

Variable Acceleration

- 1 A particle moves on the x-axis. At time t seconds its displacement from the origin is x metres, where $x = 2t^3 - 5t$.
 - (a) Find the velocity of the particle when $t = 2$. [2]
 - (b) Find the acceleration of the particle when $t = 2$. [1]
- 2 The velocity of a particle moving in a straight line is $v = 4t - 3t^2 \text{ m s}^{-1}$ at time t seconds. Find its acceleration when $t = 1$. [2]
- 3 A particle moves in a straight line with acceleration $a = 6t - 2 \text{ m s}^{-2}$ at time t seconds. When $t = 0$ its velocity is 3 m s^{-1} . Find its velocity when $t = 2$. [3]
- 4 A particle starts at the origin O and moves along the x-axis with velocity $6t^2 \text{ m s}^{-1}$ at time t seconds. Find its distance from O when $t = 3$. [2]
- 5 The velocity of a particle at time t seconds is $v = 2 \sin 3t \text{ m s}^{-1}$. Find the acceleration of the particle when $t = \frac{\pi}{3}$. [2]
- 6 A particle moves on the x-axis so that its displacement from the origin at time t seconds is $x = t^3 - 6t^2$ metres. Find the non-zero time at which the particle is instantaneously at rest. [2]
- 7 A particle moves in a straight line. At time t seconds its velocity is $v = t^2 - 6t + 5 \text{ m s}^{-1}$.
 - (a) Find the times at which the particle is instantaneously at rest. [2]
 - (b) Find the acceleration of the particle at each of these times. [2]
- 8 A particle starts from rest and moves in a straight line. Its acceleration at time t seconds is $a = 4 - 2t \text{ m s}^{-2}$ for $0 \leq t \leq 4$. Find the greatest speed of the particle in this interval. [4]
- 9 A particle moves in a straight line with acceleration $e^{-t} \text{ m s}^{-2}$ at time t seconds. When $t = 0$ its velocity is 3 m s^{-1} . Find an expression for v in terms of t and state the value that v approaches as t becomes large. [3]
- 10 A particle moves along the x-axis. When it is at displacement x metres from the origin its acceleration is $4x \text{ m s}^{-2}$. When $x = 1$ the particle is moving away from the origin with speed 2 m s^{-1} . Find the speed of the particle when $x = 3$. [3]
- 11 A particle moves on the x-axis with velocity $v = 12t - 3t^2 \text{ m s}^{-1}$ at time t seconds. At $t = 0$ it is at the origin. Find the total distance the particle travels in the first 6 seconds. [4]
- 12 A particle moves on the x-axis. At time t seconds its displacement from the origin is x metres, where $x = t^3 - 6t^2 + pt + q$ and p and q are constants. The particle is at the origin and instantaneously at rest when $t = 1$.
 - (a) Find the value of p and the value of q . [3]
 - (b) Find the other time at which the particle is at the origin. [2]
- 13 A car travelling at 18 m s^{-1} brakes at time $t = 0$. For the next few seconds its deceleration is $(2 + \frac{t}{3}) \text{ m s}^{-2}$, where t is the time in seconds, until the car stops.
 - (a) Find the time taken for the car to stop. [3]
 - (b) Find the distance travelled by the car while braking. [2]

- 14** A particle moves along the x-axis with velocity $v = 3 \sin 2t \text{ m s}^{-1}$ at time t seconds, $t \geq 0$. When $t = 0$ the particle is at the point where $x = 1$.
 (a) Find an expression for x in terms of t . [3]
 (b) Find the greatest and least values of x . [2]
- 15** A particle moves in a straight line. At time t seconds its acceleration is $-0.4v \text{ m s}^{-2}$, where $v \text{ m s}^{-1}$ is its velocity. When $t = 0$, $v = 20$. Find the time at which the speed has halved, giving your answer to 3 significant figures. [5]
- 16** A particle moves on the x-axis with acceleration $\frac{8}{(x+1)^2} \text{ m s}^{-2}$ directed away from the origin when its displacement from O is x metres ($x > 0$). The particle passes through $x = 1$ with speed 2 m s^{-1} . Find v^2 in terms of x , and hence find the speed that the particle approaches as x becomes very large. [5]
- 17** A particle moves on the x-axis with velocity $4te^{-2t} \text{ m s}^{-1}$ at time t seconds, $t \geq 0$.
 (a) Find the greatest speed of the particle, giving your answer to 3 significant figures. [3]
 (b) Given that $x = 0$ when $t = 0$, show that the displacement of the particle at time t is $x = 1 - (1 + 2t)e^{-2t}$. [4]
- 18** A particle moves along the x-axis with acceleration $a = 12 - 6t \text{ m s}^{-2}$, $t \geq 0$. When $t = 0$ the particle is at the origin with velocity 9 m s^{-1} . Show that the particle is next at the origin when $t = 3 + 3\sqrt{2}$, and find the velocity of the particle at that instant, to 3 significant figures. [8]
- 19** A particle moves along the x-axis with acceleration $a = 3 \cos t \text{ m s}^{-2}$, where $t \geq 0$ is the time in seconds. At $t = 0$ the particle is at the origin O at rest. Find the first time at which the particle is 4.5 m from O, and its speed at that time. [7]
- 20** A particle moves on the x-axis and passes through the origin O at time $t = 0$. Its velocity at time t seconds is $v = \frac{1}{2}(t - 2)(t - 6) \text{ m s}^{-1}$. Find the displacement of the particle from O at the instant when it has travelled a total distance of 20 m. [7]
- 21** A particle P moves on the positive x-axis. When P is at displacement x metres from the origin O its acceleration is $\frac{16}{x^3} \text{ m s}^{-2}$ directed towards O. P passes through the point where $x = 2$ moving away from O with speed 3 m s^{-1} .
 (a) Show that $v^2 = 5 + \frac{16}{x^2}$. [4]
 (b) Find the speed of P when $x = 4$, and describe what happens to the speed of P as x becomes large. [2]
- 22** Two particles P and Q move on the x-axis. At time $t = 0$, P is at the origin O, at rest, with constant acceleration 2 m s^{-2} . At $t = 0$, Q is at the point where $x = 100$ with velocity -8 m s^{-1} and acceleration $-2t \text{ m s}^{-2}$, where t is the time in seconds.
 Show that P and Q are at the same position when $t^3 + 3t^2 + 24t - 300 = 0$, and show that this equation has a root between $t = 4$ and $t = 5$. [7]
- 23** A particle moves on the x-axis. At time $t = 0$ it is at the origin and at rest, and its acceleration at time t seconds is $(2 - 2t) \text{ m s}^{-2}$. A student is asked to find the total distance travelled in the first 4 seconds and writes:
 "v = $2t - t^2$, $x = t^2 - \frac{t^3}{3}$. When $t = 4$, $x = 16 - \frac{64}{3} = -\frac{16}{3}$, so the distance travelled is $\frac{16}{3} \text{ m}$."
 Explain the error in the student's method and find the correct total distance travelled. [6]

- 24** A particle moves on the x-axis. When its displacement from the origin is x metres its acceleration is $-4x \text{ m s}^{-2}$. When $x = 3$ the speed of the particle is 8 m s^{-1} .
- Find v^2 in terms of x . [3]
 - Find the greatest distance of the particle from O. [2]
 - Write down the greatest speed of the particle. [1]
 - The particle moves according to $x = 5 \sin 2t$, where t is the time in seconds. Find the first positive time at which $x = 3$, giving your answer to 3 significant figures. [3]
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- 25** A train leaves station A at time $t = 0$ from rest and moves along a straight horizontal track. For $0 \leq t \leq 60$ its acceleration is $(0.6 - 0.01t) \text{ m s}^{-2}$. The train then travels at constant speed until it is brought to rest at station B by a constant deceleration of 1.5 m s^{-2} .
- Show that the speed of the train when $t = 60$ is 18 m s^{-1} . [2]
 - Find the distance travelled by the train in the first 60 seconds. [2]
 - The distance AB is 2556 m. Find the total time taken for the journey from A to B. [4]
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- 26** A particle moves on the x-axis. Starting from the origin at $t = 0$, its velocity at time t seconds is $v \text{ m s}^{-1}$, where
 $v = t(8 - t)$ for $0 \leq t \leq 6$, $v = 36 - 4t$ for $t > 6$.
- Show that the velocity is continuous at $t = 6$ and find the greatest speed of the particle for $0 \leq t \leq 6$. [3]
 - Find the total distance travelled by the particle in the first 10 seconds. [4]
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- 27** A particle P moves on the positive x-axis. When P is at displacement x metres from the origin O, its acceleration is $\frac{100}{x^2} \text{ m s}^{-2}$ directed towards O. At the instant when $x = 2$, P is moving away from O with speed $U \text{ m s}^{-1}$.
- Show that $v^2 = U^2 - 100 + \frac{200}{x}$, and hence find the least value of U for which P never comes to rest. [6]
 - Given that U has the value found in part (a), find the time taken for P to move from $x = 2$ to $x = 18$. [6]
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- 28** A particle P moves along the x-axis with acceleration $\frac{4}{(1+t)^2} \text{ m s}^{-2}$ at time t seconds, $t \geq 0$. At $t = 0$, P is at the origin O at rest.
- Show that the velocity of P at time t is $\frac{4t}{1+t} \text{ m s}^{-1}$. [3]
 - Show that the displacement of P from O at time t is $4t - 4 \ln(1+t)$. [3]
 - Find the exact displacement of P when $t = 3$. [2]
 - Show that the speed of P is always less than 4 m s^{-1} . Find the displacement of P at the instant when its speed is 3.6 m s^{-1} , giving your answer to 3 significant figures. [4]
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- 29** A particle P moves on the x-axis. When P is at displacement x metres from the origin O its acceleration is $(3x^2 - 12x + 9) \text{ m s}^{-2}$. At the instant when $x = 1$, P is moving towards O with speed 2 m s^{-1} .
- Show that $v^2 = 2x^3 - 12x^2 + 18x - 4$. [4]
 - Show that P does not reach O, and find the least value of x that P reaches. Give your answer in exact form. [5]
 - Describe the subsequent motion of P, justifying your answer. [2]

- 30** A cyclist starts from rest at the point O on a straight road. For $0 \leq t \leq T$, where t is the time in seconds, the cyclist's acceleration is $k(T - t) \text{ m s}^{-2}$, where k and T are positive constants. At $t = T$ the cyclist has speed 30 m s^{-1} and has travelled 200 m from O. After time T the cyclist continues at constant speed 30 m s^{-1} .
- (a) Show that $v = k(Tt - \frac{1}{2}t^2)$ for $0 \leq t \leq T$, and find the displacement of the cyclist from O at time T in terms of k and T . [4]
- (b) Find the values of k and T . [5]
- (c) Find the time at which the cyclist is 500 m from O. [3]

Vector Kinematics

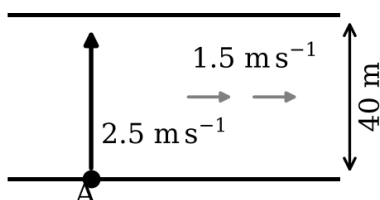
- 1** A particle moves in the x-y plane with position vector $\mathbf{r} = (3t^2 - 2)\mathbf{i} + t^3\mathbf{j}$ metres at time t seconds. Find the velocity of the particle when $t = 2$. [2]
- 2** A particle moves with velocity $\mathbf{v} = (4t\mathbf{i} - 3\mathbf{j}) \text{ m s}^{-1}$ at time t seconds.
- (a) Find the acceleration of the particle. [1]
- (b) Find the speed of the particle when $t = 1$. [2]
- 3** A particle has acceleration $(6t\mathbf{i} + 2\mathbf{j}) \text{ m s}^{-2}$ at time t seconds. When $t = 0$ its velocity is $(\mathbf{i} - \mathbf{j}) \text{ m s}^{-1}$. Find its velocity when $t = 2$. [3]
- 4** A particle moves so that its position vector at time t seconds is $\mathbf{r} = (3 \cos 2t\mathbf{i} + 3 \sin 2t\mathbf{j}) \text{ m}$. Find the velocity of the particle when $t = 0$, and its speed. [2]
- 5** A particle moves with velocity $(2t\mathbf{i} - 6\mathbf{j}) \text{ m s}^{-1}$ at time t seconds. When $t = 0$ its position vector is $4\mathbf{i} \text{ m}$. Find the position vector of the particle when $t = 3$. [3]
- 6** A particle P of mass 3 kg moves in the x-y plane under the action of a single force F newtons. At time t seconds the position vector of P is $\mathbf{r} = (t^3 - 2t)\mathbf{i} + 4t^2\mathbf{j}$ metres. Find F when $t = 2$, and find its magnitude, to 3 significant figures. [4]
- 7** A particle moves with velocity $\mathbf{v} = ((t^2 - 4)\mathbf{i} + 2t\mathbf{j}) \text{ m s}^{-1}$ at time t seconds, $t \geq 0$.
- (a) Find the time at which the particle is moving parallel to the vector $\mathbf{j}$. [2]
- (b) Find the acceleration of the particle at this time. [2]
- 8** A particle moves with acceleration $(-2\mathbf{i} + 2t\mathbf{j}) \text{ m s}^{-2}$, $t \geq 0$. When $t = 0$ its velocity is $8\mathbf{i} \text{ m s}^{-1}$. Find the time at which the particle is moving at 45° to the vector $\mathbf{i}$. [4]
- 9** A particle moves with position vector $\mathbf{r} = ((t^2 - 4t)\mathbf{i} + (3t - 1)\mathbf{j}) \text{ m}$ at time t seconds. Find the times at which the particle is on the y-axis, and its position vector at each of these times. [4]
- 10** A particle moves with velocity $((2t - 4)\mathbf{i} + 3\mathbf{j}) \text{ m s}^{-1}$ at time t seconds. Find the least speed of the particle and the time at which it occurs. [4]
- 11** A particle moves with velocity $(3t^2\mathbf{i} + 4t\mathbf{j}) \text{ m s}^{-1}$ at time t seconds. Find the angle between the direction of motion and the vector $\mathbf{i}$ when $t = 1$, and find the speed at this time. [4]
- 12** Two small remote-controlled vehicles P and Q move on a flat surface. At time t seconds their position vectors, in metres, are $\mathbf{r}_P = (3t\mathbf{i} + t^2\mathbf{j})$ and $\mathbf{r}_Q = ((10 - 2t)\mathbf{i} + 2t\mathbf{j})$. Show that P and Q collide, and find the position vector of the point of collision. [5]

- 13** A particle moves so that its position vector at time t seconds is $\mathbf{r} = ((2t + 1)\mathbf{i} + (t^2 - 3)\mathbf{j})$ m.
- (a) Find the Cartesian equation of the path of the particle. [2]
- (b) Find the distance between the positions of the particle at $t = 1$ and $t = 4$. [3]
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- 14** A particle moves with position vector $\mathbf{r} = (4 \cos 3t\mathbf{i} + 4 \sin 3t\mathbf{j})$ m at time t seconds.
- (a) Show that the acceleration of the particle is $-9\mathbf{r}$. [3]
- (b) Show that the speed of the particle is constant and find its value. [2]
- (c) Show that the velocity of the particle is always perpendicular to its position vector. [2]
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- 15** A particle has acceleration $(6t\mathbf{i} + 4\mathbf{j})$ m s⁻² at time t seconds, $t \geq 0$. At $t = 0$ its velocity is $-12\mathbf{i}$ m s⁻¹. Find the non-zero time at which the velocity of the particle is perpendicular to its acceleration. Give your answer to 3 significant figures. [6]
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- 16** Two boats P and Q move on a calm lake with position vectors, in metres, relative to a fixed origin O at time t seconds, $\mathbf{r}_P = ((1 + 2t)\mathbf{i} + (3 - t)\mathbf{j})$ and $\mathbf{r}_Q = ((7 - t)\mathbf{i} + (2t - 1)\mathbf{j})$. Find the least distance between the boats and the time at which it occurs. [6]
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- 17** A particle moves so that its position vector at time t seconds is $\mathbf{r} = ((t - \sin t)\mathbf{i} + (1 - \cos t)\mathbf{j})$ m, $0 \leq t \leq 2\pi$. Show that the speed of the particle is $2 \sin \frac{t}{2}$ m s⁻¹, and find the greatest speed and the time at which it is reached. [6]
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- 18** A particle moves with constant acceleration $(-2\mathbf{i} + 6\mathbf{j})$ m s⁻². At time $t = 0$ its position vector is $(5\mathbf{i} + 2\mathbf{j})$ m and its velocity is $(6\mathbf{i} - 9\mathbf{j})$ m s⁻¹. Find the position vector of the particle at the instant when it is moving parallel to the vector $\mathbf{i}$. [6]
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- 19** A ball is hit so that its position vector, relative to the point O of projection, at time t seconds is $\mathbf{r} = (12t\mathbf{i} + (16t - 4.9t^2)\mathbf{j})$ m, where $\mathbf{i}$ is horizontal and $\mathbf{j}$ is vertical.
- (a) Find the speed of the ball when $t = 3$, giving your answer to 3 significant figures. [3]
- (b) Find the time at which the ball is moving at 45° below the horizontal. [3]
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- 20** A particle moves with velocity $((t^2 + 2t)\mathbf{i} + (6t - 3)\mathbf{j})$ m s⁻¹ at time t seconds. When $t = 0$ the particle is at the point with position vector $\mathbf{i}$ m.
- (a) Find the two times at which the particle is moving parallel to the vector $\mathbf{i} + \mathbf{j}$. [3]
- (b) Find the position vector of the particle at each of these times. [4]
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- 21** Two aircraft P and Q fly at constant velocity. At time t minutes their position vectors, in km, relative to a fixed origin are $\mathbf{r}_P = ((1 + 3t)\mathbf{i} + (2 - t)\mathbf{j})$ and $\mathbf{r}_Q = ((25 - 3t)\mathbf{i} + (\lambda t - 18)\mathbf{j})$, where λ is a constant. The aircraft collide.
- (a) Find the time of the collision and the value of λ . [4]
- (b) Find the speed of each aircraft. [3]
- (c) Find the position vector of the point of collision. [1]
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- 22** A particle moves so that its position vector at time t seconds is $\mathbf{r} = (5 \cos t\mathbf{i} + 3 \sin t\mathbf{j})$ m.
- (a) Find the Cartesian equation of the path of the particle. [2]
- (b) Show that the acceleration of the particle is always directed towards the origin and has magnitude $|\mathbf{r}|$. [2]
- (c) Find the greatest and least speeds of the particle. [4]

- 23** A particle moves with acceleration $(6t\mathbf{i} - 2 \sin t\mathbf{j}) \text{ m s}^{-2}$ at time t seconds. When $t = 0$ the particle is at the point with position vector $\mathbf{i} \text{ m}$ and has velocity $2\mathbf{j} \text{ m s}^{-1}$.
- (a) Find the velocity of the particle at time t . [3]
- (b) Find the position vector of the particle at time t . [2]
- (c) Find the speed of the particle and its distance from the origin when $t = \frac{\pi}{2}$, giving answers to 3 significant figures. [4]
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- 24** A particle moves with velocity $((t^2 - 1)\mathbf{i} + (2t - 4)\mathbf{j}) \text{ m s}^{-1}$ at time t seconds. A student is asked to find the time at which the particle is moving parallel to $\mathbf{i}$, and the speed at that time. The student writes:
- "Parallel to $\mathbf{i}$, so the $\mathbf{j}$ component is zero: $2t - 4 = 0$, so $t = 2$. At $t = 2$, $\mathbf{v} = -2\mathbf{j}$, so the speed is 2 m s^{-1} ."
- Identify the error and find the correct time and speed. [5]
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- 25** A particle moves with position vector $\mathbf{r} = ((2t^2 - 8t)\mathbf{i} + (t^2 - 4t + 3)\mathbf{j}) \text{ m}$ at time t seconds. Show that the particle moves along a straight line and comes instantaneously to rest, and find the total distance it travels from $t = 0$ to $t = 3$. Give your answer in exact form. [7]
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- 26** A particle moves in a circle with centre the origin so that its position vector at time t seconds is $\mathbf{r} = R(\cos \omega t\mathbf{i} + \sin \omega t\mathbf{j}) \text{ m}$, where R and ω are positive constants. Show that the particle moves with constant speed $R\omega$ and that the magnitude of its acceleration is $R\omega^2$. When the speed is 20 m s^{-1} the magnitude of the acceleration is 50 m s^{-2} . Find R and ω . [7]
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- 27** A particle P moves in the x - y plane so that its position vector $\mathbf{r}$ metres at time t seconds satisfies $\frac{d^2\mathbf{r}}{dt^2} = -4\mathbf{r}$. At $t = 0$, $\mathbf{r} = 3\mathbf{i}$ and the velocity of P is $8\mathbf{j} \text{ m s}^{-1}$.
- (a) Show that $x = 3 \cos 2t$ and $y = 4 \sin 2t$. [6]
- (b) Find the Cartesian equation of the path of P and the speed of P at the instant when it first crosses the x -axis after $t = 0$. [4]
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- 28** A particle moves in a plane so that its position vector at time t seconds is $\mathbf{r} = e^{-t}(\cos t\mathbf{i} + \sin t\mathbf{j}) \text{ m}$, $t \geq 0$.
- (a) Show that the velocity of the particle is $\mathbf{v} = e^{-t}((-\cos t - \sin t)\mathbf{i} + (\cos t - \sin t)\mathbf{j})$ and that its speed is $\sqrt{2}e^{-t} \text{ m s}^{-1}$. [5]
- (b) Show that the angle between the position vector and the velocity of the particle is constant, and find its value. [4]
- (c) Find the exact total distance travelled by the particle as $t \rightarrow \infty$. [2]
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- 29** A target T moves at constant velocity with position vector $\mathbf{r}_T = ((40 - 8t)\mathbf{i} + (4 + 3t)\mathbf{j}) \text{ km}$ at time t minutes. A missile M is launched from the origin O at $t = 0$ and travels in a straight line at constant speed 13 km per minute , intercepting T at time $t = T$.
- (a) Show that $12T^2 + 77T - 202 = 0$. [5]
- (b) Solve this equation to find T . [3]
- (c) Find the velocity of M , giving the direction as an angle to $\mathbf{i}$, and the distance of the point of interception from O . [4]
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- 30** A particle P moves so that its position vector at time t seconds, relative to the origin O , is $\mathbf{r} = (t\mathbf{i} + t^2\mathbf{j}) \text{ m}$, $t \geq 0$. A beacon B is fixed at the point with position vector $4\mathbf{j} \text{ m}$. Find the exact least distance between P and B , and show that when P is closest to B the speed of P is twice this distance. [9]

Relative Motion

- 1 At a certain instant the velocity of car A is $(3\mathbf{i} + 2\mathbf{j}) \text{ m s}^{-1}$ and the velocity of car B is $(\mathbf{i} - 4\mathbf{j}) \text{ m s}^{-1}$. Find the velocity of A relative to B. [2]
- 2 A boat can travel at 5 m s^{-1} in still water. It is steered directly across a river which flows at 3 m s^{-1} . Find the speed of the boat relative to the bank, giving your answer to 3 significant figures. [3]
- 3 Car A travels due east at 25 m s^{-1} and car B travels due north at 20 m s^{-1} . Find the speed of A relative to B, to 3 significant figures. [3]
- 4 An aircraft flies due north at 240 km h^{-1} in still air. A steady wind blows from the south at 40 km h^{-1} . Find the speed of the aircraft relative to the ground. [2]
- 5 Two drones A and B have position vectors $(2\mathbf{i} + 5\mathbf{j}) \text{ m}$ and $(8\mathbf{i} - 3\mathbf{j}) \text{ m}$ respectively. Find the position vector of B relative to A, and the distance AB. [3]
- 6 The velocity of ship B is $(2\mathbf{i} + 3\mathbf{j}) \text{ km h}^{-1}$, where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors due east and due north. The velocity of ship A relative to B is $(5\mathbf{i} - \mathbf{j}) \text{ km h}^{-1}$. Find the velocity of A, and the speed of A to 3 significant figures. [3]
- 7 The diagram shows a river of width 40 m flowing at 1.5 m s^{-1} . A swimmer who can swim at 2.5 m s^{-1} in still water starts at A and swims at right angles to the bank. Find the time taken to cross and the distance downstream from the point directly opposite A at which the swimmer lands. [4]



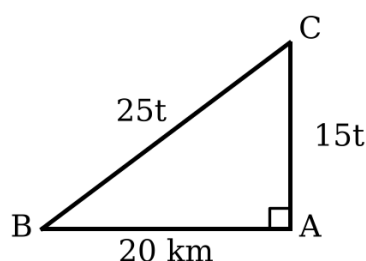
- 8 A boat can travel at 5 m s^{-1} in still water. It crosses a river of width 80 m , flowing at 3 m s^{-1} , and lands at the point directly opposite its starting point. Find the angle at which the boat must be steered, measured from the perpendicular to the bank, and the time taken to cross. [4]
- 9 An aircraft has a velocity relative to the air of 200 km h^{-1} on a bearing of 000° . The wind is blowing from the west at 50 km h^{-1} . Find the speed and bearing of the aircraft relative to the ground, giving answers to 3 significant figures. [5]
- 10 Ship A moves with velocity $(6\mathbf{i} + 8\mathbf{j}) \text{ km h}^{-1}$ and ship B moves with velocity $(-2\mathbf{i} + 3\mathbf{j}) \text{ km h}^{-1}$, where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors due east and due north. Find the velocity of A relative to B, its magnitude, and its bearing, to 3 significant figures. [5]
- 11 A cyclist rides due north at 8 m s^{-1} . The wind blows from the west at 6 m s^{-1} . Find the speed of the wind relative to the cyclist and the bearing from which the cyclist feels the wind to come. [5]

- 12** Ship P travels at 12 km h^{-1} on a bearing of 060° and ship Q travels at 9 km h^{-1} on a bearing of 150° . Find the speed of P relative to Q and the bearing of this relative velocity, giving the bearing to the nearest degree. [5]

- 13** A goods train of length 180 m and a passenger train of length 120 m travel on parallel straight tracks at 20 m s^{-1} and 30 m s^{-1} respectively. Find the time for the passenger train to pass the goods train completely (from the front of the passenger train being level with the back of the goods train until the back of the passenger train is level with the front of the goods train) when they travel (i) in the same direction, (ii) in opposite directions. [5]

- 14** Two ships P and Q are moving with constant velocities. At noon, Q is 20 km east and 18 km north of P. The velocity of Q relative to P is $(-4\mathbf{i} - 3\mathbf{j}) \text{ km h}^{-1}$, where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors due east and due north. Find the shortest distance between the ships and the time after noon at which it occurs. [6]

- 15** Ship A is moving due north at 15 km h^{-1} . At noon ship B, which can travel at 25 km h^{-1} , is 20 km due west of A. B sets a course to intercept A. The diagram shows the positions of A and B at noon and the point of interception C, t hours after noon. Find t , and the bearing on which B must sail.



[6]

- 16** A pilot flies from A to B, where B is 300 km due east of A. The aircraft has a speed of 250 km h^{-1} in still air and there is a wind of 40 km h^{-1} blowing from the north. Find the bearing on which the pilot should steer, and the time taken for the journey, giving the time in hours and minutes to the nearest minute. [6]

- 17** Two cyclists P and Q move on level ground with constant velocities $(3\mathbf{i} + 2\mathbf{j}) \text{ m s}^{-1}$ and $(-\mathbf{i} + 5\mathbf{j}) \text{ m s}^{-1}$ respectively. At time $t = 0$ the position vectors of P and Q relative to a fixed origin are $(4\mathbf{i} + 10\mathbf{j}) \text{ m}$ and $(30\mathbf{i} - 2\mathbf{j}) \text{ m}$. Find the time at which Q is due east of P, and the distance between them at that time. [6]

- 18** A cyclist riding due east at 5 m s^{-1} feels the wind to come from due south. When she rides due north at 10 m s^{-1} she feels the wind to come from the north-west. Find the speed of the true wind and the bearing from which it blows. [6]

- 19** A swimmer who can swim at 1.2 m s^{-1} in still water crosses a river by swimming at right angles to the bank. She reaches the opposite bank after 50 s at a point 30 m downstream of her starting point. Find the speed of the river current, the width of the river, and the speed of the swimmer relative to the bank, to 3 significant figures. [6]

- 20** Two drones P and Q fly at constant velocities. At time $t = 0$ seconds their position vectors relative to a fixed origin O are $(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \text{ m}$ and $(7\mathbf{i} - \mathbf{j} + 7\mathbf{k}) \text{ m}$, and their velocities are $(2\mathbf{i} - \mathbf{j} + \mathbf{k}) \text{ m s}^{-1}$ and $(-\mathbf{i} + \mathbf{j}) \text{ m s}^{-1}$ respectively.

- (a) Show that the distance PQ at time t seconds satisfies $PQ^2 = 14t^2 - 56t + 61$. [4]
(b) Find the least distance between the drones and the time at which it occurs. [3]

- 21** A boat can travel at 5 m s^{-1} in still water. It is to cross a river of width 120 m, which flows at 6 m s^{-1} , and land at the point on the opposite bank 90 m downstream of the point directly opposite its starting point. Find the two possible headings of the boat, each measured as an angle upstream of the perpendicular to the bank, and the crossing time for each. [9]
- 22** An aircraft has an airspeed of 300 km h^{-1} . There is a wind of 60 km h^{-1} blowing from the north. The pilot flies from A to B, 400 km due east of A, and then back to A, steering so that the aircraft travels directly along AB in each direction.
- (a) Find the time taken for the outward journey from A to B, to 3 significant figures. [4]
- (b) Show that the total time for the round trip is longer than it would be in still air, and find the difference to the nearest minute. [4]
- 23** A boat sails at 8 m s^{-1} relative to the water, heading due north, in a current that flows due east at 6 m s^{-1} . A student calculates the boat's speed over the ground as follows:
- "Speed over the ground = $8 + 6 = 14 \text{ m s}^{-1}$. So the boat takes $\frac{2800}{14} = 200 \text{ s}$ to travel 2800 m over the ground."
- Explain the error in the student's method, and find the correct speed over the ground and its bearing. Find also the time taken to travel 2800 m over the ground. [6]
- 24** A cyclist rides along a straight horizontal road at a constant speed of 6 m s^{-1} . To her the rain appears to fall at an angle α to the vertical, where $\tan \alpha = \frac{1}{2}$, coming from in front of her.
- When she increases her speed to 10 m s^{-1} the rain appears to fall at 45° to the vertical, still from in front. The rain has constant velocity, with horizontal component $p \text{ m s}^{-1}$ in the direction in which she is travelling and vertical component $q \text{ m s}^{-1}$ downwards. Find p and q , and the speed of the rain. [7]
- 25** Two aircraft P and Q fly at constant velocities. At time $t = 0$ hours, P is at the origin O and Q is at the point with position vector $(24\mathbf{i} - 7\mathbf{j}) \text{ km}$. The velocities of P and Q are $(2\mathbf{i} + 5\mathbf{j}) \text{ km h}^{-1}$ and $(-4\mathbf{i} + 13\mathbf{j}) \text{ km h}^{-1}$ respectively.
- (a) Show that the distance PQ at time t satisfies $PQ^2 = 100t^2 - 400t + 625$. [4]
- (b) Find the least distance between the aircraft and the time at which it occurs. [2]
- (c) The aircraft can communicate by radio only while they are less than 17 km apart. Find the length of time, in hours, for which they can communicate. [3]
- 26** Ship A is 30 km east and 40 km north of ship B, and moves with constant velocity $(-12\mathbf{i} - 5\mathbf{j}) \text{ km h}^{-1}$, where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors due east and due north. Ship B has a maximum speed of 5 km h^{-1} , so it cannot intercept A. B sets a straight course at its maximum speed so that the distance between the ships becomes as small as possible.
- (a) Show that A moves at 13 km h^{-1} , and find the angle between the direction of A's motion and the line from A to B. [3]
- (b) Explain why the relative velocity of A with respect to B must be directed along a tangent to a circle of radius 5 km h^{-1} about the end of $\mathbf{v}_A$, and use this to find the least distance between the ships, to 3 significant figures. [5]
- (c) Find the bearing on which B should steer, to the nearest degree. [4]
- 27** A freighter F is sailing at constant velocity $(-3\mathbf{i} + 4\mathbf{j}) \text{ km h}^{-1}$, where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors due east and due north. At noon its position vector relative to a coastguard cutter C is $(20\mathbf{i} + 40\mathbf{j}) \text{ km}$. The cutter can travel at 25 km h^{-1} and sets off at noon on a straight course at constant speed to intercept F, meeting it t hours after noon.
- (a) Show that $3t^2 - t - 10 = 0$. [5]
- (b) Solve the equation to find t . [2]
- (c) Find the velocity of the cutter, and the bearing on which it sails, to 3 significant figures. [5]

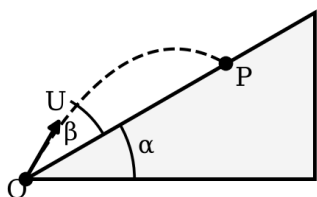
- 28** Ship Q is moving relative to ship P. At time $t = 0$ the distance PQ is d , and the velocity of Q relative to P has magnitude u and makes an angle α ($0 < \alpha < 90^\circ$) with the direction QP.
- (a) Show that the least distance between the ships is $d \sin \alpha$. [4]
- (b) At $t = 0$ (in hours) the position of Q relative to P is $(24\mathbf{i} + 10\mathbf{j})$ km and the velocity of Q relative to P is $(-15\mathbf{i} - 8\mathbf{j})$ km h^{-1} . Find the least distance between the ships, in km to 3 significant figures, and the time at which it occurs. [6]
-
- 29** At noon ship Q is 12 km east and 9 km north of ship P. The velocity of Q relative to P is $(-3\mathbf{i} + \lambda\mathbf{j})$ km h^{-1} , where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors due east and due north and λ is a constant.
- (a) Show that, t hours after noon, $PQ^2 = (9 + \lambda^2)t^2 + (18\lambda - 72)t + 225$. [3]
- (b) Show that the least value of PQ^2 is $225 - \frac{81(\lambda - 4)^2}{9 + \lambda^2}$. [3]
- (c) The least distance between the ships is 9 km. Find the possible values of λ . [4]
- (d) For each value of λ , find the time after noon at which the ships are closest. [2]
-
- 30** A river of width w flows at a constant speed u . A boat has speed v relative to the water, where $v < u$. The boat is steered at a constant angle θ to the upstream bank ($0 < \theta < 180^\circ$) so that it crosses the river, and lands a distance D downstream of the point directly opposite its starting point.
- (a) Show that $D = \frac{w(u - v \cos \theta)}{v \sin \theta}$. [4]
- (b) Show that D is least when $\cos \theta = \frac{v}{u}$, and that the least value of D is $\frac{w\sqrt{u^2 - v^2}}{v}$. [6]
- (c) A boat with $v = 5 \text{ m s}^{-1}$ crosses a river of width 120 m with $u = 6 \text{ m s}^{-1}$. Find the least possible value of D , to 3 significant figures. [2]

Advanced Projectiles

- 1** A stone is thrown horizontally at 12 m s^{-1} from the top of a vertical cliff 44.1 m high. Take $g = 9.8 \text{ m s}^{-2}$.
- (a) Find the time taken for the stone to reach the sea. [2]
- (b) Find the horizontal distance of the point of impact from the foot of the cliff. [1]
-
- 2** A ball is projected from level ground with speed 20 m s^{-1} at 30° above the horizontal. Take $g = 9.8 \text{ m s}^{-2}$. Find the time of flight, to 3 significant figures. [2]
-
- 3** A particle is projected from level ground at 14 m s^{-1} at 30° above the horizontal. Take $g = 9.8 \text{ m s}^{-2}$. Find the greatest height reached. [3]
-
- 4** A golf ball is hit from level ground at 14 m s^{-1} at 45° to the horizontal. Take $g = 9.8 \text{ m s}^{-2}$. Find the horizontal range of the ball. [3]
-
- 5** A ball is thrown horizontally at 8 m s^{-1} from a window 19.6 m above level ground. Take $g = 9.8 \text{ m s}^{-2}$. Find the speed of the ball when it hits the ground, to 3 significant figures. [3]
-
- 6** A ball is thrown from the top of a vertical wall of height 10 m. Its initial velocity has a horizontal component of 8 m s^{-1} and a downward component of 6 m s^{-1} . Take $g = 9.8 \text{ m s}^{-2}$.
- (a) Find the time taken for the ball to reach the ground. [2]
- (b) Find the horizontal distance from the foot of the wall to the point where the ball lands. [1]
-
- 7** A ball is hit from level ground with speed 28 m s^{-1} at 40° above the horizontal. Take $g = 9.8 \text{ m s}^{-2}$.
- (a) Find the horizontal range of the ball, to 3 significant figures. [3]
- (b) Find the greatest height reached by the ball, to 3 significant figures. [2]

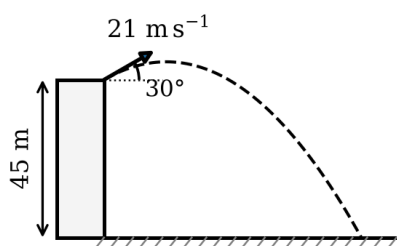
- 8** A ball is thrown horizontally at 15 m s^{-1} from the top of a tower of height 30 m. Take $g = 9.8 \text{ m s}^{-2}$. Find the angle that the velocity of the ball makes with the horizontal as it hits the ground, to the nearest degree. [4]
-
- 9** A ball is thrown from level ground at 20 m s^{-1} at 60° above the horizontal. Take $g = 9.8 \text{ m s}^{-2}$. Find the height of the ball above the ground when it has travelled a horizontal distance of 10 m. [4]
-
- 10** A ball is kicked from level ground at 20 m s^{-1} at 45° above the horizontal towards a wall 3 m high which is 25 m from the point of projection on level ground. Take $g = 9.8 \text{ m s}^{-2}$. Show that the ball passes over the wall, and find how far above the top of the wall the ball is as it passes it, to 3 significant figures. [5]
-
- 11** A ball is thrown from a point 1.5 m above level ground with speed 14 m s^{-1} at 30° above the horizontal. Take $g = 9.8 \text{ m s}^{-2}$.
 (a) Find the time taken for the ball to hit the ground. [3]
 (b) Find the horizontal distance travelled by the ball before it hits the ground. [2]
-
- 12** A footballer kicks a ball from level ground with speed 21 m s^{-1} at 30° above the horizontal, towards a goal whose crossbar is 2.44 m high and whose goal line is 30 m away horizontally. The ball moves in a vertical plane perpendicular to the goal line. Take $g = 9.8 \text{ m s}^{-2}$. Show that the ball is descending when it reaches the goal line, and find the distance by which it clears the crossbar, to 3 significant figures. [7]
-
- 13** A stone is thrown from the top of a tower of height 30 m with speed 12 m s^{-1} at an angle of 25° below the horizontal. Take $g = 9.8 \text{ m s}^{-2}$.
 (a) Find the time taken for the stone to reach the ground, to 3 significant figures. [3]
 (b) Find the horizontal distance of the landing point from the foot of the tower. [2]
 (c) Find the speed of the stone as it hits the ground. [2]
-
- 14** A particle is projected from the origin O with speed U at an angle θ above the horizontal, with horizontal and vertical axes through O. Show that the path of the particle has equation $y = x \tan \theta - \frac{g x^2}{2U^2 \cos^2 \theta}$.
 Hence show that the equation can be written $y = x \tan \theta - \frac{g x^2}{2U^2} (1 + \tan^2 \theta)$. [6]
-
- 15** A particle is projected from the origin O with speed 14 m s^{-1} at an angle θ above the horizontal and passes through the point with coordinates (14, 3), where the x-axis is horizontal and the y-axis is vertically upwards, in metres. Take $g = 9.8 \text{ m s}^{-2}$. Show that $49 \tan^2 \theta - 140 \tan \theta + 79 = 0$ and find the two possible values of θ , to the nearest 0.1° . [6]
-
- 16** A ball is projected from level ground. At its highest point the ball is 20 m above the ground, and at that point it has travelled a horizontal distance of 30 m from the point of projection. Take $g = 9.8 \text{ m s}^{-2}$. Find the speed and the angle above the horizontal at which the ball was projected, to 3 significant figures. [7]
-

- 17** A particle is projected from a point O on a plane inclined at 30° to the horizontal. The speed of projection is 14 m s^{-1} and the direction of projection makes an angle of 30° with the plane, in the vertical plane containing the line of greatest slope through O, upwards. The particle lands at the point P on the plane. Take $g = 9.8 \text{ m s}^{-2}$. By resolving parallel and perpendicular to the plane, find the distance OP.



[6]

- 18** A stone is thrown from the top of a vertical cliff of height 45 m at a speed of 21 m s^{-1} at 30° above the horizontal, as shown, and lands in the sea. Take $g = 9.8 \text{ m s}^{-2}$.



- (a) Find the time taken for the stone to reach the sea. [4]
 (b) Find the horizontal distance from the foot of the cliff to the point where the stone enters the sea, and the speed of the stone at that instant. [5]

- 19** A ball is projected from the origin O with speed $U \text{ m s}^{-1}$ at an angle θ above the horizontal, where the x-axis is horizontal and the y-axis is vertically upwards. The ball passes through the point A(24, 7), with distances in metres. Take $g = 9.8 \text{ m s}^{-2}$.

- (a) Show that $kT^2 - 24T + (k + 7) = 0$, where $T = \tan \theta$ and $k = \frac{2822.4}{U^2}$. [3]
 (b) Use the discriminant of this equation to find the least possible value of U, to 3 significant figures. [4]
 (c) Find the angle of projection when U takes this least value. [2]

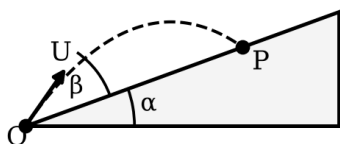
- 20** A tennis player serves the ball horizontally from a point 2.4 m above level ground. The net is 12 m horizontally from the server and 0.9 m high, and the ball just clears the top of the net. Take $g = 9.8 \text{ m s}^{-2}$ and treat the ball as a particle.

- (a) Find the speed of the ball as it leaves the racket, to 3 significant figures. [3]
 (b) The ball must land within 18.4 m of the server, horizontally, to be in. Determine whether the serve is in. [4]

- 21** A projectile is launched from level ground with speed U at an angle θ above the horizontal. Take $g = 9.8 \text{ m s}^{-2}$.

- (a) Show that the horizontal range is $\frac{U^2 \sin 2\theta}{g}$. [4]
 (b) Hence show that two projections with the same speed U and angles θ and $90^\circ - \theta$ have the same range, and state the angle that gives the greatest range for a given U. [3]
 (c) For $U = 28 \text{ m s}^{-1}$, find the two angles that give a range of 60 m, to 1 decimal place. [3]

- 22** A small ball is projected horizontally at 10 m s^{-1} from a point A which is 19.6 m above level ground. The ball strikes a smooth vertical wall, at right angles, at a point 15 m horizontally from A. The collision is perfectly elastic, so the horizontal component of velocity reverses and the vertical component is unchanged. Take $g = 9.8 \text{ m s}^{-2}$.
- (a) Find the time taken for the ball to reach the wall, and its height above the ground at that instant. [4]
- (b) Find the distance of the point where the ball lands from the foot of the wall. [4]
-
- 23** A particle P is projected from a point O on level ground with speed 25 m s^{-1} at an angle θ above the horizontal, where $\tan \theta = \frac{4}{3}$. At the same instant a particle Q is released from rest at a point 40 m vertically above the point on the ground that is 30 m horizontally from O, so that P and Q move in the same vertical plane. Take $g = 9.8 \text{ m s}^{-2}$. Show that P and Q collide, and find the time of the collision and the height of the point of collision above the ground. [7]
-
- 24** A ball is projected from the origin O, the x-axis being horizontal and the y-axis vertically upwards, and passes through the points A(16, 7.1) and B(32, 4.4), the coordinates being in metres. Take $g = 9.8 \text{ m s}^{-2}$. Find the angle of projection and the speed of projection, and the greatest height reached by the ball. [8]
-
- 25** A particle is projected from level ground with speed 24.5 m s^{-1} at 30° above the horizontal. A student is asked to find the greatest height reached by the particle, and the speed of the particle at that height. The student writes:
- "At the highest point the velocity is zero, so $0 = 24.5^2 - 2 \times 9.8 \times H$, giving $H = 30.6 \text{ m}$. The speed there is 0."
- Explain the two errors in the student's work and find the correct values. Take $g = 9.8 \text{ m s}^{-2}$. [6]
-
- 26** A ball is projected from a point O on level ground with speed U at an angle θ above the horizontal. It just touches the top of a vertical wall of height H , whose base is a horizontal distance d from O, at the instant it is at the highest point of its path.
- (a) Show that $\tan \theta = \frac{2H}{d}$. [4]
- (b) Given that $d = 24 \text{ m}$ and $H = 7 \text{ m}$, find U , giving your answer to 3 significant figures. [4]
- Take $g = 9.8 \text{ m s}^{-2}$.
-
- 27** A particle is projected from a point O on a plane inclined at an angle α to the horizontal, with speed U at an angle β above the plane, in a vertical plane containing a line of greatest slope. The particle moves up the plane and strikes it again at the point P. Take $g = 9.8 \text{ m s}^{-2}$.

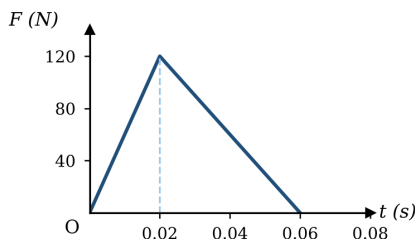


- (a) Show that $OP = \frac{2U^2 \sin \beta \cos(\alpha + \beta)}{g \cos^2 \alpha}$. [5]
- (b) Show that, for a given U , the greatest value of OP is $\frac{U^2}{g(1 + \sin \alpha)}$, and state the value of β for which this occurs. [5]
- (c) Given $U = 21 \text{ m s}^{-1}$ and $\alpha = 20^\circ$, find the value of β for the greatest range and the greatest range, to 3 significant figures. [2]

- 28** A ball is hit from a point O on level ground with velocity $(16\mathbf{i} + 19.6\mathbf{j}) \text{ m s}^{-1}$, where $\mathbf{i}$ is horizontal and $\mathbf{j}$ is vertically upwards. A strong wind causes the ball to have a constant horizontal acceleration of -8 m s^{-2} , in addition to the acceleration due to gravity of 9.8 m s^{-2} downwards.
- Show that the ball lands at O. [4]
 - Find the speed of the ball just before it lands, and the angle its path then makes with the horizontal, to 3 significant figures. [5]
 - Find the greatest horizontal distance of the ball from O. [2]
-
- 29** A small ball is projected horizontally at 6 m s^{-1} from a point 4.9 m above level ground. It bounces repeatedly on the ground. At each bounce the horizontal component of velocity is unchanged and the vertical component of velocity is reduced to half its value just before the bounce (coefficient of restitution 0.5). Take $g = 9.8 \text{ m s}^{-2}$.
- Find the time taken for the ball to reach the ground for the first time. [2]
 - Show that the time between the n -th and the $(n + 1)$ -th bounces is $2(0.5)^n$ seconds. [3]
 - Find the total time until the ball stops bouncing, and the total horizontal distance travelled by the ball in that time. [3]
 - Find the total vertical distance travelled by the ball. [4]
-
- 30** A stone is thrown from the top of a cliff of height h above level horizontal ground, with speed U at an angle θ above the horizontal. The stone lands at a horizontal distance R from the foot of the cliff. Take $g = 9.8 \text{ m s}^{-2}$.
- Show that $\frac{gR^2}{2U^2} \tan^2 \theta - R \tan \theta + \left(\frac{gR^2}{2U^2} - h\right) = 0$. [2]
 - Show that, for a given U , the greatest possible value of R is $\frac{U}{g} \sqrt{U^2 + 2gh}$. [5]
 - Given $U = 28 \text{ m s}^{-1}$ and $h = 22.5 \text{ m}$, find the greatest possible R and the angle of projection that achieves it. [4]

Motion Under Variable Forces

- A particle of mass 2 kg is at rest at $t = 0$ and is acted on by a single force of $6t \text{ N}$, where t is the time in seconds. Find the speed of the particle when $t = 3$. [2]
- A particle of mass 0.5 kg moves in a straight line under a force of $(4 + 3t) \text{ N}$ at time t seconds. Find its acceleration when $t = 2$. [2]
- A force of $12t \text{ N}$ acts on a particle for the time interval $0 \leq t \leq 2$, where t is the time in seconds. Find the magnitude of the impulse of the force. [2]
- A particle moves along the x -axis from $x = 1$ to $x = 4$ under a force of $6x \text{ N}$ in the direction of x increasing. Find the work done by the force. [2]
- A particle of mass 3 kg moves on the x -axis under a single force of $12x \text{ N}$ directed away from the origin. Find the acceleration of the particle when $x = 2$. [2]
- A particle of mass 3 kg moves in a straight line so that its displacement from a fixed point at time t seconds is $x = 2t^3 - t^2$ metres. Find the magnitude of the resultant force acting on the particle when $t = 2$. [3]
- A particle of mass 2 kg moves in a straight line under a force of $(6 - 2t) \text{ N}$, where t is the time in seconds. When $t = 0$ the speed of the particle is 1 m s^{-1} . Find its speed when $t = 3$. [4]

- 8** A particle of mass 0.4 kg is at rest. From time $t = 0$, a force of $20e^{-2t}$ N, where t is the time in seconds, acts on the particle for as long as the particle moves. Find the total impulse of the force, and the limiting speed of the particle. [4]
- 9** A particle of mass 2 kg moves along the x -axis. When it is at $x = 0$ its speed is 2 m s^{-1} in the direction of x increasing. It is acted on by a force of $(12 - 2x)$ N in the direction of x increasing. Find the speed of the particle when $x = 4$. [4]
- 10** A particle of mass 1.5 kg moves on a smooth horizontal table along the x -axis under a single force of $6x$ N directed towards the origin O . The particle is released from rest at the point where $x = 2$. Find the speed of the particle as it passes through the point where $x = 1$. [4]
- 11** A particle of mass 3 kg is at rest at $t = 0$ and moves in a straight line under a force of $(12 - 6t)$ N, where t is the time in seconds, for $0 \leq t \leq 4$. Find the greatest speed of the particle in this time and the time at which it occurs. [4]
- 12** A particle of mass 1.5 kg moves along the x -axis. When its displacement from the origin is x metres, its velocity is $v = (4 - x^2) \text{ m s}^{-1}$.
 (a) Find the resultant force on the particle when $x = 1$, stating its direction. [3]
 (b) Find the resultant force on the particle when $x = 3$. [2]
- 13** A particle of mass 2 kg is released from rest at the origin and moves along the x -axis. When its displacement is x metres, the only force acting on it is $8e^{-x}$ N in the direction of x increasing.
 (a) Show that the speed of the particle satisfies $v^2 = 8(1 - e^{-x})$. [3]
 (b) Find the speed of the particle when $x = \ln 4$, and the speed that it approaches as x becomes large. [3]
- 14** A particle of mass 2 kg moves along the positive x -axis. The only force acting on the particle is a force of magnitude $\frac{72}{x^2}$ N directed towards the origin O , when the particle is at a distance x metres from O . The particle is released from rest at the point where $x = 6$. Find the speed of the particle when $x = 2$. [5]
- 15** A force acts on a ball of mass 0.4 kg, initially at rest, along a straight line. The magnitude F newtons of the force varies with time t seconds as shown in the graph: F increases uniformly from 0 to 120 between $t = 0$ and $t = 0.02$, then decreases uniformly to 0 at $t = 0.06$. Find the magnitude of the impulse of the force on the ball and the speed of the ball immediately after the force has stopped acting.
- 
- [5]
- 16** A particle of mass 3 kg is at rest at $t = 0$ and acted on by a force whose magnitude is $3t$ newtons for $0 \leq t \leq 4$ and 12 newtons for $t > 4$, in a constant direction. Find the speed of the particle when $t = 6$. [5]
- 17** A particle of mass 2 kg is at rest at the origin at $t = 0$ and moves in a straight line under a force of $10 \sin 2t$ N, where t is the time in seconds. Find the first time after $t = 0$ at which the particle is instantaneously at rest, and its displacement from the origin at that time. [6]

- 18** A car of mass 1200 kg starts from rest and moves along a straight horizontal road. The resultant force on the car in its direction of motion is $(3000 - 20x)$ N, when the car has travelled x metres from its starting point.
- (a) Find the speed of the car when $x = 100$, to 3 significant figures. [3]
- (b) Find the greatest speed of the car, to 3 significant figures. [3]
-
- 19** A particle of mass 3 kg moves on the positive x -axis. When its displacement from the origin is x metres, the only force acting on it has magnitude $\frac{6}{2x+1}$ N, in the direction of x increasing. When $x = 0$ the speed of the particle is 1 m s^{-1} . Find the speed of the particle when $x = 4$, to 3 significant figures. [5]
-
- 20** A particle of mass 2 kg is at rest at the origin at time $t = 0$ and then moves in a straight line. When its speed is $v \text{ m s}^{-1}$, the resultant force on it in the direction of motion is $(4 - v)$ N.
- (a) Show that the time taken to reach a speed of 3 m s^{-1} is $2 \ln 4$ seconds, and evaluate this to 3 significant figures. [4]
- (b) Find the distance travelled by the particle in reaching this speed, to 3 significant figures. [4]
-
- 21** The gravitational force on a particle of mass m at distance x from the centre of the Earth ($x \geq R$, where $R = 6.4 \times 10^6 \text{ m}$ is the radius of the Earth) is $\frac{mgR^2}{x^2}$ directed towards the centre, where $g = 9.8 \text{ m s}^{-2}$. A particle is projected vertically upwards from the surface of the Earth with speed U . Ignore air resistance and the rotation of the Earth.
- (a) Show that when the particle is at distance x from the centre of the Earth its speed v satisfies $v^2 = U^2 - 2gR + \frac{2gR^2}{x}$. [4]
- (b) Given that $U = 8000 \text{ m s}^{-1}$, find the greatest height above the surface of the Earth reached by the particle, to 3 significant figures. [3]
- (c) Show that the particle escapes from the Earth (never returns) if $U \geq \sqrt{2gR}$, and find this speed in m s^{-1} to 3 significant figures. [3]
-
- 22** A particle of mass 2 kg moves in a horizontal plane under the action of a single force $F = (12t\mathbf{i} + 8 \cos 2t\mathbf{j})$ N at time t seconds, where $\mathbf{i}$ and $\mathbf{j}$ are perpendicular unit vectors. At $t = 0$ the particle is at rest at the origin O.
- (a) Find the velocity and position vector of the particle at time t . [4]
- (b) Find the speed of the particle when $t = \frac{\pi}{2}$ and the work done by the force during the first $\frac{\pi}{2}$ seconds. Give answers to 3 significant figures. [4]
-
- 23** A particle of mass 2 kg is at rest at $t = 0$ and moves in a straight line under a force of $6t$ N, where t is the time in seconds. A student is asked to find the speed of the particle when $t = 4$ and writes:
- "Force at $t = 4$ is 24 N, so $a = \frac{24}{2} = 12 \text{ m s}^{-2}$. Using $v = u + at$ gives $v = 0 + 12 \times 4 = 48 \text{ m s}^{-1}$."
- Explain the error in the student's method and find the correct speed. [5]
-
- 24** A block of mass 5 kg is at rest on a rough horizontal surface at the point where $x = 0$. A horizontal force of $(40 - 4x)$ N, where x is the displacement of the block in metres, acts on the block in the direction of x increasing. The surface exerts a constant friction force of 10 N on the block while it is moving.
- (a) Find the greatest speed of the block, to 3 significant figures. [5]
- (b) Find the distance moved by the block before it first comes to rest. [2]

- 25** A boat of mass 400 kg is initially at rest on calm water. Its engine produces a thrust of $(600 - 20t)$ N in the direction of motion at time t seconds, for $0 \leq t \leq 30$. Ignore any resistance to the motion of the boat.
- Find the speed of the boat when $t = 30$. [4]
 - Find the distance travelled by the boat in the first 30 seconds. [3]
-
- 26** A particle of mass 1 kg moves along the x -axis under a single force of $-(4x + 2x^3)$ N, where x is the displacement in metres from the origin O. The particle is released from rest at the point where $x = 1.5$.
- Find the greatest speed of the particle. [4]
 - Find the speed of the particle when $x = 1$, to 3 significant figures. [2]
 - Explain why the particle oscillates between $x = 1.5$ and $x = -1.5$. [1]
-
- 27** A particle of mass 2 kg moves along the positive x -axis towards the origin O under the action of a single force of magnitude $\frac{16}{x^2}$ N directed towards O, where x is the distance of the particle from O in metres. The particle is released from rest at the point where $x = 4$.
- Show that when the particle is at distance x from O its speed v satisfies $v^2 = 16(\frac{1}{x} - \frac{1}{4})$. [4]
 - Show that the time taken for the particle to reach O is π seconds. [7]
 - Find the speed of the particle when $x = 1$. [1]
-
- 28** A particle of mass 2 kg is at rest on a rough horizontal surface, the coefficient of friction between the particle and the surface being 0.5. A horizontal force of magnitude $5t$ N, where t is the time in seconds, is applied to the particle from $t = 0$ until $t = 6$, when it is removed. Take $g = 9.8 \text{ m s}^{-2}$.
- Find the time at which the particle starts to move. [2]
 - Find the speed of the particle at $t = 6$. [4]
 - Find the total distance travelled by the particle from the start of its motion until it comes to rest, to 3 significant figures. [6]
-
- 29** A particle of mass 2 kg moves along the positive x -axis. When it is at distance x metres from the origin O, the only force acting on it is a force of magnitude $\frac{4}{x^3}$ N directed away from O. At time $t = 0$ the particle is at rest at the point where $x = 1$.
- Show that $v^2 = 2(1 - \frac{1}{x^2})$. [4]
 - Hence show that $x^2 = 1 + 2t^2$. [5]
 - Find the speed of the particle when $t = 2$, and the speed that it approaches as t becomes large. [3]
-
- 30** A particle of mass 2 kg is at rest at a point O on a smooth horizontal surface. From time $t = 0$ to $t = 4$ a horizontal force of magnitude $12 \sin(\frac{\pi t}{4})$ N, where t is the time in seconds, acts on the particle in a constant direction. At $t = 4$ the particle moves onto a rough part of the surface, with coefficient of friction 0.3, and the force ceases to act. Take $g = 9.8 \text{ m s}^{-2}$.
- Show that for $0 \leq t \leq 4$ the speed of the particle is $\frac{24}{\pi}(1 - \cos \frac{\pi t}{4}) \text{ m s}^{-1}$. [3]
 - Find the distance of the particle from O at $t = 4$. [3]
 - Find the further distance travelled by the particle on the rough surface before it comes to rest. [4]

Resistance and Drag

- 1 A particle of mass 2 kg moves in a straight line. The only horizontal force acting on it is a resistance of $0.5v$ N, where $v \text{ m s}^{-1}$ is its speed. Find the magnitude of its deceleration when $v = 8$. [2]
- 2 A particle moves in a straight line with acceleration $-0.2v \text{ m s}^{-2}$ where $v \text{ m s}^{-1}$ is its velocity at time t seconds. When $t = 0$, $v = 10$. Find v when $t = 5$, to 3 significant figures. [3]
- 3 A particle moves in a straight line with acceleration $-0.05v^2 \text{ m s}^{-2}$ where $v \text{ m s}^{-1}$ is its velocity. Find the magnitude of the acceleration when $v = 20$. [2]
- 4 A particle of mass m falls vertically through the air. At speed $v \text{ m s}^{-1}$ the air resistance has magnitude $0.5mv$ N. Take $g = 9.8 \text{ m s}^{-2}$. Find the acceleration of the particle when $v = 4$. [2]
- 5 A boat of mass 150 kg is moving at 6 m s^{-1} when its engine is switched off. The water exerts a constant resistance of 30 N. Find the distance the boat travels before coming to rest. [3]
- 6 A car of mass 1000 kg is driven along a straight horizontal road by an engine working at a constant rate of 30 kW. The resistance to motion is 800 N. Find the acceleration of the car when its speed is 15 m s^{-1} . [3]
- 7 A particle moves in a straight line with acceleration $-kv$, where k is a positive constant and $v \text{ m s}^{-1}$ is the velocity at time t seconds. The speed is 20 m s^{-1} when $t = 0$ and 10 m s^{-1} when $t = 4$. Find the exact value of k , and the speed when $t = 10$, to 3 significant figures. [5]
- 8 A particle moves along the x -axis with acceleration $-0.1v \text{ m s}^{-2}$ where $v \text{ m s}^{-1}$ is its velocity. When $x = 0$, $v = 25$. Find the value of x at which the particle comes to rest. [3]
- 9 A particle of mass 2 kg moves in a straight line under a resistance of $0.5v^2$ N, where $v \text{ m s}^{-1}$ is its speed at time t seconds. When $t = 0$, $v = 12$. Find the speed of the particle when $t = 4$, to 3 significant figures. [4]
- 10 A particle falls from rest through the air, with acceleration $(9.8 - 0.5v) \text{ m s}^{-2}$ where $v \text{ m s}^{-1}$ is its speed after t seconds. Show that $v = 19.6(1 - e^{-0.5t})$ and find the speed after 2 seconds, to 3 significant figures. [5]
- 11 A cyclist and bicycle have total mass 80 kg. The cyclist stops pedalling while travelling at 10 m s^{-1} on a straight horizontal road. The resistance to motion is $4v$ N, where $v \text{ m s}^{-1}$ is the speed. Find the time taken for the speed to fall to 5 m s^{-1} . [4]
- 12 A cyclist and bicycle of total mass 80 kg travel along a straight horizontal road. The cyclist stops pedalling at time $t = 0$ when travelling at 6 m s^{-1} . The resistance to motion is proportional to the speed of the cyclist, and when the speed is 6 m s^{-1} the resistance is 12 N.
 - (a) Show that, while the cyclist is freewheeling, $80 \frac{dv}{dt} = -2v$, where $v \text{ m s}^{-1}$ is the speed at time t seconds. [3]
 - (b) Find the time taken for the speed to fall to 3 m s^{-1} , to 3 significant figures. [2]
- 13 A car of mass 1500 kg starts from rest on a straight horizontal road. The engine works at a constant rate of 45 kW and any resistance to motion may be ignored. The speed of the car at time t seconds is $v \text{ m s}^{-1}$.
 - (a) Show that $v \frac{dv}{dt} = 30$ and deduce that $v^2 = 60t$. [4]
 - (b) Find the time taken for the car to reach a speed of 24 m s^{-1} . [1]

- 14** A particle moves in a straight line with acceleration $-0.05v^2 \text{ m s}^{-2}$, where $v \text{ m s}^{-1}$ is its velocity at time t seconds. When $t = 0$ the particle is at the origin O with velocity 20 m s^{-1} . Find v in terms of t , and the distance of the particle from O when $t = 3$, to 3 significant figures. [6]
- 15** A particle of mass m falls from rest through the air. The resistance has magnitude $0.098mv^2 \text{ N}$ when its speed is $v \text{ m s}^{-1}$. Take $g = 9.8 \text{ m s}^{-2}$. Show that when the particle has fallen x metres, $v^2 = 100(1 - e^{-0.196x})$, and find the speed when $x = 20$, to 3 significant figures. [6]
- 16** A sledge of mass 1 kg slides on level ice with initial speed 12 m s^{-1} . The total resistance to its motion is $(2 + 0.5v) \text{ N}$, where $v \text{ m s}^{-1}$ is its speed. Find the time taken for the sledge to come to rest, and the distance it travels in this time, giving both answers to 3 significant figures. [6]
- 17** A particle of unit mass moves in a straight line under a resistance of $2\sqrt{v} \text{ N}$, where $v \text{ m s}^{-1}$ is its speed. The speed at time $t = 0$ is 16 m s^{-1} . Find the time taken to come to rest and the distance travelled in this time, in exact form. [6]
- 18** A particle of mass 3 kg is projected horizontally along a smooth surface at 10 m s^{-1} . The only horizontal force acting on it is a resistance of $6v \text{ N}$, where $v \text{ m s}^{-1}$ is its speed at time t seconds. Find the velocity at time t , and show that the particle never travels further than 5 m from its starting point. [6]
- 19** A car of mass m coasts with its engine switched off. The resistance is mkv^2 where v is its speed and k is a constant. The speed falls from 30 m s^{-1} to 20 m s^{-1} over a distance of 100 m . Find the value of k , and the speed of the car after it has travelled 150 m from the point where its speed was 30 m s^{-1} . Give answers to 3 significant figures. [6]
- 20** A particle falls vertically from rest through the air. When its speed is $v \text{ m s}^{-1}$ its acceleration is $(9.8 - 0.5v) \text{ m s}^{-2}$. The particle reaches the ground with speed 10 m s^{-1} . Find the height from which it fell, to 3 significant figures. [6]
- 21** A particle moves in a straight line with acceleration $-(1 + v^2) \text{ m s}^{-2}$, where $v \text{ m s}^{-1}$ is its velocity. When $t = 0$ the particle is at the origin with velocity 3 m s^{-1} .
 (a) Show that when the velocity is v the displacement is $x = \frac{1}{2} \ln\left(\frac{10}{1 + v^2}\right)$. [4]
 (b) Find the distance travelled before the particle comes to rest. [1]
 (c) Find the time taken for the particle to come to rest. [3]
- 22** A particle falls from rest through the air with acceleration $(9.8 - 0.5v) \text{ m s}^{-2}$, where $v \text{ m s}^{-1}$ is its speed after t seconds, so that $v = 19.6(1 - e^{-0.5t})$.
 (a) Show that the distance fallen after t seconds is $x = 19.6(t - 2 + 2e^{-0.5t})$. [3]
 (b) Find the distance fallen in the first 4 seconds. [2]
 (c) Find the distance fallen during the fifth second (between $t = 4$ and $t = 5$). [2]
- 23** A particle moves in a straight line with acceleration $-0.5v \text{ m s}^{-2}$, where $v \text{ m s}^{-1}$ is its velocity at time t seconds, and $v = 8$ when $t = 0$. A student is asked to find v when $t = 2$ and writes:
 "Acceleration is $\frac{dv}{dt} = -0.5v$, so integrating with respect to t gives $v = 8 - 0.5vt$. When $t = 2$, $v = 8 - 0.5v \times 2$, so $2v = 8$ and $v = 4 \text{ m s}^{-1}$."
 Explain the error in the student's method and find the correct value of v when $t = 2$, to 3 significant figures. [5]

- 24** A motor boat is moving with its engine switched off. The resistance to its motion produces a deceleration of κv^2 , where $v \text{ m s}^{-1}$ is its speed and κ is a constant. The speed falls from 12 m s^{-1} to 6 m s^{-1} in 10 seconds.
- Find the value of κ . [4]
 - Find the distance travelled by the boat in these 10 seconds, to 3 significant figures. [2]
 - Find the further time needed for the speed to fall from 6 m s^{-1} to 3 m s^{-1} . [2]
-
- 25** A particle of unit mass moves along a straight line. At first it moves under a resistance $0.04v^2 \text{ N}$, where $v \text{ m s}^{-1}$ is its speed, with initial speed 20 m s^{-1} . When its speed has fallen to 10 m s^{-1} the resistance changes to a constant 2 N , and the particle moves under this until it comes to rest. Find the total distance travelled and the total time taken, to 3 significant figures. [7]
-
- 26** A car of mass 1000 kg moves along a straight horizontal road. The engine works at a constant rate of 40 kW and the resistance to motion is $40v \text{ N}$, where $v \text{ m s}^{-1}$ is the speed of the car at time t seconds.
- Show that $1000 v \frac{dv}{dt} = 40\,000 - 40v^2$. [3]
 - Find the maximum speed of the car, to 3 significant figures. [2]
 - The car starts from rest. Find the time taken to reach 20 m s^{-1} , to 3 significant figures. [4]
-
- 27** A particle of unit mass is projected vertically upwards with speed $u \text{ m s}^{-1}$. The air resistance has magnitude kv , where v is the speed of the particle and k is a positive constant. Take $g = 9.8 \text{ m s}^{-2}$.
- Show that the time taken to reach the greatest height is $\frac{1}{k} \ln(1 + \frac{ku}{g})$. [4]
 - Show that the greatest height is $\frac{u}{k} - \frac{g}{k^2} \ln(1 + \frac{ku}{g})$. [5]
 - For $u = 30$ and $k = 0.1$, find the time to reach the greatest height and the greatest height, to 3 significant figures. [3]
-
- 28** A particle is projected vertically upwards with speed 20 m s^{-1} . The air resistance per unit mass is $\frac{g}{400}v^2$, where $v \text{ m s}^{-1}$ is the speed of the particle, so that while moving upwards the acceleration is $-g(1 + \frac{v^2}{400}) \text{ m s}^{-2}$. Take $g = 9.8 \text{ m s}^{-2}$.
- Show that the greatest height reached is $\frac{200}{g} \ln 2 \text{ m}$ and find its value to 3 significant figures. [5]
 - Find the time taken to reach the greatest height, to 3 significant figures. [4]
 - The particle then falls back with acceleration $g(1 - \frac{v^2}{400}) \text{ m s}^{-2}$, where v is its speed. Find its speed when it returns to the point of projection. [3]
-
- 29** A particle of unit mass moves in a straight line with acceleration $-(v + v^2)$, where $v \text{ m s}^{-1}$ is its velocity. At time $t = 0$ the particle is at the origin O with velocity 3 m s^{-1} .
- Show that the velocity of the particle when its displacement from O is x metres is $v = 4e^{-x} - 1$. [4]
 - Show that the particle comes to rest after travelling a distance $\ln 4$ from O . [2]
 - Show that $v = \frac{3}{4e^t - 3}$, and find the time at which the speed of the particle is 1 m s^{-1} . [6]

- 30** A meteorite of unit mass enters the atmosphere moving vertically downwards with speed U at the instant it is at height 0 (measured downwards, x metres being the distance it has descended). As the air thickens, the air resistance is $0.0002e^{x/2000}v^2$ newtons when the speed is v m s⁻¹. Ignore the weight of the meteorite.
- (a) Show that $v = U \exp(-0.4(e^{x/2000} - 1))$. [5]
- (b) Given that $U = 600$, find the speed after the meteorite has descended 2000 m, to 3 significant figures. [2]
- (c) Find the distance descended when the speed has fallen to half its initial value, to 3 significant figures. [3]

Terminal Velocity

- 1** A particle of mass m falls vertically through the air against a resistance of $0.5mv$, where v m s⁻¹ is its speed. Take $g = 9.8$ m s⁻². Find the terminal velocity of the particle. [2]
- 2** A car has an engine that produces a constant driving force of 1800 N. The resistance to motion is $30v$ N, where v m s⁻¹ is the speed of the car. Find the greatest speed of the car on a straight horizontal road. [2]
- 3** A particle of mass 2 kg falls vertically through the air against a resistance of $0.04v^2$ N, where v m s⁻¹ is its speed. Take $g = 9.8$ m s⁻². Find the terminal velocity, to 3 significant figures. [2]
- 4** A body falls from rest with acceleration $(9.8 - 0.2v)$ m s⁻², where v m s⁻¹ is its speed. Find the terminal velocity of the body. [2]
- 5** A particle falls from rest through the air against a resistance that increases with speed, and eventually reaches a terminal velocity V . Sketch a velocity-time graph for the motion, marking V . [2]
- 6** A particle of mass 3 kg falls vertically through the air against a resistance of kv N, where v m s⁻¹ is its speed and k is a constant. The terminal velocity of the particle is 14.7 m s⁻¹. Take $g = 9.8$ m s⁻². Find the value of k . [3]
- 7** A particle falls from rest through the air with acceleration $(9.8 - 0.5v)$ m s⁻², where v m s⁻¹ is its speed after t seconds, so that $v = 19.6(1 - e^{-0.5t})$. Find the time taken to reach 90% of its terminal velocity, to 3 significant figures. [4]
- 8** A skydiver of mass 80 kg, with parachute open, falls against an air resistance of $16v^2$ N, where v m s⁻¹ is the speed. Take $g = 9.8$ m s⁻². Find the terminal velocity of the skydiver, and the magnitude of the acceleration when the speed is 14 m s⁻¹. [4]
- 9** A particle falls from rest with acceleration $9.8(1 - \frac{v}{19.6})$ m s⁻², where v m s⁻¹ is its speed after t seconds. Find the time taken for the speed to reach 95% of the terminal velocity, to 3 significant figures. [5]
- 10** A sphere of mass 0.5 kg falls vertically through a liquid. The liquid exerts an upward buoyancy force of 1.5 N and a resistance of $2v$ N, where v m s⁻¹ is the speed of the sphere. Take $g = 9.8$ m s⁻². Find the terminal velocity of the sphere. [3]

- 11** A body falls from rest under gravity against a resistance. Its terminal velocity is V . (i) When the resistance is proportional to the speed, find, in terms of V , the speed at which the acceleration is $\frac{1}{2}g$. (ii) When the resistance is proportional to the square of the speed, find, in terms of V , the speed at which the acceleration is $\frac{1}{2}g$. [4]
-
- 12** A particle falls from rest through a liquid. Its speed $v \text{ m s}^{-1}$ after t seconds is given by $v = V(1 - e^{-kt})$, where V and k are positive constants. After 2 seconds the speed of the particle is 60% of its terminal velocity.
 (a) Find the value of k , to 3 significant figures. [3]
 (b) Find the time taken for the speed to reach 90% of the terminal velocity, to 3 significant figures. [2]
-
- 13** A boat of mass 250 kg is driven through the water by a constant thrust of 500 N. The resistance of the water is $25v \text{ N}$, where $v \text{ m s}^{-1}$ is the speed of the boat. The boat reaches its terminal velocity, and the engine is then switched off.
 (a) Find the terminal velocity of the boat. [2]
 (b) Find the distance travelled by the boat after the engine is switched off until its speed has fallen to 5 m s^{-1} . [3]
 (c) Find the time taken for the speed to fall from 20 m s^{-1} to 5 m s^{-1} , to 3 significant figures. [2]
-
- 14** A particle of mass m falls from rest through the air against a resistance of $0.05mv^2$, where $v \text{ m s}^{-1}$ is its speed. Take $g = 9.8 \text{ m s}^{-2}$. Show that after falling a distance x metres, $v^2 = 196(1 - e^{-0.1x})$, and find the distance fallen when the speed is 7 m s^{-1} , to 3 significant figures. [6]
-
- 15** A particle falls from rest against a resistance such that its acceleration is $9.8(1 - \frac{v^2}{196}) \text{ m s}^{-2}$, where $v \text{ m s}^{-1}$ is its speed after t seconds. Use partial fractions to find the time taken for the speed to reach 7 m s^{-1} , to 3 significant figures. [6]
-
- 16** A particle of mass m falls from rest through a liquid against a resistance of mgv/V , where v is its speed and V is a positive constant.
 (a) Show that $v = V(1 - e^{-gt/V})$. [4]
 (b) Find the distance fallen in time t . [3]
-
- 17** A car of mass 1000 kg starts from rest on a straight horizontal road. The engine produces a constant driving force of 3000 N and the resistance to motion is $50v \text{ N}$, where $v \text{ m s}^{-1}$ is the speed. Find the speed and the distance travelled when $t = 20$, to 3 significant figures. [7]
-
- 18** A ball is projected vertically downwards with speed 40 m s^{-1} through the air. Its acceleration is $(9.8 - 0.5v) \text{ m s}^{-2}$, where $v \text{ m s}^{-1}$ is its speed. Find the terminal velocity and the time taken for the speed to fall to 25 m s^{-1} , to 3 significant figures. [6]
-
- 19** A block of mass 2 kg slides down a smooth plane inclined at 30° to the horizontal. The block is subject to an air resistance of $0.8v \text{ N}$, where $v \text{ m s}^{-1}$ is its speed. The block is released from rest. Take $g = 9.8 \text{ m s}^{-2}$. Find the terminal velocity of the block, and the time taken for its speed to reach half the terminal velocity. [6]
-
- 20** A car of mass 1200 kg has an engine working at a constant rate of 30 kW. The resistance to motion at speed $v \text{ m s}^{-1}$ is $15v \text{ N}$. Find the greatest speed of the car on a horizontal road, to 3 significant figures, and the acceleration of the car when its speed is 20 m s^{-1} . [5]
-

- 21** A particle falls from rest through the air with acceleration $g(1 - \frac{v^2}{V^2})$, where v is its speed and V is a positive constant.
- (a) Show that $v = V \tanh(\frac{gt}{V})$ satisfies this equation, with $v = 0$ at $t = 0$. [5]
- (b) Explain why V is the terminal velocity, and find the speed at $t = \frac{V}{g}$ as a multiple of V , to 3 significant figures. [2]
-
- 22** A skydiver of mass 80 kg falls from rest. Before the parachute opens, the air resistance is $8v$ N, where v m s⁻¹ is the speed. At $t = 5$ the parachute opens and the resistance becomes $160v$ N. Take $g = 9.8$ m s⁻².
- (a) Find the speed of the skydiver when the parachute opens, to 3 significant figures. [4]
- (b) Find the terminal velocity after the parachute has opened, and the time after opening taken for the speed to fall to 6 m s⁻¹, to 3 significant figures. [6]
-
- 23** A particle falls from rest through the air with acceleration $(g - kv^2)$ where v is its speed and k is a positive constant. A student is asked to find the terminal velocity when $k = 0.05$ and $g = 9.8$, and to find the time taken to reach half of this speed. The student writes: "At terminal velocity the acceleration is zero, so $9.8 - 0.05V = 0$, giving $V = 196$ m s⁻¹." Explain the error in the student's method, and find the correct terminal velocity. Find also the time taken to reach half of the terminal velocity, to 3 significant figures. [6]
-
- 24** A small steel ball of mass 0.02 kg falls vertically from rest through oil. The oil exerts an upward buoyancy force of 0.049 N and a resistance of $0.6v$ N, where v m s⁻¹ is the speed of the ball. Take $g = 9.8$ m s⁻².
- (a) Find the terminal velocity of the ball. [2]
- (b) Show that $v = 0.245(1 - e^{-30t})$. [4]
- (c) Find the time taken for the ball to reach 99% of its terminal velocity and the distance it has fallen in that time, to 3 significant figures. [4]
-
- 25** A car of mass 1500 kg freewheels, with its engine off, from rest down a long straight hill inclined at an angle α to the horizontal where $\sin \alpha = \frac{1}{10}$. The resistance to motion is $3v^2$ N, where v m s⁻¹ is the speed. Take $g = 9.8$ m s⁻². Find the terminal velocity of the car, and the distance the car has travelled when its speed is 20 m s⁻¹, to 3 significant figures. [7]
-
- 26** A particle falls from rest through the air with acceleration $g(1 - \frac{v}{V})$ where v is its speed and V is the terminal velocity, a positive constant.
- (a) Show that the distance fallen when the speed of the particle is $\frac{1}{2}V$ is $\frac{V^2}{g}(\ln 2 - \frac{1}{2})$. [5]
- (b) Given that $V = 19.6$ m s⁻¹ and $g = 9.8$ m s⁻², find the time taken to reach a speed of $\frac{1}{2}V$, and the distance fallen in that time, to 3 significant figures. [4]
-
- 27** A skydiver falls from rest. The acceleration of the skydiver is $g(1 - \frac{v^2}{V^2})$ where v is the speed and V is the terminal velocity, a positive constant. Take $g = 9.8$ m s⁻².
- (a) Use partial fractions to show that $v = V \tanh(\frac{gt}{V})$. [6]
- (b) Show that the distance fallen in time t is $x = \frac{V^2}{g} \ln \cosh(\frac{gt}{V})$. [3]
- (c) For $V = 50$ m s⁻¹, find the time taken to fall 500 m, to 3 significant figures. [3]

- 28** A particle falls from rest through the air with acceleration $(g - \frac{g v}{V})$, where $V = 19.6 \text{ m s}^{-1}$ is the terminal velocity and $g = 9.8 \text{ m s}^{-2}$, v being the speed after t seconds.
- (a) Show that $v = V(1 - e^{-0.5t})$ and that the distance fallen is $x = V t - 2V(1 - e^{-0.5t})$. [5]
- (b) Show that, after a long time, the particle falls a further 19.6 m in each second, to a very good approximation. [2]
- (c) Find the time taken to fall 400 m, giving your answer to 3 significant figures, and justify any approximation made. [3]
-
- 29** A car of mass 1000 kg starts from rest on a straight horizontal road. The engine produces a constant driving force of 2500 N. The resistance to motion is $4v^2 \text{ N}$, where $v \text{ m s}^{-1}$ is the speed of the car.
- (a) Find the terminal velocity of the car. [1]
- (b) Show that $t = 5 \ln(\frac{25 + v}{25 - v})$. [6]
- (c) Find the time taken, and the distance travelled, as the car's speed increases from rest to 20 m s^{-1} , to 3 significant figures. [5]
-
- 30** A box of mass m slides from rest down a rough plane inclined at an angle α to the horizontal, where $\tan \alpha = \frac{3}{4}$. The coefficient of friction between the box and the plane is 0.25. The air resistance on the box has magnitude $0.02mv^2$, where $v \text{ m s}^{-1}$ is the speed. Take $g = 9.8 \text{ m s}^{-2}$.
- (a) Show that the acceleration of the box is $0.02(196 - v^2) \text{ m s}^{-2}$, and write down the terminal velocity. [4]
- (b) Find the speed of the box after it has travelled 10 m down the plane, to 3 significant figures. [4]
- (c) Find the time taken for the box to reach a speed of 7 m s^{-1} , to 3 significant figures. [3]

Dimensional Analysis

- 1** Using M, L and T for the dimensions of mass, length and time, write down the dimensions of
- (a) speed, [1]
- (b) acceleration, [1]
- (c) force. [1]
-
- 2** Find the dimensions of energy (work done), using the formula work done = force $\times$ distance. [2]
-
- 3** Find the dimensions of
- (a) momentum, [1]
- (b) impulse, using impulse = force $\times$ time. [1]
-
- 4** Find the dimensions of
- (a) density, mass per unit volume, [1]
- (b) pressure, force per unit area. [1]
-
- 5** Find the dimensions of power, given that power is the rate at which work is done. [2]
-
- 6** Show that the equation $v = u + at$ is dimensionally consistent, where v and u are speeds, a is an acceleration and t is a time. [2]
-
- 7** A particle moves under a resistive force F . Find the dimensions of the constant k in the model
- (a) $F = kv$, where v is speed, [2]
- (b) $F = kv^2$. [2]

- 8** Newton's law of gravitation states that the force between two particles of masses m_1 and m_2 at distance r is $F = \frac{G m_1 m_2}{r^2}$. Find the dimensions of the gravitational constant G . [3]
-
- 9** In each case, state whether the equation could be correct, justifying your answer by dimensions. The symbols have their usual meanings (u , v speeds; a acceleration; s distance; t time). (i) $v^2 = u^2 + 2as$ (ii) $v = u + at^2$ (iii) $s = ut + \frac{1}{2}at^2$ [4]
-
- 10** A light spring exerts a tension T when it is extended by x , where $T = kx$. Find the dimensions of the spring constant k . The modulus of elasticity λ of an elastic string is defined by $T = \frac{\lambda x}{l}$, where l is the natural length. Find the dimensions of λ . [4]
-
- 11** Find the dimensions of
 (a) angular velocity ω , [1]
 (b) the moment of a force (torque) about a point, [1]
 (c) angular momentum, given by the moment of momentum about a point. [1]
-
- 12** The period of a simple pendulum of length l in a gravitational field of strength g is claimed to be $T = 2\pi\sqrt{\frac{l}{g}}$. Show by dimensions that this formula is consistent. [3]
-
- 13** The displacement of a particle at time t is given by $x = Ae^{-bt}\sin \omega t$, where A , b and ω are constants. Find the dimensions of A , b and ω . [4]
-
- 14** The speed v of waves on deep water is assumed to depend only on the wavelength λ and the gravitational field strength g , in the form $v = k g^a \lambda^b$, where k is a dimensionless constant. Use dimensional analysis to find a and b . [5]
-
- 15** The time t taken for a stone to fall from rest through a height h in a gravitational field of strength g is assumed to have the form $t = k h^a g^b$, where k is a dimensionless constant. Find a and b . A stone dropped from a height of 19.6 m takes 2 s to reach the ground. Take $g = 9.8 \text{ m s}^{-2}$ and find the exact value of k . [6]
-
- 16** The escape speed from the surface of a planet of mass M and radius R is claimed to be $v = \sqrt{\frac{2GM}{R}}$, where G is the gravitational constant, with $[G] = \text{M}^{-1}\text{L}^3\text{T}^{-2}$.
 (a) Show that this formula is dimensionally consistent. [3]
 (b) Explain why $v = \sqrt{\frac{2GM}{R^2}}$ cannot be correct. [2]
-
- 17** The drag force on a cyclist is modelled by $D = \frac{1}{2}C\rho A v^2$, where C is a dimensionless constant, ρ is the density of air, A is the frontal area of the cyclist and v is the speed.
 (a) Show that this model is dimensionally consistent. [2]
 (b) When $\rho = 1.2 \text{ kg m}^{-3}$, $A = 0.5 \text{ m}^2$ and $v = 10 \text{ m s}^{-1}$, the drag is 15 N. Find C . [2]
-
- 18** Bernoulli's equation for a moving fluid of density ρ , at speed v and pressure p , at height h above a datum, is $p + \frac{1}{2}\rho v^2 + \rho gh = \text{constant}$. Show by dimensions that the three terms on the left-hand side are consistent with each other. [4]
-

- 19** A particle of mass m moves in a horizontal circle of radius r with angular speed ω . The magnitude F of the force needed is assumed to have the form $F = km^a\omega^br^c$, where k is a dimensionless constant. Find a , b and c . When $m = 2$ kg, $\omega = 3$ rad s^{-1} and $r = 0.5$ m, the force required is 9 N. Find k . [6]
- 20** The gravitational constant is $G = 6.67 \times 10^{-11}$ in SI units (metres, kilograms and seconds), and has dimensions $M^{-1}L^3T^{-2}$.
 (a) Write down the units of G in terms of m, kg and s. [1]
 (b) Find the value of G when lengths are measured in kilometres, masses in tonnes (1 tonne = 1000 kg) and times in hours, giving your answer in standard form to 3 significant figures. [4]
- 21** The coefficient of viscosity η of a fluid is defined by $F = \eta A \frac{v}{d}$, where F is the tangential force needed to slide a flat plate of area A at speed v across a layer of the fluid of thickness d .
 (a) Find the dimensions of η . [2]
 (b) A sphere of radius r moves at speed v through the fluid. The drag force is assumed to have the form $F = k\eta^ar^bv^c$, where k is a dimensionless constant. Find a , b and c . [4]
 (c) It is given that $k = 6\pi$. Find the drag on a ball of radius 0.01 m moving at 0.2 m s^{-1} through oil with $\eta = 0.9$ kg $m^{-1} s^{-1}$. Give your answer to 3 significant figures. [2]
- 22** The period P of a satellite in a circular orbit of radius r about a planet of mass M is assumed to have the form $P = kG^aM^br^c$, where G is the gravitational constant, with $[G] = M^{-1}L^3T^{-2}$, and k is a dimensionless constant.
 (a) Find a , b and c . [5]
 (b) Taking $k = 2\pi$, $G = 6.67 \times 10^{-11}$ (SI units), $M = 6.0 \times 10^{24}$ kg and $r = 7.0 \times 10^6$ m, find P in minutes, correct to 3 significant figures. [2]
 (c) State the factor by which P changes if r is doubled. [2]
- 23** A student is asked to find the time of flight t of a projectile launched from level ground with speed u at an angle θ above the horizontal. The student writes $t = \frac{u^2 \sin \theta}{g}$, where g is the gravitational field strength. Take $g = 9.8$ m s^{-2} .
 (a) Use dimensions to show that the student's formula is wrong. [2]
 (b) Assume that $t = ku^ag^b \sin \theta$, where k is a dimensionless constant. Find a and b . [2]
 (c) A projectile launched at 14.7 m s^{-1} at 30° above the horizontal has a time of flight of 1.5 s. Find k . [2]
- 24** A particle of mass m falls from rest under gravity against a resistance of magnitude kv^2 , where v is its speed, so that $m \frac{dv}{dt} = mg - kv^2$.
 (a) Find the dimensions of k . [2]
 (b) Show that $V = \sqrt{\frac{mg}{k}}$ has the dimensions of speed. [2]
 (c) Show that $\tau = \sqrt{\frac{m}{gk}}$ has the dimensions of time. [2]
 (d) The solution of the equation is $v = V \tanh(\frac{t}{\tau})$. Show that this is dimensionally consistent. [2]

- 25** The speed v of small ripples on the surface of a liquid is assumed to depend on the surface tension σ (the force per unit length acting in the surface), the density ρ of the liquid and the wavelength λ , in the form $v = k\sigma^a\rho^b\lambda^c$, where k is a dimensionless constant.
- (a) Find a , b and c . [5]
- (b) Taking $k = \sqrt{2\pi}$, find v when $\sigma = 0.073 \text{ N m}^{-1}$, $\rho = 1000 \text{ kg m}^{-3}$ and $\lambda = 0.01 \text{ m}$. Give your answer to 3 significant figures. [2]
- (c) Find the factor by which v changes if λ is divided by 4. [2]
-
- 26** A particle of mass m , attached to a spring, moves on a horizontal table against a resistance cv , where v is its speed. The spring has stiffness k , so that the equation of motion of the displacement x is $m \frac{d^2x}{dt^2} + c \frac{dx}{dt} + kx = 0$.
- (a) Find the dimensions of c and of k . [4]
- (b) Show that $\sqrt{k/m}$ has the dimensions of angular velocity, and that the critical value $c = 2\sqrt{mk}$ is dimensionally consistent. [4]
- (c) For $c < 2\sqrt{mk}$ the angular frequency of the motion is $\omega = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}}$. Show that this formula is dimensionally consistent, and find ω to 3 significant figures when $m = 0.5 \text{ kg}$, $k = 18 \text{ N m}^{-1}$ and $c = 3 \text{ kg s}^{-1}$. [4]
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- 27** The radius r of the shock wave from an explosion is assumed to depend on the energy E released, the density ρ of the surrounding air and the time t since the explosion, in the form $r = kE^a\rho^b t^c$, where k is a dimensionless constant.
- (a) Use dimensional analysis to find a , b and c . [6]
- (b) At 0.01 s after a particular explosion the radius is 50 m. Find the radius at 0.04 s after the same explosion, to 3 significant figures. [2]
- (c) A second explosion in the same air releases 32 times as much energy. Find the ratio of the radii of the two shock waves at the same time after each explosion. [2]
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- 28** A particle moves in a straight line with speed v and decelerates according to $\frac{dv}{dt} = -kv^{3/2}$, where k is a positive constant. The initial speed is u .
- (a) Find the dimensions of k . [2]
- (b) Solve the equation to show that $v = \frac{4u}{(2 + kt\sqrt{u})^2}$. [5]
- (c) Show that the result in part (b) is dimensionally consistent. [2]
- (d) Given that $k = 0.5$ and $u = 4$ in SI units, find the time taken for the speed to fall to $\frac{1}{2}u$, correct to 3 significant figures. [2]
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- 29** A particle of mass m is attached to one end of a light elastic string of natural length l and modulus of elasticity λ . The other end of the string is fixed, and the particle performs small vertical oscillations with period P . Assume that P depends only on m , l and λ .
- (a) Using the definition of the tension $T = \frac{\lambda x}{l}$ for an extension x , show that the dimensions of λ are MLT^{-2} . [1]
- (b) Assuming that $P = km^a l^b \lambda^c$, where k is a dimensionless constant, show that $P = k\sqrt{\frac{ml}{\lambda}}$. [5]
- (c) In an experiment, $m = 0.4 \text{ kg}$, $l = 0.5 \text{ m}$ and $\lambda = 30 \text{ N}$ give $P = 0.513 \text{ s}$. Find k to 3 significant figures. [2]
- (d) State the factor by which P changes if m is halved and l is tripled, giving your answer to 3 significant figures. [2]

- 30** A sphere of radius r moves at speed v through a fluid of density ρ and viscosity η , where $[\eta] = \text{ML}^{-1}\text{T}^{-1}$. The drag force D is modelled by $D = \rho v^2 r^2 f(\Pi)$, where f is a function and $\Pi = \frac{\rho v r}{\eta}$.
- (a) Show that Π is dimensionless. [2]
- (b) Show that $\rho v^2 r^2$ has the dimensions of force, and explain why this makes the model consistent. [2]
- (c) For slow motion, the drag is given by Stokes' law, $D = 6\pi\eta r v$. Express $6\pi\eta r v$ in the form $\rho v^2 r^2 f(\Pi)$ and hence find $f(\Pi)$. [2]
- (d) For fast motion, $f(\Pi)$ is a constant, equal to 0.2. Find the value of Π at which the slow-motion and fast-motion models give the same drag. Give your answer to 3 significant figures. [2]
- (e) A sphere of radius 0.01 m moves through water, with $\rho = 1000 \text{ kg m}^{-3}$ and $\eta = 1.0 \times 10^{-3} \text{ kg m}^{-1} \text{ s}^{-1}$. Find the speed at which the two models agree, to 3 significant figures. [2]