

Permutations and Combinations

1 (2 marks)

(a) $7 \times 6 \times 5 = 210$

Answer: 210

B1

1 mark

(b) $\frac{210}{3!} = 35$

Answer: 35

B1

1 mark

2 (1 mark)

$5! = 120$

Answer: 120

B1

1 mark

3 (2 marks)

${}^{10}C_3$

$= \frac{10 \times 9 \times 8}{3 \times 2 \times 1} = 120$

Answer: 120

M1

A1

2 marks

4 (1 mark)

8 different letters, so $8! = 40320$

Answer: 40320

B1

1 mark

5 (2 marks)

5 letters with L repeated twice and E repeated twice

$\frac{5!}{2! \times 2!} = 30$

Answer: 30

M1

A1

2 marks

6 (2 marks)

Order does not matter: ${}^6C_2$

$= 15$

Answer: 15

M1

A1

2 marks

7 (2 marks)

6 letters with A three times and N twice

$\frac{6!}{3! \times 2!} = 60$

Answer: 60

M1

A1

2 marks

8 (3 marks)

Men: ${}^4C_2 = 6$ ways

Women: ${}^3C_2 = 3$ ways

Total $6 \times 3 = 18$

Answer: 18

B1

B1

B1

3 marks

9 (2 marks)

Fix one person; arrange the other five

$5! = 120$

Answer: 120

M1

A1

2 marks

10 (3 marks)

The last digit is 1, 3, 5, 7 or 9: 5 choices

The other three digits are chosen in order from the 8 remaining digits: $8 \times 7 \times 6 = 336$

$5 \times 336 = 1680$

Answer: 1680

B1

M1

A1

3 marks

11 (3 marks)

The only repeated letter is E (three times), so the first and last letters must both be E

B1

The four letters between them are E, L, V, N, all different

M1

$$4! = 24$$

A1

Answer: 24

3 marks

12 (3 marks)

10 letters: S three times, T three times, I twice

B1

$$\frac{10!}{3! \times 3! \times 2!}$$

M1

$$= 50400$$

A1

Answer: 50400

3 marks

13 (3 marks)

$$\frac{n(n-1)}{2} = 45$$

M1

$$n^2 - n - 90 = 0, (n - 10)(n + 9) = 0$$

M1

$$n = 10 \text{ (n is positive)}$$

A1

Answer: n = 10

3 marks

14 (4 marks)

The letters B and I each occur twice, so there are $\frac{11!}{2! \times 2!} = 9979200$ arrangements

M1 A1

Treat the two Bs as one block: 10 objects with I twice, giving $\frac{10!}{2!} = 1814400$ arrangements

M1

$$\text{Probability} = \frac{1814400}{9979200} = \frac{2}{11}$$

A1

Answer: $\frac{2}{11}$

4 marks

15 (4 marks)

(a) M, T and E are each repeated twice

M1

$$\frac{9!}{2! \times 2! \times 2!} = 45360$$

A1

Answer: 45360

2 marks

(b) Treat MM as one object: 8 objects with T and E each twice

M1

$$\frac{8!}{2! \times 2!} = 10080$$

A1

Answer: 10080

2 marks

16 (4 marks)

$$\text{Total } {}^{11}C_5 = 462$$

B1

$$\text{Remove 0 women: } {}^6C_5 = 6$$

M1

$$\text{Remove 1 woman: } 5 \times {}^6C_4 = 75$$

M1

$$462 - 6 - 75 = 381$$

A1

Answer: 381

4 marks

17 (5 marks)

Last digit 2, 4 or 6

B1

First digit is from 3, 4, 5, 6, 7 and differs from the last digit; the middle digits are chosen from those remaining

M1

$$\text{Last digit 2: } 5 \times 5 \times 4 = 100; \text{ last digit 4: } 4 \times 5 \times 4 = 80; \text{ last digit 6: } 4 \times 5 \times 4 = 80$$

M1 A1

$$\text{Total} = 260$$

A1

Answer: 260

5 marks

18 (4 marks)All arrangements: $8! = 40320$

B1

Asha and Ben together: treat as a block, $7! \times 2 = 10080$

M1 A1

 $40320 - 10080 = 30240$

A1

Answer: 30240

4 marks

19 (4 marks)

Treat Cara and Dev as a block: 6 objects round a table

M1

 $(6 - 1)! = 120$ arrangements of the objects

A1

Block can be ordered 2 ways

M1

 $120 \times 2 = 240$

A1

Answer: 240

4 marks

20 (5 marks)(a) Choose the group of 6: $^{10}C_6$

M1

 $= 210$

A1

Answer: 210

2 marks

(b) Boys: $^4C_3 = 4$

B1

Girls: 6 ways to choose 1 girl

B1

 $4 \times 6 = 24$

B1

Answer: 24

3 marks

21 (4 marks)

$$r \times {}^nC_r = r \times \frac{n!}{r!(n-r)!}$$

M1

$$= \frac{n!}{(r-1)!(n-r)!}$$

M1 A1

$$= n \times \frac{(n-1)!}{(r-1)!((n-1)-(r-1))!} = n \times {}^{n-1}C_{r-1}$$

A1

Answer: $r \times {}^nC_r = n \times {}^{n-1}C_{r-1}$

4 marks

22 (6 marks)There are $4! = 24$ numbers

B1

Each digit appears in each place value $24 \div 4 = 6$ times

M1 A1

Sum of digits in one place = $6 \times (1 + 2 + 3 + 4) = 60$

M1

Total = $60 \times (1000 + 100 + 10 + 1)$

M1

 $= 66660$

A1

Answer: 66660

6 marks

23 (6 marks)

Arrange the 7 non-I letters (M, S, S, S, S, P, P)

M1

$$\frac{7!}{4! \times 2!} = 105$$

A1

There are 8 gaps (including the ends); choose 4 for the Is

M1 A1

$${}^8C_4 = 70$$

A1

$$105 \times 70 = 7350$$

A1

Answer: 7350

6 marks

24 (6 marks)Committees with at least one man: ${}^{13}C_5 - {}^6C_5 = 1287 - 6 = 1281$

M1 A1

Committees containing both Mr Ali and Ms Bell: choose the other 3 from 11

M1

$${}^{11}C_3 = 165 \text{ (all of these contain a man)}$$

A1

$$1281 - 165$$

M1

$$= 1116$$

A1

Answer: 1116

6 marks

25 (5 marks)

Lay the 10 sweets in a row: there are 9 gaps

B1

Three dividers are needed to make 4 non-empty groups

M1

Choose 3 gaps from 9

M1 A1

$${}^9C_3 = 84$$

A1

Answer: 84

5 marks

26 (4 marks)

The 4! counts the two Os as different letters

E1

Swapping the two Os gives the same arrangement, so each arrangement has been counted twice

E1

Correct number $\frac{4!}{2!}$

M1

$$= 12$$

A1

Answer: 12 (the student counted each arrangement twice)

4 marks

27 (8 marks)

(a) $(1 + x)^n = \sum_{r=0}^n {}^nC_r x^r$

M1

Put $x = 1$: $2^n = {}^nC_0 + {}^nC_1 + \dots + {}^nC_n$

A1 E1

Answer: Sum of ${}^nC_r = 2^n$

3 marks

(b) Put $x = -1$: $0 = {}^nC_0 - {}^nC_1 + {}^nC_2 - \dots$

M1

So (sum over even r) = (sum over odd r)

E1

Each sum is half of 2^n , that is 2^{n-1}

M1 A1

For $n = 10$: $2^9 = 512$

A1

Answer: 512

5 marks

28 (9 marks)

The seven letters are B, A, L, L, O, O, N; one letter is left out

M1

Leave out B, A or N: the six letters are L, L, O, O and two different letters

M1

$$\frac{6!}{2! \times 2!} = 180 \text{ each, so } 3 \times 180 = 540$$

M1 A1

Leave out one L or one O: letters contain one repeated pair only

M1

$$\frac{6!}{2!} = 360 \text{ each, so } 2 \times 360 = 720$$

M1 A1

Total $540 + 720$

M1

$$= 1260$$

A1

Answer: 1260

9 marks

29 (8 marks)

An ascending number is determined by the multiset of its six digits from 1 to 9

M1 E1

Let a_k be the number of times digit k is used, so $a_1 + a_2 + \dots + a_9 = 6$, each $a_k \geq 0$

M1 A1

Arrange 6 identical objects and 8 dividers in a row

M1

Number of arrangements ${}^{14}C_6$

M1 A1

$$= 3003$$

A1

Answer: 3003

8 marks

30 (10 marks)Let A_i be the event that letter i is in the correct envelope

M1

 $|A_i| = 4! = 24$, five such events

A1

 $|A_i \cap A_j| = 3! = 6$, ${}^5C_2 = 10$ pairs

A1

 $|A_i \cap A_j \cap A_k| = 2! = 2$, 10 triples; four letters correct: 5×1 ; all five: 1

A1

At least one correct = $5(24) - 10(6) + 10(2) - 5(1) + 1$

M1 A1

 $= 120 - 60 + 20 - 5 + 1 = 76$

A1

None correct = $5! - 76 = 44$

M1 A1

Probability = $\frac{44}{120} = \frac{11}{30}$

A1

Answer: 44 ways; probability $\frac{11}{30}$

10 marks

Conditional Probability**1 (2 marks)**

$$P(A | B) = \frac{P(A \cap B)}{P(B)}$$

M1

$$= \frac{0.3}{0.5} = 0.6$$

A1

Answer: 0.6

2 marks

2 (2 marks)

Scores greater than 3: 4, 5, 6; of these 4 and 6 are even

M1

Probability $\frac{2}{3}$

A1

Answer: $\frac{2}{3}$

2 marks

3 (2 marks)

$$P(B | A) = \frac{0.2}{0.5}$$

M1

$$= 0.4$$

A1

Answer: 0.4

2 marks

4 (2 marks)

$$(a) 0.4 \times 0.3 = 0.12$$

B1

Answer: 0.12

1 mark

(b) Independent, so $P(A | B) = P(A) = 0.4$

B1

Answer: 0.4

1 mark

5 (2 marks)

$$(a) \frac{23}{40}$$

B1

Answer: $\frac{23}{40}$

1 mark

$$(b) \frac{11}{20}$$

B1

Answer: $\frac{11}{20}$

1 mark

6 (2 marks)

$$P(A \cap B) = 0.3 \times 0.5 = 0.15$$

M1

$$P(A \cup B) = 0.3 + 0.5 - 0.15 = 0.65$$

A1

Answer: 0.65

2 marks

7 (4 marks)

(a) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

M1

$P(A \cap B) = 0.5 + 0.4 - 0.7 = 0.2$

A1

Answer: 0.2

2 marks

(b) $P(A | B) = \frac{0.2}{0.4} = 0.5$

B1

$P(B | A) = \frac{0.2}{0.5} = 0.4$

B1

Answer: 0.5 and 0.4

2 marks

8 (3 marks)

(a) $\frac{4}{11}$

B1

Answer: $\frac{4}{11}$

1 mark

(b) $\frac{5}{12} \times \frac{4}{11}$

M1

$= \frac{5}{33}$

A1

Answer: $\frac{5}{33}$

2 marks

9 (5 marks)

(a) $80 - (36 + 28 - 15) = 31$, so the probability is $\frac{31}{80}$

B1

Answer: $\frac{31}{80}$

1 mark

(b) Tennis only $36 - 15 = 21$, squash only $28 - 15 = 13$

M1

$\frac{21 + 13}{80} = \frac{17}{40}$

A1

Answer: $\frac{17}{40}$

2 marks

(c) At least one sport: 49 members

M1

$\frac{15}{49}$

A1

Answer: $\frac{15}{49}$

2 marks

10 (4 marks)

(a) $0.4 \times 0.5 = 0.2$

B1

Answer: 0.2

1 mark

(b) $\frac{0.2}{0.35}$

M1

$= \frac{4}{7}$

A1

Answer: $\frac{4}{7}$

2 marks

(c) $0.35 + 0.5 - 0.2 = 0.65$

B1

Answer: 0.65

1 mark

11 (4 marks)

(a) $\frac{60}{100} = \frac{3}{5}$

B1

Answer: $\frac{3}{5}$

1 mark

(b) $\frac{42}{50} = \frac{21}{25}$

B1

Answer: $\frac{21}{25}$

1 mark

(c) $\frac{32}{40}$

M1

$$= \frac{4}{5}$$

A1

Answer: $\frac{4}{5}$

2 marks

12 (3 marks)

$$P(A' | B) = \frac{P(A' \cap B)}{P(B)}$$

M1

$$P(A' \cap B) = P(B) - P(A \cap B)$$

M1

$$\text{So } P(A' | B) = 1 - \frac{P(A \cap B)}{P(B)} = 1 - P(A | B)$$

A1

Answer: $P(A' | B) = 1 - P(A | B)$

3 marks

13 (3 marks)

At least one 6: 11 of the 36 outcomes

B1

Total at least 10 among these: (4,6), (5,6), (6,6), (6,4), (6,5) which is 5

M1

Probability $\frac{5}{11}$

A1

Answer: $\frac{5}{11}$

3 marks

14 (3 marks)

(a) $P(A \cap B) = 0$, so $P(A | B) = 0$

B1

Answer: 0

1 mark

(b) If independent then $P(A | B) = P(A) = 0.4$

E1

But $P(A | B) = 0$, so they are not independent

E1

Answer: Not independent

2 marks

15 (4 marks)

Product odd needs both odd: $3 \times 3 = 9$ outcomes, so product even in $36 - 9 = 27$

M1 A1

Sum odd means one odd and one even; this always gives an even product: $2 \times 3 \times 3 = 18$

M1

Probability $\frac{18}{27} = \frac{2}{3}$

A1

Answer: $\frac{2}{3}$

4 marks

16 (5 marks)

Let $P(B) = p$, then $P(A) = 2p$ and $P(A \cap B) = 2p^2$

M1

$$P(A \cup B) = 3p - 2p^2 = 0.52$$

M1

$$2p^2 - 3p + 0.52 = 0, \text{ so } p = \frac{3 \pm \sqrt{9 - 4.16}}{4} = \frac{3 \pm 2.2}{4}$$

M1 A1

$p = 0.2$ (the other root, 1.3, is greater than 1)

A1

Answer: $P(B) = 0.2$, $P(A) = 0.4$

5 marks

17 (5 marks)

(a) Exhaustive means $P(A \cup B) = 1$

M1

$$P(A \cap B) = 0.72 + 0.56 - 1 = 0.28$$

A1

Answer: 0.28

2 marks

(b) $\frac{0.28}{0.56} = \frac{1}{2}$

B1

Answer: $\frac{1}{2}$

1 mark

(c) $P(A)P(B) = 0.72 \times 0.56 = 0.4032$

M1

This is not equal to $P(A \cap B) = 0.28$, so A and B are not independent

E1

Answer: Not independent

2 marks

18 (4 marks)

Number of ways to choose 5 red cards: ${}^{26}C_5 = 65780$

B1

Five hearts: ${}^{13}C_5 = 1287$

B1

$$P = \frac{1287}{65780}$$

M1

$$= \frac{9}{460}$$

A1

Answer: $\frac{9}{460}$

4 marks

19 (5 marks)

$P(\text{bus and not late}) = 0.7 \times 0.9 = 0.63$

M1 A1

$P(\text{no bus and not late}) = 0.3 \times 0.55 = 0.165$

M1

$$P(\text{bus} \mid \text{not late}) = \frac{0.63}{0.63 + 0.165}$$

M1

$$= \frac{0.63}{0.795} = \frac{42}{53}$$

A1

Answer: $\frac{42}{53}$

5 marks

20 (5 marks)

(a) $P(A \cap B) = 1 - 0.3 - 0.2 - 0.1 = 0.4$

M1 A1

$P(B) = 0.4 + 0.2 = 0.6$, so $P(A \mid B) = \frac{0.4}{0.6} = \frac{2}{3}$

A1

Answer: $\frac{2}{3}$

3 marks

(b) $P(A) = 0.7$, $P(A)P(B) = 0.7 \times 0.6 = 0.42$

M1

$0.42 \neq 0.4$, so A and B are not independent

E1

Answer: Not independent

2 marks

21 (4 marks)

$P(A' \cap B') = 1 - P(A \cup B)$

M1

$= 1 - P(A) - P(B) + P(A)P(B)$

M1 A1

$= (1 - P(A))(1 - P(B)) = P(A')P(B')$, so A' and B' are independent

A1

Answer: $P(A' \cap B') = P(A')P(B')$, so A' and B' are independent

4 marks

22 (6 marks)

(a) $P(A) = 0.28 + x$ and $P(B) = 0.18 + x$

B1

Independence: $P(A \cap B) = P(A)P(B)$, so $x = (0.28 + x)(0.18 + x)$

M1

$x = 0.0504 + 0.46x + x^2$, giving $x^2 - 0.54x + 0.0504 = 0$

A1

Answer: $x^2 - 0.54x + 0.0504 = 0$

3 marks

(b) $(x - 0.12)(x - 0.42) = 0$, so $x = 0.12$ or $x = 0.42$

M1 A1

$P(\text{neither}) = 1 - (0.28 + x + 0.18)$: 0.42 when $x = 0.12$ and 0.12 when $x = 0.42$

A1

Answer: $x = 0.12$ ($P(\text{neither}) = 0.42$) or $x = 0.42$ ($P(\text{neither}) = 0.12$)

3 marks

23 (5 marks)

$$P(A \mid B) > P(A) \iff \frac{P(A \cap B)}{P(B)} > P(A)$$

M1

$$\iff P(A \cap B) > P(A)P(B)$$

A1

$$\text{Dividing by } P(A) > 0: \frac{P(A \cap B)}{P(A)} > P(B)$$

M1 A1

which is $P(B \mid A) > P(B)$. All steps are reversible, so the statement holds in both directions

E1

Answer: Both statements are equivalent to $P(A \cap B) > P(A)P(B)$

5 marks

24 (4 marks)

Mutually exclusive means $P(A \cap B) = 0$, which is not true here as $P(A \cap B) = 0.2$ **E1**

$P(A \cap B) = P(A)P(B)$ is the condition for independence **E1**

$$P(A \cap B) = 0.5 + 0.4 - 0.7 = 0.2$$
 M1

So the events are independent and not mutually exclusive **A1**

Answer: A and B are independent, not mutually exclusive **4 marks**

25 (7 marks)

(a) Use $P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$ **M1**

$$= 1.2 - 0.45 + 0.05 = 0.8$$
 A1

Answer: 0.8 **2 marks**

(b) $P(A | B \cap C) = \frac{0.05}{0.1}$ **M1**

$$= 0.5$$
 A1

Answer: 0.5 **2 marks**

(c) $P(A \cap B \cap C') = 0.2 - 0.05 = 0.15$ **M1 A1**

$$P(C') = 0.7, \text{ so } P = \frac{0.15}{0.7} = \frac{3}{14}$$
 A1

Answer: $\frac{3}{14}$ **3 marks**

26 (6 marks)

(a) Moved ball red ($\frac{3}{8}$): Y has 5 red of 7 **M1**

Moved ball green ($\frac{5}{8}$): Y has 4 red of 7 **M1**

$$\frac{3}{8} \times \frac{5}{7} + \frac{5}{8} \times \frac{4}{7} = \frac{35}{56} = \frac{5}{8}$$
 A1

Answer: $\frac{5}{8}$ **3 marks**

(b) $P(\text{moved green and red from Y}) = \frac{5}{8} \times \frac{4}{7} = \frac{5}{14}$ **M1 A1**

$$\text{Divide by } \frac{5}{8}: \frac{5}{14} \times \frac{8}{5} = \frac{4}{7}$$
 A1

Answer: $\frac{4}{7}$ **3 marks**

27 (8 marks)

Let the contestant pick door 1. Let W be the event that the prize is behind door 1 **B1**

$$P(W) = \frac{1}{3}; \text{ if W, switching loses}$$
 M1

If the prize is behind door 2 (prob $\frac{1}{3}$), the host must open door 3, so switching wins **M1 A1**

If the prize is behind door 3 (prob $\frac{1}{3}$), the host must open door 2, so switching wins **A1**

$$P(\text{win by switching}) = \frac{1}{3} + \frac{1}{3}$$
 M1

$$= \frac{2}{3}$$
 A1

$$\frac{2}{3} > \frac{1}{3}, \text{ so switching is better}$$
 E1

Answer: $\frac{2}{3}$, so switching is better than staying (probability $\frac{1}{3}$) **8 marks**

28 (12 marks)

(a) Outcomes (older, younger): BB, BG, GB, GG **B1**

At least one boy: BB, BG, GB **M1**

$$P(BB) = \frac{1}{3}$$
 A1

Answer: $\frac{1}{3}$ **3 marks**

(b) Remaining outcomes BB, BG**M1**

$$P = \frac{1}{2}$$

A1**Answer:** $\frac{1}{2}$ **2 marks****(c)** In (a) the information "at least one boy" leaves three equally likely cases**E1**

In (b) it fixes which child is a boy, leaving two cases

E1**Answer: Different conditioning events leave different sample spaces****2 marks****(d)** $P(\text{a given child is a boy born on a Tuesday}) = \frac{1}{14}$, so $P(\text{at least one}) = 1 - \left(\frac{13}{14}\right)^2 = \frac{27}{196}$ **M1 A1**

$$P(\text{both boys}) = \frac{1}{4}; P(\text{both boys, neither born on Tuesday}) = \left(\frac{6}{14}\right)^2 = \frac{9}{49}$$

M1

$$P(\text{both boys and at least one Tuesday boy}) = \frac{1}{4} - \frac{9}{49} = \frac{13}{196}$$

A1

$$\text{Probability} = \frac{13}{196} \div \frac{27}{196} = \frac{13}{27}$$

A1**Answer:** $\frac{13}{27}$ **5 marks****29 (10 marks)****(a)** First red ($\frac{2}{5}$): urn has 3 red of 6**M1**First blue ($\frac{3}{5}$): urn has 2 red of 6**M1**

$$\frac{2}{5} \times \frac{3}{6} + \frac{3}{5} \times \frac{2}{6} = \frac{1}{5} + \frac{1}{5} = \frac{2}{5}$$

A1**Answer:** $\frac{2}{5}$ **3 marks****(b)** $P(\text{both red}) = \frac{2}{5} \times \frac{3}{6} = \frac{1}{5}$ **M1 A1**

$$\frac{1}{5} \div \frac{2}{5} = \frac{1}{2}$$

A1**Answer:** $\frac{1}{2}$ **3 marks****(c)** After two red draws the urn holds $2 + 2 = 4$ red and 3 blue balls**M1 A1**

Total 7 balls

A1

$$P(\text{red}) = \frac{4}{7}$$

A1**Answer:** $\frac{4}{7}$ **4 marks****30 (8 marks)****(a)** Anna wins on roll 1, 3, 5, ... of the game**M1**

$$P(\text{Anna wins}) = \frac{1}{6} + \left(\frac{5}{6}\right)^2 \frac{1}{6} + \left(\frac{5}{6}\right)^4 \frac{1}{6} + \dots$$

M1 A1Geometric series, first term $\frac{1}{6}$, common ratio $\frac{25}{36}$: $\frac{1}{6} \div \left(1 - \frac{25}{36}\right) = \frac{6}{11}$ **A1****Answer:** $\frac{6}{11}$ **4 marks****(b)** $P(\text{Anna wins on the third roll}) = \left(\frac{5}{6}\right)^2 \times \frac{1}{6} = \frac{25}{216}$ **M1 A1**Divide by $\frac{6}{11}$ **M1**

$$= \frac{275}{1296}$$

A1**Answer:** $\frac{275}{1296}$ **4 marks**

Bayes' Theorem

1 (2 marks)

$$P(B) = P(B | A)P(A) + P(B | A')P(A') = 0.8 \times 0.3 + 0.2 \times 0.7$$

$$= 0.38$$

M1

A1

Answer: 0.38

2 marks

2 (2 marks)

$$0.6 \times 0.02 + 0.4 \times 0.05$$

$$= 0.032$$

M1

A1

Answer: 0.032

2 marks

3 (2 marks)

$$P(A | B) = \frac{P(B | A)P(A)}{P(B)} = \frac{0.5 \times 0.4}{0.3}$$

$$= \frac{2}{3}$$

M1

A1

Answer: $\frac{2}{3}$

2 marks

4 (2 marks)

$$0.2 \times 0.9 + 0.8 \times 0.05$$

$$= 0.22$$

M1

A1

Answer: 0.22

2 marks

5 (2 marks)

$$P(\text{head}) = \frac{3}{4} \times \frac{1}{2} + \frac{1}{4} \times 1$$

$$= \frac{5}{8}$$

M1

A1

Answer: $\frac{5}{8}$

2 marks

6 (2 marks)

$$\frac{3}{5} \times \frac{1}{6} + \frac{2}{5} \times \frac{1}{2}$$

$$= \frac{1}{10} + \frac{1}{5} = \frac{3}{10}$$

M1

A1

Answer: $\frac{3}{10}$

2 marks

7 (3 marks)

$$P(\text{faulty}) = 0.6 \times 0.02 + 0.4 \times 0.05 = 0.032$$

B1

$$P(A | \text{faulty}) = \frac{0.6 \times 0.02}{0.032}$$

M1

$$= \frac{3}{8}$$

A1

Answer: $\frac{3}{8}$

3 marks

8 (4 marks)

$$P(\text{positive and disease}) = 0.01 \times 0.95 = 0.0095$$

M1

$$P(\text{positive and no disease}) = 0.99 \times 0.04 = 0.0396$$

M1

$$P(\text{disease} | \text{positive}) = \frac{0.0095}{0.0095 + 0.0396}$$

M1

$$= 0.1935$$

A1

Answer: 0.193

4 marks

9 (5 marks)

$$(a) 0.5 \times 0.01 + 0.3 \times 0.02 + 0.2 \times 0.04 = 0.019$$

M1

A1

Answer: 0.019

2 marks

$$(b) P(Z | \text{defective}) = \frac{0.2 \times 0.04}{0.019}$$

M1 A1

$$= \frac{8}{19}$$

A1

Answer: $\frac{8}{19}$

3 marks

10 (3 marks)

$$P(\text{red}) = \frac{1}{2} \times \frac{3}{5} + \frac{1}{2} \times \frac{2}{7} = \frac{31}{70}$$

M1 A1

$$P(\text{Bag 1} | \text{red}) = \frac{3}{10} \div \frac{31}{70} = \frac{21}{31}$$

A1

Answer: $\frac{21}{31}$

3 marks

11 (3 marks)

$$P(A | B) = \frac{P(A \cap B)}{P(B)} \text{ and } P(A \cap B) = P(B | A)P(A)$$

M1

$$P(B) = P(B \cap A) + P(B \cap A') = P(B | A)P(A) + P(B | A')P(A')$$

M1

Substituting gives the result

A1

$$\text{Answer: } P(A | B) = \frac{P(B | A)P(A)}{P(B | A)P(A) + P(B | A')P(A')}$$

3 marks

12 (4 marks)

$$P(\text{train and late}) = 0.6 \times 0.1 = 0.06$$

M1

$$P(\text{bus and late}) = 0.4 \times 0.3 = 0.12$$

M1

$$P(\text{train} | \text{late}) = \frac{0.06}{0.18}$$

M1

$$= \frac{1}{3}$$

A1

Answer: $\frac{1}{3}$

4 marks

13 (3 marks)

P(A | B) and P(B | A) have different denominators, P(B) and P(A)

E1

$$P(A \cap B) = P(A | B)P(B) = P(B | A)P(A)$$

M1

$$\text{So } P(B | A) = \frac{P(A | B)P(B)}{P(A)}, \text{ which equals 0.9 only if } P(A) = P(B)$$

A1

Answer: They are equal only when $P(A) = P(B)$

3 marks

14 (5 marks)

$$P(\text{two positives and disease}) = 0.01 \times 0.95^2 = 0.009025$$

M1 A1

$$P(\text{two positives and no disease}) = 0.99 \times 0.04^2 = 0.001584$$

M1

$$P(\text{disease} | \text{two positives}) = \frac{0.009025}{0.009025 + 0.001584}$$

M1

$$= 0.8507$$

A1

Answer: 0.851

5 marks

15 (4 marks)

$$P(\text{HHH and two-headed}) = \frac{1}{2} \times 1 = \frac{1}{2} \quad \text{M1}$$

$$P(\text{HHH and fair}) = \frac{1}{2} \times \left(\frac{1}{2}\right)^3 = \frac{1}{16} \quad \text{M1}$$

$$P(\text{two-headed} \mid \text{HHH}) = \frac{\frac{1}{2}}{\frac{1}{2} + \frac{1}{16}} \quad \text{M1}$$

$$= \frac{8}{9} \quad \text{A1}$$

Answer: $\frac{8}{9}$ 4 marks

16 (5 marks)

$$P(\text{removed heart and drawn heart}) = \frac{13}{52} \times \frac{12}{51} \quad \text{M1 A1}$$

$$P(\text{removed not heart and drawn heart}) = \frac{39}{52} \times \frac{13}{51} \quad \text{M1}$$

$$P(\text{drawn heart}) = \frac{156 + 507}{2652} = \frac{1}{4} \quad \text{A1}$$

$$P(\text{removed heart} \mid \text{drawn heart}) = \frac{156}{2652} \div \frac{1}{4} = \frac{4}{17} \quad \text{A1}$$

Answer: $\frac{4}{17}$ 5 marks

17 (5 marks)

$$P(\text{condition} \mid \text{positive}) = \frac{0.9p}{0.9p + 0.1(1 - p)} \quad \text{M1 A1}$$

$$\frac{0.9p}{0.8p + 0.1} = \frac{1}{2} \quad \text{M1}$$

$$1.8p = 0.8p + 0.1 \quad \text{M1}$$

$$p = 0.1 \quad \text{A1}$$

Answer: $p = 0.1$ 5 marks

18 (6 marks)

$$\text{Rain on Wed via Mon rain, Tue rain: } 0.4 \times 0.7 \times 0.7 = 0.196 \quad \text{M1 A1}$$

$$\text{Mon rain, Tue dry, Wed rain: } 0.4 \times 0.3 \times 0.2 = 0.024 \quad \text{M1}$$

$$\text{Mon dry, Tue rain, Wed rain: } 0.6 \times 0.2 \times 0.7 = 0.084; \text{ Mon dry, Tue dry: } 0.6 \times 0.8 \times 0.2 = 0.096 \quad \text{M1 A1}$$

$$P(\text{Wed rain}) = 0.4; P(\text{Mon rain} \mid \text{Wed rain}) = \frac{0.22}{0.4} \quad \text{A1}$$

Answer: $\frac{11}{20}$ 6 marks

19 (4 marks)

$$P(\text{RR} \mid \text{A}) = \frac{3}{5} \times \frac{2}{4} = \frac{3}{10} \quad \text{M1 A1}$$

$$P(\text{RR} \mid \text{B}) = \frac{2}{5} \times \frac{1}{4} = \frac{1}{10} \quad \text{A1}$$

$$P(\text{A} \mid \text{RR}) = \frac{\frac{3}{10}}{\frac{3}{10} + \frac{1}{10}} = \frac{3}{4} \quad \text{A1}$$

Answer: $\frac{3}{4}$ 4 marks

20 (5 marks)

$$P(1 \text{ defective} | \text{good}) = 5(0.02)(0.98)^4 = 0.09224$$

M1 A1

$$P(1 \text{ defective} | \text{poor}) = 5(0.1)(0.9)^4 = 0.32805$$

A1

$$P(\text{good} | 1 \text{ defective}) = \frac{0.8 \times 0.09224}{0.8 \times 0.09224 + 0.2 \times 0.32805}$$

M1

$$= 0.5293$$

A1

Answer: 0.529

5 marks

21 (7 marks)

$$P(\text{disease} | n \text{ positives}) = \frac{0.01(0.95)^n}{0.01(0.95)^n + 0.99(0.04)^n}$$

M1 A1

$$\text{This exceeds } 0.99 \text{ when } 0.01(0.95)^n > 99 \times 0.99(0.04)^n$$

M1

$$\left(\frac{0.95}{0.04}\right)^n > 9801$$

A1

$$n > \frac{\ln 9801}{\ln 23.75} = 2.90$$

M1 A1

$$\text{Least } n = 3$$

A1

Answer: n = 3

7 marks

22 (6 marks)

$$(a) P(\text{culprit}) = \frac{1}{20000}, P(\text{match} | \text{culprit}) = 1$$

B1

$$P(\text{match and innocent}) = \frac{19999}{20000} \times \frac{1}{1000}$$

M1

$$P(\text{culprit} | \text{match}) = \frac{\frac{1}{20000}}{\frac{1}{20000} + \frac{19999}{20000000}}$$

M1

$$= \frac{1000}{1000 + 19999} = \frac{1000}{20999} \approx 0.048$$

A1

Answer: ≈ 0.048

4 marks

(b) The 1 in 1000 is $P(\text{match} | \text{innocent})$, not $P(\text{innocent} | \text{match})$

E1

It ignores the very low prior probability of guilt (about 20 000 people, so around 20 innocent matches are expected)

E1

Answer: Confuses $P(\text{match} | \text{innocent})$ with $P(\text{innocent} | \text{match})$

2 marks

23 (7 marks)

$$(a) \text{Likelihoods: } \left(\frac{1}{5}\right)^2 = \frac{1}{25}, \left(\frac{1}{2}\right)^2 = \frac{1}{4}, \left(\frac{4}{5}\right)^2 = \frac{16}{25}$$

M1 A1

$$\text{Multiply each by } \frac{1}{3}; \text{ ratio } 4 : 25 : 64 \text{ (scaling by 100)}$$

M1

$$P(\text{Urn 3}) = \frac{64}{93}$$

A1

$$\text{Answer: } \frac{64}{93}$$

4 marks

$$(b) P(\text{red}) = \sum P(\text{Urn } i | \text{two reds}) \times P(\text{red} | \text{Urn } i)$$

M1

$$\frac{4}{93} \times \frac{1}{5} + \frac{25}{93} \times \frac{1}{2} + \frac{64}{93} \times \frac{4}{5}$$

M1

$$= \frac{0.8 + 12.5 + 51.2}{93} = \frac{64.5}{93} = \frac{43}{62}$$

A1

$$\text{Answer: } \frac{43}{62}$$

3 marks

24 (6 marks)

$$P(A | B) = \frac{P(B | A)P(A)}{P(B)} \text{ and } P(A' | B) = \frac{P(B | A')P(A')}{P(B)} \quad \text{M1}$$

Dividing, $P(B)$ cancels, giving the result A1

$$\text{Posterior odds} = \frac{1}{49} \times 8 = \frac{8}{49} \quad \text{M1 A1}$$

$$\text{So } P(A | B) = \frac{8}{8 + 49} \quad \text{M1}$$

$$= \frac{8}{57} \quad \text{A1}$$

$$\text{Answer: } \frac{8}{57} \quad \text{6 marks}$$

25 (6 marks)

$$\text{(a) } P(\text{says blue and blue}) = 0.15 \times 0.8 = 0.12; P(\text{says blue and green}) = 0.85 \times 0.2 = 0.17 \quad \text{M1 A1}$$

$$P = \frac{0.12}{0.29} = \frac{12}{29} \quad \text{A1}$$

$$\text{Answer: } \frac{12}{29} \quad \text{3 marks}$$

$$\text{(b) } 0.15 \times 0.8^2 = 0.096; 0.85 \times 0.2^2 = 0.034 \quad \text{M1 A1}$$

$$P = \frac{0.096}{0.13} = \frac{48}{65} \quad \text{A1}$$

$$\text{Answer: } \frac{48}{65} \quad \text{3 marks}$$

26 (5 marks)

The 95% is $P(\text{positive} | \text{condition})$, not $P(\text{condition} | \text{positive})$; the student has ignored how rare the condition is E1

$$P(\text{positive and condition}) = 0.005 \times 0.95 = 0.00475 \quad \text{M1}$$

$$P(\text{positive and no condition}) = 0.995 \times 0.05 = 0.04975 \quad \text{M1}$$

$$P(\text{condition} | \text{positive}) = \frac{0.00475}{0.0545} \quad \text{M1}$$

$$= 0.0872 \quad \text{A1}$$

$$\text{Answer: } 0.0872 \quad \text{5 marks}$$

27 (9 marks)

(a) It cannot land tails, but a tail occurred E1

Answer: It never shows a tail 1 mark

$$\text{(b) } P(\text{HHT} | \text{fair}) = \left(\frac{1}{2}\right)^3 = \frac{1}{8} \quad \text{B1}$$

$$P(\text{HHT} | \text{biased}) = \left(\frac{1}{3}\right)^2 \times \frac{2}{3} = \frac{2}{27} \quad \text{B1}$$

$$P(\text{fair} | \text{HHT}) = \frac{\frac{1}{8}}{\frac{1}{8} + \frac{2}{27}} \quad \text{M1}$$

$$= \frac{27}{27 + 16} = \frac{27}{43} \quad \text{A1}$$

$$\text{Answer: } \frac{27}{43} \quad \text{4 marks}$$

$$\text{(c) } P(\text{biased} | \text{HHT}) = \frac{16}{43} \quad \text{M1}$$

$$P(\text{head}) = \frac{27}{43} \times \frac{1}{2} + \frac{16}{43} \times \frac{1}{3} \quad \text{M1 A1}$$

$$= \frac{81 + 32}{258} = \frac{113}{258} \quad \text{A1}$$

$$\text{Answer: } \frac{113}{258} \quad \text{4 marks}$$

28 (8 marks)

Let W be the warden's statement "B will not be pardoned"

B1

$$P(W | A \text{ pardoned}) = \frac{1}{2}, P(W | B \text{ pardoned}) = 0, P(W | C \text{ pardoned}) = 1$$

M1 A1

$$P(W) = \frac{1}{3}\left(\frac{1}{2}\right) + \frac{1}{3}(0) + \frac{1}{3}(1) = \frac{1}{2}$$

M1 A1

$$P(A \text{ pardoned} | W) = \frac{\frac{1}{3} \times \frac{1}{2}}{\frac{1}{2}} = \frac{1}{3}$$

M1 A1

A's chance is unchanged because the warden would say B or C whichever was the case; the information is in the fact that C's chance has risen to $\frac{2}{3}$

E1

Answer: $\frac{1}{3}$

8 marks

29 (9 marks)

(a) $P(\text{positive and condition}) = sp$

B1

$P(\text{positive and no condition}) = (1 - t)(1 - p)$

M1

Bayes' theorem gives the result

A1

Answer: $\frac{sp}{sp + (1 - t)(1 - p)}$

3 marks

(b) $\frac{0.98p}{0.98p + 0.05(1 - p)} > \frac{1}{2}$, so $0.98p > 0.05(1 - p)$

M1

$0.98p > 0.05 - 0.05p$, so $1.03p > 0.05$

M1 A1

$p > \frac{5}{103} \approx 0.0485$

A1

Answer: $p > 0.0485$

4 marks

(c) If p is below about 0.0485, the posterior probability is below $\frac{1}{2}$

E1

Among a large number of positives, false positives from the many healthy people outnumber true positives

E1

Answer: Most positives are false when p is small

2 marks

30 (10 marks)

(a) $P(2 \text{ heads} | n) = {}^nC_2\left(\frac{1}{2}\right)^n$: $\frac{1}{4}, \frac{3}{8}, \frac{3}{8}, \frac{5}{16}, \frac{15}{64}$ for $n = 2, 3, 4, 5, 6$ (and 0 for $n = 1$)

M1 A1

Each n has probability $\frac{1}{6}$

M1

Sum = $\frac{1}{6}\left(\frac{1}{4} + \frac{3}{8} + \frac{3}{8} + \frac{5}{16} + \frac{15}{64}\right) = \frac{33}{128}$

A1

Answer: $\frac{33}{128}$

4 marks

(b) $P(n = 4 \text{ and } 2 \text{ heads}) = \frac{1}{6} \times \frac{6}{16}$

M1 A1

Divide by the total: $\frac{8}{33}$

A1

Answer: $\frac{8}{33}$

3 marks

(c) $P(n | 2 \text{ heads})$ is proportional to ${}^nC_2/2^n$

M1

Values for $n = 2$ to 6 : 0.25, 0.375, 0.375, 0.3125, 0.234 (to 3 d.p.)

M1

$n = 3$ and $n = 4$ are equally most likely

A1

Answer: $n = 3$ and $n = 4$ (equally likely)

3 marks

Probability Generating Functions**1 (1 mark)**

$G(t) = 0.2 + 0.5t + 0.3t^2$

B1

Answer: $0.2 + 0.5t + 0.3t^2$

1 mark

2 (3 marks)**(a)** Coefficient of t : 0.4**B1****Answer: 0.4****1 mark****(b)** $G'(t) = 0.4 + t$, so $E(X) = G'(1) = 1.4$ **M1****A1****Answer: 1.4****2 marks****3 (2 marks)** $G(1) = 1$, so $k(2 + 3 + 1) = 1$ **M1**

$$k = \frac{1}{6}$$

A1**Answer: $\frac{1}{6}$** **2 marks****4 (2 marks)**

$$G'(t) = 3 \times \frac{3}{4} \left(\frac{1}{4} + \frac{3}{4}t \right)^2$$

M1

$$G'(1) = \frac{9}{4} \times 1 = \frac{9}{4}$$

A1**Answer: $\frac{9}{4}$** **2 marks****5 (3 marks)**

(a) $G(t) = \frac{1}{6}(t + t^2 + t^3 + t^4 + t^5 + t^6)$

B1**Answer: $\frac{1}{6}(t + t^2 + \dots + t^6)$** **1 mark**

(b) $G'(t) = \frac{1}{6}(1 + 2t + 3t^2 + 4t^3 + 5t^4 + 6t^5)$

M1

$$G'(1) = \frac{21}{6} = 3.5$$

A1**Answer: $E(X) = \frac{21}{6} = 3.5$** **2 marks****6 (2 marks)**

$$G(t) = \frac{t}{2} \left(1 - \frac{t}{2} \right)^{-1} = \frac{t}{2} \left(1 + \frac{t}{2} + \frac{t^2}{4} + \dots \right)$$

M1

$$P(X = 1) = \frac{1}{2}, P(X = 2) = \frac{1}{4}$$

A1**Answer: $\frac{1}{2}$ and $\frac{1}{4}$** **2 marks****7 (4 marks)**

$$G'(t) = \frac{2 + 6t}{6}, G'(1) = \frac{4}{3}$$

M1 A1

$$G''(t) = \frac{6}{6} = 1$$

B1

$$\text{Var}(X) = G''(1) + G'(1) - [G'(1)]^2 = 1 + \frac{4}{3} - \frac{16}{9} = \frac{5}{9}$$

A1**Answer: $E(X) = \frac{4}{3}$, $\text{Var}(X) = \frac{5}{9}$** **4 marks****8 (3 marks)**

(a) $G(1) = 8k = 1$, so $k = \frac{1}{8}$

B1**Answer: $\frac{1}{8}$** **1 mark**

(b) $P(X = 0) = k = \frac{1}{8}$

B1

$$(1 + t)^3 = 1 + 3t + 3t^2 + t^3, \text{ so } P(X = 2) = \frac{3}{8}$$

B1

Answer: $\frac{1}{8}$ and $\frac{3}{8}$ **2 marks****9 (3 marks)**

$$G_{X+Y}(t) = G_X(t)G_Y(t)$$

M1

$$= \left(\frac{1}{2} + \frac{1}{2}t\right)\left(\frac{1}{3} + \frac{2}{3}t\right) = \frac{1}{6} + \frac{1}{2}t + \frac{1}{3}t^2$$

A1

$$P(X + Y = 1) = \frac{1}{2}$$

A1**Answer: $\frac{1}{2}$** **3 marks****10 (4 marks)**

$$(a) G(t) = \frac{t}{3}\left(1 - \frac{2t}{3}\right)^{-1} = \frac{t}{3}\left(1 + \frac{2t}{3} + \left(\frac{2t}{3}\right)^2 + \dots\right)$$

M1

$$\text{Coefficient of } t^3: \frac{1}{3} \times \frac{4}{9} = \frac{4}{27}$$

A1**Answer: $\frac{4}{27}$** **2 marks**

$$(b) G'(t) = \frac{3}{(3 - 2t)^2}$$

M1

$$G'(1) = 3$$

A1**Answer: 3****2 marks****11 (4 marks)**

$$G'(t) = \lambda e^{\lambda(t-1)}, \text{ so } E(X) = G'(1) = \lambda$$

M1 A1

$$G''(t) = \lambda^2 e^{\lambda(t-1)}, \text{ so } G''(1) = \lambda^2$$

B1

$$\text{Var}(X) = \lambda^2 + \lambda - \lambda^2 = \lambda$$

A1**Answer: $E(X) = \text{Var}(X) = \lambda$** **4 marks****12 (4 marks)**

$$(a) G(t) = \frac{1}{4}(t + t^2 + t^3 + t^4)$$

B1**Answer: $\frac{1}{4}(t + t^2 + t^3 + t^4)$** **1 mark**

$$(b) G'(1) = \frac{1}{4}(1 + 2 + 3 + 4) = \frac{5}{2}$$

M1 A1

$$G''(1) = \frac{1}{4}(0 + 2 + 6 + 12) = 5, \text{ so } \text{Var}(X) = 5 + \frac{5}{2} - \frac{25}{4} = \frac{5}{4}$$

A1**Answer: $E(X) = \frac{5}{2}, \text{Var}(X) = \frac{5}{4}$** **3 marks****13 (3 marks)**

$$G'(1) = E(X) \text{ and } G''(t) = E[X(X-1)t^{X-2}], \text{ so } G''(1) = E[X(X-1)] = E(X^2) - E(X)$$

M1 A1

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = G''(1) + G'(1) - [G'(1)]^2$$

A1**Answer: $\text{Var}(X) = G''(1) + G'(1) - [G'(1)]^2$** **3 marks****14 (4 marks)**

$$\text{One trial: } G(t) = 0.6 + 0.4t$$

B1

$$\text{Three independent trials: } [G(t)]^3 = (0.6 + 0.4t)^3$$

M1

$$\text{Coefficient of } t^2: 3 \times (0.6)(0.4)^2$$

M1

$$= 0.288$$

A1**Answer: 0.288****4 marks**

15 (5 marks)

Let $G(t) = p_0 + p_1t + p_2t^2$. Then $p_0 + p_1 + p_2 = 1$

B1

$$G'(1) = p_1 + 2p_2 = 1.2$$

M1

$$E(X^2) = 0.56 + 1.44 = 2, \text{ and } G''(1) + G'(1) = E(X^2) \text{ so } p_1 + 4p_2 = 2$$

M1 A1

$$\text{Solving: } p_2 = 0.4, p_1 = 0.4, p_0 = 0.2, \text{ so } G(t) = 0.2 + 0.4t + 0.4t^2$$

A1

$$\text{Answer: } 0.2 + 0.4t + 0.4t^2$$

5 marks

16 (5 marks)

$$G'(t) = \frac{(4 - 3t) + 3t}{(4 - 3t)^2} = \frac{4}{(4 - 3t)^2}$$

M1 A1

$$E(X) = G'(1) = 4$$

A1

$$G''(t) = \frac{24}{(4 - 3t)^3}, G''(1) = 24$$

M1

$$\text{Var}(X) = 24 + 4 - 16 = 12$$

A1

$$\text{Answer: } E(X) = 4, \text{ Var}(X) = 12$$

5 marks

17 (5 marks)

$$G_{X+Y}(t) = \left(\frac{t}{2-t}\right)^2 = \frac{t^2}{4} \left(1 - \frac{t}{2}\right)^{-2}$$

M1 A1

$$\left(1 - \frac{t}{2}\right)^{-2} = 1 + 2\left(\frac{t}{2}\right) + 3\left(\frac{t}{2}\right)^2 + \dots = 1 + t + \dots$$

M1 A1

$$\text{Coefficient of } t^3 \text{ in } \frac{t^2}{4}(1 + t + \dots) \text{ is } \frac{1}{4}, \text{ so } P(X + Y = 3) = \frac{1}{4}$$

A1

$$\text{Answer: } \frac{1}{4}$$

5 marks

18 (4 marks)

$$(a) G_Y(t) = E(t^{2X+3}) = t^3 E((t^2)^X)$$

M1

$$= t^3 G_X(t^2) = \frac{t^3(1+t^2)}{2}$$

A1

$$\text{Answer: } G_Y(t) = \frac{t^3(1+t^2)}{2}$$

2 marks

$$(b) G_Y(t) = \frac{1}{2}(t^3 + t^5), G_Y'(1) = \frac{1}{2}(3 + 5)$$

M1

$$= 4$$

A1

$$\text{Answer: } 4$$

2 marks

19 (6 marks)

$$(a) G(1) = 1 \text{ gives } (a + b)^2 = 16, \text{ so } a + b = 4 \text{ (a, b positive)}$$

M1

$$\text{The coefficient of } t^4 \text{ is } \frac{b^2}{16} = \frac{9}{16}, \text{ so } b = 3$$

M1

$$a = 1$$

A1

$$\text{Answer: } a = 1, b = 3$$

3 marks

$$(b) G(t) = \frac{1}{16}(1 + 6t^2 + 9t^4), \text{ so } P(X = 2) = \frac{6}{16} = \frac{3}{8}$$

B1

$$G'(t) = \frac{1}{16}(12t + 36t^3)$$

M1

$$E(X) = G'(1) = \frac{48}{16} = 3$$

A1

$$\text{Answer: } P(X = 2) = \frac{3}{8}, E(X) = 3$$

3 marks

20 (5 marks)

$$(a) G'(t) = \frac{3(2+t)^2}{27}$$

M1

$$G'(1) = \frac{27}{27} = 1$$

A1

$$\text{Answer: } E(X) = 1$$

2 marks

$$(b) G''(t) = \frac{6(2+t)}{27}, G''(1) = \frac{18}{27} = \frac{2}{3}$$

M1 A1

$$\text{Var}(X) = \frac{2}{3} + 1 - 1 = \frac{2}{3}$$

A1

$$\text{Answer: } \frac{2}{3}$$

3 marks

21 (4 marks)

$$G_{X+Y}(t) = E(t^{X+Y}) = E(t^X t^Y)$$

M1

$$X \text{ and } Y \text{ independent, so } E(t^X t^Y) = E(t^X)E(t^Y) \\ = G_X(t)G_Y(t)$$

M1 A1

A1

$$\text{Answer: } G_{X+Y}(t) = G_X(t)G_Y(t)$$

4 marks

22 (8 marks)

$$(a) G_X(t) = \frac{1}{3}(1+t+t^2)$$

B1

$$G_S(t) = \frac{1}{27}(1+t+t^2)^3$$

B1

$$\text{Answer: } \frac{1}{27}(1+t+t^2)^3$$

2 marks

$$(b) (1+t+t^2)^3 = 1 + 3t + 6t^2 + 7t^3 + 6t^4 + 3t^5 + t^6$$

M1 A1

$$P(S=4) = \frac{6}{27} = \frac{2}{9}$$

A1

$$\text{Answer: } \frac{2}{9}$$

3 marks

$$(c) G_S'(t) = \frac{3(1+2t)(1+t+t^2)^2}{27}, G_S'(1) = \frac{3 \times 3 \times 9}{27} = 3$$

M1 A1

$$\text{Var}(S) = 3 \text{ Var}(X) = 3 \times \frac{2}{3} = 2 \text{ (or by } G''(1) + G'(1) - [G'(1)]^2)$$

A1

$$\text{Answer: } E(S) = 3, \text{ Var}(S) = 2$$

3 marks

23 (6 marks)

$$(a) G'(t) = \frac{p}{(1-qt)^2} \text{ where } q = 1 - p, \text{ so } G'(1) = \frac{p}{p^2} = \frac{1}{p}$$

M1 A1

$$G''(t) = \frac{2pq}{(1-qt)^3}, G''(1) = \frac{2q}{p^2}$$

M1

$$\text{Var}(X) = \frac{2q}{p^2} + \frac{1}{p} - \frac{1}{p^2} = \frac{2q+p-1}{p^2} = \frac{q}{p^2}$$

A1

$$\text{Answer: } E(X) = \frac{1}{p}, \text{ Var}(X) = \frac{1-p}{p^2}$$

4 marks

$$(b) \frac{1-p}{p^2} = 12, \text{ so } 12p^2 + p - 1 = 0$$

M1

$$(4p-1)(3p+1) = 0, \text{ so } p = \frac{1}{4}$$

A1

$$\text{Answer: } \frac{1}{4}$$

2 marks

24 (5 marks)

$$\text{Dev's formula gives } G_{X+Y}(1) = 1 + 1 = 2$$

M1 A1

$$\text{but every probability generating function satisfies } G(1) = 1$$

E1

$$\text{The correct result is the product: } G_{X+Y}(t) = \left(\frac{1}{2} + \frac{1}{2}t\right)^2 = \frac{1}{4} + \frac{1}{2}t + \frac{1}{4}t^2$$

M1

$$\text{which equals 1 when } t = 1$$

A1

$$\text{Answer: Correct: } G_{X+Y}(t) = G_X(t)G_Y(t)$$

5 marks

25 (6 marks)

$$G(t) = \sum_{r=0}^n {}^nC_r p^r q^{n-r} t^r = (q + pt)^n, \text{ where } q = 1 - p$$

M1 A1

$$G'(t) = np(q + pt)^{n-1}, \text{ so } G'(1) = np(q + p)^{n-1} = np$$

M1 A1

$$G''(t) = n(n-1)p^2(q + pt)^{n-2}, G''(1) = n(n-1)p^2$$

M1

$$\text{Var}(X) = n(n-1)p^2 + np - n^2p^2 = np - np^2 = np(1 - p)$$

A1

$$\text{Answer: } E(X) = np, \text{ Var}(X) = np(1 - p)$$

6 marks

26 (7 marks)

$$(a) G(1) = 1 \text{ gives } k \ln 2 = 1, \text{ so } k = \frac{1}{\ln 2}$$

B1

$$\text{Answer: } k = \frac{1}{\ln 2}$$

1 mark

$$(b) G(t) = k[\ln 2 - \ln(2 - t)], \text{ so } G'(t) = \frac{k}{2 - t}$$

M1 A1

$$E(X) = G'(1) = k = \frac{1}{\ln 2}$$

A1

$$G''(t) = \frac{k}{(2 - t)^2}, \text{ so } G''(1) = k$$

M1 A1

$$\text{Var}(X) = G''(1) + G'(1) - [G'(1)]^2 = k + k - k^2 = k(2 - k)$$

A1

$$\text{Answer: } E(X) = \frac{1}{\ln 2}; \text{ Var}(X) = k(2 - k) = 0.8040$$

6 marks

27 (9 marks)

$$(a) G_X(t) = e^{\lambda(t-1)}, G_Y(t) = e^{\mu(t-1)}$$

B1

$$G_{X+Y}(t) = e^{(\lambda + \mu)(t-1)}$$

M1

which is the PGF of $\text{Po}(\lambda + \mu)$

A1

$$\text{Answer: } G_{X+Y}(t) = e^{(\lambda + \mu)(t-1)}, \text{ so } X + Y \sim \text{Po}(\lambda + \mu)$$

3 marks

$$(b) P(X = r | X + Y = n) = \frac{P(X = r)P(Y = n - r)}{P(X + Y = n)}$$

M1 A1

$$= \frac{\frac{e^{-\lambda} \lambda^r}{r!} \times \frac{e^{-\mu} \mu^{n-r}}{(n-r)!}}{\frac{e^{-(\lambda + \mu)} (\lambda + \mu)^n}{n!}}$$

M1 A1

$$= \frac{n!}{r!(n-r)!} \times \frac{\lambda^r \mu^{n-r}}{(\lambda + \mu)^n}$$

M1

which gives the result; X given $X + Y = n$ is binomial $B(n, \frac{\lambda}{\lambda + \mu})$

A1

$$\text{Answer: } B(n, \frac{\lambda}{\lambda + \mu})$$

6 marks

28 (8 marks)

$$G_N(s) = e^{2(s-1)}$$

B1

$$g(t) = \frac{1}{2} + \frac{1}{2}t$$

B1

$$G_Y(t) = e^{2(\frac{1}{2} + \frac{1}{2}t - 1)}$$

M1 A1

$$= e^{t-1}$$

A1

This is the PGF of a Poisson distribution with mean 1, so $Y \sim \text{Po}(1)$

E1

$$P(Y = 0) = e^{-1}$$

M1

$$= 0.368$$

A1

$$\text{Answer: } Y \sim \text{Po}(1); P(Y = 0) = 0.368$$

8 marks

29 (8 marks)

$$(a) G(t) = \sum_{r=1}^{\infty} (1-p)^{r-1} p t^r = p t \sum_{k=0}^{\infty} ((1-p)t)^k \quad \text{M1}$$

$$= \frac{pt}{1 - (1-p)t} \quad (\text{sum of a geometric series}) \quad \text{A1}$$

$$\text{Answer: } G(t) = \frac{pt}{1 - (1-p)t} \quad \text{2 marks}$$

$$(b) G(-1) = P(\text{even}) - P(\text{odd}) \text{ and } G(1) = P(\text{even}) + P(\text{odd}) = 1 \quad \text{M1 A1}$$

$$\text{So } P(\text{even}) = \frac{1 + G(-1)}{2} \quad \text{M1}$$

$$G(-1) = \frac{-p}{1 + (1-p)} = \frac{-p}{2-p}, \text{ so } P(\text{even}) = \frac{1}{2} \left(1 - \frac{p}{2-p} \right) = \frac{1-p}{2-p} \quad \text{A1}$$

$$\text{Answer: } P(X \text{ is even}) = \frac{1-p}{2-p} \quad \text{4 marks}$$

$$(c) \frac{1 - \frac{1}{3}}{2 - \frac{1}{3}} = \frac{\frac{2}{3}}{\frac{5}{3}} \quad \text{M1}$$

$$= \frac{2}{5} \quad \text{A1}$$

$$\text{Answer: } \frac{2}{5} \quad \text{2 marks}$$

30 (10 marks)

$$\text{If the first toss is T (probability } \frac{1}{2} \text{), the process restarts, giving a factor } tG(t) \quad \text{M1 A1}$$

$$\text{If the first two tosses are HT (probability } \frac{1}{4} \text{), the process restarts, giving } t^2 G(t) \quad \text{M1 A1}$$

$$\text{If the first two tosses are HH (probability } \frac{1}{4} \text{) the game ends after 2 tosses, giving } t^2 \quad \text{A1}$$

$$G = \frac{t}{2} G + \frac{t^2}{4} G + \frac{t^2}{4}, \text{ so } G(4 - 2t - t^2) = t^2 \quad \text{M1}$$

$$G(t) = \frac{t^2}{4 - 2t - t^2} \quad \text{A1}$$

$$G'(t) = \frac{2t(4 - 2t - t^2) + t^2(2 + 2t)}{(4 - 2t - t^2)^2} \quad \text{M1}$$

$$G'(1) = \frac{2 \times 1 + 4}{1} = 6 \quad \text{A1 A1}$$

$$\text{Answer: } E(N) = 6 \quad \text{10 marks}$$

Discrete Random Variables**1 (2 marks)**

$$0.1 + k + 0.4 + 0.2 = 1 \quad \text{M1}$$

$$k = 0.3 \quad \text{A1}$$

$$\text{Answer: } 0.3 \quad \text{2 marks}$$

2 (2 marks)

$$(a) 0.30 + 0.20 = 0.5 \quad \text{B1}$$

$$\text{Answer: } 0.5 \quad \text{1 mark}$$

$$(b) 0.35 + 0.30 = 0.65 \quad \text{B1}$$

$$\text{Answer: } 0.65 \quad \text{1 mark}$$

3 (2 marks)

$$\text{Each probability is between 0 and 1} \quad \text{B1}$$

$$\text{Sum} = \frac{1 + 2 + 3 + 4}{10} = 1 \quad \text{B1}$$

$$\text{Answer: } \text{Valid} \quad \text{2 marks}$$

4 (2 marks)

Outcomes HH, HT, TH, TT equally likely

M1

$$P(X = 0) = \frac{1}{4}, P(X = 1) = \frac{1}{2}, P(X = 2) = \frac{1}{4}$$

A1

$$\text{Answer: } P(0) = \frac{1}{4}, P(1) = \frac{1}{2}, P(2) = \frac{1}{4}$$

2 marks

5 (2 marks)

$$P(X = 3) = F(3) - F(2)$$

M1

$$= 0.8 - 0.5 = 0.3$$

A1

Answer: 0.3

2 marks

6 (2 marks)

$$P(X \leq 4) = \frac{4}{6} = \frac{2}{3}$$

B1

$$P(X > 4) = \frac{2}{6} = \frac{1}{3}$$

B1

$$\text{Answer: } \frac{2}{3} \text{ and } \frac{1}{3}$$

2 marks

7 (3 marks)

$$\text{(a) } k(1 + 2 + 3 + 4 + 5) = 1$$

M1

$$k = \frac{1}{15}$$

A1

$$\text{Answer: } \frac{1}{15}$$

2 marks

$$\text{(b) } \frac{4+5}{15} = \frac{3}{5}$$

B1

$$\text{Answer: } \frac{3}{5}$$

1 mark

8 (4 marks)

$$\text{(a) } k(4 + 3 + 2 + 1) = 1$$

M1

$$10k = 1, k = \frac{1}{10}$$

A1

$$\text{Answer: } k = \frac{1}{10}$$

2 marks

$$\text{(b) } P(X = 2) + P(X = 4) = \frac{3}{10} + \frac{1}{10}$$

M1

$$= \frac{2}{5}$$

A1

$$\text{Answer: } \frac{2}{5}$$

2 marks

9 (3 marks)

$$\text{(a) } P(X = 1) = 0.1, P(X = 2) = 0.2, P(X = 3) = 0.25$$

M1

$$P(X = 4) = 0.3, P(X = 5) = 0.15$$

A1

Answer: 0.1, 0.2, 0.25, 0.3, 0.15

2 marks

$$\text{(b) } F(4) - F(1) = 0.85 - 0.1 = 0.75$$

B1

Answer: 0.75

1 mark

10 (4 marks)

Possible values 0, 1, 2, 3, 4, 5

B1

$$P(X = 0) = \frac{6}{36} = \frac{1}{6}$$

M1

$$P(X = d) = \frac{2(6-d)}{36} \text{ for } d = 1, \dots, 5$$

M1

$$P(1) = \frac{5}{18}, P(2) = \frac{2}{9}, P(3) = \frac{1}{6}, P(4) = \frac{1}{9}, P(5) = \frac{1}{18}$$

A1

$$\text{Answer: } P(0) = \frac{1}{6}, P(1) = \frac{5}{18}, P(2) = \frac{2}{9}, P(3) = \frac{1}{6}, P(4) = \frac{1}{9}, P(5) = \frac{1}{18}$$

4 marks

–	1	2	3	4	5	6
1	0	1	2	3	4	5
2	1	0	1	2	3	4
3	2	1	0	1	2	3
4	3	2	1	0	1	2
5	4	3	2	1	0	1
6	5	4	3	2	1	0

11 (4 marks)

$X = 0, 1, 2, 3$ with ${}^9C_3 = 84$ equally likely selections

B1

$$P(X = x) = \frac{{}^4C_x \times {}^5C_{3-x}}{84}$$

M1

$$P(0) = \frac{10}{84}, P(1) = \frac{40}{84}, P(2) = \frac{30}{84}, P(3) = \frac{4}{84}$$

A1

$$= \frac{5}{42}, \frac{10}{21}, \frac{5}{14}, \frac{1}{21}$$

A1

$$\text{Answer: } \frac{5}{42}, \frac{10}{21}, \frac{5}{14}, \frac{1}{21}$$

4 marks

12 (3 marks)

$$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}, \text{ so the sum telescopes}$$

M1

$$k(1 - \frac{1}{5}) = 1$$

M1

$$k = \frac{5}{4}$$

A1

$$\text{Answer: } \frac{5}{4}$$

3 marks

13 (3 marks)

(a) Highest probability 0.3 at $x = 2$

B1

Answer: 2

1 mark

(b) $F(1) = 0.35 < 0.5$ and $F(2) = 0.65 > 0.5$

M1

Median = 2

A1

Answer: 2

2 marks

14 (5 marks)

$$(a) P(X \leq x) = (\frac{x}{6})^2 = \frac{x^2}{36}$$

M1 A1

$$P(X = x) = \frac{x^2 - (x-1)^2}{36} = \frac{2x-1}{36}$$

A1

$$\text{Answer: } P(X = x) = \frac{2x-1}{36}$$

3 marks

$$(b) \frac{9+11}{36}$$

M1

$$= \frac{5}{9}$$

A1

$$\text{Answer: } \frac{5}{9}$$

2 marks

15 (6 marks)

(a) $a + 0.3 + b + 0.1 = 1$, so $a + b = 0.6$

M1 A1

$a = 2b$

M1

$3b = 0.6$, $b = 0.2$, $a = 0.4$

A1

Answer: $a = 0.4$, $b = 0.2$

4 marks

$$(b) \frac{P(X=2) + P(X=3)}{P(X \geq 2)} = \frac{0.5}{0.6}$$

$$= \frac{5}{6}$$

M1

A1

Answer: $\frac{5}{6}$

2 marks

16 (5 marks)

$$(a) F(1) = \frac{1}{6}, F(2) = \frac{1}{2}$$

B1

$$F(3) = \frac{5}{6}, F(4) = 1$$

B1

$$\text{Answer: } F(1) = \frac{1}{6}, F(2) = \frac{1}{2}, F(3) = \frac{5}{6}, F(4) = 1$$

2 marks

$$(b) P(X \leq 2.5) = F(2) = \frac{1}{2}$$

B1

$$P(X > 1.5) = 1 - F(1) = \frac{5}{6}$$

M1 A1

$$\text{Answer: } \frac{1}{2} \text{ and } \frac{5}{6}$$

3 marks

17 (5 marks)

$$X = 1: \text{head first, } P = \frac{1}{2}$$

B1

$$X = 2: \text{TH, } P = \frac{1}{4}; X = 3: \text{TTH, } P = \frac{1}{8}$$

M1 A1

$$X = 4: \text{stops after the fourth toss whatever happens, so } P = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

A1

$$P(X > 2) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

A1

$$\text{Answer: } P(1) = \frac{1}{2}, P(2) = \frac{1}{4}, P(3) = \frac{1}{8}, P(4) = \frac{1}{8}; P(X > 2) = \frac{1}{4}$$

5 marks

18 (5 marks)

12 equally likely outcomes

B1

$$X = 6: (A, B) = (2, 3) \text{ or } (3, 2), 2 \text{ outcomes, so } P = \frac{1}{6}$$

M1 A1

$$X = 4: (A, B) = (1, 4) \text{ or } (2, 2), 2 \text{ outcomes, so } P = \frac{1}{6}$$

A1

$$X \geq 8: (2, 4) = 8, (3, 3) = 9, (3, 4) = 12 \text{ is 3 outcomes, } P = \frac{1}{4}$$

A1

$$\text{Answer: } \frac{1}{6}, \frac{1}{6}, \frac{1}{4}$$

5 marks

19 (5 marks)

$$(a) k(1 + 2 + 5 + 10) = 1$$

M1

$$k = \frac{1}{18}$$

A1

$$\text{Answer: } \frac{1}{18}$$

2 marks

$$(b) P(X > 0) = \frac{2 + 5 + 10}{18} = \frac{17}{18}$$

M1

$$P(0 < X < 2) = P(X = 1) = \frac{2}{18}$$

M1

$$\text{Probability } \frac{2}{17}$$

A1

$$\text{Answer: } \frac{2}{17}$$

3 marks

20 (4 marks) ${}^5C_2 = 10$ equally likely pairs

B1

If the larger number is x , the other card is one of $1, 2, \dots, x - 1$

M1 A1

So there are $x - 1$ pairs and $P(X = x) = \frac{x - 1}{10}$

A1

Answer: $P(X = x) = \frac{x - 1}{10}$

4 marks

21 (5 marks)The events $X \leq x - 1$ and $X = x$ are mutually exclusive with union $X \leq x$

M1

So $F(x) = F(x - 1) + P(X = x)$, giving the result

A1

$$P(X = x) = \frac{x(x + 1) - (x - 1)x}{n(n + 1)}$$

M1

$$= \frac{x(x + 1 - x + 1)}{n(n + 1)}$$

M1

$$= \frac{2x}{n(n + 1)}$$

A1

Answer: $P(X = x) = F(x) - F(x - 1) = \frac{2x}{n(n + 1)}$

5 marks

22 (6 marks)(a) $P(0) = 3k$, $P(1) = 4k$, $P(2) = 3k$, so $10k = 1$, $k = \frac{1}{10}$

M1 A1

Mode = 1

A1

Answer: $k = \frac{1}{10}$; mode 1

3 marks

$$(b) \left(\frac{3}{10}\right)^2 + \left(\frac{4}{10}\right)^2 + \left(\frac{3}{10}\right)^2$$

M1 A1

$$= \frac{34}{100} = \frac{17}{50}$$

A1

Answer: $\frac{17}{50}$

3 marks

23 (6 marks)

(a) 6 equally likely pairs (X, Y)

B1

 $M = 1$: (1, 1) only

M1

 $M = 2$: (1, 2), (2, 1), (2, 2) is 3 pairs; $M = 3$: (3, 1), (3, 2) is 2 pairs

M1

$$P(M = 1) = \frac{1}{6}, P(M = 2) = \frac{1}{2}, P(M = 3) = \frac{1}{3}$$

A1

Answer: $\frac{1}{6}, \frac{1}{2}, \frac{1}{3}$

4 marks

(b) Only (2, 2) of the 3 pairs with $M = 2$

M1

$$= \frac{1}{3}$$

A1

Answer: $\frac{1}{3}$

2 marks

24 (7 marks)(a) Net gains: -2 (scores 1, 2, 3), 1 (score 4), 3 (score 5), 6 (score 6)

M1 A1

$$P(-2) = \frac{1}{2}, P(1) = P(3) = P(6) = \frac{1}{6}$$

A1

Answer: $P(-2) = \frac{1}{2}$; $P(1) = P(3) = P(6) = \frac{1}{6}$

3 marks

$$(b) \frac{3}{6} = \frac{1}{2}$$

B1

Answer: $\frac{1}{2}$

1 mark

(c) Totals -4 ($-2, -2$) and -1 ($-2, 1$) are not positive; -2 with 3 gives 1 and -2 with 6 gives 4, both positive; any two gains from 1, 3, 6 give a positive total

M1

$$P(\text{one gain is } -2 \text{ and the other is 3 or 6}) = 2 \times \frac{1}{2} \times \frac{1}{3} = \frac{1}{3}$$

M1

$$P(\text{both gains from 1, 3, 6}) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}, \text{ so } P = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$$

A1

Answer: $\frac{7}{12}$

3 marks

25 (5 marks)

$$p^2 + p + 0.44 = 1$$

M1

$$p^2 + p - 0.56 = 0, \text{ so } (p - 0.4)(p + 1.4) = 0$$

M1 A1

$$p = 0.4 \text{ or } p = -1.4$$

A1

$p = -1.4$ gives a negative probability, which is impossible, so $p = 0.4$

E1

Answer: $p = 0.4$

5 marks

26 (5 marks)

(a) $Y = 1$ when $X = -1$ or 1 : $0.3 + 0.5 = 0.8$

M1

$$P(Y = 0) = 0.2, P(Y = 1) = 0.8$$

A1

Answer: $P(Y = 0) = 0.2, P(Y = 1) = 0.8$

2 marks

(b) $X = -1$ gives $Z = 2$; $X = 0$ gives $Z = 1$; $X = 1$ gives $Z = 0$

M1 A1

$$P(Z = 0) = 0.5, P(Z = 1) = 0.2, P(Z = 2) = 0.3$$

A1

Answer: $P(0) = 0.5, P(1) = 0.2, P(2) = 0.3$

3 marks

27 (8 marks)

216 equally likely outcomes

B1

$$X = 1: 6 \text{ outcomes}, P = \frac{6}{216} = \frac{1}{36}$$

M1 A1

$$X = 3: 6 \times 5 \times 4 = 120 \text{ outcomes}, P = \frac{120}{216} = \frac{5}{9}$$

M1 A1

$X = 2$: choose the repeated score (6 ways), the other score (5 ways) and the position of the single one (3 ways), giving 90 outcomes

M1 A1

$$P = \frac{90}{216} = \frac{5}{12}; \text{ check: } \frac{1}{36} + \frac{5}{12} + \frac{5}{9} = 1$$

A1

Answer: $P(1) = \frac{1}{36}, P(2) = \frac{5}{12}, P(3) = \frac{5}{9}$

8 marks

28 (9 marks)

(a) Total selections ${}^{52}C_5 = 2598960$

B1

$$\text{Favourable: } {}^4C_2 \times {}^{48}C_3 = 6 \times 17296 = 103776$$

M1

$$\frac{103776}{2598960} = \frac{2162}{54145}$$

A1

Answer: $P(X = 2) = \frac{2162}{54145}$

3 marks

(b) $P(X = x) = \frac{{}^4C_x \times {}^{48}C_{5-x}}{{}^{52}C_5}$

M1

$$P(0) = 0.6588, P(1) = 0.2995$$

A1

$$P(2) = 0.03993, P(3) = 0.001736$$

A1

$$P(4) = 1.847\text{e-}05$$

A1

Answer: $P(0) = 0.6588, P(1) = 0.2995, P(2) = 0.03993, P(3) = 0.001736, P(4) = 1.847\text{e-}05$

4 marks

(c) $1 - P(X = 0)$

M1

$$= 0.3412$$

A1

Answer: 0.3412

2 marks

29 (10 marks)

(a) $P(X = x) = F(x) - F(x - 1)$

M1

$$= \frac{x^2 - (x - 1)^2}{n^2} = \frac{2x - 1}{n^2}$$

A1

Answer: $P(X = x) = \frac{2x - 1}{n^2}$

2 marks

(b) $\frac{5}{n^2} = \frac{5}{49}$

M1

$$n = 7 \text{ (n positive)}$$

Answer: $n = 7$

A1

2 marks

(c) $F(5) = \frac{25}{49} > 0.5$ and $F(4) = \frac{16}{49} < 0.5$

M1 A1

Median = 5

A1

Answer: 5

3 marks

(d) $P(2) + P(4) + P(6) = \frac{3 + 7 + 11}{49}$

M1 A1

$$= \frac{21}{49} = \frac{3}{7}$$

A1

Answer: $\frac{3}{7}$

3 marks

30 (9 marks)

After 2 jumps the frog is at -2 , 0 or 2

B1

After 2 jumps: reach 2 with $P = \frac{1}{4}$, reach -2 with $P = \frac{1}{4}$, and be at 0 with $P = \frac{1}{2}$

M1 A1

From 0 after 2 jumps, the 3rd jump: at ± 1 , each with $P = \frac{1}{4}$

M1

From ± 1 the 4th jump reaches ± 2 (prob $\frac{1}{2}$) or 0 (prob $\frac{1}{2}$)

M1

$$P(X = 2) = \frac{1}{4} + \frac{1}{4} \times \frac{1}{2} = \frac{3}{8}$$

A1

$$P(X = -2) = \frac{3}{8} \text{ by symmetry}$$

A1

$$P(X = 0) = 2 \times \frac{1}{4} \times \frac{1}{2} = \frac{1}{4}$$

A1

The frog cannot be at ± 1 after 4 jumps (even number), so $P(X = \pm 1) = 0$, and the probabilities sum to 1

A1

Answer: $P(-2) = \frac{3}{8}$, $P(0) = \frac{1}{4}$, $P(2) = \frac{3}{8}$

9 marks

Expectation and Variance

1 (2 marks)

$$E(X) = 1(0.2) + 2(0.3) + 3(0.5)$$

M1

$$= 2.3$$

A1

Answer: 2.3

2 marks

2 (3 marks)

$$E(X) = 0 + \frac{1}{2} + 1 = \frac{3}{2}$$

B1

$$E(X^2) = 0 + \frac{1}{2} + 4 = \frac{9}{2}$$

B1

$$\text{Var}(X) = \frac{9}{2} - \left(\frac{3}{2}\right)^2 = \frac{9}{4}$$

B1

Answer: $\frac{3}{2}$, $\frac{9}{2}$, $\frac{9}{4}$

3 marks

3 (3 marks)

$$E(2X + 3) = 2 \times 4 + 3 = 11$$

B1

$$\text{Var}(2X + 3) = 2^2 \times 5$$

M1

$$= 20$$

A1

Answer: 11 and 20

3 marks

4 (3 marks)

$$E(X) = \frac{1 + 2 + 3 + 4 + 5}{5} = 3$$

B1

$$E(X^2) = \frac{1 + 4 + 9 + 16 + 25}{5} = 11$$

B1

$$\text{Var}(X) = 11 - 9 = 2$$

B1

Answer: Mean 3, variance 2

3 marks

5 (2 marks)

$$E = 5(0.2) + (-1)(0.8)$$

M1

$$= £0.20$$

A1

Answer: £0.20

2 marks

6 (2 marks)

$$E(X) = \frac{1 \times 1 + 2 \times 2 + 3 \times 3 + 4 \times 4}{10}$$

M1

$$= \frac{30}{10} = 3$$

A1

Answer: 3

2 marks

7 (3 marks)

$$0.3 + a + k = 1 \text{ so } a + k = 0.7$$

B1

$$E(X) = 0.3 + 2a + 3k = 2.1 \text{ so } 2a + 3k = 1.8$$

M1

$$k = 0.4, a = 0.3$$

A1

Answer: a = 0.3, k = 0.4

3 marks

8 (4 marks)

$$E(X) = \frac{1 + 4 + 9 + 16}{10} = 3$$

M1 A1

$$E(X^2) = \frac{1 + 8 + 27 + 64}{10} = 10$$

M1

$$\text{Var}(X) = 10 - 9 = 1$$

A1

Answer: 1

4 marks

9 (3 marks)

$$E(X^2) = 0 + 0.4 + 1.6 = 2$$

M1 A1

$$E(3X^2 - 2) = 3 \times 2 - 2 = 4$$

A1

Answer: 4

3 marks

10 (4 marks)

$$\text{Cost } C = 4 + 2.5X$$

B1

$$E(C) = 4 + 2.5 \times 6.4 = 20$$

M1 A1

$$\text{SD}(C) = 2.5 \times 3 = 7.5$$

A1

Answer: Mean £20, SD £7.50

4 marks

11 (4 marks)

$$\text{Var}(Y) = a^2 \times 4 = 1, \text{ so } a = \frac{1}{2}$$

M1 A1

$$E(Y) = \frac{1}{2} \times 10 + b = 0$$

M1

$$b = -5$$

A1

Answer: a = $\frac{1}{2}$, b = -5

4 marks

12 (3 marks)

$$E(\text{receipt}) = 6 \times \frac{1}{6}$$

M1

$$= £1$$

A1

A fair game has expected gain 0 so the stake is £1

A1

Answer: £1

3 marks

13 (3 marks)

$$E(aX + b) = aE(X) + b$$

B1

$$\text{Var}(aX + b) = E[(aX + b - aE(X) - b)^2] = E[a^2(X - E(X))^2]$$

M1

$$= a^2 \text{Var}(X)$$

A1

$$\text{Answer: } \text{Var}(aX + b) = a^2 \text{Var}(X)$$

3 marks

14 (5 marks)

$$E(X) = \frac{1 \times 1 + 2 \times 3 + 3 \times 5 + 4 \times 7 + 5 \times 9 + 6 \times 11}{36}$$

M1

$$= \frac{161}{36}$$

A1

$$E(X^2) = \frac{1 + 12 + 45 + 112 + 225 + 396}{36} = \frac{791}{36}$$

M1 A1

$$\text{Var}(X) = \frac{791}{36} - \left(\frac{161}{36}\right)^2 = \frac{2555}{1296}$$

A1

$$\text{Answer: } E(X) = \frac{161}{36}, \text{Var}(X) = \frac{2555}{1296}$$

5 marks

15 (6 marks)

Let the probabilities be p, q, r with $p + q + r = 1$

B1

$$p + 2q + 3r = 2.1$$

M1

$$E(X^2) = 0.49 + 2.1^2 = 4.9, \text{ so } p + 4q + 9r = 4.9$$

M1 A1

Subtracting equations: $q + 2r = 1.1$ and $2q + 6r = 2.8$

M1

Solving: $r = 0.3, q = 0.5, p = 0.2$

A1

$$\text{Answer: } P(1) = 1/5, P(2) = 1/2, P(3) = 3/10$$

6 marks

16 (6 marks)

$$(a) k\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) = 1, \text{ so } k \times \frac{25}{12} = 1$$

M1

$$k = \frac{12}{25}$$

A1

$$\text{Answer: } k = \frac{12}{25}$$

2 marks

$$(b) E(X) = \sum_{x=1}^4 x \times \frac{k}{x} = 4k$$

M1

$$= \frac{48}{25}$$

A1

$$\text{Answer: } \frac{48}{25}$$

2 marks

$$(c) E\left(\frac{1}{X}\right) = k\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16}\right)$$

M1

$$= \frac{12}{25} \times \frac{205}{144} = \frac{41}{60}$$

A1

$$\text{Answer: } \frac{41}{60}$$

2 marks

17 (5 marks)

$$E(X) = \frac{1}{2} + \frac{1}{2} + \frac{3}{8} + \frac{1}{2} = \frac{15}{8}$$

M1 A1

$$E(X^2) = \frac{1}{2} + 1 + \frac{9}{8} + 2 = \frac{37}{8}$$

M1

$$\text{Var}(X) = \frac{37}{8} - \left(\frac{15}{8}\right)^2 = \frac{296 - 225}{64}$$

M1

$$= \frac{71}{64}$$

A1

$$\text{Answer: } E(X) = \frac{15}{8}, \text{Var}(X) = \frac{71}{64}$$

5 marks

18 (5 marks)

$$P(X = 1) = \frac{1}{6}, P(X = 2) = \frac{5}{6} \times \frac{1}{6} = \frac{5}{36}$$

M1 A1

$$P(X = 3) = \frac{5}{6} \times \frac{5}{6} = \frac{25}{36} \text{ (the third roll is made whatever happened before it is rolled)}$$

M1

$$E(X) = \frac{1}{6} + \frac{10}{36} + \frac{75}{36} = \frac{91}{36}$$

M1

$$\text{Expected cost} = \text{£}2.53$$

A1

$$\text{Answer: £}2.53 \text{ (= £} \frac{91}{36} \text{)}$$

5 marks

19 (5 marks)

$$E(\text{payout}) = 2000(0.02) + 500(0.08) = 40 + 40 = 80$$

M1 A1

$$\text{Expected profit} = 60 - 80 = -\text{£}20 \text{ (a loss of £}20\text{)}$$

A1

$$E(\text{payout}^2) = 4\,000\,000(0.02) + 250\,000(0.08) = 100\,000$$

M1

$$\text{Var} = 100\,000 - 6400 = 93\,600, \text{SD} = \text{£}306$$

A1

$$\text{Answer: Expected profit } -\text{£}20 \text{ (loss); SD £}306$$

5 marks

20 (4 marks)

$$\text{(a) } X = -3 \text{ gives } Y = 9, X = -1 \text{ gives } Y = 1, X = 2 \text{ gives } Y = 5, X = 4 \text{ gives } Y = 9$$

M1

$$P(Y < 9) = P(Y = 1) + P(Y = 5) = 0.25 + 0.25 = 0.5$$

A1

$$\text{Answer: } 0.5$$

2 marks

$$\text{(b) } E(Y) = 9(0.125) + 1(0.25) + 5(0.25) + 9(0.375)$$

M1

$$= 1.125 + 0.25 + 1.25 + 3.375 = 6$$

A1

$$\text{Answer: } 6$$

2 marks

21 (5 marks)

$$\text{Var}(X) = \sum_x (x - \mu)^2 P(X = x), \text{ where } \mu = E(X)$$

M1 A1

$$\text{Each term is a product of non-negative numbers, so } \text{Var}(X) \geq 0$$

E1

$$\text{Var}(X) = 0 \text{ requires every term to be } 0$$

M1

$$\text{So } P(X = x) = 0 \text{ unless } x = \mu, \text{ giving } P(X = \mu) = 1$$

A1

$$\text{Answer: } \text{Var}(X) \geq 0, \text{ with equality only when } X \text{ is constant}$$

5 marks

22 (7 marks)

$$P(W = 2^n) = \left(\frac{1}{2}\right)^n \text{ for } n = 1, \dots, 4; \text{ the remaining probability is } 1 - \frac{15}{16} = \frac{1}{16}, \text{ paying £}32$$

M1 A1

$$E(W) = \sum_{n=1}^4 \left(\frac{1}{2}\right)^n 2^n + \frac{1}{16} \times 32 = 4 + 2 = 6$$

M1 A1

$$\text{Expected net gain} = 6 - 5 = \text{£}1$$

A1

$$E(W^2) = \sum_{n=1}^4 \left(\frac{1}{2}\right)^n 4^n + \frac{1}{16} \times 1024 = 2 + 4 + 8 + 16 + 64 = 94$$

M1

$$\text{Var}(W) = 94 - 36 = 58, \text{ so SD} = \text{£}7.62$$

A1

$$\text{Answer: Expected net gain £}1; \text{SD £}7.62$$

7 marks

23 (5 marks)

$$\text{Var}(2X) = 2^2 \text{Var}(X)$$

M1

$$= 12, \text{ not } 6 \text{ (the constant is squared)}$$

A1

$$\text{For independent } X \text{ and } Y, \text{Var}(X - Y) = \text{Var}(X) + (-1)^2 \text{Var}(Y)$$

M1

$$= 3 + 2 = 5, \text{ not } 1 \text{ (variances add, never subtract)}$$

A1

$$\text{Both errors arise from treating variance as if it were linear}$$

E1

$$\text{Answer: } \text{Var}(2X) = 12 \text{ and } \text{Var}(X - Y) = 5$$

5 marks

24 (8 marks)**(a)** Both scores at most x : $(\frac{x}{n})^2$

M1 A1

$$P(X = x) = \frac{x^2 - (x-1)^2}{n^2} = \frac{2x-1}{n^2}$$

A1

$$\text{Answer: } P(X = x) = \frac{2x-1}{n^2}$$

3 marks

$$\text{(b)} E(X) = \sum_{x=1}^n \frac{x(2x-1)}{n^2} = \frac{1}{n^2} (2 \sum_{x=1}^n x^2 - \sum_{x=1}^n x)$$

M1

$$= \frac{1}{n^2} \left(\frac{n(n+1)(2n+1)}{3} - \frac{n(n+1)}{2} \right)$$

M1 A1

$$= \frac{(n+1)}{n} \left(\frac{2n+1}{3} - \frac{1}{2} \right) = \frac{(n+1)(4n-1)}{6n}$$

A1

$$\text{Answer: } E(X) = \frac{(n+1)(4n-1)}{6n}$$

4 marks

$$\text{(c)} \frac{7 \times 23}{36} = \frac{161}{36} \square$$

B1

$$\text{Answer: } \frac{161}{36}$$

1 mark

25 (6 marks)

$$P(X = x) = \frac{1}{n}; E(X) = \frac{1}{n} \times \frac{n(n+1)}{2} = \frac{n+1}{2}$$

M1 A1

$$E(X^2) = \frac{1}{n} \times \frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6}$$

M1 A1

$$\text{Var}(X) = \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4} = \frac{(n+1)(4n+2-3n-3)}{12} = \frac{n^2-1}{12}$$

A1

$$n = 20: \text{Var} = \frac{399}{12} = 33.25$$

A1

$$\text{Answer: } 33.25$$

6 marks

26 (6 marks)**(a)** Two red: 8; one of each: $4 - 3 = 1$; two green: -6

B1

$$P(RR) = \frac{3}{5} \times \frac{2}{4} = \frac{3}{10}, P(GG) = \frac{2}{5} \times \frac{1}{4} = \frac{1}{10}$$

M1

$$P(\text{one of each}) = 1 - \frac{3}{10} - \frac{1}{10} = \frac{3}{5}$$

A1

$$\text{Answer: } P(8) = \frac{3}{10}, P(1) = \frac{3}{5}, P(-6) = \frac{1}{10}$$

3 marks

$$\text{(b)} E(X) = 8(0.3) + 1(0.6) - 6(0.1) = 2.4$$

M1 A1

$$E(X^2) = 64(0.3) + 0.6 + 36(0.1) = 23.4, \text{Var}(X) = 23.4 - 5.76 = 17.64$$

A1

$$\text{Answer: } E(X) = 2.4, \text{Var}(X) = 17.64$$

3 marks

27 (8 marks)

$$\text{(a)} \sum_{k=1}^n P(X \geq k) = \sum_{k=1}^n \sum_{x=k}^n P(X = x)$$

M1 A1

Swap the order of summation: each $P(X = x)$ appears for $k = 1, \dots, x$, that is x times

M1

$$= \sum_{x=1}^n x P(X = x) = E(X)$$

A1

$$\text{Answer: } E(X) = \sum_{k=1}^n P(X \geq k)$$

4 marks

$$\text{(b)} P(X \geq k) = 1 - \left(\frac{k-1}{6}\right)^2$$

M1 A1

$$\sum_{k=1}^6 = 6 - \frac{0+1+4+9+16+25}{36} = 6 - \frac{55}{36}$$

M1

$$= \frac{161}{36}$$

A1

Answer: $E(X) = \frac{161}{36}$

4 marks

28 (8 marks)

(a) The first box gives a first type

B1

G_2 : boxes until a new type, success probability $\frac{2}{3}$

M1

G_3 : boxes until the last type, success probability $\frac{1}{3}$

A1

Answer: $G_2 \sim \text{Geo}(\frac{2}{3})$, $G_3 \sim \text{Geo}(\frac{1}{3})$

3 marks

(b) $E(G) = \frac{1}{p}$: $E(G_2) = \frac{3}{2}$, $E(G_3) = 3$

M1

$E(N) = 1 + \frac{3}{2} + 3 = \frac{11}{2}$

A1

Answer: $\frac{11}{2}$

2 marks

(c) $\text{Var}(G) = \frac{1-p}{p^2}$: $\text{Var}(G_2) = \frac{\frac{1}{3}}{\frac{4}{9}} = \frac{3}{4}$, $\text{Var}(G_3) = \frac{\frac{2}{3}}{\frac{1}{9}} = 6$

M1 A1

G_2 and G_3 independent, so $\text{Var}(N) = \frac{3}{4} + 6 = \frac{27}{4}$

A1

Answer: $\frac{27}{4}$

3 marks

29 (9 marks)

(a) $E = 3.5$

B1

Answer: 3.5

1 mark

(b) Rejecting gives an expected prize of 3.5 from the second roll

M1

Reject if the score is below 3.5, that is 1, 2 or 3

M1

Accept if the score is 4, 5 or 6 (above 3.5)

A1

Answer: Reject the first score if it is 1, 2 or 3

3 marks

(c) $E = \frac{1}{6}(4 + 5 + 6) + \frac{1}{2} \times 3.5 = 2.5 + 1.75 = 4.25$

M1 A1

$E(\text{prize}^2) = \frac{1}{6}(16 + 25 + 36) + \frac{1}{2}E(X^2)$

M1

$E(X^2) = \frac{91}{6}$, so $E(\text{prize}^2) = \frac{77}{6} + \frac{91}{12} = \frac{245}{12}$

A1

$\text{Var} = \frac{245}{12} - (\frac{17}{4})^2 = \frac{245}{12} - \frac{289}{16} = \frac{113}{48}$

A1

Answer: $E = 4.25$, $\text{Var} = \frac{113}{48}$

5 marks

30 (11 marks)

(a) $4! = 24$ arrangements; all four fixed: 1; exactly two fixed: ${}^4C_2 \times 1 = 6$; exactly one fixed: $4 \times$

M1 A1

$2 = 8$; none: $24 - 1 - 6 - 8 = 9$

Probabilities $\frac{9}{24}$, $\frac{8}{24}$, $\frac{6}{24}$, $\frac{1}{24}$

A1

If three positions are fixed the fourth must also be fixed, so $X = 3$ is impossible

E1

Answer: $P(X = 0) = \frac{3}{8}$, $P(X = 1) = \frac{1}{3}$, $P(X = 2) = \frac{1}{4}$, $P(X = 4) = \frac{1}{24}$, $P(X = 3) = 0$

4 marks

(b) $P(I_k = 1) = \frac{1}{n}$, so $E(I_k) = \frac{1}{n}$

M1 A1

$E(X) = n \times \frac{1}{n} = 1$

A1

Answer: $E(X) = 1$

3 marks

(c) $X(X - 1)$ counts ordered pairs of distinct fixed points: $E[X(X - 1)] = \sum_{j \neq k} P(I_j I_k = 1)$

M1

$P(I_j I_k = 1) = \frac{1}{n(n - 1)}$ and there are $n(n - 1)$ ordered pairs, so $E[X(X - 1)] = 1$

M1 A1

$$E(X^2) = 1 + 1 = 2, \text{Var}(X) = 2 - 1 = 1$$

$$\text{Answer: } E[X(X - 1)] = 1, \text{Var}(X) = 1$$

A1

4 marks

Linear Combinations of Random Variables

1 (1 mark)

$$2 \times 5 + 3 = 13$$

$$\text{Answer: } 13$$

B1

1 mark

2 (2 marks)

$$\text{Var}(X + Y) = 4 + 9 = 13$$

$$\text{Var}(X - Y) = 4 + 9 = 13$$

$$\text{Answer: } 13 \text{ and } 13$$

B1

B1

2 marks

3 (2 marks)

$$\text{Var}(3X) = 9 \times 2 = 18$$

$$\text{Var}(S) = 3 \times 2 = 6$$

$$\text{Answer: } 18 \text{ and } 6$$

B1

B1

2 marks

4 (2 marks)

$$\text{Mean } 10 + 6 = 16$$

$$\text{Variance } 4 + 9 = 13, \text{ so } X + Y \sim N(16, 13)$$

$$\text{Answer: } N(16, 13)$$

B1

B1

2 marks

5 (2 marks)

$$5 \times 4 - 2 \times 7$$

$$= 6$$

$$\text{Answer: } 6$$

M1

A1

2 marks

6 (3 marks)

$$E(X_1 + X_2) = 3.5 + 3.5 = 7$$

The scores are independent, so variances add

$$\text{Var} = \frac{35}{12} + \frac{35}{12} = \frac{35}{6}$$

$$\text{Answer: Mean } 7, \text{ variance } \frac{35}{6}$$

B1

M1

A1

3 marks

7 (4 marks)

$$X + Y \sim N(80, 25)$$

$$P(X + Y > 90) = P(Z > \frac{90 - 80}{5}) = P(Z > 2)$$

$$= 0.0228$$

$$\text{Answer: } 0.0228$$

M1 A1

M1

A1

4 marks

8 (4 marks)

$$\text{Let } D = X - Y; D \sim N(2, 9)$$

$$P(X > Y) = P(D > 0) = P(Z > -\frac{2}{3})$$

$$= 0.7475$$

$$\text{Answer: } 0.7475$$

M1 A1

M1

A1

4 marks

9 (4 marks)

$$\text{Total } T \sim N(4 \times 250, 4 \times 100) = N(1000, 400)$$

$$P(T > 1050) = P(Z > \frac{50}{20}) = P(Z > 2.5)$$

$$= 0.0062$$

$$\text{Answer: } 0.0062$$

M1 A1

M1

A1

4 marks

10 (4 marks)

$$E = 2 \times 6 - 3 \times 4 + 5 = 5$$

M1 A1

$$\text{Var} = 2^2 \times 2 + (-3)^2 \times 3$$

M1

$$= 8 + 27 = 35$$

A1

Answer: 5 and 35

4 marks

11 (3 marks)

$$X + Y \sim \text{Po}(7)$$

M1 A1

$$P(X + Y = 5) = \frac{e^{-7} \times 7^5}{5!} = 0.128$$

A1

Answer: 0.128

3 marks

12 (3 marks)

$$2X \sim N(20, 2^2 \times 4) = N(20, 16)$$

B1

$$X_1 + X_2 \sim N(20, 4 + 4) = N(20, 8)$$

B1

$2X$ doubles a single observation, so the variance is multiplied by 4, but $X_1 + X_2$ adds independent variations which partly cancel

E1

Answer: N(20, 16) and N(20, 8)

3 marks

13 (4 marks)

$$\bar{X} \sim N(20, \frac{25}{25}) = N(20, 1)$$

M1 A1

$$P(\bar{X} > 21) = P(Z > 1)$$

M1

$$= 0.1587$$

A1

Answer: 0.1587

4 marks

14 (5 marks)

$$\text{Total } T \sim N(8 \times 75, 8 \times 12^2) = N(600, 1152)$$

M1 A1

$$\sigma = \sqrt{1152} = 33.94$$

B1

$$P(T > 640) = P(Z > \frac{40}{33.94}) = P(Z > 1.1785)$$

M1

$$= 0.1193$$

A1

Answer: 0.1193

5 marks

15 (5 marks)

Let $D = B - (C_1 + C_2 + C_3)$, where B is the contents of the bottle

M1

$$E(D) = 455 - 450 = 5$$

B1

$$\text{Var}(D) = 4 + 3 \times 9 = 31$$

M1

$$P(D < 0) = P(Z < -\frac{5}{\sqrt{31}}) = P(Z < -0.898)$$

M1

$$= 0.1846$$

A1

Answer: 0.1846

5 marks

16 (5 marks)

$$X + Y \sim N(50, 25)$$

M1 A1

$$P(Z > \frac{k - 50}{5}) = 0.05$$

M1

$$\frac{k - 50}{5} = 1.6449$$

A1

$$k = 58.2$$

A1

Answer: 58.2

5 marks

17 (5 marks)

$$E(X) = 0.2 + 1.0 + 0.9 = 2.1$$

M1 A1

$$E(X^2) = 0.2 + 2.0 + 2.7 = 4.9$$

M1

$$\text{Var}(X) = 4.9 - 4.41 = 0.49, E(Y) = 3 \times 2.1 - 2 = 4.3$$

A1

$$\text{Var}(Y) = 9 \times 0.49 = 4.41$$

B1

Answer: E(Y) = 4.3, Var(Y) = 4.41

5 marks

18 (5 marks)

$$E(X) = \frac{1}{2}(1 + 2) = \frac{3}{2} \quad \text{B1}$$

$$E(Y) = 0 + \frac{1}{2} + \frac{2}{4} = 1 \quad \text{B1}$$

$$E(XY) = \sum_{x,y} xy P(X = x, Y = y), \text{ with } P(X = x, Y = y) = P(X = x)P(Y = y) \quad \text{M1}$$

$$= \frac{1}{2} \left[1 \times \left(0 + \frac{1}{2} + \frac{2}{4} \right) + 2 \times \left(0 + \frac{1}{2} + \frac{2}{4} \right) \right] \quad \text{M1}$$

$$= \frac{1}{2}(1 + 2) = \frac{3}{2} = E(X)E(Y) \quad \text{A1}$$

Answer: $E(X) = \frac{3}{2}$, $E(Y) = 1$, $E(XY) = \frac{3}{2}$ 5 marks

19 (5 marks)

Total rate 4 per hour, so in 2.5 hours the number of contacts $T \sim \text{Po}(10)$ M1 A1

$$P(T \geq 8) = 1 - P(T \leq 7) \quad \text{M1}$$

$$P(T \leq 7) = 0.2202 \quad \text{A1}$$

$$P(T \geq 8) = 0.7798 \quad \text{A1}$$

Answer: 0.7798 5 marks

20 (5 marks)

X is the number of successes in 10 independent trials with $p = 0.3$ and Y is the number in 15 independent trials with $p = 0.3$ M1

X + Y is the number of successes in 25 independent trials with the same p, so $X + Y \sim B(25, 0.3)$ A1 A1

X + Z combines trials with different success probabilities (0.3 and 0.6) M1

so X + Z is not binomial A1

Answer: $X + Y \sim B(25, 0.3)$; X + Z is not binomial 5 marks

21 (5 marks)

$$\text{Var}(aX + bY) = E[(aX + bY)^2] - [E(aX + bY)]^2 \quad \text{M1}$$

$$E[(aX + bY)^2] = a^2E(X^2) + 2abE(XY) + b^2E(Y^2) \quad \text{M1 A1}$$

Independence gives $E(XY) = E(X)E(Y)$, so the cross terms $2abE(X)E(Y)$ cancel with those from $[E(aX + bY)]^2$ M1

leaving $a^2\text{Var}(X) + b^2\text{Var}(Y)$ A1

Answer: $\text{Var}(aX + bY) = a^2\text{Var}(X) + b^2\text{Var}(Y)$ 5 marks

22 (9 marks)

$$\text{Total } T = A_1 + A_2 + C_1 + C_2 + C_3, E(T) = 160 + 120 = 280 \quad \text{M1 A1}$$

$$\text{Var}(T) = 2 \times 100 + 3 \times 36 = 308 \quad \text{M1 A1}$$

$$P(T > 300) = P\left(Z > \frac{20}{\sqrt{308}}\right) = P(Z > 1.1396) = 0.1272 \quad \text{M1 A1}$$

$$\text{Let } D = A - C_1 - C_2; E(D) = 80 - 80 = 0, \text{Var}(D) = 100 + 36 + 36 = 172 \quad \text{M1 A1}$$

$$P(D > 5) = P\left(Z > \frac{5}{\sqrt{172}}\right) = P(Z > 0.3812) = 0.3515 \quad \text{A1}$$

Answer: 0.1272; 0.3515 9 marks

23 (7 marks)

$$\begin{aligned} \text{(a) } E(T) &= a\theta + (1 - a)\theta \\ &= \theta \end{aligned} \quad \begin{array}{l} \text{M1} \\ \text{A1} \end{array}$$

Answer: $E(T) = \theta$ for every a 2 marks

$$\text{(b) } \text{Var}(T) = a^2\sigma^2 + (1 - a)^2(4\sigma^2) = \sigma^2(5a^2 - 8a + 4) \quad \text{M1 A1}$$

$$\frac{d}{da}: \sigma^2(10a - 8) = 0, a = \frac{4}{5} \quad \text{M1 A1}$$

$$\text{Var}(T) = \sigma^2\left(5 \times \frac{16}{25} - \frac{32}{5} + 4\right) = \frac{4}{5}\sigma^2 \quad \text{A1}$$

Answer: $a = \frac{4}{5}$; minimum variance $\frac{4}{5}\sigma^2$

5 marks

24 (6 marks)

(a) $\bar{X} \sim N(500, \frac{36}{10}) = N(500, 3.6)$

B1

$\bar{Y} \sim N(497, \frac{25}{12})$

B1

Answer: $\bar{X} \sim N(500, 3.6)$, $\bar{Y} \sim N(497, \frac{25}{12})$

2 marks

(b) $\bar{X} - \bar{Y} \sim N(3, 3.6 + \frac{25}{12}) = N(3, 5.6833)$

M1 A1

$P(\bar{X} - \bar{Y} > 5) = P(Z > \frac{2}{\sqrt{5.6833}}) = P(Z > 0.8389)$

M1

$= 0.201$

A1

Answer: 0.201

4 marks

25 (5 marks)

5X is five times one parcel, not the total of five different parcels

E1

Total $T = X_1 + \dots + X_5$ with $E(T) = 5\mu$ and $\text{Var}(T) = 5\sigma^2$

M1 A1

For 5X: $E(5X) = 5\mu$ but $\text{Var}(5X) = 25\sigma^2$

M1 A1

Answer: T: mean 5μ , variance $5\sigma^2$; 5X: mean 5μ , variance $25\sigma^2$

5 marks

26 (8 marks)

$\frac{12 - \mu}{\sigma} = -0.8416$ and $\frac{20 - \mu}{\sigma} = 1.2816$

M1 A1 A1

Subtract: $\frac{8}{\sigma} = 2.1232$, $\sigma = 3.77$

M1 A1

$\mu = 12 + 0.8416 \times 3.768 = 15.2$

A1

$\bar{X}_2 \sim N(15.17, \frac{3.768^2}{2})$ so $P(\bar{X}_2 > 17) = P(Z > \frac{17 - 15.17}{2.664})$

M1

$= 0.246$

A1

Answer: $\mu = 15.2$, $\sigma = 3.77$; 0.246

8 marks

27 (9 marks)

(a) $UV = X^2 - Y^2$

M1

$E(UV) = E(X^2) - E(Y^2) = \sigma_X^2 + \mu_X^2 - \sigma_Y^2 - \mu_Y^2$

M1 A1

$E(U)E(V) = (\mu_X + \mu_Y)(\mu_X - \mu_Y) = \mu_X^2 - \mu_Y^2$

M1

Difference: $\sigma_X^2 - \sigma_Y^2$

A1

Answer: $E(UV) - E(U)E(V) = \sigma_X^2 - \sigma_Y^2$

5 marks

(b) Uncorrelated when $\sigma_X^2 = \sigma_Y^2$

B1

Equal variances (3 and 3) so U and V are independent and the condition $V = 0$ is irrelevant

M1

$U \sim N(9, 6)$, $P(U > 10) = P(Z > \frac{1}{\sqrt{6}}) = P(Z > 0.408)$

M1

$= 0.342$

A1

Answer: 0.342

4 marks

28 (8 marks)

(a) Let $D = W - (T_1 + T_2 + T_3 + T_4)$

M1

$E(D) = 121 - 120 = 1$

B1

$\text{Var}(D) = 0.25 + 4 \times 0.09 = 0.61$

M1 A1

$P(D > 0) = P(Z > -\frac{1}{\sqrt{0.61}}) = P(Z < 1.280) = 0.8998$

A1

Answer: 0.8998

5 marks

(b) $P(\text{not fit}) = 1 - 0.8998 = 0.1002$

M1

$P(\text{at least one fails}) = 1 - (0.8998)^5$

M1

$= 0.4102$

A1

Answer: 0.4102**3 marks****29 (10 marks)**

(a) $E = 0.5 \times 12 + 0.8 \times 8 = 12.4$

M1 A1

$\text{Var} = 0.25 \times 12 + 0.64 \times 8 = 3 + 5.12$

M1

$= 8.12$

A1**Answer: Mean £12.40, variance 8.12****4 marks**

(b) $A + R \sim \text{Po}(20)$

M1 A1

$P(A + R > 25) = 1 - P(A + R \leq 25) = 0.1122$

A1**Answer: 0.1122****3 marks**

(c) $P(A = 12 | A + R = 20) = \frac{P(A = 12)P(R = 8)}{P(A + R = 20)}$, with $A + R \sim \text{Po}(20)$

M1

$$= \frac{e^{-12} 12^{12}}{12! \times e^{-8} 8^8 / 8! \times e^{-20} 20^{20} / 20!} = {}^{20}C_{12} \left(\frac{12}{20}\right)^{12} \left(\frac{8}{20}\right)^8 \text{ (the factors } e^{-20} \text{ cancel)}$$

M1

$$= {}^{20}C_{12} (0.6)^{12} (0.4)^8 = 0.1797$$

A1**Answer: 0.1797****3 marks****30 (10 marks)**

(a) $E(\bar{X}) = \frac{1}{n}(\mu + \dots + \mu) = \mu$

M1 A1

$\text{Var}(\bar{X}) = \frac{1}{n^2} \text{Var}(X_1 + \dots + X_n)$

M1

$$= \frac{1}{n^2} \times n\sigma^2 = \frac{\sigma^2}{n}$$

A1

Answer: $E(\bar{X}) = \mu$, $\text{Var}(\bar{X}) = \frac{\sigma^2}{n}$

4 marks

(b) $\bar{X} - \mu \sim N(0, \frac{9}{n})$

B1

$P(|\bar{X} - \mu| < 0.5) = P(|Z| < \frac{0.5\sqrt{n}}{3}) > 0.95$

M1 A1

$$\frac{0.5\sqrt{n}}{3} > 1.96$$

M1

$$\sqrt{n} > 11.76, n > 138.3$$

A1

$$\text{Smallest } n = 139$$

A1**Answer: $n = 139$** **6 marks**