

Three-Dimensional Vectors

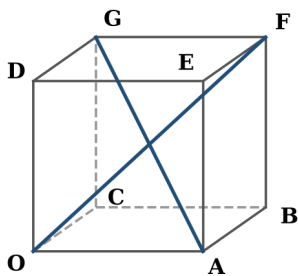
- 1 Find the magnitude of the vector $2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$. [2]
- 2 The points A and B have position vectors $\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$ and $3\mathbf{i} + \mathbf{j} - \mathbf{k}$ respectively. Find the vector $\mathbf{AB}$. [2]
- 3 Find a unit vector in the direction of $2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$. [2]
- 4 Find the coordinates of the midpoint of the line segment joining A(2, -4, 6) and B(8, 0, -2). [2]
- 5 The vectors $2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ and $\begin{pmatrix} 6 \\ p \\ q \end{pmatrix}$ are parallel.
Find the values of p and q. [3]
- 6 Find the distance between the points P(1, 2, 3) and Q(4, -2, 15). [2]
- 7 Find the vector of magnitude 10 that is in the same direction as $2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$. [3]
- 8 The points A, B and C have coordinates (1, 2, 3), (3, -1, 5) and (6, 1, 5).
Show that triangle ABC is right-angled, stating which angle is the right angle. [4]
- 9 The point P lies on the line segment AB, where A is (1, -4, 6) and B is (11, 1, -4), such that $AP : PB = 2 : 3$.
Find the coordinates of P. [4]
- 10 The points A, B and C have position vectors $2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$, $5\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$ and $8\mathbf{i} + 7\mathbf{j} + 9\mathbf{k}$.
Show that A, B and C are collinear and find the ratio AB : BC. [4]
- 11 ABCD is a parallelogram, where A, B and C have coordinates (1, 2, 0), (4, 3, 2) and (6, 6, 5) (in that order round the parallelogram).
Find the coordinates of D. [3]
- 12 The vector $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ makes an angle α with the positive x-axis.
Find the exact value of $\cos \alpha$ and the value of α in degrees, correct to 1 decimal place. [4]
- 13 The vector $\mathbf{a} = p\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ has magnitude 3. Find the possible values of p. [3]
- 14 The points A, B and C have coordinates (1, 0, 2), (3, 2, 3) and (5, 0, 4).
(a) Show that triangle ABC is isosceles. [3]
(b) Find the exact perimeter of the triangle. [3]
- 15 The triangle ABC has vertices with position vectors $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$, $\mathbf{b} = 4\mathbf{i} - 5\mathbf{j} + \mathbf{k}$ and $\mathbf{c} = 7\mathbf{j} + 5\mathbf{k}$. The point M is the midpoint of BC.
(a) Find the position vector of M. [2]
(b) The centroid G of the triangle has position vector $\frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c})$. Find $\mathbf{g}$, and show that G lies on AM with $AG : GM = 2 : 1$. [4]
- 16 The points A and B have coordinates (3, 1, 2) and (-1, 5, 6). The point P lies on the y-axis and is equidistant from A and B.
Find the coordinates of P. [6]
- 17 The points A and B have coordinates A(t + 1, 6, t) and B(2t + 1, t + 1, 4), where t is a constant.
(a) Show that $AB^2 = 3t^2 - 18t + 41$. [4]
(b) Find the value of t for which AB is least, and the least value of AB. [4]

- 18** The vectors $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{c} = -\mathbf{i} + 8\mathbf{j} - 9\mathbf{k}$ are given. Find the constants p and q such that $\mathbf{c} = p\mathbf{a} + q\mathbf{b}$.
Verify your values satisfy all three components. [5]
- 19** A drone flies in a straight line from a launch point P to a landing point Q . Relative to a base O , the position vectors (in metres) of P and Q are $120\mathbf{i} + 40\mathbf{j} + 15\mathbf{k}$ and $300\mathbf{i} + 190\mathbf{j} + 60\mathbf{k}$, where $\mathbf{i}$ and $\mathbf{j}$ point east and north and $\mathbf{k}$ points vertically up.
- (a) Find the distance PQ , giving your answer correct to the nearest metre. [3]
- (b) The drone has covered $\frac{2}{5}$ of the journey from P to Q . Find its position vector and its height above the base at that moment. [3]
- 20** $OABC$ is a parallelogram with O the origin. The points A, B, C have position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$.
- (a) Write $\mathbf{b}$ in terms of $\mathbf{a}$ and $\mathbf{c}$. [2]
- (b) Hence prove that the diagonals OB and AC bisect each other. [3]
- 21** The points A and B have position vectors $2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ and $4\mathbf{i} + \mathbf{j}$.
- (a) The point C lies on AB extended beyond B such that $AC : CB = 3 : 1$. Find the position vector of C . [4]
- (b) The point D lies on BA extended beyond A such that $BD = 3 BA$. Find the position vector of D . [2]
- (c) Hence find the ratio $AD : DC$ in its simplest form. [3]
- 22** $ABCD$ is a parallelogram with $A(1, -2, 3)$, $B(4, 0, 5)$ and $C(2, 3, 7)$, in that order.
- (a) Find the coordinates of D . [2]
- (b) Show that $ABCD$ is a rhombus. [3]
- (c) Find the exact lengths of the two diagonals. [3]
- 23** The points A and B have position vectors $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $3\mathbf{i} - \mathbf{k}$. The point P lies on the line AB , with $\mathbf{AP} = \lambda\mathbf{AB}$.
Find, in terms of λ , the position vector of P , and show that $OP^2 = 24\lambda^2 - 28\lambda + 14$.
Hence find the least possible distance from O to a point on the line AB , and the value of λ for which it occurs. [8]
- 24** A vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ makes angles α, β and γ with the positive x, y and z axes respectively.
- (a) Prove that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. [3]
- (b) A vector of magnitude 12 has $\cos\alpha = \frac{1}{3}$ and $\cos\beta = \frac{2}{3}$, and makes an obtuse angle with the z -axis. Find the vector. [5]
- 25** A student is given the points $A(1, 2, 3)$, $B(4, 5, 6)$ and $C(7, 8, k)$. The student writes: " $\mathbf{AB} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$ and $\mathbf{BC} = \begin{pmatrix} 3 \\ 3 \\ k-6 \end{pmatrix}$ are both made of the number 3, so A, B and C are collinear whatever k is."
- (a) Explain the error in the student's reasoning. [2]
- (b) Find the value of k for which A, B and C are collinear. [2]
- (c) For $k \neq 9$ the triangle ABC has $AB = BC$ for one value of k . Find it. [4]
- 26** $OABC$ is a tetrahedron with O the origin and A, B, C having position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$. The point G is the centroid of triangle ABC .
- (a) Write down the position vector of G . [1]
- (b) The point P lies on OG with $OP : PG = 3 : 1$. Show that $\mathbf{p} = \frac{1}{4}(\mathbf{a} + \mathbf{b} + \mathbf{c})$. [2]
- (c) Given that A, B, C are $(4, 0, 0)$, $(0, 8, 0)$, $(0, 0, 12)$, find P and the distance OP . [3]

- 27** ABCD is a quadrilateral in three dimensions (the four vertices need not lie in one plane). P, Q, R, S are the midpoints of AB, BC, CD, DA respectively.
- (a) Show that $\mathbf{PQ} = \frac{1}{2}\mathbf{AC}$ and find a similar expression for $\mathbf{SR}$. [4]
- (b) Hence prove that PQRS is a parallelogram. [2]
- (c) Given A(0, 0, 0), B(6, 2, 0), C(8, 8, 4), D(2, 6, 10), find the coordinates of P, Q, R, S and verify $\mathbf{PQ} = \mathbf{SR}$. [4]
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- 28** The points A, B and C have coordinates A(1, 2, 3), B(3, 0, 4) and C(5, 6, 1).
- (a) Find $\mathbf{AB}$ and $\mathbf{AC}$, and the magnitude of each. [2]
- (b) The unit vectors in the directions of AB and AC are $\mathbf{e}_1$ and $\mathbf{e}_2$. Find $\mathbf{e}_1 + \mathbf{e}_2$, and explain why it is along the bisector of angle BAC. [3]
- (c) The bisector of angle BAC meets BC at D. Find the coordinates of D. [5]
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- 29** The points A and B have coordinates A(-1, 2, 0) and B(5, 2, 0). The point P(x, y, z) moves so that $PA = 2PB$.
- (a) Show that P lies on the surface $(x - 7)^2 + (y - 2)^2 + z^2 = 16$. [5]
- (b) Write down the geometrical description of the surface. [2]
- (c) Find the coordinates of the two points of the surface that lie on the line AB. [3]
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- 30** A regular tetrahedron ABCD has edges of length 2. Its base ABC lies in the plane $z = 0$, with A(0, 0, 0), B(2, 0, 0) and C(1, $\sqrt{3}$, 0), and D is above the base.
- (a) Show that triangle ABC is equilateral. [2]
- (b) Let D be (p, q, h) with $h > 0$. Use $DA = DB$ to show that $p = 1$, and use $DA = DC$ to show that $q = \frac{\sqrt{3}}{3}$. [4]
- (c) Hence find h, giving the exact coordinates of D. [3]

Scalar Product

- 1** Find $\mathbf{a} \cdot \mathbf{b}$ when $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ and $\mathbf{b} = 4\mathbf{i} + 5\mathbf{k}$. [2]
- 2** Show that the vectors $3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ are perpendicular. [2]
- 3** Find the angle between the vectors $\mathbf{i} + \mathbf{j}$ and $\mathbf{j} + \mathbf{k}$. [3]
- 4** The vectors $p\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $4\mathbf{i} + p\mathbf{j} + 2\mathbf{k}$ are perpendicular. Find p. [2]
- 5** The vectors $\mathbf{a}$ and $\mathbf{b}$ have $|\mathbf{a}| = 3$ and $|\mathbf{b}| = 4$, and the angle between them is 60° . Find $\mathbf{a} \cdot \mathbf{b}$. [2]
- 6** Without finding the angle itself, state whether the angle between $\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$ is acute, obtuse or a right angle. Justify your answer. [2]
- 7** The vectors $\mathbf{a} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 4\mathbf{j} - \mathbf{k}$ are given. Find the value of λ for which $\mathbf{a} + \lambda\mathbf{b}$ is perpendicular to $\mathbf{b}$. [4]
- 8** The points A, B and C have coordinates (1, 2, 3), (2, 0, 5) and (4, 1, 4). Find angle ABC, correct to 1 decimal place. [4]
- 9** The vectors $\mathbf{a}$ and $\mathbf{b}$ are $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$.
- (a) Find the scalar projection of $\mathbf{a}$ onto $\mathbf{b}$. [2]
- (b) Find the vector projection of $\mathbf{a}$ onto $\mathbf{b}$. [2]
- 10** Find a vector that is perpendicular to both $\mathbf{i} + 2\mathbf{j}$ and $\mathbf{j} + 3\mathbf{k}$. [4]

- 11** The vectors $\mathbf{a}$ and $\mathbf{b}$ satisfy $|\mathbf{a}| = 2$, $|\mathbf{b}| = 3$ and $\mathbf{a} \cdot \mathbf{b} = -1$. Find the exact values of $|\mathbf{a} + \mathbf{b}|$ and $|\mathbf{a} - \mathbf{b}|$. [4]
- 12** A constant force of $3\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$ N moves a particle from the point $A(1, 1, 1)$ to the point $B(4, 3, -1)$ (distances in metres). Find the work done by the force, and explain the sign of your answer. [4]
- 13** The vectors $\mathbf{a} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = \mathbf{i} - 2\mathbf{j} + p\mathbf{k}$ satisfy $\mathbf{a} \cdot \mathbf{b} = 6$. Find p , and hence find the angle between $\mathbf{a}$ and $\mathbf{b}$ to the nearest tenth of a degree. [4]
- 14** OABCDEFG is a unit cube. O is the origin and A, C, D lie on the x, y, z axes respectively. The lines OF and AG are two of the space diagonals.
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- (a) Find the angle between OF and the face diagonal OB. [3]
(b) Find the acute angle between the space diagonals OF and AG. [3]
- 15** The points A, B and C have coordinates $(1, 0, 2)$, $(3, 1, 0)$ and $(2, 4, 3)$. Find the three angles of triangle ABC, each to the nearest 0.1° , and check that they sum to 180° . [7]
- 16** Find the vectors of magnitude 6 that are perpendicular to both $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $2\mathbf{i} - \mathbf{j} + \mathbf{k}$. [5]
- 17** A particle has velocity $\mathbf{v} = 4\mathbf{i} - 3\mathbf{j} + 12\mathbf{k}$ m s⁻¹. Find the components of $\mathbf{v}$ parallel and perpendicular to the direction of $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$. Give each component as a vector. [6]
- 18** OABC is a rhombus, with O the origin and A, B, C having position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$.
(a) Write down $\mathbf{b}$ in terms of $\mathbf{a}$ and $\mathbf{c}$, and state the relationship between $|\mathbf{a}|$ and $|\mathbf{c}|$. [2]
(b) Prove that the diagonals OB and AC are perpendicular. [4]
- 19** A particle moves in a straight line from $A(2, 1, 0)$ to $B(8, 5, p)$ under a constant force $\mathbf{F} = 6\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ N (distances in metres). The work done by $\mathbf{F}$ is 40 J. Find p , and then find the component of $\mathbf{F}$ in the direction AB, correct to 3 significant figures. [6]
- 20** The points A, B and C have coordinates $(2, 0, 1)$, $(4, 2, 2)$ and $(6, -1, 4)$. The quadrilateral ABCD satisfies $\mathbf{AD} = \mathbf{BC}$. Find the coordinates of D, show that ABCD is a rectangle and find its area. [6]
- 21** The vectors $\mathbf{a} = p\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{b} = \mathbf{i} + p\mathbf{j} + \mathbf{k}$, where p is a constant, are at an angle of 60° to each other.
(a) Show that $p^2 - 4p = 0$ and hence find the possible values of p . [5]
(b) For the larger value of p , show that $\mathbf{a} + \mathbf{b}$ makes equal angles with $\mathbf{a}$ and with $\mathbf{b}$. [3]
- 22** A student writes: " $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$, so $\mathbf{b} = \mathbf{c}$."
(a) Show that the claim is false by finding a vector $\mathbf{c}$, different from $\mathbf{b}$, such that $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$, where $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{b} = \mathbf{i}$. [3]
(b) Explain what $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$ does tell us about the vector $\mathbf{b} - \mathbf{c}$. [2]
(c) Prove that if $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$, $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{b} - \mathbf{c}$ is parallel to $\mathbf{a}$, then $\mathbf{b} = \mathbf{c}$. [3]

- 23** The vectors $\mathbf{a}$ and $\mathbf{b}$ are two adjacent sides of a parallelogram.
- Prove that $|\mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2 = 2|\mathbf{a}|^2 + 2|\mathbf{b}|^2$. [3]
 - A parallelogram has sides of length 5 and 7. One diagonal has length 8. Find the exact length of the other diagonal. [3]
 - The diagonal of length 8 is the one given by $|\mathbf{a} - \mathbf{b}|$. Find the angle between the sides of lengths 5 and 7, to the nearest 0.1° . [3]
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- 24** A unit vector $\mathbf{u}$ makes an angle of 60° with the positive x -axis and an angle of 60° with the positive y -axis.
- Show that $\mathbf{u} = \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} + w\mathbf{k}$, where $w^2 = \frac{1}{2}$. [4]
 - Find the two possible angles that $\mathbf{u}$ makes with the z -axis. [2]
 - Find the angle between the two possible vectors $\mathbf{u}$. [3]
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- 25** The points A , B and C have coordinates $(3, 1, 3)$, $(2, 0, 1)$ and $(4, 2, 2)$. The point D lies on BC such that AD is perpendicular to BC .
- Find the coordinates of D . [5]
 - Find the exact area of triangle ABC . [4]
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- 26** The points A and B have coordinates $A(2t, 3, t - 1)$ and $B(t + 2, -t - 1, 3)$, where t is a constant, and O is the origin.
- Find the values of t for which OA is perpendicular to OB . [4]
 - Find angle AOB when $t = 2$, correct to 1 decimal place. [5]
 - Find the exact area of triangle OAB when $t = 1$. [3]
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- 27** Let $\mathbf{a}$ and $\mathbf{b}$ be vectors with $\mathbf{b} \neq \mathbf{0}$ and let λ be a real number.
- By considering $|\mathbf{a} + \lambda\mathbf{b}|^2$, which is non-negative for all λ , prove that $(\mathbf{a} \cdot \mathbf{b})^2 \leq |\mathbf{a}|^2|\mathbf{b}|^2$. [5]
 - State, with a reason, when equality holds. [1]
 - Real numbers x, y, z satisfy $x + 2y + 3z = 28$. Use the result of (a) with $\mathbf{a} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ to show that $x^2 + y^2 + z^2 \geq 56$, and find the values of x, y, z for which $x^2 + y^2 + z^2 = 56$. [4]
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- 28** The points A and B have coordinates $A(1, 2, 3)$ and $B(5, -2, 7)$. The point $P(x, y, z)$ satisfies $\mathbf{AP} \cdot \mathbf{BP} = 0$.
- Show that P lies on a sphere with equation $(x - 3)^2 + y^2 + (z - 5)^2 = 12$. [4]
 - Show that the centre of the sphere is the midpoint of AB and that AB is a diameter. [3]
 - The point Q has coordinates $(5, 2, 3)$. Verify that Q is on the sphere by evaluating $\mathbf{QA} \cdot \mathbf{QB}$, and state the size of angle AQB . [2]
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- 29** The points $P_1(1, 1, 1)$, $P_2(1, -1, -1)$, $P_3(-1, 1, -1)$ and $P_4(-1, -1, 1)$ are four vertices of a cube centred at the origin O .
- Show that every edge of the tetrahedron $P_1P_2P_3P_4$ has length $2\sqrt{2}$. [3]
 - Find the angle between $\mathbf{OP}_1$ and $\mathbf{OP}_2$, to the nearest 0.1° . [2]
 - Find the angle between the edges P_1P_2 and P_1P_3 . [2]
 - Show that $\mathbf{OP}_1 + \mathbf{OP}_2 + \mathbf{OP}_3 + \mathbf{OP}_4 = \mathbf{0}$, and explain what this tells you about O . [2]
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- 30** The vectors $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $\mathbf{c} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ are given.
- Show that $|\mathbf{a}| = |\mathbf{c}|$ and that $\mathbf{a}$ and $\mathbf{c}$ are perpendicular. [3]
 - The vector $\mathbf{b} = p\mathbf{i} + q\mathbf{j}$ is horizontal, has magnitude 15, has $p > 0$ and makes equal angles with $\mathbf{a}$ and with $\mathbf{c}$. Find p and q exactly. [4]
 - Find the angle between $\mathbf{b}$ and $\mathbf{a}$, and comment on your answer. [3]

Vector Product

- 1** Find $\mathbf{a} \times \mathbf{b}$ when $\mathbf{a} = 2\mathbf{i} + \mathbf{j}$ and $\mathbf{b} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$. [2]

- 2 Evaluate $\mathbf{i} \times \mathbf{j} + \mathbf{j} \times \mathbf{k} + \mathbf{k} \times \mathbf{i}$. [2]
- 3 The vectors $\mathbf{a}$ and $\mathbf{b}$ are not parallel. Simplify $(2\mathbf{a} + \mathbf{b}) \times (\mathbf{a} - 2\mathbf{b})$ in terms of $\mathbf{a} \times \mathbf{b}$. [3]
- 4 Find the area of the parallelogram with adjacent sides $\mathbf{a} = 3\mathbf{i} + \mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 2\mathbf{j}$. [2]
- 5 The vectors $\mathbf{a}$ and $\mathbf{b}$ have magnitudes 4 and 5 and the angle between them is 30° . Find $|\mathbf{a} \times \mathbf{b}|$. [2]
- 6 The points A and B have coordinates (2, 0, 1) and (0, 3, 1) and O is the origin. Find the area of triangle OAB. [2]
- 7 Find a unit vector that is perpendicular to both $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $3\mathbf{i} + 2\mathbf{k}$. [4]
- 8 The vectors $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{c} = 3\mathbf{i} + p\mathbf{j} - \mathbf{k}$ are three edges of a parallelepiped of volume 14, where p is a constant. Find the possible values of p . [5]
- 9 Use the vector product to find the acute angle between the vectors $\mathbf{i} + \mathbf{k}$ and $\mathbf{j} + \mathbf{k}$. [4]
- 10 The points A, B and C have coordinates (1, 1, 0), (2, 3, 1) and (0, 2, 3). Find a vector that is perpendicular to the plane containing A, B and C. [4]
- 11 Find the scalar triple product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ for $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = \mathbf{j} + 4\mathbf{k}$ and $\mathbf{c} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$. State the volume of the parallelepiped with edges $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. [4]
- 12 The vectors $\mathbf{p} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$, $\mathbf{q} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + \lambda\mathbf{k}$ are given, where λ is a constant.
 - (a) Given that $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ are coplanar, find the value of λ . [4]
 - (b) Hence express $\mathbf{p}$ in terms of $\mathbf{q}$ and $\mathbf{r}$. [2]
- 13 Given $\mathbf{a} = \mathbf{i} + 2\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + \mathbf{j}$ and $\mathbf{c} = 3\mathbf{j} + \mathbf{k}$, find $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$. Verify your answer using the identity $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$. [4]
- 14 The tetrahedron OABC has O at the origin and A, B, C with position vectors $\mathbf{a} = \mathbf{i} + 2\mathbf{j}$, $\mathbf{b} = 3\mathbf{j} + \mathbf{k}$ and $\mathbf{c} = 2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$. Find the volume of the tetrahedron. [5]
- 15 The points A, B, C and D have coordinates (1, 0, 1), (2, 1, 3), (0, 2, 2) and (4, 0, 4).
 - (a) Show that A, B, C and D are coplanar. [4]
 - (b) Find constants α and β such that $\mathbf{AD} = \alpha\mathbf{AB} + \beta\mathbf{AC}$. [3]
- 16 The vectors $\mathbf{a}$ and $\mathbf{b}$ make an angle θ with each other.
 - (a) Prove that $|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2$. [3]
 - (b) Verify the identity for $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$. [3]
- 17 The triangle OAB has O at the origin, A(1, 2, 0) and B(k, 1, 2), where k is a constant. The area of the triangle is 3. Find the possible values of k . [5]
- 18 A force $\mathbf{F} = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$ N acts at the point with position vector $\mathbf{r} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ m relative to the origin O. The moment of $\mathbf{F}$ about O is $\mathbf{r} \times \mathbf{F}$. Find the moment as a vector, and its magnitude to 3 significant figures. [5]
- 19 The vectors $\mathbf{a}$ and $\mathbf{b}$ are any two vectors.
 - (a) Prove that $(\mathbf{a} - \mathbf{b}) \times (\mathbf{a} + \mathbf{b}) = 2(\mathbf{a} \times \mathbf{b})$. [3]
 - (b) Interpret this result in terms of the parallelogram with adjacent sides $\mathbf{a}$ and $\mathbf{b}$. [2]
- 20 A student claims that for all vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$: (i) $\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{a}$ and (ii) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$.
 - (a) Use $\mathbf{a} = \mathbf{i}$ and $\mathbf{b} = \mathbf{j}$ to show that (i) is false. [2]
 - (b) Use $\mathbf{a} = \mathbf{i}$, $\mathbf{b} = \mathbf{i}$, $\mathbf{c} = \mathbf{j}$ to show that (ii) is false. [3]

- 21** The vectors $\mathbf{a}$ and $\mathbf{b}$ are $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{b} = -\mathbf{j} - 2\mathbf{k}$.
- (a) Show that the equation $\mathbf{a} \times \mathbf{v} = \mathbf{b}$ can only have solutions if $\mathbf{a} \cdot \mathbf{b} = 0$, and check this condition holds for the given $\mathbf{a}$ and $\mathbf{b}$. [2]
- (b) Find all vectors $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ such that $\mathbf{a} \times \mathbf{v} = \mathbf{b}$. [5]
- (c) Find the vector $\mathbf{v}$ of smallest magnitude satisfying $\mathbf{a} \times \mathbf{v} = \mathbf{b}$. [2]
-
- 22** A parallelepiped has edges represented by the vectors $\mathbf{a} = \mathbf{i} + 2\mathbf{j}$, $\mathbf{b} = \mathbf{j} + 3\mathbf{k}$ and $\mathbf{c} = 2\mathbf{i} + \mathbf{k}$.
- (a) Find the volume of the parallelepiped. [4]
- (b) Find the area of the face spanned by $\mathbf{a}$ and $\mathbf{b}$. [2]
- (c) Find the perpendicular distance between the face spanned by $\mathbf{a}$ and $\mathbf{b}$ and the opposite face, to 3 significant figures. [2]
-
- 23** The points P , Q and R have coordinates $P(1, 0, 0)$, $Q(0, 2, 0)$ and $R(0, 0, 3)$.
- (a) Find $\mathbf{PQ} \times \mathbf{PR}$. [3]
- (b) Find the area of triangle PQR . [2]
- (c) Find the perpendicular distance from P to the line QR , to 3 significant figures. [3]
-
- 24** The vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are any three vectors. You may use the identity $\mathbf{x} \times (\mathbf{y} \times \mathbf{z}) = \mathbf{y}(\mathbf{x} \cdot \mathbf{z}) - \mathbf{z}(\mathbf{x} \cdot \mathbf{y})$.
- (a) Write out $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$, $\mathbf{b} \times (\mathbf{c} \times \mathbf{a})$ and $\mathbf{c} \times (\mathbf{a} \times \mathbf{b})$ using the identity. [3]
- (b) Hence prove that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}$. [3]
-
- 25** A rigid body rotates with constant angular velocity $\boldsymbol{\omega} = 3\mathbf{k} \text{ rad s}^{-1}$ about an axis through the origin. A point P of the body has position vector $\mathbf{r} = 2\mathbf{i} + \mathbf{j} + 5\mathbf{k} \text{ m}$. The velocity of P is $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$ and its acceleration is $\mathbf{a} = \boldsymbol{\omega} \times \mathbf{v}$.
- (a) Find $\mathbf{v}$ and the speed of P . [3]
- (b) Find $\mathbf{a}$. [2]
- (c) Show that $\mathbf{a}$ is directed towards the axis of rotation and state its magnitude in terms of $\boldsymbol{\omega}$ and the distance of P from the axis. [4]
-
- 26** $ABCD$ is a plane quadrilateral with position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$.
- (a) By splitting $ABCD$ into triangles ABC and ACD , show that its area is $\frac{1}{2}|\mathbf{AC} \times \mathbf{BD}|$. [5]
- (b) Hence find the area of the quadrilateral with $A(1, 0, 0)$, $B(3, 1, 2)$, $C(4, 4, 5)$ and $D(2, 3, 3)$. [4]
-
- 27** The vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are three vectors. You may use $\mathbf{x} \times (\mathbf{y} \times \mathbf{z}) = \mathbf{y}(\mathbf{x} \cdot \mathbf{z}) - \mathbf{z}(\mathbf{x} \cdot \mathbf{y})$ and $(\mathbf{x} \times \mathbf{y}) \cdot \mathbf{z} = \mathbf{x} \cdot (\mathbf{y} \times \mathbf{z})$.
- (a) Prove that $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{a} \times \mathbf{c}) = [\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})] \mathbf{a}$. [5]
- (b) Verify the result when $\mathbf{a} = \mathbf{i} + 2\mathbf{j}$, $\mathbf{b} = \mathbf{j} + \mathbf{k}$ and $\mathbf{c} = \mathbf{i} + 2\mathbf{k}$. [4]
- (c) Deduce that if $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$ (with $\mathbf{a} \neq \mathbf{0}$) then $\mathbf{a} \times \mathbf{b}$ and $\mathbf{a} \times \mathbf{c}$ are parallel. [2]
-
- 28** Given three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ with $\mathbf{V} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \neq 0$, the reciprocal vectors are $\mathbf{a}' = \frac{\mathbf{b} \times \mathbf{c}}{\mathbf{V}}$, $\mathbf{b}' = \frac{\mathbf{c} \times \mathbf{a}}{\mathbf{V}}$ and $\mathbf{c}' = \frac{\mathbf{a} \times \mathbf{b}}{\mathbf{V}}$. Let $\mathbf{a} = \mathbf{i} + \mathbf{k}$, $\mathbf{b} = 2\mathbf{j}$ and $\mathbf{c} = \mathbf{i} + \mathbf{j} + 3\mathbf{k}$.
- (a) Find $\mathbf{V}$. [3]
- (b) Find $\mathbf{a}'$, $\mathbf{b}'$ and $\mathbf{c}'$. [4]
- (c) Verify that $\mathbf{a} \cdot \mathbf{a}' = \mathbf{b} \cdot \mathbf{b}' = \mathbf{c} \cdot \mathbf{c}' = 1$ and that $\mathbf{a} \cdot \mathbf{b}' = \mathbf{a} \cdot \mathbf{c}' = 0$. [3]
-
- 29** A force $\mathbf{F} = 3\mathbf{i} - 4\mathbf{k} \text{ N}$ acts along a line through the point $P(0, 2, 5)$ (distances in metres).
- (a) Find the moment $\mathbf{M}$ of $\mathbf{F}$ about the origin O , and $|\mathbf{M}|$. [3]
- (b) Find the perpendicular distance from O to the line of action of $\mathbf{F}$. [2]
- (c) Find the moment of $\mathbf{F}$ about the axis through O in the direction of the unit vector $\frac{1}{3}(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$. [3]
- (d) The force acts at the point $Q = P + \lambda \mathbf{F}$ instead. Show that the moment about O is unchanged. [2]

- 30** The tetrahedron OABC has O at the origin and A, B, C with position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$.
- (a) Show that the vector area of triangle ABC is $\frac{1}{2}(\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a})$. [3]
- (b) Given $\mathbf{a} = 2\mathbf{i}$, $\mathbf{b} = 3\mathbf{j}$ and $\mathbf{c} = 6\mathbf{k}$, find the exact area of triangle ABC. [4]
- (c) Find the areas of triangles OAB, OBC and OCA, and show that $(\text{area OAB})^2 + (\text{area OBC})^2 + (\text{area OCA})^2 = (\text{area ABC})^2$. [3]

Vector Equations of Lines

- 1** Write down a vector equation of the line through the point A(1, 2, 3) with direction vector $2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$. [2]
- 2** Find a vector equation of the line through A(3, -1, 2) and B(5, 2, -2). [2]
- 3** Write the line $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + \lambda(2\mathbf{i} - \mathbf{j} + 4\mathbf{k})$ in Cartesian form. [2]
- 4** Determine whether the point (5, 0, 11) lies on the line $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + \lambda(2\mathbf{i} - \mathbf{j} + 4\mathbf{k})$. [2]
- 5** The line l has equation $\frac{x-3}{2} = \frac{y+1}{4} = \frac{5-z}{1}$. Write down a point on l and a direction vector of l. [2]
- 6** Show that the lines $\mathbf{r} = \mathbf{i} + 2\mathbf{k} + \lambda(2\mathbf{i} - 4\mathbf{j} + 6\mathbf{k})$ and $\mathbf{r} = \mathbf{j} + \mathbf{k} + \mu(-\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})$ are parallel. [2]
- 7** The line l has equation $\mathbf{r} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$. Find the coordinates of the point where l crosses the plane $z = 0$. [3]
- 8** A line has equation $(\mathbf{r} - (2\mathbf{i} + \mathbf{j} - 3\mathbf{k})) \times (3\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = \mathbf{0}$. Write a vector equation of the line in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, and give its Cartesian equations. [4]
- 9** The point (7, p, q) lies on the line $\mathbf{r} = \mathbf{i} - \mathbf{j} + 2\mathbf{k} + \lambda(3\mathbf{i} + 2\mathbf{j} + \mathbf{k})$. Find p and q. [3]
- 10** A drone's position vector, in metres, at time t seconds is $\mathbf{r} = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k} + t(3\mathbf{i} + 2\mathbf{j} - \mathbf{k})$ for $t \geq 0$.
- (a) Find its position at $t = 3$ and its speed. [3]
- (b) Find the time when the drone's x-coordinate is 17. [2]
- 11** The lines l_1 and l_2 have equations $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} + 2\mathbf{k})$ and $\mathbf{r} = 5\mathbf{i} + 2\mathbf{k} + \mu(-2\mathbf{i} + 4\mathbf{j} + 5\mathbf{k})$.
Show that l_1 and l_2 intersect and find the coordinates of the point of intersection. [5]
- 12** The lines $\mathbf{r} = \mathbf{i} + \lambda(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$ and $\mathbf{r} = \mathbf{j} + \mathbf{k} + \mu(2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k})$ meet at an acute angle θ .
Find θ to the nearest tenth of a degree. [4]
- 13** The points A(2, 5, -1), B(5, 2, 2) and C(-1, p, q) lie on a straight line. Find a vector equation of the line and the values of p and q. [4]
- 14** The lines l_1 and l_2 have equations $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + 2\mathbf{k})$ and $\mathbf{r} = \mathbf{j} + 5\mathbf{k} + \mu(2\mathbf{i} + \mathbf{j} - \mathbf{k})$.
Show that l_1 and l_2 are skew. [6]
- 15** The lines l_1 and l_2 have equations $\mathbf{r} = \mathbf{j} + 2\mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} - \mathbf{k})$ and $\mathbf{r} = 4\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} + \mu(3\mathbf{i} - \mathbf{j} + \mathbf{k})$.
- (a) Show that the lines intersect and find the coordinates of the point of intersection. [4]
- (b) Show that l_1 and l_2 are perpendicular. [2]

- 16** The lines $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j} - \mathbf{k})$ and $\mathbf{r} = p\mathbf{j} + \mathbf{k} + \mu(\mathbf{i} - \mathbf{j} + 3\mathbf{k})$ intersect. Find the value of p and the coordinates of the point of intersection. [6]
- 17** Two aircraft fly in straight lines at constant velocity. Relative to an airport, with $\mathbf{i}$ and $\mathbf{j}$ pointing east and north and $\mathbf{k}$ vertically up, their positions in km at time t minutes are $\mathbf{r}_P = 10\mathbf{k} + t(2\mathbf{i} + \mathbf{j} - \mathbf{k})$ and $\mathbf{r}_Q = 10\mathbf{i} - \mathbf{j} + 5\mathbf{k} + t(-2\mathbf{i} + \mathbf{j} + \mathbf{k})$.
 (a) Show that the flight paths cross, and find the coordinates of the crossing point. [3]
 (b) Explain why the aircraft do not collide. [2]
 (c) Find the distance between the aircraft when $t = 2$. [2]
- 18** The line l has equation $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$. Find the coordinates of the points where l meets the xy -plane, the yz -plane and the xz -plane. [6]
- 19** Show that the lines $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-2}{1}$ and $\frac{x-2}{1} = \frac{y-4}{-2} = \frac{z-1}{2}$ intersect, and find the point of intersection. [5]
- 20** A student is given the lines $l_1: \mathbf{r} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + 2\mathbf{k})$ and $l_2: \mathbf{r} = 3\mathbf{i} + 7\mathbf{k} + \mu(2\mathbf{i} - 2\mathbf{j} + 4\mathbf{k})$ and writes: "The direction vectors are different, so the lines are not the same line."
 (a) Identify the error in the student's statement. [2]
 (b) Determine whether l_1 and l_2 are the same line, parallel distinct lines, or intersecting lines. [3]
- 21** The lines l_1 and l_2 have equations $\mathbf{r} = 2\mathbf{i} + \mathbf{j} + \lambda(\mathbf{i} + \mathbf{j} + 2\mathbf{k})$ and $\mathbf{r} = \mathbf{i} + 3\mathbf{j} + 2\mathbf{k} + \mu(2\mathbf{i} - \mathbf{j})$.
 (a) Show that l_1 and l_2 intersect and find the point of intersection. [3]
 (b) Find the acute angle between the lines, to the nearest tenth of a degree. [3]
 (c) Find a vector equation of the line through the point of intersection that is perpendicular to both l_1 and l_2 . [2]
- 22** Two particles A and B move with constant velocities. Their position vectors, in metres, at time t seconds are $\mathbf{r}_A = \mathbf{i} + 2\mathbf{j} + t(2\mathbf{i} - \mathbf{j} + 3\mathbf{k})$ and $\mathbf{r}_B = 11\mathbf{i} - 6\mathbf{j} + t(-3\mathbf{i} + 3\mathbf{j} + 3\mathbf{k})$.
 (a) Show that the particles collide and find the position vector of the collision. [4]
 (b) Find the speed of each particle. [2]
 (c) Find the angle between the paths of the particles. [2]
- 23** The triangle ABC has vertices A(1, 0, 2), B(5, 4, -2) and C(3, 8, 6). M and N are the midpoints of BC and AC respectively.
 (a) Find vector equations of the medians AM and BN. [2]
 (b) Show that AM and BN meet at the centroid G of the triangle and find G. [4]
 (c) Show that the line CG passes through the midpoint of AB. [3]
- 24** The lines l_1 and l_2 have equations $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} + 2\mathbf{k})$ and $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} + p\mathbf{k} + \mu(2\mathbf{i} + \mathbf{j} - \mathbf{k})$, where p is a constant.
 (a) Show that the lines can only intersect if $p = 5$. [4]
 (b) State, with a reason, the relationship between the lines when $p \neq 5$. [1]
 (c) When $p = 5$, find the point of intersection and the acute angle between the lines. [4]
- 25** ABCD is a parallelogram with A(0, 1, 2), B(3, 4, 0) and C(5, 2, 3).
 (a) Write down vector equations of the line through A parallel to BC and the line through C parallel to AB. [3]
 (b) By solving the equations of these lines, find the coordinates of D. [4]
 (c) Verify that your answer agrees with $\mathbf{d} = \mathbf{a} + \mathbf{c} - \mathbf{b}$. [1]
- 26** The lines l_1 and l_2 have equations $\mathbf{r} = 2\mathbf{i} + \lambda(\mathbf{j} + \mathbf{k})$ and $\mathbf{r} = 3\mathbf{i} + \mathbf{j} + \mathbf{k} + \mu(\mathbf{j} - \mathbf{k})$. The point P is (1, 1, 1). Find a vector equation of the line through P that meets both l_1 and l_2 . [6]

- 27** An aircraft is tracked by radar at O. Relative to O, its position in km at time t minutes is $\mathbf{r}_A = 6\mathbf{i} + 4\mathbf{j} + 8\mathbf{k} + t(\mathbf{i} - 2\mathbf{j} - \mathbf{k})$. At $t = 0$ a missile is launched from O and travels in a straight line at a constant speed of 5 km per minute to intercept the aircraft at time T .
- Show that T satisfies $19T^2 + 20T - 116 = 0$. [4]
 - Solve the equation to find T and the position of the interception. [3]
 - Find the velocity of the missile and check that its speed is 5 km per minute. [2]
 - Find the angle of elevation of the missile's path, to the nearest tenth of a degree. [2]
-
- 28** Two boats P and Q move on a calm lake with constant velocities. Relative to a marker buoy, their position vectors in metres at time t seconds are $\mathbf{r}_P = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + t(2\mathbf{i} - \mathbf{j} + 2\mathbf{k})$ and $\mathbf{r}_Q = 10\mathbf{i} - 4\mathbf{j} + 6\mathbf{k} + t(-\mathbf{i} + \mathbf{j} + 2\mathbf{k})$.
- Find the vector from P to Q at time t seconds. [2]
 - Show that the distance d between the boats satisfies $d^2 = 13t^2 - 78t + 126$. [2]
 - Find the least distance between the boats and the time at which it occurs. [4]
 - Hence explain why the boats never collide. [1]
-
- 29** The line l has equation $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$. The sphere S_R has centre at the origin and radius R , so its equation is $|\mathbf{r}| = R$.
- Show that $|\mathbf{r}|^2 = 9\lambda^2 + 6\lambda + 14$ for a point on l . [2]
 - For $R = 7$, find the coordinates of the two points where l meets S_7 and the length of the chord between them. [6]
 - Find the value of R for which l is a tangent to S_R , and the point of tangency. [4]
-
- 30** The lines $l_1: \mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{d}_1$ and $l_2: \mathbf{r} = \mathbf{a}_2 + \mu\mathbf{d}_2$ are not parallel.
- Prove that if l_1 and l_2 intersect then $(\mathbf{a}_2 - \mathbf{a}_1) \cdot (\mathbf{d}_1 \times \mathbf{d}_2) = 0$. [4]
 - Explain what the condition $(\mathbf{a}_2 - \mathbf{a}_1) \cdot (\mathbf{d}_1 \times \mathbf{d}_2) = 0$ means geometrically, and why it does not by itself distinguish intersecting lines from parallel ones. [2]
 - The lines $\mathbf{r} = \mathbf{i} + \lambda(\mathbf{i} + \mathbf{j})$ and $\mathbf{r} = \mathbf{j} + p\mathbf{k} + \mu(\mathbf{i} + \mathbf{k})$ intersect. Find p and the point of intersection. [6]

Vector Equations of Planes

- Find the Cartesian equation of the plane through the point $A(1, 2, 3)$ with normal vector $2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$. [2]
- Write the plane $\mathbf{r} \cdot (3\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = 5$ in Cartesian form. [2]
- The plane Π has equation $4x - y + 3z = 7$. Write down a normal vector to Π , and determine whether the point $(1, -3, 0)$ lies on Π . [2]
- Determine which of the points $A(1, 0, 2)$ and $B(2, 1, 0)$ lie on the plane $2x - y + z = 4$. [2]
- Find the coordinates of the points where the plane $2x + 3y + 6z = 12$ meets the x , y and z axes. [2]
- Show that the planes $x + 2y - z = 3$ and $2x + 4y - 2z = 11$ are parallel. [2]
- Find the Cartesian equation of the plane through $A(1, 0, 2)$, $B(3, 1, 1)$ and $C(2, 3, 0)$. [5]
- The plane Π has vector equation $\mathbf{r} = \mathbf{i} + 2\mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j}) + \mu(\mathbf{j} + 3\mathbf{k})$. Find the Cartesian equation of Π . [4]
- The plane Π has equation $x + 2y - z = 4$. Find a vector equation of Π in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$. [4]

- 10** The plane Π passes through the points $A(2, 1, 1)$, $B(4, 0, 3)$ and $C(3, 3, 0)$.
 (a) Write down parametric equations for Π , using the parameters λ and μ . [3]
 (b) Eliminate λ and μ to find the Cartesian equation of Π . [4]
- 11** The points A and B have coordinates $A(1, 2, 3)$ and $B(5, 0, -1)$. Find the Cartesian equation of the plane that is the perpendicular bisector of AB . [4]
- 12** The point $(k, 2, -1)$ lies on the plane $2x - y + 3z = 7$. Find k . [2]
- 13** The lines l_1 and l_2 have equations $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + \mathbf{k} + \lambda(2\mathbf{i} - \mathbf{j} + \mathbf{k})$ and $\mathbf{r} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k} + \mu(\mathbf{i} + \mathbf{j} - 2\mathbf{k})$. Show that l_1 and l_2 intersect, and find the Cartesian equation of the plane that contains both lines. [6]
- 14** The line l has equation $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + t(\mathbf{i} + \mathbf{j} + \mathbf{k})$ and the point P has coordinates $(0, 0, 3)$. Find the Cartesian equation of the plane that contains l and P . [5]
- 15** The lines l_1 and l_2 have equations $\mathbf{r} = \mathbf{i} + \lambda(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$ and $\mathbf{r} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k} + \mu(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$. Explain why the lines are parallel, and find the Cartesian equation of the plane that contains both of them. [5]
- 16** The planes $\Pi_1: 2x - y + z = 3$ and $\Pi_2: x + y - z = 1$ are perpendicular. Show this, and find the Cartesian equation of the plane through $(1, 1, 1)$ that is perpendicular to both Π_1 and Π_2 . [5]
- 17** A plane meets the coordinate axes at $A(6, 0, 0)$, $B(0, 4, 0)$ and $C(0, 0, 3)$.
 (a) Show that the plane has equation $2x + 3y + 4z = 12$. [2]
 (b) Find the exact area of triangle ABC . [3]
 (c) Find the volume of the tetrahedron $OABC$. [2]
- 18** The plane Π has vector equation $\mathbf{r} = \mathbf{i} + \mathbf{j} + \lambda(\mathbf{i} + 2\mathbf{k}) + \mu(-\mathbf{i} + \mathbf{j} + \mathbf{k})$.
 (a) Find the Cartesian equation of Π . [4]
 (b) The point $(3, -1, -2)$ lies on Π . Find the values of λ and μ that give this point. [3]
- 19** Find the Cartesian equation of the plane that contains the points $A(1, 0, 0)$ and $B(0, 1, 0)$ and is parallel to the vector $\mathbf{i} + \mathbf{j} + \mathbf{k}$. [5]
- 20** A student tries to find the plane through $A(1, 2, 3)$, $B(2, 3, 5)$ and $C(0, 1, 1)$ by calculating $\mathbf{AB} \times \mathbf{AC}$ and gets the zero vector.
 (a) Explain why the zero vector arises and what it tells you about A , B and C . [2]
 (b) Find the Cartesian equation of a plane containing A , B , C and the point $D(1, 0, 0)$. [4]
- 21** The line l has equation $\mathbf{r} = 2\mathbf{i} + \mathbf{k} + t(\mathbf{i} - \mathbf{j} + 2\mathbf{k})$ and the plane Π_1 has equation $x + 2y - z = 4$. Find the Cartesian equation of the plane Π_2 that contains l and is perpendicular to Π_1 . [7]
- 22** The plane Π has equation $x + 2y + 2z = 12$ and meets the coordinate axes at A , B and C .
 (a) Find the coordinates of A , B and C . [1]
 (b) Find the area of triangle ABC . [4]
 (c) Find the volume of the tetrahedron $OABC$. [2]
 (d) The distance from O to Π is the scalar projection of $\mathbf{OA}$ onto the unit normal. Find it, and confirm that $\text{volume} = \frac{1}{3} \times \text{area } ABC \times \text{this distance}$. [2]

- 23** The points $A(1, 2, 0)$ and $B(3, -2, 4)$ are given. The point $P(x, y, z)$ moves so that $PA = PB$.
- Show that P lies on the plane $x - 2y + 2z = 6$. [4]
 - Show that this plane is the perpendicular bisector of AB . [2]
 - Hence write down the distance from A to the plane in terms of the length AB . [2]
-
- 24** $OABC$ is a tetrahedron with O the origin, $A(2, 0, 0)$, $B(0, 3, 0)$ and $C(1, 1, 4)$. M is the midpoint of AB .
- Find the Cartesian equation of the plane containing O , C and M . [4]
 - Show that the line through A parallel to OC never meets this plane, and explain why. [3]
-
- 25** A sphere has centre $C(1, 2, 3)$ and passes through the point $P(3, 0, 4)$.
- Find the radius of the sphere. [2]
 - The tangent plane to a sphere at P is perpendicular to CP . Find its Cartesian equation. [3]
 - Find the coordinates of the point where this tangent plane meets the x -axis. [2]
-
- 26** The points $A(1, 1, 1)$, $B(2, p, 3)$ and $C(0, 2, p)$ lie on a plane Π_p , where p is a constant.
- Show that a normal to Π_p is $(p^2 - 2p - 1)\mathbf{i} - (p + 1)\mathbf{j} + p\mathbf{k}$. [3]
 - Find the values of p for which Π_p passes through the origin. [4]
 - Find the Cartesian equation of Π_p when $p = 0$. [2]
-
- 27** The points A , B and C have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ and are not collinear.
- Show that $\mathbf{n} = \mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}$ is perpendicular to the plane ABC . [3]
 - Hence prove that the plane ABC has equation $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$. [3]
 - Use this to find the Cartesian equation of the plane through $A(1, 1, 0)$, $B(0, 1, 2)$ and $C(3, 0, 1)$. [4]
 - Deduce a condition on a , b , c for the plane ABC to pass through the origin. [2]
-
- 28** The lines l_1 and l_2 have equations $\mathbf{r} = \mathbf{i} + \mathbf{k} + \lambda(\mathbf{i} + \mathbf{j})$ and $\mathbf{r} = 2\mathbf{j} + \mu(\mathbf{j} + \mathbf{k})$.
- Find a vector that is perpendicular to both lines. [2]
 - Find the Cartesian equation of the plane Π_1 that contains l_1 and is parallel to l_2 . [3]
 - Find the Cartesian equation of the plane Π_2 that contains l_2 and is parallel to l_1 . [2]
 - Hence explain why l_1 and l_2 are skew. [2]
-
- 29** The plane Π passes through three points A , B and C with position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.
- Show that every point of Π can be written as $\alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$ with $\alpha + \beta + \gamma = 1$. [4]
 - For $A(1, 0, 0)$, $B(0, 2, 0)$ and $C(0, 0, 3)$, show that Π has equation $6x + 3y + 2z = 6$. [2]
 - Find the position vector of the centroid of ABC and verify that it lies on Π . [2]
 - The point $D = \frac{1}{2}\mathbf{a} + \frac{1}{3}\mathbf{b} + \frac{1}{6}\mathbf{c}$. Show that D lies on Π and find its coordinates. [2]
-
- 30** A plane Π passes through the point $P(1, 2, 3)$ and meets the positive x , y and z axes at $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$.
- Show that $\frac{1}{a} + \frac{2}{b} + \frac{3}{c} = 1$. [2]
 - Write down an expression for the volume V of the tetrahedron $OABC$. [1]
 - Use the AM-GM inequality, that for positive u , v , w : $u + v + w \geq 3(uvw)^{\frac{1}{3}}$, to show that $V \geq 27$, and find the values of a , b , c for which $V = 27$. [6]
 - Hence write down the equation of this plane of least volume in the form $px + qy + rz = s$. [1]

Intersections of Lines and Planes

- 1** Find the coordinates of the point where the line $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} + 2\mathbf{k})$ meets the plane $x + y + z = 12$. [2]

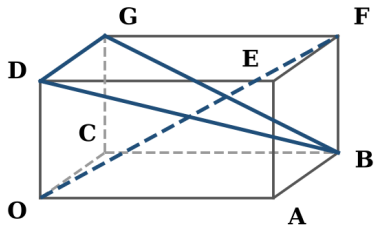
- 2** Show that the line $\mathbf{r} = \mathbf{i} + 2\mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j})$ is parallel to the plane $x - 2y + z = 5$ but does not lie in it. [2]
- 3** Show that the line $\mathbf{r} = 2\mathbf{i} + \mathbf{j} + \lambda(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$ lies in the plane $x + y - z = 3$. [2]
- 4** Find the point of intersection of the three planes $x + y + z = 6$, $x - y = 0$ and $z = 2$. [2]
- 5** The line l has equation $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z}{1}$. Find the point where l meets the plane $x + y - 2z = 6$. [2]
- 6** Find the point where the line through $A(1, 1, 1)$ and $B(2, 3, 4)$ meets the plane $3x - y + 2z = 18$. [3]
- 7** Find a vector equation of the line of intersection of the planes $x + y + z = 3$ and $x - y + 2z = 1$. [4]
- 8** The line l has equation $\mathbf{r} = 2\mathbf{i} - \mathbf{j} + \lambda(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$ and the plane Π has equation $\mathbf{r} \cdot (\mathbf{i} + 3\mathbf{j} - \mathbf{k}) = c$, where c is a constant. The line meets Π at the point P where $\lambda = 2$. Find the position vector of P and the value of c . [4]
- 9** Solve the simultaneous equations $x + y + z = 6$, $2x - y + z = 3$, $x - y + 2z = 5$, and state the geometrical meaning of your solution. [5]
- 10** Find the coordinates of the foot of the perpendicular from the point $P(1, 2, 3)$ to the plane $x + 2y + 2z = 2$. [4]
- 11** The planes $2x - y + z = 4$ and $x + y - z = 2$ meet in a line l . Find the Cartesian equations of l . [4]
- 12** The line l has equation $\mathbf{r} = \mathbf{i} + 2\mathbf{k} + \lambda(p\mathbf{i} + \mathbf{j} + 2\mathbf{k})$ and the plane Π has equation $x - 2y + z = 3$.
 (a) Find the value of p for which l is parallel to Π . [2]
 (b) Show that for this value of p the line lies in Π . [2]
- 13** Describe the relationship between each pair of planes.
 (a) $2x - y + z = 3$ and $4x - 2y + 2z = 6$ [2]
 (b) $2x - y + z = 3$ and $4x - 2y + 2z = 7$ [2]
 (c) $2x - y + z = 3$ and $x + y - z = 2$ [2]
- 14** The plane Π has equation $x + 2y - z = 4$. For each line, state whether it meets Π at a single point, is parallel to Π (not in it), or lies in Π . Find the point of intersection where there is one.
 (a) $\mathbf{r} = 2\mathbf{i} + \mathbf{j} + \lambda(\mathbf{i} + \mathbf{j} + 3\mathbf{k})$ [2]
 (b) $\mathbf{r} = \mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} + 3\mathbf{k})$ [2]
 (c) $\mathbf{r} = \lambda(\mathbf{i} + \mathbf{j} + \mathbf{k})$ [2]
- 15** The planes $\Pi_1: 2x + y - z = 5$ and $\Pi_2: x - 2y + 3z = -1$ intersect in the line l .
 (a) Find a vector equation of l . [5]
 (b) Verify that the point $(3, -7, -6)$ lies on both planes. [1]
- 16** Consider the planes $\Pi_1: x + 2y + z = 4$, $\Pi_2: 2x - y + z = 1$ and $\Pi_3: 3x + y + 2z = k$.
 (a) Show that the three planes have no common point unless $k = 5$. [3]
 (b) Describe the arrangement of the planes when $k \neq 5$. [2]
 (c) When $k = 5$, find the line of intersection. [2]
- 17** Find the coordinates of the image of the point $P(1, 2, 3)$ after reflection in the plane $2x + y - 2z = 7$. [6]

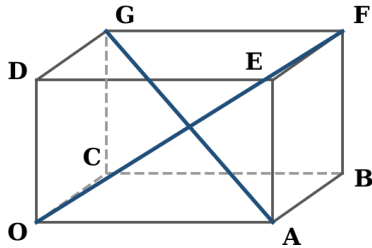
- 18** The line l has equation $\mathbf{r} = 2\mathbf{i} + \mathbf{j} + \lambda(\mathbf{i} - \mathbf{j} + 3\mathbf{k})$ and the plane Π has equation $\mathbf{r} = \mathbf{k} + s(\mathbf{i} + \mathbf{k}) + t(\mathbf{j} - \mathbf{k})$.
Find the position vector of the point where l meets Π . [6]
-
- 19** A student finds where the line $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j} - \mathbf{k})$ meets the plane $3x - 2y + z = 5$. The student writes: " $3(1 + 2\lambda) - 2(2 + \lambda) + (3 - \lambda) = 5$, so $3 + 6\lambda - 4 + 2\lambda + 3 - \lambda = 5$, so $2 + 7\lambda = 5$, so $\lambda = \frac{3}{7}$."
(a) Identify the error in the student's working. [2]
(b) Find the correct point of intersection. [2]
-
- 20** A laser beam is fired from the point $A(2, 1, 8)$ in the direction $\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$ towards a horizontal mirror lying in the plane $z = 0$ (all lengths in metres).
(a) Find the coordinates of the point P where the beam hits the mirror. [3]
(b) The mirror reverses the vertical component of the direction, so the reflected beam has direction $\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$. Find the point at which the reflected beam reaches the height $z = 12$. [3]
-
- 21** Consider the system $x + 2y + z = 4$, $2x + y + 3z = 7$ and $x - y + kz = 3$, where k is a constant.
(a) Show that the system has a unique solution when $k \neq 2$, and find it. [5]
(b) Show that when $k = 2$ the system has infinitely many solutions, and find a vector equation of the line of solutions. [4]
(c) Describe the geometrical arrangement of the three planes in each case. [1]
-
- 22** The planes $x = 0$, $y = 0$, $z = 0$ and $x + 2y + 3z = 6$ bound a tetrahedron $OABC$.
(a) Write down the coordinates of A , B and C , the points where the slanted plane meets the axes. [1]
(b) Find the coordinates of the foot F of the perpendicular from O to the slanted plane. [4]
(c) Find the length OF and, given that the area of face ABC is $3\sqrt{14}$, verify that the volume of the tetrahedron is $\frac{1}{3} \times \text{area } ABC \times OF$. [3]
-
- 23** A light ray travels from the point $A(1, -2, 0)$ in the direction $\mathbf{i} + 4\mathbf{j}$ and is reflected by a plane mirror with equation $x + 2y + 2z = 6$.
(a) Show that the ray hits the mirror at $P(2, 2, 0)$. [2]
(b) The reflected direction is $\mathbf{d}' = \mathbf{d} - 2\frac{\mathbf{d} \cdot \mathbf{n}}{|\mathbf{n}|^2}\mathbf{n}$, where $\mathbf{n}$ is the normal to the mirror. Find $\mathbf{d}'$. [3]
(c) Find the coordinates of the point where the reflected ray meets the plane $x = 0$. [2]
-
- 24** The planes $\Pi_1: x - 2y + z = 3$ and $\Pi_2: 2x + y - z = 4$ meet in the line l .
(a) Find a vector equation of l . [4]
(b) Find the point where l meets the plane $x + y + z = 9$. [2]
(c) Show that the plane $(x - 2y + z - 3) + k(2x + y - z - 4) = 0$ contains l for every value of k , and find the value of k for which this plane passes through the point $(1, 1, 1)$. [3]
-
- 25** A drone flies in a straight line with position vector, in metres, $\mathbf{r} = 50\mathbf{k} + t(3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k})$ at time t seconds. A hillside is modelled by the plane $x + 2y - 2z = -16$, with the drone above the plane at $t = 0$.
(a) Show that the drone reaches the hillside at $t = 4$ and find its position then. [3]
(b) Find the speed of the drone and the distance it flies before reaching the hillside. [3]
-
- 26** The plane Π_1 has equation $\mathbf{r} = \mathbf{i} + s(\mathbf{i} + \mathbf{j}) + t(\mathbf{j} + \mathbf{k})$ and the plane Π_2 has equation $x + y - z = 3$.
(a) Find the Cartesian equation of Π_1 . [3]
(b) Find a vector equation of the line of intersection of Π_1 and Π_2 . [4]

- 27** The plane Π has equation $\mathbf{r} \cdot \mathbf{n} = d$.
- (a) Prove that the image of a point with position vector $\mathbf{p}$ after reflection in Π is $\mathbf{p}' = \mathbf{p} - 2 \frac{\mathbf{p} \cdot \mathbf{n} - d}{|\mathbf{n}|^2} \mathbf{n}$. [4]
- (b) Use the formula to find the image of $P(2, 1, 6)$ in the plane $x - 2y + 2z = 3$. [3]
- (c) The line l passes through P with direction $\mathbf{i} + \mathbf{j} + 5\mathbf{k}$. Find the point where l meets the plane, and hence find a vector equation of the reflection of l in the plane. [5]
-
- 28** Consider the planes $\Pi_1: x + y + z = 6$, $\Pi_2: x + 2y + 3z = 14$ and $\Pi_3: 2x + 3y + az = b$, where a and b are constants.
- (a) Find a vector equation of the line of intersection of Π_1 and Π_2 . [4]
- (b) Find the conditions on a and b so that the three planes meet (i) in a single point, (ii) in a line, (iii) in no common point. Describe the arrangement in (iii). [6]
- (c) Find the point of intersection when $a = 0$ and $b = 12$. [2]
-
- 29** The plane Π has equation $x + 2y + 2z = 9$ and the sphere S has equation $|\mathbf{r}| = 5$, with centre at the origin O .
- (a) Find the foot F of the perpendicular from O to Π , and the distance OF . [3]
- (b) Hence show that Π meets S in a circle, and find its centre and radius. [4]
- (c) The line through F with direction $2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ lies in Π . Find the coordinates of the two points where it meets the circle. [4]
-
- 30** The planes $\Pi_1: \mathbf{n}_1 \cdot \mathbf{r} = d_1$ and $\Pi_2: \mathbf{n}_2 \cdot \mathbf{r} = d_2$ meet in a line l .
- (a) Prove that any plane containing l has an equation of the form $\lambda(\mathbf{n}_1 \cdot \mathbf{r} - d_1) + \mu(\mathbf{n}_2 \cdot \mathbf{r} - d_2) = 0$, where λ and μ are constants, not both zero. [5]
- (b) Find the Cartesian equation of the plane that contains the line of intersection of $2x - y + z = 3$ and $x + y - 2z = 1$ and is perpendicular to the plane $x + 2y + z = 7$. [6]

Angles Between Lines and Planes

- 1** Find the acute angle between the planes $x + y + z = 3$ and $x = 2$. [3]
- 2** Find the acute angle between the line $\mathbf{r} = \lambda(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$ and the plane $x = 0$. [3]
- 3** Find the angle between the lines with direction vectors $2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$. [2]
- 4** Find the acute angle between the planes $2x - y + 2z = 5$ and $x + 2y + 2z = 1$. [3]
- 5** Find the angle between the line $\frac{x-1}{2} = \frac{y}{1} = \frac{z+2}{2}$ and the plane $z = 0$. [3]
- 6** Show that the planes $2x + y - z = 3$ and $x - 2y = 4$ are perpendicular. [2]
- 7** Find the acute angle between the line $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + \lambda(2\mathbf{i} - \mathbf{j} + 2\mathbf{k})$ and the plane $x + 2y + 2z = 7$. [4]
- 8** Find the acute angle between the planes $\mathbf{r} \cdot (\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) = 3$ and $\mathbf{r} \cdot (2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}) = 5$, to the nearest tenth of a degree. [4]
- 9** The lines l_1 and l_2 have equations $\mathbf{r} = \mathbf{i} - 2\mathbf{j} + \lambda(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$ and $\mathbf{r} = \mathbf{j} + 5\mathbf{k} + \mu(3\mathbf{i} - 4\mathbf{j})$. Find the acute angle between them. [4]
- 10** Find the acute angles that the line $\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(2\mathbf{i} - 2\mathbf{j} + \mathbf{k})$ makes with the x , y and z axes, to the nearest tenth of a degree. [4]
- 11** The planes $kx + 2y - z = 4$ and $3x - y + 2z = 1$ are perpendicular. Find k . [2]

- 12** Find the acute angle between the plane $x + 2y + 2z = 6$ and each of the coordinate planes, to the nearest tenth of a degree. [4]
- 13** Find the acute angle between the line through $A(0, 1, 1)$ and $B(2, -1, 2)$ and the plane $6x + 2y - 3z = 1$, giving your answer to the nearest tenth of a degree. [4]
- 14** OABCDEFG is a cuboid with O at the origin, $A(4, 0, 0)$, $C(0, 3, 0)$ and $D(0, 0, 2)$.

 - Find the angle between the diagonal OF and the base OABC. [2]
 - Find the Cartesian equation of the plane BDG. [3]
 - Find the acute angle between the plane BDG and the base OABC. [2]
- 15** The line l has direction vector $\mathbf{i} + p\mathbf{j} + 2\mathbf{k}$, where $p > 0$. The angle between l and the plane $z = 0$ is 30° . Find the exact value of p . [5]
- 16** The plane $\Pi: 6x + 3y + 2z = 6$ meets the coordinate axes at A, B and C.
 - Write down the coordinates of A, B and C. [1]
 - Find the acute angle between the face ABC and the face OAB (the plane $z = 0$), to the nearest tenth of a degree. [2]
 - Find the acute angle between the face ABC and the face OBC (the plane $x = 0$). [2]
- 17** The planes $x + 2y + z = 4$ and $2x - y + z = 1$ meet in the line l . Find the acute angle between l and the plane $x + y + z = 2$, to the nearest tenth of a degree. [6]
- 18** A light ray with direction $\mathbf{i} + \mathbf{j} + 4\mathbf{k}$ is reflected by the plane mirror $2x - y + 2z = 4$. The normal to the mirror is $\mathbf{n} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$.
 - Find the acute angle between the incoming ray and the mirror. [3]
 - The reflected ray has direction $\mathbf{d}' = \mathbf{d} - 2\frac{\mathbf{d} \cdot \mathbf{n}}{|\mathbf{n}|^2}\mathbf{n}$. Find $\mathbf{d}'$ and show that the incoming and reflected rays are perpendicular. [3]
- 19** A student is asked to find the acute angle between the line $\mathbf{r} = \lambda(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$ and the plane $x + y + z = 0$ and writes: " $\cos \theta = \frac{\mathbf{d} \cdot \mathbf{n}}{|\mathbf{d}||\mathbf{n}|} = \frac{5}{3\sqrt{3}}$, so $\theta = 15.8^\circ$."
 - Explain the error. [2]
 - Find the correct angle between the line and the plane. [2]
- 20** The line l_1 has equation $\mathbf{r} = \mathbf{i} + 2\mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + 2\mathbf{k})$. The line l_2 is the intersection of the planes $x + y + z = 3$ and $x - y + 2z = 1$. Show that l_1 and l_2 are perpendicular. [4]
- 21** A rectangular hip roof has horizontal base ABCD with $A(0, 0, 0)$, $B(8, 0, 0)$, $C(8, 6, 0)$ and $D(0, 6, 0)$ (metres, z vertical). The ridge runs from $R_1(2, 3, 2)$ to $R_2(6, 3, 2)$. The front face is ABR_2R_1 and the end face is ADR_1 .
 - Find the angle that the front face makes with the horizontal. [3]
 - Find the angle that the end face makes with the horizontal. [2]
 - Find the angle between the hip edge AR_1 and the horizontal. [2]
 - Find the angle between the front face and the end face, measured inside the roof. [2]

- 22** The plane Π passes through $A(2, 0, 0)$, $B(0, 2, 0)$ and $C(0, 0, c)$, where $c > 0$. Π makes an angle of 45° with the plane $z = 0$. Find the exact value of c . [6]
- 23** OABCDEFG is a cuboid with O the origin, $A(5, 0, 0)$, $C(0, 4, 0)$ and $D(0, 0, 3)$.
- 
- (a) Show that the space diagonals OF and AG are perpendicular. [3]
- (b) Find the acute angle between the face diagonals OB and AF, which do not intersect, to the nearest tenth of a degree. [4]
- 24** A regular tetrahedron has vertices $P_1(1, 1, 1)$, $P_2(1, -1, -1)$, $P_3(-1, 1, -1)$ and $P_4(-1, -1, 1)$.
- (a) Find a normal to the face $P_1P_2P_3$ and a normal to the face $P_1P_2P_4$. [3]
- (b) Find the angle between these two faces, to the nearest tenth of a degree. [2]
- (c) Show that the line from the centre O to P_1 is perpendicular to the face $P_2P_3P_4$. [2]
- 25** A hillside is modelled by the plane $3x + 4y + 12z = 0$, where the z -axis is vertical.
- (a) Find the angle that the hillside makes with the horizontal. [2]
- (b) Show that the vector $36\mathbf{i} + 48\mathbf{j} - 25\mathbf{k}$ lies in the hillside, and find the angle it makes with the horizontal. [4]
- (c) A path on the hillside runs in the direction $4\mathbf{i} - 3\mathbf{j}$. Show that it is horizontal. [2]
- 26** The planes $\Pi_1: x + 2y + 2z = 3$ and $\Pi_2: 2x + y - 2z = 6$ meet in a line l .
- (a) Show that the point $(3, 0, 0)$ lies on both planes. [1]
- (b) The two planes that bisect the angles between Π_1 and Π_2 both contain l and have normals along $\hat{n}_1 + \hat{n}_2$ and $\hat{n}_1 - \hat{n}_2$, where $\hat{n}_1$ and $\hat{n}_2$ are unit normals. Find the Cartesian equations of the two bisecting planes. [5]
- (c) Verify that the plane $x + y = 3$ makes equal angles with Π_1 and Π_2 . [2]
- 27** The plane Π passes through the origin and contains the line $\mathbf{r} = \lambda(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$. Π makes an angle θ with the plane $z = 0$, where $\cos \theta = \frac{2}{3}$. Let a normal to Π be $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + \mathbf{k}$.
- (a) Show that $a = -2b - 2$. [2]
- (b) Show that $20b^2 + 32b + 11 = 0$. [3]
- (c) Hence find the Cartesian equations of the two possible planes Π . [4]
- 28** A plane Π is inclined at an angle α to the horizontal. Take the origin in Π , the x -axis horizontal along a contour of Π , the z -axis vertically upwards and the y -axis so that the line of greatest slope has direction $\mathbf{g} = (0, \cos \alpha, -\sin \alpha)$. A line in Π makes an angle φ with the line of greatest slope and has direction $\mathbf{u}$.
- (a) Show that Π has unit normal $\mathbf{n} = (0, \sin \alpha, \cos \alpha)$, and write down $\mathbf{u}$ in terms of α , φ and the unit vector $\mathbf{e} = (1, 0, 0)$. [3]
- (b) Prove that the angle θ between this line and the horizontal satisfies $\sin \theta = \sin \alpha \cos \varphi$. [2]
- (c) Find θ when $\alpha = 40^\circ$ and $\varphi = 30^\circ$, to the nearest tenth of a degree. [2]
- (d) A path in a plane inclined at 50° to the horizontal rises at 20° to the horizontal. Find the angle φ between the path and the line of greatest slope, to the nearest tenth of a degree. [3]

- 29** The point P moves along the line $\mathbf{r} = \mathbf{i} + t\mathbf{j} + 2\mathbf{k}$, where t is a real parameter. The angle between OP and the plane $x + y + z = 0$ is θ .
- (a) Show that $\sin^2 \theta = \frac{(t+3)^2}{3(t^2+5)}$. [4]
- (b) Find the value of t for which θ is greatest. [4]
- (c) Find the greatest value of θ , to the nearest tenth of a degree. [2]
-
- 30** VABCD is a pyramid with a horizontal square base A(0, 0, 0), B(6, 0, 0), C(6, 6, 0), D(0, 6, 0) and with apex V(3, 3, h), where $h > 0$ (units in metres). Each triangular face is inclined at 60° to the base.
- (a) Show that a normal to the face VAB is $(0, -h, 3)$ and hence find the exact value of h . [4]
- (b) Find the angle between the edge VA and the base, to the nearest tenth of a degree. [3]
- (c) Find the angle between the adjacent faces VAB and VBC, measured inside the pyramid, to the nearest tenth of a degree. [4]

Shortest Distances in 3D

- 1** Find the perpendicular distance from the origin to the plane $2x - y + 2z = 9$. [2]
- 2** Find the perpendicular distance from the point (1, 2, 3) to the plane $x + 2y + 2z = 2$. [2]
- 3** Find the distance between the parallel planes $x + 2y + 2z = 3$ and $x + 2y + 2z = 12$. [2]
- 4** Find the perpendicular distance from the point (1, 2, 0) to the plane $3x + 4y = 10$. [2]
- 5** Find the distance of the point A(2, -1, 1) from the line $\mathbf{r} = \lambda(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$. [3]
- 6** Find the distance from the point (3, 1, -2) to the plane $6x - 2y + 3z = 4$. [2]
- 7** The distance from the point (p, 1, 2) to the plane $2x - 2y + z = 7$ is 3. Find the possible values of p . [4]
- 8** The parallel lines l_1 and l_2 have equations $\mathbf{r} = \mathbf{i} + 2\mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$ and $\mathbf{r} = 3\mathbf{i} + \mathbf{j} + \mu(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$. Find the distance between them. [4]
- 9** Find the shortest distance from the point P(2, -1, 3) to the line through A(1, 1, 0) and B(3, 2, 2). [5]
- 10** The planes $2x - y + 2z = 3$ and $4x - 2y + 4z = -6$ are parallel. Find the distance between them. [3]
- 11** The line l has equation $\mathbf{r} = 3\mathbf{i} + 3\mathbf{j} + t(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$. Find the position vector of the point on l that is closest to the origin, and the distance from the origin to l . [4]
- 12** The lines l_1 and l_2 have equations $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + \mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j} - 2\mathbf{k})$ and $\frac{x-3}{2} = \frac{y+1}{1} = \frac{z-2}{-2}$. Show that l_1 and l_2 are parallel and find the distance between them. [5]
- 13** The point P(1, 1, 1) is reflected in the plane $x + y + z = 6$ to give the point P'. Find the distance PP'. [3]
- 14** Find the shortest distance between the skew lines $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + 2\mathbf{k})$ and $\mathbf{r} = \mathbf{j} + 5\mathbf{k} + \mu(2\mathbf{i} + \mathbf{j} - \mathbf{k})$. Give your answer to 3 significant figures. [6]

- 15** The line l has equation $\mathbf{r} = \mathbf{i} + \mathbf{k} + t(2\mathbf{i} + \mathbf{j} - \mathbf{k})$ and the point P has coordinates $(4, -1, 0)$. F is the foot of the perpendicular from P to l .
 (a) Find the coordinates of F . [4]
 (b) Hence find the shortest distance from P to l . [1]
-
- 16** The line l has equation $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + \lambda(2\mathbf{i} + \mathbf{j} - 2\mathbf{k})$ and the plane Π has equation $x + 2y + 2z = 10$.
 Show that l is parallel to Π and find the distance between l and Π . [4]
-
- 17** The plane Π passes through $A(1, 0, 0)$, $B(0, 2, 0)$ and $C(0, 0, 3)$.
 (a) Show that Π has equation $6x + 3y + 2z = 6$. [1]
 (b) Find the perpendicular distance from the origin to Π , and hence confirm that the volume of the tetrahedron $OABC$ is 1 given that the area of triangle ABC is $\frac{7}{2}$. [3]
-
- 18** A student is finding the distance from the point $(3, 1, 2)$ to the plane $2x - 2y + z = 1$ and writes: "Distance = $|2(3) - 2(1) + 2 - 1| = 5$."
 (a) Explain what is missing from the student's calculation. [2]
 (b) Find the correct distance. [1]
-
- 19** A sphere has centre $C(1, 2, -1)$ and touches the plane $2x - y + 2z = 7$.
 (a) Find the radius of the sphere. [2]
 (b) Find the coordinates of the point where the sphere touches the plane. [2]
-
- 20** The point X moves on the line $\mathbf{r} = \mathbf{i} - \mathbf{j} + t(2\mathbf{i} + 2\mathbf{j} + \mathbf{k})$ and P is the point $(4, 3, -1)$.
 Show that $PX^2 = 9t^2 - 26t + 26$, and use completing the square to find the shortest distance from P to the line. [6]
-
- 21** The lines l_1 and l_2 have equations $\mathbf{r} = \lambda(\mathbf{i} + \mathbf{j})$ and $\mathbf{r} = 2\mathbf{k} + \mu(\mathbf{i} - \mathbf{j} + \mathbf{k})$.
 (a) Find the shortest distance between l_1 and l_2 . [4]
 (b) Find the position vectors of the points X on l_1 and Y on l_2 that are closest together. [3]
 (c) Find a vector equation of the common perpendicular. [2]
-
- 22** A zip wire runs in a straight line from $A(0, 0, 10)$ to $B(30, 40, 0)$, where distances are in metres and the z -axis is vertical. A tree top is at $P(20, 5, 4)$.
 (a) Find the shortest distance from P to the wire, to 3 significant figures. [5]
 (b) Find how far along the wire from A the closest point to P is, to 3 significant figures. [2]
-
- 23** The lines l_1 and l_2 have equations $\mathbf{r} = \mathbf{i} + \lambda(\mathbf{i} + \mathbf{j})$ and $\mathbf{r} = \mathbf{j} + \mu(\mathbf{j} + \mathbf{k})$. The point X is on l_1 and Y is on l_2 .
 (a) Find $\mathbf{XY}$ in terms of λ and μ . [2]
 (b) Given that XY is the common perpendicular, find λ and μ . [4]
 (c) Hence find the shortest distance between the lines. [2]
-
- 24** The plane Π_0 has equation $x + 2y + 2z = 3$. The point P has coordinates $(1, 1, 1)$.
 (a) Find the equations of the two planes that are parallel to Π_0 and at a distance of 2 from P . [3]
 (b) Find the coordinates of the points where the line through P perpendicular to these planes meets them. [3]
 (c) Find the distance between the two planes you found. [1]
-
- 25** The line l has equation $\mathbf{r} = \mathbf{j} + 2\mathbf{k} + t(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$ and P is the point $(3, 2, 4)$.
 (a) Find the coordinates of the foot F of the perpendicular from P to l . [4]
 (b) Find the shortest distance from P to l . [1]
 (c) Find the coordinates of the image P' of P after reflection in l . [2]

- 26** The tetrahedron OABC has O at the origin, A(2, 0, 0), B(0, 3, 0) and C(0, 0, 6).
 (a) Show that the plane ABC has equation $3x + 2y + z = 6$. [3]
 (b) Find the exact distance from O to the plane ABC. [2]
 (c) Find the exact area of triangle ABC and use it to confirm that the volume of the tetrahedron is 6. [3]
-
- 27** The plane Π has equation $\mathbf{r} \cdot \mathbf{n} = d$ and P is a point with position vector $\mathbf{p}$.
 (a) Prove that the shortest distance from P to Π is $\frac{|\mathbf{p} \cdot \mathbf{n} - d|}{|\mathbf{n}|}$. [3]
 (b) The line l has equation $\mathbf{r} = \mathbf{a} + t\mathbf{d}$. Prove that the shortest distance from P to l is $\frac{|(\mathbf{p} - \mathbf{a}) \times \mathbf{d}|}{|\mathbf{d}|}$. [3]
 (c) For P(1, 1, 3) and the line through A(0, 0, 1) with direction $\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$, find the distance from P to the line, and the distance from P to the plane through A perpendicular to the line. Verify that the squares of these distances sum to AP^2 . [5]
-
- 28** Two skew lines have equations $\mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{d}_1$ and $\mathbf{r} = \mathbf{a}_2 + \mu\mathbf{d}_2$.
 (a) Prove that the shortest distance between them is $\frac{|(\mathbf{a}_2 - \mathbf{a}_1) \cdot (\mathbf{d}_1 \times \mathbf{d}_2)|}{|\mathbf{d}_1 \times \mathbf{d}_2|}$. [4]
 (b) The lines $\mathbf{r} = \lambda(\mathbf{i} + \mathbf{j})$ and $\mathbf{r} = \mathbf{i} + p\mathbf{k} + \mu(\mathbf{j} + \mathbf{k})$ are a distance 2 apart. Find the possible values of p . [7]
-
- 29** The planes $\Pi_1: x + 2y + 2z = 6$ and $\Pi_2: x + 2y + 2z = -12$ are parallel. A sphere S of radius 3 lies between them and touches both.
 (a) Show that the distance between the planes is 6, and explain why S has its centre on the plane $x + 2y + 2z = -3$. [4]
 (b) The centre of S also lies on the line $\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + t(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})$. Find the coordinates of the centre and the equation of S. [4]
 (c) Find the coordinates of the points where S touches the two planes. [3]
-
- 30** A and B are points on the same side of the plane $\Pi: x + y + z = 0$, with A(2, 1, 1) and B(3, 0, 0). A point X on Π is chosen to make the path length AX + XB as small as possible.
 (a) Find the coordinates of the image B' of B after reflection in Π . [3]
 (b) Explain why $AX + XB = AX + XB'$ for every point X of Π , and why the path is shortest when X lies on AB'. [3]
 (c) Find the coordinates of X and the least value of AX + XB. [5]

Vector Geometry Problems

- 1** In triangle ABC, M and N are the midpoints of AB and AC. Show that $\mathbf{MN} = \frac{1}{2}\mathbf{BC}$. [2]
-
- 2** In triangle OAB, with $\mathbf{OA} = \mathbf{a}$ and $\mathbf{OB} = \mathbf{b}$, M is the midpoint of AB. Find $\mathbf{OM}$ and $\mathbf{AM}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. [2]
-
- 3** ABCD is a parallelogram with $\mathbf{AB} = \mathbf{a}$ and $\mathbf{AD} = \mathbf{b}$. Find $\mathbf{AC}$ and $\mathbf{BD}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. [2]
-
- 4** Show that the points A(1, 2, 3), B(3, 5, 7) and C(7, 11, 15) are collinear. [2]
-
- 5** The vectors $\mathbf{a}$ and $\mathbf{b}$ have equal magnitudes. Show that $\mathbf{a} + \mathbf{b}$ is perpendicular to $\mathbf{a} - \mathbf{b}$. [2]
-
- 6** The point P lies on the line segment AB with $AP : PB = 1 : 3$. Show that $\mathbf{p} = \frac{1}{4}(3\mathbf{a} + \mathbf{b})$. [2]

- 7 In triangle OAB, $\mathbf{OA} = \mathbf{a}$ and $\mathbf{OB} = \mathbf{b}$. The point P lies on OA with $OP : PA = 2 : 1$ and Q is the midpoint of OB.
 (a) Find $\mathbf{PQ}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. [2]
 (b) Find $\mathbf{AQ}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. [1]
-
- 8 Prove that the diagonals of a rectangle ABCD are equal in length. (Take $\mathbf{AB} = \mathbf{a}$ and $\mathbf{AD} = \mathbf{b}$.) [3]
-
- 9 In triangle OAB, with $\mathbf{OA} = \mathbf{a}$ and $\mathbf{OB} = \mathbf{b}$, the point X lies on AB with $AX : XB = 2 : 3$. Find $\mathbf{OX}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. [3]
-
- 10 OABC is a tetrahedron with $\mathbf{OA} = \mathbf{a}$, $\mathbf{OB} = \mathbf{b}$ and $\mathbf{OC} = \mathbf{c}$. M is the midpoint of BC and G is the point on AM with $AG : GM = 2 : 1$. Show that $\mathbf{OG} = \frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c})$. [3]
-
- 11 The plane $x + y + z = 6$ meets the line segment joining A(1, 1, 1) and B(4, 3, 5) at the point P. Find the ratio AP : PB. [4]
-
- 12 M is the midpoint of AB and P is any point. Prove that $PA^2 + PB^2 = 2PM^2 + \frac{1}{2}AB^2$. [4]
-
- 13 AB is a diameter of a circle with centre O, and P is any other point on the circle. Prove that angle APB is a right angle. [4]
-
- 14 In triangle OAB, $\mathbf{OA} = \mathbf{a}$ and $\mathbf{OB} = \mathbf{b}$. The midpoint of OA is M and the midpoint of OB is N. The lines AN and BM meet at G.
 (a) Write the position vectors of points on AN and BM using parameters λ and μ . [2]
 (b) Show that $\mathbf{OG} = \frac{1}{3}(\mathbf{a} + \mathbf{b})$. [4]
-
- 15 In triangle OAB, $\mathbf{OA} = \mathbf{a}$ and $\mathbf{OB} = \mathbf{b}$. The point P lies on OA with $OP : PA = 1 : 2$ and Q lies on OB with $OQ : QB = 2 : 1$. The lines AQ and BP meet at X. Find $\mathbf{OX}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. [7]
-
- 16 OABCDEFG is a cuboid with O at the origin, A(6, 0, 0), C(0, 4, 0) and D(0, 0, 3). Show that the diagonals OF and AG bisect each other. [4]
-
- 17 A regular tetrahedron has vertices $P_1(1, 1, 1)$, $P_2(1, -1, -1)$, $P_3(-1, 1, -1)$ and $P_4(-1, -1, 1)$. Show that each pair of opposite edges is perpendicular. [4]
-
- 18 The points O(0, 0, 0), A(4, 0, 0), B(0, 6, 0) and C(0, 0, 8) lie on a sphere. Find the coordinates of its centre and its radius. [5]
-
- 19 OABC is a tetrahedron with $\mathbf{OA} = \mathbf{a}$, $\mathbf{OB} = \mathbf{b}$, $\mathbf{OC} = \mathbf{c}$ and OABC has opposite edge pairs (OA, BC), (OB, AC) and (OC, AB). Let M_1, N_1 be the midpoints of OA and BC, M_2, N_2 the midpoints of OB and AC, and M_3, N_3 the midpoints of OC and AB.
 (a) Find the midpoint of M_1N_1 . [3]
 (b) Hence show that the three lines M_1N_1 , M_2N_2 and M_3N_3 are concurrent. [2]
-
- 20 A student states that for triangle ABC with M the midpoint of BC, $\mathbf{AM} = \frac{1}{2}(\mathbf{b} + \mathbf{c})$, where $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are position vectors of A, B, C.
 (a) Explain the error in the student's statement. [2]
 (b) Write down the correct expression for $\mathbf{AM}$, and use it to show that $\mathbf{AM} = \frac{1}{2}(\mathbf{AB} + \mathbf{AC})$. [2]
-

- 21** In triangle ABC the points D and E lie on BC and CA with $BD : DC = 1 : 2$ and $CE : EA = 1 : 3$. The lines AD and BE meet at X. Take $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ as the position vectors of A, B, C.
- Find the position vectors of D and E. [2]
 - Use the fact that X lies on both AD and BE to find the position vector of X. [5]
 - Hence find the ratios $AX : XD$ and $BX : XE$. [2]
-
- 22** OABCDEFG is a cuboid with O at the origin, $A(6, 0, 0)$, $C(0, 4, 0)$ and $D(0, 0, 3)$. M is the midpoint of the edge EF.
- Write down the coordinates of M. [1]
 - Find the Cartesian equation of the plane ACG. [4]
 - The line OM meets the plane ACG at P. Find the coordinates of P and the ratio $OP : PM$. [4]
-
- 23** OABC is a tetrahedron with $\mathbf{OA} = \mathbf{a}$, $\mathbf{OB} = \mathbf{b}$ and $\mathbf{OC} = \mathbf{c}$. The points P, Q and R lie on OA, OB and OC with $OP = \frac{1}{2}\mathbf{OA}$, $OQ = \frac{1}{3}\mathbf{OB}$ and $OR = \frac{1}{4}\mathbf{OC}$. G is the centroid of triangle ABC.
- Show that a point with position vector $\alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$ lies in the plane PQR if $2\alpha + 3\beta + 4\gamma = 1$. [3]
 - The line OG meets the plane PQR at X. Find the ratio $OX : XG$. [4]
-
- 24** ABCD is a rectangle and P is any point in space. Prove that $PA^2 + PC^2 = PB^2 + PD^2$.
- Take the centre O of the rectangle as the origin. Explain why $\mathbf{c} = -\mathbf{a}$, $\mathbf{d} = -\mathbf{b}$ and $|\mathbf{a}| = |\mathbf{b}|$. [2]
 - Show that $PA^2 + PC^2 = 2|\mathbf{p}|^2 + 2|\mathbf{a}|^2$. [2]
 - Hence complete the proof. [2]
-
- 25** Triangle OAB has O as the origin and $\mathbf{OA} = \mathbf{a}$, $\mathbf{OB} = \mathbf{b}$. The point H has position vector $\mathbf{h}$ and satisfies $\mathbf{h} \cdot (\mathbf{b} - \mathbf{a}) = 0$ and $(\mathbf{h} - \mathbf{a}) \cdot \mathbf{b} = 0$.
- Explain what these two conditions tell you about H. [2]
 - Show that $\mathbf{h} \cdot \mathbf{a} = \mathbf{h} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{b}$. [2]
 - Hence prove that BH is perpendicular to OA, so that the three altitudes of a triangle are concurrent. [2]
-
- 26** The tetrahedron OABC has O at the origin and $\mathbf{OA} = \mathbf{a}$, $\mathbf{OB} = \mathbf{b}$, $\mathbf{OC} = \mathbf{c}$. Its volume is $\frac{1}{6}|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$. The points P, Q, R lie on OA, OB, OC with $OP = \frac{1}{2}\mathbf{OA}$, $OQ = \frac{1}{3}\mathbf{OB}$ and $OR = \frac{1}{4}\mathbf{OC}$.
- Show that the volume of tetrahedron OPQR is $\frac{1}{24}$ of the volume of OABC. [4]
 - Given that $A(2, 0, 0)$, $B(0, 3, 0)$ and $C(0, 0, 6)$, find the volumes of OABC and OPQR. [2]
-
- 27** Triangle ABC is inscribed in a circle with centre O, the origin, so $|\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}|$. The point H has position vector $\mathbf{h} = \mathbf{a} + \mathbf{b} + \mathbf{c}$ and G is the centroid of ABC.
- Show that AH is perpendicular to BC. [3]
 - Deduce that H is the orthocentre of the triangle. [2]
 - Prove that O, G and H are collinear and find the ratio $OG : GH$. [3]
 - The circle has radius 5 and $A(3, 4, 0)$, $B(-4, 3, 0)$, $C(5, 0, 0)$ lie on it. Find H and verify that AH is perpendicular to BC. [2]

- 28** ABCD is a tetrahedron with position vectors **a**, **b**, **c**, **d**. The centroid of triangle BCD is G_A , with $\mathbf{g}_A = \frac{1}{3}(\mathbf{b} + \mathbf{c} + \mathbf{d})$. Similar points G_B , G_C , G_D are the centroids of the faces opposite B, C and D.
- (a) Show that the point G with position vector $\frac{1}{4}(\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d})$ lies on AG_A and that $AG : GG_A = 3 : 1$. [4]
- (b) Explain why the four lines AG_A , BG_B , CG_C , DG_D are concurrent. [1]
- (c) For $A(0, 0, 0)$, $B(4, 0, 0)$, $C(0, 4, 0)$, $D(0, 0, 4)$, find G_A and G, and verify $AG : GG_A = 3 : 1$. [3]
- (d) Show that G is also the midpoint of the line joining the midpoints of AB and CD. [2]
-
- 29** ABCD is a quadrilateral (not necessarily planar) with position vectors **a**, **b**, **c**, **d**. M and N are the midpoints of the diagonals AC and BD.
- (a) Prove that $AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4MN^2$. [6]
- (b) Deduce that $AC^2 + BD^2 \leq AB^2 + BC^2 + CD^2 + DA^2$, with equality only for a parallelogram. [2]
- (c) Verify the identity for $A(0, 0, 0)$, $B(4, 0, 0)$, $C(5, 3, 1)$ and $D(1, 2, 2)$. [2]
-
- 30** The line l has equation $\mathbf{r} = \mathbf{i} + \mathbf{j} + t(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$ and the points A and B have coordinates $A(3, 2, 1)$ and $B(6, 5, 4)$. The point P lies on l and angle $APB = 90^\circ$.
- (a) Show that the position of P satisfies $t^2 - 3t + 2 = 0$. [4]
- (b) Find the coordinates of the two possible positions of P. [2]
- (c) Show that both positions of P lie on the sphere with diameter AB, and give its centre and radius. [3]