j from 1 to 4. The payoff to A is $ij - 6$. Write down the 4×4 payoff matrix. Explain, without calculating minima or maxima for every row and column, why A should choose $i = 4$ and B should choose $j = 1$, and find the value of the game. State which player the game favours. [8]

Dominance

- 1** In a zero-sum game with payoff matrix for A, state what it means for row i to dominate row k . [1]
- 2** In a zero-sum game with payoff matrix for A, state what it means for column j to dominate column k (from B's point of view). [1]
- 3** For the payoff matrix $\begin{pmatrix} 3 & 5 \\ 2 & 4 \end{pmatrix}$ for A, state which row dominates the other. [1]
- 4** For the payoff matrix $\begin{pmatrix} 3 & 5 \\ 2 & 4 \end{pmatrix}$ for A, state which column dominates the other from B's point of view. [2]
- 5** Explain why player A would never play a dominated row. [1]
- 6** The payoff matrix for A has rows 1 and 2 equal to (2, 1, 3) and (4, 1, 3). State whether one row dominates the other, and which. [1]
- 7** The payoff matrix for A is $\begin{pmatrix} 3 & 1 & 4 \\ 2 & 0 & 3 \\ 1 & 5 & 2 \end{pmatrix}$. Show that one of A's rows is dominated and write down the matrix after deleting it. [3]
- 8** The payoff matrix for A is $\begin{pmatrix} 3 & 1 & 4 \\ 1 & 5 & 2 \end{pmatrix}$ after a dominated row has been deleted from a 3×3 game, leaving columns B_1 , B_2 , B_3 . Show that B has a dominated column and write down the matrix after deleting it. [3]

- 9** The payoff matrix for A is $\begin{pmatrix} 4 & 2 & 6 \\ 3 & 1 & 5 \\ 2 & 3 & 4 \end{pmatrix}$.
Use dominance to reduce the game to a 2×2 matrix, stating clearly each row or column that you delete and why. [4]
-
- 10** The payoff matrix for A is $\begin{pmatrix} 2 & 5 & 3 \\ 1 & 4 & 2 \\ 3 & 2 & 6 \end{pmatrix}$.
Show that the game can be reduced to a 2×2 game, and write down the reduced matrix. [4]
-
- 11** The payoff matrix for A is $\begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$. Explain why neither row dominates the other, and neither column dominates the other. [2]
-
- 12** Two rows of a payoff matrix for A are (3, 5, 2) and (3, 5, 2). Explain what can be said about the rows, and whether one can be deleted. [2]
-
- 13** A company can choose strategy S_1 , S_2 or S_3 against a rival's two possible responses. The gains in profit (in £1000s) are given in the matrix $\begin{pmatrix} 4 & 6 \\ 5 & 3 \\ 4 & 2 \end{pmatrix}$ (rows S_1 , S_2 , S_3).
Show that strategy S_3 should never be chosen. [2]
-
- 14** The payoff matrix for A is $\begin{pmatrix} 3 & 2 & 4 \\ 1 & 3 & 2 \\ 2 & 1 & 5 \\ 0 & 2 & 1 \end{pmatrix}$.
Use dominance repeatedly to reduce the matrix to 2×2 , stating each deletion and the reason. [6]
-
- 15** The payoff matrix for A is $\begin{pmatrix} 5 & 2 & 3 \\ 4 & 1 & 6 \\ 2 & 0 & 2 \end{pmatrix}$. Use dominance to show that the game has a saddle point, and state the optimal strategies and the value. [6]
-
- 16** The payoff matrix for A is $\begin{pmatrix} 7 & 0 & 1 \\ 1 & 3 & 1 \\ 2 & 4 & 7 \end{pmatrix}$.
Show that there is no dominated column to begin with, but that after a dominated row is deleted one of B's columns becomes dominated. Write down the 2×2 matrix that remains. [6]
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- 17** The payoff matrix for A is $\begin{pmatrix} 4 & 2 & 6 \\ 3 & 1 & 5 \\ 2 & 3 & 4 \end{pmatrix}$. Use dominance to reduce the game to 2×2 , and then find the optimal strategy for A in the reduced game and the value of the game. [7]
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- 18** Two petrol stations, A and B, each choose one of three advertising strategies. The matrix shows the gain in customers per day for A (in hundreds), with A's strategies as rows:
 $\begin{pmatrix} 2 & 5 & 3 \\ 1 & 4 & 2 \\ 3 & 2 & 6 \end{pmatrix}$.
Use dominance to reduce the game and find A's optimal mixed strategy and the value of the game. Interpret the value. [6]

- 19** A student has the matrix $\begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$ for A and writes: 'The minimum of row 1 is 1 and the minimum of row 2 is 2, so row 2 is better and row 1 is dominated, so A should always play row 2.'
Explain the error in the student's reasoning. [4]
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- 20** The table shows the payoffs to B, with B's strategies as the rows and A's as the columns:
 $\begin{pmatrix} -3 & -1 \\ -2 & 0 \\ -5 & -4 \end{pmatrix}$.
 Find B's dominated strategies, and state B's best strategy. Hence write down A's best response and the value of the game to A. [5]
-
- 21** The payoff matrix for A is $\begin{pmatrix} 3 & 1 & 4 \\ 2 & 0 & 3 \\ 1 & 5 & 2 \end{pmatrix}$. Use dominance to reduce the game to 2×2 . Find the optimal strategy for each player in the original game and the value of the game. [8]
-
- 22** The payoff matrix for A is $\begin{pmatrix} 6 & 2 & 5 & 3 \\ 3 & 1 & 4 & 2 \\ 5 & 4 & 3 & 6 \\ 4 & 3 & 6 & 1 \end{pmatrix}$.
 Use dominance to reduce the game to 3×3 (rows 1, 3, 4 and columns 2, 3, 4 remain). It is claimed that the optimal strategies in the reduced game are: A plays rows 1, 3, 4 with probabilities $0, \frac{3}{4}, \frac{1}{4}$; B plays columns 2, 3, 4 with probabilities $\frac{3}{4}, \frac{1}{4}, 0$.
 Verify that the value of the game is $\frac{15}{4}$, by checking A's expected payoff against each column and B's expected loss against each row. [9]
-
- 23** The payoff matrix for A is $\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$ (rock, paper, scissors).
 Show that no row or column is dominated. Explain why the value of the game is 0 and give the optimal strategy for A. [7]
-
- 24** The payoff matrix for A is $\begin{pmatrix} 5 & 3 & 6 & 2 \\ 4 & 2 & 5 & 1 \\ 3 & 4 & 2 & 6 \end{pmatrix}$.
 Show that row 2 is dominated. Let A play row 1 with probability p and row 3 with probability $1 - p$, and by considering the expected payoff against each column, find the optimal value of p and the value of the game. [7]
-
- 25** Two rival shops, A and B, each choose one of three promotions. The matrix gives A's weekly gain in sales (in £100s), with A's promotions as rows: $\begin{pmatrix} 4 & 2 & 6 \\ 3 & 1 & 5 \\ 2 & 3 & 4 \end{pmatrix}$.
 Use dominance to reduce the game, and find B's optimal strategy and the value of the game. State what B should do and how much A gains per week with best play. [7]
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- 26** The payoff matrix for A is $\begin{pmatrix} 5 & 2 & 3 \\ 4 & 1 & 6 \\ 2 & 0 & 2 \end{pmatrix}$.
 Find the value of the game in two ways: first by using dominance, and then by finding maximin and minimax values. Comment on the two results. [7]

27

A zero-sum game has payoff matrix $\begin{pmatrix} 3 & 2 & 4 \\ 1 & 3 & 2 \\ 2 & 1 & 5 \\ 0 & 2 & 1 \end{pmatrix}$ for A.

Reduce the matrix by dominance in two different orders, first starting by deleting a row and then starting by deleting a column, and show that both orders lead to the same 2×2 matrix. Hence find the value of the game.

[9]

28

The payoff matrix for A is $\begin{pmatrix} 4 & 0 \\ 0 & 4 \\ 1 & 1 \end{pmatrix}$.

Show that no row is dominated by another single row, but that row 3 is dominated by a mixture of rows 1 and 2. Hence find the value of the game and the optimal strategies.

[9]

29

The payoff matrix for A is $\begin{pmatrix} 7 & 4 & 6 & 5 & 8 \\ 3 & 2 & 4 & 1 & 5 \\ 6 & 5 & 4 & 3 & 7 \\ 2 & 1 & 3 & 0 & 4 \\ 5 & 6 & 2 & 4 & 6 \end{pmatrix}$.

Use dominance repeatedly to reduce the game to 3×3 , stating every deletion in order. Given that the optimal strategy for A in the reduced game uses the remaining rows with probabilities $\frac{2}{3}$, 0, $\frac{1}{3}$, verify that A's expected payoff is at least $\frac{14}{3}$ against each of B's remaining columns.

[8]

30

The payoff matrix for A is $\begin{pmatrix} 3 & 3 \\ 1 & 3 \end{pmatrix}$.

Show that row 2 is weakly dominated by row 1. Find the maximin and minimax values, and all the saddle points of the game. State optimal strategies for both players.

[9]

Optimal Strategies

1 State what is meant by a saddle point in a zero-sum game with a payoff matrix for A. [1]

2 State what is meant by the value of a zero-sum game that has a saddle point. [1]

3 For the payoff matrix $\begin{pmatrix} 3 & 5 \\ 2 & 4 \end{pmatrix}$ for A, find the saddle point and the value of the game. [2]

4 For the payoff matrix $\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$ for A, state the maximin and minimax values and say whether there is a saddle point. [2]

5 In a zero-sum game, A's maximin value is 4 and B's minimax value is 4. State the value of the game and whether the game is stable. [2]

6 At a saddle point of a zero-sum game, explain why neither player gains by changing to a different pure strategy while the other keeps to theirs. [2]

7 The payoff matrix for A is $\begin{pmatrix} 6 & 4 & 5 \\ 3 & 2 & 7 \end{pmatrix}$. Determine whether the game has a saddle point. If it does, state it and the value of the game. [5]

- 8** The payoff matrix for A is $\begin{pmatrix} 2 & 7 \\ 4 & 5 \\ 3 & 1 \end{pmatrix}$.
Determine whether the game has a saddle point. If it does, state it and the value of the game. [5]
-
- 9** The payoff matrix for A is $\begin{pmatrix} -1 & 3 & 2 \\ -2 & 4 & -2 \\ -3 & 1 & 5 \end{pmatrix}$.
Determine whether the game has a saddle point. If it does, state it and the value of the game. [5]
-
- 10** The payoff matrix for A is $\begin{pmatrix} 5 & 2 \\ 1 & 4 \end{pmatrix}$.
Find the maximin and minimax values and state whether the game has a saddle point. [5]
-
- 11** The payoff matrix for A is $\begin{pmatrix} 5 & 3 & 5 \\ 2 & 3 & 4 \\ 5 & 6 & 5 \end{pmatrix}$. Find all the saddle points and the value of the game. [5]
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- 12** The payoff matrix for A is $\begin{pmatrix} 4 & 6 & 5 \\ 2 & 3 & 1 \\ 7 & 9 & 8 \end{pmatrix}$. Verify that (A_3, B_1) is a saddle point and state the value of the game. [3]
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- 13** Two players each choose a number from 1, 2 and 3 at the same time. If A's number a is at least B's number b , A wins $a - b$ points from B. If B's number is larger, B wins $2(b - a)$ points from A.
Write down the payoff matrix for A, and find the saddle point and the value of the game. [4]
-
- 14** The table shows the payoffs to A. Find the saddle point, the optimal strategy for each player and the value of the game.
- | | B1 | B2 | B3 |
|-----------|-----------|-----------|-----------|
| A1 | 2 | 5 | 3 |
| A2 | 4 | 6 | 6 |
| A3 | 1 | 7 | 5 |
- [5]
-
- 15** Two chess players, A and B, each choose one of three openings. The matrix gives A's score advantage (positive means A is ahead) at the end of the opening: $\begin{pmatrix} -2 & 1 & 3 \\ 0 & 2 & 1 \\ -1 & -3 & 0 \end{pmatrix}$.
Find the optimal opening for each player. Is the game fair? Give a reason. [7]
-
- 16** The table gives the payoffs to B, with B's strategies as the rows and A's strategies as the columns: $\begin{pmatrix} -4 & -2 \\ -6 & -7 \\ -5 & -3 \end{pmatrix}$.
Find the optimal strategy for each player and the value of the game to B. [6]
-

17

The payoff matrix for A is $\begin{pmatrix} 3 & 6 & 2 & 5 \\ 4 & 3 & 7 & 1 \\ 5 & 4 & 3 & 6 \\ 2 & 7 & 4 & 8 \end{pmatrix}$.

Show that the game has no saddle point. State the range within which the value of the game must lie.

[6]

18

The payoff matrix for A is $\begin{pmatrix} 2 & 5 & 3 \\ 4 & x & 6 \\ 1 & 7 & 5 \end{pmatrix}$, where x is a constant. Given that (A_2, B_1) is a saddle point, find the range of values of x and state the value of the game.

[5]

19

A student looks at the payoff matrix $\begin{pmatrix} 4 & 1 & 6 \\ 3 & 5 & 2 \\ 7 & 4 & 3 \end{pmatrix}$ for A and says: 'The smallest entry in the matrix is 1, and it is in the top row, so the saddle point is at (A_1, B_2) .' Explain why this is wrong and find the saddle point, if there is one.

[4]

20

The payoff matrix for A is $\begin{pmatrix} 2 & 5 \\ 6 & 3 \end{pmatrix}$. Show that there is no saddle point. Given that, when both players use their best mixed strategies, the value of the game is 4, show that this value lies between the maximin and minimax values.

[6]

21

The payoff matrix for A is $\begin{pmatrix} k & 4 \\ 3 & 5 \end{pmatrix}$, where k is a constant.

Find the range of values of k for which the game has a saddle point, and give the value of the game in each case.

[8]

22

The payoff matrix for A is $\begin{pmatrix} 2 & k \\ 5 & 3 \end{pmatrix}$, where k is a constant.

Show that the game has a saddle point if and only if $k \leq 3$, and find the saddle point and the value in that case.

[6]

23

The payoff matrix for A is $\begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}$.

Starting with A playing A_1 , trace the sequence in which each player in turn changes to the best reply to the other's current choice. Use your answer to explain why the game has no stable pair of pure strategies.

[6]

24

Suppose a zero-sum game has two saddle points, at (A_i, B_j) with payoff a_{ij} and at (A_k, B_l) with payoff a_{kl} .

Show that $a_{ij} = a_{kl}$, so the value of the game is unique, and that (A_i, B_l) is also a saddle point.

[6]

25

A company (A) and its rival (B) each choose a pricing policy. The matrix gives the change in A's market share (in %) for A's policies (rows) against B's policies (columns):

$$\begin{pmatrix} 8 & 3 & 6 & 9 \\ 5 & 4 & 7 & 6 \\ 2 & 1 & 9 & 3 \end{pmatrix}.$$

Show that the game has a saddle point. State each company's optimal policy and the value of the game, and say who gains.

[7]

- 26** The payoff matrix for A in game 1 is $\begin{pmatrix} 3 & 5 & 4 \\ 2 & 6 & 3 \end{pmatrix}$. In game 2 every payoff in game 1 is doubled and then increased by 3. Find the saddle point and value of game 1. Write down the payoff matrix for game 2 and find its saddle point and value. What do you notice? [6]
- 27** Let the payoff matrix for A have entries a_{ij} . Prove that the maximin value is never greater than the minimax value, that is $\max \text{ over } i \text{ of } (\min \text{ over } j \text{ of } a_{ij}) \leq \min \text{ over } j \text{ of } (\max \text{ over } i \text{ of } a_{ij})$. Explain what this tells you about a game with a saddle point. [8]
- 28** The payoff matrix for A is $\begin{pmatrix} 4 & k & 2 \\ 3 & 5 & k \\ 6 & 2 & 3 \end{pmatrix}$, where k is a constant. Show that the game has a saddle point only when $k = 3$, and find the saddle point and the value of the game in that case. [8]
- 29** The payoff matrix for A in a zero-sum game is $\begin{pmatrix} 4 & 6 & 5 \\ 2 & 7 & 3 \end{pmatrix}$. Write down the table of payoffs to B, with B's strategies as the rows and A's as the columns. Find the maximin value for B and the minimax value for A in this table, and show that the value of the game to B is the negative of the value of the game to A. [8]
- 30** The payoff matrix for A is $\begin{pmatrix} 6 & 2 & 6 \\ 3 & 4 & 5 \\ 6 & 7 & 6 \end{pmatrix}$. Show that the game has two saddle points and state the value. Show that B has infinitely many optimal strategies by considering B playing B_1 with probability q and B_3 with probability $1 - q$. Then find all the optimal strategies for A that use only rows 1 and 3. [10]

Mixed Strategies

- 1** State what is meant by a mixed strategy for player A. [1]
- 2** Explain why mixed strategies are needed when a zero-sum game has no saddle point. [1]
- 3** The payoff matrix for A is $\begin{pmatrix} 4 & 1 \\ 2 & 5 \end{pmatrix}$. A plays A_1 with probability 0.3 and A_2 with probability 0.7, and B plays B_1 . Find A's expected payoff. [2]
- 4** The payoff matrix for A is $\begin{pmatrix} 3 & -1 \\ -2 & 4 \end{pmatrix}$. A plays A_1 with probability p and A_2 with probability $1 - p$, and B plays B_1 . Write down A's expected payoff in terms of p . [2]
- 5** The payoff matrix for A is $\begin{pmatrix} 6 & 2 \\ 0 & 4 \end{pmatrix}$. A plays each row with probability $\frac{1}{2}$. Find A's expected payoff when B plays B_2 . [2]
- 6** The payoff matrix for A is $\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$. A plays A_1 with probability $\frac{1}{2}$ and A_2 with probability $\frac{1}{2}$. Find A's expected payoff against each of B's pure strategies, and say what B gains by choosing between them. [3]

7 The payoff matrix for A is $\begin{pmatrix} 5 & 1 \\ 2 & 4 \end{pmatrix}$, and the game has no saddle point. Find A's optimal mixed strategy and the value of the game. [4]

8 The payoff matrix for A is $\begin{pmatrix} 3 & -1 \\ -2 & 4 \end{pmatrix}$, and the game has no saddle point. Find A's optimal mixed strategy and the value of the game. [4]

9 The payoff matrix for A is $\begin{pmatrix} 2 & 6 \\ 5 & 3 \end{pmatrix}$, and the game has no saddle point. Find B's optimal mixed strategy. [3]

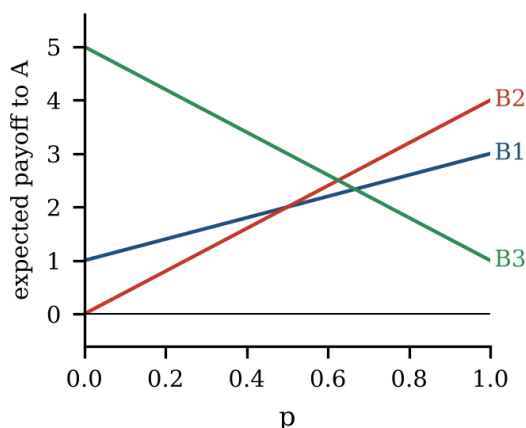
10 For a 2×2 game with payoff matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and no saddle point, A should play A_1 with probability $p = (d - c) / (a - b - c + d)$. Use this to find A's optimal strategy and the value of the game when the matrix is $\begin{pmatrix} 7 & 2 \\ 3 & 6 \end{pmatrix}$. [4]

11 The payoff matrix for A is $\begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}$. A plays A_1 with probability $\frac{1}{4}$. Show that A's expected payoff is the same whichever strategy B plays, and state its value. [3]

12 The payoff matrix for A is $\begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$, and the game has no saddle point. Find the value of the game and say which player it favours. [4]

13 The payoff matrix for A is $\begin{pmatrix} 4 & 2 \\ 1 & 5 \end{pmatrix}$. A plays A_1 with probability $\frac{1}{4}$ and B plays B_1 with probability $\frac{1}{3}$. Find A's expected payoff. [4]

14 The graph shows A's expected payoff against each of B's pure strategies when A plays A_1 with probability p and A_2 with probability $1 - p$, for the game with payoff matrix $\begin{pmatrix} 3 & 4 & 1 \\ 1 & 0 & 5 \end{pmatrix}$. Find A's optimal strategy and the value of the game. Show that B_2 is never needed by B.



[6]

- 15** A striker (A) chooses to shoot left or right, and a goalkeeper (B) chooses to dive left or right. The matrix gives the percentage chance that the striker scores, with the striker's choices as rows (left, right) and the goalkeeper's as columns (left, right): $\begin{pmatrix} 60 & 90 \\ 80 & 30 \end{pmatrix}$.
Show that there is no saddle point. Find the striker's optimal strategy and the percentage chance of scoring with optimal play. [6]
-
- 16** The payoff matrix for A is $\begin{pmatrix} 3 & 1 \\ 4 & 0 \\ 1 & 5 \end{pmatrix}$. B plays B_1 with probability q and B_2 with probability $1 - q$.
Find B's expected loss against each of A's three strategies in terms of q . Hence find B's optimal strategy and the value of the game. [6]
-
- 17** The payoff matrix for A is $\begin{pmatrix} 6 & 2 \\ 1 & 4 \end{pmatrix}$. Show that there is no saddle point, find A's optimal strategy and the value of the game, and explain why A's expected payoff does not depend on what B does when A uses this strategy. [7]
-
- 18** The payoff matrix for A is $\begin{pmatrix} -2 & 3 \\ 4 & -1 \end{pmatrix}$, and the game has no saddle point. Find the optimal mixed strategy for each player and the value of the game. [7]
-
- 19** A student says: 'In the game with payoff matrix $\begin{pmatrix} 1 & 5 \\ 4 & 2 \end{pmatrix}$ there are two rows, so A should play each with probability $\frac{1}{2}$.'
Show that B can hold A below the value of the game if A does this, and find A's correct optimal strategy and the value of the game. [6]
-
- 20** The payoff matrix for A is $\begin{pmatrix} 3 & 5 \\ 7 & 2 \end{pmatrix}$, and the game has no saddle point.
A plays the optimal mixed strategy. B instead plays B_1 with probability 0.9 and B_2 with probability 0.1. Find A's expected payoff, and explain why it is the same as the value of the game. [5]
-
- 21** The payoff matrix for A is $\begin{pmatrix} 2 & 1 & 5 \\ 4 & 7 & 3 \end{pmatrix}$.
A plays A_1 with probability p . By finding A's expected payoff against each of B's strategies, find A's optimal strategy. Then find B's optimal strategy and the value of the game. [8]
-
- 22** The payoff matrix for A is $\begin{pmatrix} 0 & 2 & -1 \\ -2 & 0 & 3 \\ 1 & -3 & 0 \end{pmatrix}$.
Explain why the value of this game is 0. Find A's optimal strategy by making A's expected payoff 0 against every column of B, and write down B's optimal strategy. [8]
-
- 23** The payoff matrix for A is $\begin{pmatrix} 2 & 5 & 3 \\ 8 & 3 & 5 \\ 3 & 4 & 6 \end{pmatrix}$. It is claimed that A's optimal strategy is $(\frac{1}{2}, \frac{1}{3}, \frac{1}{6})$ and B's optimal strategy is $(\frac{1}{6}, \frac{2}{3}, \frac{1}{6})$.
Verify the claim, and state the value of the game. [8]

- 24** In a betting game, player A either bluffs (row 1) or plays honestly (row 2), and player B either calls (column 1) or folds (column 2). The payoffs to A, in pounds, are $\begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}$.
Show that there is no saddle point. Find the optimal strategy for each player, the value of the game, and A's expected profit over 100 games with optimal play. [9]

- 25** The payoff matrix for A is $\begin{pmatrix} k & 2 \\ 1 & 3 \end{pmatrix}$, where $k > 2$.
Show that the game has no saddle point. Find, in terms of k , A's optimal strategy and the value of the game. Hence find the value of k for which the value of the game is $\frac{5}{2}$. [8]

- 26** The payoff matrix for A is $\begin{pmatrix} 6 & 2 & 7 \\ 3 & 5 & 4 \\ 2 & 1 & 3 \end{pmatrix}$.
Use dominance to reduce the game to 2×2 . Hence find the optimal strategy for each player and the value of the game. [9]

- 27** The payoff matrix for A is $\begin{pmatrix} 2 & 5 & 3 \\ 8 & 3 & 5 \\ 3 & 4 & 6 \end{pmatrix}$. A plays A_1, A_2, A_3 with probabilities p_1, p_2, p_3 .
Given that the optimal strategies of both players use all three of their strategies, find A's optimal strategy, B's optimal strategy and the value of the game. [11]

- 28** The payoff matrix for A is $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and the game has no saddle point, so that a, d are both greater than b, c or both less than b, c .
A plays A_1 with probability p . Show that A's optimal strategy has $p = (d - c) / (a - b - c + d)$ and that the value of the game is $(ad - bc) / (a - b - c + d)$. Use these results to solve the game with matrix $\begin{pmatrix} 8 & 3 \\ 2 & 6 \end{pmatrix}$. [10]

- 29** The payoff matrix for A is $\begin{pmatrix} k & -3 \\ -2 & 4 \end{pmatrix}$, where k is a positive constant. The game is fair if its value is zero.
Assuming the game has no saddle point, find the value of the game in terms of k , and hence find the value of k for which the game is fair. Verify that for this value of k there is no saddle point, and find each player's optimal strategy. [11]

- 30** The payoff matrix for A is $\begin{pmatrix} 3 & 6 & 2 \\ 0 & 5 & 5 \\ 4 & 1 & 7 \end{pmatrix}$. It is suspected that A should never play A_2 and that B should never play B_3 .
Solve the 2×2 game obtained by deleting row 2 and column 3. Then show that the strategies you find are optimal in the full 3×3 game, and state the value of the game. [12]

Game Theory and Linear Programming

- 1** Before a zero-sum game is converted into a linear programming problem, the same constant is added to every payoff so that all entries are positive. Explain why this is done. [2]

2 The payoff matrix for A is $\begin{pmatrix} -2 & 3 \\ 4 & -1 \end{pmatrix}$. Find the smallest positive integer that can be added to every entry so that all the entries are positive, and write down the new matrix. [2]

3 A constant of 3 was added to every payoff in a game. The value of the new game is 5. State the value of the original game. [1]

4 In a game with payoff matrix $\begin{pmatrix} 6 & 2 \\ 1 & 7 \end{pmatrix}$, B plays B_1 with probability q_1 and B_2 with probability q_2 , and the value of the game is V . Write down the two inequalities for B's expected loss when A plays A_1 and when A plays A_2 . [2]

5 In the game with payoff matrix $\begin{pmatrix} 6 & 2 \\ 1 & 7 \end{pmatrix}$, B plays B_1 and B_2 with probabilities q_1 and q_2 . Let $y_1 = q_1/V$ and $y_2 = q_2/V$, where V is the value of the game. Show that $y_1 + y_2 = 1/V$. [2]

6 Write the constraint $6y_1 + 2y_2 \leq 1$ as an equation by introducing a slack variable r . [1]

7 The payoff matrix for A is $\begin{pmatrix} 2 & -1 \\ -3 & 4 \end{pmatrix}$. Add 4 to every entry, and then formulate the linear programming problem for B, in which $y_j = q_j/V$ and $Z = y_1 + y_2$ is to be maximised. [4]

8 The payoff matrix for A is $\begin{pmatrix} 3 & 5 & 2 \\ 4 & 1 & 6 \end{pmatrix}$. Formulate the linear programming problem for B, using $y_j = q_j/V$, with Z to be maximised. [3]

9 The payoff matrix for A is $\begin{pmatrix} 2 & 4 \\ 5 & 1 \\ 3 & 3 \end{pmatrix}$. Formulate the linear programming problem for B, using $y_j = q_j/V$, with Z to be maximised. [4]

10 Maximise $Z = y_1 + y_2$ subject to $6y_1 + 2y_2 \leq 1$, $y_1 + 7y_2 \leq 1$, $y_1, y_2 \geq 0$. Introduce slack variables r and s and set up the initial simplex tableau. [3]

11 The final simplex tableau for the linear programming problem for B in a game with payoff matrix $\begin{pmatrix} 6 & 2 \\ 1 & 7 \end{pmatrix}$ (no constant has been added) is shown. Find the value of the game and B's optimal strategy.

b.v.	y_1	y_2	r	s	Value
y_1	1	0	$\frac{7}{40}$	$-\frac{1}{20}$	$\frac{1}{8}$
y_2	0	1	$-\frac{1}{40}$	$\frac{3}{20}$	$\frac{1}{8}$
Z	0	0	$\frac{3}{20}$	$\frac{1}{10}$	$\frac{1}{4}$

[4]

- 12** The final simplex tableau for the linear programming problem for B in a game with payoff matrix $\begin{pmatrix} 6 & 2 \\ 1 & 7 \end{pmatrix}$ is shown.
- Read from the tableau A's optimal strategy.

b.v.	y_1	y_2	r	s	Value
y_1	1	0	$\frac{7}{40}$	$-\frac{1}{20}$	$\frac{1}{8}$
y_2	0	1	$-\frac{1}{40}$	$\frac{3}{20}$	$\frac{1}{8}$
Z	0	0	$\frac{3}{20}$	$\frac{1}{10}$	$\frac{1}{4}$

[4]

- 13** The payoffs in a game were each increased by 3 and the linear programming problem for B was solved, giving an optimal value $Z = \frac{1}{4}$.
- Find the value of the original game.

[2]

- 14** The initial simplex tableau for the linear programming problem for B in a game with payoff matrix $\begin{pmatrix} 6 & 2 \\ 1 & 7 \end{pmatrix}$ is shown.
- Carry out the first iteration of the simplex algorithm, showing the pivot and the row operations, and state the value of Z after this iteration.

b.v.	y_1	y_2	r	s	Value
r	6	2	1	0	1
s	1	7	0	1	1
Z	-1	-1	0	0	0

[5]

- 15** The tableau shown is obtained after the first iteration of the simplex algorithm for the linear programming problem for B in the game with payoff matrix $\begin{pmatrix} 6 & 2 \\ 1 & 7 \end{pmatrix}$.
- Carry out the next iteration to reach the optimal tableau. Hence find B's optimal strategy and the value of the game.

b.v.	y_1	y_2	r	s	Value
y_1	1	$\frac{1}{3}$	$\frac{1}{6}$	0	$\frac{1}{6}$
s	0	$\frac{20}{3}$	$-\frac{1}{6}$	1	$\frac{5}{6}$
Z	0	$-\frac{2}{3}$	$\frac{1}{6}$	0	$\frac{1}{6}$

[7]

- 16** Two companies compete for market share. The matrix gives the change in percentage share for company A for each pair of strategies: $\begin{pmatrix} 3 & -1 \\ -2 & 4 \end{pmatrix}$.
- Show how to turn the game into a linear programming problem for company B, by adding a suitable constant. Given that the optimal value of Z is $\frac{1}{4}$, find the value of the game and interpret it.

[7]

- 17** A student adds 5 to every payoff of a game, solves the linear programming problem for B and obtains $Z = \frac{1}{9}$. The student states: 'The value of the game is 9.'
- Explain the mistake and give the correct value of the game.

[4]

18

The payoff matrix for A is $\begin{pmatrix} 4 & -2 & 1 \\ 0 & 3 & -1 \\ 2 & 5 & 6 \end{pmatrix}$.

Add a suitable constant to make every entry positive and formulate the linear programming problem for B, in which $Z = y_1 + y_2 + y_3$ is to be maximised.

[5]

19

The final simplex tableau for the linear programming problem for B in a game with payoff

matrix $\begin{pmatrix} 3 & 5 & 2 \\ 4 & 1 & 6 \end{pmatrix}$ is shown.

Find the value of the game and the optimal strategy for each player.

b.v.	y_1	y_2	y_3	r	s	Value
y_2	0	1	$-\frac{10}{17}$	$\frac{4}{17}$	$-\frac{3}{17}$	$\frac{1}{17}$
y_1	1	0	$\frac{28}{17}$	$-\frac{1}{17}$	$\frac{5}{17}$	$\frac{4}{17}$
Z	0	0	$\frac{1}{17}$	$\frac{3}{17}$	$\frac{2}{17}$	$\frac{5}{17}$

[6]

20

The game with payoff matrix $\begin{pmatrix} 3 & -1 \\ -2 & 4 \end{pmatrix}$ is converted to a linear programming problem by

adding 3 to each payoff. In the final tableau the Z row has entries $\frac{3}{20}$ and $\frac{1}{10}$ under the slack variables r and s , and $Z = \frac{1}{4}$.

Use the formula $p = (d - c) / (a - b - c + d)$ for a 2×2 game with matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to check

A's optimal strategy obtained from the tableau.

[4]

21

The payoff matrix for A is $\begin{pmatrix} 3 & 5 & 2 \\ 4 & 1 & 6 \end{pmatrix}$. The linear programming problem for B is to

maximise $Z = y_1 + y_2 + y_3$ subject to $3y_1 + 5y_2 + 2y_3 \leq 1$, $4y_1 + y_2 + 6y_3 \leq 1$, $y_1, y_2, y_3 \geq 0$. The initial tableau is shown.

Solve the problem using the simplex algorithm. Hence find B's optimal strategy and the value of the game.

b.v.	y_1	y_2	y_3	r	s	Value
r	3	5	2	1	0	1
s	4	1	6	0	1	1
Z	-1	-1	-1	0	0	0

[9]

22

The payoff matrix for A is $\begin{pmatrix} 3 & 5 \\ 2 & 4 \end{pmatrix}$. The linear programming problem for B is to maximise Z

$= y_1 + y_2$ subject to $3y_1 + 5y_2 \leq 1$, $2y_1 + 4y_2 \leq 1$, $y_1, y_2 \geq 0$.

Solve it by the simplex algorithm, and show that the solution agrees with the saddle point of the game.

b.v.	y_1	y_2	r	s	Value
r	3	5	1	0	1
s	2	4	0	1	1
Z	-1	-1	0	0	0

[8]

23

The payoff matrix for A is $\begin{pmatrix} 1 & -2 \\ -3 & 2 \end{pmatrix}$. The payoffs were increased by 4 and the linear programming problem for B was solved. The final tableau is shown. Find the value of the original game and the optimal strategy for each player. Which player does the game favour?

b.v.	y_1	y_2	r	s	Value
y_1	1	0	$\frac{3}{14}$	$-\frac{1}{14}$	$\frac{1}{7}$
y_2	0	1	$-\frac{1}{28}$	$\frac{5}{28}$	$\frac{1}{7}$
Z	0	0	$\frac{5}{28}$	$\frac{3}{28}$	$\frac{2}{7}$

[7]

24

The payoff matrix for A is $\begin{pmatrix} 6 & 2 \\ 1 & 7 \end{pmatrix}$. Player A plays A_1 with probability p_1 and A_2 with probability p_2 . The value of the game is V. Explain why A's problem can be written as: maximise $P = V$ subject to $V - 6p_1 - p_2 \leq 0$, $V - 2p_1 - 7p_2 \leq 0$, $p_1 + p_2 \leq 1$, $V, p_1, p_2 \geq 0$. Write the problem as an initial simplex tableau using slack variables r, s and t .

[6]

25

The payoff matrix for A is $\begin{pmatrix} 0 & 2 & -1 \\ -2 & 0 & 3 \\ 1 & -3 & 0 \end{pmatrix}$. Each payoff was increased by 4 and the linear programming problem for B was solved. The final tableau is shown. Find the value of the original game and the optimal strategy for each player. What does the value tell you about the game?

b.v.	y_1	y_2	y_3	r	s	t	Value
y_2	0	1	0	$\frac{3}{16}$	$\frac{1}{144}$	$-\frac{11}{72}$	$\frac{1}{24}$
y_3	0	0	1	$-\frac{1}{8}$	$\frac{13}{72}$	$\frac{1}{36}$	$\frac{1}{12}$
y_1	1	0	0	$\frac{1}{16}$	$-\frac{7}{48}$	$\frac{5}{24}$	$\frac{1}{8}$
Z	0	0	0	$\frac{1}{8}$	$\frac{1}{24}$	$\frac{1}{12}$	$\frac{1}{4}$

[7]

26

The payoff matrix for A is $\begin{pmatrix} 4 & -2 & 1 \\ 0 & 3 & -1 \\ 2 & 5 & 6 \end{pmatrix}$. Each payoff was increased by 3 and the linear programming problem for B was solved. The final tableau is shown. Find the value of the original game and the optimal strategy for each player. State which of A's strategies is never used.

b.v.	y_1	y_2	y_3	r	s	t	Value
y_1	1	0	$\frac{23}{51}$	$\frac{8}{51}$	0	$-\frac{1}{51}$	$\frac{7}{51}$
s	0	0	$-\frac{75}{17}$	$\frac{2}{17}$	1	$-\frac{13}{17}$	$\frac{6}{17}$
y_2	0	1	$\frac{43}{51}$	$-\frac{5}{51}$	0	$\frac{7}{51}$	$\frac{2}{51}$
Z	0	0	$\frac{5}{17}$	$\frac{1}{17}$	0	$\frac{2}{17}$	$\frac{3}{17}$

[7]

27

The payoff matrix for A is $\begin{pmatrix} 6 & 5 & 1 \\ 4 & 6 & 5 \\ 5 & 4 & 6 \end{pmatrix}$. The linear programming problem for B is to

maximise $Z = y_1 + y_2 + y_3$ subject to $6y_1 + 5y_2 + y_3 \leq 1$, $4y_1 + 6y_2 + 5y_3 \leq 1$, $5y_1 + 4y_2 + 6y_3 \leq 1$, $y_1, y_2, y_3 \geq 0$. The initial tableau is shown.

Solve the problem using the simplex algorithm. Hence find B's optimal strategy and the value of the game.

b.v.	y_1	y_2	y_3	r	s	t	Value
r	6	5	1	1	0	0	1
s	4	6	5	0	1	0	1
t	5	4	6	0	0	1	1
Z	-1	-1	-1	0	0	0	0

[11]

28

In a zero-sum game with an $m \times n$ payoff matrix, all the payoffs b_{ij} are positive. B plays column j with probability q_j and the value of the game is V .

Show that B's optimal strategy can be found by maximising $Z = y_1 + \dots + y_n$ subject to $b_{i1}y_1 + \dots + b_{in}y_n \leq 1$ for each row i , where $y_j = q_j/V$. State how V and q are recovered from the solution.

[9]

29

The payoff matrix for A is $\begin{pmatrix} 4 & -2 & 1 \\ 0 & 3 & -1 \\ 2 & 5 & 6 \end{pmatrix}$. The initial tableau of the linear programming

problem for B after adding a constant to every payoff is shown.

Determine the constant that was added, and the linear programming problem. Complete the simplex algorithm. Hence find the value of the original game and the optimal strategy for each player.

b.v.	y_1	y_2	y_3	r	s	t	Value
r	7	1	4	1	0	0	1
s	3	6	2	0	1	0	1
t	5	8	9	0	0	1	1
Z	-1	-1	-1	0	0	0	0

[12]

30

The final simplex tableau for the linear programming problem for B in a game with payoff

matrix $\begin{pmatrix} 6 & 2 \\ 1 & 7 \end{pmatrix}$ has $Z = \frac{1}{4}$, $y_1 = y_2 = \frac{1}{8}$, and Z row entries $\frac{3}{20}$ and $\frac{1}{10}$ under the slack variables r and s .

Show that the sum of the y values equals the sum of the Z row entries under the slack variables. Explain why this guarantees that the probabilities found for B and for A each sum to 1, and find both strategies.

[8]