

## Permutations and Combinations

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|-----------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----|
| <b>1</b>  | Evaluate                                                                                                                                                                                                 |     |
|           | (a) ${}^7P_3$ ,                                                                                                                                                                                          | [1] |
|           | (b) ${}^7C_3$ .                                                                                                                                                                                          | [1] |
| <hr/>     |                                                                                                                                                                                                          |     |
| <b>2</b>  | Five different books are placed in a row on a shelf. In how many different orders can this be done?                                                                                                      | [1] |
| <hr/>     |                                                                                                                                                                                                          |     |
| <b>3</b>  | A committee of 3 people is chosen from a group of 10 people. How many different committees are possible?                                                                                                 | [2] |
| <hr/>     |                                                                                                                                                                                                          |     |
| <b>4</b>  | Find the number of different arrangements of all the letters of the word TRIANGLE.                                                                                                                       | [1] |
| <hr/>     |                                                                                                                                                                                                          |     |
| <b>5</b>  | Find the number of different arrangements of all the letters of the word LEVEL.                                                                                                                          | [2] |
| <hr/>     |                                                                                                                                                                                                          |     |
| <b>6</b>  | A pizza is made with exactly 2 different toppings chosen from a list of 6 toppings. How many different pizzas are possible?                                                                              | [2] |
| <hr/>     |                                                                                                                                                                                                          |     |
| <b>7</b>  | Find the number of different arrangements of all the letters of the word BANANA.                                                                                                                         | [2] |
| <hr/>     |                                                                                                                                                                                                          |     |
| <b>8</b>  | A team of 4 is chosen from 4 men and 3 women. Find the number of teams that contain exactly 2 men and 2 women.                                                                                           | [3] |
| <hr/>     |                                                                                                                                                                                                          |     |
| <b>9</b>  | Six people sit at a round table. Two seatings are the same if each person has the same neighbours in the same positions (rotations are not counted as different). Find the number of different seatings. | [2] |
| <hr/>     |                                                                                                                                                                                                          |     |
| <b>10</b> | Four-digit numbers are formed using four different digits chosen from 1, 2, 3, 4, 5, 6, 7, 8, 9. Find how many of these numbers are odd.                                                                 | [3] |
| <hr/>     |                                                                                                                                                                                                          |     |
| <b>11</b> | The letters of the word ELEVEN are arranged in a line. Find the number of arrangements that begin and end with the same letter.                                                                          | [3] |
| <hr/>     |                                                                                                                                                                                                          |     |
| <b>12</b> | Find the number of distinct arrangements of all the letters of the word STATISTICS.                                                                                                                      | [3] |
| <hr/>     |                                                                                                                                                                                                          |     |
| <b>13</b> | Given that ${}^nC_2 = 45$ , find the value of $n$ .                                                                                                                                                      | [3] |
| <hr/>     |                                                                                                                                                                                                          |     |
| <b>14</b> | The eleven letters of the word PROBABILITY are arranged in a random order. Find the probability that the two Bs are next to each other.                                                                  | [4] |
| <hr/>     |                                                                                                                                                                                                          |     |
| <b>15</b> | Consider all the arrangements of the nine letters of the word COMMITTEE.                                                                                                                                 |     |
|           | (a) Find the total number of different arrangements.                                                                                                                                                     | [2] |
|           | (b) Find the number of these arrangements in which the two Ms are next to each other.                                                                                                                    | [2] |
| <hr/>     |                                                                                                                                                                                                          |     |
| <b>16</b> | A panel of 5 people is to be chosen from 6 men and 5 women. The panel must contain at least 2 women. Find the number of possible panels.                                                                 | [4] |
| <hr/>     |                                                                                                                                                                                                          |     |
| <b>17</b> | Four-digit numbers are formed using four different digits chosen from 1, 2, 3, 4, 5, 6, 7. How many such numbers are even and at least 3000?                                                             | [5] |
| <hr/>     |                                                                                                                                                                                                          |     |
| <b>18</b> | Eight people stand in a line. Two of them, Asha and Ben, refuse to stand next to each other. Find the number of possible arrangements.                                                                   | [4] |
| <hr/>     |                                                                                                                                                                                                          |     |
| <b>19</b> | Seven people, including Cara and Dev, sit at a round table. Find the number of seatings in which Cara and Dev sit next to each other (rotations of a seating are not counted as different).              | [4] |

- 20** Ten students are to be split into a group of 6 and a group of 4.  
 (a) Find the number of ways this can be done. [2]  
 (b) Of the 10 students, 4 are boys. Find the number of ways in which the group of 4 contains exactly 3 boys. [3]
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- 21** Prove that  $r \times {}^nC_r = n \times {}^{n-1}C_{r-1}$  for  $1 \leq r \leq n$ . [4]
- 
- 22** All the four-digit numbers that can be made using each of the digits 1, 2, 3, 4 exactly once are written down. Find the sum of all these numbers. [6]
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- 23** Find the number of arrangements of the letters of the word MISSISSIPPI in which no two Is are next to each other. [6]
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- 24** A committee of 5 is chosen from 7 men and 6 women. It must contain at least one man. Two of the people, Mr Ali (a man) and Ms Bell (a woman), will not serve together. Find the number of possible committees. [6]
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- 25** Ten identical sweets are shared among four children so that each child receives at least one sweet. By considering the gaps between sweets laid in a row, or otherwise, find the number of ways this can be done. [5]
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- 26** A student says: "The number of arrangements of the letters of the word BOOK is  $4! = 24$ ." Explain the error in the student's working, and find the correct number of arrangements. [4]
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- 27** A set has  $n$  elements.  
 (a) Using the binomial theorem for  $(1 + x)^n$ , or otherwise, prove that  ${}^nC_0 + {}^nC_1 + \dots + {}^nC_n = 2^n$ . [3]  
 (b) By also putting  $x = -1$ , show that the number of subsets with an even number of elements is  $2^{n-1}$  for  $n \geq 1$ . Hence find the number of such subsets when  $n = 10$ . [5]
- 
- 28** Six letters are selected from the seven letters of the word BALLOON and arranged in a row. Find the number of different six-letter arrangements that can be made. [9]
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- 29** A six-digit number is called ascending if each digit is at least as large as the digit before it, for example 112359. No digit is zero. Find the number of ascending six-digit numbers, explaining your method. [8]
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- 30** Five letters, each addressed to a different person, are put at random into five envelopes, one letter in each envelope, each envelope addressed to one of the five people. Use the inclusion-exclusion principle to find the number of ways in which no letter is placed in its correct envelope. Hence find the probability that no letter is correctly placed. [10]
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## Conditional Probability

- 1** Events A and B are such that  $P(A) = 0.6$ ,  $P(B) = 0.5$  and  $P(A \cap B) = 0.3$ . Find  $P(A | B)$ . [2]
- 
- 2** A fair six-sided die is rolled. Given that the score is greater than 3, find the probability that it is even. [2]
- 
- 3**  $P(A) = 0.5$  and  $P(A \cap B) = 0.2$ . Find  $P(B | A)$ . [2]
- 
- 4** A and B are independent events with  $P(A) = 0.4$  and  $P(B) = 0.3$ .  
 (a) Find  $P(A \cap B)$ . [1]  
 (b) Write down  $P(A | B)$ . [1]
-

- 5 The table shows how 40 students travel to school.

|       | Cycle | Walk |
|-------|-------|------|
| Boys  | 14    | 6    |
| Girls | 9     | 11   |

- (a) One student is chosen at random. Find the probability that the student cycles. [1]  
 (b) Given that the student chosen is a girl, find the probability that she walks. [1]

- 6 A and B are independent events with  $P(A) = 0.3$  and  $P(B) = 0.5$ . Find  $P(A \cup B)$ . [2]

- 7 For events A and B,  $P(A) = 0.5$ ,  $P(B) = 0.4$  and  $P(A \cup B) = 0.7$ .  
 (a) Find  $P(A \cap B)$ . [2]  
 (b) Find  $P(A | B)$  and  $P(B | A)$ . [2]

- 8 A bag contains 5 red and 7 blue counters. Two counters are drawn at random, one at a time, without replacement.  
 (a) Find the probability that the second counter is red given that the first is red. [1]  
 (b) Find the probability that both counters are red. [2]

- 9 Of the 80 members of a club, 36 play tennis, 28 play squash and 15 play both sports. A member is chosen at random.  
 (a) Find the probability that the member plays neither sport. [1]  
 (b) Find the probability that the member plays exactly one of the two sports. [2]  
 (c) Given that the member plays at least one of the sports, find the probability that the member plays both. [2]

- 10  $P(A | B) = 0.4$ ,  $P(B) = 0.5$  and  $P(A) = 0.35$ . Find  
 (a)  $P(A \cap B)$ , [1]  
 (b)  $P(B | A)$ , [2]  
 (c)  $P(A \cup B)$ . [1]

- 11 100 candidates sat a test. The table shows the numbers who studied beforehand and the results.

|               | Pass | Fail |
|---------------|------|------|
| Studied       | 42   | 8    |
| Did not study | 18   | 32   |

- (a) Find the probability that a randomly chosen candidate passed. [1]  
 (b) Find the probability that a candidate who studied passed. [1]  
 (c) Find the probability that a candidate who failed did not study. [2]

- 12 Show that, for any events A and B with  $P(B) > 0$ ,  $P(A' | B) = 1 - P(A | B)$ . [3]

- 13 Two fair dice are rolled. Given that at least one of them shows a 6, find the probability that the total score is at least 10. [3]

- 14  $P(A) = 0.4$  and  $P(B) = 0.3$ , and A and B are mutually exclusive.  
 (a) Find  $P(A | B)$ . [1]  
 (b) Explain why A and B are not independent. [2]

- 15 Two fair dice are rolled. Given that the product of the scores is even, find the probability that the sum of the scores is odd. [4]

- 16 A and B are independent events with  $P(A) = 2P(B)$  and  $P(A \cup B) = 0.52$ . Find  $P(A)$  and  $P(B)$ . [5]

- 17** The events A and B are exhaustive, with  $P(A) = 0.72$  and  $P(B) = 0.56$ .  
 (a) Find  $P(A \cap B)$ . [2]  
 (b) Find  $P(A | B)$ . [1]  
 (c) Determine whether A and B are independent. [2]

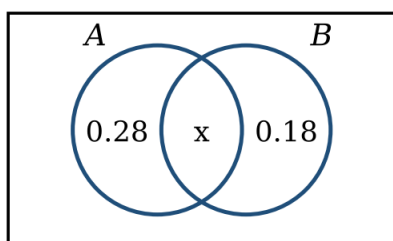
- 18** Five cards are dealt from a standard pack of 52 cards. Given that all five are red, find the probability that all five are hearts. Give your answer as a fraction in its lowest terms. [4]

- 19** The probability that Ravi catches a bus to work is 0.7. If he catches the bus he is late with probability 0.1; if he does not catch the bus he is late with probability 0.45. Given that Ravi is not late, find the probability that he caught the bus. [5]

- 20** For events A and B:  $P(A \cap B') = 0.3$ ,  $P(A' \cap B) = 0.2$  and  $P(A' \cap B') = 0.1$ .  
 (a) Find  $P(A | B)$ . [3]  
 (b) Determine whether A and B are independent. [2]

- 21** Events A and B are independent. Prove that  $A'$  and  $B'$  are also independent. [4]

- 22** The Venn diagram shows the probabilities of three of the four regions for events A and B, where x is the probability of  $A \cap B$ . The events A and B are independent.



- (a) Show that  $x^2 - 0.54x + 0.0504 = 0$ . [3]  
 (b) Hence find the two possible values of x, and the probability that neither A nor B occurs in each case. [3]

- 23** For events A and B with  $0 < P(A) < 1$  and  $0 < P(B) < 1$ , prove that  $P(A | B) > P(A)$  if and only if  $P(B | A) > P(B)$ . [5]

- 24** Two events satisfy  $P(A) = 0.5$ ,  $P(B) = 0.4$  and  $P(A \cup B) = 0.7$ . Hannah says, " $P(A \cap B) = 0.2 = 0.5 \times 0.4$ , and so A and B must be mutually exclusive."  
 Identify the error in Hannah's statement, and state, with a reason, the correct relationship between A and B. [4]

- 25** For events A, B and C:  $P(A) = 0.5$ ,  $P(B) = 0.4$ ,  $P(C) = 0.3$ ,  $P(A \cap B) = 0.2$ ,  $P(A \cap C) = 0.15$ ,  $P(B \cap C) = 0.1$  and  $P(A \cap B \cap C) = 0.05$ .  
 (a) Find  $P(A \cup B \cup C)$ . [2]  
 (b) Find  $P(A | B \cap C)$ . [2]  
 (c) Find  $P(A \cap B | C')$ . [3]

- 26** Bag X contains 3 red and 5 green balls. Bag Y contains 4 red and 2 green balls. A ball is moved at random from X to Y, and then a ball is taken at random from Y.  
 (a) Find the probability that the ball taken from Y is red. [3]  
 (b) Given that the ball taken from Y is red, find the probability that the ball moved from X was green. [3]

- 27** In a game show there are three closed doors. A prize is behind one door, chosen at random. A contestant picks a door. The host, who knows where the prize is, then opens one of the other two doors that does not hide the prize (choosing at random if there is a choice) and offers the contestant the chance to switch to the remaining closed door.  
Find the probability that the contestant wins the prize if they always switch, justifying your answer carefully. Hence state whether switching is better than staying. [8]
- 28** A family has two children. Assume that each child is equally likely to be a boy or a girl, independently of the other.
- (a) Given that at least one of the children is a boy, find the probability that both are boys. [3]
  - (b) Given that the older child is a boy, find the probability that both are boys. [2]
  - (c) Explain briefly why the answers to parts (a) and (b) differ. [2]
  - (d) Each child is also equally likely to be born on any of the 7 days of the week. Given that at least one child is a boy born on a Tuesday, find the probability that both children are boys. [5]
- 29** An urn contains 2 red balls and 3 blue balls. A ball is drawn at random, its colour is noted, and it is returned to the urn together with one extra ball of the same colour. A second ball is then drawn at random.
- (a) Show that the probability that the second ball is red is  $\frac{2}{5}$ . [3]
  - (b) Given that the second ball is red, find the probability that the first ball was red. [3]
  - (c) Given that the first two balls drawn were both red, find the probability that a third ball drawn from the urn is red. [4]
- 30** Anna and Ben take turns to roll a fair die, with Anna rolling first. The first person to roll a six wins the game.
- (a) Show that the probability that Anna wins is  $\frac{6}{11}$ . [4]
  - (b) Given that Anna wins, find the probability that the game ends on the third roll of the die. [4]

## Bayes' Theorem

- 1**  $P(A) = 0.3$ ,  $P(B | A) = 0.8$  and  $P(B | A') = 0.2$ . Find  $P(B)$ . [2]
- 2** A factory has two machines. Machine A makes 60% of the items and 2% of its items are faulty. Machine B makes the rest and 5% of its items are faulty. An item is chosen at random. Find the probability that it is faulty. [2]
- 3**  $P(A) = 0.4$ ,  $P(B | A) = 0.5$  and  $P(B) = 0.3$ . Use Bayes' theorem to find  $P(A | B)$ . [2]
- 4** 20% of emails received are spam. A filter flags 90% of spam emails and 5% of emails that are not spam. Find the probability that a randomly chosen email is flagged. [2]
- 5** A box contains 3 fair coins and 1 coin with a head on both sides. A coin is chosen at random from the box and tossed once. Find the probability that it shows a head. [2]
- 6** A bag holds 3 red and 2 blue counters. One counter is drawn at random. If it is red, a die is rolled; if it is blue, a fair coin is tossed. Find the probability that the outcome is a six or a head, as appropriate (the die shows 6, or the coin shows a head). [2]
- 7** A factory has two machines. Machine A makes 60% of the items and 2% of its items are faulty. Machine B makes the rest and 5% of its items are faulty. An item is chosen at random and is found to be faulty. Find the probability that it was made by Machine A. [3]

- 8** 1% of a population have a disease. A test gives a positive result for 95% of people with the disease and for 4% of people without the disease. A person is chosen at random and tests positive. Find the probability that the person has the disease, correct to 3 significant figures. [4]
- 9** Three suppliers X, Y and Z provide 50%, 30% and 20% of the components used by a company. The proportions of components that are defective are 1% for X, 2% for Y and 4% for Z.  
(a) Find the probability that a component chosen at random is defective. [2]  
(b) Given that a component is defective, find the probability that it came from Z. [3]
- 10** Bag 1 contains 3 red and 2 blue balls; Bag 2 contains 2 red and 5 blue balls. A bag is chosen at random and a ball is drawn from it. The ball is red. Find the probability that it came from Bag 1. [3]
- 11** Prove Bayes' theorem in the form  $P(A | B) = \frac{P(B | A)P(A)}{P(B | A)P(A) + P(B | A')P(A')}$ , where  $P(B) > 0$ . [3]
- 12** Priya travels to work by train with probability 0.6, otherwise by bus. She is late 10% of the time when she takes the train and 30% of the time when she takes the bus. On one day Priya is late. Find the probability that she took the train. [4]
- 13** A student says, " $P(A | B) = 0.9$ , so  $P(B | A)$  must also be 0.9." Explain why this is wrong, and show that  $P(B | A) = \frac{P(A | B)P(B)}{P(A)}$ . [3]
- 14** 1% of a population have a disease. A test is positive for 95% of people with the disease and for 4% of people without it. A person is chosen at random and tested twice, the two results being independent given the person's disease status. Both tests are positive. Find the probability that the person has the disease, to 3 significant figures. [5]
- 15** A bag contains two coins. One coin is fair and the other has a head on both sides. A coin is chosen at random and tossed three times, giving three heads. Find the probability that the coin chosen is the two-headed coin. [4]
- 16** One card is removed at random from a standard pack of 52 cards and is not looked at. A second card is then drawn at random from the remaining 51 cards and is found to be a heart. Find the probability that the removed card was a heart. [5]
- 17** A test for a condition gives a positive result with probability 0.9 for a person with the condition and with probability 0.1 for a person without it. A proportion  $p$  of a population has the condition. A person from the population tests positive. Given that the probability that the person has the condition is exactly  $\frac{1}{2}$ , find  $p$ . [5]
- 18** The probability that it rains on Monday is 0.4. After a rainy day, the probability of rain the next day is 0.7; after a dry day it is 0.2. It rains on Wednesday. Find the probability that it also rained on Monday, as a fraction. [6]
- 19** A fair coin is tossed. If it shows a head, two counters are taken without replacement from Bag A, which holds 3 red and 2 white counters; otherwise two counters are taken without replacement from Bag B, which holds 2 red and 3 white counters. Both counters taken are red. Find the probability that they came from Bag A. [4]
- 20** 80% of the batches delivered to a shop are good, with each item independently defective with probability 0.02; the other 20% of batches are poor, with each item independently defective with probability 0.1. A random sample of 5 items from one batch contains exactly one defective item. Find the probability that the batch is a good batch, to 3 significant figures. [5]

- 21** 1% of a population have a disease. A test is positive for 95% of people with the disease and for 4% of people without it, and repeated tests are independent given the person's disease status. A person chosen at random has  $n$  positive results in  $n$  tests. Find the least value of  $n$  for which the probability that the person has the disease exceeds 0.99. [7]
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- 22** A crime was committed by one person from a town of 20 000 people, and each person is equally likely to be the culprit. DNA found at the scene matches the culprit with certainty, and matches a particular innocent person with probability  $\frac{1}{1000}$ . A person selected at random from the town is found to match the DNA.
- (a) Show that the probability that this person is the culprit is approximately 0.048. [4]
- (b) A juror says, "There is only a 1 in 1000 chance that an innocent person would match, so there is a 99.9% chance that this person is guilty." Explain the error in this reasoning. [2]
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- 23** Three urns contain red and blue balls: Urn 1 has 2 red and 8 blue, Urn 2 has 5 red and 5 blue, and Urn 3 has 8 red and 2 blue. An urn is chosen at random and two balls are drawn from it with replacement. Both balls are red.
- (a) Find the probability that the urn chosen was Urn 3. [4]
- (b) A third ball is drawn from the same urn, again with replacement. Find the probability that it is red. [3]
- 
- 24** For events  $A$  and  $B$  with  $0 < P(A) < 1$  and  $P(B | A') > 0$ , prove that  $\frac{P(A | B)}{P(A' | B)} = \frac{P(A)}{P(A')} \times \frac{P(B | A)}{P(B | A')}$ . A test result is 8 times as likely when the condition is present as when it is absent. Before testing, the odds of the condition being present are 1 : 49 (that is,  $P(A)/P(A') = \frac{1}{49}$ ). Find the probability that the condition is present after the test result. [6]
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- 25** In a city, 85% of taxis are green and 15% are blue. A witness to a hit-and-run accident says the taxi was blue. Tests show the witness identifies the colour of a taxi correctly 80% of the time (whichever colour it is).
- (a) Find the probability that the taxi really was blue. [3]
- (b) A second independent witness, with the same reliability, also says the taxi was blue. Find the new probability that the taxi was blue. [3]
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- 26** A student reads that a screening test is "95% accurate", meaning it gives the correct result for 95% of people with the condition and 95% of people without it. The condition affects 1 person in 200. The student tests positive and concludes, "There is a 95% chance that I have the condition." Explain why the student is wrong, and find the correct probability to 3 significant figures. [5]
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- 27** A box contains three coins: a fair coin, a coin with a head on both sides, and a biased coin for which  $P(\text{head}) = \frac{1}{3}$ . A coin is chosen at random and tossed three times, giving head, head, tail in that order.
- (a) Explain why the two-headed coin cannot have been chosen. [1]
- (b) Find the probability that the coin chosen was the fair coin. [4]
- (c) The same coin is tossed a fourth time. Find the probability that it shows a head. [4]
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- 28** Three prisoners A, B and C are told that exactly one of them, chosen at random, is to be pardoned. A asks the warden, who knows the outcome, to name one of the other two prisoners who will not be pardoned. If both B and C will not be pardoned, the warden names one of them at random. The warden says, "B will not be pardoned." Find the probability that A is to be pardoned, given what the warden says. Explain carefully why this is different from the  $\frac{1}{2}$  that many people expect. [8]

- 29** A test for a condition has sensitivity  $s$  (the probability of a positive result for a person with the condition) and specificity  $t$  (the probability of a negative result for a person without it). The condition has prevalence  $p$ , where  $0 < p < 1$ .
- (a) Show that the probability that a person who tests positive has the condition is  $\frac{sp}{sp + (1 - t)(1 - p)}$ . [3]
- (b) Given that  $s = 0.98$  and  $t = 0.95$ , find the smallest prevalence  $p$ , correct to 3 significant figures, for which a positive result means the person is more likely than not to have the condition. [4]
- (c) Hence explain why a very accurate test can still produce mostly false positives when used on a rare condition. [2]
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- 30** A fair six-sided die is rolled once. If it shows  $n$ , then a fair coin is tossed  $n$  times. It is observed that exactly 2 heads occur.
- (a) Show that the probability that exactly 2 heads occur is  $\frac{33}{128}$ . [4]
- (b) Given that exactly 2 heads occurred, find the probability that the die showed 4. [3]
- (c) Find the value or values of  $n$  that are most likely, given that exactly 2 heads occurred, justifying your answer. [3]

## Probability Generating Functions

- 1** The discrete random variable  $X$  takes the values 0, 1 and 2 with probabilities 0.2, 0.5 and 0.3 respectively. Write down the probability generating function of  $X$ . [1]
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- 2** The random variable  $X$  has probability generating function  $G(t) = 0.1 + 0.4t + 0.5t^2$ .
- (a) Write down  $P(X = 1)$ . [1]
- (b) Find  $E(X)$ . [2]
- 
- 3** The random variable  $X$  has probability generating function  $G(t) = k(2 + 3t + t^2)$ . Find the value of  $k$ . [2]
- 
- 4** The random variable  $X$  has probability generating function  $G(t) = (\frac{1}{4} + \frac{3}{4}t)^3$ . Find  $E(X)$ . [2]
- 
- 5** The random variable  $X$  is the score when a fair six-sided die is rolled.
- (a) Write down the probability generating function of  $X$ . [1]
- (b) Use  $G'(1)$  to show that  $E(X) = 3.5$ . [2]
- 
- 6** The random variable  $X$  has probability generating function  $G(t) = \frac{t}{2 - t}$ . By expanding  $G(t)$  as a series in  $t$ , find  $P(X = 1)$  and  $P(X = 2)$ . [2]
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- 7** The random variable  $X$  has probability generating function  $G(t) = \frac{1 + 2t + 3t^2}{6}$ . Find  $E(X)$  and  $\text{Var}(X)$ . [4]
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- 8** The random variable  $X$  has probability generating function  $G(t) = k(1 + t)^3$ .
- (a) Find the value of  $k$ . [1]
- (b) Find  $P(X = 0)$  and  $P(X = 2)$ . [2]
- 
- 9**  $X$  and  $Y$  are independent random variables with probability generating functions  $G_X(t) = \frac{1}{2} + \frac{1}{2}t$  and  $G_Y(t) = \frac{1}{3} + \frac{2}{3}t$ . Find the probability generating function of  $X + Y$ , and hence  $P(X + Y = 1)$ . [3]

- 10** The random variable  $X$  has probability generating function  $G(t) = \frac{t}{3 - 2t}$ .
- (a) Find  $P(X = 3)$ . [2]  
(b) Find  $E(X)$ . [2]
- 
- 11** The random variable  $X \sim \text{Po}(\lambda)$  has probability generating function  $G(t) = e^{\lambda(t-1)}$ . Use  $G(t)$  to show that  $E(X) = \text{Var}(X) = \lambda$ . [4]
- 
- 12**  $X$  takes the values 1, 2, 3 and 4 with equal probability.
- (a) Write down the probability generating function of  $X$ . [1]  
(b) Use differentiation to find  $E(X)$  and  $\text{Var}(X)$ . [3]
- 
- 13** The random variable  $X$  has probability generating function  $G(t)$ . Show that  $\text{Var}(X) = G''(1) + G'(1) - [G'(1)]^2$ . [3]
- 
- 14** Each of three independent trials is a success with probability 0.4. By writing down the probability generating function of the number of successes in one trial, and using it to find the probability generating function of the total number of successes in the three trials, find the probability of exactly two successes. [4]
- 
- 15** The random variable  $X$  takes the values 0, 1 and 2 only.  $E(X) = 1.2$  and  $\text{Var}(X) = 0.56$ . Find the probability generating function of  $X$ . [5]
- 
- 16** The random variable  $X$  has probability generating function  $G(t) = \frac{t}{4 - 3t}$ . Use differentiation to find  $E(X)$  and  $\text{Var}(X)$ . [5]
- 
- 17**  $X$  and  $Y$  are independent random variables, each with probability generating function  $G(t) = \frac{t}{2 - t}$ . Find the probability generating function of  $X + Y$ , and use it to find  $P(X + Y = 3)$ . [5]
- 
- 18** The random variable  $X$  has probability generating function  $G_X(t) = \frac{1+t}{2}$ . The random variable  $Y$  is defined by  $Y = 2X + 3$ .
- (a) Show that the probability generating function of  $Y$  is  $G_Y(t) = \frac{t^3(1+t^2)}{2}$ . [2]  
(b) Hence find  $E(Y)$ . [2]
- 
- 19** The discrete random variable  $X$  has probability generating function  $G(t) = \frac{(a + bt^2)^2}{16}$ , where  $a$  and  $b$  are positive constants. It is given that  $P(X = 4) = \frac{9}{16}$ .
- (a) Find the values of  $a$  and  $b$ . [3]  
(b) Find  $P(X = 2)$  and  $E(X)$ . [3]
- 
- 20** The random variable  $X$  has probability generating function  $G(t) = \frac{(2+t)^3}{27}$ .
- (a) Show that  $E(X) = 1$ . [2]  
(b) Find  $\text{Var}(X)$ . [3]
- 
- 21** The independent discrete random variables  $X$  and  $Y$  take non-negative integer values and have probability generating functions  $G_X(t)$  and  $G_Y(t)$ . Prove that the probability generating function of  $X + Y$  is  $G_X(t)G_Y(t)$ . [4]
-

- 22** The random variable  $X$  takes the values 0, 1 and 2 with equal probability.  $X_1$ ,  $X_2$  and  $X_3$  are three independent observations of  $X$ .
- (a) Write down the probability generating function of  $X$ , and hence the probability generating function of  $S = X_1 + X_2 + X_3$ . [2]
- (b) Find  $P(S = 4)$ . [3]
- (c) Use  $G_S(t)$  to find  $E(S)$  and  $\text{Var}(S)$ . [3]
- 
- 23** The random variable  $X$  has probability generating function  $G(t) = \frac{pt}{1 - (1 - p)t}$ , where  $0 < p < 1$ .
- (a) Show that  $E(X) = \frac{1}{p}$  and  $\text{Var}(X) = \frac{1 - p}{p^2}$ . [4]
- (b) Given that  $\text{Var}(X) = 12$ , find  $p$ . [2]
- 
- 24** Dev says, "If  $X$  and  $Y$  are independent, then  $G_{X+Y}(t) = G_X(t) + G_Y(t)$ ." Show that Dev is wrong by considering the case in which  $X$  and  $Y$  both have probability generating function  $G(t) = \frac{1}{2} + \frac{1}{2}t$ , and state the correct result. [5]
- 
- 25** The random variable  $X \sim B(n, p)$ . Use the probability generating function of  $X$  to prove that  $E(X) = np$  and  $\text{Var}(X) = np(1 - p)$ . [6]
- 
- 26** The discrete random variable  $X$  has probability generating function  $G(t) = k \ln\left(\frac{2}{2 - t}\right)$ , where  $k$  is a constant.
- (a) Find the exact value of  $k$ . [1]
- (b) Find the exact value of  $E(X)$  and show that  $\text{Var}(X) = k(2 - k)$ . [6]
- 
- 27**  $X \sim \text{Po}(\lambda)$  and  $Y \sim \text{Po}(\mu)$  are independent random variables.
- (a) Using probability generating functions, show that  $X + Y \sim \text{Po}(\lambda + \mu)$ . [3]
- (b) Show that  $P(X = r \mid X + Y = n) = {}^nC_r \left(\frac{\lambda}{\lambda + \mu}\right)^r \left(\frac{\mu}{\lambda + \mu}\right)^{n-r}$ , and name the distribution of  $X$  given  $X + Y = n$ . [6]
- 
- 28** The number  $N$  of customers entering a shop in an hour has a Poisson distribution with mean 2. Each customer independently buys something with probability  $\frac{1}{2}$ . Let  $Y$  be the number of customers in the hour who buy something. Given that  $G_Y(t) = G_N(g(t))$ , where  $g(t)$  is the probability generating function of the indicator for one customer buying something, find the probability generating function of  $Y$ . Hence identify the distribution of  $Y$  and find  $P(Y = 0)$  to 3 significant figures. [8]
- 
- 29** The random variable  $X$  has a geometric distribution with  $P(X = r) = (1 - p)^{r-1}p$  for  $r = 1, 2, 3, \dots$
- (a) Show that the probability generating function of  $X$  is  $G(t) = \frac{pt}{1 - (1 - p)t}$ . [2]
- (b) By considering  $G(1)$  and  $G(-1)$ , show that  $P(X \text{ is even}) = \frac{1 - p}{2 - p}$ . [4]
- (c) Find  $P(X \text{ is even})$  when  $p = \frac{1}{3}$ . [2]
- 
- 30** A fair coin is tossed repeatedly until two consecutive heads appear. Let  $N$  be the total number of tosses, and let  $G(t) = E(t^N)$ . By considering the outcome of the first one or two tosses, show that  $G(t) = \frac{t^2}{4 - 2t - t^2}$ . Hence find  $E(N)$ . [10]

## Discrete Random Variables

- 1 The discrete random variable  $X$  has the probability distribution shown. Find the value of  $k$ .

| $x$        | 1   | 2   | 3   | 4   |
|------------|-----|-----|-----|-----|
| $P(X = x)$ | 0.1 | $k$ | 0.4 | 0.2 |

[2]

- 2 The discrete random variable  $X$  has the probability distribution shown.

| $x$        | 0    | 1    | 2    | 3    |
|------------|------|------|------|------|
| $P(X = x)$ | 0.15 | 0.35 | 0.30 | 0.20 |

- (a) Find  $P(X \geq 2)$ . [1]  
 (b) Find  $P(1 \leq X < 3)$ . [1]

- 3 The random variable  $X$  has  $P(X = x) = \frac{x}{10}$  for  $x = 1, 2, 3, 4$ . Show that this is a valid probability distribution. [2]

- 4 Two fair coins are tossed and  $X$  is the number of heads. Write down the probability distribution of  $X$  in a table. [2]

- 5 The cumulative distribution function  $F(x) = P(X \leq x)$  of a discrete random variable  $X$  is shown in the table. Find  $P(X = 3)$ .

| $x$    | 1   | 2   | 3   | 4 |
|--------|-----|-----|-----|---|
| $F(x)$ | 0.2 | 0.5 | 0.8 | 1 |

[2]

- 6 A fair die is rolled and  $X$  is the score. Write down  $P(X \leq 4)$  and  $P(X > 4)$ . [2]

- 7 The random variable  $X$  has  $P(X = x) = kx$  for  $x = 1, 2, 3, 4, 5$ .

- (a) Find the value of  $k$ . [2]  
 (b) Find  $P(X > 3)$ . [1]

- 8 The random variable  $X$  has  $P(X = x) = k(5 - x)$  for  $x = 1, 2, 3, 4$ .

- (a) Show that  $k = \frac{1}{10}$ . [2]  
 (b) Find the probability that  $X$  is even. [2]

- 9 The cumulative distribution function of  $X$  is given in the table.

| $x$    | 1   | 2   | 3    | 4    | 5 |
|--------|-----|-----|------|------|---|
| $F(x)$ | 0.1 | 0.3 | 0.55 | 0.85 | 1 |

- (a) Find the probability distribution of  $X$ . [2]  
 (b) Find  $P(2 \leq X \leq 4)$ . [1]

- 10 Two fair dice are rolled and  $X$  is the positive difference between the scores (0 if they are equal). Find the probability distribution of  $X$ . [4]

- 11 A bag contains 4 red and 5 blue counters. Three counters are taken at random without replacement. Let  $X$  be the number of red counters taken. Find the probability distribution of  $X$ . [4]

- 12** The random variable  $X$  has  $P(X = x) = \frac{k}{x(x+1)}$  for  $x = 1, 2, 3, 4$ . Find the value of  $k$ . [3]
- 
- 13** The probability distribution of  $X$  is shown.
- | $x$        | 0   | 1    | 2   | 3    | 4   |
|------------|-----|------|-----|------|-----|
| $P(X = x)$ | 0.1 | 0.25 | 0.3 | 0.25 | 0.1 |
- (a) State the mode of  $X$ . [1]  
(b) Find the median of  $X$ . [2]
- 
- 14** Two fair dice are rolled and  $X$  is the larger of the two scores.
- (a) Show that  $P(X = x) = \frac{2x-1}{36}$  for  $x = 1, 2, \dots, 6$ . [3]  
(b) Find  $P(X \geq 5)$ . [2]
- 
- 15** The random variable  $X$  has the probability distribution shown, where  $a$  and  $b$  are constants.  $P(X = 1)$  is twice  $P(X = 3)$ .
- | $x$        | 1   | 2   | 3   | 4   |
|------------|-----|-----|-----|-----|
| $P(X = x)$ | $a$ | 0.3 | $b$ | 0.1 |
- (a) Find the values of  $a$  and  $b$ . [4]  
(b) Find  $P(X \leq 3 | X \geq 2)$ . [2]
- 
- 16** The random variable  $X$  has probability distribution  $P(X = 1) = \frac{1}{6}$ ,  $P(X = 2) = \frac{1}{3}$ ,  $P(X = 3) = \frac{1}{3}$ ,  $P(X = 4) = \frac{1}{6}$ .
- (a) Write down the cumulative distribution function  $F(x)$  for  $x = 1, 2, 3, 4$ . [2]  
(b) Find  $P(X \leq 2.5)$  and  $P(X > 1.5)$ . [3]
- 
- 17** A fair coin is tossed up to four times and the tossing stops as soon as a head appears. Let  $X$  be the number of tosses made. Find the probability distribution of  $X$  and  $P(X > 2)$ . [5]
- 
- 18** Spinner A shows 1, 2 or 3 and spinner B shows 1, 2, 3 or 4, each number being equally likely on its spinner. Both are spun and  $X$  is the product of the two numbers shown. Find  $P(X = 6)$ ,  $P(X = 4)$  and  $P(X \geq 8)$ . [5]
- 
- 19** The random variable  $X$  has  $P(X = x) = k(x^2 + 1)$  for  $x = 0, 1, 2, 3$ .
- (a) Find the value of  $k$ . [2]  
(b) Find  $P(X < 2 | X > 0)$ . [3]
- 
- 20** Two cards are taken at random, without replacement, from five cards numbered 1, 2, 3, 4, 5. Let  $X$  be the larger of the two numbers. Show that  $P(X = x) = \frac{x-1}{10}$  for  $x = 2, 3, 4, 5$ . [4]
- 
- 21** The cumulative distribution function of a discrete random variable  $X$  taking values 1, 2, ...,  $n$  is  $F(x) = P(X \leq x)$ . Prove that  $P(X = x) = F(x) - F(x - 1)$  for  $x = 2, 3, \dots, n$ , and show that, if  $F(x) = \frac{x(x+1)}{n(n+1)}$ , then  $P(X = x) = \frac{2x}{n(n+1)}$ . [5]
- 
- 22** The random variable  $X$  has  $P(X = x) = k(3 - x)(x + 1)$  for  $x = 0, 1, 2$  and zero otherwise.
- (a) Find the value of  $k$  and the mode of  $X$ . [3]  
(b) Two independent observations of  $X$  are made. Find the probability that they are equal. [3]

- 23** Independent random variables  $X$  and  $Y$  each have equal probabilities on their possible values:  $X$  takes the values 1, 2, 3 and  $Y$  takes the values 1, 2. Let  $M$  be the larger of  $X$  and  $Y$  ( $M = X$  if  $X = Y$ ).
- (a) Find the probability distribution of  $M$ . [4]  
 (b) Find  $P(X = Y \mid M = 2)$ . [2]
- 
- 24** In a game a player pays £2 to roll a fair die. If the score is 1, 2 or 3 the player wins nothing; if the score is 4 the player receives £3; if 5 the player receives £5; and if 6 the player receives £8. Let  $X$  be the player's net gain in pounds (receipts minus the £2 stake).
- (a) Find the probability distribution of  $X$ . [3]  
 (b) Find the probability that the player makes a profit. [1]  
 (c) The player plays the game twice. Find the probability that the total net gain is positive. [3]
- 
- 25** The random variable  $X$  has the probability distribution  $P(X = 1) = p^2$ ,  $P(X = 2) = p$  and  $P(X = 3) = 0.44$ , where  $p$  is a constant. Find the value of  $p$ , explaining why the other root of your equation is rejected. [5]
- 
- 26** The random variable  $X$  takes the values  $-1$ ,  $0$  and  $1$  with probabilities  $0.3$ ,  $0.2$  and  $0.5$  respectively. Let  $Y = X^2$  and  $Z = |X - 1|$ .
- (a) Find the probability distribution of  $Y$ . [2]  
 (b) Find the probability distribution of  $Z$ . [3]
- 
- 27** Three fair dice are rolled. Let  $X$  be the number of different scores that appear (so  $X = 1$  if all three scores are the same,  $X = 3$  if all are different). Find the probability distribution of  $X$ , giving each probability as a fraction in lowest terms. [8]
- 
- 28** Five cards are dealt from a well-shuffled standard pack of 52 cards. Let  $X$  be the number of aces among the cards dealt.
- (a) Show that  $P(X = 2) = \frac{2162}{54145}$ . [3]  
 (b) Find the full probability distribution of  $X$ , as decimals correct to 4 significant figures. [4]  
 (c) Find the probability that at least one ace is dealt. [2]
- 
- 29** The discrete random variable  $X$  takes the values  $1, 2, \dots, n$  and has cumulative distribution function  $F(x) = \frac{x^2}{n^2}$  for  $x = 1, 2, \dots, n$ .
- (a) Show that  $P(X = x) = \frac{2x - 1}{n^2}$ . [2]  
 (b) Given that  $P(X = 3) = \frac{5}{49}$ , find  $n$ . [2]  
 (c) For this value of  $n$ , find the median of  $X$ , justifying your answer. [3]  
 (d) Find  $P(X \text{ is even})$ . [3]
- 
- 30** A frog starts at 0 on a number line. Each second it jumps 1 unit to the right or 1 unit to the left with equal probability. It stops as soon as it reaches 2 or  $-2$ , or after 4 jumps, whichever happens first. Let  $X$  be the position of the frog when it stops. Find the probability distribution of  $X$ . [9]

## Expectation and Variance

- 1** The random variable  $X$  has the distribution shown. Find  $E(X)$ .

| $x$        | 1   | 2   | 3   |
|------------|-----|-----|-----|
| $P(X = x)$ | 0.2 | 0.3 | 0.5 |

[2]

- 2 The random variable  $X$  has the distribution shown. Find  $E(X)$ ,  $E(X^2)$  and  $\text{Var}(X)$ .

| $x$        | 0    | 1   | 4    |
|------------|------|-----|------|
| $P(X = x)$ | 0.25 | 0.5 | 0.25 |

[3]

- 3 The random variable  $X$  has  $E(X) = 4$  and  $\text{Var}(X) = 5$ . Find  $E(2X + 3)$  and  $\text{Var}(2X + 3)$ .

[3]

- 4 A fair spinner is equally likely to show 1, 2, 3, 4 or 5. Find the mean and variance of the number shown.

[3]

- 5 In a game a player wins £5 with probability 0.2 and otherwise loses £1. Find the expected gain of the player in one game.

[2]

- 6 The random variable  $X$  has  $P(X = x) = \frac{x}{10}$  for  $x = 1, 2, 3, 4$ . Find  $E(X)$ .

[2]

- 7 The random variable  $X$  has the distribution shown, where  $a$  and  $k$  are constants and  $E(X) = 2.1$ . Find  $a$  and  $k$ .

| $x$        | 1   | 2   | 3   |
|------------|-----|-----|-----|
| $P(X = x)$ | 0.3 | $a$ | $k$ |

[3]

- 8 The random variable  $X$  has  $P(X = x) = \frac{x}{10}$  for  $x = 1, 2, 3, 4$ . Find  $\text{Var}(X)$ .

[4]

- 9 The random variable  $X$  has  $P(X = 0) = 0.2$ ,  $P(X = 1) = 0.4$  and  $P(X = 2) = 0.4$ . Find  $E(3X^2 - 2)$ .

[3]

- 10 A taxi firm charges £4 plus £2.50 per mile. The distance  $X$  miles of a journey has mean 6.4 and variance 9. Find the mean and standard deviation of the cost of a journey.

[4]

- 11 The random variable  $X$  has mean 10 and variance 4. Find constants  $a$  and  $b$  ( $a > 0$ ) such that  $Y = aX + b$  has mean 0 and variance 1.

[4]

- 12 A game consists of rolling a fair die. The player receives £6 if a six is rolled and nothing otherwise. Find the stake that makes the game fair.

[3]

- 13 Prove that, for constants  $a$  and  $b$ ,  $\text{Var}(aX + b) = a^2\text{Var}(X)$ .

[3]

- 14 Two fair dice are rolled and  $X$  is the larger of the two scores, so that  $P(X = x) = \frac{2x - 1}{36}$  for  $x = 1, 2, \dots, 6$ . Find  $E(X)$  and  $\text{Var}(X)$ .

[5]

- 15 The random variable  $X$  takes only the values 1, 2 and 3.  $E(X) = 2.1$  and  $\text{Var}(X) = 0.49$ . Find the probability distribution of  $X$ .

[6]

- 16 The random variable  $X$  has  $P(X = x) = \frac{k}{x}$  for  $x = 1, 2, 3, 4$ .

(a) Show that  $k = \frac{12}{25}$ .

[2]

(b) Find  $E(X)$ .

[2]

(c) Find  $E\left(\frac{1}{X}\right)$ .

[2]

- 17** A fair coin is tossed up to four times, stopping as soon as a head appears. Let  $X$  be the number of tosses. The probability distribution is  $P(X = 1) = \frac{1}{2}$ ,  $P(X = 2) = \frac{1}{4}$ ,  $P(X = 3) = P(X = 4) = \frac{1}{8}$ . Find  $E(X)$  and  $\text{Var}(X)$ . [5]

- 18** A player rolls a fair die, up to a maximum of three times, stopping as soon as a six is rolled. Each roll costs £1. Let  $X$  be the number of rolls made. Find the expected cost of the game. [5]

- 19** An insurer sells a policy for £60. With probability 0.02 the insurer must pay out £2000, with probability 0.08 it pays out £500, and otherwise it pays nothing. Find the insurer's expected profit per policy and the standard deviation of the payout, correct to the nearest £. [5]

- 20** The discrete random variable  $X$  has the probability distribution shown. The random variable  $Y$  is defined by  $Y = X^2$  when  $X$  is negative and  $Y = 2X + 1$  when  $X$  is positive.

| $x$        | -3    | -1   | 2    | 4     |
|------------|-------|------|------|-------|
| $P(X = x)$ | 0.125 | 0.25 | 0.25 | 0.375 |

- (a) Find  $P(Y < 9)$ . [2]  
 (b) Find  $E(Y)$ . [2]

- 21** Prove that  $\text{Var}(X) \geq 0$  for any discrete random variable  $X$ , and that  $\text{Var}(X) = 0$  only if  $X$  takes one value with probability 1. [5]

- 22** A coin is tossed until a head appears, up to a maximum of 5 tosses. If the first head occurs on toss  $n$  ( $n = 1, 2, 3, 4$ ), the player wins £ $2^n$ . Otherwise (first head on toss 5, or no head at all) the player wins £32. The player pays £5 to play. Find the expected net gain, and the standard deviation of the winnings  $W$  (not the net gain), correct to 3 significant figures. [7]

- 23** Sam says, "If  $\text{Var}(X) = 3$  then  $\text{Var}(2X) = 6$ , and if  $X$  and  $Y$  are independent with  $\text{Var}(X) = 3$  and  $\text{Var}(Y) = 2$  then  $\text{Var}(X - Y) = 1$ ." Identify both errors in Sam's statements and give the correct values. [5]

- 24** Two fair  $n$ -sided dice, with faces numbered 1 to  $n$ , are rolled and  $X$  is the larger score.
- (a) Show that  $P(X \leq x) = \frac{x^2}{n^2}$  and hence find  $P(X = x)$ . [3]  
 (b) Show that  $E(X) = \frac{(n+1)(4n-1)}{6n}$ . [4]  
 (c) Verify your formula for a standard six-sided die. [1]

- 25** The random variable  $X$  takes each of the values 1, 2, ...,  $n$  with equal probability. Show that  $E(X) = \frac{n+1}{2}$  and  $\text{Var}(X) = \frac{n^2-1}{12}$ . Hence write down the variance of the score on a fair 20-sided die. [6]

- 26** A bag contains 3 red and 2 green counters. Two counters are taken at random without replacement. A player receives £4 for each red counter taken and pays £3 for each green counter taken. Let  $X$  be the player's net gain in pounds.
- (a) Show that  $X$  takes the values 8, 1 and  $-6$ , and find their probabilities. [3]  
 (b) Find  $E(X)$  and  $\text{Var}(X)$ . [3]

- 27** X is a discrete random variable taking values in 0, 1, 2, ..., n.
- (a) Prove that  $E(X) = \sum_{k=1}^n P(X \geq k)$ . [4]
- (b) Two fair dice are rolled and X is the larger score. Use the result in (a) to show that  $E(X) = \frac{161}{36}$ . [4]
- 
- 28** A cereal manufacturer puts one of three different collectable cards in each box, each type being equally likely and independent of other boxes. Let N be the number of boxes bought until a collector has all three types.
- (a) Explain why N can be written as  $N = 1 + G_2 + G_3$ , where  $G_2$  and  $G_3$  are geometric random variables, stating their parameters. [3]
- (b) Find  $E(N)$ . [2]
- (c) Find  $\text{Var}(N)$ . [3]
- 
- 29** In a game a fair die is rolled. The player may either accept the score as their prize in pounds, or reject it and roll the die a second time, in which case the second score must be accepted as the prize. The player aims to maximise their expected prize.
- (a) Find the expected prize if the player always accepts the first roll. [1]
- (b) Show that the player should reject the first score if and only if it is 1, 2 or 3. [3]
- (c) Find the expected prize with this strategy, and its variance. [5]
- 
- 30** The numbers 1, 2, ..., n are arranged in a random order. Let X be the number of positions in which the number equals its position (a fixed point).
- (a) For  $n = 4$ , show that  $P(X = 0) = \frac{3}{8}$ ,  $P(X = 1) = \frac{1}{3}$ ,  $P(X = 2) = \frac{1}{4}$  and  $P(X = 4) = \frac{1}{24}$ , and explain why X cannot equal 3. [4]
- (b) Write  $X = I_1 + I_2 + \dots + I_n$ , where  $I_k = 1$  if position k is a fixed point and 0 otherwise. Show that  $E(X) = 1$  for every n. [3]
- (c) Show that  $E[X(X - 1)] = 1$  for  $n \geq 2$ , and hence that  $\text{Var}(X) = 1$ . [4]

## Linear Combinations of Random Variables

- 1**  $E(X) = 5$  and  $E(Y) = 3$ . Find  $E(2X + Y)$ . [1]
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- 2** X and Y are independent random variables with  $\text{Var}(X) = 4$  and  $\text{Var}(Y) = 9$ . Find  $\text{Var}(X + Y)$  and  $\text{Var}(X - Y)$ . [2]
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- 3** X has variance 2. Let  $Y = 3X$ , and let  $S = X_1 + X_2 + X_3$ , where  $X_1$ ,  $X_2$  and  $X_3$  are independent observations of X. Find  $\text{Var}(Y)$  and  $\text{Var}(S)$ . [2]
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- 4**  $X \sim N(10, 4)$  and  $Y \sim N(6, 9)$  are independent. State the distribution of  $X + Y$ . [2]
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- 5**  $E(X) = 4$  and  $E(Y) = 7$ . Find  $E(5X - 2Y)$ . [2]
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- 6** Two fair dice are rolled and the scores are  $X_1$  and  $X_2$ . Given that each score has mean 3.5 and variance  $\frac{35}{12}$ , find the mean and variance of  $X_1 + X_2$ . [3]
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- 7**  $X \sim N(50, 16)$  and  $Y \sim N(30, 9)$  are independent. Find  $P(X + Y > 90)$ . [4]
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- 8**  $X \sim N(12, 4)$  and  $Y \sim N(10, 5)$  are independent. Find  $P(X > Y)$ . [4]
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- 9** The mass of a box of cereal is  $N(250, 100)$  grams. Four boxes are chosen at random. Find the probability that their total mass exceeds 1050 grams. [4]

- 10** X and Y are independent random variables with  $E(X) = 6$ ,  $\text{Var}(X) = 2$ ,  $E(Y) = 4$  and  $\text{Var}(Y) = 3$ . Find  $E(2X - 3Y + 5)$  and  $\text{Var}(2X - 3Y + 5)$ . [4]
- 11**  $X \sim \text{Po}(3)$  and  $Y \sim \text{Po}(4)$  are independent. Find  $P(X + Y = 5)$ , correct to 3 significant figures. [3]
- 12**  $X \sim N(10, 4)$ .  $X_1$  and  $X_2$  are independent observations of X. Find the distributions of  $2X$  and of  $X_1 + X_2$ , and explain why they are different. [3]
- 13** The random variable  $X \sim N(20, 25)$ . A random sample of 25 observations is taken and the sample mean is  $\bar{X}$ . State the distribution of  $\bar{X}$  and find  $P(\bar{X} > 21)$ . [4]
- 14** The masses of adult passengers are normally distributed with mean 75 kg and standard deviation 12 kg. A lift is safe for a total mass of up to 640 kg. Eight passengers, chosen at random, enter the lift. Find the probability that the lift is overloaded. [5]
- 15** A bottle contains a volume of drink that is normally distributed with mean 455 ml and standard deviation 2 ml. A cup holds a volume of drink that is normally distributed with mean 150 ml and standard deviation 3 ml. Find the probability that the contents of a bottle are not enough to fill three cups, if the amounts in the cups are independent. [5]
- 16**  $X \sim N(30, 9)$  and  $Y \sim N(20, 16)$  are independent. Find the value of k such that  $P(X + Y > k) = 0.05$ , correct to 3 significant figures. [5]
- 17** The random variable X has the distribution shown. Let  $Y = 3X - 2$ . Find  $E(Y)$  and  $\text{Var}(Y)$ .
- |                 |          |          |          |
|-----------------|----------|----------|----------|
| <b>x</b>        | <b>1</b> | <b>2</b> | <b>3</b> |
| <b>P(X = x)</b> | 0.2      | 0.5      | 0.3      |
- [5]
- 18** X takes the values 1 and 2 with equal probability. Y, independently of X, takes the values 0, 1 and 2 with probabilities  $\frac{1}{4}$ ,  $\frac{1}{2}$  and  $\frac{1}{4}$ . Find  $E(X)$ ,  $E(Y)$  and  $E(XY)$ , and verify that  $E(XY) = E(X)E(Y)$ . [5]
- 19** Telephone calls arrive at a helpline at an average rate of 2.5 per hour and emails at an average rate of 1.5 per hour. Calls and emails arrive independently, each as a Poisson process. Find the probability that in a 2.5 hour period there are at least 8 contacts in total (calls plus emails). [5]
- 20**  $X \sim B(10, 0.3)$  and  $Y \sim B(15, 0.3)$  are independent.  $Z \sim B(15, 0.6)$  is independent of X. State, with a reason, whether  $X + Y$  has a binomial distribution and give its parameters. State whether  $X + Z$  is binomial. [5]
- 21** X and Y are independent random variables. Prove that  $\text{Var}(aX + bY) = a^2\text{Var}(X) + b^2\text{Var}(Y)$ . [5]
- 22** The masses of adult passengers are  $N(80, 10^2)$  kg and the masses of child passengers are  $N(40, 6^2)$  kg, all independent. A small boat carries 2 adults and 3 children and is safe for a total mass of up to 300 kg. Find the probability that the boat is overloaded. Find also the probability that the mass of one adult exceeds the total mass of two children by more than 5 kg. [9]
- 23** X and Y are independent unbiased estimators of a parameter  $\theta$  (so  $E(X) = E(Y) = \theta$ ) with  $\text{Var}(X) = \sigma^2$  and  $\text{Var}(Y) = 4\sigma^2$ . The estimator  $T = aX + (1 - a)Y$  is formed.
- (a) Show that T is unbiased for any value of a. [2]
- (b) Find the value of a that minimises  $\text{Var}(T)$ , and the minimum variance. [5]

- 24** Boxes of cereal from machine A have mass  $X \sim N(500, 36)$  g and from machine B have mass  $Y \sim N(497, 25)$  g. A random sample of 10 boxes from A has mean  $\bar{X}$  and an independent random sample of 12 boxes from B has mean  $\bar{Y}$ .
- (a) State the distributions of  $\bar{X}$  and  $\bar{Y}$ . [2]
- (b) Find the distribution of  $\bar{X} - \bar{Y}$  and hence  $P(\bar{X} - \bar{Y} > 5)$ , correct to 3 significant figures. [4]
- 
- 25** The mass of a parcel is a random variable  $X$  with mean  $\mu$  and variance  $\sigma^2$ . An engineer says: "The total mass of 5 parcels is  $5X$ , so it has mean  $5\mu$  and variance  $5\sigma^2$ ." Identify the error in the engineer's reasoning and give the correct mean and variance of the total mass of 5 independent parcels. Find also the mean and variance of  $5X$ . [5]
- 
- 26** The random variable  $X \sim N(\mu, \sigma^2)$ , with  $P(X < 12) = 0.2$  and  $P(X > 20) = 0.1$ . Find  $\mu$  and  $\sigma$  correct to 3 significant figures. Hence find the probability that the mean of two independent observations of  $X$  exceeds 17. [8]
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- 27**  $X \sim N(\mu_X, \sigma_X^2)$  and  $Y \sim N(\mu_Y, \sigma_Y^2)$  are independent. Let  $U = X + Y$  and  $V = X - Y$ .
- (a) Show that  $E(UV) - E(U)E(V) = \sigma_X^2 - \sigma_Y^2$ . [5]
- (b) Hence state the condition under which  $U$  and  $V$  are uncorrelated, and find  $P(U > 10 \mid V = 0)$  when  $X \sim N(4, 3)$  and  $Y \sim N(5, 3)$  are independent. (For independent normal  $X$  and  $Y$  with equal variances,  $U$  and  $V$  are independent.) [4]
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- 28** The widths of tiles are independent, each normally distributed with mean 30 mm and standard deviation 0.3 mm. The internal width of a tray is normally distributed with mean 121 mm and standard deviation 0.5 mm, independently of the tiles. Four tiles are chosen at random and placed side by side in a tray.
- (a) Find the probability that the four tiles fit in the tray (that is, their total width is less than the internal width of the tray). [5]
- (b) Trays are packed with 4 tiles, one tray at a time, independently. Find the probability that at least one of 5 trays packed is too small for its tiles. [3]
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- 29** A shop's daily demand for apples (in kilograms, whole numbers only) is  $A \sim \text{Po}(12)$  and for pears is  $R \sim \text{Po}(8)$ , independently.
- (a) The profit per kilogram is £0.50 on apples and £0.80 on pears. Find the mean and variance of the daily profit  $\pounds(0.5A + 0.8R)$ . [4]
- (b) Find the probability that the total demand  $A + R$  on a day exceeds 25 kg. [3]
- (c) Given that the total demand on a day was 20 kg, find the probability that exactly 12 kg were apples. [3]
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- 30**  $X_1, X_2, \dots, X_n$  are independent observations from  $N(\mu, \sigma^2)$ .
- (a) Show that the sample mean  $\bar{X}$  has  $E(\bar{X}) = \mu$  and  $\text{Var}(\bar{X}) = \frac{\sigma^2}{n}$ . [4]
- (b) Given that  $\sigma = 3$ , find the smallest sample size  $n$  for which  $P(|\bar{X} - \mu| < 0.5) > 0.95$ . [6]
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