

Variable Acceleration

1 (3 marks)

$$(a) v = \frac{dx}{dt} = 6t^2 - 5 \quad \text{M1}$$

$$\text{When } t = 2, v = 19 \text{ m s}^{-1} \quad \text{A1}$$

Answer: 19 m s⁻¹ 2 marks

$$(b) a = \frac{dv}{dt} = 12t = 24 \text{ m s}^{-2} \quad \text{B1}$$

Answer: 24 m s⁻² 1 mark

2 (2 marks)

$$a = \frac{dv}{dt} = 4 - 6t \quad \text{M1}$$

$$\text{When } t = 1, a = -2 \text{ m s}^{-2} \quad \text{A1}$$

Answer: -2 m s⁻² 2 marks

3 (3 marks)

$$v = \int (6t - 2) dt = 3t^2 - 2t + c \quad \text{M1}$$

$$v = 3 \text{ when } t = 0, \text{ so } c = 3 \quad \text{A1}$$

$$\text{When } t = 2, v = 12 - 4 + 3 = 11 \text{ m s}^{-1} \quad \text{A1}$$

Answer: 11 m s⁻¹ 3 marks

4 (2 marks)

$$x = \int 6t^2 dt = 2t^3 + c, \text{ and } x = 0 \text{ when } t = 0 \text{ so } c = 0 \quad \text{M1}$$

$$\text{When } t = 3, x = 2 \times 27 = 54 \text{ m} \quad \text{A1}$$

Answer: 54 m 2 marks

5 (2 marks)

$$a = \frac{dv}{dt} = 6 \cos 3t \quad \text{M1}$$

$$\text{When } t = \frac{\pi}{3}, a = 6 \cos \pi = -6 \text{ m s}^{-2} \quad \text{A1}$$

Answer: -6 m s⁻² 2 marks

6 (2 marks)

$$v = \frac{dx}{dt} = 3t^2 - 12t \quad \text{M1}$$

$$v = 0 \text{ gives } 3t(t - 4) = 0, \text{ so } t = 4 \text{ s} \quad \text{A1}$$

Answer: t = 4 s 2 marks

7 (4 marks)

$$(a) v = 0: (t - 1)(t - 5) = 0 \quad \text{M1}$$

$$t = 1 \text{ s and } t = 5 \text{ s} \quad \text{A1}$$

Answer: t = 1 and t = 5 2 marks

$$(b) a = \frac{dv}{dt} = 2t - 6 \quad \text{M1}$$

$$\text{At } t = 1, a = -4 \text{ m s}^{-2}; \text{ at } t = 5, a = 4 \text{ m s}^{-2} \quad \text{A1}$$

Answer: -4 m s⁻² at t = 1; 4 m s⁻² at t = 5 2 marks

8 (4 marks)

$$v = \int (4 - 2t) dt = 4t - t^2 \text{ (} v = 0 \text{ at } t = 0) \quad \text{M1 A1}$$

$$\text{Greatest speed when } a = 0, \text{ so } t = 2 \quad \text{M1}$$

$$v = 8 - 4 = 4 \text{ m s}^{-1} \quad \text{A1}$$

Answer: 4 m s⁻¹ 4 marks

9 (3 marks)

$$v = \int e^{-t} dt = -e^{-t} + c$$

M1

$$v = 3 \text{ at } t = 0: c = 4, \text{ so } v = 4 - e^{-t}$$

A1

$$\text{As } t \rightarrow \infty, e^{-t} \rightarrow 0 \text{ so } v \rightarrow 4 \text{ m s}^{-1}$$

B1

Answer: $v = 4 - e^{-t}$; v approaches 4 m s^{-1}

3 marks

10 (3 marks)

$$v \frac{dv}{dx} = 4x, \text{ so } \int v dv = \int 4x dx$$

M1

$$\frac{1}{2}v^2 = 2x^2 + c, \text{ so } v^2 = 4x^2 + c'$$

A1

$$v = 2 \text{ at } x = 1: 4 = 4 + c', \text{ so } c' = 0 \text{ and } v^2 = 4x^2$$

When $x = 3$, $v = 6 \text{ m s}^{-1}$ (taking the positive root as the particle moves away from O)

A1

Answer: 6 m s^{-1}

3 marks

11 (4 marks)

$$v = 0 \text{ at } t = 4, \text{ so the direction changes at } t = 4$$

M1

$$x = 6t^2 - t^3 \text{ (} x = 0 \text{ at } t = 0)$$

M1

$$x(4) = 96 - 64 = 32 \text{ and } x(6) = 216 - 216 = 0$$

A1

$$\text{Total distance} = 32 + 32 = 64 \text{ m}$$

A1

Answer: 64 m

4 marks

12 (5 marks)

$$\text{(a) } v = 3t^2 - 12t + p = 0 \text{ when } t = 1, \text{ so } p = 9$$

M1 A1

$$x = 0 \text{ when } t = 1: 1 - 6 + 9 + q = 0, \text{ so } q = -4$$

A1

Answer: $p = 9$, $q = -4$

3 marks

$$\text{(b) } x = t^3 - 6t^2 + 9t - 4 = (t - 1)^2(t - 4)$$

M1

$$x = 0 \text{ again when } t = 4 \text{ s}$$

A1

Answer: $t = 4 \text{ s}$

2 marks

13 (5 marks)

$$\text{(a) } v = 18 - \int (2 + \frac{t}{3}) dt = 18 - 2t - \frac{t^2}{6}$$

M1 A1

$$v = 0: t^2 + 12t - 108 = 0, (t - 6)(t + 18) = 0, \text{ so } t = 6 \text{ s}$$

A1

Answer: 6 s

3 marks

$$\text{(b) } x = \int (18 - 2t - \frac{t^2}{6}) dt = 18t - t^2 - \frac{t^3}{18}$$

M1

$$x(6) = 108 - 36 - 12 = 60 \text{ m}$$

A1

Answer: 60 m

2 marks

14 (5 marks)

$$\text{(a) } x = \int 3 \sin 2t dt = -\frac{3}{2} \cos 2t + c$$

M1 A1

$$x = 1 \text{ at } t = 0: c = \frac{5}{2}, \text{ so } x = \frac{5}{2} - \frac{3}{2} \cos 2t$$

A1

$$\text{Answer: } x = \frac{5}{2} - \frac{3}{2} \cos 2t$$

3 marks

$$\text{(b) } -1 \leq \cos 2t \leq 1$$

M1

$$\text{Greatest } x = \frac{5}{2} + \frac{3}{2} = 4; \text{ least } x = \frac{5}{2} - \frac{3}{2} = 1$$

A1

Answer: greatest 4, least 1

2 marks

15 (5 marks)

$$\frac{dv}{dt} = -0.4v, \text{ so } \int \frac{1}{v} dv = \int -0.4 dt$$

M1

$$\ln v = -0.4t + c, \text{ and } v = 20 \text{ at } t = 0: v = 20e^{-0.4t}$$

M1 A1

$$10 = 20e^{-0.4t}, \text{ so } e^{-0.4t} = \frac{1}{2}$$

M1

$$t = \frac{\ln 2}{0.4} = 1.73 \text{ s}$$

A1

Answer: 1.73 s

5 marks

16 (5 marks)

$$v \frac{dv}{dx} = \frac{8}{(x+1)^2}, \text{ so } \frac{1}{2}v^2 = -\frac{8}{x+1} + c$$

M1 A1

$$v = 2 \text{ at } x = 1: 2 = -4 + c, \text{ so } c = 6$$

M1

$$v^2 = 12 - \frac{16}{x+1}$$

A1

$$\text{As } x \rightarrow \infty, v^2 \rightarrow 12, \text{ so } v \rightarrow 2\sqrt{3} \approx 3.46 \text{ m s}^{-1}$$

A1

Answer: $v^2 = 12 - \frac{16}{x+1}$; speed approaches $2\sqrt{3} \text{ m s}^{-1}$

5 marks

17 (7 marks)

$$(a) \frac{dv}{dt} = 4e^{-2t}(1 - 2t) = 0, \text{ so } t = \frac{1}{2}$$

M1 A1

$$v = 2e^{-1} = 0.736 \text{ m s}^{-1}$$

A1

Answer: 0.736 m s⁻¹

3 marks

$$(b) x = \int 4te^{-2t} dt = -2te^{-2t} + \int 2e^{-2t} dt$$

M1 A1

$$= -2te^{-2t} - e^{-2t} + c$$

A1

$$x = 0 \text{ at } t = 0: c = 1, \text{ so } x = 1 - (1 + 2t)e^{-2t}$$

A1

Answer: $x = 1 - (1 + 2t)e^{-2t}$

4 marks

18 (8 marks)

$$v = \int (12 - 6t) dt = 12t - 3t^2 + 9$$

M1 A1

$$x = \int v dt = 6t^2 - t^3 + 9t \text{ (} x = 0 \text{ at } t = 0)$$

M1 A1

$$x = 0: t(t^2 - 6t - 9) = 0, \text{ so } t = \frac{6 \pm \sqrt{72}}{2} = 3 \pm 3\sqrt{2}$$

M1

$$t > 0, \text{ so } t = 3 + 3\sqrt{2}$$

A1

$$\text{Using } t^2 = 6t + 9: v = 9 + 12t - 3(6t + 9) = -18 - 6t = -36 - 18\sqrt{2}, \text{ so } v = -61.5 \text{ m s}^{-1}$$

M1 A1

Answer: $v = -61.5 \text{ m s}^{-1}$

8 marks

19 (7 marks)

$$v = \int 3 \cos t dt = 3 \sin t \text{ (} v = 0 \text{ at } t = 0)$$

M1 A1

$$x = \int 3 \sin t dt = 3 - 3 \cos t \text{ (} x = 0 \text{ at } t = 0)$$

M1 A1

$$3 - 3 \cos t = 4.5, \text{ so } \cos t = -\frac{1}{2}, t = \frac{2\pi}{3} \approx 2.09 \text{ s}$$

M1 A1

$$v = 3 \sin \frac{2\pi}{3} = \frac{3\sqrt{3}}{2} \approx 2.60 \text{ m s}^{-1}$$

A1

Answer: $t = \frac{2\pi}{3} \text{ s} \approx 2.09 \text{ s}$; speed $\frac{3\sqrt{3}}{2} \approx 2.60 \text{ m s}^{-1}$

7 marks

20 (7 marks)

$$x = \int \frac{1}{2}(t^2 - 8t + 12) dt = \frac{t^3}{6} - 2t^2 + 6t \quad \text{M1 A1}$$

$$v > 0 \text{ for } t < 2, v < 0 \text{ for } 2 < t < 6, v > 0 \text{ for } t > 6 \quad \text{M1}$$

$$x(2) = \frac{16}{3}, x(6) = 0, \text{ so the distance to } t = 6 \text{ is } \frac{16}{3} + \frac{16}{3} = \frac{32}{3} \text{ m} \quad \text{M1 A1}$$

$$\text{Remaining distance} = 20 - \frac{32}{3} = \frac{28}{3} \text{ m, in the positive direction from O} \quad \text{M1}$$

$$\text{Displacement} = \frac{28}{3} \text{ m} \quad \text{A1}$$

$$\text{Answer: } \frac{28}{3} \text{ m (about 9.33 m)} \quad \text{7 marks}$$

21 (6 marks)

$$(a) v \frac{dv}{dx} = -\frac{16}{x^3}, \text{ so } \frac{1}{2}v^2 = \frac{8}{x^2} + c \quad \text{M1 A1}$$

$$v = 3 \text{ at } x = 2: \frac{9}{2} = 2 + c, \text{ so } c = \frac{5}{2} \quad \text{M1}$$

$$v^2 = 5 + \frac{16}{x^2} \quad \text{A1}$$

$$\text{Answer: } v^2 = 5 + \frac{16}{x^2} \quad \text{4 marks}$$

$$(b) x = 4: v^2 = 5 + 1 = 6, \text{ so } v = \sqrt{6} \approx 2.45 \text{ m s}^{-1} \quad \text{A1}$$

$$\text{As } x \rightarrow \infty, v^2 \rightarrow 5: \text{ the speed decreases towards } \sqrt{5} \text{ m s}^{-1} \text{ and P never comes to rest} \quad \text{A1}$$

$$\text{Answer: } \sqrt{6} \text{ m s}^{-1}; \text{ speed tends to } \sqrt{5} \text{ m s}^{-1} \quad \text{2 marks}$$

22 (7 marks)

$$x_P = \int \int 2 dt dt = t^2 \text{ (starts at rest at O)} \quad \text{M1 A1}$$

$$v_Q = \int -2t dt = -t^2 - 8, \text{ so } x_Q = 100 - 8t - \frac{t^3}{3} \quad \text{M1 A1}$$

$$x_P = x_Q: t^2 = 100 - 8t - \frac{t^3}{3}, \text{ so } t^3 + 3t^2 + 24t - 300 = 0 \quad \text{M1 A1}$$

$$f(4) = -92 < 0 \text{ and } f(5) = 20 > 0, \text{ so there is a sign change and a root in } (4, 5) \quad \text{A1}$$

$$\text{Answer: } t^3 + 3t^2 + 24t - 300 = 0; f(4) = -92 < 0 \text{ and } f(5) = 20 > 0 \quad \text{7 marks}$$

23 (6 marks)

$$\text{The student found the displacement, not the distance; the particle changes direction} \quad \text{E1}$$

$$v = t(2 - t) = 0 \text{ at } t = 2, \text{ where the direction reverses} \quad \text{M1}$$

$$x(2) = 4 - \frac{8}{3} = \frac{4}{3} \quad \text{A1}$$

$$x(4) = -\frac{16}{3}, \text{ so the return journey is } \frac{4}{3} + \frac{16}{3} = \frac{20}{3} \quad \text{A1}$$

$$\text{Total distance} = \frac{4}{3} + \frac{20}{3} \quad \text{M1}$$

$$= 8 \text{ m} \quad \text{A1}$$

$$\text{Answer: } 8 \text{ m} \quad \text{6 marks}$$

24 (9 marks)

$$(a) v \frac{dv}{dx} = -4x, \text{ so } \frac{1}{2}v^2 = -2x^2 + c \quad \text{M1 A1}$$

$$v = 8 \text{ at } x = 3: 32 = -18 + c, \text{ so } c = 50 \text{ and } v^2 = 100 - 4x^2 \quad \text{A1}$$

$$\text{Answer: } v^2 = 100 - 4x^2 \quad \text{3 marks}$$

$$(b) v = 0: 4x^2 = 100 \quad \text{M1}$$

$$x = \pm 5, \text{ so the greatest distance is } 5 \text{ m} \quad \text{A1}$$

$$\text{Answer: } 5 \text{ m} \quad \text{2 marks}$$

$$(c) \text{ Greatest speed occurs at } x = 0: v = \sqrt{100} = 10 \text{ m s}^{-1} \quad \text{B1}$$

$$\text{Answer: } 10 \text{ m s}^{-1} \quad \text{1 mark}$$

$$(d) 5 \sin 2t = 3, \text{ so } \sin 2t = 0.6 \quad \text{M1}$$

$$2t = 0.6435... \text{ (radians)}$$

M1

$$t = 0.322 \text{ s}$$

A1

Answer: $t = 0.322 \text{ s}$

3 marks

25 (8 marks)

(a) $v = \int(0.6 - 0.01t) \, dt = 0.6t - 0.005t^2$ ($v = 0$ at $t = 0$)

M1 A1

Answer: 18 m s^{-1}

2 marks

(b) $x = \int(0.6t - 0.005t^2) \, dt = 0.3t^2 - \frac{t^3}{600}$

M1

$$x(60) = 1080 - 360 = 720 \text{ m}$$

A1

Answer: 720 m

2 marks

(c) Braking: $0 = 18^2 - 2 \times 1.5 \text{ s}$, so $s = 108 \text{ m}$, and time = $\frac{18}{1.5} = 12 \text{ s}$

M1 A1

Constant speed distance = $2556 - 720 - 108 = 1728 \text{ m}$

M1

Time = $\frac{1728}{18} = 96 \text{ s}$, so total time = $60 + 96 + 12 = 168 \text{ s}$

A1

Answer: 168 s

4 marks

26 (9 marks)

(a) $t(8 - t) = 12$ and $36 - 24 = 12$ at $t = 6$

B1

Greatest of $t(8 - t)$ when $\frac{dv}{dt} = 8 - 2t = 0$, so $t = 4$

M1

Greatest speed = $4 \times 4 = 16 \text{ m s}^{-1}$

A1

Answer: 16 m s^{-1}

3 marks

(b) Distance for 0 to 6: $\int(8t - t^2) \, dt = 144 - 72 = 72 \text{ m}$

M1 A1

$v = 0$ when $t = 9$, so the particle turns round at $t = 9$

M1

Distance for 6 to 9: $[36t - 2t^2] = 162 - 144 = 18 \text{ m}$

A1

Distance for 9 to 10: $|[36t - 2t^2]| = |160 - 162| = 2 \text{ m}$

A1

Total distance = $72 + 18 + 2 = 92 \text{ m}$

A1

Answer: 92 m

6 marks

27 (12 marks)

(a) $v \frac{dv}{dx} = -\frac{100}{x^2}$, so $\frac{1}{2}v^2 = \frac{100}{x} + c$

M1 A1

$x = 2$: $\frac{1}{2}U^2 = 50 + c$, so $c = \frac{1}{2}U^2 - 50$

M1

$$v^2 = U^2 - 100 + \frac{200}{x}$$

A1

$\frac{200}{x} > 0$ and decreases to 0 as $x \rightarrow \infty$, so $v^2 > 0$ for all x if $U^2 \geq 100$

M1

Least $U = 10 \text{ m s}^{-1}$

A1

Answer: least $U = 10 \text{ m s}^{-1}$

6 marks

(b) $v^2 = \frac{200}{x}$, so $v = \frac{10\sqrt{2}}{\sqrt{x}}$ and $\frac{dx}{dt} = \frac{10\sqrt{2}}{\sqrt{x}}$

M1 A1

$$t = \int_2^{18} \frac{\sqrt{x}}{10\sqrt{2}} \, dx$$

M1

$$= \frac{1}{10\sqrt{2}} \left[\frac{2}{3} x^{3/2} \right]_2^{18} = \frac{1}{10\sqrt{2}} \cdot \frac{2}{3} (54\sqrt{2} - 2\sqrt{2})$$

M1 A1

$$= \frac{52}{15} \text{ s} \approx 3.47 \text{ s}$$

A1

Answer: $\frac{52}{15} \text{ s}$

6 marks

28 (12 marks)

$$(a) v = \int \frac{4}{(1+t)^2} dt = -\frac{4}{1+t} + c$$

M1 A1

$$v = 0 \text{ at } t = 0: c = 4, \text{ so } v = 4 - \frac{4}{1+t} = \frac{4t}{1+t}$$

A1

$$\text{Answer: } v = \frac{4t}{1+t}$$

3 marks

$$(b) x = \int \frac{4t}{1+t} dt = \int 4 - \frac{4}{1+t} dt$$

M1 A1

$$= 4t - 4 \ln(1+t) + c, \text{ and } x = 0 \text{ at } t = 0 \text{ so } c = 0$$

A1

$$\text{Answer: } x = 4t - 4 \ln(1+t)$$

3 marks

$$(c) x = 12 - 4 \ln 4$$

M1

$$= 12 - 8 \ln 2$$

A1

$$\text{Answer: } 12 - 8 \ln 2$$

2 marks

$$(d) v = 4 - \frac{4}{1+t} < 4 \text{ since } \frac{4}{1+t} > 0$$

E1

$$\frac{4t}{1+t} = 3.6, \text{ so } 4t = 3.6 + 3.6t \text{ and } t = 9$$

M1 A1

$$x = 36 - 4 \ln 10 = 26.8 \text{ m}$$

A1

$$\text{Answer: } 26.8 \text{ m}$$

4 marks

29 (11 marks)

$$(a) v \frac{dv}{dx} = 3x^2 - 12x + 9, \text{ so } \frac{1}{2}v^2 = x^3 - 6x^2 + 9x + c$$

M1 A1

$$x = 1, v = \pm 2: 2 = 1 - 6 + 9 + c, \text{ so } c = -2$$

M1

$$v^2 = 2x^3 - 12x^2 + 18x - 4$$

A1

$$\text{Answer: } v^2 = 2x^3 - 12x^2 + 18x - 4$$

4 marks

$$(b) \text{ At } x = 0, v^2 = -4 < 0, \text{ so O cannot be reached}$$

M1 A1

$$P \text{ stops when } v = 0: x^3 - 6x^2 + 9x - 2 = 0, \text{ which factorises as } (x-2)(x^2 - 4x + 1) = 0$$

M1 A1

$$x = 2 \text{ or } x = 2 \pm \sqrt{3}; P \text{ is moving towards O from } x = 1, \text{ so it first stops at } x = 2 - \sqrt{3}$$

A1

$$\text{Answer: } x = 2 - \sqrt{3} \approx 0.268$$

5 marks

$$(c) \text{ At } x = 2 - \sqrt{3}, a = 3(x-1)(x-3) > 0 \text{ since } x < 1, \text{ so P accelerates in the direction of } x \text{ increasing}$$

M1

$$P \text{ therefore returns past } x = 1, \text{ so it moves away from O}$$

A1

$$\text{Answer: P turns round and moves away from O}$$

2 marks

30 (12 marks)

$$(a) v = \int k(T-t) dt = k(Tt - \frac{1}{2}t^2) \text{ (} v = 0 \text{ at } t = 0)$$

M1 A1

$$x = k(\frac{1}{2}Tt^2 - \frac{1}{6}t^3), \text{ so } x(T) = \frac{1}{3}kT^3$$

M1 A1

$$\text{Answer: } x(T) = \frac{1}{3}kT^3$$

4 marks

$$(b) v(T) = \frac{1}{2}kT^2 = 30 \text{ and } \frac{1}{3}kT^3 = 200$$

M1 A1

$$\text{Dividing: } \frac{2}{3}T = \frac{20}{3}, \text{ so } T = 10$$

M1 A1

$$k = \frac{60}{100} = 0.6$$

A1

$$\text{Answer: } T = 10, k = 0.6$$

5 marks

$$(c) \text{ After } t = 10 \text{ the cyclist travels at } 30 \text{ m s}^{-1}: 200 + 30(t - 10) = 500$$

M1 A1

$$t = 20 \text{ s}$$

A1

$$\text{Answer: } t = 20 \text{ s}$$

3 marks

Vector Kinematics

1 (2 marks)

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = 6t\mathbf{i} + 3t^2\mathbf{j}$$

M1

When $t = 2$, $\mathbf{v} = (12\mathbf{i} + 12\mathbf{j}) \text{ m s}^{-1}$

A1

Answer: $(12\mathbf{i} + 12\mathbf{j}) \text{ m s}^{-1}$

2 marks

2 (3 marks)

(a) $\mathbf{a} = \frac{d\mathbf{v}}{dt} = 4\mathbf{i} \text{ m s}^{-2}$

B1

Answer: $4\mathbf{i} \text{ m s}^{-2}$

1 mark

(b) At $t = 1$, $\mathbf{v} = 4\mathbf{i} - 3\mathbf{j}$

M1

Speed = $\sqrt{16 + 9} = 5 \text{ m s}^{-1}$

A1

Answer: 5 m s^{-1}

2 marks

3 (3 marks)

$$\mathbf{v} = \int (6t\mathbf{i} + 2\mathbf{j}) dt = 3t^2\mathbf{i} + 2t\mathbf{j} + \mathbf{c}$$

M1

At $t = 0$, $\mathbf{c} = \mathbf{i} - \mathbf{j}$, so $\mathbf{v} = (3t^2 + 1)\mathbf{i} + (2t - 1)\mathbf{j}$

A1

When $t = 2$, $\mathbf{v} = (13\mathbf{i} + 3\mathbf{j}) \text{ m s}^{-1}$

A1

Answer: $(13\mathbf{i} + 3\mathbf{j}) \text{ m s}^{-1}$

3 marks

4 (2 marks)

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -6 \sin 2t\mathbf{i} + 6 \cos 2t\mathbf{j}$$

M1

When $t = 0$, $\mathbf{v} = 6\mathbf{j} \text{ m s}^{-1}$, so the speed is 6 m s^{-1}

A1

Answer: $\mathbf{v} = 6\mathbf{j} \text{ m s}^{-1}$; speed 6 m s^{-1}

2 marks

5 (3 marks)

$$\mathbf{r} = \int (2t\mathbf{i} - 6\mathbf{j}) dt = t^2\mathbf{i} - 6t\mathbf{j} + \mathbf{c}$$

M1

At $t = 0$, $\mathbf{c} = 4\mathbf{i}$, so $\mathbf{r} = (t^2 + 4)\mathbf{i} - 6t\mathbf{j}$

A1

When $t = 3$, $\mathbf{r} = (13\mathbf{i} - 18\mathbf{j}) \text{ m}$

A1

Answer: $(13\mathbf{i} - 18\mathbf{j}) \text{ m}$

3 marks

6 (4 marks)

$$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = 6t\mathbf{i} + 8\mathbf{j}$$

M1 A1

$\mathbf{F} = m\mathbf{a} = 3(12\mathbf{i} + 8\mathbf{j}) = (36\mathbf{i} + 24\mathbf{j}) \text{ N}$ when $t = 2$

A1

$|\mathbf{F}| = \sqrt{36^2 + 24^2} = 43.3 \text{ N}$

A1

Answer: $\mathbf{F} = (36\mathbf{i} + 24\mathbf{j}) \text{ N}$; $|\mathbf{F}| = 43.3 \text{ N}$

4 marks

7 (4 marks)

(a) Moving parallel to $\mathbf{j}$ when the $\mathbf{i}$ component is zero: $t^2 - 4 = 0$

M1

$t = 2 \text{ s}$

A1

Answer: $t = 2 \text{ s}$

2 marks

(b) $\mathbf{a} = 2t\mathbf{i} + 2\mathbf{j}$

M1

At $t = 2$, $\mathbf{a} = (4\mathbf{i} + 2\mathbf{j}) \text{ m s}^{-2}$

A1

Answer: $(4\mathbf{i} + 2\mathbf{j}) \text{ m s}^{-2}$

2 marks

8 (4 marks)

$$\mathbf{v} = \int (-2\mathbf{i} + 2t\mathbf{j}) dt = (8 - 2t)\mathbf{i} + t^2\mathbf{j}$$

M1 A1

45° to $\mathbf{i}$, so the components are equal: $8 - 2t = t^2$

M1

$t^2 + 2t - 8 = 0$, $(t - 2)(t + 4) = 0$, so $t = 2 \text{ s}$

A1

Answer: $t = 2 \text{ s}$

4 marks

9 (4 marks)

On the y-axis the **i** component is zero: $t(t - 4) = 0$

M1

$$t = 0 \text{ or } t = 4$$

A1

$$t = 0: r = -\mathbf{j} \text{ m}$$

A1

$$t = 4: r = 11\mathbf{j} \text{ m}$$

A1

Answer: $t = 0, r = -\mathbf{j}; t = 4, r = 11\mathbf{j}$

4 marks

10 (4 marks)

$$\text{Speed}^2 = (2t - 4)^2 + 9$$

M1 A1

This is least when $2t - 4 = 0$, so $t = 2 \text{ s}$

M1

$$\text{Least speed} = 3 \text{ m s}^{-1}$$

A1

Answer: 3 m s^{-1} when $t = 2 \text{ s}$

4 marks

11 (4 marks)

$$\text{At } t = 1, v = 3\mathbf{i} + 4\mathbf{j}$$

B1

$$\tan \theta = \frac{4}{3}, \text{ so } \theta = 53.1^\circ$$

M1 A1

$$\text{Speed} = \sqrt{9 + 16} = 5 \text{ m s}^{-1}$$

A1

Answer: $53.1^\circ; 5 \text{ m s}^{-1}$

4 marks

12 (5 marks)

i components equal: $3t = 10 - 2t$, so $t = 2$

M1 A1

j components at $t = 2$: $t^2 = 4$ and $2t = 4$, equal

M1 A1

So they collide when $t = 2$ at the point $6\mathbf{i} + 4\mathbf{j} \text{ m}$

A1

Answer: collide at $t = 2$ at $(6\mathbf{i} + 4\mathbf{j}) \text{ m}$

5 marks

13 (5 marks)

$$\text{(a) } x = 2t + 1, \text{ so } t = \frac{(x - 1)}{2}$$

M1

$$y = \frac{(x - 1)^2}{4} - 3$$

A1

$$\text{Answer: } y = \frac{(x - 1)^2}{4} - 3$$

2 marks

$$\text{(b) } r(1) = 3\mathbf{i} - 2\mathbf{j}, r(4) = 9\mathbf{i} + 13\mathbf{j}$$

B1

$$r(4) - r(1) = 6\mathbf{i} + 15\mathbf{j}$$

M1

$$\text{Distance} = \sqrt{36 + 225} = \sqrt{261} = 3\sqrt{29} = 16.2 \text{ m}$$

A1

Answer: $3\sqrt{29} = 16.2 \text{ m}$

3 marks

14 (7 marks)

$$\text{(a) } v = -12 \sin 3t \mathbf{i} + 12 \cos 3t \mathbf{j}$$

M1 A1

$$a = -36 \cos 3t \mathbf{i} - 36 \sin 3t \mathbf{j} = -9(4 \cos 3t \mathbf{i} + 4 \sin 3t \mathbf{j}) = -9\mathbf{r}$$

A1

Answer: $a = -9\mathbf{r}$

3 marks

$$\text{(b) Speed} = \sqrt{144 \sin^2 3t + 144 \cos^2 3t}$$

M1

$$= 12 \text{ m s}^{-1}, \text{ which is constant}$$

A1

Answer: 12 m s^{-1}

2 marks

$$\text{(c) } r \cdot v = -48 \sin 3t \cos 3t + 48 \sin 3t \cos 3t$$

M1

$$= 0, \text{ so } v \perp r$$

A1

Answer: $r \cdot v = 0$

2 marks

15 (6 marks)

$$v = (3t^2 - 12)\mathbf{i} + 4t\mathbf{j}$$

M1 A1

$$\text{Perpendicular: } v \cdot a = 0: 6t(3t^2 - 12) + 16t = 0$$

M1 A1

$$18t^3 - 56t = 0, \text{ so } t^2 = \frac{28}{9}$$

M1

$$t = 1.76 \text{ s}$$

A1

Answer: $t = 1.76 \text{ s}$

6 marks

16 (6 marks)Displacement of Q from P: $d = (6 - 3t)\mathbf{i} + (3t - 4)\mathbf{j}$

M1 A1

$$|d|^2 = (6 - 3t)^2 + (3t - 4)^2 = 18t^2 - 60t + 52$$

M1 A1

Least when $\frac{d}{dt}|d|^2 = 36t - 60 = 0$, so $t = \frac{5}{3}$ s

M1

$$|d|^2 = 18 \times \frac{25}{9} - 100 + 52 = 2, \text{ so least distance} = \sqrt{2} \approx 1.41 \text{ m}$$

A1

Answer: $\sqrt{2}$ m at $t = \frac{5}{3}$ s

6 marks

17 (6 marks)

$$v = (1 - \cos t)\mathbf{i} + \sin t\mathbf{j}$$

M1 A1

$$\text{Speed}^2 = (1 - \cos t)^2 + \sin^2 t = 2 - 2 \cos t$$

M1

Using $\cos t = 1 - 2 \sin^2 \frac{t}{2}$: $\text{speed}^2 = 4 \sin^2 \frac{t}{2}$, so $\text{speed} = 2 \sin \frac{t}{2}$ ($\sin \frac{t}{2} \geq 0$ for $0 \leq t \leq 2\pi$)

A1

Greatest speed 2 m s^{-1} when $\sin \frac{t}{2} = 1$, i.e. $t = \pi$ s

M1 A1

Answer: greatest speed 2 m s^{-1} at $t = \pi$ s

6 marks

18 (6 marks)

$$v = (6 - 2t)\mathbf{i} + (6t - 9)\mathbf{j}$$

M1 A1

Parallel to $\mathbf{i}$: $6t - 9 = 0$, so $t = \frac{3}{2}$

M1

$$r = (5 + 6t - t^2)\mathbf{i} + (2 - 9t + 3t^2)\mathbf{j}$$

M1 A1

$$\text{At } t = \frac{3}{2}: r = \left(\frac{47}{4}\mathbf{i} - \frac{19}{4}\mathbf{j}\right) \text{ m}$$

A1

Answer: $\left(\frac{47}{4}\mathbf{i} - \frac{19}{4}\mathbf{j}\right) \text{ m}$

6 marks

19 (6 marks)

$$\text{(a) } v = 12\mathbf{i} + (16 - 9.8t)\mathbf{j}$$

M1

$$\text{At } t = 3: v = 12\mathbf{i} - 13.4\mathbf{j}, \text{ speed} = \sqrt{144 + 179.56} = 18.0 \text{ m s}^{-1}$$

M1 A1

Answer: 18.0 m s^{-1}

3 marks

(b) Components equal in size: $16 - 9.8t = -12$

M1 A1

$$t = \frac{28}{9.8} = 2.86 \text{ s}$$

A1

Answer: $t = 2.86 \text{ s}$

3 marks

20 (7 marks)**(a)** Parallel to $\mathbf{i} + \mathbf{j}$ means the components are equal: $t^2 + 2t = 6t - 3$

M1 A1

$$t^2 - 4t + 3 = 0, \text{ so } t = 1 \text{ s or } t = 3 \text{ s}$$

A1

Answer: $t = 1$ and $t = 3$

3 marks

$$\text{(b) } r = \int v \, dt = \left(\frac{t^3}{3} + t^2 + c_1\right)\mathbf{i} + (3t^2 - 3t + c_2)\mathbf{j}$$

M1 A1

$$r(0) = \mathbf{i} \text{ gives } c_1 = 1, c_2 = 0$$

A1

$$t = 1: r = \left(\frac{7}{3}\mathbf{i}\right) \text{ m}; t = 3: r = (19\mathbf{i} + 18\mathbf{j}) \text{ m}$$

A1

Answer: $t = 1: \frac{7}{3}\mathbf{i} \text{ m}; t = 3: (19\mathbf{i} + 18\mathbf{j}) \text{ m}$

4 marks

21 (8 marks)**(a)** $\mathbf{i}$ components equal: $1 + 3t = 25 - 3t$, so $t = 4$

M1 A1

 $\mathbf{j}$ components equal: $2 - 4 = 4\lambda - 18$

M1

$$\lambda = 4$$

A1

Answer: $t = 4 \text{ min}, \lambda = 4$

4 marks

(b) $v_P = 3\mathbf{i} - \mathbf{j}$, speed $= \sqrt{10} \approx 3.16 \text{ km per minute}$

M1 A1

 $v_Q = -3\mathbf{i} + 4\mathbf{j}$, speed $= 5 \text{ km per minute}$

A1

Answer: P: $\sqrt{10} \text{ km per min}; Q: 5 \text{ km per min}$

3 marks

(c) r_p at $t = 4$: $(13\mathbf{i} - 2\mathbf{j})$ km

B1

Answer: $(13\mathbf{i} - 2\mathbf{j})$ km

1 mark

22 (8 marks)(a) $x = 5 \cos t$, $y = 3 \sin t$, so $\cos t = \frac{x}{5}$ and $\sin t = \frac{y}{3}$

M1

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

A1

Answer: $\frac{x^2}{25} + \frac{y^2}{9} = 1$

2 marks

(b) $\mathbf{v} = -5 \sin t \mathbf{i} + 3 \cos t \mathbf{j}$, $\mathbf{a} = -5 \cos t \mathbf{i} - 3 \sin t \mathbf{j}$

M1

 $\mathbf{a} = -\mathbf{r}$, which is directed towards the origin with magnitude $|\mathbf{r}|$

A1

Answer: $\mathbf{a} = -\mathbf{r}$

2 marks

(c) $\text{Speed}^2 = 25 \sin^2 t + 9 \cos^2 t = 9 + 16 \sin^2 t$

M1 A1

Greatest speed 5 m s^{-1} (when $\sin^2 t = 1$) and least speed 3 m s^{-1} (when $\sin t = 0$)

A1 A1

Answer: greatest 5 m s^{-1} , least 3 m s^{-1}

4 marks

23 (9 marks)(a) $\mathbf{v} = \int \mathbf{a} \, dt = 3t^2 \mathbf{i} + 2 \cos t \mathbf{j} + \mathbf{c}$

M1 A1

At $t = 0$: $\mathbf{c} = 0$, so $\mathbf{v} = 3t^2 \mathbf{i} + 2 \cos t \mathbf{j}$

A1

Answer: $\mathbf{v} = 3t^2 \mathbf{i} + 2 \cos t \mathbf{j}$

3 marks

(b) $\mathbf{r} = \int \mathbf{v} \, dt = (t^3 + c_1) \mathbf{i} + (2 \sin t + c_2) \mathbf{j}$

M1

At $t = 0$: $\mathbf{r} = \mathbf{i}$, so $\mathbf{r} = (t^3 + 1) \mathbf{i} + 2 \sin t \mathbf{j}$

A1

Answer: $\mathbf{r} = (t^3 + 1) \mathbf{i} + 2 \sin t \mathbf{j}$

2 marks

(c) $\mathbf{v} = \frac{3\pi^2}{4} \mathbf{i} + 0 \mathbf{j}$, so speed $= \frac{3\pi^2}{4} = 7.40 \text{ m s}^{-1}$

M1 A1

 $\mathbf{r} = (\frac{\pi^3}{8} + 1) \mathbf{i} + 2 \mathbf{j} = 4.88 \mathbf{i} + 2 \mathbf{j}$, so $|\mathbf{r}| = \sqrt{4.876^2 + 4} = 5.27 \text{ m}$

M1 A1

Answer: speed 7.40 m s^{-1} ; distance 5.27 m

4 marks

24 (5 marks)Moving parallel to $\mathbf{i}$ means the $\mathbf{j}$ component is zero, not the $\mathbf{i}$ component

E1

 $2t - 4 = 0$, so $t = 2$

M1 A1

At $t = 2$, $\mathbf{v} = 3\mathbf{i}$

M1

Speed $= 3 \text{ m s}^{-1}$

A1

Answer: $t = 2 \text{ s}$, speed 3 m s^{-1}

5 marks

25 (7 marks)Let $u = t^2 - 4t$. Then $x = 2u$ and $y = u + 3$, so $x = 2y - 6$, a straight line

M1 A1

 $\mathbf{v} = (4t - 8) \mathbf{i} + (2t - 4) \mathbf{j} = 0$ when $t = 2$

M1 A1

Positions: $t = 0$: $(0, 3)$; $t = 2$: $(-8, -1)$; $t = 3$: $(-6, 0)$

B1

Distance $= \sqrt{64 + 16} + \sqrt{4 + 1} = 4\sqrt{5} + \sqrt{5} = 5\sqrt{5} \text{ m}$

M1 A1

Answer: $5\sqrt{5} \text{ m} \approx 11.2 \text{ m}$

7 marks

26 (7 marks) $\mathbf{v} = R\omega(-\sin \omega t \mathbf{i} + \cos \omega t \mathbf{j})$, so speed $= R\omega$

M1 A1

 $\mathbf{a} = -R\omega^2(\cos \omega t \mathbf{i} + \sin \omega t \mathbf{j})$, so $|\mathbf{a}| = R\omega^2$

M1 A1

 $R\omega = 20$ and $R\omega^2 = 50$

B1

Dividing: $\omega = \frac{50}{20} = 2.5 \text{ rad s}^{-1}$

M1

 $R = \frac{20}{2.5} = 8 \text{ m}$

A1

Answer: $\omega = 2.5 \text{ rad s}^{-1}$, $R = 8 \text{ m}$

7 marks

27 (10 marks)

(a) $\frac{d^2x}{dt^2} = -4x$ and $\frac{d^2y}{dt^2} = -4y$, with general solutions $x = A \cos 2t + B \sin 2t$, $y = C \cos 2t + D \sin 2t$ M1 A1

$x(0) = 3$ gives $A = 3$; $\frac{dx}{dt}(0) = 0$ gives $B = 0$ M1 A1

$y(0) = 0$ gives $C = 0$; $\frac{dy}{dt}(0) = 8$ gives $2D = 8$, $D = 4$ M1 A1

Answer: $x = 3 \cos 2t$, $y = 4 \sin 2t$ 6 marks

(b) $\frac{x^2}{9} + \frac{y^2}{16} = 1$ M1 A1

$y = 0$ when $\sin 2t = 0$, first at $t = \frac{\pi}{2}$ M1

$v = -6 \sin 2t \mathbf{i} + 8 \cos 2t \mathbf{j} = -8 \mathbf{j}$, speed = 8 m s^{-1} A1

Answer: $\frac{x^2}{9} + \frac{y^2}{16} = 1$; 8 m s^{-1} 4 marks

28 (11 marks)

(a) Product rule: $v = -e^{-t}(\cos t \mathbf{i} + \sin t \mathbf{j}) + e^{-t}(-\sin t \mathbf{i} + \cos t \mathbf{j})$ M1 A1

$= e^{-t}((-\cos t - \sin t) \mathbf{i} + (\cos t - \sin t) \mathbf{j})$ A1

Speed² = $e^{-2t}((\cos t + \sin t)^2 + (\cos t - \sin t)^2) = 2e^{-2t}$, so speed = $\sqrt{2}e^{-t}$ M1 A1

Answer: speed = $\sqrt{2}e^{-t} \text{ m s}^{-1}$ 5 marks

(b) $r \cdot v = e^{-2t}(-\cos^2 t - \sin t \cos t + \sin t \cos t - \sin^2 t) = -e^{-2t}$ M1 A1

$|r| = e^{-t}$, $|v| = \sqrt{2}e^{-t}$, so $\cos \theta = \frac{-e^{-2t}}{\sqrt{2}e^{-2t}} = -\frac{1}{\sqrt{2}}$ M1

$\theta = 135^\circ$, which does not depend on t A1

Answer: 135° 4 marks

(c) Distance = $\int_0^\infty \sqrt{2}e^{-t} dt$ M1

$= \sqrt{2}[-e^{-t}]_0^\infty = \sqrt{2} \text{ m}$ A1

Answer: $\sqrt{2} \text{ m}$ 2 marks

29 (12 marks)

(a) At interception $|r_T| = 13T$ M1

$(40 - 8T)^2 + (4 + 3T)^2 = 169T^2$ M1 A1

$1600 - 640T + 64T^2 + 16 + 24T + 9T^2 = 169T^2$ M1

$96T^2 + 616T - 1616 = 0$, so $12T^2 + 77T - 202 = 0$ A1

Answer: $12T^2 + 77T - 202 = 0$ 5 marks

(b) $(T - 2)(12T + 101) = 0$ M1 A1

$T > 0$, so $T = 2$ minutes A1

Answer: $T = 2$ 3 marks

(c) Interception point $r_T(2) = 24\mathbf{i} + 10\mathbf{j} \text{ km}$, so velocity of M = $\frac{1}{2}(24\mathbf{i} + 10\mathbf{j}) = (12\mathbf{i} + 5\mathbf{j}) \text{ km per minute}$ M1 A1

Angle to $\mathbf{i}$: $\tan \theta = \frac{5}{12}$, so $\theta = 22.6^\circ$; distance = $13 \times 2 = 26 \text{ km}$ M1 A1

Answer: $(12\mathbf{i} + 5\mathbf{j}) \text{ km per min}$, 22.6° above $\mathbf{i}$; 26 km 4 marks

30 (9 marks)

$$BP = r - 4j = ti + (t^2 - 4)j \quad \text{M1}$$

$$BP^2 = t^2 + (t^2 - 4)^2 = t^4 - 7t^2 + 16 \quad \text{M1 A1}$$

$$\frac{d}{dt}BP^2 = 4t^3 - 14t = 0, \text{ so } t^2 = \frac{7}{2} \text{ (} t = 0 \text{ gives } BP^2 = 16, \text{ which is larger)} \quad \text{M1 A1}$$

$$BP^2 = \frac{49}{4} - \frac{49}{2} + 16 = \frac{15}{4}, \text{ so least distance} = \frac{\sqrt{15}}{2} \approx 1.94 \text{ m} \quad \text{A1}$$

$$v = i + 2tj, \text{ so speed}^2 = 1 + 4t^2 = 1 + 14 = 15 \quad \text{M1 A1}$$

$$\text{Speed} = \sqrt{15} = 2 \times \frac{\sqrt{15}}{2} \quad \text{A1}$$

$$\text{Answer: } \frac{\sqrt{15}}{2} \text{ m} \quad \text{9 marks}$$

Relative Motion**1 (2 marks)**

$$v_{AB} = v_A - v_B \quad \text{M1}$$

$$= (3i + 2j) - (i - 4j) = (2i + 6j) \text{ m s}^{-1} \quad \text{A1}$$

$$\text{Answer: } (2i + 6j) \text{ m s}^{-1} \quad \text{2 marks}$$

2 (3 marks)

The two velocities are perpendicular M1

$$\text{Speed} = \sqrt{5^2 + 3^2} = \sqrt{34} = 5.83 \text{ m s}^{-1} \quad \text{M1 A1}$$

$$\text{Answer: } 5.83 \text{ m s}^{-1} \quad \text{3 marks}$$

3 (3 marks)

$$v_{AB} = v_A - v_B \text{ has components 25 east and 20 south} \quad \text{M1}$$

$$\text{Speed} = \sqrt{625 + 400} = 32.0 \text{ m s}^{-1} \quad \text{M1 A1}$$

$$\text{Answer: } 32.0 \text{ m s}^{-1} \quad \text{3 marks}$$

4 (2 marks)

Wind from the south blows towards the north, so the speeds add M1

$$240 + 40 = 280 \text{ km h}^{-1} \quad \text{A1}$$

$$\text{Answer: } 280 \text{ km h}^{-1} \quad \text{2 marks}$$

5 (3 marks)

$$r_{AB} = (8i - 3j) - (2i + 5j) = (6i - 8j) \text{ m} \quad \text{M1 A1}$$

$$\text{Distance} = \sqrt{36 + 64} = 10 \text{ m} \quad \text{A1}$$

$$\text{Answer: } (6i - 8j) \text{ m; } 10 \text{ m} \quad \text{3 marks}$$

6 (3 marks)

$$v_{AB} = v_A - v_B, \text{ so } v_A = v_B + v_{AB} \quad \text{M1}$$

$$v_A = (2 + 5)i + (3 - 1)j = (7i + 2j) \text{ km h}^{-1} \quad \text{A1}$$

$$\text{Speed} = \sqrt{49 + 4} = 7.28 \text{ km h}^{-1} \quad \text{A1}$$

$$\text{Answer: } (7i + 2j) \text{ km h}^{-1}; 7.28 \text{ km h}^{-1} \quad \text{3 marks}$$

7 (4 marks)

$$\text{Time} = \frac{40}{2.5} = 16 \text{ s} \quad \text{M1 A1}$$

$$\text{Drift} = 1.5 \times 16 = 24 \text{ m} \quad \text{M1 A1}$$

$$\text{Answer: } 16 \text{ s; } 24 \text{ m} \quad \text{4 marks}$$

8 (4 marks)

The boat must head upstream so that the component along the river cancels the flow

M1

$5 \sin \theta = 3$, so $\theta = 36.9^\circ$ to the perpendicular

A1

Speed across $= 5 \cos \theta = 4 \text{ m s}^{-1}$

M1

Time $= \frac{80}{4} = 20 \text{ s}$

A1

Answer: 36.9° upstream of the perpendicular; 20 s

4 marks

9 (5 marks)

Wind velocity is 50 km h^{-1} towards the east: $v = (50, 200)$ (east, north)

M1 A1

Speed $= \sqrt{50^2 + 200^2} = 206 \text{ km h}^{-1}$

A1

Bearing $= \tan^{-1} \frac{50}{200} = 014^\circ$

M1 A1

Answer: 206 km h^{-1} on a bearing of 014°

5 marks

10 (5 marks)

$v_{AB} = (6\mathbf{i} + 8\mathbf{j}) - (-2\mathbf{i} + 3\mathbf{j}) = (8\mathbf{i} + 5\mathbf{j}) \text{ km h}^{-1}$

M1 A1

Magnitude $= \sqrt{64 + 25} = 9.43 \text{ km h}^{-1}$

A1

Bearing $= \tan^{-1} \frac{8}{5} = 058.0^\circ$

M1 A1

Answer: $(8\mathbf{i} + 5\mathbf{j}) \text{ km h}^{-1}$; 9.43 km h^{-1} on a bearing of 058.0°

5 marks

11 (5 marks)

Wind velocity (6 east) – cyclist velocity (8 north) = 6 east and 8 south

M1 A1

Speed $= \sqrt{36 + 64} = 10 \text{ m s}^{-1}$

A1

The relative wind is blowing towards a bearing of $\tan^{-1} \frac{6}{8}$ east of south, i.e. 143.1° , so it appears to come from a bearing of 323°

M1 A1

Answer: 10 m s^{-1} , from a bearing of 323°

5 marks

12 (5 marks)

The directions of travel differ by 90°

B1

Speed of P relative to Q $= \sqrt{12^2 + 9^2} = 15 \text{ km h}^{-1}$

M1 A1

$v_{PQ} = v_P - v_Q$, which is v_P plus the reverse of v_Q (bearing 330°)

M1

Bearing $= 060^\circ - \tan^{-1} \frac{9}{12} = 060^\circ - 36.9^\circ = 023^\circ$

A1

Answer: 15 km h^{-1} on a bearing of 023°

5 marks

13 (5 marks)

Distance to be covered relative to the goods train $= 180 + 120 = 300 \text{ m}$

B1

Same direction: relative speed $= 30 - 20 = 10 \text{ m s}^{-1}$, time $= \frac{300}{10} = 30 \text{ s}$

M1 A1

Opposite directions: relative speed $= 30 + 20 = 50 \text{ m s}^{-1}$, time $= \frac{300}{50} = 6 \text{ s}$

M1 A1

Answer: (i) 30 s; (ii) 6 s

5 marks

14 (6 marks)

Position of Q relative to P at time t hours: $d = (20 - 4t)\mathbf{i} + (18 - 3t)\mathbf{j}$

M1 A1

$|d|^2 = (20 - 4t)^2 + (18 - 3t)^2 = 25t^2 - 268t + 724$

M1 A1

Least when $t = \frac{268}{50} = 5.36 \text{ hours}$

M1

$|d|^2 = 724 - \frac{268^2}{100} = 5.76$, so the least distance is 2.4 km

A1

Answer: 2.4 km after 5.36 h

6 marks

15 (6 marks)

BC = 25t, AC = 15t, BA = 20 with a right angle at A

B1

$$(25t)^2 = 20^2 + (15t)^2$$

M1 A1

$$400t^2 = 400, \text{ so } t = 1 \text{ hour}$$

A1

$$\tan \theta = \frac{AC}{BA} = \frac{15}{20}, \text{ where } \theta = \text{angle ABC, so } \theta = 36.9^\circ$$

M1

$$\text{Bearing} = 090^\circ - 36.9^\circ = 053.1^\circ$$

A1

Answer: t = 1 h; bearing 053.1°

6 marks

16 (6 marks)

The wind blows towards the south at 40 km h^{-1} and the resultant must be due east

B1

North component of the aircraft's velocity in air: $250 \sin \varphi = 40$, where φ is the angle north of east

M1

$$\varphi = 9.21^\circ, \text{ so the bearing is } 090^\circ - 9.21^\circ = 080.8^\circ$$

A1

$$\text{Ground speed} = 250 \cos \varphi = 246.8 \text{ km h}^{-1}$$

M1 A1

$$\text{Time} = \frac{300}{246.8} = 1.216 \text{ h} = 1 \text{ h } 13 \text{ min}$$

A1

Answer: bearing 080.8°; 1 h 13 min

6 marks

17 (6 marks)

$$\mathbf{r}_p = (4 + 3t)\mathbf{i} + (10 + 2t)\mathbf{j}, \mathbf{r}_q = (30 - t)\mathbf{i} + (5t - 2)\mathbf{j}$$

M1 A1

Due east: the $\mathbf{j}$ components are equal: $10 + 2t = 5t - 2$

M1

$$t = 4 \text{ s}$$

A1

$$\mathbf{r}_q - \mathbf{r}_p = (26 - 4t)\mathbf{i} + (3t - 12)\mathbf{j} = 10\mathbf{i}, \text{ so the distance is } 10 \text{ m}$$

M1 A1

Answer: t = 4 s; 10 m

6 marks

18 (6 marks)

Let the wind velocity be $p\mathbf{i} + q\mathbf{j}$. Cycling east: relative wind = $(p - 5)\mathbf{i} + q\mathbf{j}$

M1

From the south means the relative wind blows north: $p - 5 = 0$, so $p = 5$

A1

Cycling north: relative wind = $p\mathbf{i} + (q - 10)\mathbf{j}$

M1

From the north-west the relative wind blows towards the south-east, so its components are equal in size: $p = -(q - 10)$, giving $q = 5$

M1 A1

Wind velocity = $5\mathbf{i} + 5\mathbf{j}$: speed $5\sqrt{2} = 7.07 \text{ m s}^{-1}$, blowing towards the north-east, so it comes from a bearing of 225°

A1

Answer: 7.07 m s^{-1} from a bearing of 225°

6 marks

19 (6 marks)

$$\text{Width} = 1.2 \times 50 = 60 \text{ m}$$

M1 A1

$$\text{Current} = \frac{30}{50} = 0.6 \text{ m s}^{-1}$$

M1 A1

$$\text{Speed relative to the bank} = \sqrt{1.2^2 + 0.6^2} = \sqrt{1.8} = 1.34 \text{ m s}^{-1}$$

M1 A1

Answer: 0.6 m s^{-1} ; 60 m; 1.34 m s^{-1}

6 marks

20 (7 marks)

$$\text{(a) PQ at time } t = (6\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}) + t(-3\mathbf{i} + 2\mathbf{j} - \mathbf{k})$$

M1 A1

$$PQ^2 = (6 - 3t)^2 + (2t - 3)^2 + (4 - t)^2$$

M1

$$= 14t^2 - 56t + 61$$

A1

Answer: $PQ^2 = 14t^2 - 56t + 61$

4 marks

$$\text{(b) } PQ^2 = 14(t - 2)^2 + 5$$

M1 A1

$$\text{Least distance} = \sqrt{5} = 2.24 \text{ m, when } t = 2 \text{ s}$$

A1

Answer: $\sqrt{5} \text{ m}$ (2.24 m) when $t = 2 \text{ s}$

3 marks

21 (9 marks)

Let θ be the angle upstream of the perpendicular. Across the river $5 \cos \theta$, downstream $6 - 5 \sin \theta$

M1 A1

The path has $\frac{\text{downstream}}{\text{across}} = \frac{90}{120} = \frac{3}{4}$: $\frac{6 - 5 \sin \theta}{5 \cos \theta} = \frac{3}{4}$

M1

$24 - 20 \sin \theta = 15 \cos \theta$, so $20 \sin \theta + 15 \cos \theta = 24$

A1

$25 \sin(\theta + 36.87^\circ) = 24$, so $\theta + 36.87^\circ = 73.74^\circ$ or 106.26°

M1 A1

$\theta = 36.9^\circ$ or $\theta = 69.4^\circ$

A1

Times = $\frac{120}{5 \cos \theta}$: 30 s and 68.2 s

M1 A1

Answer: $\theta = 36.9^\circ$ (30 s) or $\theta = 69.4^\circ$ (68.2 s)

9 marks

22 (8 marks)

(a) The resultant must be due east, so $300 \sin \varphi = 60$, $\sin \varphi = 0.2$

M1 A1

Ground speed = $300 \cos \varphi = 300\sqrt{1 - 0.04} = 293.9 \text{ km h}^{-1}$

M1

Time = $\frac{400}{293.9} = 1.36 \text{ h}$

A1

Answer: 1.36 h

4 marks

(b) The return has the same ground speed, so the total time is $\frac{800}{293.9} = 2.722 \text{ h}$

M1 A1

In still air: $\frac{800}{300} = 2.667 \text{ h} < 2.722 \text{ h}$

M1

Difference = $0.0556 \text{ h} \approx 3 \text{ minutes}$

A1

Answer: about 3 minutes longer

4 marks

23 (6 marks)

The velocities are vectors at right angles, so they cannot be added as speeds

E1

Speed = $\sqrt{8^2 + 6^2} = 10 \text{ m s}^{-1}$

M1 A1

Bearing = $\tan^{-1} \frac{6}{8} = 36.9^\circ$

M1 A1

Time = $\frac{2800}{10} = 280 \text{ s}$

A1

Answer: 10 m s^{-1} on a bearing of 36.9° ; 280 s

6 marks

24 (7 marks)

Rain relative to the cyclist has horizontal component $(p - c)$ and downward component q

M1

Speed 6: $\frac{6 - p}{q} = \frac{1}{2}$, so $q = 12 - 2p$

M1 A1

Speed 10: $\frac{10 - p}{q} = \tan 45^\circ = 1$, so $q = 10 - p$

M1 A1

$12 - 2p = 10 - p$, so $p = 2$ and $q = 8$

A1

Speed of the rain = $\sqrt{2^2 + 8^2} = 8.25 \text{ m s}^{-1}$

A1

Answer: $p = 2$, $q = 8$; speed 8.25 m s^{-1}

7 marks

25 (9 marks)

(a) Position of Q relative to P: $d = (24 - 6t)\mathbf{i} + (8t - 7)\mathbf{j}$

M1 A1

$PQ^2 = (24 - 6t)^2 + (8t - 7)^2 = 100t^2 - 400t + 625$

M1 A1

Answer: $PQ^2 = 100t^2 - 400t + 625$

4 marks

(b) $PQ^2 = 100(t - 2)^2 + 225$

M1

Least distance = 15 km when $t = 2$

A1

Answer: 15 km at $t = 2 \text{ h}$

2 marks

(c) $PQ = 17$: $100(t - 2)^2 + 225 = 289$

M1

$(t - 2)^2 = 0.64$, so $t = 1.2$ or $t = 2.8$

A1

Time = $2.8 - 1.2 = 1.6 \text{ hours}$

A1

Answer: 1.6 hours

3 marks

26 (12 marks)

(a) $|v_A| = \sqrt{144 + 25} = 13 \text{ km h}^{-1}$

B1

The direction from A to B is $(-3, -4)$, of length 5; $v_A \cdot (-3, -4) = 36 + 20 = 56$

M1

$$\cos \theta = \frac{56}{13 \times 5} = \frac{56}{65}, \text{ so } \theta = 30.5^\circ$$

A1

Answer: 13 km h⁻¹; 30.5°

3 marks

(b) $v_{AB} = v_A - v_B$, with $|v_B| = 5$, so v_{AB} lies on a circle of radius 5 centred at the end of v_A

E1

The least distance occurs when v_{AB} is as near as possible to the direction AB, i.e. along the

M1

tangent, at angle α to v_A with $\sin \alpha = \frac{5}{13}$

$$\text{Angle between } v_{AB} \text{ and } AB = \theta - \alpha, \text{ and } \sin(\theta - \alpha) = \frac{33}{65} \times \frac{12}{13} - \frac{56}{65} \times \frac{5}{13} = \frac{116}{845}$$

M1 A1

$$\text{Least distance} = 50 \times \frac{116}{845} = 6.86 \text{ km}$$

A1

Answer: 6.86 km

5 marks

(c) v_B is perpendicular to v_{AB} , with $|v_{AB}| = 12$, so $v_{AB} = 12(\cos 225.2^\circ, \sin 225.2^\circ) = (-8.45, -8.52)$

M1 A1

$$v_B = v_A - v_{AB} = (-3.55, 3.52)$$

M1

$$\text{Bearing} = 360^\circ - \arctan \frac{3.55}{3.52} = 315^\circ$$

A1

Answer: 315°

4 marks**27 (12 marks)**

(a) Position of F relative to C after t hours: $(20 - 3t)\mathbf{i} + (40 + 4t)\mathbf{j}$

M1

The cutter travels $25t$, so $(20 - 3t)^2 + (40 + 4t)^2 = (25t)^2$

M1 A1

$$2000 + 200t + 25t^2 = 625t^2$$

M1

$$600t^2 - 200t - 2000 = 0, \text{ so } 3t^2 - t - 10 = 0$$

A1

Answer: $3t^2 - t - 10 = 0$

5 marks

(b) $(3t + 5)(t - 2) = 0$

M1

$t > 0$, so $t = 2$

A1

Answer: $t = 2$

2 marks

(c) Interception point: $(20 - 6)\mathbf{i} + (40 + 8)\mathbf{j} = 14\mathbf{i} + 48\mathbf{j} \text{ km}$

B1

$$\text{Velocity of C} = \frac{1}{2}(14\mathbf{i} + 48\mathbf{j}) = (7\mathbf{i} + 24\mathbf{j}) \text{ km h}^{-1}$$

M1 A1

$$\text{Bearing} = \tan^{-1} \frac{7}{24} = 16.3^\circ$$

M1 A1

Answer: $(7\mathbf{i} + 24\mathbf{j}) \text{ km h}^{-1}$; bearing 016.3°

5 marks**28 (10 marks)**

(a) Distance after time t : $s^2 = d^2 - 2dut \cos \alpha + u^2t^2$ (cosine rule, or resolving along and perpendicular to QP)

M1 A1

$$\text{Least when } t = \frac{d \cos \alpha}{u}$$

M1

$$s_{\min}^2 = d^2 - d^2 \cos^2 \alpha = d^2 \sin^2 \alpha, \text{ so } s_{\min} = d \sin \alpha$$

A1

Answer: least distance = $d \sin \alpha$

4 marks

(b) $d = \sqrt{24^2 + 10^2} = 26 \text{ km}$, $u = \sqrt{15^2 + 8^2} = 17 \text{ km h}^{-1}$

B1

$$\cos \alpha = \frac{24 \times 15 + 10 \times 8}{26 \times 17} = \frac{440}{442}, \text{ so } \sin \alpha = \frac{\sqrt{442^2 - 440^2}}{442} = \frac{42}{442}$$

M1 A1

$$\text{Least distance} = 26 \times \frac{42}{442} = \frac{42}{17} = 2.47 \text{ km}$$

M1 A1

$$\text{Time} = \frac{d \cos \alpha}{u} = \frac{26 \times 440}{442 \times 17} = \frac{440}{289} = 1.52 \text{ h}$$

A1

Answer: 2.47 km at $t = 1.52 \text{ h}$

6 marks

29 (12 marks)**(a)** Position of Q relative to P: $(12 - 3t)\mathbf{i} + (9 + \lambda t)\mathbf{j}$ **M1**

$$PQ^2 = (12 - 3t)^2 + (9 + \lambda t)^2$$

M1

$$= 144 - 72t + 9t^2 + 81 + 18\lambda t + \lambda^2 t^2 = (9 + \lambda^2)t^2 + (18\lambda - 72)t + 225$$

A1**Answer: $PQ^2 = (9 + \lambda^2)t^2 + (18\lambda - 72)t + 225$** **3 marks****(b)** For $at^2 + bt + c$ the least value is $c - \frac{b^2}{4a}$ **M1**

$$= 225 - \frac{(18\lambda - 72)^2}{4(9 + \lambda^2)}$$

A1

$$= 225 - \frac{81(\lambda - 4)^2}{9 + \lambda^2}$$

A1**Answer: least $PQ^2 = 225 - \frac{81(\lambda - 4)^2}{9 + \lambda^2}$** **3 marks**

$$\text{(c)} \quad 225 - \frac{81(\lambda - 4)^2}{9 + \lambda^2} = 81$$

M1

$$144(9 + \lambda^2) = 81(\lambda - 4)^2, \text{ so } 16(9 + \lambda^2) = 9(\lambda - 4)^2$$

M1

$$7\lambda^2 + 72\lambda = 0$$

A1

$$\lambda = 0 \text{ or } \lambda = -\frac{72}{7}$$

A1**Answer: $\lambda = 0$ or $\lambda = -\frac{72}{7}$** **4 marks**

$$\text{(d)} \quad t = \frac{72 - 18\lambda}{2(9 + \lambda^2)} = \frac{36 - 9\lambda}{9 + \lambda^2}$$

M1

$$\lambda = 0: t = 4 \text{ hours}; \lambda = -\frac{72}{7}: t = \frac{28}{25} = 1.12 \text{ hours}$$

A1**Answer: 4 h and 1.12 h****2 marks****30 (12 marks)****(a)** Component of the boat's velocity across the river = $v \sin \theta$ **B1**Component downstream = $u - v \cos \theta$ **B1**

$$\text{Time to cross} = \frac{w}{v \sin \theta}, \text{ so } D = (u - v \cos \theta) \times \frac{w}{v \sin \theta}$$

M1 A1**Answer: $D = \frac{w(u - v \cos \theta)}{v \sin \theta}$** **4 marks**

$$\text{(b)} \quad \frac{dD}{d\theta} = \frac{w}{v} \times \frac{v \sin^2 \theta - (u - v \cos \theta) \cos \theta}{\sin^2 \theta}$$

M1 A1

$$= \frac{w}{v} \times \frac{v - u \cos \theta}{\sin^2 \theta}$$

A1

Zero when $\cos \theta = \frac{v}{u}$; the gradient is negative for smaller θ ($\cos \theta > \frac{v}{u}$) and positive for larger θ , so this is a minimum

E1

$$\sin \theta = \frac{\sqrt{u^2 - v^2}}{u}, \text{ so } D = \frac{w(u - v^2)}{u/v \sin \theta} = \frac{w\sqrt{u^2 - v^2}}{v}$$

M1 A1**Answer: $D_{\min} = \frac{w\sqrt{u^2 - v^2}}{v}$** **6 marks**

$$\text{(c)} \quad D_{\min} = \frac{120\sqrt{36 - 25}}{5} = 24\sqrt{11} = 79.6 \text{ m}$$

M1 A1**Answer: 79.6 m****2 marks****Advanced Projectiles****1 (3 marks)**

$$\text{(a)} \quad 44.1 = \frac{1}{2} \times 9.8t^2, \text{ so } t^2 = 9$$

M1

$$t = 3 \text{ s}$$

A1**Answer: 3 s****2 marks**

(b) $12 \times 3 = 36 \text{ m}$

Answer: 36 m

B1

1 mark

2 (2 marks)

Vertical: $0 = 20 \sin 30^\circ t - \frac{1}{2} \times 9.8t^2$

M1

$t = \frac{2 \times 10}{9.8} = 2.04 \text{ s}$

A1

Answer: 2.04 s

2 marks

3 (3 marks)

Vertical component of velocity = $14 \sin 30^\circ = 7 \text{ m s}^{-1}$

B1

$0 = 7^2 - 2 \times 9.8 \times H$, so $H = 2.5 \text{ m}$

M1 A1

Answer: 2.5 m

3 marks

4 (3 marks)

Time of flight = $\frac{2 \times 14 \sin 45^\circ}{9.8}$

M1

Range = $14 \cos 45^\circ \times \frac{2 \times 14 \sin 45^\circ}{9.8} = \frac{14^2 \times 2 \sin 45^\circ \cos 45^\circ}{9.8} = \frac{196}{9.8} = 20 \text{ m}$

M1 A1

Answer: 20 m

3 marks

5 (3 marks)

Vertical speed: $v^2 = 0 + 2 \times 9.8 \times 19.6 = 384.16$, so $v = 19.6 \text{ m s}^{-1}$

M1 A1

Speed = $\sqrt{8^2 + 19.6^2} = 21.2 \text{ m s}^{-1}$

A1

Answer: 21.2 m s⁻¹

3 marks

6 (3 marks)

(a) $10 = 6t + 4.9t^2$, so $4.9t^2 + 6t - 10 = 0$

M1

$t = \frac{-6 + \sqrt{36 + 196}}{9.8} = 0.942 \text{ s}$

A1

Answer: 0.942 s

2 marks

(b) Distance = $8 \times 0.942 = 7.54 \text{ m}$

B1

Answer: 7.54 m

1 mark

7 (5 marks)

(a) Time of flight $T = \frac{2 \times 28 \sin 40^\circ}{9.8} = 3.673 \text{ s}$

M1 A1

Range = $28 \cos 40^\circ \times 3.673 = 78.8 \text{ m}$

A1

Answer: 78.8 m

3 marks

(b) $0 = (28 \sin 40^\circ)^2 - 2 \times 9.8H$

M1

$H = 16.5 \text{ m}$

A1

Answer: 16.5 m

2 marks

8 (4 marks)

Vertical speed on landing: $v^2 = 2 \times 9.8 \times 30 = 588$, $v = 24.25 \text{ m s}^{-1}$

M1 A1

$\tan \alpha = \frac{24.25}{15}$

M1

$\alpha = 58^\circ$ below the horizontal

A1

Answer: 58°

4 marks

9 (4 marks)

Time: $10 = 20 \cos 60^\circ \times t$, so $t = 1 \text{ s}$

M1 A1

Height = $20 \sin 60^\circ \times 1 - \frac{1}{2} \times 9.8 \times 1^2$

M1

$= 12.4 \text{ m}$

A1

Answer: 12.4 m

4 marks

10 (5 marks)

Time to reach the wall: $25 = 20 \cos 45^\circ \times t$, so $t = \frac{25\sqrt{2}}{20} = 1.768 \text{ s}$

M1 A1

Height = $20 \sin 45^\circ \times 1.768 - 4.9 \times 1.768^2 = 25 - 15.3 = 9.69 \text{ m}$

M1 A1

$9.69 > 3$, so the ball clears the wall by 6.69 m

A1

Answer: clears the wall by 6.69 m

5 marks

11 (5 marks)

(a) $-1.5 = 14 \sin 30^\circ t - 4.9t^2$, so $4.9t^2 - 7t - 1.5 = 0$

M1 A1

$t = \frac{7 + \sqrt{49 + 29.4}}{9.8} = 1.62 \text{ s}$

A1

Answer: 1.62 s

3 marks

(b) $14 \cos 30^\circ \times 1.618 = 19.6 \text{ m}$

M1 A1

Answer: 19.6 m

2 marks

12 (7 marks)

Time to goal line: $t = \frac{30}{21 \cos 30^\circ} = 1.65 \text{ s}$

M1 A1

Time to highest point: $\frac{21 \sin 30^\circ}{9.8} = 1.07 \text{ s} < 1.65 \text{ s}$, so the ball is descending

M1 A1

Height = $21 \sin 30^\circ \times 1.65 - 4.9 \times 1.65^2 = 3.99 \text{ m}$

M1 A1

Clearance = $3.99 - 2.44 = 1.55 \text{ m}$

A1

Answer: descending (1.07 s < 1.65 s); clears the bar by 1.55 m

7 marks

13 (7 marks)

(a) Vertical (downwards): $30 = 12 \sin 25^\circ t + 4.9t^2$

M1 A1

$4.9t^2 + 5.07t - 30 = 0$, so $t = 2.01 \text{ s}$

A1

Answer: 2.01 s

3 marks

(b) Distance = $12 \cos 25^\circ \times 2.01 = 21.9 \text{ m}$

M1 A1

Answer: 21.9 m

2 marks

(c) Vertical speed = $12 \sin 25^\circ + 9.8 \times 2.01 = 24.8 \text{ m s}^{-1}$

M1

Speed = $\sqrt{(12 \cos 25^\circ)^2 + 24.8^2} = 27.1 \text{ m s}^{-1}$

A1

Answer: 27.1 m s⁻¹

2 marks

14 (6 marks)

$x = U \cos \theta t$ and $y = U \sin \theta t - \frac{1}{2} g t^2$

B1 B1

From the first equation $t = \frac{x}{U \cos \theta}$

M1

$y = U \sin \theta \times \frac{x}{U \cos \theta} - \frac{1}{2} g \frac{x^2}{U^2 \cos^2 \theta} = x \tan \theta - \frac{g x^2}{2 U^2 \cos^2 \theta}$

M1 A1

Using $\frac{1}{\cos^2 \theta} = 1 + \tan^2 \theta$ gives the second form

A1

Answer: $y = x \tan \theta - \frac{g x^2}{2 U^2 \cos^2 \theta} = x \tan \theta - \frac{g x^2}{2 U^2} (1 + \tan^2 \theta)$

6 marks

15 (6 marks)

$3 = 14 \tan \theta - \frac{9.8 \times 14^2}{2 \times 14^2} (1 + \tan^2 \theta)$

M1 A1

$3 = 14 \tan \theta - 4.9(1 + \tan^2 \theta)$, so $4.9 \tan^2 \theta - 14 \tan \theta + 7.9 = 0$

M1

$49 \tan^2 \theta - 140 \tan \theta + 79 = 0$

A1

$\tan \theta = \frac{140 \pm \sqrt{19600 - 15484}}{98} = 0.774 \text{ or } 2.083$

M1

$\theta = 37.7^\circ \text{ or } 64.4^\circ$

A1

Answer: $\theta = 37.7^\circ \text{ or } 64.4^\circ$

6 marks

16 (7 marks)

Vertical component $U \sin \theta$: $0 = (U \sin \theta)^2 - 2 \times 9.8 \times 20$, so $U \sin \theta = 19.8$

M1 A1

Time to highest point = $\frac{19.8}{9.8} = 2.02$ s

M1

Horizontal component $U \cos \theta = \frac{30}{2.02} = 14.8$

M1 A1

$U = \sqrt{19.8^2 + 14.8^2} = 24.7 \text{ m s}^{-1}$, $\tan \theta = \frac{19.8}{14.8}$, so $\theta = 53.1^\circ$

M1 A1

Answer: 24.7 m s⁻¹ at 53.1° above the horizontal

7 marks

17 (6 marks)

Perpendicular to plane: initial component $14 \sin 30^\circ = 7$, acceleration $-g \cos 30^\circ = -8.487$

M1 A1

Time of flight: $0 = 7T - \frac{1}{2} \times 8.487T^2$, so $T = 1.650$ s

M1 A1

Parallel to plane: initial component $14 \cos 30^\circ = 12.12$, acceleration $-g \sin 30^\circ = -4.9$

M1

$OP = 12.12T - \frac{1}{2} \times 4.9T^2 = 20.0 - 6.67 = 13.3$ m

A1

Answer: 13.3 m

6 marks

18 (9 marks)

(a) $-45 = 21 \sin 30^\circ t - 4.9t^2$, so $4.9t^2 - 10.5t - 45 = 0$

M1 A1

$t = \frac{10.5 + \sqrt{110.25 + 882}}{9.8} = \frac{10.5 + 31.5}{9.8} = \frac{30}{7} = 4.29$ s

M1 A1

Answer: 4.29 s

4 marks

(b) Distance = $21 \cos 30^\circ \times \frac{30}{7} = 77.9$ m

M1 A1

Vertical velocity = $10.5 - 9.8 \times \frac{30}{7} = -31.5 \text{ m s}^{-1}$

M1

Speed = $\sqrt{(21 \cos 30^\circ)^2 + 31.5^2} = 36.4 \text{ m s}^{-1}$

M1 A1

Answer: 77.9 m; 36.4 m s⁻¹

5 marks

19 (9 marks)

(a) $y = x \tan \theta - \frac{g x^2}{2U^2}(1 + \tan^2 \theta)$, so $7 = 24T - \frac{9.8 \times 576}{2U^2}(1 + T^2)$

M1 A1

With $k = \frac{2822.4}{U^2}$: $kT^2 - 24T + (k + 7) = 0$

A1

Answer: $kT^2 - 24T + (k + 7) = 0$

3 marks

(b) For a real T: $576 - 4k(k + 7) \geq 0$, so $k^2 + 7k - 144 \leq 0$

M1 A1

$(k - 9)(k + 16) \leq 0$, so $k \leq 9$ ($k > 0$)

M1

$U^2 \geq \frac{2822.4}{9} = 313.6$, so least $U = 17.7 \text{ m s}^{-1}$

A1

Answer: 17.7 m s⁻¹

4 marks

(c) $k = 9$ gives a repeated root $T = \frac{24}{2k} = \frac{4}{3}$

M1

$\theta = 53.1^\circ$

A1

Answer: 53.1°

2 marks

20 (7 marks)

(a) Drop to the net = $2.4 - 0.9 = 1.5$ m, so $1.5 = 4.9t^2$ and $t = 0.553$ s

M1 A1

$U = \frac{12}{0.553} = 21.7 \text{ m s}^{-1}$

A1

Answer: 21.7 m s⁻¹

3 marks

(b) Time to fall 2.4 m: $2.4 = 4.9t^2$, so $t = 0.700$ s

M1 A1

Horizontal distance = $21.7 \times 0.700 = 15.2$ m < 18.4 m, so the serve is in

M1 A1

Answer: 15.2 m; the serve is in

4 marks

21 (10 marks)

$$(a) T = \frac{2U \sin \theta}{g}$$

M1 A1

$$R = U \cos \theta \times T = \frac{2U^2 \sin \theta \cos \theta}{g} = \frac{U^2 \sin 2\theta}{g}$$

M1 A1

$$\text{Answer: } R = \frac{U^2 \sin 2\theta}{g}$$

4 marks

$$(b) \sin 2(90^\circ - \theta) = \sin(180^\circ - 2\theta) = \sin 2\theta$$

M1 A1

Range is greatest when $\sin 2\theta = 1$, i.e. $\theta = 45^\circ$

A1

Answer: equal ranges; 45°

3 marks

$$(c) \sin 2\theta = \frac{9.8 \times 60}{784} = 0.75$$

M1

 $2\theta = 48.6^\circ$ or 131.4° , so $\theta = 24.3^\circ$ or 65.7°

A1 A1

Answer: 24.3° or 65.7°

3 marks

22 (8 marks)

$$(a) 15 = 10t, \text{ so } t = 1.5 \text{ s}$$

M1 A1

$$\text{Height} = 19.6 - 4.9 \times 1.5^2 = 8.575 \text{ m}$$

M1 A1

Answer: 1.5 s; 8.575 m

4 marks

$$(b) \text{Total fall time: } 19.6 = 4.9T^2, \text{ so } T = 2 \text{ s}$$

M1 A1

After the rebound the ball travels back at 10 m s^{-1} for $2 - 1.5 = 0.5 \text{ s}$: distance = 5 m

M1 A1

Answer: 5 m from the wall, on the side of A

4 marks

23 (7 marks)P: horizontal velocity 15 m s^{-1} , vertical velocity 20 m s^{-1}

B1

$$\text{Horizontal distance } 30 \text{ m at } t = \frac{30}{15} = 2 \text{ s}$$

M1 A1

$$\text{Height of P at } t = 2: 20 \times 2 - 4.9 \times 4 = 20.4 \text{ m}$$

M1

$$\text{Height of Q at } t = 2: 40 - 4.9 \times 4 = 20.4 \text{ m, so P and Q are at the same point}$$

M1 A1

They collide after 2 s at a height of 20.4 m

A1

Answer: collide at $t = 2 \text{ s}$, 20.4 m above the ground

7 marks

24 (8 marks)

$$\text{Path: } y = ax - bx^2 \text{ where } a = \tan \theta \text{ and } b = \frac{g}{2U^2 \cos^2 \theta}$$

M1 A1

$$7.1 = 16a - 256b \text{ and } 4.4 = 32a - 1024b$$

M1

$$\text{Subtract twice the first from the second: } -9.8 = -512b, \text{ so } b = \frac{9.8}{512}, \text{ and } a = 0.75$$

M1 A1

$$\tan \theta = 0.75, \text{ so } \theta = 36.9^\circ; U^2 = \frac{g(1 + \tan^2 \theta)}{2b} = \frac{9.8 \times 1.5625 \times 512}{2 \times 9.8} = 400, \text{ so } U = 20 \text{ m s}^{-1}$$

M1 A1

$$\text{Greatest height} = \frac{(20 \sin \theta)^2}{2 \times 9.8} = \frac{144}{19.6} = 7.35 \text{ m}$$

A1

Answer: $\theta = 36.9^\circ$, $U = 20 \text{ m s}^{-1}$, greatest height 7.35 m

8 marks

25 (6 marks)

Only the vertical component of velocity is zero at the highest point; the horizontal component is unchanged and non-zero

E1

The student used the full speed, not the vertical component $24.5 \sin 30^\circ = 12.25$

E1

$$0 = 12.25^2 - 2 \times 9.8H$$

M1

$$H = 7.66 \text{ m (3 s.f.)}$$

A1

$$\text{Speed at the highest point} = 24.5 \cos 30^\circ = 21.2 \text{ m s}^{-1}$$

M1 A1

Answer: $H = 7.66 \text{ m}$; speed 21.2 m s^{-1}

6 marks

26 (8 marks)

(a) At the highest point: vertical velocity zero, so time $t = \frac{U \sin \theta}{g}$ M1

$$d = U \cos \theta \times t = \frac{U^2 \sin \theta \cos \theta}{g} \text{ and } H = \frac{U^2 \sin^2 \theta}{2g} \quad \text{M1 A1}$$

$$\text{Dividing: } \frac{H}{d} = \frac{\sin \theta}{2 \cos \theta} = \frac{1}{2} \tan \theta, \text{ so } \tan \theta = \frac{2H}{d} \quad \text{A1}$$

$$\text{Answer: } \tan \theta = \frac{2H}{d} \quad \text{4 marks}$$

(b) $\tan \theta = \frac{14}{24} = \frac{7}{12}$, so $\sin \theta = \frac{7}{\sqrt{193}}$ M1

$$H = \frac{U^2 \sin^2 \theta}{2g}: 7 = \frac{U^2 \times 49}{193 \times 19.6}, \text{ so } U^2 = \frac{7 \times 19.6 \times 193}{49} = 540.4 \quad \text{M1 A1}$$

$$U = 23.2 \text{ m s}^{-1} \quad \text{A1}$$

$$\text{Answer: } 23.2 \text{ m s}^{-1} \quad \text{4 marks}$$

27 (12 marks)

(a) Perpendicular to the plane: $0 = U \sin \beta T - \frac{1}{2} g \cos \alpha T^2$, so $T = \frac{2U \sin \beta}{g \cos \alpha}$ M1 A1

Parallel: $OP = U \cos \beta T - \frac{1}{2} g \sin \alpha T^2$ M1

$$= \frac{2U^2}{g \cos^2 \alpha} (\sin \beta \cos \beta \cos \alpha - \sin^2 \beta \sin \alpha) \quad \text{A1}$$

$$= \frac{2U^2 \sin \beta \cos(\alpha + \beta)}{g \cos^2 \alpha} \quad \text{A1}$$

$$\text{Answer: } OP = \frac{2U^2 \sin \beta \cos(\alpha + \beta)}{g \cos^2 \alpha} \quad \text{5 marks}$$

(b) $2 \sin \beta \cos(\alpha + \beta) = \sin(\alpha + 2\beta) - \sin \alpha$ M1

$$OP = \frac{U^2}{g \cos^2 \alpha} (\sin(\alpha + 2\beta) - \sin \alpha), \text{ which is greatest when } \sin(\alpha + 2\beta) = 1 \quad \text{M1 A1}$$

$$OP_{\max} = \frac{U^2(1 - \sin \alpha)}{g \cos^2 \alpha} = \frac{U^2(1 - \sin \alpha)}{g(1 - \sin \alpha)(1 + \sin \alpha)} = \frac{U^2}{g(1 + \sin \alpha)} \quad \text{M1 A1}$$

$$\text{Answer: } OP_{\max} = \frac{U^2}{g(1 + \sin \alpha)} \quad \text{5 marks}$$

(c) $\alpha + 2\beta = 90^\circ$, so $\beta = 35^\circ$ B1

$$OP_{\max} = \frac{441}{9.8(1 + \sin 20^\circ)} = 33.5 \text{ m} \quad \text{B1}$$

$$\text{Answer: } \beta = 35^\circ; 33.5 \text{ m} \quad \text{2 marks}$$

28 (11 marks)

(a) Vertical: $0 = 19.6T - 4.9T^2$, so $T = 4 \text{ s}$ M1 A1

Horizontal displacement at $T = 4$: $16 \times 4 - \frac{1}{2} \times 8 \times 4^2 = 64 - 64 = 0$ M1 A1

$$\text{Answer: lands at O after 4 s} \quad \text{4 marks}$$

(b) Velocity at $t = 4$: horizontal $16 - 8 \times 4 = -16$, vertical $19.6 - 9.8 \times 4 = -19.6$ M1 A1

$$\text{Speed} = \sqrt{16^2 + 19.6^2} = 25.3 \text{ m s}^{-1} \quad \text{A1}$$

$$\tan \alpha = \frac{19.6}{16}, \text{ so } \alpha = 50.8^\circ \text{ below the horizontal} \quad \text{M1 A1}$$

$$\text{Answer: } 25.3 \text{ m s}^{-1}; 50.8^\circ \quad \text{5 marks}$$

(c) The horizontal velocity is zero when $16 - 8t = 0$, so $t = 2$ M1

$$x = 16 \times 2 - 4 \times 2^2 = 16 \text{ m} \quad \text{A1}$$

$$\text{Answer: } 16 \text{ m} \quad \text{2 marks}$$

29 (12 marks)

(a) $4.9 = 4.9t^2$ M1

$$t = 1 \text{ s} \quad \text{A1}$$

$$\text{Answer: } 1 \text{ s} \quad \text{2 marks}$$

(b) Vertical speed just before the first bounce = 9.8 m s^{-1} ; just after the n-th bounce it is $9.8(0.5)^n$ upwards

M1 A1

$$\text{Time between bounces} = \frac{2 \times 9.8(0.5)^n}{9.8} = 2(0.5)^n$$

A1

Answer: time between bounces = $2(0.5)^n \text{ s}$

3 marks

(c) Total time = $1 + 2(0.5) + 2(0.5)^2 + \dots = 1 + \frac{1}{1 - 0.5} = 3 \text{ s}$

M1 A1

Horizontal distance = $6 \times 3 = 18 \text{ m}$

A1

Answer: 3 s; 18 m

3 marks

(d) After the n-th bounce the ball rises to height $\frac{(9.8 \times 0.5^n)^2}{2 \times 9.8} = 4.9(0.25)^n \text{ m}$

M1 A1

Total = $4.9 + 2(4.9)(0.25 + 0.25^2 + \dots) = 4.9 + 9.8 \times \frac{0.25}{0.75}$

M1

= 8.17 m

A1

Answer: 8.17 m

4 marks

30 (11 marks)

(a) $x = U \cos \theta t$, so $R = U \cos \theta t$ and $-h = U \sin \theta t - \frac{1}{2}g t^2$

M1

$-h = R \tan \theta - \frac{g R^2}{2U^2}(1 + \tan^2 \theta)$, which rearranges to the given equation

A1

Answer: $\frac{g R^2}{2U^2} \tan^2 \theta - R \tan \theta + (\frac{g R^2}{2U^2} - h) = 0$

2 marks

(b) The equation must have real roots for $\tan \theta$: $R^2 - 4(\frac{g R^2}{2U^2})(\frac{g R^2}{2U^2} - h) \geq 0$

M1

Divide by R^2 : $1 - \frac{g^2 R^2}{U^4} + \frac{2g h}{U^2} \geq 0$

M1 A1

$R^2 \leq \frac{U^2(U^2 + 2g h)}{g^2}$, so $R_{\max} = \frac{U}{g} \sqrt{U^2 + 2g h}$

M1 A1

Answer: $R_{\max} = \frac{U}{g} \sqrt{U^2 + 2g h}$

5 marks

(c) $R_{\max} = \frac{28}{9.8} \sqrt{784 + 441} = \frac{28}{9.8} \times 35 = 100 \text{ m}$

M1 A1

At the greatest R the roots coincide: $\tan \theta = \frac{R}{2 \times g R^2 / 2U^2} = \frac{U^2}{g R} = \frac{784}{980} = 0.8$

M1

$\theta = 38.7^\circ$

A1

Answer: 100 m at 38.7° above the horizontal

4 marks

Motion Under Variable Forces

1 (2 marks)

$a = \frac{F}{m} = 3t$

M1

$v = \int 3t \, dt = \frac{3}{2}t^2$, so when $t = 3$, $v = 13.5 \text{ m s}^{-1}$

A1

Answer: 13.5 m s^{-1}

2 marks

2 (2 marks)

$F = 4 + 6 = 10 \text{ N}$

M1

$a = \frac{10}{0.5} = 20 \text{ m s}^{-2}$

A1

Answer: 20 m s^{-2}

2 marks

3 (2 marks)

$$\text{Impulse} = \int_0^2 12t \, dt$$

M1

$$= [6t^2]_0^2 = 24 \, \text{N s}$$

A1

Answer: 24 N s

2 marks

4 (2 marks)

$$\text{Work done} = \int_1^4 6x \, dx$$

M1

$$= [3x^2]_1^4 = 48 - 3 = 45 \, \text{J}$$

A1

Answer: 45 J

2 marks

5 (2 marks)

$$F = 12 \times 2 = 24 \, \text{N}$$

M1

$$a = \frac{24}{3} = 8 \, \text{m s}^{-2}$$

A1

Answer: 8 m s⁻²

2 marks

6 (3 marks)

$$v = \frac{dx}{dt} = 6t^2 - 2t \text{ and } a = \frac{dv}{dt} = 12t - 2$$

M1 A1

$$\text{When } t = 2, a = 22 \, \text{m s}^{-2}, \text{ so } F = ma = 3 \times 22 = 66 \, \text{N}$$

A1

Answer: 66 N

3 marks

7 (4 marks)

$$a = \frac{6 - 2t}{2} = 3 - t$$

M1

$$v = 3t - \frac{1}{2}t^2 + c, \text{ and } v = 1 \text{ at } t = 0, \text{ so } c = 1$$

M1 A1

$$\text{When } t = 3, v = 9 - 4.5 + 1 = 5.5 \, \text{m s}^{-1}$$

A1

Answer: 5.5 m s⁻¹

4 marks

8 (4 marks)

$$\text{Impulse} = \int_0^\infty 20e^{-2t} \, dt$$

M1

$$= [-10e^{-2t}]_0^\infty = 10 \, \text{N s}$$

A1

$$\text{Impulse} = \text{change in momentum: } 10 = 0.4v$$

M1

$$v = 25 \, \text{m s}^{-1}$$

A1

Answer: 10 N s; 25 m s⁻¹

4 marks

9 (4 marks)

$$\text{Work done} = \int_0^4 (12 - 2x) \, dx = [12x - x^2]_0^4 = 32 \, \text{J}$$

M1 A1

$$\text{Work-energy: } \frac{1}{2} \times 2v^2 - \frac{1}{2} \times 2 \times 2^2 = 32$$

M1

$$v^2 = 36, \text{ so } v = 6 \, \text{m s}^{-1}$$

A1

Answer: 6 m s⁻¹

4 marks

10 (4 marks)

$$\text{Work done by the force} = \int_1^2 6x \, dx = [3x^2]_1^2 = 9 \, \text{J}$$

M1 A1

$$\frac{1}{2} \times 1.5v^2 = 9$$

M1

$$v^2 = 12, \text{ so } v = 3.46 \, \text{m s}^{-1}$$

A1

Answer: 3.46 m s⁻¹

4 marks

11 (4 marks)

$$a = \frac{12 - 6t}{3} = 4 - 2t$$

M1

$$v = 4t - t^2$$

A1

Greatest speed when $a = 0$, so $t = 2$ s

M1

$$v = 8 - 4 = 4 \text{ m s}^{-1}$$

A1

Answer: 4 m s⁻¹ at t = 2 s

4 marks

12 (5 marks)

$$(a) a = v \frac{dv}{dx} = (4 - x^2)(-2x)$$

M1 A1

When $x = 1$, $a = 3 \times (-2) = -6 \text{ m s}^{-2}$, so $F = 1.5 \times (-6) = -9 \text{ N}$, i.e. 9 N towards the origin

A1

Answer: 9 N in the direction of x decreasing

3 marks

$$(b) \text{ When } x = 3, v = -5 \text{ and } \frac{dv}{dx} = -6, \text{ so } a = 30 \text{ m s}^{-2}$$

M1

$F = 1.5 \times 30 = 45 \text{ N}$ in the direction of x increasing

A1

Answer: 45 N in the direction of x increasing

2 marks

13 (6 marks)

$$(a) \frac{1}{2} \times 2 \times v^2 = \int_0^x 8e^{-x} dx$$

M1 A1

$$v^2 = [-8e^{-x}]_0^x = 8(1 - e^{-x})$$

A1

$$\text{Answer: } v^2 = 8(1 - e^{-x})$$

3 marks

$$(b) e^{-\ln 4} = \frac{1}{4}, \text{ so } v^2 = 8 \times \frac{3}{4} = 6 \text{ and } v = \sqrt{6} = 2.45 \text{ m s}^{-1}$$

M1 A1

As $x \rightarrow \infty$, $e^{-x} \rightarrow 0$, so $v^2 \rightarrow 8$ and $v \rightarrow 2\sqrt{2} = 2.83 \text{ m s}^{-1}$

A1

$$\text{Answer: } 2.45 \text{ m s}^{-1}; 2\sqrt{2} = 2.83 \text{ m s}^{-1}$$

3 marks

14 (5 marks)

$$\text{Newton's second law: } 2v \frac{dv}{dx} = -\frac{72}{x^2}$$

M1

$$\int v dv = \int -\frac{36}{x^2} dx \text{ gives } \frac{1}{2}v^2 = \frac{36}{x} + c$$

M1 A1

$$v = 0 \text{ at } x = 6: c = -6, \text{ so } v^2 = \frac{72}{x} - 12$$

A1

When $x = 2: v^2 = 36 - 12 = 24$, so $v = 4.90 \text{ m s}^{-1}$

A1

$$\text{Answer: } 4.90 \text{ m s}^{-1}$$

5 marks

15 (5 marks)

Impulse = area under the graph

M1

$$= \frac{1}{2} \times 0.06 \times 120 = 3.6 \text{ N s}$$

M1 A1

Impulse = change in momentum: $3.6 = 0.4v$

M1

$$v = 9 \text{ m s}^{-1}$$

A1

$$\text{Answer: } 3.6 \text{ N s; } 9 \text{ m s}^{-1}$$

5 marks

16 (5 marks)

$$\text{Impulse for } 0 \leq t \leq 4: \int_0^4 3t dt = \left[\frac{3}{2}t^2\right]_0^4 = 24 \text{ N s}$$

M1 A1

$$\text{Impulse for } 4 < t \leq 6: 12 \times 2 = 24 \text{ N s}$$

M1

$$\text{Total impulse} = 48 \text{ N s} = 3v$$

M1

$$v = 16 \text{ m s}^{-1}$$

A1

$$\text{Answer: } 16 \text{ m s}^{-1}$$

5 marks

17 (6 marks)

$$a = 5 \sin 2t, \text{ so } v = -\frac{5}{2} \cos 2t + c; v = 0 \text{ at } t = 0 \text{ gives } c = \frac{5}{2}, \text{ so } v = \frac{5}{2}(1 - \cos 2t)$$

M1 A1

$$v = 0 \text{ when } \cos 2t = 1, \text{ so the first time after } t = 0 \text{ is } t = \pi \text{ s}$$

M1 A1

$$x = \frac{5}{2}(t - \frac{1}{2} \sin 2t), \text{ so at } t = \pi, x = \frac{5\pi}{2}$$

M1

$$x = 7.85 \text{ m}$$

A1

Answer: $t = \pi$ s; $x = 7.85$ m

6 marks

18 (6 marks)

$$(a) \text{ Work-energy: } \frac{1}{2} \times 1200v^2 = \int_0^{100} (3000 - 20x) dx$$

M1

$$= [3000x - 10x^2]_0^{100} = 200\,000$$

A1

$$v^2 = \frac{1000}{3}, \text{ so } v = 18.3 \text{ m s}^{-1}$$

A1

Answer: 18.3 m s^{-1}

3 marks

$$(b) \text{ Greatest speed when the force is zero: } x = 150$$

M1

$$\frac{1}{2} \times 1200v^2 = 3000 \times 150 - 10 \times 150^2 = 225\,000, \text{ so } v^2 = 375$$

M1

$$v = 19.4 \text{ m s}^{-1}$$

A1

Answer: 19.4 m s^{-1}

3 marks

19 (5 marks)

$$3v \frac{dv}{dx} = \frac{6}{2x+1}, \text{ so } \int v dv = \int \frac{2}{2x+1} dx$$

M1

$$\frac{1}{2}v^2 = \ln(2x+1) + c$$

A1

$$v = 1 \text{ at } x = 0: c = \frac{1}{2}, \text{ so } v^2 = 2 \ln(2x+1) + 1$$

M1 A1

$$\text{When } x = 4: v^2 = 2 \ln 9 + 1 = 5.394, \text{ so } v = 2.32 \text{ m s}^{-1}$$

A1

Answer: 2.32 m s^{-1}

5 marks

20 (8 marks)

$$(a) 2 \frac{dv}{dt} = 4 - v, \text{ so } t = \int_0^3 \frac{2}{4-v} dv$$

M1 A1

$$= [-2 \ln(4-v)]_0^3 = -2 \ln 1 + 2 \ln 4 = 2 \ln 4$$

M1

$$= 2.77 \text{ s}$$

A1

Answer: $2 \ln 4 = 2.77 \text{ s}$

4 marks

$$(b) 2v \frac{dv}{dx} = 4 - v, \text{ so } x = \int_0^3 \frac{2v}{4-v} dv$$

M1

$$= \int_0^3 (-2 + \frac{8}{4-v}) dv = [-2v - 8 \ln(4-v)]_0^3$$

M1 A1

$$= -6 + 8 \ln 4 = 5.09 \text{ m}$$

A1

Answer: 5.09 m

4 marks

21 (10 marks)

$$(a) m v \frac{dv}{dx} = -\frac{mgR^2}{x^2}, \text{ so } \int v dv = -gR^2 \int x^{-2} dx$$

M1 A1

$$\frac{1}{2}v^2 = \frac{gR^2}{x} + c$$

A1

$$v = U \text{ at } x = R: c = \frac{1}{2}U^2 - gR, \text{ so } v^2 = U^2 - 2gR + \frac{2gR^2}{x}$$

A1

$$\text{Answer: } v^2 = U^2 - 2gR + \frac{2gR^2}{x}$$

4 marks

$$(b) v = 0: \frac{2gR^2}{x} = 2gR - U^2, \text{ so } x = \frac{2gR^2}{2gR - U^2} = 1.307 \times 10^7 \text{ m}$$

M1 A1

$$\text{Height} = x - R = 6.67 \times 10^6 \text{ m}$$

A1

Answer: $6.67 \times 10^6 \text{ m}$

3 marks

(c) $v^2 > 0$ for all x requires $U^2 - 2gR \geq 0$, since $\frac{2gR^2}{x} > 0$ tends to 0

M1 A1

$$U = \sqrt{2 \times 9.8 \times 6.4 \times 10^6} = 11\,200 \text{ m s}^{-1}$$

A1

Answer: 11 200 m s⁻¹

3 marks

22 (8 marks)

(a) $\mathbf{a} = \frac{\mathbf{F}}{m} = 6t\mathbf{i} + 4 \cos 2t\mathbf{j}$

M1

$$\mathbf{v} = 3t^2\mathbf{i} + 2 \sin 2t\mathbf{j} \quad (\mathbf{v} = 0 \text{ at } t = 0)$$

M1 A1

$$\mathbf{r} = t^3\mathbf{i} + (1 - \cos 2t)\mathbf{j} \quad (\mathbf{r} = 0 \text{ at } t = 0)$$

A1

Answer: $\mathbf{v} = 3t^2\mathbf{i} + 2 \sin 2t\mathbf{j}$, $\mathbf{r} = t^3\mathbf{i} + (1 - \cos 2t)\mathbf{j}$

4 marks

(b) At $t = \frac{\pi}{2}$: $\mathbf{v} = \frac{3\pi^2}{4}\mathbf{i} + 0\mathbf{j}$, so speed = $\frac{3\pi^2}{4} = 7.40 \text{ m s}^{-1}$

M1 A1

$$\text{Work done} = \text{change in kinetic energy} = \frac{1}{2} \times 2 \times \left(\frac{3\pi^2}{4}\right)^2 = \frac{9\pi^4}{16} = 54.8 \text{ J}$$

M1 A1

Answer: 7.40 m s⁻¹; 54.8 J

4 marks

23 (5 marks)

The acceleration is not constant, so the constant acceleration formula $\mathbf{v} = \mathbf{u} + \mathbf{a}t$ cannot be used

E1

$$\mathbf{a} = \frac{6t}{2} = 3t$$

M1

$$\mathbf{v} = \int 3t \, dt = \frac{3}{2}t^2 \quad (\mathbf{v} = 0 \text{ at } t = 0)$$

M1 A1

$$\text{When } t = 4, \mathbf{v} = 24 \text{ m s}^{-1}$$

A1

Answer: 24 m s⁻¹

5 marks

24 (7 marks)

(a) Net force = $30 - 4x$, which is zero when $x = 7.5$

M1 A1

$$\frac{1}{2} \times 5v^2 = \int_0^{7.5} (30 - 4x) \, dx = 225 - 112.5 = 112.5$$

M1 A1

$$v^2 = 45, \text{ so } v = 6.71 \text{ m s}^{-1}$$

A1

Answer: 6.71 m s⁻¹

5 marks

(b) $\frac{1}{2} \times 5v^2 = 30x - 2x^2 = 0$ at rest

M1

$$x = 15 \text{ m}$$

A1

Answer: 15 m

2 marks

25 (7 marks)

(a) $\mathbf{a} = \frac{600 - 20t}{400} = 1.5 - 0.05t$

M1

$$\mathbf{v} = 1.5t - 0.025t^2 \quad (\mathbf{v} = 0 \text{ at } t = 0)$$

M1 A1

$$\text{When } t = 30: \mathbf{v} = 45 - 22.5 = 22.5 \text{ m s}^{-1}$$

A1

Answer: 22.5 m s⁻¹

4 marks

(b) $x = \int (1.5t - 0.025t^2) \, dt = 0.75t^2 - \frac{t^3}{120}$

M1 A1

$$\text{When } t = 30: x = 675 - 225 = 450 \text{ m}$$

A1

Answer: 450 m

3 marks

26 (7 marks)

(a) The speed is greatest at O, where the force is zero

B1

$$\frac{1}{2}v^2 = \int_0^{1.5} (4x + 2x^3) \, dx = [2x^2 + \frac{1}{2}x^4]_0^{1.5} = 4.5 + 2.53125$$

M1 A1

$$v = 3.75 \text{ m s}^{-1}$$

A1

Answer: 3.75 m s⁻¹

4 marks

(b) $\frac{1}{2}v^2 = (2 \times 2.25 + 2.53125) - (2 \times 1 + \frac{1}{2})$

M1

$$\frac{1}{2}v^2 = 4.53125, \text{ so } v = 3.01 \text{ m s}^{-1}$$

A1
Answer: 3.01 m s⁻¹
2 marks

(c) The force always acts towards O, and the force is odd in x, so the motion on the negative side mirrors that on the positive side; the particle reaches rest at $x = -1.5$ and returns

E1
Answer: force always towards O; symmetric
1 mark

27 (12 marks)

(a) $2v \frac{dv}{dx} = -\frac{16}{x^2}$, so $\int v dv = -8 \int x^{-2} dx$

M1 A1

$$\frac{1}{2}v^2 = \frac{8}{x} + c, \text{ with } v = 0 \text{ at } x = 4: c = -2$$

M1

$$v^2 = \frac{16}{x} - 4 = 16\left(\frac{1}{x} - \frac{1}{4}\right)$$

A1

Answer: $v^2 = 16\left(\frac{1}{x} - \frac{1}{4}\right)$

4 marks

(b) $v = 4\sqrt{\frac{1}{x} - \frac{1}{4}} = 2\sqrt{\frac{4-x}{x}}$, so the time is $t = \int_0^4 \frac{1}{v} dx = \frac{1}{2} \int_0^4 \sqrt{\frac{x}{4-x}} dx$

M1 A1

Let $x = 4 \sin^2 \theta$, so $dx = 8 \sin \theta \cos \theta d\theta$ and $\sqrt{\frac{x}{4-x}} = \tan \theta$

M1 A1

$$t = \frac{1}{2} \int_0^{\pi/2} \tan \theta \times 8 \sin \theta \cos \theta d\theta = \int_0^{\pi/2} 4 \sin^2 \theta d\theta$$

M1

$$= \int_0^{\pi/2} 2(1 - \cos 2\theta) d\theta = [2\theta - \sin 2\theta]_0^{\pi/2} = \pi$$

M1 A1
Answer: π s
7 marks

(c) $v^2 = \frac{16}{1} - 4 = 12$, so $v = 2\sqrt{3} = 3.46 \text{ m s}^{-1}$

A1

Answer: $2\sqrt{3} \text{ m s}^{-1}$

1 mark

28 (12 marks)

(a) Limiting friction = $0.5 \times 2 \times 9.8 = 9.8 \text{ N}$

M1

$$5t = 9.8, \text{ so } t = 1.96 \text{ s}$$

A1
Answer: $t = 1.96 \text{ s}$
2 marks

(b) For $t > 1.96$: $2a = 5t - 9.8$, so $a = 2.5t - 4.9$

M1

$$v = \int_{1.96}^t (2.5s - 4.9) ds = 1.25(t^2 - 1.96^2) - 4.9(t - 1.96)$$

M1 A1

$$v(6) = 1.25(36 - 3.8416) - 4.9(4.04) = 20.4 \text{ m s}^{-1}$$

A1
Answer: 20.4 m s^{-1}
4 marks

(c) Distance for $1.96 \leq t \leq 6$: $\int_{1.96}^6 (1.25(t^2 - 1.96^2) - 4.9(t - 1.96)) dt = 27.47 \text{ m}$

M1 A1

After $t = 6$ the deceleration is $\mu g = 4.9 \text{ m s}^{-2}$

B1

$$\text{Distance} = \frac{20.402^2}{2 \times 4.9} = 42.47 \text{ m}$$

M1 A1

$$\text{Total} = 27.47 + 42.47 = 69.9 \text{ m}$$

A1
Answer: 69.9 m
6 marks

29 (12 marks)

(a) $2v \frac{dv}{dx} = \frac{4}{x^3}$, so $\int v dv = \int 2x^{-3} dx$

M1 A1

$$\frac{1}{2}v^2 = -x^{-2} + c, \text{ and } v = 0 \text{ at } x = 1 \text{ gives } c = 1$$

M1

$$v^2 = 2\left(1 - \frac{1}{x^2}\right)$$

A1

Answer: $v^2 = 2\left(1 - \frac{1}{x^2}\right)$

4 marks

(b) $v = \frac{dx}{dt} = \frac{\sqrt{2}\sqrt{x^2 - 1}}{x}$

M1 A1

$$\int \frac{x}{\sqrt{x^2 - 1}} dx = \int \sqrt{2} dt$$

M1

$$\sqrt{x^2 - 1} = \sqrt{2}t + c, \text{ with } x = 1 \text{ at } t = 0 \text{ so } c = 0$$

A1

$$x^2 = 1 + 2t^2$$

A1

Answer: $x^2 = 1 + 2t^2$

5 marks

(c) $v = \frac{dx}{dt} = \frac{2t}{\sqrt{1+2t^2}}$, so at $t = 2$, $v = \frac{4}{3} \text{ m s}^{-1}$

M1 A1

As $t \rightarrow \infty$: $v \rightarrow \frac{2t}{\sqrt{2t}} = \sqrt{2} \text{ m s}^{-1}$

A1

Answer: $\frac{4}{3} \text{ m s}^{-1}$; $\sqrt{2} \text{ m s}^{-1}$

3 marks

30 (10 marks)

(a) $a = 6 \sin\left(\frac{\pi t}{4}\right)$

M1

$v = -\frac{24}{\pi} \cos\left(\frac{\pi t}{4}\right) + c$

A1

$v = 0$ at $t = 0$, so $c = \frac{24}{\pi}$, giving the required result

A1

Answer: $v = \frac{24}{\pi}(1 - \cos \frac{\pi t}{4})$

3 marks

(b) $x = \frac{24}{\pi}(t - \frac{4}{\pi} \sin \frac{\pi t}{4})$ ($x = 0$ at $t = 0$)

M1 A1

At $t = 4$: $x = \frac{24}{\pi} \times 4 = \frac{96}{\pi} = 30.6 \text{ m}$

A1

Answer: $\frac{96}{\pi} \text{ m} = 30.6 \text{ m}$

3 marks

(c) Speed at $t = 4$: $v = \frac{24}{\pi}(1 - \cos \pi) = \frac{48}{\pi} = 15.28 \text{ m s}^{-1}$

B1

Deceleration $= 0.3 \times 9.8 = 2.94 \text{ m s}^{-2}$

M1

Distance $= \frac{(48)^2}{\pi^2/2 \times 2.94} = 39.7 \text{ m}$

M1 A1

Answer: 39.7 m

4 marks

Resistance and Drag

1 (2 marks)

Resistance $= 0.5 \times 8 = 4 \text{ N}$

M1

Deceleration $= \frac{4}{2} = 2 \text{ m s}^{-2}$

A1

Answer: 2 m s^{-2}

2 marks

2 (3 marks)

$\frac{dv}{dt} = -0.2v$, so $\int \frac{1}{v} dv = \int -0.2 dt$

M1

$\ln v = -0.2t + c$, so $v = 10e^{-0.2t}$

A1

When $t = 5$, $v = 10e^{-1} = 3.68 \text{ m s}^{-1}$

A1

Answer: 3.68 m s^{-1}

3 marks

3 (2 marks)

$a = -0.05 \times 20^2$

M1

$= -20 \text{ m s}^{-2}$, so the magnitude is 20 m s^{-2}

A1

Answer: 20 m s^{-2}

2 marks

4 (2 marks)

Newton's second law: $ma = mg - 0.5mv$

M1

$a = 9.8 - 0.5 \times 4 = 7.8 \text{ m s}^{-2}$

A1

Answer: 7.8 m s^{-2}

2 marks

5 (3 marks)

$$\text{Deceleration} = \frac{30}{150} = 0.2 \text{ m s}^{-2}$$

M1

$$0 = 6^2 - 2 \times 0.2 s, \text{ so } s = 90 \text{ m}$$

M1 A1

Answer: 90 m

3 marks

6 (3 marks)

$$\text{Driving force} = \frac{P}{v} = \frac{30000}{15} = 2000 \text{ N}$$

M1 A1

$$2000 - 800 = 1000a, \text{ so } a = 1.2 \text{ m s}^{-2}$$

A1

Answer: 1.2 m s⁻²

3 marks

7 (5 marks)

$$v = 20e^{-kt}$$

M1 A1

$$10 = 20e^{-4k}, \text{ so } k = \frac{\ln 2}{4}$$

A1

$$v = 20e^{-10 \ln 2/4} = 3.54 \text{ m s}^{-1}$$

M1 A1

Answer: k = $\frac{\ln 2}{4}$; 3.54 m s⁻¹

5 marks

8 (3 marks)

$$v \frac{dv}{dx} = -0.1v, \text{ so } \frac{dv}{dx} = -0.1$$

M1

$$v = -0.1x + 25$$

A1

$$v = 0 \text{ when } x = 250 \text{ m}$$

A1

Answer: 250 m

3 marks

9 (4 marks)

$$2 \frac{dv}{dt} = -0.5v^2, \text{ so } \int \frac{1}{v^2} dv = \int -0.25 dt$$

M1

$$-\frac{1}{v} = -0.25t + c$$

A1

$$t = 0, v = 12: c = -\frac{1}{12}, \text{ so } \frac{1}{v} = 0.25t + \frac{1}{12}$$

M1

$$t = 4: \frac{1}{v} = \frac{13}{12}, v = 0.923 \text{ m s}^{-1}$$

A1

Answer: 0.923 m s⁻¹

4 marks

10 (5 marks)

$$\frac{dv}{dt} = 9.8 - 0.5v, \text{ so } \int \frac{dv}{9.8 - 0.5v} = \int dt$$

M1

$$-2 \ln(9.8 - 0.5v) = t + c$$

A1

$$t = 0, v = 0: c = -2 \ln 9.8, \text{ so } \ln\left(\frac{9.8 - 0.5v}{9.8}\right) = -0.5t$$

M1

$$v = 19.6(1 - e^{-0.5t}) \text{ and when } t = 2, v = 12.4 \text{ m s}^{-1}$$

A1 A1

Answer: 12.4 m s⁻¹

5 marks

11 (4 marks)

$$80 \frac{dv}{dt} = -4v, \text{ so } \frac{dv}{dt} = -0.05v$$

M1

$$v = 10e^{-0.05t}$$

M1 A1

$$5 = 10e^{-0.05t}, \text{ so } t = 20 \ln 2 = 13.9 \text{ s}$$

A1

Answer: 13.9 s

4 marks

12 (5 marks)

(a) Resistance = kv with $12 = 6k$, so $k = 2$

M1 A1

Newton's second law: $80 \frac{dv}{dt} = -2v$

A1

Answer: $80 \frac{dv}{dt} = -2v$

3 marks

(b) $t = -40 \int_6^3 \frac{dv}{v}$

M1

$= 40 \ln 2 = 27.7 \text{ s}$

A1

Answer: $40 \ln 2 = 27.7 \text{ s}$

2 marks

13 (5 marks)

(a) Driving force = $\frac{45000}{v}$, so $1500 \frac{dv}{dt} = \frac{45000}{v}$

M1 A1

$v \frac{dv}{dt} = 30$; integrating, $\frac{1}{2}v^2 = 30t$ ($v = 0$ at $t = 0$), so $v^2 = 60t$

M1 A1

Answer: $v^2 = 60t$

4 marks

(b) $576 = 60t$, so $t = 9.6 \text{ s}$

A1

Answer: 9.6 s

1 mark

14 (6 marks)

$\int \frac{1}{v^2} dv = \int -0.05 dt$, so $\frac{1}{v} = 0.05t + c$

M1 A1

$v = 20$ at $t = 0$: $c = 0.05$, so $v = \frac{20}{1+t}$

M1 A1

$x = \int \frac{20}{1+t} dt = 20 \ln(1+t)$ ($x = 0$ at $t = 0$)

M1

$t = 3$: $x = 20 \ln 4 = 27.7 \text{ m}$

A1

Answer: $v = \frac{20}{1+t}$; 27.7 m

6 marks

15 (6 marks)

$v \frac{dv}{dx} = 9.8 - 0.098v^2$

M1

$\int \frac{v}{9.8 - 0.098v^2} dv = \int dx$, so $-\frac{1}{0.196} \ln(9.8 - 0.098v^2) = x + c$

M1 A1

$x = 0$, $v = 0$: $\ln\left(\frac{9.8 - 0.098v^2}{9.8}\right) = -0.196x$

M1

$v^2 = 100(1 - e^{-0.196x})$

A1

$x = 20$: $v^2 = 100(1 - e^{-3.92}) = 98.0$, so $v = 9.90 \text{ m s}^{-1}$

A1

Answer: 9.90 m s^{-1}

6 marks

16 (6 marks)

$\frac{dv}{dt} = -(2 + 0.5v)$, so $t = \int_0^{12} \frac{dv}{2 + 0.5v}$

M1

$= 2 \ln 8 - 2 \ln 2$

A1

$t = 2 \ln 4 = 2.77 \text{ s}$

A1

$v \frac{dv}{dx} = -(2 + 0.5v)$, so $x = \int_0^{12} \frac{v}{2 + 0.5v} dv = \int_0^{12} \left(2 - \frac{8}{4+v}\right) dv$

M1 A1

$x = 24 - 8 \ln 4 = 12.9 \text{ m}$

A1

Answer: 2.77 s ; 12.9 m

6 marks

17 (6 marks)

$$\frac{dv}{dt} = -2v^{1/2}, \text{ so } \int v^{-1/2} dv = \int -2 dt$$

M1

$$2v^{1/2} = -2t + 8 \text{ (using } v = 16 \text{ at } t = 0)$$

A1

$$v = 0 \text{ when } t = 4 \text{ s}$$

A1

$$v \frac{dv}{dx} = -2v^{1/2}, \text{ so } \int v^{1/2} dv = -2x + c$$

M1

$$\frac{2}{3}v^{3/2} = -2x + \frac{2}{3} \times 64, \text{ so at rest } x = \frac{64}{3} \text{ m}$$

A1 A1

$$\text{Answer: } 4 \text{ s; } \frac{64}{3} \text{ m}$$

6 marks

18 (6 marks)

$$3 \frac{dv}{dt} = -6v, \text{ so } \frac{dv}{dt} = -2v$$

M1

$$v = 10e^{-2t}$$

M1 A1

$$x = \int 10e^{-2t} dt = 5(1 - e^{-2t})$$

M1 A1

$$e^{-2t} > 0, \text{ so } x < 5 \text{ for all } t$$

A1

$$\text{Answer: } v = 10e^{-2t}; x < 5$$

6 marks

19 (6 marks)

$$v \frac{dv}{dx} = -kv^2, \text{ so } v = 30e^{-kx}$$

M1 A1

$$20 = 30e^{-100k}, \text{ so } k = \frac{\ln 1.5}{100} = 0.00405$$

M1 A1

$$x = 150: v = 30e^{-1.5 \ln 1.5} = 30 \times 1.5^{-1.5}$$

M1

$$v = 16.3 \text{ m s}^{-1}$$

A1

$$\text{Answer: } k = 0.00405; 16.3 \text{ m s}^{-1}$$

6 marks

20 (6 marks)

$$v \frac{dv}{dx} = 9.8 - 0.5v, \text{ so } x = \int_0^{10} \frac{v}{9.8 - 0.5v} dv$$

M1 A1

$$\frac{v}{9.8 - 0.5v} = -2 + \frac{19.6}{9.8 - 0.5v}$$

M1

$$x = [-2v - 39.2 \ln(9.8 - 0.5v)]_0^{10}$$

A1

$$= -20 - 39.2 \ln 4.8 + 39.2 \ln 9.8 = -20 + 39.2 \ln \frac{9.8}{4.8}$$

M1

$$= 7.98 \text{ m}$$

A1

$$\text{Answer: } 7.98 \text{ m}$$

6 marks

21 (8 marks)

$$(a) v \frac{dv}{dx} = -(1 + v^2), \text{ so } \int \frac{v}{1 + v^2} dv = -\int dx$$

M1

$$\frac{1}{2} \ln(1 + v^2) = -x + c$$

A1

$$x = 0, v = 3: c = \frac{1}{2} \ln 10, \text{ so } x = \frac{1}{2} \ln 10 - \frac{1}{2} \ln(1 + v^2) = \frac{1}{2} \ln \left(\frac{10}{1 + v^2} \right)$$

M1 A1

$$\text{Answer: } x = \frac{1}{2} \ln \left(\frac{10}{1 + v^2} \right)$$

4 marks

$$(b) v = 0: x = \frac{1}{2} \ln 10 = 1.15 \text{ m}$$

A1

$$\text{Answer: } \frac{1}{2} \ln 10 = 1.15 \text{ m}$$

1 mark

$$(c) \frac{dv}{dt} = -(1 + v^2), \text{ so } t = \int_0^3 \frac{dv}{1 + v^2}$$

M1

$$= [\arctan v]_0^3$$

A1

$$= \arctan 3 = 1.25 \text{ s}$$

A1

$$\text{Answer: } \arctan 3 = 1.25 \text{ s}$$

3 marks

22 (7 marks)

$$(a) x = \int 19.6(1 - e^{-0.5t}) dt = 19.6(t + 2e^{-0.5t}) + c$$

M1 A1

$$x = 0 \text{ at } t = 0: c = -39.2, \text{ so } x = 19.6(t - 2 + 2e^{-0.5t})$$

A1

$$\text{Answer: } x = 19.6(t - 2 + 2e^{-0.5t})$$

3 marks

$$(b) x = 19.6(2 + 2e^{-2}) = 44.5 \text{ m}$$

M1 A1

$$\text{Answer: } 44.5 \text{ m}$$

2 marks

$$(c) x(5) = 19.6(3 + 2e^{-2.5}) = 62.0 \text{ m}$$

M1

$$\text{Distance} = 62.0 - 44.5 = 17.5 \text{ m}$$

A1

$$\text{Answer: } 17.5 \text{ m}$$

2 marks

23 (5 marks)

The student has treated v as a constant when integrating, but v changes with t

E1

$$\text{Separate the variables: } \int \frac{1}{v} dv = \int -0.5 dt$$

M1

$$\ln v = -0.5t + c$$

A1

$$v = 8 \text{ at } t = 0: v = 8e^{-0.5t}$$

M1

$$t = 2: v = 8e^{-1} = 2.94 \text{ m s}^{-1}$$

A1

$$\text{Answer: } 2.94 \text{ m s}^{-1}$$

5 marks

24 (8 marks)

$$(a) \frac{dv}{dt} = -Kv^2, \text{ so } \frac{1}{v} = Kt + c$$

M1 A1

$$\text{At } t = 0: c = \frac{1}{12}; \text{ at } t = 10: \frac{1}{6} = 10K + \frac{1}{12}$$

M1

$$K = \frac{1}{120}$$

A1

$$\text{Answer: } K = \frac{1}{120}$$

4 marks

$$(b) v \frac{dv}{dx} = -Kv^2 \text{ gives } v = 12e^{-Kx}$$

M1

$$6 = 12e^{-Kx}, \text{ so } x = \frac{\ln 2}{K} = 120 \ln 2 = 83.2 \text{ m}$$

A1

$$\text{Answer: } 83.2 \text{ m}$$

2 marks

$$(c) \frac{1}{3} = \frac{1}{6} + \frac{t}{120}$$

M1

$$t = 20 \text{ s}$$

A1

$$\text{Answer: } 20 \text{ s}$$

2 marks

25 (7 marks)

$$\text{Stage 1: } v \frac{dv}{dx} = -0.04v^2, \text{ so } v = 20e^{-0.04x}$$

M1 A1

$$10 = 20e^{-0.04x}, \text{ so } x_1 = 25 \ln 2 = 17.3 \text{ m}$$

A1

$$\frac{dv}{dt} = -0.04v^2: \frac{1}{v} = 0.04t + \frac{1}{20}, \text{ so } \frac{1}{10} = 0.04t_1 + \frac{1}{20} \text{ and } t_1 = 1.25 \text{ s}$$

M1 A1

$$\text{Stage 2: deceleration } 2 \text{ m s}^{-2}: x_2 = \frac{10^2}{2 \times 2} = 25 \text{ m}, t_2 = \frac{10}{2} = 5 \text{ s}$$

M1

$$\text{Total distance} = 42.3 \text{ m, total time} = 6.25 \text{ s}$$

A1

$$\text{Answer: } 42.3 \text{ m; } 6.25 \text{ s}$$

7 marks

26 (9 marks)

$$(a) \text{Driving force} = \frac{40\,000}{v}$$

B1

$$1000 \frac{dv}{dt} = \frac{40\,000}{v} - 40v, \text{ so } 1000 v \frac{dv}{dt} = 40\,000 - 40v^2$$

M1 A1

$$\text{Answer: } 1000 v \frac{dv}{dt} = 40\,000 - 40v^2$$

3 marks

$$(b) \text{Maximum speed when } \frac{dv}{dt} = 0: 40v^2 = 40\,000$$

M1

$$v = \sqrt{1000} = 31.6 \text{ m s}^{-1}$$

A1

Answer: 31.6 m s^{-1} **2 marks**

$$(c) t = \int_0^{20} \frac{1000 v}{40\,000 - 40v^2} dv$$

M1

$$= \left[-\frac{1000}{80} \ln(40\,000 - 40v^2) \right]_0^{20} = 12.5 \ln \frac{40\,000}{24\,000}$$

M1 A1

$$= 6.39 \text{ s}$$

A1**Answer: 6.39 s****4 marks****27 (12 marks)**

$$(a) \frac{dv}{dt} = -(g + kv), \text{ so } \int \frac{dv}{g + kv} = -\int dt$$

M1

$$\frac{1}{k} \ln(g + kv) = -t + c$$

A1

$$t = 0, v = u: c = \frac{1}{k} \ln(g + ku)$$

M1

$$v = 0: t = \frac{1}{k} \ln\left(\frac{g + ku}{g}\right) = \frac{1}{k} \ln\left(1 + \frac{ku}{g}\right)$$

A1

$$\text{Answer: } t = \frac{1}{k} \ln\left(1 + \frac{ku}{g}\right)$$

4 marks

$$(b) v \frac{dv}{dx} = -(g + kv), \text{ so } x = \int_0^u \frac{v}{g + kv} dv$$

M1

$$\frac{v}{g + kv} = \frac{1}{k} \left(1 - \frac{g}{g + kv}\right)$$

M1 A1

$$x = \frac{1}{k} \left[v - \frac{g}{k} \ln(g + kv) \right]_0^u$$

M1

$$= \frac{u}{k} - \frac{g}{k^2} \ln\left(\frac{g + ku}{g}\right), \text{ as required}$$

A1

$$\text{Answer: } H = \frac{u}{k} - \frac{g}{k^2} \ln\left(1 + \frac{ku}{g}\right)$$

5 marks

$$(c) t = 10 \ln\left(1 + \frac{3}{9.8}\right) = 2.67 \text{ s}$$

B1

$$H = 300 - 980 \ln(1.3061) = 38.3 \text{ m}$$

M1 A1**Answer: 2.67 s; 38.3 m****3 marks****28 (12 marks)**

$$(a) v \frac{dv}{dx} = -g\left(1 + \frac{v^2}{400}\right), \text{ so } \int \frac{v}{1 + v^2/400} dv = -g \int dx$$

M1

$$200 \ln\left(1 + \frac{v^2}{400}\right) = -gx + c$$

A1

$$x = 0, v = 20: c = 200 \ln 2; v = 0 \text{ at } x = H: gH = 200 \ln 2$$

M1 A1

$$H = \frac{200 \ln 2}{9.8} = 14.1 \text{ m}$$

A1

$$\text{Answer: } H = \frac{200}{g} \ln 2 = 14.1 \text{ m}$$

5 marks

$$(b) \frac{dv}{dt} = -g\left(1 + \frac{v^2}{400}\right), \text{ so } \int \frac{dv}{1 + v^2/400} = -g \int dt$$

M1

$$20 \arctan\left(\frac{v}{20}\right) = -gt + c$$

A1

$$c = 20 \arctan 1 = 5\pi; v = 0: gt = 5\pi$$

M1

$$t = \frac{5\pi}{9.8} = 1.60 \text{ s}$$

A1**Answer: 1.60 s****4 marks**

$$(c) v \frac{dv}{dx} = g\left(1 - \frac{v^2}{400}\right) \text{ with } x \text{ measured downwards from the highest point}$$

M1

$$-200 \ln\left(1 - \frac{v^2}{400}\right) = gx, \text{ so at } x = H: \ln\left(1 - \frac{v^2}{400}\right) = -\ln 2$$

M1

$$v^2 = 200, \text{ so } v = 10\sqrt{2} = 14.1 \text{ m s}^{-1}$$

A1

$$\text{Answer: } 10\sqrt{2} = 14.1 \text{ m s}^{-1}$$

3 marks

29 (12 marks)

$$(a) v \frac{dv}{dx} = -v(1+v), \text{ so } \int \frac{1}{1+v} dv = -\int dx \quad \text{M1 A1}$$

$$\ln(1+v) = -x + c, \text{ and } v = 3 \text{ at } x = 0 \text{ gives } c = \ln 4 \quad \text{M1}$$

$$1+v = 4e^{-x}, \text{ so } v = 4e^{-x} - 1 \quad \text{A1}$$

$$\text{Answer: } v = 4e^{-x} - 1 \quad \text{4 marks}$$

$$(b) v = 0: 4e^{-x} = 1, \text{ so } x = \ln 4 \quad \text{M1 A1}$$

$$\text{Answer: } \ln 4 = 1.39 \text{ m} \quad \text{2 marks}$$

$$(c) \frac{dv}{dt} = -v(1+v), \text{ so } \int \frac{1}{v(1+v)} dv = -\int dt, \text{ i.e. } \int \left(\frac{1}{v} - \frac{1}{1+v} \right) dv = -t + c \quad \text{M1 A1}$$

$$\ln\left(\frac{v}{1+v}\right) = -t + c, \text{ and } v = 3 \text{ at } t = 0 \text{ gives } c = \ln \frac{3}{4} \quad \text{M1}$$

$$\frac{v}{1+v} = \frac{3}{4} e^{-t}, \text{ so } v = \frac{3e^{-t}}{4 - 3e^{-t}} = \frac{3}{4e^t - 3} \quad \text{A1}$$

$$v = 1: 4e^t - 3 = 3, \text{ so } t = \ln \frac{3}{2} = 0.405 \text{ s} \quad \text{M1 A1}$$

$$\text{Answer: } v = \frac{3}{4e^t - 3}; t = \ln 1.5 = 0.405 \text{ s} \quad \text{6 marks}$$

30 (10 marks)

$$(a) v \frac{dv}{dx} = -0.0002e^{x/2000}v^2, \text{ so } \int \frac{1}{v} dv = -0.0002 \int e^{x/2000} dx \quad \text{M1 A1}$$

$$\ln v = -0.4e^{x/2000} + c \quad \text{A1}$$

$$x = 0, v = U: c = \ln U + 0.4, \text{ so } \ln\left(\frac{v}{U}\right) = -0.4(e^{x/2000} - 1) \quad \text{M1 A1}$$

$$\text{Answer: } v = U \exp(-0.4(e^{x/2000} - 1)) \quad \text{5 marks}$$

$$(b) v = 600 \exp(-0.4(e - 1)) \quad \text{M1}$$

$$= 302 \text{ m s}^{-1} \quad \text{A1}$$

$$\text{Answer: } 302 \text{ m s}^{-1} \quad \text{2 marks}$$

$$(c) \frac{1}{2} = \exp(-0.4(e^{x/2000} - 1)), \text{ so } 0.4(e^{x/2000} - 1) = \ln 2 \quad \text{M1}$$

$$e^{x/2000} = 1 + \frac{\ln 2}{0.4} = 2.733 \quad \text{A1}$$

$$x = 2000 \ln 2.733 = 2010 \text{ m} \quad \text{A1}$$

$$\text{Answer: } 2010 \text{ m} \quad \text{3 marks}$$

Terminal Velocity**1 (2 marks)**

$$\text{At terminal velocity the acceleration is zero: } mg = 0.5mV \quad \text{M1}$$

$$V = \frac{9.8}{0.5} = 19.6 \text{ m s}^{-1} \quad \text{A1}$$

$$\text{Answer: } 19.6 \text{ m s}^{-1} \quad \text{2 marks}$$

2 (2 marks)

$$\text{Greatest speed when driving force} = \text{resistance: } 30v = 1800 \quad \text{M1}$$

$$v = 60 \text{ m s}^{-1} \quad \text{A1}$$

$$\text{Answer: } 60 \text{ m s}^{-1} \quad \text{2 marks}$$

3 (2 marks)

$$\text{At terminal velocity: } 2 \times 9.8 = 0.04V^2 \quad \text{M1}$$

$$V^2 = 490, \text{ so } V = 22.1 \text{ m s}^{-1} \quad \text{A1}$$

$$\text{Answer: } 22.1 \text{ m s}^{-1} \quad \text{2 marks}$$

4 (2 marks)

Terminal velocity when the acceleration is zero: $9.8 - 0.2V = 0$

M1

$$V = 49 \text{ m s}^{-1}$$

A1

Answer: 49 m s^{-1}

2 marks

5 (2 marks)

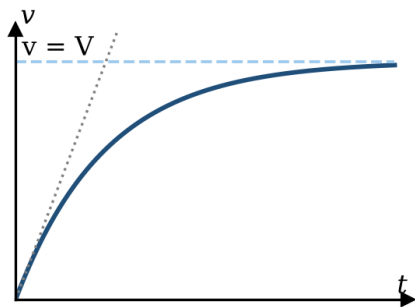
The graph is a curve through the origin, starting with gradient g (initial acceleration g)

B1

The gradient decreases as v increases and the curve approaches the horizontal line $v = V$ from below (asymptote)

B1

Answer: curve from the origin with initial gradient g , increasing and tending to the asymptote $v = V$ 2 marks

**6 (3 marks)**

At terminal velocity the acceleration is zero: $3 \times 9.8 = k \times 14.7$

M1 A1

$$k = \frac{29.4}{14.7} = 2$$

A1

Answer: $k = 2$

3 marks

7 (4 marks)

$$\text{Terminal velocity} = \frac{9.8}{0.5} = 19.6 \text{ m s}^{-1}$$

B1

$$90\% \text{ of this: } 0.9 \times 19.6 = 19.6(1 - e^{-0.5t}), \text{ so } e^{-0.5t} = 0.1$$

M1 A1

$$t = 2 \ln 10 = 4.61 \text{ s}$$

A1

Answer: 4.61 s

4 marks

8 (4 marks)

$$\text{Terminal velocity: } 80 \times 9.8 = 16V^2, \text{ so } V^2 = 49 \text{ and } V = 7 \text{ m s}^{-1}$$

M1 A1

$$\text{Resistance at } 14 \text{ m s}^{-1} = 16 \times 196 = 3136 \text{ N upwards}$$

M1

$$80a = 3136 - 784, \text{ so } a = 29.4 \text{ m s}^{-2} \text{ (upwards, a deceleration)}$$

A1

Answer: 7 m s^{-1} ; 29.4 m s^{-2}

4 marks

9 (5 marks)

$$\text{The terminal velocity is } 19.6 \text{ m s}^{-1} \text{ and } \frac{dv}{dt} = 0.5(19.6 - v)$$

B1

$$\int \frac{dv}{19.6 - v} = \int 0.5 dt, \text{ so } \ln\left(\frac{19.6 - v}{19.6}\right) = -0.5t$$

M1 A1

$$v = 0.95 \times 19.6: \ln 0.05 = -0.5t$$

M1

$$t = 2 \ln 20 = 5.99 \text{ s}$$

A1

Answer: 5.99 s

5 marks

10 (3 marks)

$$\text{At terminal velocity the forces balance: } 0.5 \times 9.8 = 1.5 + 2V$$

M1 A1

$$4.9 - 1.5 = 2V, \text{ so } V = 1.7 \text{ m s}^{-1}$$

A1

Answer: 1.7 m s^{-1}

3 marks

11 (4 marks)

$$(i) a = g(1 - \frac{v}{V}) = \frac{1}{2}g, \text{ so } v = \frac{1}{2}V$$

M1 A1

$$(ii) a = g(1 - \frac{v^2}{V^2}) = \frac{1}{2}g$$

M1

$$v^2 = \frac{1}{2}V^2, \text{ so } v = \frac{V}{\sqrt{2}} \approx 0.707V$$

A1

$$\text{Answer: (i) } \frac{1}{2}V; \text{ (ii) } \frac{V}{\sqrt{2}}$$

4 marks

12 (5 marks)

$$(a) 1 - e^{-2k} = 0.6, \text{ so } e^{-2k} = 0.4$$

M1 A1

$$k = \frac{-\ln 0.4}{2} = 0.458$$

A1

$$\text{Answer: } k = 0.458$$

3 marks

$$(b) e^{-kt} = 0.1, \text{ so } t = \frac{\ln 10}{k}$$

M1

$$t = 5.03 \text{ s}$$

A1

$$\text{Answer: } 5.03 \text{ s}$$

2 marks

13 (7 marks)

$$(a) \text{ At terminal velocity } 500 = 25v$$

M1

$$v = 20 \text{ m s}^{-1}$$

A1

$$\text{Answer: } 20 \text{ m s}^{-1}$$

2 marks

$$(b) 250 v \frac{dv}{dx} = -25v, \text{ so } \frac{dv}{dx} = -0.1$$

M1 A1

$$x = \frac{20 - 5}{0.1} = 150 \text{ m}$$

A1

$$\text{Answer: } 150 \text{ m}$$

3 marks

$$(c) 250 \frac{dv}{dt} = -25v, \text{ so } t = 10 \int_5^{20} \frac{dv}{v}$$

M1

$$= 10 \ln 4 = 13.9 \text{ s}$$

A1

$$\text{Answer: } 13.9 \text{ s}$$

2 marks

14 (6 marks)

$$v \frac{dv}{dx} = 9.8 - 0.05v^2$$

M1

$$\int \frac{v}{196 - v^2} dv = \int 0.05 dx, \text{ so } -\frac{1}{2} \ln(196 - v^2) = 0.05x + c$$

M1 A1

$$x = 0, v = 0: \ln\left(\frac{196 - v^2}{196}\right) = -0.1x, \text{ so } v^2 = 196(1 - e^{-0.1x})$$

A1

$$v = 7: 49 = 196(1 - e^{-0.1x}), \text{ so } e^{-0.1x} = \frac{3}{4}$$

M1

$$x = 10 \ln \frac{4}{3} = 2.88 \text{ m}$$

A1

$$\text{Answer: } 2.88 \text{ m}$$

6 marks

15 (6 marks)

$$\frac{dv}{dt} = \frac{9.8}{196}(196 - v^2), \text{ so } \int \frac{dv}{196 - v^2} = \int 0.05 dt$$

M1 A1

$$\frac{1}{196 - v^2} = \frac{1}{28} \left(\frac{1}{14 - v} + \frac{1}{14 + v} \right)$$

M1 A1

$$\frac{1}{28} \ln\left(\frac{14 + v}{14 - v}\right) = 0.05t \text{ (} v = 0 \text{ at } t = 0\text{)}$$

M1

$$v = 7: t = \frac{1}{1.4} \ln 3 = 0.785 \text{ s}$$

A1

$$\text{Answer: } 0.785 \text{ s}$$

6 marks

16 (7 marks)

$$(a) m \frac{dv}{dt} = m g - \frac{m g v}{V}, \text{ so } \int \frac{dv}{V-v} = \int \frac{g}{V} dt \quad \text{M1}$$

$$-\ln(V-v) = \frac{g t}{V} + c; v = 0 \text{ at } t = 0 \text{ gives } \ln\left(\frac{V-v}{V}\right) = -\frac{g t}{V} \quad \text{M1 A1}$$

$$v = V(1 - e^{-g t/V}) \quad \text{A1}$$

$$\text{Answer: } v = V(1 - e^{-g t/V}) \quad \text{4 marks}$$

$$(b) x = \int V(1 - e^{-g t/V}) dt = V(t + \frac{V}{g} e^{-g t/V}) + c \quad \text{M1 A1}$$

$$x = 0 \text{ at } t = 0: x = V t - \frac{V^2}{g}(1 - e^{-g t/V}) \quad \text{A1}$$

$$\text{Answer: } x = V t - \frac{V^2}{g}(1 - e^{-g t/V}) \quad \text{3 marks}$$

17 (7 marks)

$$1000 \frac{dv}{dt} = 3000 - 50v, \text{ so } \frac{dv}{dt} = 0.05(60 - v) \quad \text{M1 A1}$$

$$v = 60(1 - e^{-0.05t}) \quad \text{M1 A1}$$

$$t = 20: v = 60(1 - e^{-1}) = 37.9 \text{ m s}^{-1} \quad \text{A1}$$

$$x = \int_0^{20} 60(1 - e^{-0.05t}) dt = 60[t + 20e^{-0.05t}]_0^{20} = 60(20 + 20e^{-1} - 20) = 441 \text{ m} \quad \text{M1 A1}$$

$$\text{Answer: } 37.9 \text{ m s}^{-1}; 441 \text{ m} \quad \text{7 marks}$$

18 (6 marks)

$$\text{Terminal velocity} = \frac{9.8}{0.5} = 19.6 \text{ m s}^{-1} \quad \text{B1}$$

$$\frac{dv}{dt} = -0.5(v - 19.6), \text{ so } \ln(v - 19.6) = -0.5t + c \quad \text{M1 A1}$$

$$v = 40 \text{ at } t = 0: v = 19.6 + 20.4e^{-0.5t} \quad \text{A1}$$

$$25 = 19.6 + 20.4e^{-0.5t}, \text{ so } e^{-0.5t} = \frac{5.4}{20.4} \quad \text{M1}$$

$$t = 2 \ln\left(\frac{20.4}{5.4}\right) = 2.66 \text{ s} \quad \text{A1}$$

$$\text{Answer: } 19.6 \text{ m s}^{-1}; 2.66 \text{ s} \quad \text{6 marks}$$

19 (6 marks)

$$\text{At terminal velocity: } 2 \times 9.8 \sin 30^\circ = 0.8V \quad \text{M1}$$

$$V = 12.25 \text{ m s}^{-1} \quad \text{A1}$$

$$2 \frac{dv}{dt} = 9.8 - 0.8v, \text{ so } \frac{dv}{dt} = 0.4(12.25 - v) \quad \text{M1}$$

$$v = 12.25(1 - e^{-0.4t}) \quad \text{A1}$$

$$\frac{1}{2} = 1 - e^{-0.4t}, \text{ so } t = \frac{\ln 2}{0.4} = 1.73 \text{ s} \quad \text{M1 A1}$$

$$\text{Answer: } 12.25 \text{ m s}^{-1}; 1.73 \text{ s} \quad \text{6 marks}$$

20 (5 marks)

$$\text{Greatest speed: driving force } \frac{P}{v} \text{ equals resistance: } \frac{30000}{v} = 15v \quad \text{M1 A1}$$

$$v^2 = 2000, \text{ so } v = 44.7 \text{ m s}^{-1} \quad \text{A1}$$

$$\text{At } 20 \text{ m s}^{-1}: \text{driving force} = \frac{30000}{20} = 1500 \text{ N, resistance} = 300 \text{ N} \quad \text{M1}$$

$$1200a = 1500 - 300, \text{ so } a = 1 \text{ m s}^{-2} \quad \text{A1}$$

$$\text{Answer: } 44.7 \text{ m s}^{-1}; 1 \text{ m s}^{-2} \quad \text{5 marks}$$

21 (7 marks)

$$(a) \frac{dv}{dt} = V \times \frac{g}{V} \operatorname{sech}^2\left(\frac{gt}{V}\right) = g \operatorname{sech}^2\left(\frac{gt}{V}\right)$$

M1 A1

$$\text{Since } 1 - \tanh^2 x = \operatorname{sech}^2 x: \frac{dv}{dt} = g(1 - \tanh^2(\frac{gt}{V})) = g(1 - \frac{v^2}{V^2})$$

M1 A1

$$\tanh 0 = 0, \text{ so } v = 0 \text{ when } t = 0$$

B1

Answer: verified

5 marks

$$(b) \text{ As } t \rightarrow \infty, \tanh\left(\frac{gt}{V}\right) \rightarrow 1, \text{ so } v \rightarrow V$$

E1

$$\text{At } t = \frac{V}{g}: v = V \tanh 1 = 0.762V$$

A1

Answer: V; 0.762V

2 marks

22 (10 marks)

$$(a) 80 \frac{dv}{dt} = 784 - 8v, \text{ so } \frac{dv}{dt} = 0.1(98 - v)$$

M1 A1

$$v = 98(1 - e^{-0.1t}), \text{ so at } t = 5: v = 98(1 - e^{-0.5}) = 38.6 \text{ m s}^{-1}$$

M1 A1

Answer: 38.6 m s⁻¹

4 marks

$$(b) \text{ After opening: } 80 \frac{dv}{dt} = 784 - 160v, \text{ so } \frac{dv}{dt} = 2(4.9 - v)$$

M1

$$\text{Terminal velocity} = 4.9 \text{ m s}^{-1}$$

A1

$$v - 4.9 = (38.56 - 4.9)e^{-2\tau}, \text{ where } \tau \text{ is the time after opening}$$

M1 A1

$$6 - 4.9 = 33.66e^{-2\tau}, \text{ so } \tau = \frac{1}{2} \ln\left(\frac{33.66}{1.1}\right) = 1.71 \text{ s}$$

M1 A1

Answer: 4.9 m s⁻¹; 1.71 s

6 marks

23 (6 marks)The resistance is proportional to v^2 , but the student used kV instead of kV^2

E1

$$9.8 - 0.05V^2 = 0, \text{ so } V^2 = 196 \text{ and } V = 14 \text{ m s}^{-1}$$

M1 A1

$$\frac{dv}{dt} = 0.05(196 - v^2), \text{ so } \int \frac{dv}{196 - v^2} = \int 0.05 dt, \text{ giving } \frac{1}{28} \ln\left(\frac{14+v}{14-v}\right) = 0.05t$$

M1 A1

$$v = 7: t = \frac{\ln 3}{1.4} = 0.785 \text{ s}$$

A1

Answer: V = 14 m s⁻¹; t = 0.785 s

6 marks

24 (10 marks)

$$(a) \text{ At terminal velocity: } 0.02 \times 9.8 = 0.049 + 0.6V$$

M1

$$V = \frac{0.147}{0.6} = 0.245 \text{ m s}^{-1}$$

A1

Answer: 0.245 m s⁻¹

2 marks

$$(b) 0.02 \frac{dv}{dt} = 0.196 - 0.049 - 0.6v, \text{ so } \frac{dv}{dt} = 30(0.245 - v)$$

M1 A1

$$\ln(0.245 - v) = -30t + c; v = 0 \text{ at } t = 0: v = 0.245(1 - e^{-30t})$$

M1 A1

Answer: v = 0.245(1 - e^{-30t})

4 marks

$$(c) e^{-30t} = 0.01, \text{ so } t = \frac{\ln 100}{30} = 0.154 \text{ s}$$

M1 A1

$$x = 0.245\left(t - \frac{1 - e^{-30t}}{30}\right) = 0.245(0.1535 - 0.033) = 0.0295 \text{ m}$$

M1 A1

Answer: 0.154 s; 0.0295 m

4 marks

25 (7 marks)

$$1500 \times 9.8 \times \frac{1}{10} = 1470 \text{ N down the slope; at terminal velocity } 1470 = 3V^2, \text{ so } V^2 = 490, V =$$

M1 A1

$$22.1 \text{ m s}^{-1}$$

$$1500v \frac{dv}{dx} = 1470 - 3v^2, \text{ so } v \frac{dv}{dx} = 0.002(490 - v^2)$$

M1 A1

$$\int \frac{v}{490 - v^2} dv = \int 0.002 dx, \text{ so } -\frac{1}{2} \ln(490 - v^2) = 0.002x + c$$

M1

$$v = 0 \text{ at } x = 0: x = 250 \ln\left(\frac{490}{490 - v^2}\right)$$

M1

$$v = 20: x = 250 \ln\left(\frac{490}{90}\right) = 424 \text{ m}$$

A1

Answer: 22.1 m s⁻¹; 424 m

7 marks

26 (9 marks)

$$(a) v \frac{dv}{dx} = g\left(1 - \frac{v}{V}\right), \text{ so } x = \int_0^{V/2} \frac{v}{g(1 - v/V)} dv$$

M1 A1

$$\text{With } u = \frac{v}{V}: x = \frac{V^2}{g} \int_0^{1/2} \frac{u}{1 - u} du = \frac{V^2}{g} \int_0^{1/2} \left(-1 + \frac{1}{1 - u}\right) du$$

M1 A1

$$= \frac{V^2}{g} [-u - \ln(1 - u)]_0^{1/2} = \frac{V^2}{g} \left(-\frac{1}{2} + \ln 2\right)$$

A1

$$\text{Answer: } x = \frac{V^2}{g} \left(\ln 2 - \frac{1}{2}\right)$$

5 marks

$$(b) \frac{dv}{dt} = \frac{g}{V}(V - v), \text{ so } t = \frac{V}{g} \int_0^{V/2} \frac{dv}{V - v} = \frac{V}{g} \ln 2$$

M1 A1

$$t = 2 \ln 2 = 1.39 \text{ s}$$

A1

$$x = \frac{19.6^2}{9.8} \left(\ln 2 - \frac{1}{2}\right) = 7.57 \text{ m}$$

A1

Answer: 1.39 s; 7.57 m

4 marks

27 (12 marks)

$$(a) \int \frac{dv}{1 - v^2/V^2} = \int g dt, \text{ and } \frac{1}{1 - v^2/V^2} = \frac{V^2}{(V - v)(V + v)} = \frac{V}{2} \left(\frac{1}{V - v} + \frac{1}{V + v}\right)$$

M1 A1

$$\frac{V}{2} [-\ln(V - v) + \ln(V + v)] = gt + c, \text{ and } v = 0 \text{ at } t = 0 \text{ gives } c = 0$$

M1 A1

$$\frac{V}{2} \ln\left(\frac{V + v}{V - v}\right) = gt, \text{ so } \frac{V + v}{V - v} = e^{2gt/V}$$

M1

$$v = V \frac{e^{2gt/V} - 1}{e^{2gt/V} + 1} = V \tanh\left(\frac{gt}{V}\right)$$

A1

$$\text{Answer: } v = V \tanh\left(\frac{gt}{V}\right)$$

6 marks

$$(b) x = \int V \tanh\left(\frac{gt}{V}\right) dt$$

M1

$$= \frac{V^2}{g} \ln \cosh\left(\frac{gt}{V}\right) + c, \text{ with } x = 0 \text{ at } t = 0 \text{ so } c = 0$$

M1 A1

$$\text{Answer: } x = \frac{V^2}{g} \ln \cosh\left(\frac{gt}{V}\right)$$

3 marks

$$(c) 500 = \frac{2500}{9.8} \ln \cosh(0.196t), \text{ so } \cosh(0.196t) = e^{1.96} = 7.10$$

M1

$$0.196t = \operatorname{arcosh} 7.10 = 2.648$$

M1

$$t = 13.5 \text{ s}$$

A1

Answer: 13.5 s

3 marks

28 (10 marks)

$$(a) \frac{dv}{dt} = 0.5(19.6 - v), \text{ so } v = 19.6(1 - e^{-0.5t})$$

M1 A1

$$x = \int 19.6(1 - e^{-0.5t}) dt = 19.6(t + 2e^{-0.5t}) + c$$

M1 A1

$$x = 0 \text{ at } t = 0: c = -39.2, \text{ so } x = 19.6t - 39.2(1 - e^{-0.5t}) = Vt - 2V(1 - e^{-0.5t})$$

A1

$$\text{Answer: } v = V(1 - e^{-0.5t}); x = Vt - 2V(1 - e^{-0.5t})$$

5 marks

(b) As t grows, $e^{-0.5t} \rightarrow 0$, so $x \approx 19.6t - 39.2$

M1

$x(t+1) - x(t) \approx 19.6$ m, since the particle then moves at its terminal velocity of 19.6 m s^{-1}

A1

Answer: 19.6 m per second

2 marks

(c) For $t > 10$, $e^{-0.5t} < 0.007$ and so $39.2e^{-0.5t} < 0.3$ m

M1

$$400 \approx 19.6t - 39.2, \text{ so } t = \frac{439.2}{19.6} = 22.4 \text{ s}$$

A1

At $t = 22.4$, $e^{-11.2} \approx 1.4 \times 10^{-5}$, so the approximation changes x by less than 0.001 m and is valid

E1

Answer: 22.4 s

3 marks

29 (12 marks)

(a) $2500 = 4V^2$, so $V = 25 \text{ m s}^{-1}$

B1

Answer: 25 m s⁻¹

1 mark

(b) $1000 \frac{dv}{dt} = 2500 - 4v^2$, so $\frac{dv}{dt} = 0.004(625 - v^2)$

M1 A1

$$\int \frac{dv}{625 - v^2} = \int 0.004 dt, \text{ and } \frac{1}{625 - v^2} = \frac{1}{50} \left(\frac{1}{25 - v} + \frac{1}{25 + v} \right)$$

M1 A1

$$\frac{1}{50} \ln \left(\frac{25 + v}{25 - v} \right) = 0.004t \quad (v = 0 \text{ at } t = 0)$$

M1

$$t = 5 \ln \left(\frac{25 + v}{25 - v} \right)$$

A1

Answer: $t = 5 \ln \left(\frac{25 + v}{25 - v} \right)$

6 marks

(c) $t = 5 \ln 9 = 11.0$ s

A1

$$1000v \frac{dv}{dx} = 2500 - 4v^2, \text{ so } \int \frac{v}{625 - v^2} dv = \int 0.004 dx$$

M1

$$-\frac{1}{2} \ln(625 - v^2) = 0.004x + c, \text{ so } x = 125 \ln \left(\frac{625}{625 - v^2} \right)$$

M1 A1

$$x = 125 \ln \left(\frac{625}{225} \right) = 128 \text{ m}$$

A1

Answer: 11.0 s; 128 m

5 marks

30 (11 marks)

(a) $\sin \alpha = \frac{3}{5}$, $\cos \alpha = \frac{4}{5}$, $R = mg \cos \alpha$

B1

$$ma = mg \sin \alpha - 0.25 \times mg \cos \alpha - 0.02mv^2$$

M1

$$a = 9.8(0.6 - 0.2) - 0.02v^2 = 3.92 - 0.02v^2 = 0.02(196 - v^2)$$

A1

Terminal velocity 14 m s^{-1}

A1

Answer: $V = 14 \text{ m s}^{-1}$

4 marks

(b) $v \frac{dv}{dx} = 0.02(196 - v^2)$, so $\int \frac{v}{196 - v^2} dv = \int 0.02 dx$

M1

$$-\frac{1}{2} \ln(196 - v^2) = 0.02x + c; x = 0, v = 0 \text{ gives } v^2 = 196(1 - e^{-0.04x})$$

M1 A1

$$x = 10: v^2 = 196(1 - e^{-0.4}) = 64.6, v = 8.04 \text{ m s}^{-1}$$

A1

Answer: 8.04 m s⁻¹

4 marks

(c) $\int \frac{dv}{196 - v^2} = \int 0.02 dt$, so $\frac{1}{28} \ln \left(\frac{14 + v}{14 - v} \right) = 0.02t$

M1 A1

$$v = 7: t = \frac{\ln 3}{0.56} = 1.96 \text{ s}$$

A1

Answer: 1.96 s

3 marks

Dimensional Analysis

1 (3 marks)

(a) $[v] = \text{distance} \div \text{time} = \text{LT}^{-1}$

B1

Answer: LT^{-1}

1 mark

(b) $[a] = \text{LT}^{-1} \div T = \text{LT}^{-2}$

B1

Answer: LT^{-2} **1 mark****(c) $[F] = [m][a] = MLT^{-2}$** **B1****Answer: MLT^{-2}** **1 mark****2 (2 marks)** $[F] = MLT^{-2}$, [distance] = L**M1** $[energy] = ML^2T^{-2}$ **A1****Answer: ML^2T^{-2}** **2 marks****3 (2 marks)****(a) [momentum] = $[m][v] = MLT^{-1}$** **B1****Answer: MLT^{-1}** **1 mark****(b) [impulse] = $MLT^{-2} \times T = MLT^{-1}$** **B1****Answer: MLT^{-1}** **1 mark****4 (2 marks)****(a) $[\rho] = \frac{M}{L^3} = ML^{-3}$** **B1****Answer: ML^{-3}** **1 mark****(b) $[p] = \frac{MLT^{-2}}{L^2} = ML^{-1}T^{-2}$** **B1****Answer: $ML^{-1}T^{-2}$** **1 mark****5 (2 marks)** $[power] = \frac{[energy]}{[time]} = \frac{ML^2T^{-2}}{T}$ **M1** $= ML^2T^{-3}$ **A1****Answer: ML^2T^{-3}** **2 marks****6 (2 marks)** $[v] = [u] = LT^{-1}$ **B1** $[a t] = LT^{-2} \times T = LT^{-1}$, so every term has dimensions LT^{-1} **B1****Answer: all terms have dimensions LT^{-1}** **2 marks****7 (4 marks)****(a) $[k] = \frac{[F]}{[v]} = \frac{MLT^{-2}}{LT^{-1}}$** **M1** $= MT^{-1}$ **A1****Answer: MT^{-1}** **2 marks****(b) $[k] = \frac{MLT^{-2}}{L^2T^{-2}}$** **M1** $= ML^{-1}$ **A1****Answer: ML^{-1}** **2 marks****8 (3 marks)** $G = \frac{F r^2}{m_1 m_2}$ **M1** $[G] = \frac{MLT^{-2} \times L^2}{M^2}$ **M1** $= M^{-1}L^3T^{-2}$ **A1****Answer: $M^{-1}L^3T^{-2}$** **3 marks**

9 (4 marks)

(i) $[v^2] = [u^2] = L^2 T^{-2}$, $[as] = LT^{-2} \times L = L^2 T^{-2}$: consistent

B1

(ii) $[at^2] = LT^{-2} \times T^2 = L$ but $[v] = LT^{-1}$: not consistent, so it cannot be correct

M1 A1

(iii) $[s] = L$, $[ut] = LT^{-1} \times T = L$, $[at^2] = L$: consistent

B1**Answer: (i) consistent; (ii) inconsistent; (iii) consistent****4 marks****10 (4 marks)**

$$[k] = \frac{[T]}{[x]} = \frac{MLT^{-2}}{L} = MT^{-2}$$

M1 A1

$\frac{x}{l}$ is dimensionless, so $[\lambda] = [T] = MLT^{-2}$

M1 A1**Answer: $[k] = MT^{-2}$; $[\lambda] = MLT^{-2}$** **4 marks****11 (3 marks)**

(a) ω is an angle per unit time; angle is dimensionless: $[\omega] = T^{-1}$

B1**Answer: T^{-1}** **1 mark**

(b) $[\text{moment}] = [\text{force}][\text{distance}] = MLT^{-2} \times L = ML^2 T^{-2}$

B1**Answer: $ML^2 T^{-2}$** **1 mark**

(c) $[\text{angular momentum}] = MLT^{-1} \times L = ML^2 T^{-1}$

B1**Answer: $ML^2 T^{-1}$** **1 mark****12 (3 marks)**

$[l] = L$ and $[g] = LT^{-2}$

B1

$$\left[\frac{1}{g}\right] = \frac{L}{LT^{-2}} = T^2$$

M1

$$\left[\sqrt{\frac{1}{g}}\right] = T$$
, which is the dimension of a period (2π is dimensionless)

A1**Answer: consistent****3 marks****13 (4 marks)**

x has dimension L and $\sin \omega t$ has none, so $[A] = L$

B1

The argument of an exponential function must be dimensionless: bt is dimensionless, so $[b] = T^{-1}$

M1 A1

Similarly ωt is dimensionless, so $[\omega] = T^{-1}$

A1**Answer: $[A] = L$, $[b] = [\omega] = T^{-1}$** **4 marks****14 (5 marks)**

$[v] = LT^{-1}$, $[g] = LT^{-2}$, $[\lambda] = L$

B1

$[g^a \lambda^b] = L^{a+b} T^{-2a}$; equate with LT^{-1}

M1

T: $-2a = -1$, so $a = \frac{1}{2}$

A1

L: $a + b = 1$, so $b = \frac{1}{2}$

A1

$v = k\sqrt{g\lambda}$

A1**Answer: $a = \frac{1}{2}$, $b = \frac{1}{2}$** **5 marks**

15 (6 marks)

$$[t] = T, [h] = L, [g] = LT^{-2}$$

B1

$$T: -2b = 1, \text{ so } b = -\frac{1}{2}; L: a + b = 0, \text{ so } a = \frac{1}{2}$$

M1 A1

$$t = k\sqrt{\frac{h}{g}}$$

A1

$$2 = k\sqrt{\frac{19.6}{9.8}} = k\sqrt{2}$$

M1

$$k = \sqrt{2}$$

A1

$$\text{Answer: } a = \frac{1}{2}, b = -\frac{1}{2}, k = \sqrt{2}$$

6 marks

16 (5 marks)

$$(a) [GM] = M^{-1}L^3T^{-2} \times M = L^3T^{-2}$$

B1

$$\left[\frac{GM}{R}\right] = \frac{L^3T^{-2}}{L} = L^2T^{-2}$$

M1

$$\left[\sqrt{\frac{2GM}{R}}\right] = LT^{-1}, \text{ the dimensions of speed (2 is dimensionless)}$$

A1

Answer: consistent

3 marks

$$(b) \left[\frac{GM}{R^2}\right] = \frac{L^3T^{-2}}{L^2} = LT^{-2}$$

M1

so the square root has dimensions $L^{1/2}T^{-1}$, which are not those of speed

E1

Answer: dimensions $L^{1/2}T^{-1}$, not LT^{-1}

2 marks

17 (4 marks)

$$(a) [\rho Av^2] = ML^{-3} \times L^2 \times L^2T^{-2}$$

M1

$$= MLT^{-2}, \text{ the dimensions of force}$$

A1

Answer: consistent

2 marks

$$(b) 15 = \frac{1}{2} \times C \times 1.2 \times 0.5 \times 10^2 = 30C$$

M1

$$C = 0.5$$

A1

Answer: C = 0.5

2 marks

18 (4 marks)

$$[p] = \frac{MLT^{-2}}{L^2} = ML^{-1}T^{-2}$$

B1

$$\left[\frac{1}{2}\rho v^2\right] = ML^{-3} \times L^2T^{-2} = ML^{-1}T^{-2}$$

B1

$$[\rho gh] = ML^{-3} \times LT^{-2} \times L = ML^{-1}T^{-2}$$

B1

All three terms have the same dimensions, so they can be added

E1

Answer: each term has dimensions $ML^{-1}T^{-2}$

4 marks

19 (6 marks)

$$[F] = MLT^{-2}, [m] = M, [\omega] = T^{-1}, [r] = L$$

B1

$$M: a = 1; T: -b = -2; L: c = 1$$

M1 A1

$$F = km\omega^2r$$

A1

$$9 = k \times 2 \times 3^2 \times 0.5 = 9k$$

M1

$$k = 1$$

A1

Answer: a = 1, b = 2, c = 1, k = 1

6 marks

20 (5 marks)

$$(a) \text{ Replacing M, L and T by kg, m and s gives } m^3 kg^{-1} s^{-2}$$

B1

Answer: $m^3 kg^{-1} s^{-2}$

1 mark

$$(b) 1 m = 10^{-3} km, 1 kg = 10^{-3} \text{ tonne}, 1 s = \frac{1}{3600} \text{ hour}$$

M1

$$G = 6.67 \times 10^{-11} \times (10^{-3})^3 \times (10^{-3})^{-1} \times (3600)^2 km^3 tonne^{-1} h^{-2}$$

M1 A1

$$= 6.67 \times 10^{-11} \times 12.96 = 8.64 \times 10^{-10}$$

A1

$$\text{Answer: } G = 8.64 \times 10^{-10} \text{ km}^3 \text{ tonne}^{-1} \text{ h}^{-2}$$

4 marks

21 (8 marks)

$$(a) [\eta] = \frac{[F][d]}{[A][v]} = \frac{MLT^{-2} \times L}{L^2 \times LT^{-1}}$$

M1

$$= ML^{-1}T^{-1}$$

A1

$$\text{Answer: } ML^{-1}T^{-1}$$

2 marks

$$(b) M: a = 1; T: -a - c = -2; L: -a + b + c = 1$$

M1 A1

$$\text{so } a = 1, c = 1 \text{ and } b = 1$$

A1

$$F = k\eta rv$$

A1

$$\text{Answer: } a = b = c = 1$$

4 marks

$$(c) F = 6\pi \times 0.9 \times 0.01 \times 0.2$$

M1

$$= 0.0339 \text{ N}$$

A1

$$\text{Answer: } 0.0339 \text{ N}$$

2 marks

22 (9 marks)

$$(a) [P] = T, [G] = M^{-1}L^3T^{-2}, [M] = M, [r] = L$$

B1

$$T: -2a = 1; M: -a + b = 0; L: 3a + c = 0$$

M1 A1

$$a = b = -\frac{1}{2} \text{ and } c = \frac{3}{2}$$

A1

$$P = k\sqrt{\frac{r^3}{GM}}$$

A1

$$\text{Answer: } a = b = -\frac{1}{2}, c = \frac{3}{2}$$

5 marks

$$(b) P = 2\pi\sqrt{\frac{(7.0 \times 10^6)^3}{6.67 \times 10^{-11} \times 6.0 \times 10^{24}}}$$

M1

$$= 5820 \text{ s} = 96.9 \text{ minutes}$$

A1

$$\text{Answer: } 96.9 \text{ minutes}$$

2 marks

$$(c) P \propto r^{3/2}, \text{ so the factor is } 2^{3/2}$$

M1

$$= 2\sqrt{2}$$

A1

$$\text{Answer: } 2\sqrt{2}$$

2 marks

23 (6 marks)

$$(a) \left[\frac{u^2}{g}\right] = \frac{L^2T^{-2}}{LT^{-2}} = L$$

M1

$$\sin \theta \text{ is dimensionless, so the right-hand side has dimension } L, \text{ not } T$$

A1

$$\text{Answer: right-hand side has dimension } L, \text{ not } T$$

2 marks

$$(b) T: -a - 2b = 1; L: a + b = 0$$

M1

$$a = 1, b = -1, \text{ so } t = \frac{ku \sin \theta}{g}$$

A1

$$\text{Answer: } a = 1, b = -1$$

2 marks

$$(c) 1.5 = \frac{k \times 14.7 \times \sin 30^\circ}{9.8}$$

M1

$$k = 2$$

A1

$$\text{Answer: } k = 2$$

2 marks

24 (8 marks)

$$(a) [kv^2] = MLT^{-2}, \text{ so } [k] = \frac{MLT^{-2}}{L^2T^{-2}}$$

M1

$$= ML^{-1}$$

A1

$$\text{Answer: } ML^{-1}$$

2 marks

$$(b) \left[\frac{mg}{k}\right] = \frac{MLT^{-2}}{ML^{-1}} = L^2T^{-2}$$

M1

$$\text{so } [V] = LT^{-1}$$

A1

$$\text{Answer: } [V] = LT^{-1}$$

2 marks

$$(c) \left[\frac{m}{gk} \right] = \frac{M}{LT^{-2} \times ML^{-1}} = T^2$$

M1

so $[\tau] = T$

A1

Answer: $[\tau] = T$

2 marks

(d) t and τ are both times, so $\frac{t}{\tau}$ is dimensionless

B1

 $\tanh$ is dimensionless, so v and V both have dimensions of speed

E1

Answer: consistent

2 marks

25 (9 marks)

$$(a) [\sigma] = \frac{MLT^{-2}}{L} = MT^{-2}, [\rho] = ML^{-3}, [\lambda] = L$$

B1

M: $a + b = 0$; T: $-2a = -1$; L: $-3b + c = 1$

M1 A1

$$a = \frac{1}{2}, b = -\frac{1}{2}, c = -\frac{1}{2}$$

A1

$$v = k\sqrt{\frac{\sigma}{\rho\lambda}}$$

A1

$$\text{Answer: } a = \frac{1}{2}, b = -\frac{1}{2}, c = -\frac{1}{2}$$

5 marks

$$(b) v = \sqrt{\frac{2\pi \times 0.073}{1000 \times 0.01}}$$

M1

$$= 0.214 \text{ m s}^{-1}$$

A1

Answer: 0.214 m s^{-1}

2 marks

$$(c) v \propto \lambda^{-1/2}$$

M1

 v is multiplied by $4^{1/2} = 2$

A1

Answer: v is doubled

2 marks

26 (12 marks)

$$(a) [kx] = [\text{force}] = MLT^{-2}, \text{ so } [k] = MLT^{-2} \div L = MT^{-2}$$

M1 A1

$$[cv] = MLT^{-2}, \text{ so } [c] = MLT^{-2} \div LT^{-1} = MT^{-1}$$

M1 A1

Answer: $[c] = MT^{-1}$; $[k] = MT^{-2}$

4 marks

$$(b) [\sqrt{k/m}] = \sqrt{MT^{-2}/M} = T^{-1}$$

M1 A1

$$[\sqrt{mk}] = \sqrt{M \times MT^{-2}} = MT^{-1} = [c]$$

M1 A1

Answer: T^{-1} ; $MT^{-1} = [c]$

4 marks

$$(c) \left[\frac{c^2}{4m^2} \right] = M^2T^{-2} \div M^2 = T^{-2} = \left[\frac{k}{m} \right]$$

M1 A1

$$\omega = \sqrt{36 - 9} = \sqrt{27} = 5.20 \text{ rad s}^{-1}$$

M1 A1

Answer: 5.20 rad s^{-1}

4 marks

27 (10 marks)

$$(a) [E] = ML^2T^{-2}, [\rho] = ML^{-3}, [t] = T, [r] = L$$

B1

M: $a + b = 0$; L: $2a - 3b = 1$; T: $-2a + c = 0$

M1 A1

$$b = -a, \text{ so } 5a = 1 \text{ and } a = \frac{1}{5}, b = -\frac{1}{5}$$

A1

$$c = 2a = \frac{2}{5}$$

A1

$$r = k\left(\frac{Et^2}{\rho}\right)^{1/5}$$

A1

$$\text{Answer: } a = \frac{1}{5}, b = -\frac{1}{5}, c = \frac{2}{5}$$

6 marks

$$(b) r \propto t^{2/5}, \text{ so } r = 50 \times 4^{2/5}$$

M1

$$= 87.1 \text{ m}$$

A1

Answer: 87.1 m

2 marks

$$(c) r \propto E^{1/5}, \text{ so the ratio is } 32^{1/5}$$

M1

$$= 2$$

A1

Answer: ratio 2 : 1

2 marks

28 (11 marks)

$$(a) [k] = \frac{[dv]}{dt/[v^{3/2}]} = \frac{LT^{-2}}{L^{3/2}T^{-3/2}} = L^{-1/2}T^{-1/2}$$

M1
A1

Answer: $L^{-1/2}T^{-1/2}$

2 marks

(b) Separate variables: $\int v^{-3/2} dv = -\int k dt$

M1

$$-2v^{-1/2} + 2u^{-1/2} = -kt \text{ (using } v = u \text{ when } t = 0)$$

M1 A1

$$v^{-1/2} = u^{-1/2} + \frac{kt}{2} = \frac{2 + kt\sqrt{u}}{2\sqrt{u}}$$

A1

$$v = \frac{4u}{(2 + kt\sqrt{u})^2}$$

A1

Answer: $v = \frac{4u}{(2 + kt\sqrt{u})^2}$

5 marks

(c) $[kt\sqrt{u}] = L^{-1/2}T^{-1/2} \times T \times L^{1/2}T^{-1/2} = 1$, so it can be added to 2

B1

the right-hand side is then $u \times$ (a dimensionless number), which has the dimensions of speed

E1

Answer: consistent

2 marks

(d) $(2 + kt\sqrt{u})^2 = 8$, so $kt\sqrt{u} = 2\sqrt{2} - 2$

M1

$$t = \frac{2(\sqrt{2} - 1)}{0.5 \times 2} = 0.828 \text{ s}$$

A1

Answer: 0.828 s

2 marks

29 (10 marks)

(a) $\frac{x}{l}$ is dimensionless, so $[\lambda] = [T] = MLT^{-2}$

B1

Answer: $[\lambda] = MLT^{-2}$

1 mark

(b) $[m^a l^b \lambda^c] = M^{a+c} L^{b+c} T^{-2c}$

M1

T: $-2c = 1$, so $c = -\frac{1}{2}$

A1

M: $a + c = 0$, so $a = \frac{1}{2}$

A1

L: $b + c = 0$, so $b = \frac{1}{2}$

A1

$$P = k\sqrt{\frac{ml}{\lambda}}$$

A1

Answer: $P = k\sqrt{\frac{ml}{\lambda}}$

5 marks

(c) $\frac{P}{\sqrt{\frac{ml}{\lambda}}} = \frac{0.513}{\sqrt{\frac{0.4 \times 0.5}{30}}}$

M1

$$k = 6.28$$

A1

Answer: $k = 6.28$

2 marks

(d) $P \propto \sqrt{ml}$, so the factor is $\sqrt{\frac{3}{2}}$

M1

$$= 1.22$$

A1

Answer: 1.22

2 marks

30 (10 marks)

(a) $[\Pi] = \frac{ML^{-3} \times LT^{-1} \times L}{ML^{-1}T^{-1}}$

M1

$$= \frac{ML^{-1}T^{-1}}{ML^{-1}T^{-1}} = 1$$

A1

Answer: dimensionless

2 marks

(b) $[\rho v^2 r^2] = ML^{-3} \times L^2 T^{-2} \times L^2 = MLT^{-2}$

M1

$f(\Pi)$ has no dimensions because Π has none, so both sides are forces

E1

Answer: $\rho v^2 r^2$ has dimensions MLT^{-2}

2 marks

$$(c) 6\pi\eta r v = \rho v^2 r^2 \times \frac{6\pi\eta}{\rho v r}$$

M1

$$f(\Pi) = \frac{6\pi}{\Pi}$$

A1

$$\text{Answer: } f(\Pi) = \frac{6\pi}{\Pi}$$

2 marks

$$(d) \frac{6\pi}{\Pi} = 0.2$$

M1

$$\Pi = 30\pi = 94.2$$

A1

$$\text{Answer: } \Pi = 94.2$$

2 marks

$$(e) v = \frac{\Pi\eta}{\rho r} = \frac{30\pi \times 1.0 \times 10^{-3}}{1000 \times 0.01}$$

M1

$$= 0.00942 \text{ m s}^{-1}$$

A1

$$\text{Answer: } 0.00942 \text{ m s}^{-1}$$

2 marks