

Three-Dimensional Vectors

1 (2 marks)

$$|v| = \sqrt{2^2 + (-3)^2 + 6^2} = \sqrt{49}$$

$$= 7$$

M1

A1

Answer: 7

2 marks

2 (2 marks)

$$AB = b - a$$

M1

$$= 2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$$

A1

Answer: $2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$

2 marks

3 (2 marks)

$$|w| = \sqrt{2^2 + (-2)^2 + 1^2} = 3$$

M1

$$\text{Unit vector} = \frac{1}{3}(2\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = \frac{2}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}$$

A1

Answer: $\frac{2}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}$

2 marks

4 (2 marks)

$$\text{Midpoint} = \frac{1}{2}(a + b)$$

M1

$$= (5, -2, 2)$$

A1

Answer: $(5, -2, 2)$

2 marks

5 (3 marks)

$$\begin{pmatrix} 6 \\ -3 \\ 12 \end{pmatrix} = 3 \times 2\mathbf{i} - \mathbf{j} + 4\mathbf{k}, \text{ so the scale factor is } 3$$

M1

$$p = -3$$

A1

$$q = 12$$

A1

Answer: $p = -3, q = 12$

3 marks

6 (2 marks)

$$PQ = \sqrt{(4-1)^2 + (-2-2)^2 + (15-3)^2}$$

$$= \sqrt{9 + 16 + 144} = \sqrt{169} = 13$$

M1

A1

Answer: 13

2 marks

7 (3 marks)

$$|u| = \sqrt{2^2 + (-1)^2 + 2^2} = 3$$

M1

$$\text{Unit vector} = \frac{1}{3}(2\mathbf{i} - \mathbf{j} + 2\mathbf{k})$$

A1

$$\text{Required vector} = \frac{10}{3}(2\mathbf{i} - \mathbf{j} + 2\mathbf{k}) = \frac{20}{3}\mathbf{i} - \frac{10}{3}\mathbf{j} + \frac{20}{3}\mathbf{k}$$

A1

Answer: $\frac{20}{3}\mathbf{i} - \frac{10}{3}\mathbf{j} + \frac{20}{3}\mathbf{k}$

3 marks

8 (4 marks)

$$AB^2 = 2^2 + (-3)^2 + 2^2 = 17, BC^2 = 3^2 + 2^2 + 0^2 = 13, AC^2 = 5^2 + (-1)^2 + 2^2 = 30$$

M1 A1

$$AB^2 + BC^2 = 30 = AC^2$$

M1

By the converse of Pythagoras the triangle is right-angled at B

A1

Answer: Right angle at B

4 marks

9 (4 marks)

$$\mathbf{p} = \mathbf{a} + \frac{2}{5}(\mathbf{b} - \mathbf{a})$$

M1 A1

$$= 5\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$$

M1

P is (5, -2, 2)

A1

Answer: (5, -2, 2)

4 marks

10 (4 marks)

$$\mathbf{AB} = 3\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}, \mathbf{BC} = 3\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$$

M1 A1

BC = AB, so AB is parallel to BC with the common point B, so A, B, C are collinear

E1

$$\mathbf{AB} : \mathbf{BC} = 1 : 1$$

A1

Answer: Collinear; AB : BC = 1 : 1

4 marks

11 (3 marks)

$$\mathbf{AD} = \mathbf{BC}$$

M1

$$\mathbf{d} = \mathbf{a} + \mathbf{c} - \mathbf{b} = 3\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}$$

M1

D is (3, 5, 3)

A1

Answer: (3, 5, 3)

3 marks

12 (4 marks)

$$|\mathbf{r}| = \sqrt{1^2 + 2^2 + 2^2} = 3$$

B1

$$\cos \alpha = \frac{\text{x-component}}{|\mathbf{r}|} = \frac{1}{3}$$

M1 A1

$$\alpha = 70.5^\circ$$

A1

Answer: $\cos \alpha = 1/3$, $\alpha = 70.5^\circ$

4 marks

13 (3 marks)

$$|\mathbf{a}|^2 = p^2 + 2^2 + (-1)^2 = 9$$

M1

$$p^2 = 4$$

A1

$$p = 2 \text{ or } p = -2$$

A1

Answer: $p = \pm 2$

3 marks

14 (6 marks)

$$\text{(a) } \mathbf{AB} = \sqrt{2^2 + 2^2 + 1^2} = 3$$

M1 A1

$$\mathbf{BC} = \sqrt{2^2 + (-2)^2 + 1^2} = 3, \text{ so } \mathbf{AB} = \mathbf{BC}$$

A1

Answer: $\mathbf{AB} = \mathbf{BC} = 3$

3 marks

$$\text{(b) } \mathbf{AC} = \sqrt{4^2 + 0^2 + 2^2} = \sqrt{20} = 2\sqrt{2}$$

M1 A1

$$\text{Perimeter} = 3 + 3 + 2\sqrt{2} = 6 + 2\sqrt{2}$$

A1

Answer: $6 + 2\sqrt{2}$

3 marks

15 (6 marks)

$$\text{(a) } \mathbf{m} = \frac{1}{2}(\mathbf{b} + \mathbf{c})$$

M1

$$= 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$$

A1

Answer: $2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$

2 marks

$$\text{(b) } \mathbf{g} = \frac{1}{3}6\mathbf{i} + 3\mathbf{j} + 3\mathbf{k} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$$

M1 A1

$$\mathbf{AM} = 6\mathbf{k}, \mathbf{AG} = 4\mathbf{k}$$

M1

$$\mathbf{AG} = \frac{2}{3}\mathbf{AM}, \text{ so G is on AM and } \mathbf{AG} : \mathbf{GM} = 2 : 1$$

A1

Answer: $\mathbf{g} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$; $\mathbf{AG} : \mathbf{GM} = 2 : 1$

4 marks

16 (6 marks)

Let P be (0, p, 0)

B1

$$PA^2 = 3^2 + (p - 1)^2 + 2^2, PB^2 = (-1)^2 + (p - 5)^2 + 6^2$$

M1 A1

$$PA^2 = PB^2: p^2 - 2p + 14 = p^2 - 10p + 62$$

M1

$$8p = 48, \text{ so } p = 6$$

A1

P is (0, 6, 0)

A1

Answer: (0, 6, 0)

6 marks

17 (8 marks)

$$(a) \mathbf{AB} = (2t + 1 - t - 1, t + 1 - 6, 4 - t) = (t, t - 5, 4 - t)$$

M1 A1

$$AB^2 = t^2 + (t - 5)^2 + (4 - t)^2$$

M1

$$= 3t^2 - 18t + 41$$

A1

$$\text{Answer: } AB^2 = 3t^2 - 18t + 41$$

4 marks

$$(b) AB^2 = 3(t - 3)^2 + 14$$

M1 A1

Least when $t = 3$

A1

$$\text{Least } AB = \sqrt{14}$$

A1

$$\text{Answer: } t = 3, \text{ least } AB = \sqrt{14}$$

4 marks

18 (5 marks)

$$i: p + 2q = -1; j: 2p - q = 8; k: -p + 3q = -9$$

M1 A1

$$\text{Solve the first two: from } 2p - q = 8, q = 2p - 8, \text{ so } p + 4p - 16 = -1$$

M1

$$p = 3, q = -2$$

A1

$$\text{Check } k: -3 + 3(-2) = -9 \text{ and } -9 = -9, \text{ so consistent}$$

B1

$$\text{Answer: } p = 3, q = -2$$

5 marks

19 (6 marks)

$$(a) \mathbf{PQ} = 180\mathbf{i} + 150\mathbf{j} + 45\mathbf{k}$$

M1

$$PQ = \sqrt{180^2 + 150^2 + 45^2} = \sqrt{56925} = 239 \text{ m}$$

M1 A1

$$\text{Answer: } 239 \text{ m}$$

3 marks

$$(b) \mathbf{r} = \mathbf{p} + \frac{2}{5}\mathbf{PQ}$$

M1

$$= 192\mathbf{i} + 100\mathbf{j} + 33\mathbf{k}$$

A1

$$\text{Height} = 33 \text{ m}$$

A1

$$\text{Answer: } 192\mathbf{i} + 100\mathbf{j} + 33\mathbf{k}; \text{ height } 33 \text{ m}$$

3 marks

20 (5 marks)

$$(a) \text{ OABC parallelogram so } \mathbf{AB} = \mathbf{OC}$$

M1

$$\mathbf{b} = \mathbf{a} + \mathbf{c}$$

A1

$$\text{Answer: } \mathbf{b} = \mathbf{a} + \mathbf{c}$$

2 marks

$$(b) \text{ Midpoint of OB: } \frac{1}{2}\mathbf{b} = \frac{1}{2}(\mathbf{a} + \mathbf{c})$$

M1

$$\text{Midpoint of AC: } \frac{1}{2}(\mathbf{a} + \mathbf{c})$$

M1

The midpoints coincide, so the diagonals bisect each other

A1

$$\text{Answer: Both midpoints are } \frac{1}{2}(\mathbf{a} + \mathbf{c})$$

3 marks

21 (9 marks)

$$(a) BC = \frac{1}{2}AB, \text{ so } AC = \frac{3}{2}AB$$

M1 A1

$$\mathbf{c} = \mathbf{a} + \frac{3}{2}(\mathbf{b} - \mathbf{a}) = 5\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$$

M1 A1

$$\text{Answer: } 5\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$$

4 marks

$$(b) \mathbf{d} = \mathbf{b} + 3(\mathbf{a} - \mathbf{b})$$

M1

$$= -2\mathbf{i} - 5\mathbf{j} + 12\mathbf{k}$$

A1

$$\text{Answer: } -2\mathbf{i} - 5\mathbf{j} + 12\mathbf{k}$$

2 marks

$$(c) \mathbf{AD} = -4\mathbf{i} - 4\mathbf{j} + 8\mathbf{k}, \mathbf{DC} = 7\mathbf{i} + 7\mathbf{j} - 14\mathbf{k}$$

M1

D, A, B, C are collinear; $|AD| : |DC| = 2 : 3.5 = 4 : 7$

Answer: AD : DC = 4 : 7

M1 A1

3 marks

22 (8 marks)

(a) $\mathbf{d} = \mathbf{a} + \mathbf{c} - \mathbf{b}$

M1

$= -\mathbf{i} + \mathbf{j} + 5\mathbf{k}$

A1

Answer: D(-1, 1, 5)

2 marks

(b) $AB^2 = 3^2 + 2^2 + 2^2 = 17$

M1 A1

$BC^2 = (-2)^2 + 3^2 + 2^2 = 17$, so $AB = BC$ and adjacent sides are equal

A1

Answer: AB = BC = $\sqrt{17}$

3 marks

(c) $AC = \sqrt{1^2 + 5^2 + 4^2} = \sqrt{42}$

M1 A1

$BD = \sqrt{(-5)^2 + 1^2 + 0^2} = \sqrt{26}$

A1

Answer: AC = $\sqrt{42}$, BD = $\sqrt{26}$

3 marks

23 (8 marks)

$$\mathbf{p} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}) = \begin{pmatrix} 2\lambda + 1 \\ 2 - 2\lambda \\ 3 - 4\lambda \end{pmatrix}$$

M1 A1

$OP^2 = (1 + 2\lambda)^2 + (2 - 2\lambda)^2 + (3 - 4\lambda)^2$

M1

$= 24\lambda^2 - 28\lambda + 14$

A1

Complete the square: $24\left(\lambda - \frac{7}{12}\right)^2 + \frac{35}{6}$

M1 A1

Least value of OP is $\sqrt{\frac{35}{6}} = 2.42$, when $\lambda = \frac{7}{12}$

M1 A1

Answer: Least distance 2.42 when $\lambda = \frac{7}{12}$

8 marks

24 (8 marks)

(a) $\cos \alpha = \frac{x}{|r|}$, $\cos \beta = \frac{y}{|r|}$, $\cos \gamma = \frac{z}{|r|}$

M1

Sum of squares = $\frac{x^2 + y^2 + z^2}{|r|^2}$

M1

$= 1$ since $|r|^2 = x^2 + y^2 + z^2$

A1

Answer: Proved: $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$

3 marks

(b) $\cos^2 \gamma = 1 - \frac{1}{9} - \frac{4}{9} = \frac{4}{9}$

M1 A1

γ obtuse so $\cos \gamma = -\frac{2}{3}$

A1

$$\mathbf{r} = 12(\cos \alpha, \cos \beta, \cos \gamma) = \begin{pmatrix} 4 \\ 8 \\ -8 \end{pmatrix}$$

M1 A1

Answer: $4\mathbf{i} + 8\mathbf{j} - 8\mathbf{k}$

5 marks

25 (8 marks)

(a) Collinear needs **BC** to be a scalar multiple of **AB**

E1

That needs $k - 6 = 3$, so the third components must match, not just two of them

E1

Answer: Parallel vectors must be scalar multiples in every component

2 marks

(b) $\mathbf{BC} = \lambda \mathbf{AB}$ requires $\lambda = 1$ from the first component

M1

$k - 6 = 3$, so $k = 9$

A1

Answer: $k = 9$

2 marks

(c) $AB^2 = 27$, $BC^2 = 3^2 + 3^2 + (k - 6)^2$

M1

$(k - 6)^2 = 9$

M1

$k = 9$ (collinear, rejected) or $k = 3$

A1 A1

Answer: $k = 3$

4 marks

26 (6 marks)

(a) $\mathbf{g} = \frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c})$

B1

Answer: $\mathbf{g} = (\mathbf{a} + \mathbf{b} + \mathbf{c})/3$

1 mark

(b) $\mathbf{OP} = \frac{3}{4}\mathbf{OG}$

M1

$$\mathbf{p} = \frac{3}{4} \times \frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c}) = \frac{1}{4}(\mathbf{a} + \mathbf{b} + \mathbf{c})$$

A1

Answer: Shown: $\mathbf{p} = \frac{1}{4}(\mathbf{a} + \mathbf{b} + \mathbf{c})$

2 marks

(c) $\mathbf{p} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

M1

$$\mathbf{OP} = \sqrt{1 + 4 + 9} = \sqrt{14} = 3.74 \text{ (3 s.f.)}$$

M1 A1

Answer: $\mathbf{P} = (1, 2, 3), \mathbf{OP} = \sqrt{14}$

3 marks**27 (10 marks)**

(a) $\mathbf{PQ} = \frac{1}{2}(\mathbf{b} + \mathbf{c}) - \frac{1}{2}(\mathbf{a} + \mathbf{b}) = \frac{1}{2}(\mathbf{c} - \mathbf{a})$

M1 A1

$$\mathbf{SR} = \frac{1}{2}(\mathbf{c} + \mathbf{d}) - \frac{1}{2}(\mathbf{d} + \mathbf{a}) = \frac{1}{2}(\mathbf{c} - \mathbf{a})$$

M1 A1

Answer: $\mathbf{SR} = (1/2)\mathbf{AC}$

4 marks

(b) $\mathbf{PQ} = \mathbf{SR}$

M1

so PQ and SR are equal and parallel, so PQRS is a parallelogram (and it is planar)

A1

Answer: Proved: **PQRS is a parallelogram**

2 marks

(c) $\mathbf{P}(3, 1, 0), \mathbf{Q}(7, 5, 2), \mathbf{R}(5, 7, 7), \mathbf{S}(1, 3, 5)$

M1 A1

$$\mathbf{PQ} = \begin{pmatrix} 4 \\ 4 \\ 2 \end{pmatrix}, \mathbf{SR} = \begin{pmatrix} 4 \\ 4 \\ 2 \end{pmatrix}, \text{equal}$$

M1 A1

Answer: $\mathbf{P}(3, 1, 0), \mathbf{Q}(7, 5, 2), \mathbf{R}(5, 7, 7), \mathbf{S}(1, 3, 5)$

4 marks**28 (10 marks)**

(a) $\mathbf{AB} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}, |\mathbf{AB}| = 3$

B1

$$\mathbf{AC} = 4\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}, |\mathbf{AC}| = 6$$

B1

Answer: $|\mathbf{AB}| = 3, |\mathbf{AC}| = 6$

2 marks

(b) $\mathbf{e}_1 = \frac{2}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}, \mathbf{e}_2 = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}$

M1

$$\mathbf{e}_1 + \mathbf{e}_2 = \frac{4}{3}\mathbf{i}$$

A1 $\mathbf{e}_1$ and $\mathbf{e}_2$ have equal length, so $\mathbf{e}_1, \mathbf{e}_2$ and their sum are the sides and diagonal of a rhombus; the diagonal bisects the angle**E1**

Answer: $\mathbf{e}_1 + \mathbf{e}_2 = \frac{4}{3}\mathbf{i}$

3 marks

(c) Bisector: $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + s \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

M1

$$\mathbf{BC}: \mathbf{r} = 3\mathbf{i} + 4\mathbf{k} + m2\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}$$

M1

$$y: 2 = 6m \text{ so } m = \frac{1}{3}$$

M1 A1

$$\mathbf{D} = \left(\frac{11}{3}, 2, 3\right)$$

A1

Answer: $\left(\frac{11}{3}, 2, 3\right)$

5 marks

29 (10 marks)

$$(a) PA^2 = (x + 1)^2 + (y - 2)^2 + z^2, PB^2 = (x - 5)^2 + (y - 2)^2 + z^2 \quad M1$$

$$PA^2 = 4PB^2: (x + 1)^2 + (y - 2)^2 + z^2 = 4[(x - 5)^2 + (y - 2)^2 + z^2] \quad M1$$

$$\text{Expand: } x^2 + 2x + 1 + (y - 2)^2 + z^2 = 4x^2 - 40x + 100 + 4(y - 2)^2 + 4z^2 \quad A1$$

$$0 = 3x^2 - 42x + 99 + 3(y - 2)^2 + 3z^2, \text{ so } x^2 - 14x + 33 + (y - 2)^2 + z^2 = 0 \quad M1$$

$$(x - 7)^2 - 49 + 33 + (y - 2)^2 + z^2 = 0, \text{ giving the result} \quad A1$$

Answer: Shown: P lies on the surface $(x - 7)^2 + (y - 2)^2 + z^2 = 16$ 5 marks

(b) A sphere B1

with centre (7, 2, 0) and radius 4 B1

Answer: Sphere, centre (7, 2, 0), radius 4 2 marks

(c) On AB, $y = 2$ and $z = 0$, so $(x - 7)^2 = 16$ M1

$x = 11$ or $x = 3$, giving (11, 2, 0) and (3, 2, 0) M1 A1

Answer: (3, 2, 0) and (11, 2, 0) 3 marks

30 (9 marks)

$$(a) AB = 2; AC = \sqrt{1 + 3} = 2; BC = \sqrt{(1 - 2)^2 + 3} = 2 \quad M1 A1$$

Answer: $AB = AC = BC = 2$ 2 marks

$$(b) DA^2 = DB^2: p^2 + q^2 + h^2 = (p - 2)^2 + q^2 + h^2, \text{ so } p = 1 \quad M1 A1$$

$$DA^2 = DC^2: 1 + q^2 + h^2 = 0 + (q - \sqrt{3})^2 + h^2 \quad M1$$

$$1 + q^2 = q^2 - 2\sqrt{3}q + 3, \text{ so } q = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \quad A1$$

Answer: $p = 1, q = \sqrt{3}/3$ 4 marks

$$(c) DA^2 = 4: 1 + \frac{1}{3} + h^2 = 4 \quad M1$$

$$h^2 = \frac{8}{3}, h = \frac{2\sqrt{6}}{3} \quad A1$$

$$D \text{ is } (1, \frac{\sqrt{3}}{3}, \frac{2\sqrt{6}}{3}) \quad A1$$

Answer: $D(1, \sqrt{3}/3, 2\sqrt{6}/3)$ 3 marks

Scalar Product**1 (2 marks)**

$$a \cdot b = 2 \times 4 + 3 \times 0 + (-1) \times 5 \quad M1$$

$$= 3 \quad A1$$

Answer: 3 2 marks

2 (2 marks)

$$\text{Scalar product} = 3 \times 1 + (-2) \times 2 + 1 \times 1 = 0 \quad M1$$

The scalar product is zero, so the vectors are perpendicular A1

Answer: Scalar product = 0 2 marks

3 (3 marks)

$$a \cdot b = 1 \text{ and } |a| = |b| = \sqrt{2} \quad B1$$

$$\cos \theta = \frac{1}{\sqrt{2} \times \sqrt{2}} = \frac{1}{2}, \text{ so } \theta = 60^\circ \quad M1 A1$$

Answer: 60° 3 marks

4 (2 marks)

$$4p + 2p - 6 = 0 \quad M1$$

$$p = 1 \quad A1$$

Answer: $p = 1$ 2 marks

5 (2 marks)

$$a \cdot b = |a||b| \cos 60^\circ$$

M1

$$= 3 \times 4 \times \frac{1}{2} = 6$$

A1

Answer: 6

2 marks

6 (2 marks)

$$a \cdot b = 1 \times 2 + 2 \times 1 + (-3) \times 4 = -8$$

M1

The scalar product is negative, so the angle is obtuse

A1

Answer: Obtuse

2 marks

7 (4 marks)

$$a \cdot b = 3 - 4 - 2 = -3 \text{ and } b \cdot b = 1 + 16 + 1 = 18$$

M1 A1

$$(a + \lambda b) \cdot b = 0, \text{ so } -3 + 18\lambda = 0$$

M1

$$\lambda = \frac{1}{6}$$

A1

$$\text{Answer: } \lambda = \frac{1}{6}$$

4 marks

8 (4 marks)

$$BA = -i + 2j - 2k, BC = 2i + j - k$$

M1

$$BA \cdot BC = 2, |BA| = 3, |BC| = \sqrt{6}$$

M1 A1

$$\cos ABC = \frac{2}{3\sqrt{6}}, \text{ so } ABC = 74.2^\circ$$

A1

Answer: 74.2°

4 marks

9 (4 marks)

$$(a) a \cdot b = 6 \text{ and } |b| = 3$$

M1

$$\text{Scalar projection} = \frac{a \cdot b}{|b|} = 2$$

A1

Answer: 2

2 marks

$$(b) \frac{a \cdot b}{|b|^2} b = \frac{6}{9}(2i + j + 2k)$$

M1

$$= \frac{4}{3}i + \frac{2}{3}j + \frac{4}{3}k$$

A1

$$\text{Answer: } \frac{4}{3}i + \frac{2}{3}j + \frac{4}{3}k$$

2 marks

10 (4 marks)

$$\text{Let } n = xi + yj + zk. \text{ Then } x + 2y = 0 \text{ and } y + 3z = 0$$

M1 A1

$$\text{Take } z = 1, \text{ so } y = -3 \text{ and } x = 6$$

M1

$$n = 6i - 3j + k \text{ (or any non-zero multiple)}$$

A1

Answer: 6i - 3j + k

4 marks

11 (4 marks)

$$|a + b|^2 = |a|^2 + 2a \cdot b + |b|^2 = 4 - 2 + 9 = 11$$

M1 A1

$$|a - b|^2 = |a|^2 - 2a \cdot b + |b|^2 = 4 + 2 + 9 = 15$$

M1

$$|a + b| = \sqrt{11} \text{ and } |a - b| = \sqrt{15}$$

A1

Answer: $\sqrt{11}$ and $\sqrt{15}$

4 marks

12 (4 marks)

$$\text{Displacement} = 3i + 2j - 2k$$

B1

$$\text{Work done} = F \cdot d = 3 \times 3 + (-2) \times 2 + 5 \times (-2) = -5 \text{ J}$$

M1 A1

Negative: the force has a component against the motion, so the force removes energy from the particle

E1

Answer: -5 J

4 marks

13 (4 marks)

$$\mathbf{a} \cdot \mathbf{b} = 2 - 2 + 2p = 6, \text{ so } p = 3$$

M1 A1

$$|\mathbf{b}| = \sqrt{1 + 4 + 9} = \sqrt{14} \text{ and } |\mathbf{a}| = 3$$

M1

$$\cos \theta = \frac{6}{3\sqrt{14}} = \frac{2}{\sqrt{14}}, \text{ so } \theta = 57.7^\circ$$

A1

Answer: $p = 3$, $\theta = 57.7^\circ$

4 marks

14 (6 marks)

(a) $\mathbf{OF} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{OB} = \mathbf{i} + \mathbf{j}$, $\cos \theta = \frac{2}{\sqrt{3}\sqrt{2}} = \frac{2}{\sqrt{6}}$

M1 A1

$$\theta = 35.3^\circ$$

A1

Answer: 35.3°

3 marks

(b) $\mathbf{AG} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$, $|\cos \phi| = \frac{|-1 + 1 + 1|}{\sqrt{3}\sqrt{3}} = \frac{1}{3}$

M1 A1

$$\phi = 70.5^\circ$$

A1

Answer: 70.5°

3 marks

15 (7 marks)

$\mathbf{AB} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, $\mathbf{AC} = \mathbf{i} + 4\mathbf{j} + \mathbf{k}$, $\mathbf{BC} = -\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$

B1

$$\cos A = \frac{4}{\sqrt{9}\sqrt{18}}, \text{ so } A = 71.7^\circ$$

M1 A1

$$\cos B = \frac{5}{\sqrt{9}\sqrt{19}}, \text{ so } B = 67.5^\circ$$

M1 A1

$$\cos C = \frac{14}{\sqrt{18}\sqrt{19}}, \text{ so } C = 40.8^\circ$$

M1

$$A + B + C = 180.0^\circ = 180^\circ$$

A1

Answer: $A = 71.7^\circ$, $B = 67.5^\circ$, $C = 40.8^\circ$

7 marks

16 (5 marks)

Let $\mathbf{n} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$: $x + 2y + 3z = 0$ and $2x - y + z = 0$

M1 A1

Eliminate y : $5x + 5z = 0$, so $x = -z$ and $y = -z$

M1

$\mathbf{n}$ is parallel to $-\mathbf{i} - \mathbf{j} + \mathbf{k}$ and $|\mathbf{n}| = \sqrt{3}$

A1

Vectors: $\pm \frac{6}{\sqrt{3}}(-\mathbf{i} - \mathbf{j} + \mathbf{k}) = \pm 2\sqrt{3}(-\mathbf{i} - \mathbf{j} + \mathbf{k})$

A1

Answer: $\pm 2\sqrt{3}(-\mathbf{i} - \mathbf{j} + \mathbf{k})$

5 marks

17 (6 marks)

$$\mathbf{v} \cdot \mathbf{u} = 4 \times 1 + (-3) \times 2 + 12 \times 2 = 22 \text{ and } |\mathbf{u}|^2 = 9$$

M1 A1

$$\text{Parallel component} = \frac{\mathbf{v} \cdot \mathbf{u}}{|\mathbf{u}|^2} \mathbf{u} = \frac{22}{9}(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) = \frac{22}{9}\mathbf{i} + \frac{44}{9}\mathbf{j} + \frac{44}{9}\mathbf{k}$$

M1 A1

$$\text{Perpendicular component} = \mathbf{v} - \frac{22}{9}\mathbf{i} + \frac{44}{9}\mathbf{j} + \frac{44}{9}\mathbf{k}$$

M1

$$= \frac{14}{9}\mathbf{i} - \frac{71}{9}\mathbf{j} + \frac{64}{9}\mathbf{k}$$

A1

Answer: Parallel $\frac{22}{9}\mathbf{i} + \frac{44}{9}\mathbf{j} + \frac{44}{9}\mathbf{k}$, perpendicular $\frac{14}{9}\mathbf{i} - \frac{71}{9}\mathbf{j} + \frac{64}{9}\mathbf{k}$

6 marks

18 (6 marks)

(a) Parallelogram: $\mathbf{b} = \mathbf{a} + \mathbf{c}$

B1

Rhombus: all sides are equal, so $|\mathbf{a}| = |\mathbf{c}|$

B1

Answer: $\mathbf{b} = \mathbf{a} + \mathbf{c}$, $|\mathbf{a}| = |\mathbf{c}|$

2 marks

(b) $\mathbf{OB} = \mathbf{a} + \mathbf{c}$ and $\mathbf{AC} = \mathbf{c} - \mathbf{a}$

M1

$$\mathbf{OB} \cdot \mathbf{AC} = |\mathbf{c}|^2 - |\mathbf{a}|^2$$

M1 A1

This is 0 because $|\mathbf{a}| = |\mathbf{c}|$, so the diagonals are perpendicular

A1

Answer: Proved: the diagonals $\mathbf{OB}$ and $\mathbf{AC}$ are perpendicular

4 marks

19 (6 marks)

$$\mathbf{AB} = 6\mathbf{i} + 4\mathbf{j} + p\mathbf{k}$$

B1

$$\text{Work done} = \mathbf{F} \cdot \mathbf{AB} = 36 - 8 + 3p = 40$$

M1 A1

$$p = 4$$

A1

$$|\mathbf{AB}| = \sqrt{36 + 16 + 16} = 2\sqrt{17}$$

M1

$$\text{Component} = \frac{40}{2\sqrt{17}} = \frac{20}{\sqrt{17}} = 4.85 \text{ N}$$

A1

Answer: $p = 4$, component 4.85 N

6 marks

20 (6 marks)

$$\mathbf{d} = \mathbf{a} + \mathbf{BC} = 4\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}, \text{ so D is } (4, -3, 3)$$

M1 A1

$$\mathbf{AD} = \mathbf{BC}, \text{ so ABCD is a parallelogram}$$

M1

$$\mathbf{AB} \cdot \mathbf{BC} = 2 \times 2 + 2 \times (-3) + 1 \times 2 = 0, \text{ so angle } \angle ABC = 90^\circ \text{ and it is a rectangle}$$

M1 A1

$$\text{Area} = |\mathbf{AB}| \times |\mathbf{BC}| = 3 \times \sqrt{17} = 3\sqrt{17}$$

A1

Answer: D(4, -3, 3), rectangle, area $3\sqrt{17}$

6 marks

21 (8 marks)

$$(a) \mathbf{a} \cdot \mathbf{b} = 2p + 1 \text{ and } |\mathbf{a}|^2 = |\mathbf{b}|^2 = p^2 + 2$$

M1 A1

$$\cos 60^\circ = \frac{2p + 1}{p^2 + 2} = \frac{1}{2}$$

M1

$$4p + 2 = p^2 + 2, \text{ so } p^2 - 4p = 0$$

A1

$$p = 0 \text{ or } p = 4$$

A1

Answer: $p = 0$ or $p = 4$

5 marks

$$(b) \mathbf{a} + \mathbf{b} = 5\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}; |\mathbf{a}| = |\mathbf{b}| = \sqrt{18}$$

B1

$$(\mathbf{a} + \mathbf{b}) \cdot \mathbf{a} = 18 + 9 = 27 \text{ and } (\mathbf{a} + \mathbf{b}) \cdot \mathbf{b} = 27$$

M1

Equal scalar products and equal magnitudes, so the cosines are equal

A1

Answer: Shown: $\mathbf{a} + \mathbf{b}$ makes equal angles with $\mathbf{a}$ and with $\mathbf{b}$

3 marks

22 (8 marks)

$$(a) \mathbf{a} \cdot \mathbf{b} = 2$$

B1

$$\text{Try } \mathbf{c} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}; \mathbf{a} \cdot \mathbf{c} = 2 + 2 - 2 = 2 = \mathbf{a} \cdot \mathbf{b}, \text{ but } \mathbf{c} \neq \mathbf{b}$$

M1 A1

Answer: $\mathbf{c} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ works

3 marks

$$(b) \mathbf{a} \cdot (\mathbf{b} - \mathbf{c}) = 0$$

M1

so $\mathbf{b} - \mathbf{c}$ is perpendicular to $\mathbf{a}$, or is the zero vector

E1

Answer: $\mathbf{b} - \mathbf{c}$ is perpendicular to $\mathbf{a}$ (or zero)

2 marks

$$(c) \mathbf{b} - \mathbf{c} = \lambda \mathbf{a} \text{ for some } \lambda$$

M1

$$\mathbf{a} \cdot (\mathbf{b} - \mathbf{c}) = \lambda |\mathbf{a}|^2 = 0 \text{ and } |\mathbf{a}| \neq 0, \text{ so } \lambda = 0$$

M1 A1

Answer: Proved: if $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$, $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{b} - \mathbf{c}$ is parallel to $\mathbf{a}$, then $\mathbf{b} = \mathbf{c}$

3 marks

23 (9 marks)

$$(a) |\mathbf{a} + \mathbf{b}|^2 = |\mathbf{a}|^2 + 2\mathbf{a} \cdot \mathbf{b} + |\mathbf{b}|^2$$

M1

$$|\mathbf{a} - \mathbf{b}|^2 = |\mathbf{a}|^2 - 2\mathbf{a} \cdot \mathbf{b} + |\mathbf{b}|^2$$

M1

Adding gives the result

A1

$$\text{Answer: Proved: } |\mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2 = 2|\mathbf{a}|^2 + 2|\mathbf{b}|^2$$

3 marks

$$(b) 2 \times 25 + 2 \times 49 = 64 + d^2$$

M1 A1

$$d^2 = 84, \text{ so } d = 2\sqrt{21}$$

A1

Answer: $2\sqrt{21}$

3 marks

$$(c) 64 = 25 + 49 - 2\mathbf{a} \cdot \mathbf{b}, \text{ so } \mathbf{a} \cdot \mathbf{b} = 5$$

M1

$$\cos \theta = \frac{5}{5 \times 7} = \frac{1}{7}$$

M1

$$\theta = 81.8^\circ$$

A1

Answer: 81.8°

3 marks

24 (9 marks)

(a) $u = (\cos 60^\circ, \cos 60^\circ, w)$ and $|u| = 1$ M1

$$\frac{1}{4} + \frac{1}{4} + w^2 = 1$$
 M1 A1

$$w^2 = \frac{1}{2}$$
 A1

Answer: Shown: $u = \frac{1}{2}i + \frac{1}{2}j + wk$, where $w^2 = \frac{1}{2}$ 4 marks

(b) $\cos \gamma = \pm \frac{1}{\sqrt{2}}$ M1

$\gamma = 45^\circ$ or 135° A1

Answer: 45° or 135° 2 marks

(c) $u_1 \cdot u_2 = \frac{1}{4} + \frac{1}{4} - \frac{1}{2} = 0$ M1 A1

The angle is 90° A1

Answer: 90° 3 marks

25 (9 marks)

(a) $BD = tBC$ so $d = (2t + 2)i + (2t)j + (t + 1)k$ M1

$AD = (2t - 1)i + (2t - 1)j + (t - 2)k$ A1

$AD \cdot BC = 0: 2(2t - 1) + 2(2t - 1) + (t - 2) = 0$, so $9t = 6$ and $t = \frac{2}{3}$ M1 A1

D is $(\frac{10}{3}, \frac{4}{3}, \frac{5}{3})$ A1

Answer: $(\frac{10}{3}, \frac{4}{3}, \frac{5}{3})$ 5 marks

(b) $AD^2 = 2 = 2$, so $AD = \sqrt{2}$ M1 A1

$\text{Area} = \frac{1}{2} \times |BC| \times |AD| = \frac{1}{2} \times 3 \times \sqrt{2} = \frac{3\sqrt{2}}{2}$ M1 A1

Answer: $\frac{3\sqrt{2}}{2}$ 4 marks

26 (12 marks)

(a) $OA \cdot OB = 2t(t + 2) + 3(-t - 1) + 3(t - 1)$ M1 A1

$= 2t^2 + 4t - 6 = 2(t - 1)(t + 3)$ A1

$t = 1$ or $t = -3$ A1

Answer: $t = 1$ or $t = -3$ 4 marks

(b) $OA = 4i + 3j + k$, $OB = 4i - 3j + 3k$ B1

$OA \cdot OB = 16 - 9 + 3 = 10$, $|OA| = \sqrt{26}$, $|OB| = \sqrt{34}$ M1 A1

$\cos AOB = \frac{10}{\sqrt{26}\sqrt{34}}$, so $AOB = 70.3^\circ$ M1 A1

Answer: 70.3° 5 marks

(c) $OA = 2i + 3j$ and $OB = 3i - 2j + 3k$ are perpendicular by (a) M1

$\text{Area} = \frac{1}{2} \times \sqrt{13} \times \sqrt{22}$ M1

$= \frac{\sqrt{286}}{2}$ A1

Answer: $\frac{\sqrt{286}}{2}$ 3 marks

27 (10 marks)

(a) $|a + \lambda b|^2 = |b|^2 \lambda^2 + 2(a \cdot b)\lambda + |a|^2 \geq 0$ for all λ M1 A1

$|b|^2 > 0$ so this quadratic in λ has no distinct real roots M1

Discriminant $\leq 0: 4(a \cdot b)^2 - 4|a|^2|b|^2 \leq 0$ M1

so $(a \cdot b)^2 \leq |a|^2|b|^2$ A1

Answer: Proved: $(a \cdot b)^2 \leq |a|^2|b|^2$ 5 marks

(b) Equality holds when the discriminant is 0, so $|a + \lambda b| = 0$ for some λ , so $a = -\lambda b$: a and b are parallel E1

Answer: When a and b are parallel**1 mark****(c)** Take $\mathbf{b} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$: $(x + 2y + 3z)^2 \leq 14(x^2 + y^2 + z^2)$ **M1** $784 \leq 14(x^2 + y^2 + z^2)$, so $x^2 + y^2 + z^2 \geq 56$ **A1**Equality needs $(x, y, z) = \lambda(1, 2, 3)$: $14\lambda = 28$, so $\lambda = 2$ **M1** $x = 2, y = 4, z = 6$ (check: $2 + 8 + 18 = 28$ and $4 + 16 + 36 = 56$)**A1****Answer: $x = 2, y = 4, z = 6$** **4 marks****28 (9 marks)****(a)** $\mathbf{AP} \cdot \mathbf{BP} = (x - 1)(x - 5) + (y - 2)(y + 2) + (z - 3)(z - 7) = 0$ **M1 A1** $x^2 - 6x + 5 + y^2 - 4 + z^2 - 10z + 21 = 0$ **M1** $(x - 3)^2 - 9 + y^2 + (z - 5)^2 - 25 + 22 = 0$, so $(x - 3)^2 + y^2 + (z - 5)^2 = 12$ **A1****Answer: Shown: P lies on a sphere with equation $(x - 3)^2 + y^2 + (z - 5)^2 = 12$** **4 marks****(b)** Centre $(3, 0, 5) = \frac{1}{2}(\mathbf{A} + \mathbf{B})$ **B1**Radius $\sqrt{12} = 2\sqrt{3}$ and $AB = \sqrt{16 + 16 + 16} = 4\sqrt{3} = 2 \times \text{radius}$ **M1 A1****Answer: Centre $(3, 0, 5)$; $AB = 2 \times \text{radius}$** **3 marks****(c)** $\mathbf{QA} = -4\mathbf{i}$ and $\mathbf{QB} = -4\mathbf{j} + 4\mathbf{k}$ **M1** $\mathbf{QA} \cdot \mathbf{QB} = 0$, so angle $\text{AQB} = 90^\circ$ **A1****Answer: Angle $\text{AQB} = 90^\circ$** **2 marks****29 (9 marks)****(a)** The six edge vectors are $-2\mathbf{j} - 2\mathbf{k}, -2\mathbf{i} - 2\mathbf{k}, -2\mathbf{i} - 2\mathbf{j}, -2\mathbf{i} + 2\mathbf{j}, -2\mathbf{i} + 2\mathbf{k}, -2\mathbf{j} + 2\mathbf{k}$ **M1 A1**Each has two non-zero components of size 2, so each has length $\sqrt{8} = 2\sqrt{2}$ **A1****Answer: All edges $2\sqrt{2}$** **3 marks****(b)** $\mathbf{OP}_1 \cdot \mathbf{OP}_2 = 1 - 1 - 1 = -1$ and $|\mathbf{OP}_1| = |\mathbf{OP}_2| = \sqrt{3}$ **M1** $\cos \theta = -\frac{1}{3}$, so $\theta = 109.5^\circ$ **A1****Answer: 109.5°** **2 marks****(c)** $\mathbf{P}_1\mathbf{P}_2 \cdot \mathbf{P}_1\mathbf{P}_3 = 0 + 0 + 4 = 4$ **M1** $\cos \varphi = \frac{4}{8} = \frac{1}{2}$, so $\varphi = 60^\circ$ **A1****Answer: 60°** **2 marks****(d)** Sum of components: $x: 1 + 1 - 1 - 1 = 0, y: 1 - 1 + 1 - 1 = 0, z: 1 - 1 - 1 + 1 = 0$ **M1**

O is the centroid of the tetrahedron

E1**Answer: O is the centroid****2 marks****30 (10 marks)****(a)** $|\mathbf{a}| = |\mathbf{c}| = 3$ **M1 A1** $\mathbf{a} \cdot \mathbf{c} = 2 + 2 - 4 = 0$ **A1****Answer: Both have magnitude 3; $\mathbf{a} \cdot \mathbf{c} = 0$** **3 marks****(b)** Equal angles and $|\mathbf{a}| = |\mathbf{c}|$: $\mathbf{b} \cdot \mathbf{a} = \mathbf{b} \cdot \mathbf{c}$ **M1** $\mathbf{p} + 2\mathbf{q} = 2\mathbf{p} + \mathbf{q}$, so $\mathbf{p} = \mathbf{q}$ **A1** $|\mathbf{b}| = \sqrt{2}\mathbf{p} = 15$ **M1** $\mathbf{p} = \mathbf{q} = \frac{15}{\sqrt{2}} = \frac{15\sqrt{2}}{2}$ **A1****Answer: $\mathbf{p} = \mathbf{q} = 15\sqrt{2}/2$** **4 marks****(c)** $\cos \theta = \frac{\mathbf{b} \cdot \mathbf{a}}{|\mathbf{b}||\mathbf{a}|} = \frac{3\mathbf{p}}{15 \times 3} = \frac{\mathbf{p}}{15} = \frac{1}{\sqrt{2}}$ **M1 A1** $\theta = 45^\circ$, which is half of the 90° between a and c, so b bisects the angle between them**A1****Answer: 45°** **3 marks**

Vector Product

1 (2 marks)

$$a \times b = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 0 \\ 1 & -1 & 3 \end{vmatrix}$$

M1

$$= 3\mathbf{i} - 6\mathbf{j} - 3\mathbf{k}$$

A1

$$\text{Answer: } 3\mathbf{i} - 6\mathbf{j} - 3\mathbf{k}$$

2 marks

2 (2 marks)

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \mathbf{j} \times \mathbf{k} = \mathbf{i}, \mathbf{k} \times \mathbf{i} = \mathbf{j}$$

M1

$$\text{Sum} = \mathbf{i} + \mathbf{j} + \mathbf{k}$$

A1

$$\text{Answer: } \mathbf{i} + \mathbf{j} + \mathbf{k}$$

2 marks

3 (3 marks)

$$(2\mathbf{a} + \mathbf{b}) \times (\mathbf{a} - 2\mathbf{b}) = 2\mathbf{a} \times \mathbf{a} - 4\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{a} - 2\mathbf{b} \times \mathbf{b}$$

M1

$$\mathbf{a} \times \mathbf{a} = \mathbf{b} \times \mathbf{b} = \mathbf{0} \text{ and } \mathbf{b} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{b})$$

M1

$$= -4(\mathbf{a} \times \mathbf{b}) - (\mathbf{a} \times \mathbf{b}) = -5(\mathbf{a} \times \mathbf{b})$$

A1

$$\text{Answer: } -5(\mathbf{a} \times \mathbf{b})$$

3 marks

4 (2 marks)

$$\mathbf{a} \times \mathbf{b} = -2\mathbf{i} + \mathbf{j} + 6\mathbf{k}$$

M1

$$\text{Area} = |\mathbf{a} \times \mathbf{b}| = \sqrt{(-2)^2 + 1^2 + 6^2} = \sqrt{41}$$

A1

$$\text{Answer: } \sqrt{41}$$

2 marks

5 (2 marks)

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin 30^\circ$$

M1

$$= 4 \times 5 \times \frac{1}{2} = 10$$

A1

$$\text{Answer: } 10$$

2 marks

6 (2 marks)

$$\mathbf{OA} \times \mathbf{OB} = -3\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$$

M1

$$\text{Area} = \frac{1}{2}|\mathbf{OA} \times \mathbf{OB}| = \frac{1}{2}\sqrt{(-3)^2 + (-2)^2 + 6^2} = \frac{7}{2}$$

A1

$$\text{Answer: } \frac{7}{2}$$

2 marks

7 (4 marks)

$$\mathbf{a} \times \mathbf{b} = 4\mathbf{i} - 5\mathbf{j} - 6\mathbf{k}$$

M1 A1

$$|\mathbf{a} \times \mathbf{b}| = \sqrt{4^2 + (-5)^2 + (-6)^2} = \sqrt{77}$$

M1

$$\text{Unit vector} = \pm \frac{1}{\sqrt{77}}(4\mathbf{i} - 5\mathbf{j} - 6\mathbf{k})$$

A1

$$\text{Answer: } \pm \frac{1}{\sqrt{77}}(4\mathbf{i} - 5\mathbf{j} - 6\mathbf{k})$$

4 marks

8 (5 marks)

$$\mathbf{b} \times \mathbf{c} = (1 - 3p)\mathbf{i} + 11\mathbf{j} + (2p + 3)\mathbf{k}$$

M1 A1

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (1 - 3p) + 22 + (2p + 3) = 26 - p$$

M1 A1

$$\text{Volume} = |26 - p| = 14, \text{ so } p = 12 \text{ or } p = 40$$

A1

$$\text{Answer: } p = 12 \text{ or } p = 40$$

5 marks

9 (4 marks)

$$\mathbf{a} \times \mathbf{b} = -\mathbf{i} - \mathbf{j} + \mathbf{k} \text{ so } |\mathbf{a} \times \mathbf{b}| = \sqrt{3}$$

M1 A1

$$|\mathbf{a}| = |\mathbf{b}| = \sqrt{2}, \text{ so } \sin \theta = \frac{\sqrt{3}}{2}$$

M1

$$\theta = 60^\circ$$

A1

Answer: 60°

4 marks

10 (4 marks)

$$\mathbf{AB} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}, \mathbf{AC} = -\mathbf{i} + \mathbf{j} + 3\mathbf{k}$$

B1

 $\mathbf{AB} \times \mathbf{AC}$ is perpendicular to both

M1

$$= 5\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}$$

A1 A1

Answer: $5\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}$

4 marks

11 (4 marks)

$$\mathbf{b} \times \mathbf{c} = 7\mathbf{i} + 8\mathbf{j} - 2\mathbf{k}$$

M1 A1

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 1 \times 7 + 2 \times 8 + 3 \times (-2) = 17$$

A1

$$\text{Volume} = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| = 17$$

A1

Answer: $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 17$; volume 17

4 marks

12 (6 marks)**(a)** Coplanar means $\mathbf{p} \cdot (\mathbf{q} \times \mathbf{r}) = 0$

M1

$$\mathbf{q} \times \mathbf{r} = (-\lambda - 4)\mathbf{i} + (2 - \lambda)\mathbf{j} + 3\mathbf{k}$$

M1 A1

$$\mathbf{p} \cdot (\mathbf{q} \times \mathbf{r}) = 2(-\lambda - 4) + (2 - \lambda) + 9 = 3 - 3\lambda = 0, \text{ so } \lambda = 1$$

A1

Answer: $\lambda = 1$

4 marks

(b) $\mathbf{p} = \alpha\mathbf{q} + \beta\mathbf{r}$: i: $\alpha + \beta = 2$, j: $-\alpha + 2\beta = 1$

M1

$$\beta = 1, \alpha = 1 \text{ (k: } 2 + 1 = 3 \text{ checks)}$$

A1

Answer: $\mathbf{p} = \mathbf{q} + \mathbf{r}$

2 marks

13 (4 marks)

$$\mathbf{b} \times \mathbf{c} = \mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$$

M1

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = 4\mathbf{i} - 4\mathbf{j} - 2\mathbf{k}$$

M1 A1

$$\mathbf{a} \cdot \mathbf{c} = 2 \text{ and } \mathbf{a} \cdot \mathbf{b} = 2: 2(2\mathbf{i} + \mathbf{j}) - 2(3\mathbf{j} + \mathbf{k}) = 4\mathbf{i} - 4\mathbf{j} - 2\mathbf{k}$$

A1

Answer: $4\mathbf{i} - 4\mathbf{j} - 2\mathbf{k}$

4 marks

14 (5 marks)

$$\text{Volume} = \frac{1}{6} |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$

B1

$$\mathbf{b} \times \mathbf{c} = 11\mathbf{i} + 2\mathbf{j} - 6\mathbf{k}$$

M1 A1

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 1 \times 11 + 2 \times 2 + 0 \times (-6) = 15$$

A1

$$\text{Volume} = \frac{15}{6} = \frac{5}{2}$$

A1

Answer: $\frac{5}{2}$

5 marks

15 (7 marks)**(a)** $\mathbf{AB} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}, \mathbf{AC} = -\mathbf{i} + 2\mathbf{j} + \mathbf{k}, \mathbf{AD} = 3\mathbf{i} + 3\mathbf{k}$

B1

$$\mathbf{AC} \times \mathbf{AD} = 6\mathbf{i} + 6\mathbf{j} - 6\mathbf{k}$$

M1 A1

$$\mathbf{AB} \cdot (\mathbf{AC} \times \mathbf{AD}) = 0 = 0, \text{ so the points are coplanar}$$

A1

Answer: Scalar triple product = 0

4 marks

(b) Equate i and j components: $\alpha - \beta = 3$ and $\alpha + 2\beta = 0$

M1 A1

$$\alpha = 2, \beta = -1 \text{ (k: } 4 - 1 = 3 \text{ checks)}$$

A1

Answer: $\alpha = 2, \beta = -1$

3 marks

16 (6 marks)

$$\text{(a) } |\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2 |\mathbf{b}|^2 \sin^2 \theta$$

M1

$$= |\mathbf{a}|^2 |\mathbf{b}|^2 (1 - \cos^2 \theta)$$

M1

$$= |\mathbf{a}|^2 |\mathbf{b}|^2 - (|\mathbf{a}| |\mathbf{b}| \cos \theta)^2 = |\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2$$

A1

Answer: Proved: $|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2$

3 marks

(b) $\mathbf{a} \times \mathbf{b} = 5\mathbf{i} + 5\mathbf{j} - 5\mathbf{k}$ so $|\mathbf{a} \times \mathbf{b}|^2 = 75$

M1 A1

$|\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2 = 14 \times 6 - 3^2 = 75$

A1

Answer: Both sides equal 75

3 marks

17 (5 marks)

$\mathbf{OA} \times \mathbf{OB} = 4\mathbf{i} - 2\mathbf{j} + (1 - 2k)\mathbf{k}$

M1 A1

Area = $\frac{1}{2}|\mathbf{OA} \times \mathbf{OB}| = 3$, so $|\mathbf{OA} \times \mathbf{OB}|^2 = 36$

M1

$16 + 4 + (1 - 2k)^2 = 36$, so $(1 - 2k)^2 = 16$

M1

$1 - 2k = \pm 4$, so $k = -\frac{3}{2}$ or $k = \frac{5}{2}$

A1

Answer: $k = -3/2$ or $k = 5/2$

5 marks

18 (5 marks)

$$\mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ 2 & -3 & 4 \end{vmatrix}$$

M1

$= 5\mathbf{i} - 6\mathbf{j} - 7\mathbf{k} \text{ N m}$

M1 A1

Magnitude = $\sqrt{5^2 + (-6)^2 + (-7)^2} = \sqrt{110} = 10.5 \text{ N m}$

M1 A1

Answer: $5\mathbf{i} - 6\mathbf{j} - 7\mathbf{k} \text{ N m}$, magnitude 10.5 N m

5 marks

19 (5 marks)

(a) Expand: $\mathbf{a} \times \mathbf{a} + \mathbf{a} \times \mathbf{b} - \mathbf{b} \times \mathbf{a} - \mathbf{b} \times \mathbf{b}$

M1

$\mathbf{a} \times \mathbf{a} = \mathbf{b} \times \mathbf{b} = 0$ and $\mathbf{b} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{b})$

M1

So the result is $2(\mathbf{a} \times \mathbf{b})$

A1

Answer: Proved: $(\mathbf{a} - \mathbf{b}) \times (\mathbf{a} + \mathbf{b}) = 2(\mathbf{a} \times \mathbf{b})$

3 marks

(b) $\mathbf{a} - \mathbf{b}$ and $\mathbf{a} + \mathbf{b}$ are the two diagonals of the parallelogram with sides $\mathbf{a}$ and $\mathbf{b}$

E1

The parallelogram whose sides are these diagonals has area $2|\mathbf{a} \times \mathbf{b}|$, twice that of the original

E1

Answer: Area doubles

2 marks

20 (5 marks)

(a) $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ but $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$

M1 A1

Answer: $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$

2 marks

(b) $\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = \mathbf{i} \times \mathbf{k} = -\mathbf{j}$

M1 A1

$(\mathbf{i} \times \mathbf{i}) \times \mathbf{j} = 0 \neq -\mathbf{j}$

A1

Answer: $-\mathbf{j}$ and 0 differ

3 marks

21 (9 marks)

(a) $\mathbf{a} \times \mathbf{v}$ is perpendicular to $\mathbf{a}$, so $\mathbf{b}$ must be perpendicular to $\mathbf{a}$, i.e. $\mathbf{a} \cdot \mathbf{b} = 0$

E1

$\mathbf{a} \cdot \mathbf{b} = 0 - 2 + 2 = 0$

B1

Answer: $\mathbf{a} \cdot \mathbf{b} = 0$ holds

2 marks

$$(b) \mathbf{a} \times \mathbf{v} = \begin{pmatrix} 2z + y \\ -x - z \\ y - 2x \end{pmatrix}$$

M1 A1

$2z + y = 0$, $-x - z = -1$, $y - 2x = -2$

M1

$y = -2z$ and $x = 1 - z$ (third equation is then satisfied)

A1

$\mathbf{v} = \mathbf{i} + t(-\mathbf{i} - 2\mathbf{j} + \mathbf{k})$ with $z = t$

A1

Answer: $\mathbf{v} = \mathbf{i} + t(-\mathbf{i} - 2\mathbf{j} + \mathbf{k})$

5 marks

(c) $|\mathbf{v}|^2 = (1 - t)^2 + 4t^2 + t^2 = 6t^2 - 2t + 1$

M1

Minimum at $t = \frac{1}{6}$, giving $\mathbf{v} = \frac{5}{6}\mathbf{i} - \frac{1}{3}\mathbf{j} + \frac{1}{6}\mathbf{k}$

A1

Answer: $\frac{5}{6}\mathbf{i} - \frac{1}{3}\mathbf{j} + \frac{1}{6}\mathbf{k}$

2 marks

22 (8 marks)

(a) $\mathbf{b} \times \mathbf{c} = \mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$

M1 A1

$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| = |13| = 13$

M1 A1

Answer: 13

4 marks

(b) $\mathbf{a} \times \mathbf{b} = 6\mathbf{i} - 3\mathbf{j} + \mathbf{k}$

M1

Area = $\sqrt{6^2 + (-3)^2 + 1^2} = \sqrt{46}$

A1

Answer: $\sqrt{46}$

2 marks

(c) Volume = base area $\times$ height

M1

$h = \frac{13}{\sqrt{46}} = 1.92$

A1

Answer: 1.92

2 marks

23 (8 marks)

(a) $\mathbf{PQ} = -\mathbf{i} + 2\mathbf{j}$, $\mathbf{PR} = -\mathbf{i} + 3\mathbf{k}$

B1

$\mathbf{PQ} \times \mathbf{PR} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$

M1 A1

Answer: $6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$

3 marks

(b) Area = $\frac{1}{2}\sqrt{6^2 + 3^2 + 2^2} = \frac{1}{2} \times 7 = \frac{7}{2}$

M1 A1

Answer: $\frac{7}{2}$

2 marks

(c) $\mathbf{QR} = -2\mathbf{j} + 3\mathbf{k}$, $|\mathbf{QR}| = \sqrt{13}$

B1

Area = $\frac{1}{2} \times |\mathbf{QR}| \times h$, so $h = \frac{7}{\sqrt{13}} = 1.94$

M1 A1

Answer: 1.94

3 marks

24 (6 marks)

(a) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$

B1

$\mathbf{b} \times (\mathbf{c} \times \mathbf{a}) = \mathbf{c}(\mathbf{b} \cdot \mathbf{a}) - \mathbf{a}(\mathbf{b} \cdot \mathbf{c})$

M1

$\mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}(\mathbf{c} \cdot \mathbf{b}) - \mathbf{b}(\mathbf{c} \cdot \mathbf{a})$

A1

Answer: $\mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$; $\mathbf{c}(\mathbf{b} \cdot \mathbf{a}) - \mathbf{a}(\mathbf{b} \cdot \mathbf{c})$; $\mathbf{a}(\mathbf{c} \cdot \mathbf{b}) - \mathbf{b}(\mathbf{c} \cdot \mathbf{a})$

3 marks

(b) Add: terms in $\mathbf{a}$: $-\mathbf{a}(\mathbf{b} \cdot \mathbf{c}) + \mathbf{a}(\mathbf{c} \cdot \mathbf{b}) = 0$

M1

terms in $\mathbf{b}$: $\mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{b}(\mathbf{c} \cdot \mathbf{a}) = 0$, terms in $\mathbf{c}$: $-\mathbf{c}(\mathbf{a} \cdot \mathbf{b}) + \mathbf{c}(\mathbf{b} \cdot \mathbf{a}) = 0$

M1

using $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ etc. the sum is 0

A1

Answer: Proved: $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}$

3 marks

25 (9 marks)

(a) $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} = -3\mathbf{i} + 6\mathbf{j}$

M1 A1

Speed = $\sqrt{9 + 36} = 3\sqrt{5} = 6.71 \text{ m s}^{-1}$

A1

Answer: $-3\mathbf{i} + 6\mathbf{j}$, speed 6.71 m s^{-1}

3 marks

(b) $\mathbf{a} = \boldsymbol{\omega} \times \mathbf{v} = -18\mathbf{i} - 9\mathbf{j}$

M1 A1

Answer: $-18\mathbf{i} - 9\mathbf{j}$

2 marks

(c) $\mathbf{a} = -9(2\mathbf{i} + \mathbf{j})$ and $2\mathbf{i} + \mathbf{j}$ is the horizontal position of P, so a points directly towards the axis

M1 A1

Distance from axis $d = \sqrt{5}$, so $|\mathbf{a}| = 9\sqrt{5} = \omega^2 d$

M1 A1

Answer: $\mathbf{a} = -9(2\mathbf{i} + \mathbf{j})$, $|\mathbf{a}| = \omega^2 d = 9\sqrt{5}$

4 marks

26 (9 marks)

(a) Area = $\frac{1}{2}|\mathbf{AB} \times \mathbf{AC}| + \frac{1}{2}|\mathbf{AC} \times \mathbf{AD}|$; the two area vectors point the same way

M1 A1

Vector area = $\frac{1}{2}(\mathbf{AB} \times \mathbf{AC} + \mathbf{AC} \times \mathbf{AD}) = \frac{1}{2}\mathbf{AC} \times (\mathbf{AD} - \mathbf{AB})$

M1 A1

$\mathbf{AD} - \mathbf{AB} = \mathbf{BD}$, giving $\frac{1}{2}|\mathbf{AC} \times \mathbf{BD}|$

A1

Answer: Shown: its area is $\frac{1}{2}|\mathbf{AC} \times \mathbf{BD}|$

5 marks

(b) $\mathbf{AC} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$, $\mathbf{BD} = -\mathbf{i} + 2\mathbf{j} + \mathbf{k}$

B1

$\mathbf{AC} \times \mathbf{BD} = -6\mathbf{i} - 8\mathbf{j} + 10\mathbf{k}$

M1 A1

$$\text{Area} = \frac{1}{2}\sqrt{(-6)^2 + (-8)^2 + 10^2} = 5\sqrt{2}$$

A1

Answer: $5\sqrt{2}$

4 marks

27 (11 marks)**(a)** Let $u = a \times b$. Then $u \times (a \times c) = a(u \cdot c) - c(u \cdot a)$

M1

$$u \cdot a = (a \times b) \cdot a = 0$$

M1

$$u \cdot c = (a \times b) \cdot c = a \cdot (b \times c)$$

M1 A1

$$\text{So } (a \times b) \times (a \times c) = [a \cdot (b \times c)] a$$

A1

Answer: Proved: $(a \times b) \times (a \times c) = [a \cdot (b \times c)] a$

5 marks

(b) $a \times b = 2i - j + k$ and $a \times c = 4i - 2j - 2k$

M1 A1

$$(a \times b) \times (a \times c) = 4i + 8j$$

M1

$$a \cdot (b \times c) = 4 \text{ so } 4a = 4i + 8j$$

A1

Answer: Both sides equal $4i + 8j$

4 marks

(c) Then $(a \times b) \times (a \times c) = 0$

M1

A zero vector product means the two vectors are parallel (or one is zero)

A1

Answer: Shown: if $a \cdot (b \times c) = 0$ (with $a \neq 0$) then $a \times b$ and $a \times c$ are parallel

2 marks

28 (10 marks)**(a)** $b \times c = 6i - 2k$

M1 A1

$$V = a \cdot (b \times c) = 4$$

A1

Answer: 4

3 marks

$$\text{(b) } a' = \frac{1}{4}(6i - 2k) = \frac{3}{2}i - \frac{1}{2}k$$

M1 A1

$$b' = \frac{1}{4}(i + 2j - k) = \frac{1}{4}i + \frac{1}{2}j - \frac{1}{4}k$$

A1

$$c' = \frac{1}{4}(-2i + 2k) = -\frac{1}{2}i + \frac{1}{2}k$$

A1

$$\text{Answer: } a' = \frac{3}{2}i - \frac{1}{2}k, b' = \frac{1}{4}i + \frac{1}{2}j - \frac{1}{4}k, c' = -\frac{1}{2}i + \frac{1}{2}k$$

4 marks

$$\text{(c) } a \cdot a' = \frac{3}{2} - \frac{1}{2} = 1, b \cdot b' = 2 \times \frac{1}{2} = 1, c \cdot c' = -\frac{1}{2} + \frac{3}{2} = 1$$

M1 A1

$$a \cdot b' = \frac{1}{4} - \frac{1}{4} = 0 \text{ and } a \cdot c' = -\frac{1}{2} + \frac{1}{2} = 0$$

A1

Answer: $a \cdot a' = b \cdot b' = c \cdot c' = 1$ and $a \cdot b' = a \cdot c' = 0$ (verified)

3 marks

29 (10 marks)**(a)** $M = \mathbf{OP} \times \mathbf{F} = -8i + 15j - 6k$

M1 A1

$$|M| = \sqrt{64 + 225 + 36} = \sqrt{325} = 5\sqrt{13}$$

A1

Answer: $-8i + 15j - 6k$, $|M| = 5\sqrt{13}$

3 marks

(b) $|M| = |F| \times d$

M1

$$|F| = 5, \text{ so } d = \sqrt{13} = 3.61 \text{ m}$$

A1

Answer: $\sqrt{13} \text{ m}$

2 marks

(c) Moment about axis = $M \cdot n$

M1

$$= \frac{1}{3}(-8 + 30 - 12) = \frac{10}{3} \text{ N m}$$

M1 A1

Answer: $\frac{10}{3} \text{ N m}$

3 marks

(d) $\mathbf{OQ} \times \mathbf{F} = \mathbf{OP} \times \mathbf{F} + \lambda \mathbf{F} \times \mathbf{F}$

M1

 $\mathbf{F} \times \mathbf{F} = 0$, so the moment is the same

A1

Answer: $\mathbf{OQ} \times \mathbf{F} = \mathbf{OP} \times \mathbf{F}$, so the moment is unchanged

2 marks

30 (10 marks)

(a) Vector area = $\frac{1}{2}\mathbf{AB} \times \mathbf{AC} = \frac{1}{2}(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})$ M1

Expand: $\frac{1}{2}(\mathbf{b} \times \mathbf{c} - \mathbf{b} \times \mathbf{a} - \mathbf{a} \times \mathbf{c} + \mathbf{a} \times \mathbf{a})$ M1

= $\frac{1}{2}(\mathbf{b} \times \mathbf{c} + \mathbf{a} \times \mathbf{b} + \mathbf{c} \times \mathbf{a})$, as $\mathbf{a} \times \mathbf{a} = 0$, $-\mathbf{b} \times \mathbf{a} = \mathbf{a} \times \mathbf{b}$, $-\mathbf{a} \times \mathbf{c} = \mathbf{c} \times \mathbf{a}$ A1

Answer: Shown: the vector area of triangle ABC is $\frac{1}{2}(\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a})$ 3 marks

(b) $\mathbf{a} \times \mathbf{b} = 6\mathbf{k}$, $\mathbf{b} \times \mathbf{c} = 18\mathbf{i}$, $\mathbf{c} \times \mathbf{a} = 12\mathbf{j}$ M1 A1

Vector area = $\frac{1}{2}18\mathbf{i} + 12\mathbf{j} + 6\mathbf{k} = 9\mathbf{i} + 6\mathbf{j} + 3\mathbf{k}$ M1

Area = $\sqrt{81 + 36 + 9} = \sqrt{126} = 3\sqrt{14}$ A1

Answer: $3\sqrt{14}$ 4 marks

(c) Areas: $\frac{1}{2} \times 6 = 3$, $\frac{1}{2} \times 18 = 9$, $\frac{1}{2} \times 12 = 6$ M1 A1

$9 + 81 + 36 = 126 = (3\sqrt{14})^2$ A1

Answer: 3, 9 and 6; $126 = 126$ 3 marks

Vector Equations of Lines**1 (2 marks)**

$\mathbf{r} = \mathbf{a} + \lambda \mathbf{d}$ M1

$\mathbf{r} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + \lambda(2\mathbf{i} - \mathbf{j} + 4\mathbf{k})$ A1

Answer: $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + \lambda(2\mathbf{i} - \mathbf{j} + 4\mathbf{k})$ 2 marks

2 (2 marks)

Direction $\mathbf{AB} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$ M1

$\mathbf{r} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k})$ A1

Answer: $\mathbf{r} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k})$ 2 marks

3 (2 marks)

$x = 1 + 2\lambda$, $y = 2 - \lambda$, $z = 3 + 4\lambda$ M1

$\frac{x-1}{2} = \frac{y-2}{-1} = \frac{z-3}{4}$ A1

Answer: $\frac{x-1}{2} = \frac{y-2}{-1} = \frac{z-3}{4}$ 2 marks

4 (2 marks)

$x: 1 + 2\lambda = 5$ gives $\lambda = 2$ M1

$y: 2 - 2 = 0$ and $z: 3 + 8 = 11$ both agree, so the point lies on the line A1

Answer: Yes, at $\lambda = 2$ 2 marks

5 (2 marks)

Point $(3, -1, 5)$ B1

Rewrite $\frac{5-z}{1}$ as $\frac{z-5}{-1}$: direction $2\mathbf{i} + 4\mathbf{j} - \mathbf{k}$ B1

Answer: $(3, -1, 5); 2\mathbf{i} + 4\mathbf{j} - \mathbf{k}$ 2 marks

6 (2 marks)

$(2, -4, 6) = -2(-1, 2, -3)$ M1

The direction vectors are parallel, so the lines are parallel A1

Answer: Direction vectors are multiples 2 marks

7 (3 marks)

$$\text{z-coordinate: } -3 + 2\lambda = 0$$

M1

$$\lambda = \frac{3}{2}$$

A1

$$\text{Point: } (2 + \frac{3}{2}, 1 + 3, 0) = (\frac{7}{2}, 4, 0)$$

A1

$$\text{Answer: } (\frac{7}{2}, 4, 0)$$

3 marks

8 (4 marks)

The vector product is zero, so $\mathbf{r} - (2\mathbf{i} + \mathbf{j} - 3\mathbf{k})$ is parallel to $3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$

M1

$$\mathbf{r} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k} + \lambda(3\mathbf{i} - 2\mathbf{j} + \mathbf{k})$$

A1

$$\text{Cartesian: } \frac{x-2}{3} = \frac{y-1}{-2} = \frac{z+3}{1}$$

M1 A1

$$\text{Answer: } \mathbf{r} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k} + \lambda(3\mathbf{i} - 2\mathbf{j} + \mathbf{k}); \frac{x-2}{3} = \frac{y-1}{-2} = \frac{z+3}{1}$$

4 marks

9 (3 marks)

$$x: 1 + 3\lambda = 7, \text{ so } \lambda = 2$$

M1 A1

$$p = -1 + 4 = 3, q = 2 + 2 = 4$$

A1

$$\text{Answer: } p = 3, q = 4$$

3 marks

10 (5 marks)

$$(a) \mathbf{r}(3) = 11\mathbf{i} + 5\mathbf{j} + \mathbf{k}$$

M1 A1

$$\text{Speed} = |\mathbf{v}| = \sqrt{9 + 4 + 1} = \sqrt{14} = 3.74 \text{ m s}^{-1}$$

A1

$$\text{Answer: } 11\mathbf{i} + 5\mathbf{j} + \mathbf{k}; 3.74 \text{ m s}^{-1}$$

3 marks

$$(b) 2 + 3t = 17$$

M1

$$t = 5$$

A1

$$\text{Answer: } t = 5$$

2 marks

11 (5 marks)

$$i: 1 + \lambda = 5 - 2\mu, j: 2 + \lambda = 4\mu$$

M1

$$\text{Subtract: } 1 = 6\mu - 5, \text{ so } \mu = 1 \text{ and } \lambda = 2$$

M1 A1

$$k: 3 + 2(2) = 7 \text{ and } 2 + 5(1) = 7, \text{ so the lines intersect}$$

A1

$$\text{Point: } (3, 4, 7)$$

A1

$$\text{Answer: } (3, 4, 7)$$

5 marks

12 (4 marks)

$$d_1 \cdot d_2 = 2 - 6 + 12 = 8, |d_1| = 3, |d_2| = 7$$

M1 A1

$$\cos \theta = \frac{8}{21}$$

M1

$$\theta = 67.6^\circ$$

A1

$$\text{Answer: } 67.6^\circ$$

4 marks

13 (4 marks)

$$\mathbf{AB} = 3\mathbf{i} - 3\mathbf{j} + 3\mathbf{k} = 3(\mathbf{i} - \mathbf{j} + \mathbf{k})$$

M1

$$\mathbf{r} = 2\mathbf{i} + 5\mathbf{j} - \mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + \mathbf{k})$$

A1

$$\text{At C: } 2 + \lambda = -1, \text{ so } \lambda = -3$$

M1

$$p = 5 + 3 = 8 \text{ and } q = -1 - 3 = -4$$

A1

$$\text{Answer: } \mathbf{r} = 2\mathbf{i} + 5\mathbf{j} - \mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + \mathbf{k}); p = 8, q = -4$$

4 marks

14 (6 marks)

Direction vectors are not parallel

B1

i: $1 + \lambda = 2\mu$, j: $2 - \lambda = 1 + \mu$

M1

Adding: $3 = 1 + 3\mu$, so $\mu = \frac{2}{3}$ and $\lambda = \frac{1}{3}$

M1 A1

k: $3 + 2\lambda = \frac{11}{3}$ but $5 - \mu = \frac{13}{3}$

M1

The equations are inconsistent, so the lines do not intersect and are not parallel: they are skew

A1

Answer: Skew

6 marks

15 (6 marks)

(a) i: $\lambda = 4 + 3\mu$, j: $1 + 2\lambda = 2 - \mu$

M1

Substitute: $1 + 8 + 6\mu = 2 - \mu$, so $\mu = -1$ and $\lambda = 1$

M1 A1

k: $2 - 1 = 1$ and $2 + (-1) = 1$, consistent. Point (1, 3, 1)

A1

Answer: (1, 3, 1)

4 marks

(b) $(1, 2, -1) \cdot (3, -1, 1) = 3 - 2 - 1 = 0$

M1 A1

Answer: Scalar product of directions = 0

2 marks

16 (6 marks)

i: $1 + 2\lambda = \mu$, p: $3 - \lambda = 1 + 3\mu$

M1 A1

$3 - \lambda = 1 + 3 + 6\lambda$, so $\lambda = -\frac{1}{7}$ and $\mu = \frac{5}{7}$

M1 A1

j: $2 + \lambda = \mu$, so $\mu = 2 - \frac{1}{7} + \frac{5}{7} = \frac{18}{7}$

M1

Point: $(1 - \frac{2}{7}, 2 - \frac{1}{7}, 3 + \frac{1}{7}) = (\frac{5}{7}, \frac{13}{7}, \frac{22}{7})$

A1

Answer: p = $\frac{18}{7}$; $(\frac{5}{7}, \frac{13}{7}, \frac{22}{7})$

6 marks

17 (7 marks)

(a) $2\lambda = 10 - 2\mu$ and $\lambda = -1 + \mu$, so $2(\mu - 1) = 10 - 2\mu$, giving $\mu = 3$ and $\lambda = 2$

M1 A1

k: $10 - 2 = 8$ and $5 + 3 = 8$; the crossing point is (4, 2, 8)

A1

Answer: (4, 2, 8)

3 marks

(b) P reaches (4, 2, 8) when $t = 2$ but Q reaches it when $t = 3$

M1

They are never at the crossing point at the same time

E1

Answer: They reach the crossing point at different times

2 marks

(c) $\mathbf{r}_Q(2) = 6\mathbf{i} + \mathbf{j} + 7\mathbf{k}$, $\mathbf{r}_P(2) = 4\mathbf{i} + 2\mathbf{j} + 8\mathbf{k}$

M1

Distance = $\sqrt{4 + 1 + 1} = \sqrt{6} = 2.45$ km

A1

Answer: 2.45 km

2 marks

18 (6 marks)

xy-plane $z = 0$: $-1 + \lambda = 0$, $\lambda = 1$, point (3, 1, 0)

M1 A1

yz-plane $x = 0$: $2 + \lambda = 0$, $\lambda = -2$, point (0, 7, -3)

M1 A1

xz-plane $y = 0$: $3 - 2\lambda = 0$, $\lambda = \frac{3}{2}$, point $(\frac{7}{2}, 0, \frac{1}{2})$

M1 A1

Answer: (3, 1, 0), (0, 7, -3), $(\frac{7}{2}, 0, \frac{1}{2})$

6 marks

19 (5 marks)

Parametrise: $(1 + 2\lambda, -1 + 3\lambda, 2 + \lambda)$ and $(2 + \mu, 4 - 2\mu, 1 + 2\mu)$

M1 A1

i: $1 + 2\lambda = 2 + \mu$, so $\mu = 2\lambda - 1$

M1

j: $-1 + 3\lambda = 4 - 2(2\lambda - 1) = 6 - 4\lambda$, so $7\lambda = 7$, $\lambda = 1$ and $\mu = 1$

A1

k: $2 + 1 = 3$ and $1 + 2 = 3$, consistent, so the lines intersect at (3, 2, 3)

A1

Answer: (3, 2, 3)

5 marks

20 (5 marks)

(a) $(2, -2, 4) = 2(1, -1, 2)$ so the directions are parallel

M1

Different direction vectors can still be parallel

E1

Answer: The directions are parallel

2 marks

(b) Is $(3, 0, 7)$ on l_1 ? $1 + \lambda = 3$ gives $\lambda = 2$: $(3, 0, 7)$

M1 A1

It is, so the lines are the same line

A1

Answer: The same line

3 marks

21 (8 marks)

(a) i: $2 + \lambda = 1 + 2\mu$ and j: $1 + \lambda = 3 - \mu$

M1

Subtract: $1 = -2 + 3\mu$, so $\mu = 1$ and $\lambda = 1$

M1

k: $2\lambda = 2$ and $2 + 0\mu = 2$, so the lines intersect at $(3, 2, 2)$

A1

Answer: $(3, 2, 2)$

3 marks

(b) $(1, 1, 2) \cdot (2, -1, 0) = 1$, with magnitudes $\sqrt{6}$ and $\sqrt{5}$

M1

$\cos \theta = \frac{1}{\sqrt{30}}$, so $\theta = 79.5^\circ$

M1 A1

Answer: 79.5°

3 marks

(c) Direction $= (1, 1, 2) \times (2, -1, 0) = 2\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$

M1

$\mathbf{r} = 3\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} + s(2\mathbf{i} + 4\mathbf{j} - 3\mathbf{k})$

A1

Answer: $\mathbf{r} = 3\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} + s(2\mathbf{i} + 4\mathbf{j} - 3\mathbf{k})$

2 marks

22 (8 marks)

(a) Equate: $1 + 2t = 11 - 3t$, so $t = 2$

M1 A1

j: $2 - 2 = 0$ and $-6 + 6 = 0$; k: 6 and 6, all three agree

M1

Position $5\mathbf{i} + 6\mathbf{k}$

A1

Answer: $5\mathbf{i} + 6\mathbf{k}$

4 marks

(b) $|v_A| = \sqrt{4 + 1 + 9} = \sqrt{14}$

B1

$|v_B| = \sqrt{9 + 9 + 9} = 3\sqrt{3}$

B1

Answer: $\sqrt{14} \text{ m s}^{-1}$ and $3\sqrt{3} \text{ m s}^{-1}$

2 marks

(c) $v_A \cdot v_B = -6 - 3 + 9 = 0$

M1

The paths are perpendicular: 90°

A1

Answer: 90°

2 marks

23 (9 marks)

(a) $M = (4, 6, 2)$ and $N = (2, 4, 4)$; AM: $\mathbf{r} = \mathbf{i} + 2\mathbf{k} + \lambda(3\mathbf{i} + 6\mathbf{j})$

M1

BN: $\mathbf{r} = 5\mathbf{i} + 4\mathbf{j} - 2\mathbf{k} + \mu(-3\mathbf{i} + 6\mathbf{k})$

A1

Answer: AM: $\mathbf{r} = \mathbf{i} + 2\mathbf{k} + \lambda(3\mathbf{i} + 6\mathbf{j})$; BN: $\mathbf{r} = 5\mathbf{i} + 4\mathbf{j} - 2\mathbf{k} + \mu(-3\mathbf{i} + 6\mathbf{k})$

2 marks

(b) i: $1 + 3\lambda = 5 - 3\mu$, j: $6\lambda = 4$

M1

$\lambda = \frac{2}{3}$, $\mu = \frac{2}{3}$; k: $2 = -2 + 6\mu = 2$

M1 A1

$G = (3, 4, 2)$

A1

Answer: $G = (3, 4, 2)$

4 marks

(c) $\mathbf{CG} = -4\mathbf{j} - 4\mathbf{k}$, so line CG: $\mathbf{r} = 3\mathbf{i} + 8\mathbf{j} + 6\mathbf{k} + \nu(-4\mathbf{j} - 4\mathbf{k})$

M1

Midpoint of AB is $(3, 2, 0)$: $\nu = \frac{3}{2}$ gives $(3, 8 - 6, 6 - 6) = (3, 2, 0)$

M1 A1

Answer: The midpoint of AB, $(3, 2, 0)$, lies on CG

3 marks

24 (9 marks)

(a) i: $1 + \lambda = 2 + 2\mu$, j: $2 + \lambda = 3 + \mu$

M1

Subtract: $-1 = -1 + \mu$, so $\mu = 0$ and $\lambda = 1$

M1 A1

p: $3 + 2(1) = p + (-1)(0)$, so $p = 5$

A1

Answer: $p = 5$

4 marks

(b) Directions are not parallel and the lines do not meet, so they are skew

E1

Answer: Skew

1 mark

(c) Point: $\lambda = 1$ gives $(2, 3, 5)$

B1

$(1, 1, 2) \cdot (2, 1, -1) = 1$, magnitudes $\sqrt{6}$ and $\sqrt{6}$: $\cos \theta = \frac{1}{6}$

M1 A1

$\theta = 80.4^\circ$

A1

Answer: $(2, 3, 5)$; 80.4°

4 marks

25 (8 marks)

(a) $\mathbf{BC} = 2\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{AB} = 3\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$

B1

$\mathbf{r} = \mathbf{j} + 2\mathbf{k} + \lambda(2\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})$ and $\mathbf{r} = 5\mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + \mu(3\mathbf{i} + 3\mathbf{j} - 2\mathbf{k})$

B1 B1

Answer: $\mathbf{r} = \mathbf{j} + 2\mathbf{k} + \lambda(2\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}); \mathbf{r} = 5\mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + \mu(3\mathbf{i} + 3\mathbf{j} - 2\mathbf{k})$

3 marks

(b) i: $2\lambda = 5 + 3\mu$, j: $1 - 2\lambda = 2 + 3\mu$

M1

Add: $1 = 7 + 6\mu$, so $\mu = -1$ and $\lambda = 1$

M1 A1

k: $2 + 3 = 5$ and $3 + 2 = 5$; $\mathbf{D} = (2, -1, 5)$

A1

Answer: $\mathbf{D}(2, -1, 5)$

4 marks

(c) $\mathbf{a} + \mathbf{c} - \mathbf{b} = (0 + 5 - 3, 1 + 2 - 4, 2 + 3 - 0) = (2, -1, 5)$

B1

Answer: Agrees

1 mark

26 (6 marks)

Let $\mathbf{X} = (2, \lambda, \lambda)$ on l_1 and $\mathbf{Y} = (3, 1 + \mu, 1 - \mu)$ on l_2 ; P, X, Y collinear

M1

$\mathbf{PX} = (1, \lambda - 1, \lambda - 1)$ and $\mathbf{PY} = (2, \mu, -\mu)$

A1

$\mathbf{PY} = 2\mathbf{PX}$, so $\mu = 2\lambda - 2$ and $-\mu = 2\lambda - 2$

M1 A1

Hence $\mu = 0$, $\lambda = 1$: $\mathbf{X} = (2, 1, 1)$, $\mathbf{Y} = (3, 1, 1)$

M1

Line: $\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + s\mathbf{i}$

A1

Answer: $\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + s\mathbf{i}$

6 marks

27 (11 marks)

(a) Missile distance from O at time T is 5T; aircraft distance is $|\mathbf{r}_A(T)|$

M1

$(6 + T)^2 + (4 - 2T)^2 + (8 - T)^2 = 25T^2$

M1 A1

$116 - 20T + 6T^2 = 25T^2$, so $19T^2 + 20T - 116 = 0$

A1

Answer: Shown: T satisfies $19T^2 + 20T - 116 = 0$

4 marks

(b) $(T - 2)(19T + 58) = 0$, $T > 0$ so $T = 2$

M1 A1

Position $8\mathbf{i} + 6\mathbf{k} = (8, 0, 6)$

A1

Answer: T = 2; (8, 0, 6)

3 marks

(c) $\mathbf{v} = \frac{1}{2}(8, 0, 6) = 4\mathbf{i} + 3\mathbf{k}$

M1

Speed = $\sqrt{16 + 9} = 5$

A1

Answer: $4\mathbf{i} + 3\mathbf{k}$

2 marks

(d) $\tan \theta = \frac{3}{4}$

M1

$\theta = 36.9^\circ$

A1

Answer: 36.9°

2 marks

28 (9 marks)

(a) $\mathbf{PQ} = \mathbf{r}_Q - \mathbf{r}_P$

M1

$$= \begin{pmatrix} 9 - 3t \\ -6 + 2t \\ 3 \end{pmatrix}$$

A1

Answer: $(9 - 3t)\mathbf{i} + (-6 + 2t)\mathbf{j} + 3\mathbf{k}$

2 marks

(b) $d^2 = (9 - 3t)^2 + (-6 + 2t)^2 + 3^2$

M1

$= 81 - 54t + 9t^2 + 36 - 24t + 4t^2 + 9 = 13t^2 - 78t + 126$

A1

Answer: Shown: the distance d between the boats satisfies $d^2 = 13t^2 - 78t + 126$

2 marks

(c) $d^2 = 13(t - 3)^2 + 9$

M1 A1

Least $d = 3$ m when $t = 3$ s

A1 A1

Answer: 3 m at $t = 3$

4 marks

(d) $d^2 \geq 9$ for all t, so d is never 0

E1

Answer: $d \geq 3$ always

1 mark

29 (12 marks)

(a) $|\mathbf{r}|^2 = (1 + \lambda)^2 + (2 - 2\lambda)^2 + (3 + 2\lambda)^2$

M1

$= 1 + 2\lambda + \lambda^2 + 4 - 8\lambda + 4\lambda^2 + 9 + 12\lambda + 4\lambda^2 = 9\lambda^2 + 6\lambda + 14$

A1

Answer: Shown: $|\mathbf{r}|^2 = 9\lambda^2 + 6\lambda + 14$ for a point on l

2 marks

(b) $9\lambda^2 + 6\lambda + 14 = 49$, so $9\lambda^2 + 6\lambda - 35 = 0$

M1

$(3\lambda - 5)(3\lambda + 7) = 0$, so $\lambda = \frac{5}{3}$ or $\lambda = -\frac{7}{3}$

M1 A1

Points $(\frac{8}{3}, -\frac{4}{3}, \frac{19}{3})$ and $(-\frac{4}{3}, \frac{20}{3}, -\frac{5}{3})$

A1

Chord = $|\frac{5}{3} - (-\frac{7}{3})| \times |d| = 4 \times 3 = 12$

M1 A1

Answer: $(\frac{8}{3}, -\frac{4}{3}, \frac{19}{3})$ and $(-\frac{4}{3}, \frac{20}{3}, -\frac{5}{3})$; chord 12

6 marks

(c) Repeated root: $36 - 36(14 - R^2) = 0$

M1

$R^2 = 13$, $R = \sqrt{13}$

A1

$\lambda = -\frac{1}{3}$, point $(\frac{2}{3}, \frac{8}{3}, \frac{7}{3})$

M1 A1

Answer: $R = \sqrt{13}$; $(\frac{2}{3}, \frac{8}{3}, \frac{7}{3})$

4 marks

30 (12 marks)

(a) At intersection $\mathbf{a}_1 + \lambda \mathbf{d}_1 = \mathbf{a}_2 + \mu \mathbf{d}_2$, so $\mathbf{a}_2 - \mathbf{a}_1 = \lambda \mathbf{d}_1 - \mu \mathbf{d}_2$

M1 A1

Take the scalar product with $\mathbf{d}_1 \times \mathbf{d}_2$, which is perpendicular to both $\mathbf{d}_1$ and $\mathbf{d}_2$: the right side is 0

M1 A1

Answer: Proved: if l_1 and l_2 intersect then $(\mathbf{a}_2 - \mathbf{a}_1) \cdot (\mathbf{d}_1 \times \mathbf{d}_2) = 0$

4 marks

(b) The vectors $\mathbf{a}_2 - \mathbf{a}_1$, $\mathbf{d}_1$, $\mathbf{d}_2$ are coplanar, so the lines lie in one plane

E1

Two non-parallel lines in one plane must meet; for parallel lines $\mathbf{d}_1 \times \mathbf{d}_2 = 0$ and the condition is automatically true

E1

Answer: Coplanar; parallel lines satisfy it trivially

2 marks

(c) $\mathbf{d}_1 \times \mathbf{d}_2 = \mathbf{i} - \mathbf{j} - \mathbf{k}$

M1 A1

$\mathbf{a}_2 - \mathbf{a}_1 = (-1, 1, p)$; dot product $-1 - 1 - p = 0$, so $p = -2$

M1 A1

$z: 0 = -2 + \mu$ so $\mu = 2$; $y: \lambda = 1$; point $(2, 1, 0)$

M1 A1

Answer: $p = -2$; $(2, 1, 0)$

6 marks

Vector Equations of Planes

1 (2 marks)

$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$

M1

$1 \times 2 + 2 \times (-1) + 3 \times 4 = 12$, so $2x - y + 4z = 12$

A1

Answer: $2x - y + 4z = 12$

2 marks

2 (2 marks)

$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

M1

$3x - 2y + z = 5$

A1

Answer: $3x - 2y + z = 5$

2 marks

3 (2 marks)

Normal $4\mathbf{i} - \mathbf{j} + 3\mathbf{k}$

B1

$4(1) - (-3) + 0 = 7$, so the point lies on Π

B1

Answer: Normal $4\mathbf{i} - \mathbf{j} + 3\mathbf{k}$; yes

2 marks

4 (2 marks)

A: $2 - 0 + 2 = 4$, so A lies on the plane

B1

B: $4 - 1 + 0 = 3 \neq 4$, so B does not

B1

Answer: A only

2 marks

5 (2 marks)

$y = z = 0$: $x = 6$; $x = z = 0$: $y = 4$

M1

$x = y = 0$: $z = 2$; points $(6, 0, 0)$, $(0, 4, 0)$, $(0, 0, 2)$

A1

Answer: $(6, 0, 0)$, $(0, 4, 0)$, $(0, 0, 2)$

2 marks

6 (2 marks)

Normals $(1, 2, -1)$ and $(2, 4, -2) = 2(1, 2, -1)$

M1

The normals are parallel, so the planes are parallel

A1

Answer: Normals are parallel

2 marks

7 (5 marks)

$\mathbf{AB} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{AC} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$

B1

Normal = $\mathbf{AB} \times \mathbf{AC} = \mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$

M1 A1

$\mathbf{n} \cdot \mathbf{a} = 11$, so $x + 3y + 5z = 11$

M1 A1

Answer: $x + 3y + 5z = 11$

5 marks

8 (4 marks)

Normal = $\mathbf{i} + 2\mathbf{j} \times \mathbf{j} + 3\mathbf{k}$

M1

= $6\mathbf{i} - 3\mathbf{j} + \mathbf{k}$

A1

$\mathbf{n} \cdot \mathbf{a} = 8$, so $6x - 3y + z = 8$

M1 A1

Answer: $6x - 3y + z = 8$

4 marks

9 (4 marks)

Three points on Π : $(4, 0, 0)$, $(0, 2, 0)$, $(0, 0, -4)$

M1 A1

Direction vectors $(-4, 2, 0)$ and $(-4, 0, -4)$, simplified to $(-2, 1, 0)$ and $(1, 0, 1)$

M1

$\mathbf{r} = 4\mathbf{i} + \lambda(-2\mathbf{i} + \mathbf{j}) + \mu(\mathbf{i} + \mathbf{k})$

A1

Answer: $\mathbf{r} = 4\mathbf{i} + \lambda(-2\mathbf{i} + \mathbf{j}) + \mu(\mathbf{i} + \mathbf{k})$

4 marks

10 (7 marks)

(a) $\mathbf{AB} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{AC} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$

B1

$x = 2 + 2\lambda + \mu$, $y = 1 - \lambda + 2\mu$, $z = 1 + 2\lambda - \mu$

M1 A1

Answer: $x = 2 + 2\lambda + \mu$, $y = 1 - \lambda + 2\mu$, $z = 1 + 2\lambda - \mu$

3 marks

(b) $x + z = 3 + 4\lambda$ and $y + 2z = 3 + 3\lambda$ (μ eliminated)

M1 A1

$3(x + z) - 4(y + 2z) = 9 - 12$

M1

$3x - 4y - 5z = -3$

A1

Answer: $3x - 4y - 5z = -3$

4 marks

11 (4 marks)

Normal = $\mathbf{AB} = 4\mathbf{i} - 2\mathbf{j} - 4\mathbf{k}$, parallel to $2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$

M1

Midpoint $M = (3, 1, 1)$ lies on the plane

B1

$2(3) - 1 - 2(1) = 3$

M1

$2x - y - 2z = 3$

A1

Answer: $2x - y - 2z = 3$

4 marks

12 (2 marks)

$2k - 2 - 3 = 7$

M1

$k = 6$

A1

Answer: $k = 6$

2 marks

13 (6 marks)

i: $1 + 2\lambda = 3 + \mu$ and j: $2 - \lambda = 1 + \mu$, so $\lambda = 1$, $\mu = 0$

M1 A1

k: $1 + 1 = 2 = 2 - 0$, so the lines meet at $(3, 1, 2)$

A1

Normal = $\mathbf{d}_1 \times \mathbf{d}_2 = \mathbf{i} + 5\mathbf{j} + 3\mathbf{k}$

M1 A1

$x + 5y + 3z = 1 + 10 + 3 = 14$

A1

Answer: Intersect at $(3, 1, 2)$; $x + 5y + 3z = 14$

6 marks

14 (5 marks)

$\mathbf{AP} = -\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$

B1

Normal = $\mathbf{i} + \mathbf{j} + \mathbf{k} \times -\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$

M1

= $5\mathbf{i} - 4\mathbf{j} - \mathbf{k}$

A1

$\mathbf{n} \cdot \mathbf{a} = -3$

M1

$5x - 4y - z = -3$; check P: -3

A1

Answer: $5x - 4y - z = -3$

5 marks

15 (5 marks)

Same direction vector, so the lines are parallel

B1

A vector joining the lines: $2\mathbf{i} + \mathbf{j} + 3\mathbf{k} - \mathbf{i} = \mathbf{i} + \mathbf{j} + 3\mathbf{k}$

M1

Normal = $\mathbf{i} + 2\mathbf{j} + \mathbf{k} \times \mathbf{i} + \mathbf{j} + 3\mathbf{k} = 5\mathbf{i} - 2\mathbf{j} - \mathbf{k}$

M1 A1

 $5x - 2y - z = 5$

A1

Answer: $5x - 2y - z = 5$

5 marks

16 (5 marks) $n_1 \cdot n_2 = 2 - 1 - 1 = 0$, so the planes are perpendicular

M1 A1

Normal to the required plane = $n_1 \times n_2 = 3\mathbf{j} + 3\mathbf{k}$

M1 A1

 $3y + 3z = 6$, so $y + z = 2$

A1

Answer: $y + z = 2$

5 marks

17 (7 marks)

(a) Check: A: 12, B: 12, C: 12

M1 A1

Answer: Shown: the plane has equation $2x + 3y + 4z = 12$

2 marks

(b) $\mathbf{AB} \times \mathbf{AC} = 12\mathbf{i} + 18\mathbf{j} + 24\mathbf{k}$

M1 A1

Area = $\frac{1}{2}\sqrt{12^2 + 18^2 + 24^2} = 3\sqrt{29}$

A1

Answer: $3\sqrt{29}$

3 marks

(c) Volume = $\frac{1}{6}|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| = \frac{1}{6} \times 6 \times 4 \times 3 = 12$

M1 A1

Answer: 12

2 marks

18 (7 marks)(a) Normal = $\mathbf{i} + 2\mathbf{k} \times -\mathbf{i} + \mathbf{j} + \mathbf{k} = -2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$

M1 A1

 $-2(1) - 3(1) + 0 = -5$, so $2x + 3y - z = 5$

M1 A1

Answer: $2x + 3y - z = 5$

4 marks

(b) j: $1 + \mu = -1$, so $\mu = -2$

M1

i: $1 + \lambda - \mu = 3$, so $\lambda = 0$; check k: $2\lambda + \mu = -2$

M1 A1

Answer: $\lambda = 0, \mu = -2$

3 marks

19 (5 marks) $\mathbf{AB} = -\mathbf{i} + \mathbf{j}$

B1

Normal = $\mathbf{AB} \times \mathbf{i} + \mathbf{j} + \mathbf{k} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$

M1 A1

 $\mathbf{n} \cdot \mathbf{a} = 1$, so $x + y - 2z = 1$

M1 A1

Answer: $x + y - 2z = 1$

5 marks

20 (6 marks)(a) $\mathbf{AB} = (1, 1, 2)$ and $\mathbf{AC} = (-1, -1, -2) = -\mathbf{AB}$

M1

The vectors are parallel, so A, B, C are collinear and do not define a unique plane

E1

Answer: A, B, C are collinear

2 marks

(b) $\mathbf{AD} = -2\mathbf{j} - 3\mathbf{k}$

B1

Normal = $\mathbf{AB} \times \mathbf{AD} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$

M1 A1

 $x + 3y - 2z = 1$

A1

Answer: $x + 3y - 2z = 1$

4 marks

21 (7 marks) Π_2 contains the direction of l, so its normal n is perpendicular to $(1, -1, 2)$

M1

 Π_2 is perpendicular to Π_1 , so n is perpendicular to $(1, 2, -1)$

M1

 $\mathbf{n} = (1, -1, 2) \times (1, 2, -1) = -3\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$, parallel to $(-1, 1, 1)$

M1 A1

Point $(2, 0, 1)$: $-2 + 0 + 1 = -1$

M1

 $x - y - z = 1$

A1

Check with t = 1: $(3, -1, 3)$: $3 + 1 - 3 = 1$

B1

Answer: $x - y - z = 1$

7 marks

22 (9 marks)

(a) A (12, 0, 0), B (0, 6, 0), C (0, 0, 6)

B1

Answer: (12, 0, 0), (0, 6, 0), (0, 0, 6)

1 mark

(b) $\mathbf{AB} \times \mathbf{AC} = 36\mathbf{i} + 72\mathbf{j} + 72\mathbf{k}$

M1 A1

$$\text{Area} = \frac{1}{2}|36\mathbf{i} + 72\mathbf{j} + 72\mathbf{k}| = \frac{1}{2} \times 108 = 54$$

M1 A1

Answer: 54

4 marks

$$(c) \text{Volume} = \frac{1}{6}|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| = \frac{1}{6} \times 12 \times 6 \times 6 = 72$$

M1 A1

Answer: 72

2 marks

$$(d) \text{Unit normal} = \frac{1}{3}(1, 2, 2); \text{distance} = 12\mathbf{i} \cdot \frac{1}{3}(1, 2, 2) = 4$$

A1

$$\frac{1}{3} \times 54 \times 4 = 72, \text{ as in (c)}$$

M1

Answer: 4

2 marks

23 (8 marks)

$$(a) \text{PA}^2 = (x - 1)^2 + (y - 2)^2 + z^2, \text{PB}^2 = (x - 3)^2 + (y + 2)^2 + (z - 4)^2$$

M1

$$\text{Equate and expand: } -2x - 4y + 5 = -6x + 4y - 8z + 29$$

M1 A1

$$4x - 8y + 8z = 24, \text{ so } x - 2y + 2z = 6$$

A1

Answer: Shown: P lies on the plane $x - 2y + 2z = 6$

4 marks

(b) Normal (1, -2, 2) is parallel to $\mathbf{AB} = (2, -4, 4)$

M1

Midpoint (2, 0, 2): $2 - 0 + 4 = 6$, so it lies on the plane

A1

Answer: Shown: this plane is the perpendicular bisector of AB

2 marks

(c) The foot of the perpendicular from A is the midpoint, so the distance is $\frac{1}{2}AB = \frac{1}{2}\sqrt{4 + 16 + 16} = 3$

M1 A1

Answer: 3

2 marks

24 (7 marks)

$$(a) \text{M} = (1, \frac{3}{2}, 0)$$

B1

$$\text{Normal} = \mathbf{OC} \times \mathbf{OM} = -6\mathbf{i} + 4\mathbf{j} + \frac{1}{2}\mathbf{k}, \text{ parallel to } (-12, 8, 1)$$

M1 A1

$$\text{Plane through O: } 12x - 8y - z = 0$$

A1

Answer: $12x - 8y - z = 0$

4 marks

(b) Line: $(2 + s, s, 4s)$; $12(2 + s) - 8s - 4s = 24$

M1 A1

$24 \neq 0$ for every s, so there is no intersection; the line is parallel to a line in the plane (OC) but does not lie in the plane

E1

Answer: No intersection ($24 \neq 0$)

3 marks

25 (7 marks)

$$(a) \text{CP} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$$

M1

$$\text{Radius} = \sqrt{4 + 4 + 1} = 3$$

A1

Answer: 3

2 marks

(b) Normal $2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$

B1

$$2(3) - 2(0) + 4 = 10$$

M1

$$2x - 2y + z = 10$$

A1

Answer: $2x - 2y + z = 10$

3 marks

(c) $y = z = 0$; $2x = 10$

M1

$$(5, 0, 0)$$

A1

Answer: (5, 0, 0)

2 marks

26 (9 marks)

(a) $\mathbf{AB} = \mathbf{i} + (p - 1)\mathbf{j} + 2\mathbf{k}$ and $\mathbf{AC} = -\mathbf{i} + \mathbf{j} + (p - 1)\mathbf{k}$; $\mathbf{i} \cdot (p - 1)^2 - 2 = p^2 - 2p - 1$

M1 A1

$\mathbf{j} \cdot 2(-1) - (p - 1) = -(p + 1)$ and $\mathbf{p} \cdot 1 + (p - 1) = p$

A1

Answer: Shown: a normal to Π_p is $(p^2 - 2p - 1)\mathbf{i} - (p + 1)\mathbf{j} + p\mathbf{k}$

3 marks

(b) The origin is on Π_p if $\mathbf{n} \cdot \mathbf{a} = 0$ (the plane has $d = \mathbf{n} \cdot \mathbf{a}$)

M1

$$p^2 - 2p - 1 - p - 1 + p = p^2 - 2p - 2 = 0$$

A1

$$p = 1 \pm \sqrt{3}$$

M1 A1

$$\text{Answer: } p = 1 \pm \sqrt{3}$$

4 marks

$$(c) \mathbf{n} = (-1, -1, 0), \mathbf{d} = -1 - 1 = -2$$

M1

$$x + y = 2$$

A1

$$\text{Answer: } x + y = 2$$

2 marks

27 (12 marks)

$$(a) \mathbf{n} \text{ is parallel to } \mathbf{AB} \times \mathbf{AC} = (\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})$$

M1

$$= \mathbf{b} \times \mathbf{c} - \mathbf{b} \times \mathbf{a} - \mathbf{a} \times \mathbf{c} + \mathbf{a} \times \mathbf{a} = \mathbf{b} \times \mathbf{c} + \mathbf{a} \times \mathbf{b} + \mathbf{c} \times \mathbf{a}$$

M1 A1

$$\text{Answer: Shown: } \mathbf{n} = \mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a} \text{ is perpendicular to the plane ABC}$$

3 marks

$$(b) \text{ Plane: } \mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$$

M1

$$\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0 \text{ and } \mathbf{a} \cdot (\mathbf{c} \times \mathbf{a}) = 0, \text{ so } \mathbf{a} \cdot \mathbf{n} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$$

M1 A1

$$\text{Answer: Proved: the plane ABC has equation } \mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$$

3 marks

$$(c) \mathbf{a} \times \mathbf{b} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}, \mathbf{b} \times \mathbf{c} = \mathbf{i} + 6\mathbf{j} - 3\mathbf{k}, \mathbf{c} \times \mathbf{a} = -\mathbf{i} + \mathbf{j} + 3\mathbf{k}$$

M1 A1

$$\mathbf{n} = 2\mathbf{i} + 5\mathbf{j} + \mathbf{k} \text{ and } \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 7$$

A1

$$2x + 5y + z = 7$$

A1

$$\text{Answer: } 2x + 5y + z = 7$$

4 marks

$$(d) \text{ O lies on the plane if } 0 = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$$

M1

$$\text{i.e. } \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0: \text{ the points O, A, B, C are coplanar}$$

A1

$$\text{Answer: } \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$$

2 marks

28 (9 marks)

$$(a) (1, 1, 0) \times (0, 1, 1) = \mathbf{i} - \mathbf{j} + \mathbf{k}$$

M1 A1

$$\text{Answer: } \mathbf{i} - \mathbf{j} + \mathbf{k}$$

2 marks

$$(b) \text{ Normal } \mathbf{n} = \mathbf{i} - \mathbf{j} + \mathbf{k}$$

B1

$$\text{Through } (1, 0, 1): 1 - 0 + 1 = 2$$

M1

$$\Pi_1: x - y + z = 2$$

A1

$$\text{Answer: } x - y + z = 2$$

3 marks

$$(c) \text{ Through } (0, 2, 0): 0 - 2 + 0 = -2$$

M1

$$\Pi_2: x - y + z = -2$$

A1

$$\text{Answer: } x - y + z = -2$$

2 marks

$$(d) \Pi_1 \text{ and } \Pi_2 \text{ are parallel distinct planes, so } l_1 \text{ and } l_2 \text{ cannot meet}$$

E1

$$\text{The directions } (1, 1, 0) \text{ and } (0, 1, 1) \text{ are not parallel, so the lines are skew}$$

E1

$$\text{Answer: Skew}$$

2 marks

29 (10 marks)

$$(a) \mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}) + \mu(\mathbf{c} - \mathbf{a})$$

M1

$$= (1 - \lambda - \mu)\mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$$

M1 A1

$$\text{The coefficients sum to 1}$$

A1

$$\text{Answer: Shown: every point of } \Pi \text{ can be written as } \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} \text{ with } \alpha + \beta + \gamma = 1$$

4 marks

$$(b) \text{ A: 6, B: 6, C: 6 satisfy the equation}$$

M1 A1

$$\text{Answer: Shown: } \Pi \text{ has equation } 6x + 3y + 2z = 6$$

2 marks

$$(c) \mathbf{g} = \frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c}) = \left(\frac{1}{3}, \frac{2}{3}, 1\right)$$

M1

$$6\left(\frac{1}{3}\right) + 3\left(\frac{2}{3}\right) + 2(1) = 6$$

A1

$$\text{Answer: } \left(\frac{1}{3}, \frac{2}{3}, 1\right)$$

2 marks

$$(d) \text{ Coefficients sum to } \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1, \text{ so D lies on } \Pi$$

M1

$$\mathbf{D} = \left(\frac{1}{2}, \frac{2}{3}, \frac{1}{2}\right)$$

A1

$$\text{Answer: } \left(\frac{1}{2}, \frac{2}{3}, \frac{1}{2}\right)$$

2 marks

30 (10 marks)

(a) $\Pi: \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ M1

P lies on $\Pi: \frac{1}{a} + \frac{2}{b} + \frac{3}{c} = 1$ A1

Answer: Shown: $\frac{1}{a} + \frac{2}{b} + \frac{3}{c} = 1$ 2 marks

(b) $V = \frac{1}{6}abc$ B1

Answer: $V = abc/6$ 1 mark

(c) Take $u = \frac{1}{a}$, $v = \frac{2}{b}$, $w = \frac{3}{c}$: $1 \geq 3\left(\frac{6}{abc}\right)^{\frac{1}{3}}$ M1

$\frac{1}{27} \geq \frac{6}{abc}$, so $abc \geq 162$ M1 A1

$V = \frac{abc}{6} \geq 27$ A1

Equality when $\frac{1}{a} = \frac{2}{b} = \frac{3}{c} = \frac{1}{3}$ M1

$a = 3$, $b = 6$, $c = 9$ A1

Answer: $V \geq 27$; $a = 3$, $b = 6$, $c = 9$ 6 marks

(d) $\frac{x}{3} + \frac{y}{6} + \frac{z}{9} = 1$, so $6x + 3y + 2z = 18$ A1

Answer: $6x + 3y + 2z = 18$ 1 mark

Intersections of Lines and Planes**1 (2 marks)**

$(1 + \lambda) + (2 + \lambda) + (3 + 2\lambda) = 12$, so $6 + 4\lambda = 12$ M1

$\lambda = \frac{3}{2}$, point $\left(\frac{5}{2}, \frac{7}{2}, 6\right)$ A1

Answer: $\left(\frac{5}{2}, \frac{7}{2}, 6\right)$ 2 marks

2 (2 marks)

$(1, -2, 1) \cdot (2, 1, 0) = 0$, so the line is parallel to the plane M1

The point $(1, 0, 2)$ gives $1 + 2 = 3 \neq 5$, so the line is not in the plane A1

Answer: Parallel, not in the plane 2 marks

3 (2 marks)

$(2 + \lambda) + (1 + 2\lambda) - 3\lambda = 3$ for all λ M1 A1

Answer: $3 = 3$ for every λ 2 marks

4 (2 marks)

$z = 2$ and $x = y$, so $2x + 2 = 6$ M1

$x = y = 2$: the point is $(2, 2, 2)$ A1

Answer: $(2, 2, 2)$ 2 marks

5 (2 marks)

$(1 + 2t) + (-1 + 3t) - 2t = 3t = 6$ M1

$t = 2$, point $(5, 5, 2)$ A1

Answer: $(5, 5, 2)$ 2 marks

6 (3 marks)

Point on line $(1 + t, 1 + 2t, 1 + 3t)$ B1

$3(1 + t) - (1 + 2t) + 2(1 + 3t) = 4 + 7t = 18$, so $t = 2$; point $(3, 5, 7)$ M1 A1

Answer: $(3, 5, 7)$ 3 marks

7 (4 marks)

$$\text{Direction} = (1, 1, 1) \times (1, -1, 2) = 3\mathbf{i} - \mathbf{j} - 2\mathbf{k}$$

M1 A1

$$\text{Put } z = 0: x + y = 3, x - y = 1, \text{ so } x = 2, y = 1$$

M1

$$\mathbf{r} = 2\mathbf{i} + \mathbf{j} + \lambda(3\mathbf{i} - \mathbf{j} - 2\mathbf{k})$$

A1

$$\text{Answer: } \mathbf{r} = 2\mathbf{i} + \mathbf{j} + \lambda(3\mathbf{i} - \mathbf{j} - 2\mathbf{k})$$

4 marks

8 (4 marks)

$$\text{At } \lambda = 2: \mathbf{r} = 2\mathbf{i} - \mathbf{j} + 2(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$$

M1

$$\mathbf{OP} = 4\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$$

A1

$$c = \mathbf{OP} \cdot (\mathbf{i} + 3\mathbf{j} - \mathbf{k}) = 4 + 9 + 4$$

M1

$$c = 17$$

A1

$$\text{Answer: } \mathbf{OP} = 4\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}, c = 17$$

4 marks

9 (5 marks)

$$\text{Add the first and second: } 3x + 2z = 9. \text{ Add the first and third: } 2x + 3z = 11$$

M1 A1

$$\text{Solve: } 9x + 6z = 27 \text{ and } 4x + 6z = 22, \text{ so } 5x = 5, x = 1 \text{ and } z = 3$$

M1

$$y = 6 - 1 - 3 = 2; \text{ the three planes meet at the single point } (1, 2, 3)$$

A1 A1

$$\text{Answer: } (1, 2, 3): \text{ the planes meet at one point}$$

5 marks

10 (4 marks)

$$\text{The perpendicular is in the direction of the normal: } \mathbf{r} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$$

M1

$$(1 + \lambda) + 2(2 + 2\lambda) + 2(3 + 2\lambda) = 11 + 9\lambda = 2$$

M1 A1

$$\lambda = -1, \text{ foot } (0, 0, 1)$$

A1

$$\text{Answer: } (0, 0, 1)$$

4 marks

11 (4 marks)

$$\text{Direction} = (2, -1, 1) \times (1, 1, -1) = 3\mathbf{j} + 3\mathbf{k}, \text{ parallel to } (0, 1, 1)$$

M1 A1

$$\text{Adding the equations: } 3x = 6, \text{ so } x = 2 \text{ on } l$$

M1

$$\text{Then } y - z = 0; \text{ the line is } x = 2, y = z$$

A1

$$\text{Answer: } x = 2, y = z$$

4 marks

12 (4 marks)

$$(a) p - 2 + 2 = 0$$

M1

$$p = 0$$

A1

$$\text{Answer: } p = 0$$

2 marks

$$(b) (1, 0, 2): 1 - 0 + 2 = 3, \text{ so a point of } l \text{ lies in } \Pi; l \text{ is parallel to } \Pi, \text{ so } l \text{ lies in } \Pi$$

M1 A1

$$\text{Answer: Lies in } \Pi$$

2 marks

13 (6 marks)

$$(a) \text{ The second is twice the first}$$

M1

$$\text{They are the same plane (coincident)}$$

A1

$$\text{Answer: Coincident}$$

2 marks

$$(b) \text{ Normals are parallel but } 4x - 2y + 2z = 6 \text{ on the first plane, not } 7$$

M1

$$\text{Parallel distinct planes}$$

A1

$$\text{Answer: Parallel, distinct}$$

2 marks

$$(c) \text{ Normals } (2, -1, 1) \text{ and } (1, 1, -1) \text{ are not parallel}$$

M1

$$\text{The planes intersect in a line}$$

A1

$$\text{Answer: Intersect in a line}$$

2 marks

14 (6 marks)

$$(a) (1, 2, -1) \cdot (1, 1, 3) = 1 + 2 - 3 = 0, \text{ so parallel}$$

M1

$$\text{Point } (2, 1, 0): 2 + 2 - 0 = 4, \text{ on } \Pi, \text{ so the line lies in } \Pi$$

A1

$$\text{Answer: Lies in } \Pi$$

2 marks

$$(b) \text{ Same direction, so parallel to } \Pi$$

M1

$$\text{Point } (0, 0, 1): 0 + 0 - 1 = -1 \neq 4, \text{ so parallel but not in } \Pi$$

A1

$$\text{Answer: Parallel to } \Pi, \text{ not in it}$$

2 marks

$$(c) (1, 2, -1) \cdot (1, 1, 1) = 2 \neq 0, \text{ so it meets } \Pi \text{ once}$$

M1

$$\lambda + 2\lambda - \lambda = 4, \text{ so } \lambda = 2 \text{ and the point is } (2, 2, 2)$$

A1

Answer: Meets at (2, 2, 2)**2 marks****15 (6 marks)****(a)** Direction = $(2, 1, -1) \times (1, -2, 3) = \mathbf{i} - 7\mathbf{j} - 5\mathbf{k}$ **M1 A1**Put $y = 0$: $2x - z = 5$ and $x + 3z = -1$, so $x = 2, z = -1$ **M1 A1** $\mathbf{r} = 2\mathbf{i} - \mathbf{k} + \lambda(\mathbf{i} - 7\mathbf{j} - 5\mathbf{k})$ **A1****Answer: $\mathbf{r} = 2\mathbf{i} - \mathbf{k} + \lambda(\mathbf{i} - 7\mathbf{j} - 5\mathbf{k})$** **5 marks****(b)** Π_1 : $6 - 7 + 6 = 5$, Π_2 : $3 + 14 - 18 = -1$ **B1****Answer: On both planes****1 mark****16 (7 marks)****(a)** Adding Π_1 and Π_2 : $3x + y + 2z = 5$ **M1 A1**This conflicts with Π_3 unless $k = 5$ **A1****Answer: Common points only if $k = 5$** **3 marks****(b)** Each pair of planes meets in a line and the three lines are parallel**E1**

(a triangular prism arrangement)

E1**Answer: Triangular prism: no common point****2 marks****(c)** Direction $(1, 2, 1) \times (2, -1, 1) = 3\mathbf{i} + \mathbf{j} - 5\mathbf{k}$ **M1**Point $(0, 1, 2)$: $\mathbf{r} = \mathbf{j} + 2\mathbf{k} + \lambda(3\mathbf{i} + \mathbf{j} - 5\mathbf{k})$ **A1****Answer: $\mathbf{r} = \mathbf{j} + 2\mathbf{k} + \lambda(3\mathbf{i} + \mathbf{j} - 5\mathbf{k})$** **2 marks****17 (6 marks)**Line through P perpendicular to the plane: $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j} - 2\mathbf{k})$ **M1** $2(1 + 2\lambda) + (2 + \lambda) - 2(3 - 2\lambda) = 9\lambda - 2 = 7$, so $\lambda = 1$ **M1 A1**Foot F = $(3, 3, 1)$ **A1**Image P' = $2\mathbf{f} - \mathbf{p}$ **M1**= $(5, 4, -1)$ **A1****Answer: $(5, 4, -1)$** **6 marks****18 (6 marks)** $i: 2 + \lambda = s, j: 1 - \lambda = t, k: 3\lambda = 1 + s - t$ **M1 A1**Substitute: $3\lambda = 1 + (2 + \lambda) - (1 - \lambda) = 2 + 2\lambda$ **M1** $\lambda = 2$ **A1** $\mathbf{r} = 4\mathbf{i} - \mathbf{j} + 6\mathbf{k}$ **M1 A1****Answer: $4\mathbf{i} - \mathbf{j} + 6\mathbf{k}$** **6 marks****19 (4 marks)****(a)** The term $-2(2 + \lambda) = -4 - 2\lambda$, not $-4 + 2\lambda$ **M1 A1****Answer: $-2 \times \lambda$ was given the wrong sign****2 marks****(b)** $3 + 6\lambda - 4 - 2\lambda + 3 - \lambda = 2 + 3\lambda = 5$, so $\lambda = 1$ **M1**Point $(3, 3, 2)$; check $9 - 6 + 2 = 5$ **A1****Answer: $(3, 3, 2)$** **2 marks****20 (6 marks)****(a)** Beam: $(2 + \lambda, 1 + 2\lambda, 8 - 4\lambda)$; $z = 0$ gives $\lambda = 2$ **M1 A1** $P = (4, 5, 0)$ **A1****Answer: $(4, 5, 0)$** **3 marks****(b)** Reflected beam: $(4 + \mu, 5 + 2\mu, 4\mu)$ **M1** $4\mu = 12$, so $\mu = 3$ **M1**Point $(7, 11, 12)$ **A1****Answer: $(7, 11, 12)$** **3 marks****21 (10 marks)****(a)** 2(first) - (second): $3y - z = 1$ **M1**(first) - (third): $3y + (1 - k)z = 1$; subtract: $(2 - k)z = 0$, so $z = 0$ when $k \neq 2$ **M1 A1** $y = \frac{1}{3}, x = 4 - \frac{2}{3} = \frac{10}{3}$ **M1 A1****Answer: $(10/3, 1/3, 0)$** **5 marks**

(b) For $k = 2$ the third equation is (second) – (first), so the equations are dependent

M1 A1

Direction $(1, 2, 1) \times (2, 1, 3) = 5\mathbf{i} - \mathbf{j} - 3\mathbf{k}$

M1

Point $(0, 1, 2)$ satisfies all three: $\mathbf{r} = \mathbf{j} + 2\mathbf{k} + \lambda(5\mathbf{i} - \mathbf{j} - 3\mathbf{k})$

A1

Answer: $\mathbf{r} = \mathbf{j} + 2\mathbf{k} + \lambda(5\mathbf{i} - \mathbf{j} - 3\mathbf{k})$

4 marks

(c) $k \neq 2$: the planes meet at one point; $k = 2$: they share a common line (a sheaf)

B1

Answer: Point; common line

1 mark

22 (8 marks)

(a) A $(6, 0, 0)$, B $(0, 3, 0)$, C $(0, 0, 2)$

B1

Answer: $(6, 0, 0)$, $(0, 3, 0)$, $(0, 0, 2)$

1 mark

(b) Line OF: $\mathbf{r} = \lambda(1, 2, 3)$

M1

$$\lambda + 4\lambda + 9\lambda = 6, \text{ so } \lambda = \frac{3}{7}$$

M1 A1

$$\mathbf{F} = \left(\frac{3}{7}, \frac{6}{7}, \frac{9}{7}\right)$$

A1

Answer: $\left(\frac{3}{7}, \frac{6}{7}, \frac{9}{7}\right)$

4 marks

$$(c) \text{ OF} = \frac{3}{7}\sqrt{14}$$

M1

$$\frac{1}{3} \times 3\sqrt{14} \times \frac{3\sqrt{14}}{7} = \frac{3 \times 14}{7} = 6$$

M1

$$\text{Volume} = \frac{1}{6} \times 6 \times 3 \times 2 = 6 \text{ as well}$$

A1

Answer: $\text{OF} = 3\sqrt{14}/7$; both give 6

3 marks

23 (7 marks)

(a) Ray: $(1 + \lambda, -2 + 4\lambda, 0)$; $(1 + \lambda) + 2(-2 + 4\lambda) = 6$

M1

$9\lambda - 3 = 6$, so $\lambda = 1$ and P = $(2, 2, 0)$

A1

Answer: Shown: the ray hits the mirror at P(2, 2, 0)

2 marks

(b) $\mathbf{d} \cdot \mathbf{n} = 1 + 8 = 9$ and $|\mathbf{n}|^2 = 9$

M1

$$\mathbf{d}' = (1, 4, 0) - 2(1, 2, 2) = -\mathbf{i} - 4\mathbf{k}$$

M1 A1

Answer: $-\mathbf{i} - 4\mathbf{k}$

3 marks

(c) Reflected ray: $(2 - \mu, 2, -4\mu)$

M1

$x = 0$ gives $\mu = 2$, so the point is $(0, 2, -8)$

A1

Answer: $(0, 2, -8)$

2 marks

24 (9 marks)

(a) Direction = $(1, -2, 1) \times (2, 1, -1) = \mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$

M1 A1

Point on l: put $x = 1$: $-2y + z = 2$ and $y - z = 2$, so $y = -4$, $z = -6$

M1

$$\mathbf{r} = \mathbf{i} - 4\mathbf{j} - 6\mathbf{k} + \lambda(\mathbf{i} + 3\mathbf{j} + 5\mathbf{k})$$

A1

Answer: $\mathbf{r} = \mathbf{i} - 4\mathbf{j} - 6\mathbf{k} + \lambda(\mathbf{i} + 3\mathbf{j} + 5\mathbf{k})$

4 marks

(b) $(1 + \lambda) + (-4 + 3\lambda) + (-6 + 5\lambda) = 9$, so $9\lambda = 18$

M1

$\lambda = 2$, point $(3, 2, 4)$

A1

Answer: $(3, 2, 4)$

2 marks

(c) Points on l make both brackets 0, so they satisfy the equation for every k

M1

$$\text{At } (1, 1, 1): (1 - 2 + 1 - 3) + k(2 + 1 - 1 - 4) = -3 - 2k = 0$$

M1

$$k = -\frac{3}{2}; \text{ the plane is } 4x + 7y - 5z = 6$$

A1

Answer: $k = -3/2$; $4x + 7y - 5z = 6$

3 marks

25 (6 marks)

(a) $(3t) + 2(4t) - 2(50 - 5t) = -16$, so $21t - 100 = -16$

M1 A1

$t = 4$, position $(12, 16, 30)$

A1

Answer: $t = 4$; $(12, 16, 30)$

3 marks

$$(b) \text{ Speed} = \sqrt{9 + 16 + 25} = 5\sqrt{2} = 7.07 \text{ m s}^{-1}$$

M1 A1

$$\text{Distance} = 4 \times 5\sqrt{2} = 20\sqrt{2} = 28.3 \text{ m}$$

A1

Answer: 7.07 m s^{-1} ; 28.3 m

3 marks

26 (7 marks)

(a) Normal = $(1, 1, 0) \times (0, 1, 1) = \mathbf{i} - \mathbf{j} + \mathbf{k}$ M1 A1

$x - y + z = 1$ A1

Answer: $x - y + z = 1$ 3 marks

(b) Direction = $(1, -1, 1) \times (1, 1, -1) = 2\mathbf{j} + 2\mathbf{k}$, parallel to $(0, 1, 1)$ M1 A1

Adding the equations: $2x = 4$, so $x = 2$; point $(2, 1, 0)$ M1

$\mathbf{r} = 2\mathbf{i} + \mathbf{j} + \lambda(\mathbf{j} + \mathbf{k})$ A1

Answer: $\mathbf{r} = 2\mathbf{i} + \mathbf{j} + \lambda(\mathbf{j} + \mathbf{k})$ 4 marks

27 (12 marks)

(a) Foot of perpendicular: $\mathbf{f} = \mathbf{p} + \lambda\mathbf{n}$ M1

$\mathbf{f} \cdot \mathbf{n} = d$: $\mathbf{p} \cdot \mathbf{n} + \lambda|\mathbf{n}|^2 = d$, so $\lambda = -\frac{\mathbf{p} \cdot \mathbf{n} - d}{|\mathbf{n}|^2}$ M1 A1

$\mathbf{p}' = 2\mathbf{f} - \mathbf{p} = \mathbf{p} + 2\lambda\mathbf{n}$, giving the result A1

Answer: Proved: $\mathbf{p}' = \mathbf{p} - 2\frac{\mathbf{p} \cdot \mathbf{n} - d}{|\mathbf{n}|^2}\mathbf{n}$ 4 marks

(b) $\mathbf{p} \cdot \mathbf{n} = 2 - 2 + 12 = 12$, $|\mathbf{n}|^2 = 9$ M1

$\mathbf{p}' = (2, 1, 6) - 2 \times \frac{9}{9}(1, -2, 2)$ M1

$= (0, 5, 2)$ A1

Answer: $(0, 5, 2)$ 3 marks

(c) $(2 + \lambda) - 2(1 + \lambda) + 2(6 + 5\lambda) = 12 + 9\lambda = 3$, so $\lambda = -1$ M1 A1

Intersection $(1, 0, 1)$ A1The reflected line passes through $(1, 0, 1)$ and through $\mathbf{P}' = (0, 5, 2)$, so has direction $(-1, 5, 1)$ M1

$\mathbf{r} = \mathbf{i} + \mathbf{k} + \mu(-\mathbf{i} + 5\mathbf{j} + \mathbf{k})$ A1

Answer: $\mathbf{r} = \mathbf{i} + \mathbf{k} + \mu(-\mathbf{i} + 5\mathbf{j} + \mathbf{k})$ 5 marks

28 (12 marks)

(a) Direction = $(1, 1, 1) \times (1, 2, 3) = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$ M1 A1

$z = 0$: $x + y = 6$, $x + 2y = 14$, so $y = 8$, $x = -2$ M1

$\mathbf{r} = -2\mathbf{i} + 8\mathbf{j} + \lambda(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$ A1

Answer: $\mathbf{r} = -2\mathbf{i} + 8\mathbf{j} + \lambda(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$ 4 marks

(b) Line l meets Π_3 where $2(-2 + \lambda) + 3(8 - 2\lambda) + a\lambda = b$ M1

$(a - 4)\lambda + 20 = b$ A1

(i) $a \neq 4$: one value of λ , a single point A1

(ii) $a = 4$ and $b = 20$: every λ works, so the planes share the line l A1

(iii) $a = 4$ and $b \neq 20$: no λ ; l is parallel to Π_3 and not in it, a triangular prism A1 A1

Answer: (i) $a \neq 4$; (ii) $a = 4$, $b = 20$; (iii) $a = 4$, $b \neq 20$ 6 marks

(c) $-4\lambda + 20 = 12$, so $\lambda = 2$ M1

Point $(0, 4, 2)$ A1

Answer: $(0, 4, 2)$ 2 marks

29 (11 marks)

(a) $\mathbf{F} = \lambda(1, 2, 2)$: $9\lambda = 9$, so $\lambda = 1$ M1 A1

$\mathbf{F} = (1, 2, 2)$, $OF = 3$ A1

Answer: $\mathbf{F} = (1, 2, 2)$; $OF = 3$ 3 marks

(b) A point X of the circle satisfies $OX^2 = OF^2 + FX^2$ (OF is perpendicular to the plane) M1

$FX^2 = 25 - 9 = 16$, so $FX = 4$ for every such point M1 A1

Circle of centre $\mathbf{F}(1, 2, 2)$ and radius 4 A1

Answer: Circle, centre $(1, 2, 2)$, radius 4 4 marks

(c) Check $(2, 1, -2) \cdot (1, 2, 2) = 0$ so the line lies in Π B1

Points are $\mathbf{F} \pm 4 \times \frac{1}{3}(2, 1, -2)$ M1 A1

$(\frac{11}{3}, \frac{10}{3}, -\frac{2}{3})$ and $(-\frac{5}{3}, \frac{2}{3}, \frac{14}{3})$ A1

Answer: $(\frac{11}{3}, \frac{10}{3}, -\frac{2}{3})$ and $(-\frac{5}{3}, \frac{2}{3}, \frac{14}{3})$ 4 marks

30 (11 marks)

(a) Each point of l satisfies both brackets $= 0$, so satisfies the combined equation

M1 A1

The combined equation is that of a plane, as its normal $\lambda n_1 + \mu n_2$ is non-zero (n_1, n_2 are not parallel)

M1 A1

Such planes contain l ; a plane containing l has normal perpendicular to the direction of l , which is spanned by n_1 and n_2 , so no others exist

A1

Answer: Proved: $\lambda(n_1 \cdot r - d_1) + \mu(n_2 \cdot r - d_2) = 0$

5 marks

(b) Normal $\lambda n_1 + \mu n_2 = (2\lambda + \mu, -\lambda + \mu, \lambda - 2\mu)$

M1

Perpendicular to $(1, 2, 1)$: $(2\lambda + \mu) + 2(-\lambda + \mu) + (\lambda - 2\mu) = \lambda + \mu = 0$

M1 A1

Take $\lambda = 1, \mu = -1$: $(2x - y + z - 3) - (x + y - 2z - 1) = 0$

M1

$x - 2y + 3z = 2$

A1

Check: $(1, -2, 3) \cdot (1, 2, 1) = 0$

B1

Answer: $x - 2y + 3z = 2$

6 marks

Angles Between Lines and Planes

1 (3 marks)

Normals $(1, 1, 1)$ and $(1, 0, 0)$

B1

$\cos \theta = \frac{1}{\sqrt{3}}$, so $\theta = 54.7^\circ$

M1 A1

Answer: 54.7°

3 marks

2 (3 marks)

$\sin \theta = \frac{|d \cdot n|}{|d||n|} = \frac{1}{3}$

M1 A1

$\theta = 19.5^\circ$

A1

Answer: 19.5°

3 marks

3 (2 marks)

$d_1 \cdot d_2 = 2 + 2 - 4 = 0$

M1

The lines are perpendicular: 90°

A1

Answer: 90°

2 marks

4 (3 marks)

$n_1 \cdot n_2 = 2 - 2 + 4 = 4$ and $|n_1| = |n_2| = 3$

M1 A1

$\cos \theta = \frac{4}{9}$, so $\theta = 63.6^\circ$

A1

Answer: 63.6°

3 marks

5 (3 marks)

Direction $(2, 1, 2)$, normal $(0, 0, 1)$

B1

$\sin \theta = \frac{2}{3}$, so $\theta = 41.8^\circ$

M1 A1

Answer: 41.8°

3 marks

6 (2 marks)

Normals $(2, 1, -1)$ and $(1, -2, 0)$

B1

Scalar product $2 - 2 + 0 = 0$, so the planes are perpendicular

M1

Answer: $n_1 \cdot n_2 = 0$

2 marks

7 (4 marks)

$d \cdot n = 2 - 2 + 4 = 4$, $|d| = 3$, $|n| = 3$

M1 A1

$\sin \theta = \frac{4}{9}$

M1

$\theta = 26.4^\circ$

A1

Answer: 26.4°

4 marks

8 (4 marks)

$$\mathbf{n}_1 \cdot \mathbf{n}_2 = 2 - 6 - 12 = -16, |\mathbf{n}_1| = 3, |\mathbf{n}_2| = 7$$

M1 A1

$$\cos \theta = \frac{16}{21} \text{ (taking the acute angle)}$$

M1

$$\theta = 40.4^\circ$$

A1

Answer: 40.4°

4 marks

9 (4 marks)

$$\mathbf{d}_1 \cdot \mathbf{d}_2 = 3 - 8 + 0 = -5, |\mathbf{d}_1| = 3, |\mathbf{d}_2| = 5$$

M1 A1

$$\cos \theta = \frac{5}{15} = \frac{1}{3}$$

M1

$$\theta = 70.5^\circ$$

A1

Answer: 70.5°

4 marks

10 (4 marks)

$$|\mathbf{d}| = 3$$

B1

$$\cos \alpha = \frac{2}{3}, \cos \beta = \frac{2}{3}, \cos \gamma = \frac{1}{3} \text{ (taking acute angles)}$$

M1 A1

$$\alpha = \beta = 48.2^\circ, \gamma = 70.5^\circ$$

A1

Answer: 48.2°, 48.2°, 70.5°

4 marks

11 (2 marks)

$$\mathbf{n}_1 \cdot \mathbf{n}_2 = 3k - 2 - 2 = 0$$

M1

$$k = \frac{4}{3}$$

A1

Answer: k = 4/3

2 marks

12 (4 marks)Normal (1, 2, 2), $|\mathbf{n}| = 3$; coordinate plane normals (1, 0, 0), (0, 1, 0), (0, 0, 1)

B1

$$\cos \theta_{yz} = \frac{1}{3}, \cos \theta_{xz} = \frac{2}{3}, \cos \theta_{xy} = \frac{2}{3}$$

M1 A1

$$\theta_{yz} = 70.5^\circ, \theta_{xz} = \theta_{xy} = 48.2^\circ$$

A1

Answer: 70.5°, 48.2°, 48.2°

4 marks

13 (4 marks)

$$\mathbf{AB} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}, |\mathbf{AB}| = 3$$

B1

$$\mathbf{d} \cdot \mathbf{n} = 12 - 4 - 3 = 5 \text{ and } |\mathbf{n}| = 7$$

M1 A1

$$\sin \theta = \frac{5}{21}, \theta = 13.8^\circ$$

A1

Answer: 13.8°

4 marks

14 (7 marks)

$$\text{(a) } \mathbf{OF} = 4\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}, \text{ normal to the base } (0, 0, 1): \sin \theta = \frac{2}{\sqrt{29}}$$

M1

$$\theta = 21.8^\circ$$

A1

Answer: 21.8°

2 marks

$$\text{(b) Normal} = \mathbf{BG} \times \mathbf{DG} = -4\mathbf{i} + 2\mathbf{k} \times 3\mathbf{j}, \text{ parallel to } (1, 0, 2)$$

M1 A1

$$x + 2z = 4$$

A1

Answer: x + 2z = 4

3 marks

$$\text{(c) Normals } (1, 0, 2) \text{ and } (0, 0, 1)$$

M1

$$\cos \theta = \frac{2}{\sqrt{5}}, \theta = 26.6^\circ$$

A1

Answer: 26.6°

2 marks

15 (5 marks)

Normal to $z = 0$ is $\mathbf{k}$: $\sin 30^\circ = \frac{2}{\sqrt{1+p^2+4}}$

M1 A1

$$\frac{1}{2} = \frac{2}{\sqrt{p^2+5}}, \text{ so } \sqrt{p^2+5} = 4$$

M1

$$p^2 + 5 = 16, p^2 = 11$$

M1

$$p = \sqrt{11}$$

A1

Answer: $p = \sqrt{11}$

5 marks

16 (5 marks)

(a) A (1, 0, 0), B (0, 2, 0), C (0, 0, 3)

B1

Answer: (1, 0, 0), (0, 2, 0), (0, 0, 3)

1 mark

(b) Normals (6, 3, 2) and (0, 0, 1), $|\mathbf{n}| = 7$

M1

$$\cos \theta = \frac{2}{7}, \theta = 73.4^\circ$$

A1

Answer: 73.4°

2 marks

(c) $\cos \theta = \frac{6}{7}$

M1

$$\theta = 31.0^\circ$$

A1

Answer: 31.0°

2 marks

17 (6 marks)

Direction of $\mathbf{l} = (1, 2, 1) \times (2, -1, 1) = 3\mathbf{i} + \mathbf{j} - 5\mathbf{k}$

M1 A1

$$\mathbf{d} \cdot \mathbf{n} = 3 + 1 - 5 = -1, |\mathbf{d}| = \sqrt{35}, |\mathbf{n}| = \sqrt{3}$$

M1 A1

$$\sin \theta = \frac{1}{\sqrt{105}}$$

M1

$$\theta = 5.6^\circ$$

A1

Answer: 5.6°

6 marks

18 (6 marks)

(a) $\mathbf{d} \cdot \mathbf{n} = 2 - 1 + 8 = 9, |\mathbf{d}| = 3\sqrt{2}, |\mathbf{n}| = 3$

M1 A1

$$\sin \theta = \frac{9}{9\sqrt{2}} = \frac{1}{\sqrt{2}}, \theta = 45^\circ$$

A1

Answer: 45°

3 marks

(b) $\mathbf{d}' = (1, 1, 4) - 2(2, -1, 2) = -3\mathbf{i} + 3\mathbf{j}$

M1 A1

$$\mathbf{d} \cdot \mathbf{d}' = -3 + 3 + 0 = 0$$

A1

Answer: $-3\mathbf{i} + 3\mathbf{j}$

3 marks

19 (4 marks)

(a) The scalar product formula with cos gives the angle between the line and the normal to the plane, not the plane itself

M1 E1

Answer: 15.8° is the angle with the normal

2 marks

(b) $\sin \theta = \frac{5}{3\sqrt{3}} = 0.9623$

M1

$$\theta = 74.2^\circ (= 90^\circ - 15.8^\circ)$$

A1

Answer: 74.2°

2 marks

20 (4 marks)

Direction of $\mathbf{l}_2 = (1, 1, 1) \times (1, -1, 2) = 3\mathbf{i} - \mathbf{j} - 2\mathbf{k}$

M1 A1

$$\mathbf{d}_1 \cdot \mathbf{d}_2 = 3 + 1 - 4 = 0$$

M1

So the lines are perpendicular

A1

Answer: $\mathbf{d}_1 \cdot \mathbf{d}_2 = 0$

4 marks

21 (9 marks)

(a) Normal to front face = $\mathbf{AB} \times \mathbf{AR}_1 = -16\mathbf{j} + 24\mathbf{k}$, parallel to $(0, -2, 3)$

M1 A1

$$\cos \theta = \frac{3}{\sqrt{13}}, \theta = 33.7^\circ$$

A1

Answer: 33.7°

3 marks

(b) Normal = $\mathbf{AD} \times \mathbf{AR}_1 = 12\mathbf{i} - 12\mathbf{k}$, parallel to $(1, 0, -1)$

M1

$$\cos \theta = \frac{1}{\sqrt{2}}, \theta = 45^\circ$$

A1

Answer: 45°

2 marks

(c) $\mathbf{AR}_1 = (2, 3, 2)$, $\sin \theta = \frac{2}{\sqrt{17}}$

M1

$$\theta = 29.0^\circ$$

A1

Answer: 29.0°

2 marks

(d) Outward normals $(0, -2, 3)$ and $(-1, 0, 1)$: $\cos \varphi = \frac{3}{\sqrt{13}\sqrt{2}}$, $\varphi = 54.0^\circ$

M1

$$\text{Interior angle} = 180^\circ - \varphi = 126.0^\circ$$

A1

Answer: 126.0°

2 marks

22 (6 marks)

$\mathbf{AB} = (-2, 2, 0)$, $\mathbf{AC} = (-2, 0, c)$

B1

Normal = $\mathbf{AB} \times \mathbf{AC} = (2c, 2c, 4)$, parallel to $(c, c, 2)$

M1 A1

$$\cos 45^\circ = \frac{2}{\sqrt{2c^2 + 4}}$$

M1

$$2c^2 + 4 = 8, \text{ so } c^2 = 2$$

M1

$$c = \sqrt{2}$$

A1

Answer: $c = \sqrt{2}$

6 marks

23 (7 marks)

(a) $\mathbf{OF} = 5\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$ and $\mathbf{AG} = -5\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$

B1

$$\mathbf{OF} \cdot \mathbf{AG} = -25 + 16 + 9 = 0$$

M1 A1

Answer: $\mathbf{OF} \cdot \mathbf{AG} = 0$

3 marks

(b) $\mathbf{OB} = 5\mathbf{i} + 4\mathbf{j}$ and $\mathbf{AF} = 4\mathbf{j} + 3\mathbf{k}$

B1

$$\cos \theta = \frac{16}{\sqrt{41} \times 5}$$

M1 A1

$$\theta = 60.0^\circ$$

A1

Answer: 60.0°

4 marks

24 (7 marks)

(a) $\mathbf{P}_1\mathbf{P}_2 \times \mathbf{P}_1\mathbf{P}_3 = 4\mathbf{i} + 4\mathbf{j} - 4\mathbf{k}$

M1 A1

$\mathbf{P}_1\mathbf{P}_2 \times \mathbf{P}_1\mathbf{P}_4 = -4\mathbf{i} + 4\mathbf{j} - 4\mathbf{k}$

A1

Answer: $(1, 1, -1)$ and $(-1, 1, -1)$ (multiples)

3 marks

(b) $\mathbf{n}_1 \cdot \mathbf{n}_2 = -1 + 1 + 1 = 1$ for the simplified normals, $|\mathbf{n}| = \sqrt{3}$

M1

$$\cos \theta = \frac{1}{3}, \theta = 70.5^\circ \text{ (acute angle between the planes)}$$

A1

Answer: 70.5°

2 marks

(c) Face $\mathbf{P}_2\mathbf{P}_3\mathbf{P}_4$ is $x + y + z = -1$, with normal $(1, 1, 1)$

M1

$\mathbf{OP}_1 = (1, 1, 1)$ is parallel to this normal

A1

Answer: $\mathbf{OP}_1$ is parallel to the normal

2 marks

25 (8 marks)

(a) Normal $(3, 4, 12)$, $|\mathbf{n}| = 13$; normal to horizontal $(0, 0, 1)$

M1

$$\cos \theta = \frac{12}{13}, \theta = 22.6^\circ$$

A1

Answer: 22.6°

2 marks

(b) $\mathbf{n} \cdot \mathbf{d} = 108 + 192 - 300 = 0$, so $\mathbf{d}$ lies in the plane

M1 A1

$$|d| = 65, \sin \varphi = \frac{25}{65} = \frac{5}{13}, \varphi = 22.6^\circ$$

M1 A1

Answer: 22.6°

4 marks

(c) $n \cdot d = 12 - 12 + 0 = 0$, so it lies in the plane, and its k-component is 0

M1 A1

Answer: Lies in the plane with no vertical component

2 marks

26 (8 marks)

(a) $3 + 0 + 0 = 3$ and $6 + 0 - 0 = 6$

B1

Answer: On l

1 mark

$$(b) \hat{n}_1 = \frac{1}{3}(1, 2, 2), \hat{n}_2 = \frac{1}{3}(2, 1, -2)$$

M1

$$\hat{n}_1 + \hat{n}_2 = (1, 1, 0) \text{ and } \hat{n}_1 - \hat{n}_2 = \frac{1}{3}(-1, 1, 4)$$

M1 A1

Through $(3, 0, 0)$: $x + y = 3$

A1

and $-x + y + 4z = -3$

A1

Answer: $x + y = 3$ and $x - y - 4z = 3$

5 marks

(c) Normals $(1, 1, 0)$, $(1, 2, 2)$, $(2, 1, -2)$: $\cos = \frac{3}{\sqrt{2} \times 3}$ for both

M1 A1

Answer: Both $\cos = 1/\sqrt{2}$ (45°)

2 marks

27 (9 marks)

(a) The normal is perpendicular to the direction of the line: $a + 2b + 2 = 0$

M1 A1

Answer: $a = -2b - 2$

2 marks

$$(b) \cos \theta = \frac{|n \cdot k|}{|n|} = \frac{1}{\sqrt{a^2 + b^2 + 1}} = \frac{2}{3}$$

M1

$$a^2 + b^2 + 1 = \frac{9}{4}: (2b + 2)^2 + b^2 + 1 = \frac{9}{4}$$

M1

$$5b^2 + 8b + \frac{11}{4} = 0, \text{ so } 20b^2 + 32b + 11 = 0$$

A1

Answer: Shown: $20b^2 + 32b + 11 = 0$

3 marks

(c) $(2b + 1)(10b + 11) = 0$, so $b = -\frac{1}{2}$ or $b = -\frac{11}{10}$

M1 A1

$b = -\frac{1}{2}$: $a = -1$, $n \propto (2, 1, -2)$: $2x + y - 2z = 0$

A1

$b = -\frac{11}{10}$: $a = \frac{1}{5}$, $n \propto (2, -11, 10)$: $2x - 11y + 10z = 0$

A1

Answer: $2x + y - 2z = 0$ and $2x - 11y + 10z = 0$

4 marks

28 (10 marks)

(a) $n \cdot g = \sin \alpha \cos \alpha - \cos \alpha \sin \alpha = 0$ and $n \cdot e = 0$, and $|n| = 1$

M1 A1

$u = \cos \varphi g + \sin \varphi e = (\sin \varphi, \cos \varphi \cos \alpha, -\cos \varphi \sin \alpha)$

A1

Answer: $n = (0, \sin \alpha, \cos \alpha)$; $u = (\sin \varphi, \cos \varphi \cos \alpha, -\cos \varphi \sin \alpha)$

3 marks

(b) The angle with the horizontal plane has $\sin \theta = |u \cdot k|$ (u is a unit vector)

M1

$|u \cdot k| = \cos \varphi \sin \alpha$

A1

Answer: $\sin \theta = \sin \alpha \cos \varphi$

2 marks

(c) $\sin \theta = \sin 40^\circ \cos 30^\circ = 0.5567$

M1

$\theta = 33.8^\circ$

A1

Answer: 33.8°

2 marks

(d) $\cos \varphi = \frac{\sin 20^\circ}{\sin 50^\circ} = 0.4465$

M1 A1

$\varphi = 63.5^\circ$

A1

Answer: 63.5°

3 marks

29 (10 marks)

(a) $\mathbf{OP} = (1, t, 2); \mathbf{n} = (1, 1, 1)$

B1

$$\sin \theta = \frac{|t+3|}{\sqrt{3}\sqrt{t^2+5}}$$

M1 A1

Squaring gives the result

A1

Answer: Shown: $\sin^2 \theta = \frac{(t+3)^2}{3(t^2+5)}$

4 marks

(b) Let $f(t) = \frac{(t+3)^2}{t^2+5}; f'(t) = \frac{2(t+3)(5-3t)}{(t^2+5)^2}$

M1 A1

$$f' = 0 \text{ at } t = -3 \text{ (where } f = 0, \text{ the least value) or } t = \frac{5}{3}$$

M1

$$f' > 0 \text{ for } -3 < t < \frac{5}{3} \text{ and } f' < 0 \text{ for } t > \frac{5}{3}, \text{ so } t = \frac{5}{3} \text{ gives the greatest angle}$$

A1

Answer: $t = 5/3$

4 marks

(c) $f(\frac{5}{3}) = \frac{196}{9} \div \frac{70}{9} = \frac{14}{5}$

M1

$$\sin^2 \theta = \frac{14}{15}, \theta = 75.0^\circ$$

A1

Answer: 75.0°

2 marks**30 (11 marks)**

(a) $\mathbf{AB} \times \mathbf{AV} = (6, 0, 0) \times (3, 3, h) = (0, -6h, 18), \text{ parallel to } (0, -h, 3)$

M1 A1

$$\cos 60^\circ = \frac{3}{\sqrt{h^2+9}} = \frac{1}{2}$$

M1

$$h^2 + 9 = 36, \text{ so } h = \sqrt{27} = 3\sqrt{3}$$

A1

Answer: $h = 3\sqrt{3}$

4 marks

(b) $\mathbf{AV} = (3, 3, 3\sqrt{3}), |\mathbf{AV}| = \sqrt{45}$

M1

$$\sin \varphi = \frac{3\sqrt{3}}{\sqrt{45}} = \sqrt{\frac{3}{5}}, \varphi = 50.8^\circ$$

M1 A1

Answer: 50.8°

3 marks

(c) Outward normals $(0, -h, 3)$ and $(h, 0, 3)$

B1

$$\cos \psi = \frac{9}{36} = \frac{1}{4}, \psi = 75.5^\circ$$

M1 A1

Interior angle $= 180^\circ - \psi = 104.5^\circ$

A1

Answer: 104.5°

4 marks**Shortest Distances in 3D****1 (2 marks)**

Distance $= \frac{|d|}{|n|} = \frac{9}{\sqrt{4+1+4}}$

M1

$$= 3$$

A1

Answer: 3

2 marks**2 (2 marks)**

Distance $= \frac{|1+4+6-2|}{\sqrt{1+4+4}}$

M1

$$= \frac{9}{3} = 3$$

A1

Answer: 3

2 marks**3 (2 marks)**

Distance $= \frac{|12-3|}{|n|}$ with $|n| = 3$

M1

$$= 3$$

A1

Answer: 3

2 marks

4 (2 marks)

$$\text{Distance} = \frac{|3 + 8 - 10|}{\sqrt{9 + 16}}$$

M1

$$= \frac{1}{5}$$

A1

$$\text{Answer: } \frac{1}{5}$$

2 marks

5 (3 marks)

$$\mathbf{OA} \times \mathbf{d} = -4\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$$

M1

$$\text{Distance} = \frac{|\mathbf{OA} \times \mathbf{d}|}{|\mathbf{d}|} = \frac{\sqrt{50}}{3} = \frac{5\sqrt{2}}{3}$$

M1 A1

$$\text{Answer: } \frac{5\sqrt{2}}{3}$$

3 marks

6 (2 marks)

$$\text{Distance} = \frac{|18 - 2 - 6 - 4|}{\sqrt{36 + 4 + 9}}$$

M1

$$= \frac{6}{7}$$

A1

$$\text{Answer: } \frac{6}{7}$$

2 marks

7 (4 marks)

$$\text{Distance} = \frac{|2p - 2 + 2 - 7|}{\sqrt{4 + 4 + 1}} = \frac{|2p - 7|}{3}$$

M1 A1

$$|2p - 7| = 9$$

M1

$$p = 8 \text{ or } p = -1$$

A1

$$\text{Answer: } p = 8 \text{ or } p = -1$$

4 marks

8 (4 marks)

$$\mathbf{a}_2 - \mathbf{a}_1 = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$$

B1

$$(\mathbf{a}_2 - \mathbf{a}_1) \times \mathbf{d} = 6\mathbf{i} - 6\mathbf{j} + 3\mathbf{k}, \text{ magnitude } 9$$

M1 A1

$$\text{Distance} = \frac{9}{|\mathbf{d}|} = \frac{9}{3} = 3$$

A1

$$\text{Answer: } 3$$

4 marks

9 (5 marks)

$$\mathbf{AB} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}, \mathbf{AP} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$$

B1

$$\mathbf{AP} \times \mathbf{AB} = -7\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$$

M1 A1

$$\text{Distance} = \frac{\sqrt{90}}{|\mathbf{AB}|} = \frac{\sqrt{90}}{3} = \sqrt{10} = 3.16$$

M1 A1

$$\text{Answer: } \sqrt{10}$$

5 marks

10 (3 marks)

$$\text{Rewrite the second as } 2x - y + 2z = -3$$

M1

$$\text{Distance} = \frac{|3 - (-3)|}{\sqrt{4 + 1 + 4}} = \frac{6}{3} = 2$$

M1 A1

$$\text{Answer: } 2$$

3 marks

11 (4 marks)

$$\mathbf{OX} = (3 + t, 3 + 2t, 2t) \text{ and } \mathbf{OX} \text{ is perpendicular to } \mathbf{l}$$

M1

$$(3 + t) + 2(3 + 2t) + 2(2t) = 9 + 9t = 0, \text{ so } t = -1$$

M1 A1

$$\mathbf{X} = (2, 1, -2), \text{ distance} = \sqrt{4 + 1 + 4} = 3$$

A1

$$\text{Answer: } 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}; \text{ distance } 3$$

4 marks

12 (5 marks)

Both lines have direction $2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, so they are parallel B1

$\mathbf{P}_1\mathbf{P}_2 = (2, -3, 1)$, where $P_1(1, 2, 1)$ is on l_1 and $P_2(3, -1, 2)$ is on l_2 M1

$\mathbf{P}_1\mathbf{P}_2 \times \mathbf{d} = 5\mathbf{i} + 6\mathbf{j} + 8\mathbf{k}$ M1 A1

$$\text{Distance} = \frac{\sqrt{125}}{3} = \frac{5\sqrt{5}}{3}$$
A1

Answer: Parallel; distance $\frac{5\sqrt{5}}{3}$ 5 marks

13 (3 marks)

$$\text{Distance from P to the plane} = \frac{|1 + 1 + 1 - 6|}{\sqrt{3}} = \sqrt{3}$$
M1 A1

The plane is the perpendicular bisector of PP' , so $PP' = 2\sqrt{3}$ A1

Answer: $2\sqrt{3}$ 3 marks

14 (6 marks)

Common perpendicular direction = $\mathbf{d}_1 \times \mathbf{d}_2 = -\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}$ M1 A1

$$\mathbf{a}_2 - \mathbf{a}_1 = -\mathbf{i} - \mathbf{j} + 2\mathbf{k}$$
B1

$$\text{Distance} = \frac{|(\mathbf{a}_2 - \mathbf{a}_1) \cdot (\mathbf{d}_1 \times \mathbf{d}_2)|}{|\mathbf{d}_1 \times \mathbf{d}_2|} = \frac{|(-1) \times (-1) + (-1) \times 5 + 2 \times 3|}{\sqrt{35}}$$
M1 A1

$$= \frac{2}{\sqrt{35}} = 0.338$$
A1

Answer: 0.338 6 marks

15 (5 marks)

$$\text{(a) } \mathbf{PF} = (1 + 2t - 4, t + 1, 1 - t) = (2t - 3, t + 1, 1 - t)$$
M1

$$\mathbf{PF} \cdot \mathbf{d} = 2(2t - 3) + (t + 1) - (1 - t) = 6t - 6 = 0, \text{ so } t = 1$$
M1 A1

$$\mathbf{F} = (3, 1, 0)$$
A1

Answer: (3, 1, 0) 4 marks

$$\text{(b) } PF = \sqrt{1 + 4 + 0} = \sqrt{5}$$
A1

Answer: $\sqrt{5}$ 1 mark

16 (4 marks)

$$\mathbf{d} \cdot \mathbf{n} = 2 + 2 - 4 = 0, \text{ so } l \text{ is parallel to } \Pi$$
M1 A1

$$\text{Distance from } (1, 2, 0) \text{ to } \Pi = \frac{|1 + 4 + 0 - 10|}{\sqrt{9}}$$
M1

$$= \frac{5}{3}$$
A1

Answer: $\frac{5}{3}$ 4 marks

17 (4 marks)

(a) Each of A, B, C satisfies $6x + 3y + 2z = 6$ B1

Answer: Shown: Π has equation $6x + 3y + 2z = 6$ 1 mark

$$\text{(b) Distance} = \frac{6}{\sqrt{36 + 9 + 4}} = \frac{6}{7}$$
M1 A1

$$\text{Volume} = \frac{1}{3} \times \frac{7}{2} \times \frac{6}{7} = 1$$
A1

Answer: $\frac{6}{7}$; volume 1 3 marks

18 (3 marks)

(a) The numerator 5 must be divided by $|n| = \sqrt{4 + 4 + 1} = 3$ M1 E1

Answer: Division by $|n|$ 2 marks

$$\text{(b) Distance} = \frac{5}{3}$$
A1

Answer: $\frac{5}{3}$ 1 mark

19 (4 marks)**(a)** The radius is the distance from C to the plane**M1**

$$\frac{|2 - 2 - 2 - 7|}{3} = 3$$

A1**Answer: 3****2 marks****(b)** Contact point = C + λn : $2(1 + 2\lambda) - (2 - \lambda) + 2(-1 + 2\lambda) = 7$, so $9\lambda - 2 = 7$ **M1** $\lambda = 1$, point (3, 1, 1)**A1****Answer: (3, 1, 1)****2 marks****20 (6 marks)**

$$\mathbf{PX} = (-3 + 2t, -4 + 2t, 1 + t)$$

M1

$$\mathbf{PX}^2 = 9 - 12t + 4t^2 + 16 - 16t + 4t^2 + 1 + 2t + t^2 = 9t^2 - 26t + 26$$

M1 A1

$$= 9\left(t - \frac{13}{9}\right)^2 + 26 - \frac{169}{9} = 9\left(t - \frac{13}{9}\right)^2 + \frac{65}{9}$$

M1 A1

$$\text{Least distance} = \frac{\sqrt{65}}{3} = 2.69$$

A1

$$\text{Answer: } \frac{\sqrt{65}}{3}$$

6 marks**21 (9 marks)****(a)** $d_1 \times d_2 = \mathbf{i} - \mathbf{j} - 2\mathbf{k}$, magnitude $\sqrt{6}$ **M1 A1**

$$(a_2 - a_1) \cdot (d_1 \times d_2) = (0, 0, 2) \cdot (1, -1, -2) = -4$$

M1

$$\text{Distance} = \frac{4}{\sqrt{6}} = \frac{2\sqrt{6}}{3}$$

A1

$$\text{Answer: } \frac{2\sqrt{6}}{3}$$

4 marks**(b)** $X = (\lambda, \lambda, 0)$, $Y = (\mu, -\mu, 2 + \mu)$, so $\mathbf{XY} = (\mu - \lambda, -\mu - \lambda, 2 + \mu)$ **M1**

$$\mathbf{XY} \cdot d_1 = (\mu - \lambda) + (-\mu - \lambda) = -2\lambda = 0, \text{ so } \lambda = 0$$

M1

$$\mathbf{XY} \cdot d_2 = 3\mu + 2 = 0, \text{ so } \mu = -\frac{2}{3}; X = (0, 0, 0), Y = \left(-\frac{2}{3}, \frac{2}{3}, \frac{4}{3}\right)$$

A1**Answer: X = (0, 0, 0), Y = (-2/3, 2/3, 4/3)****3 marks****(c)** Direction (1, -1, -2)**B1** $\mathbf{r} = s(\mathbf{i} - \mathbf{j} - 2\mathbf{k})$ (through X, the origin)**A1****Answer: r = s(i - j - 2k)****2 marks****22 (7 marks)****(a)** $\mathbf{AB} = 30\mathbf{i} + 40\mathbf{j} - 10\mathbf{k}$, $\mathbf{AP} = 20\mathbf{i} + 5\mathbf{j} - 6\mathbf{k}$ **B1**

$$\mathbf{AP} \times \mathbf{AB} = 190\mathbf{i} + 20\mathbf{j} + 650\mathbf{k}$$

M1 A1

$$\text{Distance} = \frac{|\mathbf{AP} \times \mathbf{AB}|}{|\mathbf{AB}|} = \frac{\sqrt{459000}}{\sqrt{2600}} = 13.3 \text{ m}$$

M1 A1**Answer: 13.3 m****5 marks**

$$\text{(b) Distance along} = \frac{\mathbf{AP} \cdot \mathbf{AB}}{|\mathbf{AB}|} = \frac{860}{\sqrt{2600}}$$

M1

$$= 16.9 \text{ m}$$

A1**Answer: 16.9 m****2 marks****23 (8 marks)****(a)** $X = (1 + \lambda, \lambda, 0)$, $Y = (0, 1 + \mu, \mu)$ **B1**

$$\mathbf{XY} = (-1 - \lambda, 1 + \mu - \lambda, \mu)$$

A1**Answer: (-1 - λ, 1 + μ - λ, μ)****2 marks****(b)** $\mathbf{XY} \cdot d_1 = 0$: $-1 - \lambda + 1 + \mu - \lambda = \mu - 2\lambda = 0$ **M1 A1**

$$\mathbf{XY} \cdot d_2 = 0$$
: $1 + \mu - \lambda + \mu = 1 + 2\mu - \lambda = 0$

M1

$$\mu = 2\lambda \text{ so } 1 + 3\lambda = 0: \lambda = -\frac{1}{3}, \mu = -\frac{2}{3}$$

A1**Answer: λ = -1/3, μ = -2/3****4 marks**

$$\text{(c) } \mathbf{XY} = \left(-\frac{2}{3}, \frac{2}{3}, -\frac{2}{3}\right)$$

M1

$$\text{Distance} = \frac{2}{\sqrt{3}} = 1.15$$

A1

$$\text{Answer: } \frac{2}{\sqrt{3}}$$

2 marks

24 (7 marks)

(a) Planes $x + 2y + 2z = c$ with $\frac{|1 + 2 + 2 - c|}{3} = 2$

M1

$$|5 - c| = 6, \text{ so } c = 11 \text{ or } c = -1$$

A1

$$x + 2y + 2z = 11 \text{ and } x + 2y + 2z = -1$$

A1

Answer: $x + 2y + 2z = 11$ and $x + 2y + 2z = -1$

3 marks

(b) Line: $(1 + \lambda, 1 + 2\lambda, 1 + 2\lambda)$

B1

$$5 + 9\lambda = 11 \text{ gives } \lambda = \frac{2}{3}; 5 + 9\lambda = -1 \text{ gives } \lambda = -\frac{2}{3}$$

M1

$$\left(\frac{5}{3}, \frac{7}{3}, \frac{7}{3}\right) \text{ and } \left(\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}\right)$$

A1

Answer: $\left(\frac{5}{3}, \frac{7}{3}, \frac{7}{3}\right)$ and $\left(\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}\right)$

3 marks

(c) $\frac{|11 - (-1)|}{3} = 4$

A1

Answer: 4

1 mark

25 (7 marks)

(a) $PX = (t - 3, 2t - 1, 2t - 2)$ for X on l

M1

$$PX \cdot d = (t - 3) + 2(2t - 1) + 2(2t - 2) = 9t - 9 = 0, \text{ so } t = 1$$

M1 A1

$$F = (1, 3, 4)$$

A1

Answer: (1, 3, 4)

4 marks

(b) $PF = \sqrt{4 + 1 + 0} = \sqrt{5}$

A1

Answer: $\sqrt{5}$

1 mark

(c) F is the midpoint of PP': $p' = 2f - p$

M1

$$= (-1, 4, 4)$$

A1

Answer: (-1, 4, 4)

2 marks

26 (8 marks)

(a) $AB \times AC = 18i + 12j + 6k$, parallel to $(3, 2, 1)$

M1 A1

$$3(2) = 6, \text{ so the equation is } 3x + 2y + z = 6$$

A1

Answer: Shown: the plane ABC has equation $3x + 2y + z = 6$

3 marks

(b) Distance = $\frac{6}{\sqrt{9 + 4 + 1}} = \frac{6}{\sqrt{14}}$

M1 A1

Answer: $\frac{6}{\sqrt{14}}$

2 marks

(c) Area = $\frac{1}{2}|18i + 12j + 6k| = \frac{1}{2}\sqrt{504} = 3\sqrt{14}$

M1 A1

$$\text{Volume} = \frac{1}{3} \times 3\sqrt{14} \times \frac{6}{\sqrt{14}} = 6$$

A1

Answer: $3\sqrt{14}$

3 marks

27 (11 marks)

(a) The foot F of the perpendicular is $p + \lambda n$, with $(p + \lambda n) \cdot n = d$

M1

$$\lambda = \frac{d - p \cdot n}{|n|^2}$$

A1

$$PF = |\lambda||n| = \frac{|p \cdot n - d|}{|n|}$$

A1

Answer: Proved: the shortest distance from P to Π is $\frac{|p \cdot n - d|}{|n|}$

3 marks

(b) The parallelogram with sides AP and d has area $|AP \times d|$

M1

Its base is $|d|$ and its height is the required distance

M1

$$\text{Distance} = \frac{|AP \times d|}{|d|}$$

A1

Answer: Proved: the shortest distance from P to l is $\frac{|(\mathbf{p} - \mathbf{a}) \times \mathbf{d}|}{|\mathbf{d}|}$ 3 marks

(c) $\mathbf{AP} \times \mathbf{d} = -2\mathbf{i} + \mathbf{k}$, so distance to the line = $\frac{\sqrt{5}}{3}$ M1 A1

Plane through A perpendicular to l: distance = $\frac{\mathbf{AP} \cdot \mathbf{d}}{|\mathbf{d}|} = \frac{7}{3}$ M1 A1

$\frac{5}{9} + \frac{49}{9} = 6 = \mathbf{AP}^2$ A1

Answer: $\frac{\sqrt{5}}{3}$ and $\frac{7}{3}$ 5 marks

28 (11 marks)

(a) The common perpendicular has the direction of $\mathbf{n} = \mathbf{d}_1 \times \mathbf{d}_2$ M1

For any points X on l_1 and Y on l_2 , $\mathbf{XY} = (\mathbf{a}_2 - \mathbf{a}_1) + \mu\mathbf{d}_2 - \lambda\mathbf{d}_1$, and the component of this along $\mathbf{n}$ is independent of λ and μ since $\mathbf{d}_1 \cdot \mathbf{n} = \mathbf{d}_2 \cdot \mathbf{n} = 0$ M1 A1

The shortest distance is the magnitude of the component of $\mathbf{a}_2 - \mathbf{a}_1$ in the direction $\hat{\mathbf{n}}$, which is the stated expression A1

Answer: Proved: the shortest distance between them is $\frac{|(\mathbf{a}_2 - \mathbf{a}_1) \cdot (\mathbf{d}_1 \times \mathbf{d}_2)|}{|\mathbf{d}_1 \times \mathbf{d}_2|}$ 4 marks

(b) $\mathbf{d}_1 \times \mathbf{d}_2 = \mathbf{i} - \mathbf{j} + \mathbf{k}$, magnitude $\sqrt{3}$ M1 A1

$(\mathbf{a}_2 - \mathbf{a}_1) \cdot (\mathbf{d}_1 \times \mathbf{d}_2) = (1, 0, \mathbf{p}) \cdot (1, -1, 1) = 1 + \mathbf{p}$ M1 A1

$\frac{|1 + \mathbf{p}|}{\sqrt{3}} = 2$, so $|1 + \mathbf{p}| = 2\sqrt{3}$ M1

$\mathbf{p} = -1 + 2\sqrt{3}$ or $\mathbf{p} = -1 - 2\sqrt{3}$ A1 A1

Answer: $\mathbf{p} = -1 \pm 2\sqrt{3}$ 7 marks

29 (11 marks)

(a) Distance = $\frac{|6 - (-12)|}{3} = 6 = 2 \times \text{radius}$ M1 A1

The centre is at distance 3 from each plane, so it lies midway between them: $\frac{6 + (-12)}{2} = -3$ M1 A1

Answer: Distance 6; centre on $x + 2y + 2z = -3$ 4 marks

(b) $(1 + 2t) + 2(1 + t) + 2(1 + 2t) = 5 + 8t = -3$, so $t = -1$ M1 A1

Centre $(-1, 0, -1)$ A1

S: $(x + 1)^2 + y^2 + (z + 1)^2 = 9$ A1

Answer: $(-1, 0, -1)$; $(x + 1)^2 + y^2 + (z + 1)^2 = 9$ 4 marks

(c) The contact points are the centre $\pm 3\hat{\mathbf{n}}$, where $\hat{\mathbf{n}} = \frac{1}{3}(1, 2, 2)$ M1

$(0, 2, 1)$ on Π_1 and $(-2, -2, -3)$ on Π_2 A1 A1

Answer: $(0, 2, 1)$ and $(-2, -2, -3)$ 3 marks

30 (11 marks)

(a) Foot of the perpendicular from B: $(3 + \lambda, \lambda, \lambda)$, so $3 + 3\lambda = 0$, $\lambda = -1$ M1 A1

$\mathbf{B}' = 2(2, -1, -1) - (3, 0, 0) = (1, -2, -2)$ A1

Answer: $(1, -2, -2)$ 3 marks

(b) Π is the perpendicular bisector of $\mathbf{BB}'$, so $\mathbf{XB} = \mathbf{XB}'$ M1 E1

$\mathbf{AX} + \mathbf{XB}' \geq \mathbf{AB}'$ with equality when X lies on the straight line $\mathbf{AB}'$ (triangle inequality) E1

Answer: $\mathbf{XB} = \mathbf{XB}'$; $\mathbf{AX} + \mathbf{XB}' \geq \mathbf{AB}'$ 3 marks

(c) Line $\mathbf{AB}'$: $(2 - s, 1 - 3s, 1 - 3s)$ M1

On Π : $4 - 7s = 0$, so $s = \frac{4}{7}$ M1 A1

$\mathbf{X} = (\frac{10}{7}, -\frac{5}{7}, -\frac{5}{7})$ A1

Least length = $\mathbf{AB}' = \sqrt{1 + 9 + 9} = \sqrt{19}$ A1

Answer: $\mathbf{X} = (\frac{10}{7}, -\frac{5}{7}, -\frac{5}{7})$; $\sqrt{19}$ 5 marks

Vector Geometry Problems

1 (2 marks)

$$\mathbf{MN} = \frac{1}{2}(\mathbf{a} + \mathbf{c}) - \frac{1}{2}(\mathbf{a} + \mathbf{b}) \quad \text{M1}$$

$$= \frac{1}{2}(\mathbf{c} - \mathbf{b}) = \frac{1}{2}\mathbf{BC} \quad \text{A1}$$

$$\text{Answer: Shown: } \mathbf{MN} = \frac{1}{2}\mathbf{BC} \quad \text{2 marks}$$

2 (2 marks)

$$\mathbf{OM} = \frac{1}{2}(\mathbf{a} + \mathbf{b}) \quad \text{B1}$$

$$\mathbf{AM} = \mathbf{OM} - \mathbf{a} = \frac{1}{2}(\mathbf{b} - \mathbf{a}) \quad \text{B1}$$

$$\text{Answer: } \mathbf{OM} = (\mathbf{a} + \mathbf{b})/2, \mathbf{AM} = (\mathbf{b} - \mathbf{a})/2 \quad \text{2 marks}$$

3 (2 marks)

$$\mathbf{AC} = \mathbf{AB} + \mathbf{BC} = \mathbf{a} + \mathbf{b} \quad \text{B1}$$

$$\mathbf{BD} = \mathbf{BA} + \mathbf{AD} = \mathbf{b} - \mathbf{a} \quad \text{B1}$$

$$\text{Answer: } \mathbf{AC} = \mathbf{a} + \mathbf{b}, \mathbf{BD} = \mathbf{b} - \mathbf{a} \quad \text{2 marks}$$

4 (2 marks)

$$\mathbf{AB} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k} \text{ and } \mathbf{AC} = 6\mathbf{i} + 9\mathbf{j} + 12\mathbf{k} \quad \text{M1}$$

$$\mathbf{AC} = 3\mathbf{AB}, \text{ so A, B, C are collinear} \quad \text{A1}$$

$$\text{Answer: } \mathbf{AC} = 3\mathbf{AB} \quad \text{2 marks}$$

5 (2 marks)

$$(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = |\mathbf{a}|^2 - |\mathbf{b}|^2 \quad \text{M1}$$

$$= 0 \text{ since } |\mathbf{a}| = |\mathbf{b}| \quad \text{A1}$$

$$\text{Answer: Scalar product} = 0 \quad \text{2 marks}$$

6 (2 marks)

$$\mathbf{p} = \mathbf{a} + \frac{1}{4}(\mathbf{b} - \mathbf{a}) \quad \text{M1}$$

$$= \frac{1}{4}(3\mathbf{a} + \mathbf{b}) \quad \text{A1}$$

$$\text{Answer: Shown: } \mathbf{p} = \frac{1}{4}(3\mathbf{a} + \mathbf{b}) \quad \text{2 marks}$$

7 (3 marks)

$$\text{(a) } \mathbf{OP} = \frac{2}{3}\mathbf{a}, \mathbf{OQ} = \frac{1}{2}\mathbf{b} \quad \text{B1}$$

$$\mathbf{PQ} = \mathbf{OQ} - \mathbf{OP} = \frac{1}{2}\mathbf{b} - \frac{2}{3}\mathbf{a} \quad \text{M1}$$

$$\text{Answer: } \mathbf{PQ} = \mathbf{b}/2 - 2\mathbf{a}/3 \quad \text{2 marks}$$

$$\text{(b) } \mathbf{AQ} = \mathbf{OQ} - \mathbf{OA} = \frac{1}{2}\mathbf{b} - \mathbf{a} \quad \text{B1}$$

$$\text{Answer: } \mathbf{AQ} = \mathbf{b}/2 - \mathbf{a} \quad \text{1 mark}$$

8 (3 marks)

$$\mathbf{a} \cdot \mathbf{b} = 0 \text{ since the sides are perpendicular} \quad \text{B1}$$

$$|\mathbf{AC}|^2 = |\mathbf{a} + \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 \quad \text{M1}$$

$$|\mathbf{BD}|^2 = |\mathbf{b} - \mathbf{a}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2, \text{ so } \mathbf{AC} = \mathbf{BD} \quad \text{A1}$$

$$\text{Answer: Proved: the diagonals of a rectangle ABCD are equal in length} \quad \text{3 marks}$$

9 (3 marks)

$$\mathbf{AX} = \frac{2}{5}\mathbf{AB} = \frac{2}{5}(\mathbf{b} - \mathbf{a}) \quad \text{M1 A1}$$

$$\mathbf{OX} = \mathbf{a} + \frac{2}{5}(\mathbf{b} - \mathbf{a}) = \frac{1}{5}(3\mathbf{a} + 2\mathbf{b}) \quad \text{A1}$$

$$\text{Answer: } \mathbf{OX} = (3\mathbf{a} + 2\mathbf{b})/5 \quad \text{3 marks}$$

10 (3 marks)

$$\mathbf{OM} = \frac{1}{2}(\mathbf{b} + \mathbf{c}) \quad \text{B1}$$

$$\mathbf{OG} = \mathbf{a} + \frac{2}{3}(\mathbf{OM} - \mathbf{a}) \quad \text{M1}$$

$$= \frac{1}{3}\mathbf{a} + \frac{2}{3} \times \frac{1}{2}(\mathbf{b} + \mathbf{c}) = \frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c}) \quad \text{A1}$$

$$\text{Answer: Shown: } \mathbf{OG} = \frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c}) \quad \text{3 marks}$$

11 (4 marks)

$$\text{Point on AB: } (1 + 3t, 1 + 2t, 1 + 4t) \quad \text{B1}$$

$$\text{Sum of coordinates: } 3 + 9t = 6, \text{ so } t = \frac{1}{3} \quad \text{M1 A1}$$

$$\mathbf{AP} : \mathbf{PB} = \frac{1}{3} : \frac{2}{3} = 1 : 2 \quad \text{A1}$$

$$\text{Answer: } 1 : 2 \quad \text{4 marks}$$

12 (4 marks)

$$\text{Let M be the origin: } \mathbf{b} = -\mathbf{a}, \text{ so } \mathbf{AB}^2 = 4|\mathbf{a}|^2 \quad \text{B1}$$

$$\mathbf{PA}^2 + \mathbf{PB}^2 = |\mathbf{p} - \mathbf{a}|^2 + |\mathbf{p} + \mathbf{a}|^2 = 2|\mathbf{p}|^2 + 2|\mathbf{a}|^2 \quad \text{M1 A1}$$

$$= 2\mathbf{PM}^2 + \frac{1}{2}(4|\mathbf{a}|^2) = 2\mathbf{PM}^2 + \frac{1}{2}\mathbf{AB}^2 \quad \text{A1}$$

$$\text{Answer: Proved: } \mathbf{PA}^2 + \mathbf{PB}^2 = 2\mathbf{PM}^2 + \frac{1}{2}\mathbf{AB}^2 \quad \text{4 marks}$$

13 (4 marks)

$$\text{Take O as origin: } \mathbf{a} = -\mathbf{b} \text{ (so } \mathbf{b} = -\mathbf{a}) \text{ and } |\mathbf{p}| = |\mathbf{a}| \quad \text{B1}$$

$$\mathbf{PA} \cdot \mathbf{PB} = (\mathbf{a} - \mathbf{p}) \cdot (-\mathbf{a} - \mathbf{p}) \quad \text{M1}$$

$$= |\mathbf{p}|^2 - |\mathbf{a}|^2 = 0, \text{ so angle APB} = 90^\circ \quad \text{M1 A1}$$

$$\text{Answer: Proved: angle APB is a right angle} \quad \text{4 marks}$$

14 (6 marks)

$$\text{(a) AN: } \mathbf{r} = \mathbf{a} + \lambda(\frac{1}{2}\mathbf{b} - \mathbf{a}) \quad \text{B1}$$

$$\text{BM: } \mathbf{r} = \mathbf{b} + \mu(\frac{1}{2}\mathbf{a} - \mathbf{b}) \quad \text{B1}$$

$$\text{Answer: AN: } \mathbf{r} = \mathbf{a} + \lambda(\frac{1}{2}\mathbf{b} - \mathbf{a}); \text{ BM: } \mathbf{r} = \mathbf{b} + \mu(\frac{1}{2}\mathbf{a} - \mathbf{b}) \quad \text{2 marks}$$

$$\text{(b) Equate coefficients of } \mathbf{a}: 1 - \lambda = \frac{\mu}{2} \quad \text{M1}$$

$$\text{of } \mathbf{b}: \frac{\lambda}{2} = 1 - \mu \quad \text{M1}$$

$$\text{So } \lambda = \mu = \frac{2}{3} \quad \text{A1}$$

$$\mathbf{OG} = \frac{1}{3}\mathbf{a} + \frac{1}{3}\mathbf{b} \quad \text{A1}$$

$$\text{Answer: } \mathbf{OG} = (\mathbf{a} + \mathbf{b})/3 \quad \text{4 marks}$$

15 (7 marks)

$$\mathbf{OP} = \frac{1}{3}\mathbf{a} \text{ and } \mathbf{OQ} = \frac{2}{3}\mathbf{b}$$

B1

$$\text{AQ: } \mathbf{r} = \mathbf{a} + \lambda(\frac{2}{3}\mathbf{b} - \mathbf{a}); \text{BP: } \mathbf{r} = \mathbf{b} + \mu(\frac{1}{3}\mathbf{a} - \mathbf{b})$$

M1 A1

$$\text{Equate: } 1 - \lambda = \frac{\mu}{3} \text{ and } \frac{2\lambda}{3} = 1 - \mu$$

M1

$$\text{Solve: } \lambda = \frac{6}{7}, \mu = \frac{3}{7}$$

M1 A1

$$\mathbf{OX} = \frac{1}{7}\mathbf{a} + \frac{4}{7}\mathbf{b}$$

A1

$$\text{Answer: } \mathbf{OX} = (\mathbf{a} + 4\mathbf{b})/7$$

7 marks

16 (4 marks)

$$\mathbf{F} = (6, 4, 3) \text{ and } \mathbf{G} = (0, 4, 3)$$

B1

$$\text{Midpoint of OF} = (3, 2, \frac{3}{2})$$

M1

$$\text{Midpoint of AG} = (\frac{6+0}{2}, \frac{0+4}{2}, \frac{0+3}{2}) = (3, 2, \frac{3}{2})$$

M1 A1

$$\text{Answer: Both midpoints are } (3, 2, 3/2)$$

4 marks

17 (4 marks)

$$\text{Pairs of opposite edges: } \mathbf{P_1P_2} \text{ and } \mathbf{P_3P_4}; \mathbf{P_1P_3} \text{ and } \mathbf{P_2P_4}; \mathbf{P_1P_4} \text{ and } \mathbf{P_2P_3}$$

B1

$$\mathbf{P_1P_2} \cdot \mathbf{P_3P_4} = (0, -2, -2) \cdot (0, -2, 2) = 4 - 4 = 0$$

M1 A1

$$\mathbf{P_1P_3} \cdot \mathbf{P_2P_4} = (-2, 0, -2) \cdot (-2, 0, 2) = 4 - 4 = 0 \text{ and } \mathbf{P_1P_4} \cdot \mathbf{P_2P_3} = (-2, -2, 0) \cdot (-2, 2, 0) = 0$$

A1

$$\text{Answer: All three dot products are 0}$$

4 marks

18 (5 marks)

$$\text{Centre } (x, y, z) \text{ equidistant from O and A: } x^2 + y^2 + z^2 = (x - 4)^2 + y^2 + z^2, \text{ so } x = 2$$

M1 A1

$$\text{Similarly from O and B: } y = 3; \text{ from O and C: } z = 4$$

M1 A1

$$\text{Centre } (2, 3, 4), \text{ radius} = \sqrt{4 + 9 + 16} = \sqrt{29}$$

A1

$$\text{Answer: Centre } (2, 3, 4), \text{ radius } \sqrt{29}$$

5 marks

19 (5 marks)

$$(a) \mathbf{M_1} = \frac{1}{2}\mathbf{a} \text{ and } \mathbf{N_1} = \frac{1}{2}(\mathbf{b} + \mathbf{c})$$

B1

$$\text{Midpoint of } \mathbf{M_1N_1} = \frac{1}{4}(\mathbf{a} + \mathbf{b} + \mathbf{c})$$

M1 A1

$$\text{Answer: } (\mathbf{a} + \mathbf{b} + \mathbf{c})/4$$

3 marks

$$(b) \text{ By symmetry the midpoints of } \mathbf{M_2N_2} \text{ and } \mathbf{M_3N_3} \text{ are also } \frac{1}{4}(\mathbf{a} + \mathbf{b} + \mathbf{c})$$

M1

$$\text{All three lines pass through this point, so they are concurrent}$$

A1

$$\text{Answer: Concurrent at } (\mathbf{a} + \mathbf{b} + \mathbf{c})/4$$

2 marks

20 (4 marks)

$$(a) \frac{1}{2}(\mathbf{b} + \mathbf{c}) \text{ is the position vector of M, not the vector } \mathbf{AM}$$

M1 E1

$$\text{Answer: That is OM, not AM}$$

2 marks

$$(b) \mathbf{AM} = \mathbf{m} - \mathbf{a} = \frac{1}{2}(\mathbf{b} + \mathbf{c}) - \mathbf{a}$$

M1

$$= \frac{1}{2}(\mathbf{b} - \mathbf{a}) + \frac{1}{2}(\mathbf{c} - \mathbf{a}) = \frac{1}{2}(\mathbf{AB} + \mathbf{AC})$$

A1

$$\text{Answer: } \mathbf{AM} = (\mathbf{b} + \mathbf{c})/2 - \mathbf{a} = (\mathbf{AB} + \mathbf{AC})/2$$

2 marks

21 (9 marks)

$$(a) \mathbf{d} = \mathbf{b} + \frac{1}{3}(\mathbf{c} - \mathbf{b}) = \frac{1}{3}(2\mathbf{b} + \mathbf{c})$$

M1

$$\mathbf{e} = \mathbf{c} + \frac{1}{4}(\mathbf{a} - \mathbf{c}) = \frac{1}{4}(\mathbf{a} + 3\mathbf{c})$$

A1

$$\text{Answer: } \mathbf{d} = (2\mathbf{b} + \mathbf{c})/3, \mathbf{e} = (\mathbf{a} + 3\mathbf{c})/4$$

2 marks

$$(b) AD: \mathbf{r} = \mathbf{a} + \lambda(\mathbf{d} - \mathbf{a}) = (1 - \lambda)\mathbf{a} + \frac{2\lambda}{3}\mathbf{b} + \frac{\lambda}{3}\mathbf{c}$$

M1 A1

$$BE: \mathbf{r} = \mathbf{b} + \mu(\mathbf{e} - \mathbf{b}) = \frac{\mu}{4}\mathbf{a} + (1 - \mu)\mathbf{b} + \frac{3\mu}{4}\mathbf{c}$$

M1

$$\text{Equate coefficients: } 1 - \lambda = \frac{\mu}{4}, \frac{\lambda}{3} = \frac{3\mu}{4}$$

M1

$$\lambda = \frac{9}{10}, \mu = \frac{2}{5}; \text{ check } \frac{2\lambda}{3} = \frac{3}{5} = 1 - \mu$$

A1

$$\text{Answer: } \lambda = 9/10, \mu = 2/5$$

5 marks

$$(c) AX = \frac{9}{10}AD, \text{ so } AX : XD = 9 : 1$$

B1

$$BX = \frac{2}{5}BE, \text{ so } BX : XE = 2 : 3$$

B1

$$\text{Answer: } AX : XD = 9 : 1; BX : XE = 2 : 3$$

2 marks

22 (9 marks)

$$(a) E = (6, 0, 3) \text{ and } F = (6, 4, 3), \text{ so } M = (6, 2, 3)$$

B1

$$\text{Answer: } (6, 2, 3)$$

1 mark

$$(b) \mathbf{AC} = -6\mathbf{i} + 4\mathbf{j}, \mathbf{AG} = -6\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$$

B1

$$\mathbf{AC} \times \mathbf{AG} = 12\mathbf{i} + 18\mathbf{j}, \text{ parallel to } (2, 3, 0)$$

M1 A1

$$2x + 3y = 12$$

A1

$$\text{Answer: } 2x + 3y = 12$$

4 marks

$$(c) \text{ Line OM: } \mathbf{r} = t(6, 2, 3); 12t + 6t = 12$$

M1 A1

$$t = \frac{2}{3}, P = (4, \frac{4}{3}, 2)$$

A1

$$OP : PM = \frac{2}{3} : \frac{1}{3} = 2 : 1$$

A1

$$\text{Answer: } P = (4, 4/3, 2); OP : PM = 2 : 1$$

4 marks

23 (7 marks)

$$(a) \mathbf{p} = \frac{1}{2}\mathbf{a}, \mathbf{q} = \frac{1}{3}\mathbf{b}, \mathbf{r} = \frac{1}{4}\mathbf{c}; \text{ plane: } \mathbf{x} = (1 - s - t)\mathbf{p} + s\mathbf{q} + t\mathbf{r}$$

M1

$$= \frac{1 - s - t}{2}\mathbf{a} + \frac{s}{3}\mathbf{b} + \frac{t}{4}\mathbf{c}$$

A1

$$\alpha = \frac{1 - s - t}{2}, \beta = \frac{s}{3}, \gamma = \frac{t}{4}; 2\alpha + 3\beta + 4\gamma = 1 - s - t + s + t = 1$$

A1

$$\text{Answer: Shown: } 2\alpha + 3\beta + 4\gamma = 1$$

3 marks

$$(b) \mathbf{OG} = \frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c}); X = k\mathbf{OG}$$

B1

$$2(\frac{k}{3}) + 3(\frac{k}{3}) + 4(\frac{k}{3}) = 1, \text{ so } k = \frac{1}{3}$$

M1 A1

$$OX = \frac{1}{3}OG, \text{ so } OX : XG = 1 : 2$$

A1

$$\text{Answer: } OX : XG = 1 : 2$$

4 marks

24 (6 marks)

$$(a) \text{ The diagonals bisect each other at O, so } \mathbf{c} = -\mathbf{a} \text{ and } \mathbf{d} = -\mathbf{b}$$

B1

$$\text{The diagonals of a rectangle are equal, so } |\mathbf{a}| = |\mathbf{b}|$$

B1

$$\text{Answer: Diagonals bisect and are equal}$$

2 marks

$$(b) |\mathbf{p} - \mathbf{a}|^2 + |\mathbf{p} + \mathbf{a}|^2$$

M1

$$= (|\mathbf{p}|^2 - 2\mathbf{p} \cdot \mathbf{a} + |\mathbf{a}|^2) + (|\mathbf{p}|^2 + 2\mathbf{p} \cdot \mathbf{a} + |\mathbf{a}|^2) = 2|\mathbf{p}|^2 + 2|\mathbf{a}|^2$$

A1

$$\text{Answer: Shown: } \mathbf{PA}^2 + \mathbf{PC}^2 = 2|\mathbf{p}|^2 + 2|\mathbf{a}|^2$$

2 marks

$$(c) \text{ Similarly } \mathbf{PB}^2 + \mathbf{PD}^2 = 2|\mathbf{p}|^2 + 2|\mathbf{b}|^2$$

M1

$$\text{These are equal since } |\mathbf{a}| = |\mathbf{b}|$$

A1

$$\text{Answer: } \mathbf{PA}^2 + \mathbf{PC}^2 = \mathbf{PB}^2 + \mathbf{PD}^2$$

2 marks

25 (6 marks)

(a) OH is perpendicular to AB, and AH is perpendicular to OB

E1

So H is where two altitudes of the triangle meet

E1

Answer: H lies on two altitudes

2 marks

(b) First condition: $\mathbf{h} \cdot \mathbf{b} = \mathbf{h} \cdot \mathbf{a}$

M1

Second condition: $\mathbf{h} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{b}$

A1

Answer: Shown: $\mathbf{h} \cdot \mathbf{a} = \mathbf{h} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{b}$

2 marks

(c) $(\mathbf{h} - \mathbf{b}) \cdot \mathbf{a} = \mathbf{h} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b}$

M1

$= 0$ by part (b), so BH is perpendicular to OA and H lies on all three altitudes

A1

Answer: Proved: BH is perpendicular to OA, so the altitudes meet

2 marks

26 (6 marks)

(a) $\mathbf{p} = \frac{1}{2}\mathbf{a}$, $\mathbf{q} = \frac{1}{3}\mathbf{b}$, $\mathbf{r} = \frac{1}{4}\mathbf{c}$

B1

$$\mathbf{p} \cdot (\mathbf{q} \times \mathbf{r}) = \frac{1}{2} \times \frac{1}{3} \times \frac{1}{4} \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$$

M1 A1

So the volume ratio is $\frac{1}{24}$

A1

Answer: 1/24

4 marks

(b) Volume OABC = $\frac{1}{6} \times 2 \times 3 \times 6 = 6$

M1

$$\text{Volume OPQR} = \frac{6}{24} = \frac{1}{4}$$

A1

Answer: 6 and 1/4

2 marks

27 (10 marks)

(a) $\mathbf{AH} = \mathbf{h} - \mathbf{a} = \mathbf{b} + \mathbf{c}$

M1

$$\mathbf{AH} \cdot \mathbf{BC} = (\mathbf{b} + \mathbf{c}) \cdot (\mathbf{c} - \mathbf{b}) = |\mathbf{c}|^2 - |\mathbf{b}|^2$$

M1

$= 0$, since $|\mathbf{b}| = |\mathbf{c}|$

A1

Answer: Shown: AH is perpendicular to BC

3 marks

(b) By the same argument BH is perpendicular to CA and CH to AB

M1

H lies on all three altitudes, so it is the orthocentre

A1

Answer: H is the orthocentre

2 marks

(c) $\mathbf{g} = \frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c}) = \frac{1}{3}\mathbf{h}$

M1 A1

So G lies on OH with $\text{OG} : \text{GH} = 1 : 2$

A1

Answer: OG : GH = 1 : 2

3 marks

(d) $\mathbf{h} = (3 - 4 + 5, 4 + 3 + 0, 0) = (4, 7, 0)$

M1

$\mathbf{AH} = (1, 3, 0)$, $\mathbf{BC} = (9, -3, 0)$, $\mathbf{AH} \cdot \mathbf{BC} = 9 - 9 = 0$

A1

Answer: H = (4, 7, 0)

2 marks

28 (10 marks)

(a) $\mathbf{AG} = \frac{1}{4}(\mathbf{b} + \mathbf{c} + \mathbf{d} - 3\mathbf{a})$

M1 A1

$$\mathbf{AG}_A = \frac{1}{3}(\mathbf{b} + \mathbf{c} + \mathbf{d}) - \mathbf{a} = \frac{1}{3}(\mathbf{b} + \mathbf{c} + \mathbf{d} - 3\mathbf{a})$$

M1

$\mathbf{AG} = \frac{3}{4}\mathbf{AG}_A$, so G lies on AG_A with $\text{AG} : \text{GG}_A = 3 : 1$

A1

Answer: Shown: G lies on AG_A with $\text{AG} : \text{GG}_A = 3 : 1$

4 marks

(b) The expression for G is symmetric in a, b, c, d, so G lies on all four lines

E1

Answer: G is symmetric in a, b, c, d

1 mark

(c) $\mathbf{G}_A = (\frac{4}{3}, \frac{4}{3}, \frac{4}{3})$ and $\mathbf{G} = (1, 1, 1)$

M1 A1

$\mathbf{AG} = (1, 1, 1)$ and $\mathbf{GG}_A = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, ratio 3 : 1

A1

Answer: G = (1, 1, 1)

3 marks

(d) Midpoints $\frac{1}{2}(\mathbf{a} + \mathbf{b})$ and $\frac{1}{2}(\mathbf{c} + \mathbf{d})$; their midpoint is $\frac{1}{4}(\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}) = \mathbf{g}$

M1 A1

Answer: Midpoint is g

2 marks

29 (10 marks)

(a) $\text{LHS} = |\mathbf{b} - \mathbf{a}|^2 + |\mathbf{c} - \mathbf{b}|^2 + |\mathbf{d} - \mathbf{c}|^2 + |\mathbf{a} - \mathbf{d}|^2 = 2\Sigma - 2(\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{d} + \mathbf{d} \cdot \mathbf{a})$, where $\Sigma = |\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2 + |\mathbf{d}|^2$ M1 A1

$$AC^2 + BD^2 = \Sigma - 2\mathbf{a} \cdot \mathbf{c} - 2\mathbf{b} \cdot \mathbf{d} \quad \text{M1}$$

$$4MN^2 = |\mathbf{a} + \mathbf{c} - \mathbf{b} - \mathbf{d}|^2 = \Sigma + 2\mathbf{a} \cdot \mathbf{c} + 2\mathbf{b} \cdot \mathbf{d} - 2(\mathbf{a} + \mathbf{c}) \cdot (\mathbf{b} + \mathbf{d}) \quad \text{M1 A1}$$

$$\text{Adding gives } 2\Sigma - 2(\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{d} + \mathbf{c} \cdot \mathbf{b} + \mathbf{c} \cdot \mathbf{d}) = \text{LHS} \quad \text{A1}$$

$$\text{Answer: Proved: } AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4MN^2 \quad \text{6 marks}$$

(b) $MN^2 \geq 0$, so the inequality holds M1

Equality iff $M = N$, i.e. the diagonals bisect each other, i.e. ABCD is a parallelogram A1

$$\text{Answer: Proved: } AC^2 + BD^2 \leq AB^2 + BC^2 + CD^2 + DA^2 \quad \text{2 marks}$$

(c) Sides: $16 + 11 + 18 + 9 = 54$ M1

$$AC^2 = 35, BD^2 = 17, MN^2 = \frac{1}{2}: 35 + 17 + 2 = 54 \quad \text{A1}$$

Answer: Both sides equal 54 2 marks

30 (9 marks)

(a) $\mathbf{PA} = (2 - t, 1 - 2t, 1 - 2t)$ and $\mathbf{PB} = (5 - t, 4 - 2t, 4 - 2t)$ M1 A1

$$\mathbf{PA} \cdot \mathbf{PB} = 0: (2 - t)(5 - t) + 2(1 - 2t)(4 - 2t) = 0 \quad \text{M1}$$

$$9t^2 - 27t + 18 = 0, \text{ so } t^2 - 3t + 2 = 0 \quad \text{A1}$$

Answer: Shown: the position of P satisfies $t^2 - 3t + 2 = 0$ 4 marks

(b) $(t - 1)(t - 2) = 0$, so $t = 1$ or $t = 2$ M1

$P = (2, 3, 2)$ or $P = (3, 5, 4)$ A1

Answer: (2, 3, 2) or (3, 5, 4) 2 marks

(c) Centre $(\frac{9}{2}, \frac{7}{2}, \frac{5}{2})$, radius $= \frac{1}{2}AB = \frac{1}{2}\sqrt{27} = \frac{3\sqrt{3}}{2}$ M1 A1

$$\text{Distance of } (2, 3, 2) \text{ from the centre: } \sqrt{\frac{25}{4} + \frac{1}{4} + \frac{1}{4}} = \sqrt{\frac{27}{4}}, \text{ and } (3, 5, 4): \sqrt{\frac{9}{4} + \frac{9}{4} + \frac{9}{4}} = \sqrt{\frac{27}{4}} \quad \text{A1}$$

Answer: Centre (9/2, 7/2, 5/2), radius $3\sqrt{3}/2$ 3 marks