

Hypothesis Testing Principles

1 (2 marks)

$$H_0: p = 0.5$$

B1

$$H_1: p > 0.5$$

B1

Answer: $H_0: p = 0.5$, $H_1: p > 0.5$

2 marks

2 (3 marks)

(a) One-tailed, since the claim is about an increase in one direction only

B1

Answer: One-tailed

1 mark

(b) Two-tailed, since the change could be an increase or a decrease

B1

Answer: Two-tailed

1 mark

(c) One-tailed, since only a decrease is of interest

B1

Answer: One-tailed

1 mark

3 (3 marks)

$$P(X \geq 7) = 1 - P(X \leq 6) = 1 - 0.9648 = 0.0352$$

M1 A1

$0.0352 < 0.05$, so reject H_0 ; there is evidence that $p > 0.5$

B1

Answer: $P = 0.0352$; reject H_0

3 marks

4 (2 marks)

(a) The set of values of the test statistic for which H_0 is rejected

B1

Answer: Values leading to rejection of H_0

1 mark

(b) The probability of rejecting H_0 when H_0 is true (the size of the critical region)

B1

Answer: $P(\text{reject } H_0 \mid H_0 \text{ true})$

1 mark

5 (3 marks)

$$z = \frac{52.5 - 50}{\frac{4}{\sqrt{16}}} = 2.50$$

M1 A1

$2.5 > 1.645$, so reject H_0 ; evidence that $\mu > 50$

B1

Answer: $z = 2.5$; reject H_0

3 marks

6 (2 marks)

(a) $0.032 < 0.05$, so reject H_0

B1

Answer: Reject H_0

1 mark

(b) $0.032 > 0.01$, so do not reject H_0

B1

Answer: Do not reject H_0

1 mark

7 (7 marks)

(a) Lower tail: $P(X \leq 1) = 0.0076 < 0.025$ and $P(X \leq 2) = 0.0355 > 0.025$

M1 A1

Upper tail: $P(X \geq 11) = 0.0171 < 0.025$ and $P(X \geq 10) = 0.0480 > 0.025$

M1 A1

Critical region: $X \leq 1$ or $X \geq 11$

A1

Answer: $X \leq 1$ or $X \geq 11$

5 marks

(b) $0.0076 + 0.0171 = 0.0248$

M1 A1

Answer: 0.0248

2 marks

8 (5 marks)

$$H_0: p = 0.2, H_1: p > 0.2$$

B1

Under H_0 , $X \sim B(15, 0.2)$; $P(X \geq 7) = 1 - P(X \leq 6) = 0.0181$

M1 A1

$0.0181 < 0.05$, so reject H_0

M1

There is evidence that more than 20% of the seeds produce red flowers

A1

Answer: Reject H_0 , $p = 0.0181$

5 marks

9 (5 marks)

$H_0: p = 0.5, H_1: p \neq 0.5; X \sim B(100, 0.5) \approx N(50, 25)$

B1 M1

$$P(X \geq 62) \approx P(Z > \frac{61.5 - 50}{5}) = P(Z > 2.30) = 0.0107$$

M1 A1

$0.0107 < 0.025$ (two-tailed), so reject H_0 : evidence the coin is biased

A1

Answer: Reject H_0 ; $p \approx 0.0107$

5 marks

10 (5 marks)

$H_0: \mu = 500, H_1: \mu \neq 500$

B1

$$z = \frac{497.2 - 500}{\frac{6}{\sqrt{25}}} = -2.333$$

M1 A1

$|z| = 2.33 > 1.96$, so reject H_0

M1

There is evidence that the mean mass has changed

A1

Answer: $z = -2.333$; reject H_0

5 marks

11 (4 marks)

(a) $0.036 < 0.05$, so reject H_0

M1

There is evidence that the proportion recovering within a week is less than 40%

A1

Answer: Reject H_0

2 marks

(b) $0.036 > 0.01$, so H_0 is not rejected at 1%; a smaller significance level needs stronger evidence

E1 E1

Answer: p-value above 0.01

2 marks

12 (4 marks)

(a) H_0 is either true or false; the test does not give a probability for a hypothesis

E1

The p-value is calculated on the assumption that H_0 is true

E1

Answer: p-value assumes H_0 true

2 marks

(b) If H_0 were true, the probability of a result at least as extreme as the one observed is 0.03

E1 E1

Answer: $P(\text{result at least as extreme} \mid H_0 \text{ true}) = 0.03$

2 marks

13 (5 marks)

Under $H_0, \bar{X} \sim N(120, \frac{9^2}{36})$, so $sd = 1.5$

B1

Critical $z = -1.645$, so critical value = $120 - 1.645 \times 1.5$
= 117.53

M1 M1

A1

Critical region: $\bar{X} < 117.5$

A1

Answer: $\bar{X} < 117.5$

5 marks

14 (9 marks)

(a) $X \sim B(40, 0.15)$; $P(X \leq 2) = 0.0486$, $P(X \leq 3) = 0.1302$

M1 A1

$P(X \geq 11) = 0.0299$, $P(X \geq 10) = 0.0672$

M1 A1

Critical region: $X \leq 2$ or $X \geq 11$

A1

Answer: $X \leq 2$ or $X \geq 11$

5 marks

(b) $0.0486 + 0.0299 = 0.0785$

M1 A1

Answer: 0.0785

2 marks

(c) 11 lies in the critical region, so reject H_0

M1

There is evidence that the proportion of complaints has changed from 0.15

A1

Answer: Reject H_0

2 marks

15 (6 marks)

$H_0: p = 0.7, H_1: p < 0.7; X \sim B(18, 0.7)$ under H_0

B1 B1

$P(X \leq 9) = 0.0596$

M1 A1

$0.0596 > 0.05$ so do not reject H_0

M1

There is insufficient evidence that the pass rate is lower than 70%

A1

Answer: p-value 0.0596; do not reject H_0

6 marks

16 (6 marks)

$$\bar{x} = \frac{2376}{40} = 59.4$$

B1

$$H_0: \mu = 60, H_1: \mu \neq 60; z = \frac{59.4 - 60}{\frac{3.2}{\sqrt{40}}} = -1.186$$

B1 M1 A1

$$|z| = 1.19 < 1.96, \text{ so do not reject } H_0$$

M1

No evidence against the claim that the mean mass is 60 g

A1

Answer: $z = -1.186$; do not reject H_0

6 marks

17 (6 marks)

$$H_0: p = 0.4, H_1: p < 0.4; X \sim B(250, 0.4) \approx N(100, 60)$$

B1 M1

$$P(X \leq 92) \approx P(Z < \frac{92.5 - 100}{\sqrt{60}}) = P(Z < -0.968)$$

M1 A1

$$= 0.1665 > 0.05, \text{ so do not reject } H_0$$

A1

Insufficient evidence that the candidate's support is below 40%

B1

Answer: $p \approx 0.1665$; do not reject H_0

6 marks

18 (5 marks)

$$\text{If all } n \text{ tosses are heads, } p\text{-value} = P(X = n) = 0.5^n$$

M1

$$n = 4: 0.5^4 = 0.0625 > 0.05, \text{ so } H_0 \text{ is not rejected}$$

M1 A1

$$n = 5: 0.5^5 = 0.03125 < 0.05, \text{ so } H_0 \text{ is rejected; larger } n \text{ gives smaller } p\text{-value}$$

M1 A1

Answer: $n = 4$: p-value 0.0625, not rejected; $n = 5$: p-value 0.03125, rejected

5 marks

19 (5 marks)

$$\text{Lower tail: } P(X \leq 6) = 0.0172$$

M1 A1

$$\text{Two-tailed } p\text{-value} = 2 \times 0.0172 = 0.0344$$

M1 A1

$$0.0344 < 0.05, \text{ so reject } H_0; \text{ evidence that } p \neq 0.4$$

B1

Answer: $p = 0.0344$; reject H_0

5 marks

20 (4 marks)

$$\text{Under } H_0, \bar{X} \sim N(60, \frac{5^2}{25}) = N(60, 1)$$

B1

$$P(\bar{X} > c) = 0.01 \text{ gives } \frac{c - 60}{1} = 2.326$$

M1 A1

$$c = 62.326$$

A1

Answer: $c = 62.326$

4 marks

21 (9 marks)

$$\text{(a) } X \sim B(50, 0.1); P(X \geq 9) = 0.0579 > 0.05$$

M1 A1

$$P(X \geq 10) = 0.0245 < 0.05$$

M1 A1

$$\text{Critical region: } X \geq 10$$

A1

Answer: $X \geq 10$

5 marks

$$\text{(b) Actual significance level} = 0.0245$$

B1

X is discrete, so the probabilities jump and cannot be made to equal exactly 0.05; the region chosen is the largest with probability not exceeding 5%

E1

Answer: 0.0245

2 marks

$$\text{(c) 9 is not in the critical region, so do not reject } H_0$$

B1

This only shows insufficient evidence of an increase; it does not prove that the proportion is still 10%

E1

Answer: Do not reject H_0

2 marks

22 (7 marks)

$$\text{(a) } P(X \geq 10) = 0.0245$$

B1

Answer: 0.0245

1 mark

$$\text{(b) } P(\text{not rejected in a week}) = 1 - 0.0245 = 0.9755$$

M1

$$1 - (0.9755)^5 = 0.1168$$

M1 A1

Answer: 0.1168**3 marks**

$$(c) 1 - (0.9755)^w > 0.5 \text{ gives } w > \frac{\ln 0.5}{\ln 0.9755}$$

M1

$$w > 27.90, \text{ so least } w = 28$$

M1 A1**Answer: 28****3 marks****23 (7 marks)**Error 1: H_1 should be $p < 0.3$ (one-tailed, since 'fewer')**E1**Error 2: the p-value should be $P(X \leq 3)$, not $P(X = 3)$ **E1**Error 3: failing to reject H_0 does not mean H_0 is accepted as proved**E1**Correct: $P(X \leq 3) = 0.1071$ **M1 A1** $0.1071 > 0.05$, so do not reject H_0 ; no evidence that fewer than 30% prefer brand X**A1 B1****Answer: p-value 0.1071; do not reject H_0** **7 marks****24 (8 marks)**

$$\bar{t} = \frac{3984}{64} = 62.25$$

B1

$$s^2 = \frac{1}{63}(249\,000 - 64 \times 62.25^2) = 15.81$$

M1 A1

$$H_0: \mu = 60, H_1: \mu \neq 60; z = \frac{62.25 - 60}{\frac{3.976}{8}} = 4.53$$

B1 M1 A1 $|z| = 4.53 > 1.96$, so reject H_0 ; evidence that the mean journey time is not 60 minutes**A1** $n = 64$ is large, so by the central limit theorem $\bar{T}$ is approximately normal and s^2 is a good estimate of σ^2 **E1****Answer: Reject H_0 ; $z = 4.53$** **8 marks****25 (10 marks)**(a) $X \approx N(70, 45.5)$, $sd = \sqrt{45.5} = 6.745$ **B1 M1**Lower: $\frac{c + 0.5 - 70}{6.745} \leq -1.96$ gives $c \leq 56.28$, so $c = 56$ **M1 A1**Upper: $\frac{c - 0.5 - 70}{6.745} \geq 1.96$ gives $c \geq 83.72$, so $c = 84$ **M1 A1**Critical region: $X \leq 56$ or $X \geq 84$ **A1****Answer: $X \leq 56$ or $X \geq 84$** **7 marks**(b) $P(X \leq 56) = 0.0213$, $P(X \geq 84) = 0.0237$ **M1 A1**

Actual level = 0.0450

A1**Answer: 0.0450****3 marks****26 (8 marks)** $P(\text{reject at stage 1}) = P(X_1 \geq 8) = 0.0547$ **M1 A1**Second stage needed for $X_1 = 5, 6, 7$ and then $X_2 \geq 13 - X_1$, that is $X_2 \geq 8, 7, 6$ respectively**M1 A1** $P(X_1=5)P(X_2 \geq 8) + P(X_1=6)P(X_2 \geq 7) + P(X_1=7)P(X_2 \geq 6)$ **M1**

$$= 0.2461 \times 0.0547 + 0.2051 \times 0.1719 + 0.1172 \times 0.3770$$

A1

Overall significance level = 0.1476

M1 A1**Answer: 0.1476****8 marks****27 (9 marks)**(a) $N \geq k$ means the first $k - 1$ rolls are all not sixes**M1**Each has probability $\frac{5}{6}$, independently, so $P(N \geq k) = (\frac{5}{6})^{k-1}$ **A1****Answer: $P(N \geq k) = (\frac{5}{6})^{k-1}$** **2 marks**(b) $(\frac{5}{6})^{k-1} \leq 0.05$ gives $(k - 1) \ln(\frac{5}{6}) \leq \ln 0.05$ **M1**

$$k - 1 \geq \frac{\ln 0.05}{\ln(\frac{5}{6})} = 16.43$$

M1 A1

$$k = 18, \text{ actual level} = \left(\frac{5}{6}\right)^{17} = 0.0451$$

A1 B1

Answer: k = 18; level 0.0451

5 marks

(c) $20 \geq 18$, so reject H_0 ; there is evidence that sixes are less likely than for a fair die

M1 A1

Answer: Reject H_0

2 marks

28 (9 marks)

$$(a) z = \frac{104 - 100}{\frac{15}{\sqrt{n}}} = \frac{4\sqrt{n}}{15}$$

M1

Reject if $\frac{4\sqrt{n}}{15} > 1.645$, so $\sqrt{n} > 6.169$

M1 A1

 $n > 38.05$, so the least n is 39

M1 A1

Answer: n = 39

5 marks

(b) Reject if $\frac{4\sqrt{n}}{15} > 1.96$, so $\sqrt{n} > 7.350$

M1 A1

 $n > 54.02$, so least $n = 55$

A1

The 5% is split between two tails, so the critical value of z is larger and more evidence is needed

E1

Answer: n = 55

4 marks

29 (10 marks)(a) Most extreme outcomes have probability $P(X = 0) = P(X = 5) = 0.5^5 = 0.0312$

M1 A1

 $0.0312 > 0.025$, so neither tail has a non-empty critical region

E1

Answer: Smallest tail probability 0.03125 > 0.025, so no rejection

3 marks

(b) Need $0.5^n \leq 0.025$; $n = 5$ fails (0.03125) and $0.5^6 = 0.01562$

M1 A1

Smallest $n = 6$; critical region $X = 0$ or $X = 6$

A1 A1

Answer: n = 6; X = 0 or 6

4 marks

(c) $2 \times 0.5^6 = 0.03125$

M1 A1

The next outcomes (1 or 5 heads) would add 12×0.01562 , taking the level well beyond 5%, so they cannot be included

E1

Answer: 0.03125

3 marks

30 (11 marks)(a) Under H_0 , $\bar{X} \sim N(80, \frac{12^2}{n})$

B1

$$P(\bar{X} < 77) = P(Z < \frac{77 - 80}{\frac{12}{\sqrt{n}}}) = P(Z < -\frac{\sqrt{n}}{4})$$

M1 A1

Answer: Significance level = $P(Z < -\frac{\sqrt{n}}{4})$

3 marks

(b) Need $\frac{\sqrt{n}}{4} \geq 2.326$

M1

 $\sqrt{n} \geq 9.304$, so $n \geq 86.56$

A1

Least $n = 87$

A1

Answer: 87

3 marks

(c) $76.4 < 77$, so reject H_0 : evidence that mean lifetime is less than 80 hours

M1 A1

 $n = 16$ gives $sd = 3$ and level $P(Z < -1) = 0.1587$, so a 1% test would need $\bar{X} < 73.02$, and 76.4 would not be rejected

M1 A1 E1

Answer: Reject H_0

5 marks

Type I and Type II Errors**1 (2 marks)**(a) Rejecting H_0 when H_0 is true

B1

Answer: Rejecting H_0 when it is true

1 mark

(b) Accepting (not rejecting) H_0 when H_0 is false

B1

Answer: Not rejecting H_0 when it is false

1 mark

2 (2 marks)

(a) Concluding that the detector is faulty when it is actually working properly

B1

Answer: Deciding it is faulty when it works

1 mark

(b) Concluding that the detector is working properly when it is actually faulty

B1

Answer: Deciding it works when it is faulty

1 mark

3 (2 marks)

$$P(\text{Type I}) = P(X \geq 8 \mid p = 0.5) = 0.0547$$

M1 A1

Answer: 0.0547

2 marks

4 (2 marks)

(a) It equals the significance level, 0.05

B1

Answer: 0.05

1 mark

(b) It decreases to 0.01

B1

Answer: Falls to 0.01

1 mark

5 (2 marks)

$$P(\text{Type II}) = P(X \leq 8 \mid p = 0.8)$$

M1

$$= 1 - P(X \geq 9) = 1 - 0.3758 = 0.6242$$

A1

Answer: 0.6242

2 marks

6 (2 marks)

(a) H_0 true but rejected: Type I error

B1

Answer: Type I

1 mark

(b) H_0 false but not rejected: Type II error

B1

Answer: Type II

1 mark

7 (4 marks)

$$(a) P(X \geq 10 \mid p = 0.3) = 1 - P(X \leq 9) = 1 - 0.9520$$

M1

$$= 0.0480$$

A1

Answer: 0.0480

2 marks

$$(b) P(X \leq 9 \mid p = 0.5)$$

M1

$$= 0.4119$$

A1

Answer: 0.4119

2 marks

8 (4 marks)

$$P(\text{Type I}) = P(X \geq 9 \mid \lambda = 4) = 1 - P(X \leq 8) = 1 - 0.9786 = 0.0214$$

M1 A1

$$P(\text{Type II}) = P(X \leq 8 \mid \lambda = 6)$$

M1

$$= 0.8472$$

A1

Answer: Type I: 0.0214; Type II: 0.8472

4 marks

9 (4 marks)

$$\text{Under } H_0, \bar{X} \sim N(50, 1); P(\bar{X} > 52) = P(Z > 2) = 0.0228$$

M1 A1

$$\text{When } \mu = 53, \bar{X} \sim N(53, 1); P(\bar{X} \leq 52) = P(Z \leq -1)$$

M1

$$= 0.1587$$

A1

Answer: Type I: 0.0228; Type II: 0.1587

4 marks

10 (4 marks)

(a) Type I: a healthy patient is told they may have the disease (unnecessary worry and further tests)

B1

Type II: a patient with the disease is told they are clear (no treatment given)

B1

Answer: False alarm; missed case

2 marks

(b) Type II is more serious as an untreated disease can be dangerous

E1

Lower the threshold, which makes rejection of H_0 more likely and so reduces $P(\text{Type II})$, at the cost of more Type I errors

E1

Answer: Type II; lower the threshold	2 marks
11 (6 marks)	
(a) Test A: $P(X \geq 8) = 0.0547$	M1 A1
Test B: $P(X \geq 15) = 0.0207$	A1
Answer: A: 0.0547; B: 0.0207	3 marks
(b) Test A: $P(X \leq 7 p = 0.7) = 0.6172$	M1 A1
Test B: $P(X \leq 14 p = 0.7) = 0.5836$	A1
Answer: A: 0.6172; B: 0.5836	3 marks
12 (5 marks)	
(a) $P(X \leq 5) = 0.0294 \leq 0.05$ and $P(X \leq 6) = 0.0736 > 0.05$	M1 A1
Critical region: $X \leq 5$	A1
Answer: $X \leq 5$	3 marks
(b) $P(X > 5 p = 0.2) = 1 - P(X \leq 5)$	M1
$= 1 - 0.6167 = 0.3833$	A1
Answer: 0.3833	2 marks
13 (3 marks)	
(a) $1 - 0.04 = 0.96$	B1
Answer: 0.96	1 mark
(b) Power $= 1 - P(\text{Type II}) = 1 - 0.30 = 0.70$	M1 A1
Answer: 0.7	2 marks
14 (6 marks)	
Under H_0 , $\bar{X} \sim N(100, 2^2)$; critical values $100 \pm 1.96 \times 2$	B1 M1
Acceptance region: $96.08 < \bar{X} < 103.9$	A1
When $\mu = 106$, $\bar{X} \sim N(106, 2^2)$	B1
$P(96.08 < \bar{X} < 103.9) = P(-4.96 < Z < -1.04)$	M1
$= 0.1492$	A1
Answer: 0.1492	6 marks
15 (5 marks)	
Acceptance region: $2 \leq X \leq 10$	B1
$P(\text{Type II}) = P(2 \leq X \leq 10 p = 0.5) = P(X \leq 10) - P(X \leq 1)$	M1 M1
$= 0.5881 - 0.000020 = 0.5881$	A1 A1
Answer: 0.5881	5 marks
16 (8 marks)	
(a) $P(X \geq 15) = 0.0207$	M1 A1
$P(X \geq 16) = 0.0059$	A1
Answer: 0.0207; 0.0059	3 marks
(b) $P(X \leq 14) = 0.5836$	M1 A1
$P(X \leq 15) = 0.7625$	A1
Answer: 0.5836; 0.7625	3 marks
(c) The second has smaller $P(\text{Type I})$ but larger $P(\text{Type II})$ (0.7625 against 0.5836)	E1
Reducing one error probability increases the other for a fixed n , so neither is better outright	E1
Answer: Trade-off	2 marks
17 (6 marks)	
$P(\text{Type I}): X \geq 10: 0.0611, X \geq 11: 0.0256$	M1 A1
The region $X \geq 11$ has the smaller $P(\text{Type I})$	B1
$P(\text{Type II}): X \geq 10: P(X \leq 9 0.4) = 0.1763$	M1 A1
$X \geq 11: P(X \leq 10 0.4) = 0.2915$	A1
Answer: $X \geq 10: 0.1763; X \geq 11: 0.2915; X \geq 11$ has smaller Type I	6 marks

18 (6 marks)

$$P(\text{reject} | \text{good}) = P(X \geq 8 | p = 0.2) = 0.1091$$

M1 A1

$$P(\text{accept} | \text{bad}) = P(X \leq 7 | p = 0.4) = 0.1536$$

M1 A1

$$P(\text{wrong decision}) = 0.8 \times 0.1091 + 0.2 \times 0.1536$$

M1

$$= 0.1180$$

A1

Answer: 0.1180

6 marks

19 (8 marks)(a) Under H_0 , $X \approx N(40, 24)$

B1

$$P(X \geq 50) \approx P(Z > \frac{49.5 - 40}{\sqrt{24}}) = P(Z > 1.939)$$

M1 A1

$$= 0.0262$$

A1

Answer: 0.0262

4 marks

(b) When $p = 0.5$, $X \approx N(50, 25)$

B1

$$P(X \leq 49) \approx P(Z < \frac{49.5 - 50}{5}) = P(Z < -0.100)$$

M1 A1

$$= 0.4602$$

A1

Answer: 0.4602

4 marks

20 (7 marks)Test A: standard error 1, reject if $\bar{X} > 51.645$; $P(\bar{X} \leq 51.645 | \mu = 52) = P(Z \leq -0.355) = 0.3613$

M1 A1 M1 A1

Test B: standard error 0.5, reject if $\bar{X} > 50.822$; $P(Z \leq \frac{50.822 - 52}{0.5}) = P(Z \leq -2.36) = 0.0093$

M1 A1

A larger sample leaves $P(\text{Type I})$ at 5% but greatly reduces $P(\text{Type II})$

E1

Answer: A: 0.3613; B: 0.0093

7 marks

21 (10 marks)(a) $X \sim B(50, 0.15)$; $P(X \geq 12) = 0.0628 > 0.05$

M1 A1

$$P(X \geq 13) = 0.0301 \leq 0.05$$

A1

Critical region $X \geq 13$, $P(\text{Type I}) = 0.0301$

A1 B1

Answer: $X \geq 13$; 0.0301

5 marks

(b) $P(X \leq 12 | p = 0.25)$

M1

$$= 0.5110$$

A1

Answer: 0.5110

2 marks

(c) $1 - 0.5110 = 0.4890$

M1 A1

This is the power of the test

B1

Answer: 0.4890 (the power)

3 marks

22 (11 marks)(a) $P(X \leq 1) = 0.0174 \leq 0.025$ and $P(X \leq 2) = 0.0620 > 0.025$

M1 A1

$$P(X \geq 12) = 0.0201 \leq 0.025 \text{ and } P(X \geq 11) = 0.0426 > 0.025$$

M1 A1

Critical region: $X \leq 1$ or $X \geq 12$; $P(\text{Type I}) = 0.0374$

A1 A1

Answer: $X \leq 1$ or $X \geq 12$; 0.0374

6 marks

(b) $P(2 \leq X \leq 11 | \lambda = 9) = P(X \leq 11) - P(X \leq 1)$

M1

$$= 0.8030 - 0.0012 = 0.8018$$

A1

Answer: 0.8018

2 marks

(c) $P(2 \leq X \leq 11 | \lambda = 3) = 0.9999 - 0.1991 = 0.8008$

M1 A1

The test is less likely to detect a decrease to 3 than an increase to 9, since the Poisson is skewed

E1

and the lower tail is narrow

Answer: 0.8008

3 marks

23 (7 marks)

$P(X \leq 4 p = 0.4) = 0.0510$, which exceeds 0.05, so the critical region should be $X \leq 3$ ($P(X \leq 3) = 0.0160$)	E1 B1
$P(\text{Type I})$ must use $p = 0.4$ (H_0 true), not $p = 0.2$; correct value $P(X \leq 3 p = 0.4) = 0.0160$	E1 A1
$P(\text{Type II})$ must use a value of p from H_1 , e.g. 0.2, not $p = 0.4$; the student has also reused the wrong region	E1
Correct: $P(\text{Type II} p = 0.2) = P(X \geq 4 p = 0.2) = 0.5886$	M1 A1
Answer: Critical region $X \leq 3$; Type I 0.0160; Type II ($p = 0.2$) 0.5886	7 marks

24 (9 marks)

(a) $P(T > 30 \lambda = 0.1) = e^{-0.1 \times 30} = e^{-3}$ $= 0.0498$	M1 A1
Answer: 0.0498	2 marks
(b) $P(T \leq 30 \lambda = 0.05) = 1 - e^{-0.05 \times 30}$ $= 1 - e^{-1.5} = 0.7769$	M1 A1
Answer: 0.7769	2 marks
(c) $P(\text{Type I}) = (e^{-3})^2 = e^{-6} = 0.00248$ $P(\text{Type II}) = 1 - (e^{-1.5})^2 = 1 - e^{-3} = 0.9502$ $P(\text{Type I})$ has fallen but $P(\text{Type II})$ has risen, so neither error improves overall	M1 A1 M1 A1 E1
Answer: 0.00248; 0.9502	5 marks

25 (8 marks)

(a) $X \sim B(20, 0.05)$; $P(X > 2) = 1 - P(X \leq 2)$ $= 1 - 0.9245 = 0.0755$	M1 A1
Answer: 0.0755	2 marks
(b) $P(X \leq 2 p = 0.25) = 0.0913$	M1 A1
Answer: 0.0913	2 marks
(c) $P(X > c p = 0.05)$: $c = 2$ gives $0.0755 > 0.05$, $c = 3$ gives $0.0159 \leq 0.05$ So $c = 3$ and consumer's risk $= P(X \leq 3 p = 0.25) = 0.2252$	M1 A1 A1 A1
Answer: $c = 3$; consumer's risk 0.2252	4 marks

26 (8 marks)

(a) $P(N \geq 6) = P(\text{first five tosses are tails}) = 0.5^5$ $= 0.03125$	M1 A1
Answer: 0.03125	2 marks
(b) $P(N \leq 5) = 1 - P(N \geq 6) = 1 - 0.7^5$ $= 0.83193$	M1 A1
Answer: 0.83193	2 marks
(c) $0.5^{k-1} \leq 0.1$ gives $k - 1 \geq 4$ ($0.5^3 = 0.125$ fails, $0.5^4 = 0.0625$ works) $k = 5$; $P(\text{Type II}) = 1 - 0.7^4 = 0.75990$	M1 A1 M1 A1
Answer: $k = 5$; 0.75990	4 marks

27 (11 marks)

(a) Under H_0 , $\bar{X} \sim N(100, \frac{10^2}{n})$; $P(\bar{X} > c) = 0.05$	B1 M1
$\frac{c - 100}{\frac{10}{\sqrt{n}}} = 1.645$, so $c = 100 + \frac{16.45}{\sqrt{n}}$	A1
Answer: $c = 100 + \frac{16.45}{\sqrt{n}}$	3 marks
(b) When $\mu = 105$, $P(\bar{X} \leq c) \leq 0.1$	M1
$P(Z \leq \frac{c - 105}{\frac{10}{\sqrt{n}}}) \leq 0.1$ and the 10% point of Z is -1.2816	A1 B1
Answer: $1.645 - \frac{\sqrt{n}}{2} \leq -1.2816$	3 marks

$$(c) \frac{(100 + \frac{16.45}{\sqrt{n}}) - 105}{\frac{10}{\sqrt{n}}} = 1.645 - \frac{5\sqrt{n}}{10} \leq -1.2816 \quad \text{M1 A1}$$

$$\frac{\sqrt{n}}{2} \geq 2.9266 \text{ so } \sqrt{n} \geq 5.853 \quad \text{M1 A1}$$

$$n \geq 34.26, \text{ so } n = 35 \quad \text{A1}$$

Answer: 35 5 marks

28 (10 marks)

(a) Under H_0 , $X \sim \text{Po}(6)$; under H_1 , $X \sim \text{Po}(10)$ B1 B1

Answer: Po(6) and Po(10) 2 marks

(b) $P(\text{Type I}) = P(X \geq c \mid \text{Po}(6)) = 1 - P(X \leq c - 1)$ M1

$P(\text{Type II}) = P(X \leq c - 1 \mid \text{Po}(10))$ M1

Sum = $1 - P(X \leq c - 1; 6) + P(X \leq c - 1; 10)$ A1

Answer: Sum = $1 - P(X \leq c - 1; \text{Po}(6)) + P(X \leq c - 1; \text{Po}(10))$ 3 marks

(c) $c = 8$: $0.2560 + 0.2202 = 0.4762$ M1 A1

Neighbouring values give 0.5238 ($c = 7$) and 0.4856 ($c = 9$), both larger M1 A1

Minimum sum = 0.4762 at $c = 8$ A1

Answer: $c = 8$; 0.4762 5 marks

29 (12 marks)

(a) The maximum is at most x only if both observations are at most x M1

By independence, $P = \left(\frac{x}{\theta}\right)^2 = \frac{x^2}{\theta^2}$ A1

Answer: $P(\max \leq x) = \frac{x^2}{\theta^2}$ 2 marks

(b) Under H_0 : $P(\max > k) = 1 - \frac{k^2}{64} = 0.05$ M1

$k^2 = 64 \times 0.95$, so $k = 7.797$ A1

Answer: $k = 7.797$ 2 marks

(c) $P(\max \leq k \mid \theta = 10) = \frac{k^2}{100}$ M1

$= \frac{60.8}{100} = 0.608$ A1

Answer: 0.608 2 marks

(d) Under H_0 : $P(\max > k) = 1 - \left(\frac{k}{8}\right)^n = 0.05$, so $k = 8 \times 0.95^{1/n}$ M1

Type II: $\left(\frac{k}{10}\right)^n = 0.8^n \times 0.95$ M1 A1

$0.95 \times 0.8^n < 0.1$ gives $n > \frac{\ln(0.1 \div 0.95)}{\ln 0.8} = 10.09$ M1 A1

Least $n = 11$ A1

Answer: 11 6 marks

30 (12 marks)

(a) $P(X \geq 9) = 0.1133 > 0.05$, $P(X \geq 10) = 0.0480 \leq 0.05$ M1 A1

$c = 10$, $P(\text{Type I}) = 0.0480$ A1 B1

Answer: $c = 10$; 0.0480 4 marks

(b) $P(X \geq 10 \mid p = 0.5)$ M1

$= 0.5881$ A1

Answer: 0.5881 2 marks

(c) $P(\text{flagged}) = 0.9 \times 0.0480 + 0.1 \times 0.5881$ M1 A1

$P(p = 0.3 \mid \text{flagged}) = \frac{0.9 \times 0.0480}{0.9 \times 0.0480 + 0.1 \times 0.5881}$ M1

$= 0.4233$ A1

Answer: 0.4233 4 marks

(d) Most treatments tested have no effect, so even a small false-positive rate produces many false E1 E1

flags compared with the few true detections

Answer: Most treatments are ineffective

2 marks

Power of a Test

1 (2 marks)

(a) The probability of rejecting H_0 when H_0 is false (for a specified alternative value)

B1

Answer: $P(\text{reject } H_0 \mid H_0 \text{ false})$

1 mark

(b) Power = $1 - P(\text{Type II error})$

B1

Answer: Power = $1 - P(\text{Type II})$

1 mark

2 (1 mark)

Power = $1 - 0.35 = 0.65$

B1

Answer: 0.65

1 mark

3 (2 marks)

Power = $P(X \geq 8 \mid p = 0.8)$

M1

= 0.6778

A1

Answer: 0.6778

2 marks

4 (2 marks)

(a) Power increases, since a larger sample gives more information (a smaller standard error)

B1

Answer: Power increases

1 mark

(b) At the null value, $P(\text{reject}) = \text{significance level} = 0.05$

B1

Answer: 0.05

1 mark

5 (2 marks)

Power = $P(X \geq 6 \mid \lambda = 5) = 1 - P(X \leq 5)$

M1

= $1 - 0.6160 = 0.3840$

A1

Answer: 0.3840

2 marks

6 (2 marks)

(a) $1 - 0.85 = 0.15$

B1

Answer: 0.15

1 mark

(b) The probability of a Type I error equals the significance level, 0.04

B1

Answer: 0.04

1 mark

7 (4 marks)

(a) $P(X \leq 5 \mid p = 0.5) = 0.0207$

M1 A1

Answer: 0.0207

2 marks

(b) $P(X \leq 5 \mid p = 0.3) = 0.4164$

M1 A1

Answer: 0.4164

2 marks

8 (6 marks)

(a) Power = $P(X \geq 12) = 1 - P(X \leq 11)$

M1

$p = 0.6: 0.0905; p = 0.7: 0.2969; p = 0.8: 0.6482$

A1 A1 A1

Answer: 0.0905, 0.2969, 0.6482

4 marks

(b) The power increases towards 1

B1

Larger differences from the null value are easier to detect

E1

Answer: Increases towards 1

2 marks

9 (4 marks)

Standard error = $\frac{5}{\sqrt{25}} = 1$; reject H_0 if $\bar{X} > 50 + 1.645 = 51.645$

B1 M1

Power = $P(\bar{X} > 51.645 \mid \mu = 52) = P(Z > -0.355)$

M1

= 0.6387

A1

Answer: 0.6387

4 marks

10 (4 marks)

Standard error = 1; reject H_0 if $\bar{X} < 78.04$ or $\bar{X} > 81.96$

B1 M1

$$\begin{aligned}\text{Power} &= P(\bar{X} < 78.04 | 83) + P(\bar{X} > 81.96 | 83) = P(Z < -4.96) + P(Z > -1.04) \\ &= 0.0000004 + 0.8508 = 0.8508\end{aligned}$$

M1

A1

Answer: 0.8508

4 marks

11 (2 marks)

$$\text{Power} = P(T > 30 | \lambda = 0.05) = e^{-0.05 \times 30}$$

M1

$$= e^{-1.5} = 0.2231$$

A1

Answer: 0.2231

2 marks

12 (3 marks)

$$\text{Power} = P(X \leq 1 | \lambda = 12) + P(X \geq 12 | \lambda = 12)$$

M1

$$= 0.00008 + 0.5384 = 0.5385$$

M1 A1

Answer: 0.5385

3 marks

13 (6 marks)

$$(a) P(X < 0.5 | \theta = 10) = \frac{0.5}{10}$$

M1

$$= 0.05$$

A1

Answer: 0.05

2 marks

$$(b) P(X < 0.5 | \theta = 4) = \frac{0.5}{4}$$

M1

$$= 0.125$$

A1

Answer: 0.125

2 marks

$$(c) \frac{0.5}{\theta} = 0.5$$

M1

$$\theta = 1$$

A1

Answer: $\theta = 1$

2 marks

14 (6 marks)

$$P(X \geq 13) = 0.0845 > 0.05$$

M1 A1

$$P(X \geq 14) = 0.0401 \leq 0.05, \text{ so critical region } X \geq 14$$

A1 B1

$$\text{Power} = P(X \geq 14 | p = 0.5)$$

M1

$$= 0.7077$$

A1

Answer: 0.7077

6 marks

15 (6 marks)

$$\text{Reject } H_0 \text{ if } \bar{X} > \mu_0 + 1.645 \frac{\sigma}{\sqrt{n}}$$

B1 M1

$$\text{Power} = P(\bar{X} > \mu_0 + 1.645 \frac{\sigma}{\sqrt{n}} | \mu = \mu_0 + \delta) = P(Z > \frac{1.645 \frac{\sigma}{\sqrt{n}} - \delta}{\frac{\sigma}{\sqrt{n}}})$$

M1

$$= P(Z > 1.645 - \frac{\delta \sqrt{n}}{\sigma})$$

A1

$$n = 64, \sigma = 8, \delta = 3: 1.645 - \frac{3 \times 8}{8} = -1.355$$

M1

$$\text{Power} = P(Z > -1.355) = 0.9123$$

A1

Answer: 0.9123

6 marks

16 (8 marks)

$$(a) \text{ Test A: } P(X \geq 8 | 0.5) = 0.0547$$

M1 A1

$$\text{Test B: } P(X \geq 15 | 0.5) = 0.0207$$

A1

Answer: A: 0.0547; B: 0.0207

3 marks

$$(b) \text{ Test A: } P(X \geq 8 | 0.75) = 0.5256$$

M1 A1

$$\text{Test B: } P(X \geq 15 | 0.75) = 0.6172$$

A1

Answer: A: 0.5256; B: 0.6172

3 marks

(c) Test B has the smaller significance level and the greater power

M1

So Test B is better, though it needs twice as many observations

A1

Answer: Test B

2 marks

17 (7 marks)

$$P(X \leq 3) = 0.0103 \leq 0.025, P(X \leq 4) = 0.0293 > 0.025$$

M1 A1

$$P(X \geq 18) = 0.0143 \leq 0.025, P(X \geq 17) = 0.0270 > 0.025$$

M1 A1

Critical region: $X \leq 3$ or $X \geq 18$

A1

$$\text{Power} = P(X \leq 3 | 14) + P(X \geq 18 | 14) = 0.00047 + 0.1728 = 0.1733$$

M1 A1

Answer: 0.1733

7 marks

18 (7 marks)

$$\text{Power function} = P(X \geq 8 | p) = 1 - P(X \leq 7)$$

M1

$$p = 0.6: 0.1673, p = 0.7: 0.3828$$

A1 A1

$$p = 0.8: 0.6778, p = 0.9: 0.9298$$

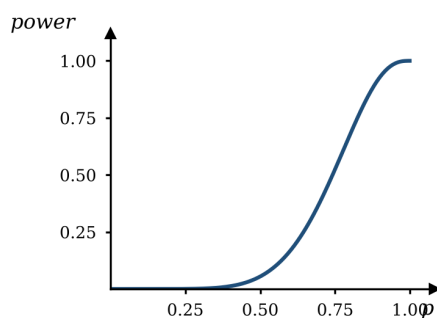
A1 A1

Increasing curve starting at 0.0547 (the significance level) when $p = 0.5$, rising to 1 at $p = 1$

B1 B1

Answer: 0.1673, 0.3828, 0.6778, 0.9298; increasing curve from 0.0547 to 1

7 marks



19 (7 marks)

(a) The graph starts at 0.05 when $\mu = 100$

B1

This is the probability of rejecting H_0 when it is true, the significance level

E1

Answer: 0.05

2 marks

(b) Reject if $\bar{X} > 100 + 1.645 = 101.64$; power = $P(Z > -0.355)$
= 0.6387

M1 M1

A1

Answer: 0.6387

3 marks

(c) $P(Z > c - \mu) = 0.9$ gives $c - \mu = -1.2816$

M1

$$\mu = 101.64 + 1.2816 = 102.93$$

A1

Answer: 102.9

2 marks

20 (5 marks)

Reject H_0 if $\bar{X} > 100 + 1.645 \times 1 = 101.64$

B1 M1

$$P(\bar{X} > 101.64 | \mu) = 0.8 \text{ gives } \frac{101.64 - \mu}{1} = -0.8416$$

M1 A1

$$\mu = 102.49$$

A1

Answer: 102.49

5 marks

21 (10 marks)

(a) Standard error $\frac{12}{\sqrt{36}} = 2$; critical values $150 \pm 1.96 \times 2$

B1 M1

$$\bar{X} < 146.08 \text{ or } \bar{X} > 153.92$$

A1

Answer: $\bar{X} < 146.08$ or $\bar{X} > 153.92$

3 marks

(b) $P(\bar{X} > 153.92 | 155) = P(Z > -0.540) = 0.7054$

M1 A1

$P(\bar{X} < 146.08 | 155) = P(Z < -4.46) = 0.000004$, so power = $0.7054 \approx 0.7$

M1 A1

Answer: 0.7054

4 marks

(c) $P(\bar{X} > 153.92 | \mu) = 0.9$ gives $\frac{153.92 - \mu}{2} = -1.2816$

M1 A1

$$\mu = 156.48$$

A1

Answer: 156.5**3 marks****22 (10 marks)****(a)** Test 1: $P(X = 5) = p^5$ **B1**Test 2: $P(X = 4) + P(X = 5) = 5p^4(1 - p) + p^5$ **M1** $= 5p^4 - 5p^5 + p^5 = 5p^4 - 4p^5$ **A1****Answer: Test 1: p^5 ; Test 2: $5p^4 - 4p^5$** **3 marks****(b)** Test 1: $0.5^5 = 0.0312$ **B1**Test 2: $5(0.5)^4 - 4(0.5)^5 = 0.1875$ **M1 A1****Answer: Test 1: 0.0312; Test 2: 0.1875****3 marks****(c)** Test 1: $0.8^5 = 0.3277$ **B1**Test 2: $5(0.8)^4 - 4(0.8)^5 = 0.7373$ **M1 A1**

Test 2 is more powerful but has a larger significance level (0.1875 against 0.03125)

E1**Answer: Test 1: 0.3277; Test 2: 0.7373****4 marks****23 (6 marks)**The student has used the null value $p = 0.4$; this is the significance level, not the power**E1**Power must be evaluated at an alternative value, here $p = 0.6$ **E1**Power = $P(X \geq 12 | p = 0.6) = 0.5956$ **M1 A1** $P(\text{Type II}) = 1 - 0.5956 = 0.4044$ **M1 A1****Answer: Power 0.5956; P(Type II) 0.4044****6 marks****24 (7 marks)**Under H_0 , $S \sim \text{Po}(10)$ **B1** $P(S \geq 15) = 0.0835 > 0.05$, $P(S \geq 16) = 0.0487 \leq 0.05$ **M1 A1**Critical region $S \geq 16$, significance level 0.0487**A1 A1**When $\lambda = 3$, $S \sim \text{Po}(15)$; power = $P(S \geq 16 | \text{Po}(15))$ **M1** $= 0.4319$ **A1****Answer: $S \geq 16$; 0.0487; power 0.4319****7 marks****25 (10 marks)****(a)** $X \approx N(50, 25)$; require $P(X \geq c) \leq 0.05$, so $\frac{c - 0.5 - 50}{5} \geq 1.645$ **B1 M1** $c \geq 58.73$, so $c = 59$; critical region $X \geq 59$ **A1 A1****Answer: $X \geq 59$** **4 marks****(b)** $X \approx N(65, 22.75)$ **B1** $P(X \geq 59) \approx P(Z > \frac{58.50 - 65}{4.770}) = P(Z > -1.36)$ **M1 A1** $= 0.9135$ **A1****Answer: 0.9135****4 marks****(c)** Exact significance level $P(X \geq 59 | 0.5) = 0.0443$ **B1**Exact power $P(X \geq 59 | 0.65) = 0.9123$ **B1****Answer: 0.0443; 0.9123****2 marks****26 (7 marks)****(a)** {5}: size 0.03, power 0.30; {4}: size 0.07, power 0.30; {4, 5}: size 0.10, power 0.60**M1 A1 A1**No other region has size ≤ 0.10 , for example {3, 5} has size 0.23**B1****Answer: {5}: 0.30; {4}: 0.30; {4, 5}: 0.60****4 marks****(b)** Region {4, 5}, with power 0.60**B1**Ratios $H1/H0$ are 4.29 for $x = 4$ and 10 for $x = 5$, the two largest of all values, so these values are the most indicative of $H1$ **M1 A1****Answer: {4, 5}, power 0.6****3 marks**

27 (5 marks)

For each n find the smallest c with $P(X \geq c | p = 0.5) \leq 0.05$, then the power at $p = 0.7$

$n = 20$: $c = 15$, $P(\text{Type I}) = 0.0207$, power = 0.4164

$n = 30$: $c = 20$, $P(\text{Type I}) = 0.0494$, power = 0.7304

$n = 40$: $c = 26$, $P(\text{Type I}) = 0.0403$, power = 0.8074

The first value of n with power at least 0.8 is 40

Answer: $n = 40$ (power 0.8074)

M1
A1
A1
A1
B1
5 marks

28 (12 marks)

(a) Standard error $\frac{3}{\sqrt{25}} = 0.6$; reject H_0 if $\bar{X} > 40 + 1.96 \times 0.6$ or $\bar{X} < 40 - 1.96 \times 0.6$

$$P(\bar{X} > 41.176 | 40 + \delta) = P(Z > \frac{1.176 - \delta}{0.6}) = P(Z > 1.96 - \frac{\delta}{0.6})$$

$$P(\bar{X} < 38.824 | 40 + \delta) = P(Z < \frac{-1.176 - \delta}{0.6}) = P(Z < -1.96 - \frac{\delta}{0.6})$$

$$\text{Answer: Power} = P(Z > 1.96 - \frac{\delta}{0.6}) + P(Z < -1.96 - \frac{\delta}{0.6})$$

B1 M1
M1 A1
A1
5 marks
M1 A1
E1

(b) $P(Z > 0) + P(Z < -3.92) = 0.5 + 0.00004 = 0.50004$

When μ equals the critical value, the sample mean exceeds it with probability 0.5; the other tail is negligible

Answer: 0.5000

3 marks

(c) Ignoring the lower tail: $P(Z > 1.96 - \frac{\delta}{0.6}) \geq 0.9$ gives $1.96 - \frac{\delta}{0.6} \leq -1.2816$

$\delta \geq 0.6 \times 3.2416 = 1.945$, so $\delta = 1.95$ to 2 d.p.

Check: the lower-tail contribution $P(Z < -5.21)$ is negligible

Answer: 1.95

M1 A1
M1 A1
4 marks

29 (11 marks)

(a) Number of rejections $R \sim B(3, 0.05)$

$$P(R \geq 2) = 3(0.05)^2(0.95) + (0.05)^3 = 0.00725$$

Answer: 0.00725

M1
M1 A1
A1
4 marks

(b) $R \sim B(3, 0.7)$: $P(R \geq 2) = 3(0.7)^2(0.3) + (0.7)^3 = 0.7840$

Answer: 0.7840

M1 A1
A1
3 marks

(c) $3q^2(1 - q) + q^3 = 3q^2 - 2q^3 = 0.9$

Solving numerically: $q = 0.804$

Answer: 0.804

M1 A1
M1 A1
4 marks

30 (11 marks)

$$(a) \Pi(p) = P(X = 3) + P(X = 4) = 4p^3(1 - p) + p^4 = 4p^3 - 4p^4 + p^4 = 4p^3 - 3p^4$$

$$\text{Answer: } \Pi(p) = 4p^3 - 3p^4$$

M1 A1
A1
3 marks

$$(b) \Pi'(p) = 12p^2 - 12p^3$$

$$= 12p^2(1 - p) > 0 \text{ for } 0 < p < 1$$

So Π is strictly increasing

Answer: $\Pi'(p) = 12p^2(1 - p) > 0$, so Π is increasing

M1 A1
M1
E1
4 marks

(c) $\Pi(0.5) = 4(0.125) - 3(0.0625) = 0.3125$

Solve $4p^3 - 3p^4 = 0.5$ numerically; Π is increasing so there is one root in $(0, 1)$: $p = 0.614$

Answer: 0.3125; $p = 0.614$

M1 A1
M1 A1
4 marks

Poisson Hypothesis Tests

1 (2 marks)

$$H_0: \lambda = 3$$

B1

$$H_1: \lambda < 3$$

B1

Answer: $H_0: \lambda = 3, H_1: \lambda < 3$

2 marks

2 (3 marks)

$$p\text{-value} = P(X \leq 1 \mid \lambda = 3.5) = 0.1359$$

M1 A1

$$0.1359 > 0.05, \text{ so do not reject } H_0$$

B1

Answer: $p = 0.1359$; do not reject H_0

3 marks

3 (2 marks)

(a) Mean = $2.4 \times 5 = 12$, so $Po(12)$

B1

Answer: $Po(12)$

1 mark

(b) Mean = $2.4 \times 0.5 = 1.2$, so $Po(1.2)$

B1

Answer: $Po(1.2)$

1 mark

4 (3 marks)

$$p\text{-value} = P(X \geq 9) = 1 - P(X \leq 8) = 1 - 0.9786 = 0.0214$$

M1 A1

$$0.0214 < 0.05, \text{ so reject } H_0$$

B1

Answer: $p = 0.0214$; reject H_0

3 marks

5 (2 marks)

Events occur independently of one another

B1

Events occur at a constant average rate, singly and at random

B1

Answer: Independent events at a constant rate

2 marks

6 (2 marks)

$$P(X \leq 1) = 0.0404 \leq 0.05, P(X \leq 2) = 0.1247 > 0.05$$

M1 A1

Answer: $X \leq 1$

2 marks

7 (5 marks)

$$H_0: \lambda = 4 \text{ per hour}, H_1: \lambda > 4 \text{ per hour}; X \sim Po(12) \text{ in 3 hours}$$

B1 B1

$$P(X \geq 18) = 1 - P(X \leq 17) = 1 - 0.9370 = 0.0630$$

M1 A1

$$0.0630 > 0.05, \text{ so do not reject } H_0; \text{ insufficient evidence of an increase}$$

A1

Answer: $p = 0.0630$; do not reject H_0

5 marks

8 (7 marks)

(a) $P(X \leq 1) = 0.0073 \leq 0.025, P(X \leq 2) = 0.0296 > 0.025$

M1 A1

$$P(X \geq 14) = 0.0128 \leq 0.025, P(X \geq 13) = 0.0270 > 0.025$$

M1 A1

Critical region: $X \leq 1$ or $X \geq 14$

A1

Answer: $X \leq 1$ or $X \geq 14$

5 marks

(b) $0.0073 + 0.0128 = 0.0201$

M1 A1

Answer: 0.0201

2 marks

9 (5 marks)

$$H_0: \lambda = 0.8 \text{ per 100 m}, H_1: \lambda < 0.8; \text{ in 300 m}, X \sim Po(2.4)$$

B1 B1

$$P(X = 0) = e^{-2.4} = 0.0907$$

M1 A1

$$0.0907 > 0.05, \text{ so do not reject } H_0; \text{ no evidence that resurfacing reduced potholes}$$

A1

Answer: $p = 0.0907$; do not reject H_0

5 marks

10 (5 marks)

$H_0: \lambda = 1.5$ per 10 minutes, $H_1: \lambda \neq 1.5$; in 40 minutes, $X \sim \text{Po}(6)$

B1 B1

$$P(X \geq 11) = 1 - P(X \leq 10) = 1 - 0.9574 = 0.0426$$

M1 A1

Two-tailed test, so compare with 0.025: $0.0426 > 0.025$, so do not reject H_0

A1

Answer: $p = 0.0426$; do not reject H_0

5 marks

11 (5 marks)

$H_0: \lambda = 30$, $H_1: \lambda > 30$; $X \approx N(30, 30)$

B1 M1

$$P(X \geq 40) \approx P(Z > \frac{39.5 - 30}{\sqrt{30}}) = P(Z > 1.734)$$

M1 A1

$$= 0.0414 < 0.05, \text{ so reject } H_0$$

A1

Answer: $p \approx 0.0414$; reject H_0

5 marks

12 (3 marks)

$$P(X \leq 2) = 0.01375$$

B1

$$P(X \geq 14) = 1 - P(X \leq 13) = 0.03418$$

B1

$$\text{Sum} = 0.0479$$

A1

Answer: 0.0479

3 marks

13 (4 marks)

$$\text{Lower tail: } P(X \leq 3 | \lambda = 9) = 0.0212$$

M1 A1

$$\text{Two-tailed p-value} = 2 \times 0.0212 = 0.0425$$

M1

$$0.0425 < 0.05, \text{ so reject } H_0$$

A1

Answer: $p = 0.0425$; reject H_0

4 marks

14 (6 marks)

Total = 4 over 8 days; under H_0 , total $\sim \text{Po}(12)$

B1 B1

$H_0: \lambda = 1.5$, $H_1: \lambda < 1.5$

B1

$$P(X \leq 4) = 0.00760$$

M1 A1

$0.00760 < 0.05$, so reject H_0 : evidence that the mean has fallen

A1

Answer: $p = 0.00760$; reject H_0

6 marks

15 (6 marks)

$$P(X \geq 18) = 0.0630 > 0.05$$

M1 A1

$$P(X \geq 19) = 0.0374 \leq 0.05, \text{ so the critical region is } X \geq 19$$

A1 A1

$$\text{Power} = P(X \geq 19 | \lambda = 16)$$

M1

$$= 0.2577$$

A1

Answer: 0.2577

6 marks

16 (9 marks)

(a) $H_0: \lambda = 0.9$, $H_1: \lambda > 0.9$; $X \sim \text{Po}(9)$

B1 B1

$$P(X \geq 14) = 0.0739$$

M1 A1

$$0.0739 > 0.05, \text{ so do not reject } H_0$$

A1

Answer: $p = 0.0739$; do not reject H_0

5 marks

(b) $X \sim \text{Po}(18)$; $P(X \geq 28) = 0.0173$

M1 A1

$$0.0173 < 0.05, \text{ so reject } H_0$$

A1

Both samples give the same rate of 1.4 calls per minute, but the longer sample gives stronger evidence

E1

Answer: $p = 0.0173$; reject H_0

4 marks

17 (5 marks)

$H_0: \lambda = 50$, $H_1: \lambda \neq 50$; $X \approx N(50, 50)$

B1 M1

$$P(X \leq 36) \approx P(Z < \frac{36.5 - 50}{\sqrt{50}}) = P(Z < -1.909)$$

M1 A1

$$= 0.0281; 0.0281 > 0.025, \text{ so do not reject } H_0 \text{ (two-tailed, 0.025 in each tail)}$$

A1

Answer: $p \approx 0.0281$; do not reject H_0

5 marks

18 (6 marks)

(a) Vehicles arrive in clusters at start and end of school, so arrivals are not independent and the rate is not constant

E1 E1

Answer: Non-constant rate; not independent

2 marks

(b) Mean = $\frac{55}{10} = 5.5$

B1

$\Sigma x^2 = 345$, variance = $\frac{345}{10} - 5.5^2 = 4.25$ (or unbiased estimate 4.72)

M1 A1

Mean and variance are of similar size, which is consistent with a Poisson model

E1

Answer: Mean 5.5, variance 4.25 (4.72): consistent

4 marks

19 (6 marks)

Total mean under H_0 : $5 \times 2.5 + 1.2 = 13.7$; total $\sim \text{Po}(13.7)$

M1 A1

H_0 : model correct, H_1 : higher mean

B1

$P(X \geq 20) = 1 - P(X \leq 19) = 0.0649$

M1 A1

$0.0649 > 0.05$, so do not reject H_0

A1

Answer: p = 0.0649

6 marks

20 (7 marks)

(a) $H_0: \lambda = 40$, $H_1: \lambda < 40$; $X \approx N(40, 40)$

B1 M1

$P(X \leq 29) \approx P(Z < \frac{29.5 - 40}{\sqrt{40}}) = P(Z < -1.660) = 0.0484$

M1 A1

$0.0484 < 0.05$, so reject H_0

A1

Answer: p $\approx$ 0.0484; reject H_0

5 marks

(b) Exact $P(X \leq 29 | \text{Po}(40)) = 0.0432$

B1

The approximation slightly overestimates the exact value, but the conclusion is the same

E1

Answer: 0.0432

2 marks

21 (11 marks)

(a) $P(X \geq 5) = 0.0527 > 0.05$, $P(X \geq 6) = 0.0166 \leq 0.05$

M1 A1

Critical region $X \geq 6$, significance level 0.0166

A1 B1

Answer: $X \geq 6$; 0.0166

4 marks

(b) $P(X \geq 6 | \lambda = 4) = 0.2149$

M1 A1

Answer: 0.2149

2 marks

(c) Power at $\lambda = 5$: $P(X \geq 6 | 5) = 0.3840 < 0.5$, at $\lambda = 6$: $0.5543 > 0.5$

M1 A1

Power increases with λ , so a solution lies between 5 and 6

E1

Solving $P(X \geq 6) = 0.5$ numerically: $\lambda = 5.67$

M1 A1

Answer: $\lambda = 5.67$

5 marks

22 (6 marks)

Error 1: for 4 rolls the distribution is $\text{Po}(4 \times 2.5) = \text{Po}(10)$, not $\text{Po}(2.5)$

E1 B1

Error 2: $P(X \geq 16) = 1 - P(X \leq 15)$, not $1 - P(X \leq 16)$

E1

Correct: $P(X \geq 16 | \text{Po}(10)) = 1 - 0.9513 = 0.0487$

M1 A1

$0.0487 < 0.05$, so reject H_0 : evidence of an increase in faults

A1

Answer: Po(10); p = 0.0487; reject H_0

6 marks

23 (6 marks)

$H_0: \lambda_A = \lambda_B$, $H_1: \lambda_A \neq \lambda_B$

B1

Under H_0 the number from A is $X \sim B(15, 0.5)$

B1

$P(X \leq 3) = 0.0176$

M1 A1

Two-tailed p-value = $2 \times 0.0176 = 0.0352$

M1

$0.0352 < 0.05$, so reject H_0 : evidence that the fault rates differ

A1

Answer: p = 0.0352; reject H_0

6 marks

24 (7 marks)

(a) The third breakdown occurs after time t only if fewer than 3 breakdowns have occurred by time t E1

So $P(T > t) = P(N \leq 2)$ where $N \sim \text{Po}(2t)$ under H_0 E1

Answer: Equivalent events 2 marks

(b) $H_0: \lambda = 2$, $H_1: \lambda < 2$; $N \sim \text{Po}(7)$ B1 B1

$p\text{-value} = P(T > 3.5) = P(N \leq 2) = 0.0296$ M1 A1

$0.0296 < 0.05$, so reject H_0 : evidence the rate has decreased A1

Answer: $p = 0.0296$; reject H_0 5 marks

25 (8 marks)

Under H_0 , $S \sim \text{Po}(12)$ B1

$P(S \leq 5) = 0.0203 \leq 0.025$, $P(S \leq 6) = 0.0458 > 0.025$ M1 A1

$P(S \geq 20) = 0.0213 \leq 0.025$, $P(S \geq 19) = 0.0374 > 0.025$ M1 A1

Critical region: $S \leq 5$ or $S \geq 20$ A1

When $\lambda = 5$, $S \sim \text{Po}(20)$: $P(\text{Type II}) = P(6 \leq S \leq 19) = 0.4703 - 0.0001 = 0.4702$ M1 A1

Answer: 0.4702 8 marks

26 (7 marks)

(a) $P(X \geq 8 | \text{Po}(3)) = 1 - 0.9881 = 0.0119$ M1 A1

Answer: 0.0119 2 marks

(b) $1 - (1 - 0.0119)^5$
 $= 0.0581$ M1
A1

Answer: 0.0581 2 marks

(c) $1 - (1 - 0.0119)^d > 0.5$, so $d > \frac{\ln 0.5}{\ln(1 - 0.0119)} = 57.88$ M1 A1

Least $d = 58$ A1

Answer: 58 3 marks

27 (10 marks)

(a) Significance level = $P(X = 0) = e^{-0.4n}$ M1

$e^{-0.4n} \leq 0.05$ gives $n \geq \frac{\ln 20}{0.4} = 7.489$ M1 A1

So the least n is 8 A1

Answer: $n = 8$ 4 marks

(b) Significance level = $e^{-3.20} = 0.0408$ M1 A1

Power = $P(X = 0 | \lambda = 0.1) = e^{-0.1 \times 8} = 0.4493$ M1 A1

Answer: 0.0408; 0.4493 4 marks

(c) Rejecting only when $X = 0$ is very restrictive, so even when λ is much lower than claimed the count is often positive E1

Allowing $X \leq 1$ (provided the significance level is still at most 5%) would give a larger critical region and greater power E1

Answer: Critical region too small 2 marks

28 (12 marks)

(a) $X \approx N(25, 25)$ B1

Lower: $\frac{c + 0.5 - 25}{5} \leq -1.96$ gives $c \leq 14.70$, so $c = 14$ M1 A1

Upper: $\frac{c - 0.5 - 25}{5} \geq 1.96$ gives $c \geq 35.30$, so $c = 36$ A1

Critical region: $X \leq 14$ or $X \geq 36$ A1

Answer: $X \leq 14$ or $X \geq 36$ 5 marks

(b) $P(X \leq 14) = 0.01240$, $P(X \geq 36) = 0.02246$ M1 A1

Actual significance level = 0.0349

Answer: 0.0349 2 marks

(c) $P(X \leq 15) = 0.02229 \leq 0.025$, $P(X \leq 16) = 0.03775 > 0.025$ M1 A1

$P(X \geq 36) = 0.02246 \leq 0.025$, $P(X \geq 35) = 0.03384 > 0.025$ A1

Exact critical region: $X \leq 15$ or $X \geq 36$; the approximation gives a different region than the exact answer because of the skewness of the Poisson distribution

A1 E1

Answer: $X \leq 15$ or $X \geq 36$

5 marks

29 (10 marks)

(a) $D > d$ means fewer than 4 faults in the first d km

M1

The number of faults in d km is $Y \sim \text{Po}(1.5d)$, so $P(D > d) = P(Y \leq 3)$

A1

Answer: $P(D > d) = P(Y \leq 3)$, $Y \sim \text{Po}(1.5d)$

2 marks

(b) A lower rate makes D larger, so the p-value is $P(D \geq 4.2)$

B1

$Y \sim \text{Po}(6.3)$; $P(Y \leq 3) = 0.1264$

M1 A1

$0.1264 > 0.05$, so do not reject H_0

A1

Answer: $p = 0.1264$; do not reject H_0

4 marks

(c) Need $P(\text{Po}(1.5d) \leq 3) \leq 0.05$

M1

$d = 5.20$: 0.0485; $d = 5.10$: 0.0536

A1 A1

Least distance = 5.20 km

A1

Answer: 5.20 km

4 marks

30 (8 marks)

$H_0: \lambda_A = \lambda_B$, $H_1: \lambda_A > \lambda_B$

B1

Total $T = 19$; under H_0 , the number from A is $X \sim B(19, \frac{10}{18})$

M1 A1

p-value = $P(X \geq 14) = 1 - P(X \leq 13)$

M1

$= 1 - 0.9150 = 0.0850$

M1 A1

$0.0850 > 0.05$, so do not reject H_0

M1

Conclusion in context: insufficient evidence that machine A has a higher fault rate

A1

Answer: $p = 0.0850$; do not reject H_0

8 marks

Chi-Squared Goodness-of-Fit Tests

1 (4 marks)

(a) H_0 : the die is fair (all scores equally likely)

B1

H_1 : the die is not fair

B1

Answer: H_0 : fair; H_1 : not fair

2 marks

(b) $\nu = 6 - 1 = 5$

B1

Critical value = 11.07

B1

Answer: $\nu = 5$; 11.07

2 marks

2 (3 marks)

$E = \frac{120}{6} = 20$ for each score

B1

$$\sum \frac{(O - E)^2}{E} = \frac{(-2)^2 + 2^2 + (-5)^2 + 5^2 + 1^2 + (-1)^2}{20} = \frac{60}{20}$$

M1 A1

Answer: 3.00

3 marks

3 (3 marks)

Expected: $\frac{3}{4} \times 200 = 150$ and $\frac{1}{4} \times 200 = 50$

B1

$$\sum \frac{(O - E)^2}{E} = \frac{15^2}{150} + \frac{15^2}{50} = 1.5 + 4.5 = 6.00$$

M1 A1

Answer: 6.00

3 marks

4 (2 marks)

(a) $\nu = 6 - 1 = 5$, since no parameter is estimated

B1

Answer: 5

1 mark

(b) Classes after pooling = 6; $\nu = 6 - 1 - 1 = 4$

B1

Answer: 4

1 mark

5 (2 marks)

Every expected frequency should be at least 5

B1

Otherwise classes are combined (pooled) with a neighbouring class

B1

Answer: $E \geq 5$; pool classes

2 marks

6 (6 marks)

(a) Total = $4 \times 45 = 180$

M1

$x = 180 - (52 + 38 + 41) = 49$

A1

Answer: 49

2 marks

(b) There are 4 classes and the total frequency is fixed, so $\nu = 4 - 1 = 3$

B1

Answer: $\nu = 3$

1 mark

(c) $\sum \frac{(O - E)^2}{E} = \frac{7^2 + 7^2 + 4^2 + 4^2}{45} = 2.889$

M1 A1

Critical value ($\nu = 3$) is 7.815; $2.889 < 7.815$, so do not reject H_0

A1

Answer: $\chi^2 = 2.89$; do not reject H_0

3 marks

7 (6 marks)

H_0 : the die is fair; H_1 : the die is not fair

B1

$E = \frac{120}{6} = 20.00$ for each score

B1

$\sum \frac{(O - E)^2}{E} = 7.700$

M1 A1

$\nu = 5$, critical value 11.07

B1

$7.700 < 11.07$, so do not reject H_0 ; no evidence at 5% that the die is unfair

A1

Answer: $\chi^2 = 7.70$; do not reject H_0

6 marks

8 (6 marks)

H_0 : the types occur in the ratio 9 : 3 : 3 : 1

B1

$E = 309.38, 103.12, 103.12, 34.375$

M1 A1

$\sum \frac{(O - E)^2}{E} = 0.7491$

M1 A1

$\nu = 3$, critical value 7.815; $0.7491 < 7.815$, so do not reject H_0

B1

Answer: $\chi^2 = 0.749$; do not reject H_0

6 marks

9 (6 marks)

H_0 : births are uniformly distributed over the days; H_1 : they are not

B1

$E = \frac{330}{7} = 47.143$

B1

$\sum \frac{(O - E)^2}{E} = 11.81$

M1 A1

$\nu = 6$, critical value 12.59; $11.81 < 12.59$, so do not reject H_0

B1 A1

Answer: $\chi^2 = 11.8$; do not reject H_0

6 marks

10 (6 marks)

H_0 : $B(4, 0.5)$ is a suitable model; H_1 : it is not

B1

$E = 160 \times (0.0625, 0.25, 0.375, 0.25, 0.0625) = 10.00, 40.00, 60.00, 40.00, 10.00$

M1 A1

All $E \geq 5$, so no pooling; $\sum \frac{(O - E)^2}{E} = 0.6917$

M1 A1

$\nu = 4$, critical value 9.488; $0.6917 < 9.488$, so do not reject H_0

B1

Answer: $\chi^2 = 0.692$; do not reject H_0

6 marks

11 (7 marks)

H_0 : Po(1.5) is a suitable model; H_1 : it is not

B1

$E = 26.78, 40.16, 30.12, 15.06, 5.648, 2.229$

M1 A1

Combine the last class with its neighbour if its E is below 5 (here classes 4 and 5 or more): new $E = 7.877, O = 10$

B1

$$\sum \frac{(O - E)^2}{E} = 1.301$$

M1 A1

$\nu = 5 - 1 = 4$, critical value 9.488; $1.301 < 9.488$, so do not reject H_0

B1

Answer: $\chi^2 = 1.30$; do not reject H_0

7 marks

12 (6 marks)

$$s = 150 \times P(X = 0) = 150 \times e^{-2.4} = 13.61$$

M1 A1

$$t = 150 \times P(X = 3) = 150 \times e^{-2.4} \times \frac{2.4^3}{3!} = 31.35$$

M1 A1

$$u = 150 - (s + 32.66 + 39.19 + t) = 33.19, \text{ or } 150 \times P(X \geq 4)$$

M1 A1

Answer: $s = 13.61, t = 31.35, u = 33.19$

6 marks

13 (5 marks)

(a) $\nu = 7 - 1 = 6$

B1

Critical values: 10.645 (10%), 12.592 (5%), 16.812 (1%)

B1 B1

Answer: 6; 10.645, 12.592, 16.812

3 marks

(b) $11.3 > 10.645$: reject H_0 at 10%

B1

$11.3 < 12.592$ and $11.3 < 16.812$: do not reject H_0 at 5% or 1%

B1

Answer: Reject at 10% only

2 marks

14 (9 marks)

Total defectives = 81, so $\hat{p} = \frac{81}{400} = 0.2025$

M1 A1

$E = 80 \times P(X = k)$ for $B(5, 0.2025)$: 25.81, 32.76, 16.64, 4.225, 0.5364, 0.02724

M1 A1

Pool so that every $E \geq 5$: classes 3, with $O = 28, 31, 21, E = 25.81, 32.76, 21.43$

B1

$$\sum \frac{(O - E)^2}{E} = 0.2899$$

M1 A1

$\nu = 3 - 1 - 1 = 1$ (p estimated); critical value 3.841; $0.2899 < 3.841$, so do not reject H_0

B1 B1

Answer: $\chi^2 = 0.290$; do not reject H_0

9 marks

15 (9 marks)

H_0 : Poisson is a suitable model; H_1 : it is not

B1

Mean = $\frac{345}{200} = 1.725$, so use Po(1.725)

M1 A1

$E = 35.63, 61.47, 53.02, 30.49, 13.15, 4.536, 1.711$

M1 A1

Pool the last classes so that $E \geq 5$: 6 classes, $\chi^2 = 0.9678$

M1 A1

$\nu = 6 - 1 - 1 = 4$; critical value 9.488; $0.9678 < 9.488$, so do not reject H_0

B1 A1

Answer: $\chi^2 = 0.968$; do not reject H_0

9 marks

16 (12 marks)

(a) $r = 150 \times 0.5^5 = 4.69, s = 150 \times 10 \times 0.5^5 = 46.88$

M1 A1

Answer: $r = 4.69, s = 46.88$

2 marks

(b) H_0 : $B(5, 0.5)$ is a suitable model; H_1 : it is not. The classes for 0 and 5 heads have $E = 4.69 < 5$, so combine each with its neighbour: $E = 28.125, 46.875, 46.875, 28.125$

B1 M1

$$\chi^2 = 7.042$$

A1

$\nu = 4 - 1 = 3$, critical value 7.815; $7.042 < 7.815$, so do not reject H_0

B1 A1

Answer: $\chi^2 = 7.04$; do not reject H_0

5 marks

(c) $\hat{p} = \frac{407}{750} = 0.5427$

M1 A1

$E = 3.001, 17.80, 42.25, 50.14, 29.75, 7.059$; after combining classes with $E < 5$, 5 classes and χ^2

M1 A1

$$= 2.697$$

$\nu = 5 - 1 - 1 = 3$, critical value 7.815; do not reject H_0 ; the statistic is smaller than for $B(5, 0.5)$, so the estimated model fits better

A1

Answer: $\hat{p} = 0.543$; $\chi^2 = 2.70$; do not reject H_0

5 marks

17 (7 marks)

$$\sum P = k(1 + 2 + 3 + 4) = 10k = 1, \text{ so } k = 0.1$$

M1 A1

$$E = 100 \times (0.1, 0.2, 0.3, 0.4) = 10.00, 20.00, 30.00, 40.00$$

M1 A1

$$\sum \frac{(O - E)^2}{E} = 0.6583$$

M1 A1

$\nu = 3$, critical value 7.815; $0.6583 < 7.815$, so do not reject H_0

A1

Answer: $k = 0.1$; $\chi^2 = 0.658$; do not reject H_0

7 marks

18 (5 marks)

$$(O - E)^2 = O^2 - 2OE + E^2, \text{ so } \sum \frac{(O - E)^2}{E} = \sum \frac{O^2}{E} - 2\sum O + \sum E$$

M1 A1

$$\sum O = \sum E = N, \text{ so the sum} = \sum \frac{O^2}{E} - 2N + N = \sum \frac{O^2}{E} - N$$

A1

$$\sum \frac{O^2}{E} = \frac{400}{25} + \frac{961}{25} + \frac{841}{30} = 82.4733$$

M1

$$\chi^2 = 82.4733 - 80 = 2.473$$

A1

Answer: 2.473

5 marks

19 (7 marks)

$$P(\text{total}) = 1, 2, 3, 4, 5, 6, 5, 4, 3, 2, 1 \div 36$$

B1

$$E = 10.00, 20.00, 30.00, 40.00, 50.00, 60.00, 50.00, 40.00, 30.00, 20.00, 10.00$$

M1 A1

$$\sum \frac{(O - E)^2}{E} = 1.325$$

M1 A1

$\nu = 10$, critical value 18.31; $1.325 < 18.31$, so do not reject H_0

B1 A1

Answer: $\chi^2 = 1.32$; do not reject H_0

7 marks

20 (7 marks)

$$H_0: \text{the die is fair, so } P(N = r) = \left(\frac{5}{6}\right)^{r-1} \left(\frac{1}{6}\right)$$

B1

$$E = 20.00, 16.67, 13.89, 11.57, 9.645, 8.038, 40.19$$

M1 A1

$$\text{All } E \geq 5 \text{ (pooling checked); } \sum \frac{(O - E)^2}{E} = 47.49$$

M1 A1

$\nu = 7 - 1 = 6$, critical value 12.59; $47.49 > 12.59$, so reject H_0

B1 A1

Answer: $\chi^2 = 47.5$; reject H_0

7 marks

21 (10 marks)

(a) Total boys = 223

M1

$$\hat{p} = \frac{223}{400} = 0.5575$$

A1

Answer: 0.5575

2 marks

(b) $E = 3.834, 19.32, 36.51, 30.67, 9.660$

M1 A1

Pool if needed so that all $E \geq 5$: 4 classes; $\chi^2 = 1.015$

B1 M1 A1

$\nu = 4 - 1 - 1 = 2$, critical value 5.991; $1.015 < 5.991$, so do not reject H_0

B1 A1

Answer: $\chi^2 = 1.02$; do not reject H_0

7 marks

(c) The estimate was calculated from the observed data, which imposes a further constraint on the frequencies and so removes a degree of freedom

E1

Answer: One more constraint

1 mark

22 (6 marks)

Error 1: H_0 should state that the die is fair (the null hypothesis is the model being tested)

E1

Error 2: ν should be $6 - 1 = 5$ (critical value 11.07), not 6

E1

Error 3: the differences must be squared in $(O - E)^2/E$; also a non-significant result does not prove the die is fair

E1

Correct: $\sum \frac{(O - E)^2}{E} = 9.333$

M1 A1

9.333: $9.333 < 11.07$; no evidence at 5% that the die is unfair

A1

Answer: $\chi^2 = 9.33$; do not reject H_0

6 marks

23 (10 marks)

(a) $P(H < 155) = P(Z < \frac{155 - 170}{8}) = 0.0304$, and similarly for the other classes

M1 A1

$E = 3.648, 15.39, 34.99, 38.77, 20.95, 6.250$

M1 A1

Answer: 3.648, 15.39, 34.99, 38.77, 20.95, 6.250

4 marks

(b) The first class has $E = 3.648 < 5$, so combine it with the next class: $O = 18, E = 19.04$

B1

$\chi^2 = 3.939$ over 5 classes

M1 A1

$\nu = 5 - 1 - 2 = 2$ (mean and standard deviation estimated); critical value 5.991

B1 B1

$3.939 < 5.991$, so do not reject H_0

A1

Answer: $\chi^2 = 3.94$; do not reject H_0

6 marks

24 (6 marks)

(a) $\nu = 10 - 1 = 9$

B1

$P(\chi^2_9 \leq 1.2) = 0.001$

B1

Answer: $\nu = 9$; 0.001

2 marks

(b) Lower 5% critical value for $\nu = 9$ is 3.325; lower 1% critical value is 2.088

B1

1.2 is below both, so such a close fit would occur with probability under 1% if the die were fair

M1 A1

The data may have been adjusted or fabricated

E1

Answer: Suspiciously close fit

4 marks

25 (8 marks)

(a) Number of A genes = $2 \times 58 + 102 = 218$ out of 400 genes

M1

$\hat{\theta} = \frac{218}{400} = 0.5450$

A1

Answer: 0.5450

2 marks

(b) $E = 59.41, 99.19, 41.40$

M1 A1

$\chi^2 = 0.1605$

M1 A1

$\nu = 3 - 1 - 1 = 1$, critical value 3.841; $0.1605 < 3.841$, so do not reject H_0

B1 A1

Answer: $\chi^2 = 0.161$; do not reject H_0

6 marks

26 (12 marks)

(a) $\hat{p} = \frac{3.020}{6} = 0.5033$; $E = 1.501, 9.127, 23.12, 31.25, 23.75, 9.627, 1.626$

M1 A1

Pool for $E \geq 5$: 5 classes, $\chi^2 = 2.851$

B1 A1

$\nu = 5 - 2 = 3$, critical value 7.815; $2.851 < 7.815$, so do not reject H_0

B1 A1

Answer: $\chi^2 = 2.85$; do not reject H_0

6 marks

(b) $E = 4.880, 14.74, 22.25, 22.40, 16.91, 10.22, 8.595$

M1 A1

Pool: 6 classes, $\chi^2 = 7.972$; $\nu = 4$, critical value 9.488

M1 A1

do not reject H_0 ; the binomial model has the smaller statistic (2.85 against 7.97) so fits better

A1 E1

Answer: $\chi^2 = 7.97$; do not reject H_0

6 marks

27 (12 marks)

(a) $O = X$, $n - X$; $E = np_0$, $n(1 - p_0)$ B1

$$\chi^2 = \frac{(X - np_0)^2}{np_0} + \frac{(n - X - n(1 - p_0))^2}{n(1 - p_0)}$$
M1

The second numerator is $(np_0 - X)^2$, so $\chi^2 = (X - np_0)^2 \left[\frac{1}{np_0} + \frac{1}{n(1 - p_0)} \right]$ M1 A1

Bracket = $\frac{1}{np_0(1 - p_0)}$, giving the result A1

Answer: $\chi^2 = \frac{(X - np_0)^2}{np_0(1 - p_0)}$ 5 marks

(b) $\frac{(X - np_0)}{\sqrt{np_0(1 - p_0)}} = Z$, so $\chi^2 = Z^2$ M1 A1

If $Z \sim N(0, 1)$, $Z^2 \sim \chi^2_1$; the 5% critical value is $3.841 = 1.96^2$, matching a two-tailed z-test E1 B1

Answer: $\chi^2 = Z^2$; **3.841 = 1.96²** 4 marks

(c) $\chi^2 = \frac{12^2}{50} + \frac{12^2}{50} = 5.76$ M1 A1

$Z = \frac{62 - 50}{5} = 2.40$, $Z^2 = 5.76$ A1

Answer: 5.76 3 marks

28 (10 marks)

(a) $E = \frac{240}{8} = 30$ for each part B1

$\chi^2 = 4.267$ M1 A1

$\nu = 7$, critical value 14.07; $4.267 < 14.07$, so do not reject H_0 B1 A1

Answer: $\chi^2 = 4.27$; do not reject H_0 5 marks

(b) Need $\frac{240}{k} \geq 5$, so $k \leq 48$ M1 A1

$\nu = 48 - 1 = 47$ A1

Answer: 48 parts; $\nu = 47$ 3 marks

(c) Small expected frequencies make the χ^2 approximation poor; too many classes also reduce the power by diluting a pattern that appears over wider ranges E1 E1

Answer: Poor approximation, lost power 2 marks

29 (10 marks)

Estimated mean = $\frac{1710.00}{140} = 12.21$, so $\hat{\lambda} = \frac{1}{12.21} = 0.08187$ M1 A1

$P(a < T < b) = e^{-\lambda a} - e^{-\lambda b}$, $P(T > 40) = e^{-40\lambda}$ B1

$P = 0.3359, 0.2231, 0.2465, 0.1567, 0.03782$ M1 A1

$E = 47.03, 31.23, 34.51, 21.93, 5.295$ A1

All $E \geq 5$ (pool otherwise); $\chi^2 = 1.088$ M1 A1

$\nu = 5 - 1 - 1 = 3$, critical value 7.815; $1.088 < 7.815$, so do not reject H_0 B1 A1

Answer: $\hat{\lambda} = 0.0819$; $\chi^2 = 1.09$; do not reject H_0 10 marks

30 (11 marks)

(a) Experiment 1: $E = \frac{100}{6}$; $\chi^2 = 5.600$; $\nu = 5$, critical value 11.07; do not reject H_0 M1 A1 B1 A1

Experiment 2: $E = \frac{400}{6}$; $\chi^2 = 22.40$; reject H_0 M1 A1 A1

Answer: 5.60 (do not reject H_0); 22.4 (reject H_0) 7 marks

(b) $O \rightarrow kO$ and $E \rightarrow kE$, so $\frac{(kO - kE)^2}{kE} = k \frac{(O - E)^2}{E}$ M1 A1

Answer: New statistic = $k \times$ old statistic 2 marks

(c) The same proportion of departures from a fair die gives stronger evidence of bias when the sample is larger E1

A larger sample gives a more powerful test: small real departures become statistically significant E1

Answer: Larger samples detect smaller departures**2 marks**

Chi-Squared Tests for Association

1 (3 marks)**(a)** H_0 : there is no association between gender and subject choice**B1** H_1 : there is an association between gender and subject choice**B1****Answer: H_0 : no association; H_1 : association****2 marks****(b)** $\nu = (2 - 1)(4 - 1) = 3$ **B1****Answer: 3****1 mark****2 (3 marks)****(a)** $(2 - 1)(2 - 1) = 1$ **B1****Answer: 1****1 mark****(b)** $(3 - 1)(5 - 1) = 8$ **B1****Answer: 8****1 mark****(c)** Now 4 rows and 2 columns: $(4 - 1)(2 - 1) = 3$ **B1****Answer: 3****1 mark****3 (2 marks)**

$$E = \frac{80 \times 45}{200}$$

M1

$$= 18$$

A1**Answer: 18****2 marks****4 (3 marks)**

Row totals: 30, 50; column totals: 40, 40

B1

$$E = \frac{30 \times 40}{80} = 15 \text{ (twice), } E = \frac{50 \times 40}{80} = 25 \text{ (twice)}$$

M1 A1**Answer: E = 15, 15, 25, 25****3 marks****5 (2 marks)**

$$\text{Each term is } \frac{5^2}{25} = 1$$

M1

$$\Sigma = 4 \times 1 = 4$$

A1**Answer: 4****2 marks****6 (2 marks)****(a)** Combine that cell's row or column with an adjacent category so all expected frequencies are at least 5**B1****Answer: Combine categories****1 mark****(b)** The number of rows or columns falls by one, so the degrees of freedom are reduced**B1****Answer: Reduced****1 mark****7 (7 marks)** H_0 : no association between method and result; H_1 : association**B1**

$$E = (42.00, 28.00); (42.00, 28.00)$$

M1 A1

$$\Sigma \frac{(O - E)^2}{E} = 4.286$$

M1 A1

$$\nu = 1, \text{ critical value } 3.841; 4.286 > 3.841, \text{ so reject } H_0$$

B1 A1**Answer: $\chi^2 = 4.29$; reject H_0** **7 marks**

8 (7 marks)

H_0 : method and age group are independent; H_1 : they are not independent

B1

$E = (43.00, 24.00, 33.00); (43.00, 24.00, 33.00)$

M1 A1

$$\sum \frac{(O - E)^2}{E} = 6.616$$

M1 A1

$\nu = (2 - 1)(3 - 1) = 2$, critical value 5.991; $6.616 > 5.991$, so reject H_0

B1 A1

Answer: $\chi^2 = 6.62$; reject H_0

7 marks

9 (7 marks)

H_0 : no association between region and colour; H_1 : association

B1

$E = (22.83, 24.35, 22.83); (27.72, 29.57, 27.72); (24.46, 26.09, 24.46)$

M1 A1

$$\sum \frac{(O - E)^2}{E} = 10.18$$

M1 A1

$\nu = 4$, critical value 9.488; $10.18 > 9.488$, so reject H_0

B1 A1

Answer: $\chi^2 = 10.2$; reject H_0

7 marks

10 (5 marks)

Chess column: $24 + b = 40$, so $b = 16$

M1 A1

Football column: $a + 24 = 60$, so $a = 36$

A1

Row totals: boys $24 + 36 = 60$, girls $16 + 24 = 40$; $E = \frac{40 \times 40}{100} = 16$

M1 A1

Answer: $a = 36$, $b = 16$, $E = 16$

5 marks

11 (5 marks)

(a) $\nu = 4$

B1

9.488 (5%), 13.28 (1%)

B1 B1

Answer: $\nu = 4$; 9.488, 13.28

3 marks

(b) $10.2 > 9.488$: reject H_0 at 5%; evidence of an association between age group and film preference

B1

$10.2 < 13.28$: do not reject H_0 at 1%; no evidence of an association at that level

B1

Answer: Significant at 5% only

2 marks

12 (8 marks)

Row totals 30, 29; column totals 10, 21, 28; E for the first column = 5.08 and 4.92, both below 5

M1 A1

Combine 'Low' and 'Medium': $O = 12, 18$ and $19, 10$; $E = (15.76, 14.24); (15.24, 13.76)$

M1 A1

$$\sum \frac{(O - E)^2}{E} = 3.851$$

M1 A1

$\nu = 1$, critical value 3.841; $3.851 > 3.841$, so reject H_0

B1 A1

Answer: $\chi^2 = 3.85$; reject H_0

8 marks

13 (8 marks)

(a) H_0 : there is no association between favourite colour and year group

B1

H_1 : there is an association

B1

Answer: H_0 : no association; H_1 : association

2 marks

(b) The smallest row total and the smallest column total give the smallest E : (Sixth form, Black)

B1

$$E = \frac{59 \times 22}{236} = 5.500 \geq 5$$

M1 A1

Answer: (Sixth form, Black), $E = 5.500$

3 marks

(c) $\nu = (3 - 1)(5 - 1) = 8$, critical value 20.09

B1 B1

$19.99 < 20.09$, so do not reject H_0 : no evidence of an association between favourite colour and year group

A1

Answer: $\nu = 8$; 20.09; do not reject H_0

3 marks

14 (7 marks)

$E = (12.73, 20.00, 14.09, 8.182); (15.27, 24.00, 16.91, 9.818)$

M1 A1

All $E \geq 5$; $\sum \frac{(O - E)^2}{E} = 12.76$

M1 A1

$\nu = 3$, critical value 7.815; $12.76 > 7.815$, so reject H_0

B1 A1

Largest contribution 2.18 from the cell with $O = 18$, $E = 12.7$

B1

Answer: $\chi^2 = 12.8$; reject H_0

7 marks

15 (10 marks)

(a) H_0 : no association; H_1 : association

B1

$E = (31.67, 35.00, 33.33); (31.67, 35.00, 33.33); (31.67, 35.00, 33.33)$

M1 A1

$\chi^2 = 17.93$

M1 A1

$\nu = 4$, critical value 9.488; $17.93 > 9.488$, so reject H_0

B1 A1

Answer: $\chi^2 = 17.9$; reject H_0

7 marks

(b) Largest contribution 5.61 from the cell (row 1, column 1), $O = 45$ against $E = 31.7$

M1 A1

O is higher than expected under independence

E1

Answer: Cell (1, 1)

3 marks

16 (8 marks)

(a) $E = (17.50, 22.50); (17.50, 22.50)$

M1 A1

$\chi^2 = 4.114$

M1 A1

Answer: 4.114

4 marks

(b) $|O - E| = 4.50$ for each cell, so each term is $\frac{(4.50 - 0.5)^2}{E}$

M1

Corrected $\chi^2 = 3.251$

A1

$\nu = 1$, critical value 3.841; $3.251 < 3.841$, so do not reject H_0

B1 A1

Answer: 3.25; do not reject H_0

4 marks

17 (5 marks)

The χ^2 statistic depends on the actual counts, not percentages, as it measures departures in relation to sample size

E1

Convert the percentages to actual frequencies and recalculate

E1

Actual frequencies = percentages $\times 2$, so the statistic is multiplied by 2: $2 \times 8.4 = 16.8$

M1 A1

$\nu = 2$, critical value 5.991; $16.8 > 5.991$ so significant at 5%

A1

Answer: 16.8

5 marks

18 (7 marks)

H_0 : the proportion recommending is the same in all stores; H_1 : it is not

B1

E : row totals are 50 each; column totals 77 and 73, so $E(\text{yes}) = 25.67$ and $E(\text{no}) = 24.33$ in every row

M1 A1

$\chi^2 = 13.18$

M1 A1

$\nu = 2$, critical value 5.991; $13.18 > 5.991$, so reject H_0

B1

p-value = $P(\chi^2_2 > 13.18) = 0.00137$

A1

Answer: $\chi^2 = 13.2$; $p = 0.00137$

7 marks

19 (8 marks)

Total = $30 + 20 + x + 25 + 35 + 40 = 150 + x = 190$ gives $x = 40$

M1 A1

Row totals 90, 100; column totals 55, 55, 80; $E = (26.05, 26.05, 37.89); (28.95, 28.95, 42.11)$

M1 A1

$\chi^2 = 4.030$

M1 A1

$\nu = 2$, critical value 5.991; $4.030 < 5.991$, so do not reject H_0

B1 A1

Answer: $x = 40$; $\chi^2 = 4.03$; do not reject H_0

8 marks

20 (10 marks)**(a)** H_0 : no association; H_1 : association**B1** $E = (19.10, 19.95, 19.95); (28.17, 29.41, 29.41); (20.72, 21.64, 21.64)$ **M1 A1** $\chi^2 = 8.577$; $\nu = 4$, critical value 9.488**M1 A1 B1** $8.577 < 9.488$, so do not reject H_0 **A1****Answer: $\chi^2 = 8.58$; do not reject H_0** **7 marks****(b)** Critical value 7.779; $8.577 > 7.779$, so reject H_0 **B1 A1**

The result is significant at 10% but not at 5%: the evidence of association is weak

E1**Answer: reject H_0** **3 marks****21 (11 marks)****(a)** Row totals 65, 60, 64; column totals 39, 56, 59, 35; $N = 189$ **B1** $E = (13.41, 19.26, 20.29, 12.04); (12.38, 17.78, 18.73, 11.11); (13.21, 18.96, 19.98, 11.85);$ **M1 A1**smallest $11.1 \geq 5$ $\chi^2 = 21.66$ **M1 A1** $\nu = (3 - 1)(4 - 1) = 6$, critical value 12.59; $21.66 > 12.59$, so reject H_0 **B1 A1****Answer: $\chi^2 = 21.7$; reject H_0** **7 marks****(b)** Largest contributions: 8.69 and 3.36**M1 A1**Cells (Year 11, daily): $O = 22$, $E = 11.9$, which is more than expected, and (Year 10, daily): $O =$ **A1 E1**5, $E = 11.1$, which is fewer than expected**Answer: Year 11, daily (8.69); Year 10, daily (3.36)****4 marks****22 (7 marks)**Expected frequency = $\frac{\text{row total} \times \text{column total}}{\text{grand total}}$, so $40 \times 30 \div 120 = 10$, not 1200**E1 B1** $\nu = (3 - 1)(4 - 1) = 6$, not 12 (the 5% critical value is 12.59)**E1 B1**The statistic is $\sum \frac{(O - E)^2}{E}$: the differences must be squared**E1** $6.1 < \text{critical value}$ means do not reject H_0 : no evidence of association, and in any case association never proves causation**E1 E1****Answer: $E = \text{row total} \times \text{column total} \div N$; $\nu = 6$; differences squared; do not reject H_0 , and association does not prove causation****23 (7 marks)****(a)** Row 1: $a + (50 - a) = 50$; row 2: $30 + 20 = 50$ **B1**Columns: $a + 30$ and $(50 - a) + 20 = 70 - a$ **B1****Answer: Row totals 50, 50; column totals $a + 30$, $70 - a$** **2 marks****(b)** Use $E_{11} = \frac{50(a + 30)}{100}$ and calculate χ^2 for successive values of a **M1** $a = 39$: $\chi^2 = 3.787$, which is below 3.841**A1** $a = 40$: $\chi^2 = 4.762$, which is above 3.841**A1** χ^2 increases as a moves away from 30, so the smallest a is 40**A1 E1****Answer: 40****5 marks****24 (8 marks)****(a)** $E = 20$ for each cell; each term $\frac{4^2}{20} = 0.8$ **M1 A1** $\chi^2 = 3.20$; critical value ($\nu = 1$) 3.841; $3.200 < 3.841$, so do not reject H_0 **A1 B1****Answer: $\chi^2 = 3.20$; do not reject H_0** **4 marks****(b)** $E = 40$, differences 8: each term $\frac{8^2}{40} = 1.6$, so $\chi^2 = 6.40 = 2 \times 3.20$ **M1 A1** $6.400 > 3.841$, so reject H_0 : evidence of an association**A1****Answer: $\chi^2 = 6.40$; reject H_0** **3 marks****(c)** With more data the same proportional departure from independence is stronger evidence against H_0 **E1****Answer: Larger sample****1 mark**

25 (10 marks)**(a)** $E = (33.33, 33.33, 33.33); (33.33, 33.33, 33.33); (33.33, 33.33, 33.33)$ **M1 A1** $\chi^2 = 17.52; \nu = 4$, critical value 9.488**M1 A1 B1**17.52 > 9.488, so reject H_0 **A1****Answer: $\chi^2 = 17.5$; reject H_0** **6 marks****(b)** Residuals: (2.540, -1.963, -0.5774); (-0.5774, 0.8083, -0.2309); (-1.963, 1.155, 0.8083)**M1 A1**

Cells with |residual| > 1.96: (row 1, column 1), (row 1, column 2), (row 3, column 1)

A1 E1**Answer: cells (1, 1), (1, 2), (3, 1)****4 marks****26 (9 marks)****(a)** Row totals are 90, 70, 60, 12; the expected frequencies in the last row are 4.13, 4.02, 3.85, all below 5**M1 A1****Answer: $E < 5$ in last row****2 marks****(b)** Combined row: 17, 24, 31; $E = (24.10, 23.44, 22.45); (24.10, 23.44, 22.45); (24.79, 24.11, 23.09)$ **M1 A1** $\chi^2 = 8.354$ **M1 A1** $\nu = (3 - 1)(3 - 1) = 4$, critical value 9.488; $8.354 < 9.488$, so do not reject H_0 **B1 A1****Answer: $\chi^2 = 8.35$; do not reject H_0** **6 marks****(c)** They are adjacent age groups, so combining gives a meaningful category ('40 and over')**E1****Answer: Adjacent categories****1 mark****27 (9 marks)****(a)** $E_{11} = \frac{(a+b)(a+c)}{N}$ **B1**

$$a - E_{11} = \frac{aN - (a+b)(a+c)}{N} = \frac{a(a+b+c+d) - a^2 - ac - ab - bc}{N}$$

M1

$$= \frac{ad - bc}{N}$$

A1**Answer: $E_{11} = \frac{(a+b)(a+c)}{N}$; $a - E_{11} = \frac{ad - bc}{N}$** **3 marks****(b)** Each cell has $|O - E| = \frac{|ad - bc|}{N}$ **M1**

$$\chi^2 = \frac{(ad - bc)^2}{N^2} \left[\frac{1}{E_{11}} + \frac{1}{E_{12}} + \frac{1}{E_{21}} + \frac{1}{E_{22}} \right]$$

M1 A1The bracket simplifies to $\frac{N^3}{(a+b)(c+d)(a+c)(b+d)}$, giving the result**A1****Answer: $\chi^2 = \frac{N(ad - bc)^2}{(a+b)(c+d)(a+c)(b+d)}$** **4 marks****(c)** $ad - bc = 432 - 1152 = -720$; $N = 110$ **M1**

$$\chi^2 = \frac{110 \times 720^2}{50 \times 60 \times 54 \times 56} = 6.286$$

A1**Answer: 6.286****2 marks****28 (9 marks)****(a)** H_0 : recovery is independent of drug**B1** $E = (5.000, 3.000); (5.000, 3.000)$; two of the expected frequencies are below 5 (3.00)**M1 A1**The χ^2 approximation requires $E \geq 5$ in all cells**E1****Answer: Two E below 5****4 marks****(b)** $P(X = 7) = \frac{C(8, 7)C(8, 3)}{C(16, 10)} = 0.05594$ **M1 A1**

$$P(X = 8) = \frac{C(8, 8)C(8, 2)}{C(16, 10)} = 0.00350$$

A1 $P(X \geq 7) = 0.05944$ **M1** $0.05944 > 0.05$, so do not reject H_0 **A1****Answer: $p = 0.05944$** **5 marks**

29 (12 marks)

(a) Hospital 1: $A \frac{81}{87} = 0.931$, $B \frac{234}{270} = 0.867$

M1 A1

Hospital 2: $A \frac{192}{263} = 0.730$, $B \frac{55}{80} = 0.688$

A1

Treatment A is better in both hospitals

E1

Answer: A better in both

4 marks

(b) A: survived 273 of 350; B: survived 289 of 350

M1

Proportions 0.780 and 0.826, so B appears better overall

A1 E1

Answer: 0.780 and 0.826

3 marks

(c) $\chi^2 = 2.311$, $\nu = 1$, critical value 3.841

M1 A1

$2.311 < 3.841$, so do not reject H_0

A1

Most patients on A were treated at Hospital 2 where survival is lower for both treatments, while most patients on B were treated at Hospital 1: the hospital is a confounding variable

E1 E1

Answer: $\chi^2 = 2.31$; do not reject H_0

5 marks

30 (10 marks)

(a) All expected frequencies are 25 since the row and column totals are all 50

B1

Each $|O - E| = d$, so $\chi^2 = 4 \times \frac{d^2}{25}$

M1 A1

Answer: $\chi^2 = \frac{4d^2}{25}$

3 marks

(b) $\frac{4d^2}{25} > 3.841$ gives $d^2 > 24.01$

M1 A1

$d > 4.900$ so the least integer is $d = 5$

A1

Answer: 5

3 marks

(c) $\frac{4(d - 0.5)^2}{25} > 3.841$ gives $d - 0.5 > 4.900$

M1 A1

$d > 5.400$, so $d = 6$

A1

The correction raises the least d from 5 to 6: it makes the test more conservative

E1

Answer: 6

4 marks

Tests for Population Means**1 (2 marks)**

$H_0: \mu = 25$

B1

$H_1: \mu > 25$

B1

Answer: $H_0: \mu = 25$, $H_1: \mu > 25$

2 marks

2 (2 marks)

The population from which the sample is taken must be normally distributed

B1

(The variance need not be known, because it is estimated from the sample)

B1

Answer: Population normally distributed

2 marks

3 (3 marks)

$s = \sqrt{12.96} = 3.6$

B1

$t = \frac{54.2 - 52}{\frac{3.6}{\sqrt{16}}} = 2.44$, with $\nu = 16 - 1 = 15$

M1 A1

Answer: $t = 2.44$, $\nu = 15$

3 marks

4 (3 marks)

(a) $\nu = 12 - 1 = 11$

B1

Answer: 11

1 mark

(b) $\nu = 10 - 1 = 9$

B1

Answer: 9

1 mark

(c) $\nu = 8 + 11 - 2 = 17$

Answer: 17

B1

1 mark

5 (3 marks)

Differences: 3, -1, 4, 2, 5

B1

Mean difference = $\frac{13}{5} = 2.6$

M1 A1

Answer: 2.6

3 marks

6 (3 marks)

(a) z-test, since the population variance is known

B1

Answer: z-test

1 mark

(b) t-test, since the variance is estimated and the sample is small

B1

Answer: t-test

1 mark

(c) z-test using s as an estimate, since the sample is large (central limit theorem)

B1

Answer: z-test (large sample)

1 mark

7 (7 marks)

$H_0: \mu = 12.0, H_1: \mu \neq 12.0$

B1

$\bar{x} = 12.225, \Sigma x^2 = 1196.16, s = 0.2816$

M1 A1

$$t = \frac{12.225 - 12}{\frac{0.2816}{\sqrt{8}}} = 2.260$$

M1 A1

$\nu = 7$, critical value ± 2.365 ; $|t| = 2.260 < 2.365$, so do not reject H_0

B1 A1

Answer: t = 2.26; do not reject H_0

7 marks

8 (7 marks)

$H_0: \mu = 60, H_1: \mu < 60$

B1

$\bar{x} = \frac{745.2}{12} = 62.100; s^2 = \frac{1}{11}(46\,321.0 - 12 \times 62.100^2) = 4.007$

M1 A1

$$t = \frac{62.100 - 60}{\frac{2.002}{\sqrt{12}}} = 3.634$$

M1 A1

$\nu = 11$, critical value -1.796 ; $3.634 > -1.796$, so do not reject H_0

B1 A1

Answer: t = 3.63; do not reject H_0

7 marks

9 (7 marks)

(a) $H_0: \mu = 100, H_1: \mu \neq 100$

B1

$$z = \frac{103.2 - 100}{\frac{15}{\sqrt{50}}} = 1.508$$

M1 A1

$|z| < 1.96$, so do not reject H_0

A1

Answer: z = 1.51; do not reject H_0

4 marks

(b) $103.2 \pm 1.96 \times \frac{15}{\sqrt{50}} = (99.04, 107.4)$

M1 A1

100 lies inside the interval, consistent with not rejecting H_0 at 5%

E1

Answer: (99.04, 107.4)

3 marks

10 (8 marks)

Differences d = before - after: 3, 1, 3, 1, 7, 2, 0, 5

B1

$\bar{d} = 2.750, s_d = 2.315$

M1 A1

$$H_0: \mu_d = 0, H_1: \mu_d > 0; t = \frac{2.750}{\frac{2.315}{\sqrt{8}}} = 3.361$$

B1 M1 A1

$\nu = 7$, critical value 1.895; $3.361 > 1.895$, so reject H_0

B1 A1

Answer: t = 3.36; reject H_0

8 marks

11 (5 marks)

$$H_0: \mu = 50, H_1: \mu < 50$$

B1

$$z = \frac{49.1 - 50}{\frac{6.2}{\sqrt{100}}} = -1.452$$

M1 A1

> -1.645 , so do not reject H_0

A1

n is large so by the central limit theorem $\bar{X}$ is approximately normal and s is a good estimate of σ

E1

Answer: $z = -1.45$; do not reject H_0

5 marks

12 (6 marks)

$$H_0: \mu_1 = \mu_2, H_1: \mu_1 \neq \mu_2$$

B1

$$\text{Variance of the difference} = \frac{36}{40} + \frac{49}{50} = 1.880$$

M1 A1

$$z = \frac{82.4 - 79.8}{\sqrt{1.880}} = 1.896$$

M1 A1

$|z| < 1.96$, so do not reject H_0

A1

Answer: $z = 1.90$; do not reject H_0

6 marks

13 (7 marks)

$$H_0: \mu_A = \mu_B, H_1: \mu_A > \mu_B$$

B1

$$s_p^2 = \frac{9 \times 4.1^2 + 11 \times 3.7^2}{20} = 15.09$$

M1 A1

$$t = \frac{54.6 - 51.2}{\sqrt{15.09(1/10 + 1/12)}} = 2.044$$

M1 A1

$\nu = 20$, critical value 1.725; $2.044 > 1.725$, so reject H_0

B1 A1

Answer: $t = 2.04$; reject H_0

7 marks

14 (10 marks)

(a) $\bar{x} = 102.42, s = 1.663$

M1 A1

$$H_0: \mu = 100, H_1: \mu > 100; t = 4.603$$

B1 M1 A1

$\nu = 9$, critical value 3.250; $4.603 > 3.250$, so reject H_0

B1 A1

Answer: $t = 4.60$; reject H_0

7 marks

(b) $102.42 \pm 2.262 \times \frac{1.663}{\sqrt{10}}$

M1 A1

(101.2, 103.6)

A1

Answer: (101.2, 103.6)

3 marks

15 (9 marks)

Assume the differences are normally distributed

B1

$$\bar{d} = \frac{138}{12} = 11.50; s_d^2 = \frac{1}{11}(2120 - 12 \times 11.50^2) = 48.45$$

M1 A1

$$H_0: \mu_d = 0, H_1: \mu_d > 0; t = \frac{11.50}{\frac{6.961}{\sqrt{12}}} = 5.723$$

B1 M1 A1

$\nu = 11$, critical value 2.718; $5.723 > 2.718$, so reject H_0

B1 A1

$$p\text{-value} = P(t_{11} > 5.723) = 0.00007$$

A1

Answer: $t = 5.72$; $p = 0.00007$

9 marks

16 (9 marks)

$$\bar{x}_A = 24.429, s_A = 1.241; \bar{x}_B = 22.388, s_B = 0.7511$$

M1 A1

$$s_p^2 = \frac{6 \times 1.241^2 + 7 \times 0.7511^2}{13} = 1.014$$

M1 A1

$$H_0: \mu_A = \mu_B, H_1: \mu_A \neq \mu_B; t = \frac{24.429 - 22.388}{\sqrt{1.014(1/7 + 1/8)}} = 3.916$$

B1 M1 A1

$\nu = 13$, critical values ± 2.160 ; $|t| = 3.916 > 2.160$, so reject H_0

B1 A1

Answer: $t = 3.92$; reject H_0

9 marks

17 (8 marks)

$$(a) s_p^2 = \frac{14 \times 7.2^2 + 17 \times 6.5^2}{31} = 46.58$$

M1 A1

$$\text{Standard error} = \sqrt{46.58(1/15 + 1/18)} = 2.386$$

M1 A1

$$4.5 \pm 2.040 \times 2.386 = (-0.368, 9.37)$$

M1 A1

Answer: (-0.368, 9.37)

6 marks

(b) 0 lies inside the interval

B1

So H_0 is not rejected at the 5% level, as the interval and the test are equivalent

E1

Answer: H_0 not rejected

2 marks

18 (4 marks)

$$\text{Reject if } \frac{\bar{x} - 40}{\frac{2.5}{\sqrt{10}}} < -1.833$$

M1 A1

$$\bar{x} < 40 - 1.833 \times 0.7906$$

M1

$$\bar{x} < 38.55$$

A1

Answer: $\bar{x} < 38.55$

4 marks

19 (9 marks)

$$(a) t = \frac{23 - 21}{\frac{6}{\sqrt{9}}} = 1.000$$

M1 A1

 $\nu = 8$, critical value 1.860; $1.000 < 1.860$, so do not reject H_0

B1 A1

Answer: $t = 1.00$; do not reject H_0

4 marks

$$(b) t = \frac{23 - 21}{\frac{6}{\sqrt{36}}} = 2.000$$

M1 A1

 $\nu = 35$, critical value 1.690; $2.000 > 1.690$, so reject H_0

B1 A1

The larger sample gives a larger test statistic and stronger evidence for the same sample mean and spread

E1

Answer: $t = 2.00$; reject H_0

5 marks

20 (5 marks)**(a)** The sample is large, so by the central limit theorem the sample mean is approximately normal

E1

Answer: Central limit theorem

1 mark

$$(b) z = \frac{148.5 - 140}{\frac{21}{\sqrt{60}}} = 3.135$$

M1 A1

Critical value 2.326; $3.135 > 2.326$, so reject H_0

B1 A1

Answer: $z = 3.14$; reject H_0

4 marks

21 (12 marks)**(a)** d (before – after): 2, 1, 3, 4, 1, 4, 3, 2; $\bar{d} = 2.500$, $s_d = 1.195$

M1 A1

$$t = \frac{2.500}{\frac{1.195}{\sqrt{8}}} = 5.916$$

M1 A1

 $\nu = 7$, critical values ± 2.365 ; $|t| = 5.916 > 2.365$, so reject H_0

B1 A1

Answer: $t = 5.92$; reject H_0

6 marks

(b) $\bar{x}_1 = 52.88$, $\bar{x}_2 = 50.38$; $s_1 = 4.612$, $s_2 = 3.662$; $s_p^2 = 17.34$

M1 A1

 $t = 1.201$ with $\nu = 14$, critical values ± 2.145 ; $|t| = 1.201 < 2.145$, so do not reject H_0

M1 A1

Answer: $t = 1.20$; do not reject H_0

4 marks

(c) The same athletes are measured twice, so the observations are not independent

E1

Pairing removes the large athlete-to-athlete variation, leaving the small variability of the differences

E1

Answer: Pairing removes between-athlete variation

2 marks

22 (9 marks)Error 1: the unbiased estimate divides by $n - 1 = 8$, not 9

E1

Error 2: the standard error is $\frac{s}{\sqrt{n}}$, which has not been used

E1

Error 3: n is small and σ unknown, so a t-test with $\nu = 8$ should be used, not a z-test

E1

$$s^2 = \frac{75.56}{8} = 9.444, s = 3.073$$

M1 A1

$$t = \frac{52.78 - 50}{\frac{3.073}{\sqrt{9}}} = 2.712; \text{critical value } 1.860; 2.712 > 1.860, \text{ so reject } H_0$$

M1 A1 B1 A1

Answer: $t = 2.71$; reject H_0

9 marks

23 (12 marks)

$$(a) s_p^2 = \frac{13 \times 11.3^2 + 19 \times 12.4^2}{32} = 143.2$$

M1 A1

$$t = \frac{142.6 - 135.1}{\sqrt{143.2(1/14 + 1/20)}} = 1.799$$

M1 A1

 $\nu = 32$, critical values ± 2.037 ; $|t| = 1.799 < 2.037$, so do not reject H_0

B1 A1

Answer: $t = 1.80$; do not reject H_0

6 marks

(b) Standard error = 4.170, $t_{32} (0.5\%) = 2.738$

B1 B1

$$7.5 \pm 2.738 \times 4.170 = (-3.92, 18.9)$$

M1 A1

Answer: $(-3.92, 18.9)$

4 marks

(c) The interval contains 0, so the test at 1% (two-tailed) does not reject H_0

E1 A1

Answer: Equivalent to the test

2 marks

24 (6 marks)

$$z = \frac{104 - 100}{\frac{12}{\sqrt{n}}} = \frac{\sqrt{n}}{3}$$

M1 A1

Reject if $\frac{\sqrt{n}}{3} > 1.96$, so $\sqrt{n} > 5.88$

M1 A1

 $n > 34.574$, so the smallest n is 35

M1 A1

Answer: 35

6 marks

25 (10 marks)

$$(a) \bar{x} = 172.01, s = 2.287$$

M1 A1

$$t = \frac{172.01 - 170}{\frac{2.287}{\sqrt{8}}} = 2.489$$

M1 A1

Answer: 2.489

4 marks

$$(b) \bar{x} = 1720.12, s = 22.869$$

M1 A1

$$t = \frac{1720.12 - 1700}{\frac{22.869}{\sqrt{8}}} = 2.489, \text{ the same value}$$

A1

Answer: 2.489

3 marks

(c) New mean is $k\bar{x}$ and new s is ks

B1

$$\text{New } t = \frac{k\bar{x} - k\mu_0}{ks/\sqrt{n}} = \frac{k(\bar{x} - \mu_0)}{k s/\sqrt{n}}$$

M1

 k cancels, so t is unchanged

A1

Answer: t is unchanged

3 marks

26 (7 marks)

$$(a) \text{Variance of the difference} = \frac{3000^2}{25} + \frac{4000^2}{25} = 1\,000\,000, \text{ so } sd = 1000$$

M1 A1

$$\text{Critical value} = 1.645 \times 1000 = 1645 \text{ km}$$

M1 A1

Answer: 1645 km

4 marks

(b) Under the alternative, the difference $\sim N(2500, 1000^2)$

B1

$$P(\text{difference} < 1645) = P(Z < \frac{1645 - 2500}{1000}) = P(Z < -0.855)$$

M1

$$= 0.1963$$

A1

Answer: 0.1963

3 marks

27 (11 marks)

$$(a) E(s_1^2) = E(s_2^2) = \sigma^2$$

B1

$$E(s_p^2) = \frac{(n_1 - 1)\sigma^2 + (n_2 - 1)\sigma^2}{n_1 + n_2 - 2}$$

M1 A1

$$= \frac{(n_1 + n_2 - 2)\sigma^2}{n_1 + n_2 - 2} = \sigma^2$$

A1

Answer: $E(s_p^2) = \sigma^2$

4 marks

$$(b) \text{Var}(\bar{X}_1) = \frac{\sigma^2}{n_1}, \text{Var}(\bar{X}_2) = \frac{\sigma^2}{n_2}, \text{ and the samples are independent}$$

B1

$$\text{Var}(\bar{X}_1 - \bar{X}_2) = \text{Var}(\bar{X}_1) + \text{Var}(\bar{X}_2) = \sigma^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)$$

M1 A1

$$\text{Answer: Var} = \sigma^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)$$

3 marks

$$(c) s_p^2 = \frac{5 \times 4.8 + 8 \times 7.2}{13} = 6.277$$

M1 A1

$$SE = \sqrt{6.277(1/6 + 1/9)} = 1.320$$

M1 A1

Answer: $s_p^2 = 6.277$; $SE = 1.320$

4 marks

28 (11 marks)

$$(a) \frac{s_1^2}{n_1} = 1.068, \frac{s_2^2}{n_2} = 3.651$$

B1

$$t = \frac{5.5}{\sqrt{4.718}} = 2.532$$

M1 A1

$$\nu = 20.34, \text{ rounded down to } 20$$

M1 A1

Answer: $t = 2.53$, $\nu = 20$

5 marks

$$(b) \text{Critical values } \pm 2.086; |t| = 2.532 > 2.086, \text{ so reject } H_0$$

B1 A1

Answer: reject H_0

2 marks

$$(c) s_p^2 = 38.34, t = 2.107 \text{ with } \nu = 22$$

M1 A1

The pooled test assumes equal variances and gives more weight to the larger sample's variance; the unequal-variance test is more reliable when the samples differ in size and spread

E1 E1

Answer: $t = 2.11$, $\nu = 22$

4 marks

29 (12 marks)

$$(a) \text{Test statistic has sd } \sigma\sqrt{2/n}; \text{ the required shift } \delta \text{ must equal } (z_\alpha + z_\beta) \times \sigma\sqrt{2/n}$$

M1 A1

$$n = \frac{2\sigma^2(z_\alpha + z_\beta)^2}{\delta^2} = \frac{2 \times 100 \times 8.5650}{16} = 107.06$$

M1 A1

$$\text{Smallest } n = 108 \text{ per group}$$

A1

Answer: $n = 108$ per group

5 marks

$$(b) \text{Var}(D) = \sigma^2 + \sigma^2 - 2\rho\sigma^2 = 2\sigma^2(1 - \rho) = 80.00$$

M1 A1

$$n = \frac{\text{Var}(D)(z_\alpha + z_\beta)^2}{\delta^2} = \frac{80 \times 8.5650}{16} = 42.825$$

M1 A1

$$\text{Smallest number of pairs} = 43$$

A1

Answer: 43 pairs

5 marks

$$(c) \text{Independent: } 2 \times 108 = 216 \text{ observations; paired: } 2 \times 43 = 86 \text{ observations}$$

B1

Positive correlation within pairs reduces the variance of the differences by the factor $(1 - \rho)$, so far fewer observations are required

E1

Answer: Pairing needs far fewer observations

2 marks

30 (12 marks)

(a) $\Sigma d = 10.80$, $\Sigma d^2 = 18.56$; $\bar{d} = 1.080$, $s_d = 0.8753$

M1 A1

$$t = \frac{1.080}{\frac{0.8753}{\sqrt{10}}} = 3.902$$

M1 A1

$\nu = 9$, critical values ± 2.262 ; $|t| = 3.902 > 2.262$, so reject H_0

B1 A1

Answer: $t = 3.90$; reject H_0

6 marks

(b) $1.080 \pm 2.262 \times \frac{0.8753}{\sqrt{10}}$

M1 A1

(0.454, 1.71)

A1

Answer: (0.454, 1.71)

3 marks

(c) Rejection occurs iff $|\frac{\bar{d}}{s_d/\sqrt{n}}| > t_9$

M1

This is equivalent to $|\bar{d}| > t_9 \frac{s_d}{\sqrt{n}}$

A1

That is, $0 < \bar{d} - t_9 \frac{s_d}{\sqrt{n}}$ or $0 > \bar{d} + t_9 \frac{s_d}{\sqrt{n}}$, i.e. 0 is outside the interval

A1

Answer: H_0 rejected if and only if 0 is outside the interval

3 marks

Tests for Population Variances

1 (2 marks)

$H_0: \sigma^2 = 4$

B1

$H_1: \sigma^2 > 4$

B1

Answer: $H_0: \sigma^2 = 4$, $H_1: \sigma^2 > 4$

2 marks

2 (2 marks)

(a) χ^2 distribution with $n - 1$ degrees of freedom

B1

Answer: χ^2_{n-1}

1 mark

(b) $\nu = 15 - 1 = 14$

B1

Answer: 14

1 mark

3 (3 marks)

$$\chi^2 = \frac{19 \times 2.4}{2} = 22.80$$

M1 A1

Critical value ($\nu = 19$) is 30.14; $22.80 < 30.14$, so do not reject H_0

B1

Answer: $\chi^2 = 22.8$; do not reject H_0

3 marks

4 (2 marks)

The population is normally distributed

B1

(The test is sensitive to departures from normality)

B1

Answer: Normal population

2 marks

5 (3 marks)

$$F = \frac{12.5}{5.0} = 2.50$$

M1 A1

Degrees of freedom (10, 8)

B1

Answer: $F = 2.5$, (10, 8)

3 marks

6 (2 marks)

$$F_{8,10} (5\%) = 3.072$$

B1

$$\chi^2_{12} (5\%) = 21.03$$

B1

Answer: 3.072; 21.03

2 marks

7 (5 marks)

$$H_0: \sigma^2 = 0.0064, H_1: \sigma^2 > 0.0064$$

B1

$$\chi^2 = \frac{24 \times 0.0098}{0.0064} = 36.75$$

M1 A1

$\nu = 24$, critical value 36.41; $36.75 > 36.41$, so reject H_0

B1 A1

Answer: $\chi^2 = 36.8$; reject H_0

5 marks

8 (5 marks)

$$H_0: \sigma^2 = 2.0, H_1: \sigma^2 < 2.0$$

B1

$$\chi^2 = \frac{15 \times 1.1}{2.0} = 8.250$$

M1 A1

$\nu = 15$, lower 5% critical value = 7.261; $8.250 > 7.261$, so do not reject H_0

B1 A1

Answer: $\chi^2 = 8.25$; do not reject H_0

5 marks

9 (7 marks)

$$\Sigma x = 71.00, \Sigma x^2 = 636.40; s^2 = \frac{1}{7}(636.40 - \frac{71.00^2}{8}) = 0.8964$$

M1 A1

$$H_0: \sigma^2 = 1.0, H_1: \sigma^2 \neq 1.0; \chi^2 = \frac{7 \times 0.8964}{1.0} = 6.275$$

B1 M1 A1

$\nu = 7$, critical values 1.690 and 16.01; 6.275 lies between 1.690 and 16.01, so do not reject H_0

B1 A1

Answer: $\chi^2 = 6.28$; do not reject H_0

7 marks

10 (5 marks)

$$\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{19}; \text{ the 2.5\% and 97.5\% points are 8.907 and 32.85}$$

B1 B1

$$\text{Lower limit} = \frac{19 \times 6.5}{32.85} = 3.759$$

M1 A1

$$\text{Upper limit} = \frac{19 \times 6.5}{8.907} = 13.87$$

A1

Answer: (3.76, 13.9)

5 marks

11 (5 marks)

$$H_0: \sigma_1^2 = \sigma_2^2, H_1: \sigma_1^2 > \sigma_2^2$$

B1

$$F = \frac{9.6}{4.2} = 2.286$$

M1 A1

Degrees of freedom (9, 11), critical value 2.896; $2.286 < 2.896$, so do not reject H_0

B1 A1

Answer: $F = 2.29$; do not reject H_0

5 marks

12 (5 marks)

$$H_0: \sigma_1^2 = \sigma_2^2, H_1: \sigma_1^2 \neq \sigma_2^2$$

B1

$$F = \frac{\text{larger}}{\text{smaller}} = \frac{11.6}{3.8} = 3.053 \text{ with degrees of freedom (12, 8)}$$

M1 A1

Two-tailed, so use the 2.5% point: 4.200; $3.053 < 4.200$, so do not reject H_0

B1 A1

Answer: $F = 3.05$; do not reject H_0

5 marks

13 (7 marks)

$$s^2 = \frac{1}{14}(3105.0 - \frac{213.0^2}{15}) = 5.743$$

M1 A1

$$H_0: \sigma^2 = 2.5, H_1: \sigma^2 > 2.5; \chi^2 = \frac{14 \times 5.743}{2.5} = 32.16$$

B1 M1 A1

Critical value ($\nu = 14$) 23.68; $32.16 > 23.68$, so reject H_0

B1 A1

Answer: $\chi^2 = 32.2$; reject H_0

7 marks

14 (8 marks)(a) $H_0: \sigma = 2.0$ ($\sigma^2 = 4$), $H_1: \sigma > 2.0$

B1

$$\chi^2 = \frac{11 \times 4.4}{4} = 12.10$$

M1 A1

 $\nu = 11$, critical value 19.68; $12.10 < 19.68$, so do not reject H_0

B1 A1

Answer: $\chi^2 = 12.1$; do not reject H_0

5 marks

(b) Interval for σ^2 : $(\frac{11 \times 4.4}{21.92}, \frac{11 \times 4.4}{3.816}) = (2.208, 12.68)$

M1 A1

Take square roots: (1.49, 3.56)

A1

Answer: (1.49, 3.56)

3 marks

15 (8 marks)

$$\bar{x} = 20.190, s^2 = 0.7743$$

M1 A1

$$H_0: \sigma^2 = 0.5, H_1: \sigma^2 \neq 0.5; \chi^2 = \frac{9 \times 0.7743}{0.5} = 13.94$$

B1 M1 A1

 $\nu = 9$, critical values 2.700 and 19.02; 13.94 lies between 2.700 and 19.02, so do not reject H_0

B1 A1

$$\text{Two-tailed p-value} = 2 \times P(\chi^2_9 > 13.94) = 0.2491$$

A1

Answer: $\chi^2 = 13.9$; do not reject H_0 ; p = 0.2491

8 marks

16 (7 marks)

$$s_A^2 = 0.9070, s_B^2 = 0.05278$$

M1 A1

$$H_0: \sigma_A^2 = \sigma_B^2, H_1: \sigma_A^2 > \sigma_B^2; F = \frac{0.9070}{0.05278} = 17.18$$

B1 M1 A1

Degrees of freedom (7, 8), critical value 3.500; $17.18 > 3.500$, so reject H_0

B1 A1

Answer: F = 17.2; reject H_0

7 marks

17 (10 marks)

$$(a) F = \frac{6.8}{5.2} = 1.308 \text{ with degrees of freedom } (9, 11)$$

M1 A1

Upper 5% point = 2.896; $1.308 < 2.896$, so do not reject H_0

B1 A1

Answer: F = 1.31; do not reject H_0

4 marks

$$(b) s_p^2 = \frac{9 \times 6.8 + 11 \times 5.2}{20} = 5.920$$

M1 A1

$$t = \frac{41.5 - 38.2}{\sqrt{5.920(1/10 + 1/12)}} = 3.168$$

M1 A1

 $\nu = 20$, critical values ± 2.086 ; reject H_0

B1 A1

Answer: t = 3.17; reject H_0

6 marks

18 (4 marks)

$$\text{Under } H_0, \frac{17s^2}{0.5} \sim \chi^2_{17}$$

B1

Critical value of χ^2_{17} at 5% is 27.59

B1

$$\text{Reject } H_0 \text{ if } \frac{17s^2}{0.5} > 27.59, \text{ so } s^2 > \frac{0.5 \times 27.59}{17} = 0.8114$$

M1 A1

Answer: $s^2 > 0.8114$

4 marks

19 (6 marks)

$$s_p^2 = \frac{7 \times 2.6 + 10 \times 3.4}{17} = 3.071$$

M1 A1

$$\frac{17s_p^2}{2.0} \sim \chi^2_{17} \text{ under } H_0; \chi^2 = 26.10$$

M1 A1

Critical value 27.59; $26.10 < 27.59$, so do not reject H_0

B1 A1

Answer: $\chi^2 = 26.1$; do not reject H_0

6 marks

20 (7 marks)

$$s^2 = \frac{1}{13}(11\,401.0 - \frac{392.0^2}{14}) = 32.69$$

M1 A1

$$H_0: \sigma^2 = 4.0, H_1: \sigma^2 \neq 4.0; \chi^2 = \frac{13 \times 32.69}{4.0} = 106.2$$

B1 M1 A1

$\nu = 13$, critical values 5.009 and 24.74; $106.2 > 24.74$, so reject H_0

B1 A1

Answer: $\chi^2 = 106$; reject H_0

7 marks

21 (8 marks)

(a) Reject if $\frac{19s^2}{5.0} > \chi^2_{19}(5\%) = 30.14$

M1 A1

$$s^2 > \frac{5.0 \times 30.14}{19} = 7.933$$

A1

Answer: $s^2 > 7.933$

3 marks

(b) When $\sigma^2 = 10$, $\frac{19s^2}{10} \sim \chi^2_{19}$

B1

$$P(s^2 > 7.933) = P(\chi^2_{19} > \frac{19 \times 7.933}{10}) = P(\chi^2_{19} > 15.07)$$

M1 A1

$$= 0.7180$$

A1

Answer: 0.7180

4 marks

(c) The power would increase, since a larger sample estimates σ^2 more precisely and so separates the two variances better

E1

Answer: Increases

1 mark

22 (6 marks)

Error 1: the statistic uses the variances: $\frac{(n-1)s^2}{\sigma_0^2}$, not $(ns)/\sigma_0$

E1

Error 2: $n - 1 = 12$ observations' worth of degrees of freedom, so $\nu = 12$, not 13

E1

Error 3: critical value for $\nu = 12$ is 21.03

E1

$$\chi^2 = \frac{12 \times 4.1^2}{3^2} = 22.41$$

M1 A1

$22.41 > 21.03$, so reject H_0

A1

Answer: $\chi^2 = 22.4$; reject H_0

6 marks

23 (7 marks)

(a) Lower 5% point of χ^2_{24} is 13.85

B1

$$k = \frac{24 \times 3.8}{13.85} = 6.586$$

M1 A1

Answer: $\sigma^2 < 6.586$

3 marks

(b) $6.0 < 6.586$, so H_0 is not rejected

M1 A1

$$\chi^2 = \frac{24 \times 3.8}{6.0} = 15.20; \text{ lower 5\% value } 13.85; 15.20 > 13.85, \text{ so do not reject } H_0$$

M1 A1

Answer: H_0 not rejected

4 marks

24 (9 marks)

(a) $\chi^2 = \frac{99 \times 1.45}{1} = 143.55$

M1 A1

$$z = \sqrt{2 \times 143.55} - \sqrt{197} = 2.908$$

M1 A1

Answer: $\chi^2 = 143.55$; $z = 2.908$

4 marks

(b) $2.908 > 1.645$, so reject H_0

M1 A1

Answer: reject H_0

2 marks

(c) Approximate p-value = $P(Z > 2.908) = 0.0018$

M1 A1

The two p-values are close, and lead to the same conclusion

E1

Answer: 0.0018 vs 0.0023

3 marks

25 (8 marks)

(a) If $F \sim F_{7,11}$ then $\frac{1}{F} \sim F_{11,7}$

M1

$P(F < c) = 0.05 \Leftrightarrow P(\frac{1}{F} > \frac{1}{c}) = 0.05$, so $\frac{1}{c}$ is the upper 5% point of $F_{11,7}$

A1

Answer: Reciprocal relation

2 marks

(b) Upper 5% point of $F_{11,7} = 3.603$

B1

Lower critical value = $\frac{1}{3.603} = 0.2775$

M1 A1

Answer: 0.2775

3 marks

(c) $F = \frac{2.1}{6.3} = 0.3333$

B1

$0.3333 > 0.2775$, so do not reject H_0

M1 A1

Answer: do not reject H_0

3 marks

26 (6 marks)

$$\chi^2 = \frac{(n-1) \times 1.5\sigma_0^2}{\sigma_0^2} = 1.5(n-1) = 1.5\nu$$

M1 A1

Need $1.5\nu > \chi^2_{\nu}(5\%)$: $\nu = 25$: 37.50 against 37.65, which fails

M1 A1

$\nu = 26$: 39.00 against 38.88, which succeeds

A1

So $n = 27$

A1

Answer: 27

6 marks

27 (10 marks)

(a) $E(\frac{(n-1)s^2}{\sigma^2}) = n-1$

M1

So $\frac{n-1}{\sigma^2}E(s^2) = n-1$ and $E(s^2) = \sigma^2$

A1

Answer: $E(s^2) = \sigma^2$

2 marks

(b) $\text{Var}(\frac{(n-1)s^2}{\sigma^2}) = 2(n-1)$

B1

$$\frac{(n-1)^2}{\sigma^4} \text{Var}(s^2) = 2(n-1)$$

M1

$$\text{Var}(s^2) = \frac{2\sigma^4}{n-1}$$

A1

Answer: $\text{Var}(s^2) = \frac{2\sigma^4}{n-1}$

3 marks

(c) $\text{sd}(s^2) = \sigma^2\sqrt{2/n-1} = \sigma^2\sqrt{2/50} = 0.200\sigma^2$

M1 A1

$\sqrt{2/n-1} < 0.1$ gives $n-1 > 200$

M1 A1

$n \geq 202$

A1

Answer: $0.2\sigma^2$; $n \geq 202$

5 marks

28 (11 marks)

(a) Upper/lower = $\frac{(n-1)s^2}{b} \div \frac{(n-1)s^2}{a}$

M1

$$= \frac{a}{b}$$

A1

Answer: Ratio = $\frac{a}{b}$

2 marks

(b) $a = 19.02$, $b = 2.700$ for $\nu = 9$; ratio = 7.05

B1 M1 A1

The upper limit is more than four times the lower limit, so a sample of 10 estimates the variance very imprecisely

E1

Answer: 7.05

4 marks

(c) Need $\frac{a}{b} \leq 2$; $n = 65$ ($\nu = 64$) gives $2.010 > 2$

M1 A1

$n = 66$ ($\nu = 65$) gives $1.999 \leq 2$

A1

ratio decreases as ν increases, so smallest $n = 66$

A1 E1

Answer: 66**5 marks****29 (11 marks)**

(a) Under the claim, $\frac{\sigma_1^2}{\sigma_2^2} = 2$, so $\frac{s_1^2}{\sigma_1^2} \div \frac{s_2^2}{\sigma_2^2} = \frac{s_1^2}{s_2^2} \times \frac{\sigma_2^2}{\sigma_1^2} = \frac{s_1^2}{2s_2^2}$

M1 A1

This has the $F_{11,9}$ distribution

A1

Answer: $\frac{s_1^2}{2s_2^2} \sim F_{11,9}$

3 marks

(b) $F = \frac{7.5}{2 \times 2.1} = 1.786$

M1 A1

Critical value $F_{11,9}$ (5%) = 3.102; $1.786 < 3.102$, so do not reject H_0

B1 A1

Answer: $F = 1.79$; do not reject H_0

4 marks

(c) $\frac{\sigma_1^2}{\sigma_2^2}$ lies between $\frac{s_1^2}{s_2^2} \div F_{11,9}(2.5\%)$ and $\frac{s_1^2}{s_2^2} \times F_{9,11}(2.5\%)$

M1

$F_{11,9}(2.5\%) = 3.912$, $F_{9,11}(2.5\%) = 3.588$

B1 B1

Interval (0.913, 12.8)

A1

Answer: (0.913, 12.8)

4 marks**30 (10 marks)**

(a) Critical value $F_{9,11}$ (5%) = 2.896

B1

$\frac{s_1^2}{\sigma_1^2} \div \frac{s_2^2}{\sigma_2^2} \sim F_{9,11}$ and $\frac{\sigma_1^2}{\sigma_2^2} = r$, so $\frac{F}{r} \sim F_{9,11}$

M1 A1

Answer: Critical value 2.896; $\frac{F}{r} \sim F_{9,11}$

3 marks

(b) Power = $P(F > 2.896) = P(\frac{F}{3} > \frac{2.896}{3}) = P(F_{9,11} > 0.9653)$

M1 A1

= 0.5130

A1

Answer: 0.5130

3 marks

(c) Need $P(F_{9,11} > \frac{2.896}{r}) = 0.9$, so $\frac{2.896}{r}$ is the lower 10% point of $F_{9,11}$, 0.4173

M1 A1

$r = \frac{2.896}{0.4173} = 6.9$

M1 A1

Answer: 6.9

4 marks**Non-Parametric Tests****1 (3 marks)**

(a) They make no assumption about the shape of the population distribution (distribution-free)

B1

They can be used with ordinal (ranked) data and with small samples

B1

Answer: Distribution-free; ordinal data

2 marks

(b) They are generally less powerful than the equivalent parametric test when the parametric assumptions hold

B1

Answer: Less powerful

1 mark**2 (2 marks)**

$H_0: m = 30$

B1

$H_1: m > 30$

B1

Answer: $H_0: m = 30$, $H_1: m > 30$

2 marks**3 (2 marks)**

Above 50: 52, 55, 58, 61, 57, so 5 positive signs

B1

Below 50: 3 negative signs; equal to 50: 2 observations discarded

B1

Answer: 5 positive, 3 negative, 2 discarded

2 marks

4 (3 marks)

Ordered: 5, 9, 9, 14, 17, 21

M1

The two 9s share ranks 2 and 3, so each has rank 2.5

A1

Ranks in the given order: 4, 2.5, 6, 2.5, 5, 1

A1

Answer: 4, 2.5, 6, 2.5, 5, 1

3 marks

5 (2 marks)Total of all ranks = $\frac{8 \times 9}{2} = 36$

M1

Negative rank sum = $36 - 26 = 10$

A1

Answer: 10

2 marks

6 (3 marks) $U_A = 33 - \frac{6 \times 7}{2} = 12$

M1 A1

 $U_B = mn - U_A = 42 - 12 = 30$, and $U_A + U_B = mn$

A1

Answer: $U_A = 12$, $U_B = 30$

3 marks

7 (6 marks) $H_0: m = 15.0$, $H_1: m > 15.0$

B1

Positive signs: 7, negative signs: 3; two values equal 15.0 are discarded, so $n = 10$

M1 A1

Under H_0 , $S \sim B(10, 0.5)$; $P(S \geq 7) = 0.1719$

M1 A1

 $0.1719 > 0.05$, so do not reject H_0

A1

Answer: $p = 0.1719$; do not reject H_0

6 marks

8 (6 marks) H_0 : median difference is 0, H_1 : median difference is not 0

B1

Differences (after – before): 4, 4, –1, 5, 5, 5, 1, 5, –1, 7; 8 positive, 2 negative, $n = 10$

M1 A1

Under H_0 , $S \sim B(10, 0.5)$; two-tailed p-value = $2 \times P(S \leq 2) = 0.1094$

M1 A1

 $0.1094 > 0.05$, so do not reject H_0

A1

Answer: $p = 0.1094$; do not reject H_0

6 marks

9 (6 marks)Differences: –7, 7, –1, 16, 2, 10, –3, 13, 4; ranks of $|d|$: 5.5, 5.5, 1, 9, 2, 7, 3, 8, 4

M1 A1

 $T_+ = 35.5$, $T_- = 9.5$ (total 45)

A1

Test statistic $T = T_- = 9.5$; critical value for $n = 9$ (one-tailed 5%) is 8

B1 B1

 $9.5 > 8$, so do not reject H_0

A1

Answer: $T = 9.5$; do not reject H_0

6 marks

10 (6 marks)

Differences (before – after): 0.6, 0.2, 1.3, 0.6, –0.1, 0.7, 0.4, 0.8

B1

Ranks of $|d|$: 4, 2, 8, 5, 1, 6, 3, 7; $T_+ = 35$, $T_- = 1$

M1 A1

 $T = 1$; two-tailed 5% critical value for $n = 8$ is 3

B1 B1

 $1 \leq 3$, so reject H_0

A1

Answer: $T = 1$; reject H_0

6 marks

11 (6 marks) H_0 : the distributions are the same, H_1 : A tends to be smaller than B

B1

Ranks of A in the combined sample: 2, 8, 1, 4, 3; $R_A = 18$

M1 A1

 $U_A = 18 - \frac{5 \times 6}{2} = 3$

M1

Critical value for (5, 6), one-tailed 5%: 5; $3 \leq 5$, so reject H_0

B1 A1

Answer: $U_A = 3$; reject H_0

6 marks

12 (4 marks)

$$S \sim B(40, 0.5) \approx N(20, 10)$$

B1

$$P(S \geq 28) \approx P(Z > \frac{27.5 - 20}{\sqrt{10}}) = P(Z > 2.372)$$

M1 A1

$$= 0.0089 < 0.05, \text{ so reject } H_0$$

A1

Answer: $p \approx 0.0089$; reject H_0

4 marks

13 (3 marks)

(a) The t-test assumes the differences are normally distributed, which is doubtful with a small skewed sample containing an outlier

E1

Answer: Non-normal differences

1 mark

(b) Sign test or Wilcoxon signed-rank test

B1

Wilcoxon uses the sizes of the differences (through ranks) as well as their signs, so it is more powerful

E1

Answer: Wilcoxon signed-rank

2 marks

14 (8 marks)

Differences from 33: -2, 3, -5, 7, 3, 0, -8, 5, 3, -4, 9; the zero difference is discarded, leaving $n = 10$

B1 B1

Ranks of $|d|$ with ties given the average rank: 1, 3, 6.5, 8, 3, 9, 6.5, 3, 5, 10

M1 A1

$T_+ = 33.5$, $T_- = 21.5$, $T = 21.5$; critical value ($n = 10$, two-tailed 5%) = 8

M1 A1 B1

$21.5 > 8$, so do not reject H_0

A1

Answer: $T = 21.5$; do not reject H_0

8 marks

15 (7 marks)

H_0 : the distributions are the same, H_1 : they differ

B1

Combined ranks of A (with 29 tied three ways): 1, 5, 8, 3, 11, 5, 2; $R_A = 35$

M1 A1

$U_A = 35 - 28 = 7$, $U_B = 56 - 7 = 49$; $U = 7$

M1 A1

Critical value (7, 8), two-tailed 5%: 10; $7 \leq 10$, so reject H_0

B1 A1

Answer: $U = 7$; reject H_0

7 marks

16 (5 marks)

$$\text{Mean} = \frac{25 \times 26}{4} = 162.5; \text{variance} = \frac{25 \times 26 \times 51}{24} = 1381.2$$

B1 B1

$$z = \frac{82.5 - 162.5}{\sqrt{1381.2}} = -2.153$$

M1 A1

$P(Z < -2.153) = 0.0157 < 0.05$, so reject H_0

A1

Answer: $z = -2.15$; reject H_0

5 marks

17 (5 marks)

$$\text{Mean} = \frac{12 \times 15}{2} = 90; \text{variance} = \frac{12 \times 15 \times 28}{12} = 420$$

B1 B1

$$z = \frac{52.5 - 90}{\sqrt{420}} = -1.830$$

M1 A1

$P(Z < -1.830) = 0.0336 < 0.05$, so reject H_0 : sample 1 tends to be smaller

A1

Answer: $z = -1.83$; reject H_0

5 marks

18 (9 marks)

(a) Positive signs: 5 out of 8; under H_0 , $S \sim B(8, 0.5)$

B1 M1

$P(S \geq 5) = 0.3633 > 0.05$, so do not reject H_0

A1 A1

Answer: $p = 0.3633$; do not reject H_0

4 marks

(b) Differences: -0.4, 0.8, -0.7, 1.5, 0.4, 1.1, -0.1, 0.6; ranks of $|d|$: 2.5, 6, 5, 8, 2.5, 7, 1, 4

M1 A1

$T_- = 8.5$, critical value ($n = 8$, one-tailed 5%) = 5; $8.5 > 5$, so do not reject H_0

B1 A1

The Wilcoxon test gives the same conclusion as the sign test, though it is the more powerful test, because it uses the magnitudes of the differences as well as their signs

E1

Answer: Sign: do not reject H_0 ; Wilcoxon: do not reject H_0

5 marks

19 (8 marks)**(a)** $T_+ = 0$: none; $T_+ = 1$: {1}; $T_+ = 2$: {2}

M1 A1

There are $2^4 = 16$ equally likely sign patterns, so $P(T_+ \leq 2) = \frac{3}{16} = 0.1875$

M1 A1

Answer: 3/16

4 marks

(b) $P(T_+ \leq 0) = \frac{1}{16} = 0.0625 \leq 0.10$

M1 A1

 $P(T_+ \leq 1) = \frac{2}{16} = 0.1250 > 0.10$

A1

Critical region $T_+ = 0$, with actual significance level 0.0625

A1

Answer: $T_+ = 0$; 0.0625

4 marks

20 (6 marks)Under H_0 , $S \sim B(15, 0.5)$

B1

 $P(S \leq 3) = 0.01758 \leq 0.025$, $P(S \leq 4) = 0.05923 > 0.025$

M1 A1

By symmetry, critical region $S \leq 3$ or $S \geq 12$; actual significance level $2 \times 0.01758 = 0.0352$

M1 A1

 $S = 12$ is in the critical region, so reject H_0

A1

Answer: $S \leq 3$ or $S \geq 12$; 0.0352

6 marks

21 (7 marks)The subtraction term must use the size $m = 4$ of sample A, $\frac{m(m+1)}{2} = 10$, not the size of sample

E1 B1

B

A negative value of U is impossible, which signals the error

E1

Ranks: 9(A) 1, 12(A) 2, 14(B) 3, 15(A) 4, 17(B) 5, 19(B) 6, 20(A) 7, 22(B) 8, 25(B) 9

M1

 $U_A = 14 - 10 = 4$, and $U_B = 20 - 4 = 16$

A1

Critical value for (4, 5), one-tailed 5%: 2; $4 > 2$ so do not reject H_0 : no evidence that A tends to be smaller

B1 A1

Answer: $U_A = 4$; do not reject H_0

7 marks

22 (7 marks)

Combined ranks: method 1: 3, 7, 11, 4, 6; method 2: 9, 13, 16, 12, 15, 14; method 3: 5, 8, 2, 1, 10

M1 A1

Rank sums 31, 79, 26 (total $136 = \frac{16 \times 17}{2}$)

B1

$$H = \frac{12}{16 \times 17} \left(\frac{31^2}{5} + \frac{79^2}{6} + \frac{26^2}{5} \right) - 51 = 9.334$$

M1 A1

 $\nu = 3 - 1 = 2$, critical value 5.991; $9.334 > 5.991$, so reject H_0

B1 A1

Answer: $H = 9.33$; reject H_0

7 marks

23 (10 marks)**(a)** $R_A - 3$ for each arrangement: AABBB:0, ABABB:1, ABBAB:2, ABBBA:3, BAABB:2,

M1 A1 A1

BABAB:3, BABBA:4, BBAAB:4, BBABA:5, BBBAA:6

Answer: AABBB: 0; ABABB: 1; ABBAB: 2; ABBBA: 3; BAABB: 2; BABAB: 3; BABBA: 4; BBAAB: 4; BBABA: 5; BBBAA: 6**(b)** $P(U_A = 0, 1, 2, 3, 4, 5, 6) = 1/10, 1/10, 2/10, 2/10, 2/10, 1/10, 1/10$

M1 A1

$$E(U_A) = 3.00 = 3 = \frac{2 \times 3}{2}$$

M1 A1

Answer: $E(U) = 3$

4 marks

(c) $P(U_A \leq 0) = 0.10 \leq 0.10$, $P(U_A \leq 1) = 0.20 > 0.10$

M1 A1

Critical region $U_A = 0$, actual significance level 0.1

A1

Answer: $U_A = 0$; 0.1

3 marks

24 (9 marks)

- (a) Each observation is below m with probability 0.5, independently, so the number below m is $S \sim B(12, 0.5)$ M1
- $x_{(k)} < m < x_{(13-k)}$ fails only if at most $k - 1$ observations are below m or at most $k - 1$ are above m (two disjoint events), each with probability $P(S \leq k - 1)$ E1 A1
- Answer: $P = 1 - 2P(S \leq k - 1)$** 3 marks
- (b) $P(S \leq 2) = 0.01929$, so confidence $= 1 - 2 \times 0.01929 = 0.9614 \geq 0.95$ M1 A1
- Check $k = 4$: confidence $0.8540 < 0.95$ B1
- Largest $k = 3$ with confidence 0.9614 A1
- Answer: $k = 3$; confidence 0.9614** 4 marks
- (c) $x_{(3)} = 20.8$ and $x_{(10)} = 26.5$ B1 B1
- Answer: (20.8, 26.5)** 2 marks

25 (8 marks)

- (a) Each of the 7 ranks is positive or negative with probability $\frac{1}{2}$, independently, so there are $2^7 = 128$ patterns M1 A1
- Answer: $2^7 = 128$** 2 marks
- (b) Sets of positive ranks with sum ≤ 5 : $\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}$ M1 A1 A1
- That is 10 patterns, so p-value $= \frac{10}{128} = 0.07812$ A1
- Answer: 0.07812** 4 marks
- (c) $0.0781 > 0.05$, so do not reject H_0 M1
- Table critical value: largest c with $P(T \leq c) \leq 0.05$ is $c = 3$ A1
- Answer: $c = 3$** 2 marks

26 (7 marks)

- (a) Number of ways of choosing the 4 ranks for sample A from 9 is $C(9, 4) = \frac{9 \times 8 \times 7 \times 6}{4!} = 126$ M1 A1
- Answer: 126** 2 marks
- (b) $P(U \leq 3) = \frac{1 + 1 + 2 + 3}{126}$ M1
- $= \frac{7}{126} = 0.0556$ A1
- Answer: 0.0556** 2 marks
- (c) $0.0556 > 0.05$, so do not reject H_0 M1 A1
- The sample sizes are too small for the normal approximation to the discrete, short-tailed distribution of U E1
- Answer: do not reject H_0** 3 marks

27 (10 marks)

- (a) $E(I_k) = \frac{1}{2}$, so $E(T_+) = \sum k \times \frac{1}{2}$ M1
- $= \frac{1}{2} \times \frac{n(n+1)}{2} = \frac{n(n+1)}{4}$ A1
- Answer: $E(T_+) = \frac{n(n+1)}{4}$** 2 marks
- (b) $\text{Var}(I_k) = \frac{1}{4}$ B1
- $\text{Var}(T_+) = \sum k^2 \times \frac{1}{4}$ (independence) M1
- $= \frac{1}{4} \times \frac{n(n+1)(2n+1)}{6} = \frac{n(n+1)(2n+1)}{24}$ A1
- Answer: $\text{Var}(T_+) = \frac{n(n+1)(2n+1)}{24}$** 3 marks
- (c) Mean 27.5; variance $\frac{10 \times 11 \times 21}{24} = 96.25$; sd = 9.811 M1 A1
- $\frac{c + 0.5 - 27.5}{9.811} \leq -1.645$ gives $c \leq 10.86$, so $c = 10$ M1 A1

Exact tabulated value is 10; the approximation is equal to it

Answer: c = 10 (exact 10)

A1

5 marks

28 (12 marks)

(a) $U = 0$: A occupies ranks 1, 2, 3 only, so 1 arrangement

M1

$U = 1$: ranks 1, 2, 4 only, 1 arrangement

A1

$U = 2$: ranks {1,2,5} or {1,3,4}: $P = \frac{2}{35}$; $U = 3$: ranks {1,2,6}, {1,3,5}, {2,3,4}: $P = \frac{3}{35}$

M1 A1

Answer: 1/35, 1/35, 2/35, 3/35

4 marks

(b) Counts for $U = 0, 1, 2, 3$ are 1, 1, 2, 3

B1

$P(U \leq 2) = \frac{4}{35} = 0.1143$; $P(U \leq 3) = \frac{7}{35} = 0.2000$

M1 A1

Largest region with probability ≤ 0.05 : $U \leq 0$, actual significance level 0.0286

A1 A1

Answer: $U \leq 0$; 0.0286

5 marks

(c) $E(U) = \frac{mn}{2} = 6$ by symmetry

E1

The smallest possible $P(U \leq u)$ is $P(U = 0) = \frac{1}{35} = 0.0286$, which exceeds 0.01

E1 A1

Answer: Minimum p-value 1/35

3 marks

29 (11 marks)

(a) $S \sim B(20, 0.5)$; $P(S \geq 15) = 0.0207 \leq 0.05$, $P(S \geq 14) = 0.0577 > 0.05$

M1 A1

Critical region $S \geq 15$, significance level 0.0207

A1

Answer: $S \geq 15$; 0.0207

3 marks

(b) $P(X > 0) = P(Z > -0.5) = 0.6915$

M1 A1

Power = $P(S \geq 15 | B(20, 0.6915)) = 0.3841$

M1 A1

Answer: 0.3841

4 marks

(c) Power = $P(Z > 1.645 - \frac{0.5\sigma}{\sigma/\sqrt{20}}) = P(Z > 1.645 - 0.5\sqrt{20}) = P(Z > -0.5911)$

M1 A1

= 0.7228, which is higher than the sign test's 0.3841: the z-test uses more information, but the sign test needs no distributional assumption

A1 E1

Answer: 0.7228

4 marks

30 (12 marks)

(a) 22 values; the median is the mean of the 11th and 12th, $\frac{56 + 57}{2} = 56.5$

M1 A1

Above: A 2, B 9; below: A 8, B 3

A1

Answer: Median 56.5; above: A 2, B 9; below: A 8, B 3

3 marks

(b) Row totals 10, 12; column totals 11, 11, $N = 22$; $E = (5.000, 5.000)$; $(6.000, 6.000)$

M1 A1

$\chi^2 = 6.600$

M1 A1

$\nu = 1$, critical value 3.841; $6.600 > 3.841$, so reject H_0

B1 A1

Answer: $\chi^2 = 6.60$; reject H_0

6 marks

(c) Corrected $\chi^2 = 4.583$

M1 A1

$4.583 > 3.841$, so the conclusion is unchanged

A1

Answer: Corrected $\chi^2 = 4.58$

3 marks