

Conic Sections

1 (3 marks)

Complete the square: $(x - 3)^2 + (y + 2)^2 = 3 + 9 + 4$
 $= 16$

M1

A1

Centre $(3, -2)$, radius 4

A1

Answer: Centre $(3, -2)$, radius 4

3 marks

2 (1 mark)

$(x - 1)^2 + (y + 2)^2 = 9$

B1

Answer: $(x - 1)^2 + (y + 2)^2 = 9$

1 mark

3 (3 marks)

(a) $0 < e < 1$: an ellipse

B1

Answer: Ellipse

1 mark

(b) $e = 1$: a parabola

B1

Answer: Parabola

1 mark

(c) $e > 1$: a hyperbola

B1

Answer: Hyperbola

1 mark

4 (3 marks)

(a) Sum of two squares equal to 1 with different denominators: ellipse

B1

Answer: Ellipse

1 mark

(b) Only one variable squared: parabola

B1

Answer: Parabola

1 mark

(c) Difference of squares equal to 1: hyperbola

B1

Answer: Hyperbola

1 mark

5 (3 marks)

Point (x, y) with distance to focus = distance to directrix: $\sqrt{(x - 3)^2 + y^2} = x + 3$

M1

$(x - 3)^2 + y^2 = (x + 3)^2$

M1

$y^2 = 12x$

A1

Answer: $y^2 = 12x$

3 marks

6 (2 marks)

$1^2 + (\sqrt{3})^2 = 1 + 3 = 4$

M1

Radius 2

A1

Answer: Radius 2

2 marks

7 (4 marks)

Angle AOB = 90° , so AB is a diameter

M1

Centre = midpoint of AB = $(2, 3)$

A1

Radius² = $2^2 + 3^2 = 13$

M1

$(x - 2)^2 + (y - 3)^2 = 13$

A1

Answer: $(x - 2)^2 + (y - 3)^2 = 13$

4 marks

8 (4 marks)

Substitute: $x^2 + (x + 1)^2 = 13$

M1

$2x^2 + 2x - 12 = 0$, so $x^2 + x - 6 = 0$

A1

$(x + 3)(x - 2) = 0$, $x = -3$ or 2

M1

A $(2, 3)$, B $(-3, -2)$

A1

Answer: $(2, 3)$ and $(-3, -2)$

4 marks

9 (3 marks)

$$\text{Gradient of the radius} = \frac{4}{3} \quad \text{B1}$$

$$\text{Gradient of the tangent} = -\frac{3}{4} \quad \text{M1}$$

$$y - 4 = -\frac{3}{4}(x - 3), \text{ so } 3x + 4y = 25 \quad \text{A1}$$

$$\text{Answer: } 3x + 4y = 25 \quad \text{3 marks}$$

10 (4 marks)

$$\text{Centre} = \text{midpoint of AB} = (2, 4) \quad \text{M1 A1}$$

$$\text{Radius}^2 = (2 + 1)^2 + (4 - 2)^2 = 13 \quad \text{M1}$$

$$(x - 2)^2 + (y - 4)^2 = 13 \quad \text{A1}$$

$$\text{Answer: } (x - 2)^2 + (y - 4)^2 = 13 \quad \text{4 marks}$$

11 (4 marks)

$$y^2 = 4ax \text{ with } a = 4 \quad \text{M1}$$

$$\text{Focus } (4, 0), \text{ directrix } x = -4 \quad \text{A1 A1}$$

$$y^2 = 144, \text{ so } y = \pm 12 \quad \text{A1}$$

$$\text{Answer: Focus } (4, 0); x = -4; y = \pm 12 \quad \text{4 marks}$$

12 (4 marks)

$$\text{Substitute: } x^2 + (2x + k)^2 = 5, \text{ so } 5x^2 + 4kx + k^2 - 5 = 0 \quad \text{M1 A1}$$

$$\text{Tangent: discriminant zero, } 16k^2 - 20(k^2 - 5) = 0 \quad \text{M1}$$

$$k^2 = 25, k = \pm 5 \quad \text{A1}$$

$$\text{Answer: } k = \pm 5 \quad \text{4 marks}$$

13 (4 marks)

$$x^2 + 4(x + 1)^2 = 4 \quad \text{M1}$$

$$5x^2 + 8x = 0, \text{ so } x(5x + 8) = 0 \quad \text{M1 A1}$$

$$(0, 1) \text{ and } \left(-\frac{8}{5}, -\frac{3}{5}\right) \quad \text{A1}$$

$$\text{Answer: } (0, 1), \left(-\frac{8}{5}, -\frac{3}{5}\right) \quad \text{4 marks}$$

14 (5 marks)

$$PF = \frac{1}{2} \times |x - 1|, \text{ so } 4[(x - 4)^2 + y^2] = (x - 1)^2 \quad \text{M1 A1}$$

$$4x^2 - 32x + 64 + 4y^2 = x^2 - 2x + 1 \quad \text{M1}$$

$$3x^2 - 30x + 4y^2 + 63 = 0, \text{ i.e. } 3(x - 5)^2 + 4y^2 = 12 \quad \text{A1}$$

$$\frac{(x - 5)^2}{4} + \frac{y^2}{3} = 1, \text{ an ellipse} \quad \text{A1}$$

$$\text{Answer: } \frac{(x - 5)^2}{4} + \frac{y^2}{3} = 1 \quad \text{5 marks}$$

15 (5 marks)

$$PF^2 = 4(x - 1)^2: (x - 4)^2 + y^2 = 4(x - 1)^2 \quad \text{M1 A1}$$

$$x^2 - 8x + 16 + y^2 = 4x^2 - 8x + 4 \quad \text{M1}$$

$$3x^2 - y^2 = 12 \quad \text{A1}$$

$$\frac{x^2}{4} - \frac{y^2}{12} = 1, \text{ a hyperbola} \quad \text{A1}$$

$$\text{Answer: } \frac{x^2}{4} - \frac{y^2}{12} = 1 \quad \text{5 marks}$$

16 (5 marks)

$$4(x^2 - 4x) + 9(y^2 + 2y) = 11$$

M1

$$4(x - 2)^2 - 16 + 9(y + 1)^2 - 9 = 11$$

M1 A1

$$\frac{(x - 2)^2}{9} + \frac{(y + 1)^2}{4} = 1: \text{an ellipse}$$

A1

Centre (2, -1), semi-axes 3 (horizontal) and 2 (vertical)

A1

Answer: Centre (2, -1), semi-axes 3 and 2

5 marks

17 (4 marks)

$$(y - 2)^2 - 4 - 8x + 20 = 0, \text{ so } (y - 2)^2 = 8x - 16 = 8(x - 2)$$

M1 A1

Vertex (2, 2)

A1

4a = 8, a = 2, so the focus is (4, 2)

A1

Answer: Vertex (2, 2), focus (4, 2)

4 marks

18 (5 marks)Centre (r, r), radius r: $(x - r)^2 + (y - r)^2 = r^2$

B1

$$(1 - r)^2 + (2 - r)^2 = r^2$$

M1

$$r^2 - 6r + 5 = 0, \text{ so } r = 1 \text{ or } r = 5$$

M1 A1

$$(x - 1)^2 + (y - 1)^2 = 1 \text{ and } (x - 5)^2 + (y - 5)^2 = 25$$

A1

Answer: $(x - 1)^2 + (y - 1)^2 = 1$; $(x - 5)^2 + (y - 5)^2 = 25$

5 marks

19 (5 marks)Centre (1, -2), radius $2\sqrt{2}$

B1

$$\text{Line: } x - y + c = 0; \text{ distance from centre} = \frac{|1 + 2 + c|}{\sqrt{2}}$$

M1 A1

$$\frac{|3 + c|}{\sqrt{2}} = 2\sqrt{2}, \text{ so } |3 + c| = 4$$

M1

$$c = 1 \text{ or } c = -7$$

A1

Answer: $c = 1$ or $c = -7$

5 marks

20 (5 marks)

$$(x - 3)^2 = 4x, \text{ so } x^2 - 10x + 9 = 0$$

M1 A1

$$x = 1 \text{ or } 9$$

A1

A(1, -2), B(9, 6)

A1

$$AB = \sqrt{8^2 + 8^2} = 8\sqrt{2}$$

A1

Answer: $8\sqrt{2}$

5 marks

21 (5 marks)

$$PA^2 = 4PB^2: x^2 + y^2 = 4[(x - 3)^2 + y^2]$$

M1 A1

$$3x^2 - 24x + 3y^2 + 36 = 0, \text{ so } x^2 - 8x + y^2 + 12 = 0$$

M1

$$(x - 4)^2 + y^2 = 4$$

A1

Circle, centre (4, 0), radius 2

A1

Answer: Circle, centre (4, 0), radius 2

5 marks

22 (6 marks)

$$PF = e \times (\text{distance from } x = -1): (x - 2)^2 + (y - 3)^2 = \frac{9}{4}(x + 1)^2$$

M1 A1

$$4(x^2 - 4x + 4) + 4(y^2 - 6y + 9) = 9(x^2 + 2x + 1)$$

M1

$$4x^2 - 16x + 16 + 4y^2 - 24y + 36 = 9x^2 + 18x + 9$$

A1

$$5x^2 + 34x - 4y^2 + 24y - 43 = 0$$

A1

 $e > 1$, so it is a hyperbola

B1

Answer: Hyperbola

6 marks

23 (6 marks)

Perpendicular bisector of AB is $x = 3$, so the centre is $(3, k)$

M1 A1

Distance to A equals distance to C: $4 + (k - 1)^2 = (k - 5)^2$

M1

$$4 + k^2 - 2k + 1 = k^2 - 10k + 25, \text{ so } 8k = 20, k = \frac{5}{2}$$

A1

$$\text{Radius}^2 = 4 + \left(\frac{3}{2}\right)^2 = \frac{25}{4}$$

M1

$$(x - 3)^2 + \left(y - \frac{5}{2}\right)^2 = \frac{25}{4}$$

A1

$$\text{Answer: } (x - 3)^2 + \left(y - \frac{5}{2}\right)^2 = \frac{25}{4}$$

6 marks

24 (3 marks)

$$(x + 1)^2 + (y + 3)^2 = 1 + 9 - 20 = -10$$

M1

The right-hand side is r^2 , which cannot be negative, so the radius is not $\sqrt{20}$

E1

No real point satisfies the equation: it represents no real curve

A1

Answer: No real points satisfy the equation

3 marks

25 (4 marks)

$$\frac{x^2}{9} = 1 - \frac{k^2}{4}$$

M1

Real solutions need $1 - \frac{k^2}{4} \geq 0$, i.e. $|k| \leq 2$

M1

$|k| < 2$: two points; $|k| = 2$: one point ($x = 0$)

A1

$|k| > 2$: no points

A1

Answer: 2 points if $|k| < 2$; 1 if $|k| = 2$; 0 if $|k| > 2$

4 marks

26 (6 marks)

$$C_a: (x - a)^2 + y^2 = a^2, \text{ so } y^2 = 2ax - x^2$$

M1 A1

$$\text{Substitute } y^2 = 4x: x^2 + (4 - 2a)x = 0$$

M1

$$x = 0 \text{ or } x = 2a - 4$$

A1

P requires $x \geq 0$, so $x = 2a - 4$ is a second point if $2a - 4 > 0$

M1

Exactly one point (O) when $a \leq 2$, i.e. $0 < a \leq 2$

A1

Answer: $0 < a \leq 2$

6 marks

27 (8 marks)

$$(a) (x + 3)^2 + y^2 = 4[(x - 3)^2 + y^2]$$

M1 A1

$$3x^2 - 30x + 3y^2 + 27 = 0, \text{ so } x^2 - 10x + y^2 + 9 = 0$$

M1

$$(x - 5)^2 + y^2 = 16$$

A1

Centre $(5, 0)$, radius 4

A1

Answer: Centre $(5, 0)$, radius 4

5 marks

$$(b) y = 0: (x - 5)^2 = 16, x = 1 \text{ or } x = 9$$

M1

$(1, 0)$: distances from A and B are 4 and 2, ratio 2 : 1 (internal); $(9, 0)$: distances 12 and 6, ratio 2

M1 A1

: 1 (external)

Answer: $(1, 0)$ and $(9, 0)$

3 marks

28 (9 marks)

$$(a) (x - 2)^2 + y^2 = \frac{1}{4}(x - 8)^2$$

M1 A1

$$4x^2 - 16x + 16 + 4y^2 = x^2 - 16x + 64$$

M1

$$3x^2 + 4y^2 = 48, \text{ so } \frac{x^2}{16} + \frac{y^2}{12} = 1$$

A1

$$\text{Answer: } \frac{x^2}{16} + \frac{y^2}{12} = 1$$

4 marks

$$(b) a = 4, b^2 = 12, \text{ so } ae = 2: \text{ foci } (\pm 2, 0), \text{ directrices } x = \pm 8$$

M1 A1

Second focus $(-2, 0)$, directrix $x = -8$

A1

$$\text{Distance from } (0, 2\sqrt{3}) \text{ to } (-2, 0) = \sqrt{4 + 12} = 4$$

M1

Distance to $x = -8$ is 8, and $4 = \frac{1}{2} \times 8$

A1

Answer: $(-2, 0)$; $x = -8$

5 marks

29 (8 marks)

C_1 has radius 5; C_2 touches C_1 and the x-axis, so its centre is (15, 5)

M1 A1

By symmetry the centre of C_3 is (10, k) with radius k

M1 A1

Distance from (10, k) to (5, 5) equals 5 + k: $25 + (5 - k)^2 = (5 + k)^2$

M1

$50 - 10k = 25 + 10k$, so $20k = 25$

A1

$$k = \frac{5}{4}$$

A1

$$(x - 10)^2 + (y - \frac{5}{4})^2 = \frac{25}{16}$$

A1

Answer: $(x - 10)^2 + (y - \frac{5}{4})^2 = \frac{25}{16}$

8 marks

30 (8 marks)

Radius = $h + 1$ (distance from the line $x = -1$)

B1

External tangency: distance between centres = 1 + radius

M1

$$\sqrt{h^2 + k^2} = h + 2$$

A1

$$h^2 + k^2 = h^2 + 4h + 4$$

M1

$k^2 = 4(h + 1)$: locus $y^2 = 4(x + 1)$, a parabola

A1

Vertex $(-1, 0)$, $a = 1$

M1

Focus $(0, 0)$

A1

Directrix $x = -2$

A1

Answer: $y^2 = 4(x + 1)$; focus $(0, 0)$; directrix $x = -2$

8 marks

Ellipses

1 (2 marks)

$y = 0$: $x^2 = 25$, so (5, 0) and $(-5, 0)$

B1

$x = 0$: $y^2 = 9$, so (0, 3) and (0, -3)

B1

Answer: $(\pm 5, 0)$, $(0, \pm 3)$

2 marks

2 (2 marks)

$$b^2 = a^2(1 - e^2): 9 = 25(1 - e^2)$$

M1

$$e^2 = \frac{16}{25}, e = \frac{4}{5}$$

A1

Answer: $\frac{4}{5}$

2 marks

3 (2 marks)

Foci $(\pm ae, 0) = (\pm 4, 0)$

B1

Directrices $x = \pm \frac{a}{e} = \pm \frac{25}{4}$

B1

Answer: Foci $(\pm 4, 0)$; $x = \pm \frac{25}{4}$

2 marks

4 (2 marks)

$$PF_1 = PF_2 = \sqrt{16 + 9} = 5$$

M1

$$PF_1 + PF_2 = 10 = 2a$$

A1

Answer: 10

2 marks

5 (1 mark)

$$\frac{x^2}{16} + \frac{y^2}{4} = 1$$

B1

Answer: $\frac{x^2}{16} + \frac{y^2}{4} = 1$

1 mark**6 (2 marks)**

$$\frac{9}{25} + \frac{(144)}{25 \cdot 9}$$

M1

$$= \frac{9}{25} + \frac{16}{25} = 1$$

A1

Answer: $\frac{9}{25} + \frac{16}{25} = 1$, so the point lies on E

2 marks**7 (4 marks)**

$$\frac{x^2}{9} + \frac{y^2}{4} = 1, \text{ so } a = 3, b = 2$$

B1

$$4 = 9(1 - e^2), e^2 = \frac{5}{9}$$

M1

$$e = \frac{\sqrt{5}}{3}$$

A1

$$\text{Foci } (\pm ae, 0) = (\pm\sqrt{5}, 0)$$

A1

Answer: $e = \frac{\sqrt{5}}{3}$; foci $(\pm\sqrt{5}, 0)$

4 marks**8 (3 marks)**

$$ae = 3 \text{ and } e = \frac{3}{5}, \text{ so } a = 5$$

M1

$$b^2 = a^2(1 - e^2) = 25 - 9 = 16$$

M1

$$\frac{x^2}{25} + \frac{y^2}{16} = 1$$

A1

Answer: $\frac{x^2}{25} + \frac{y^2}{16} = 1$

3 marks**9 (2 marks)**

$$\text{Area} = \pi ab$$

M1

$$= \pi(4)(3) = 12\pi$$

A1

Answer: 12π

2 marks**10 (4 marks)**

The larger denominator is under y, so the major axis is vertical with $a = 5$, length 10

B1

$$9 = 25(1 - e^2), e = \frac{4}{5}$$

M1 A1

$$\text{Foci } (0, \pm ae) = (0, \pm 4)$$

A1

Answer: Length 10; foci $(0, \pm 4)$; $e = \frac{4}{5}$

4 marks**11 (4 marks)**

$$\frac{(x-2)^2}{9} + \frac{(y+1)^2}{4} = 1$$

B1

$$4 = 9(1 - e^2), e = \frac{\sqrt{5}}{3}$$

M1

$$ae = \sqrt{5}$$

A1

$$\text{Foci } (2 \pm \sqrt{5}, -1)$$

A1

Answer: $\frac{(x-2)^2}{9} + \frac{(y+1)^2}{4} = 1$; foci $(2 \pm \sqrt{5}, -1)$

4 marks

12 (4 marks)

$$\frac{x^2}{4} + x^2 = 1$$

M1

$$\frac{5x^2}{4} = 1, \text{ so } x^2 = \frac{4}{5}$$

A1

$$x = \pm \frac{2}{\sqrt{5}}$$

A1

Points $(\frac{2}{\sqrt{5}}, \frac{2}{\sqrt{5}})$ and $(-\frac{2}{\sqrt{5}}, -\frac{2}{\sqrt{5}})$

A1

Answer: $(\pm \frac{2}{\sqrt{5}}, \pm \frac{2}{\sqrt{5}})$ (same signs)

4 marks

13 (3 marks)

$$a = 6, \text{ and } PF_1 + PF_2 = 2a = 12$$

M1

$$PF_2 = 12 - 4 = 8$$

A1

$$20 = 36(1 - e^2), ae = 4; \text{ foci } (\pm 4, 0)$$

A1

Answer: $PF_2 = 8$; foci $(\pm 4, 0)$

3 marks

14 (4 marks)

$$\cos \theta = \frac{x}{3}, \sin \theta = \frac{y}{2}$$

M1

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

A1

$$x = 3 \cos \frac{\pi}{3} = \frac{3}{2}$$

M1

$$y = 2 \sin \frac{\pi}{3} = \sqrt{3}, \text{ so } P(\frac{3}{2}, \sqrt{3})$$

A1

Answer: $\frac{x^2}{9} + \frac{y^2}{4} = 1$; $P(\frac{3}{2}, \sqrt{3})$

4 marks

15 (4 marks)

$$OP^2 = 9 \cos^2 \theta + 4 \sin^2 \theta$$

M1

$$= 4 + 5 \cos^2 \theta$$

A1

$\cos^2 \theta$ ranges from 0 to 1, so OP^2 ranges from 4 to 9

M1

Greatest distance 3, least distance 2

A1

Answer: Greatest 3, least 2

4 marks

16 (5 marks)

Centre $(1, -2)$, horizontal semi-axis 3 and vertical semi-axis 2

B1

$$y = 0: \frac{(x-1)^2}{9} = 1 - 1 = 0, \text{ so } x = 1$$

M1

The ellipse touches the x-axis at $(1, 0)$

A1

$$x = 0: \frac{(y+2)^2}{4} = 1 - \frac{1}{9}, \text{ so } (y+2)^2 = \frac{32}{9}$$

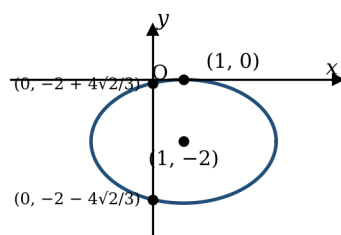
M1

It meets the y-axis at $(0, -2 \pm \frac{4\sqrt{2}}{3})$

A1

Answer: Centre $(1, -2)$; touches the x-axis at $(1, 0)$; meets the y-axis at $(0, -2 \pm \frac{4\sqrt{2}}{3})$

5 marks



17 (5 marks)

$$4x^2 + 9(x + k)^2 = 36$$

M1

$$13x^2 + 18kx + 9k^2 - 36 = 0$$

A1

$$\text{Discriminant zero: } (18k)^2 - 4(13)(9k^2 - 36) = 0$$

M1

$$-144k^2 + 1872 = 0, k^2 = 13$$

A1

$$k = \pm\sqrt{13}$$

A1

$$\text{Answer: } k = \pm\sqrt{13}$$

5 marks

18 (6 marks)

$$9(x - 2)^2 - 36 + 4(y + 1)^2 - 4 + 4 = 0$$

M1 A1

$$\frac{(x - 2)^2}{4} + \frac{(y + 1)^2}{9} = 1: \text{ an ellipse with centre } (2, -1)$$

A1

$$\text{Major axis vertical, } a = 3, b = 2: 4 = 9(1 - e^2), e = \frac{\sqrt{5}}{3}$$

M1 A1

$$\text{Foci } (2, -1 \pm \sqrt{5})$$

A1

$$\text{Answer: Centre } (2, -1); e = \frac{\sqrt{5}}{3}; \text{ foci } (2, -1 \pm \sqrt{5})$$

6 marks

19 (5 marks)

The lorry's top corners are at height $y = 3$

B1

$$\frac{x^2}{36} = 1 - \frac{9}{16} = \frac{7}{16}$$

M1 A1

$$x = \frac{6\sqrt{7}}{4} \approx 3.97$$

M1

$$\text{Width} = 2x \approx 7.94 \text{ m}$$

A1

$$\text{Answer: } 7.94 \text{ m}$$

5 marks

20 (3 marks)

The larger denominator, 9, is under y^2 , so the major axis is along the y-axis

E1

The foci lie on the major axis, so they are on the y-axis

M1

$$\text{Foci } (0, \pm\sqrt{5})$$

A1

$$\text{Answer: } (0, \pm\sqrt{5})$$

3 marks

21 (6 marks)

$$\sqrt{(x + 3)^2 + y^2} + \sqrt{(x - 3)^2 + y^2} = 10$$

M1

$$\sqrt{(x + 3)^2 + y^2} = 10 - \sqrt{(x - 3)^2 + y^2}; \text{ square: } (x + 3)^2 + y^2 = 100 - 20\sqrt{(x - 3)^2 + y^2} + (x - 3)^2 + y^2$$

M1 A1

$$12x - 100 = -20\sqrt{(x - 3)^2 + y^2}, \text{ so } 3x - 25 = -5\sqrt{(x - 3)^2 + y^2}$$

A1

$$\text{Square: } 9x^2 - 150x + 625 = 25x^2 - 150x + 225 + 25y^2$$

M1

$$16x^2 + 25y^2 = 400, \text{ so } \frac{x^2}{25} + \frac{y^2}{16} = 1$$

A1

$$\text{Answer: } \frac{x^2}{25} + \frac{y^2}{16} = 1$$

6 marks

22 (6 marks)

$$y^2 = b^2(1 - \frac{x^2}{a^2}) = a^2(1 - e^2) - (1 - e^2)x^2$$

M1 A1

$$PF_2^2 = (x - ae)^2 + y^2 = x^2 - 2aex + a^2e^2 + a^2 - a^2e^2 - x^2 + e^2x^2$$

M1

$$= a^2 - 2aex + e^2x^2 = (a - ex)^2, \text{ and } a - ex > 0 \text{ since } |x| \leq a \text{ and } e < 1, \text{ so } PF_2 = a - ex$$

A1

$$\text{Similarly (replace } e \text{ by } -e) PF_1 = a + ex$$

M1

$$PF_1 + PF_2 = 2a$$

A1

$$\text{Answer: } PF_2 = a - ex, PF_1 = a + ex, PF_1 + PF_2 = 2a$$

6 marks

23 (6 marks)

$$PF^2 = \frac{1}{4}(y - 12)^2; x^2 + (y - 3)^2 = \frac{1}{4}(y - 12)^2$$

M1 A1

$$4x^2 + 4y^2 - 24y + 36 = y^2 - 24y + 144$$

M1

$$4x^2 + 3y^2 = 108$$

A1

$$\frac{x^2}{27} + \frac{y^2}{36} = 1$$

A1

Centre at the origin

B1

Answer: $\frac{x^2}{27} + \frac{y^2}{36} = 1$; centre (0, 0)

6 marks

24 (6 marks)

Let the ends of a chord be (x_1, y_1) and (x_2, y_2) , with $h = \frac{x_1 + x_2}{2}$, $k = \frac{y_1 + y_2}{2}$

B1

$$\frac{x_1^2}{9} + \frac{y_1^2}{4} = 1 \text{ and } \frac{x_2^2}{9} + \frac{y_2^2}{4} = 1; \text{ subtract}$$

M1

$$\frac{(x_1 - x_2)(x_1 + x_2)}{9} + \frac{(y_1 - y_2)(y_1 + y_2)}{4} = 0$$

A1

Divide by $(x_1 - x_2)$ and use the gradient $\frac{y_1 - y_2}{x_1 - x_2} = 1$: $\frac{2h}{9} + \frac{2k}{4} = 0$

M1 A1

$$k = -\frac{4h}{9}; \text{ the midpoints lie on } y = -\frac{4x}{9}$$

A1

Answer: $y = -\frac{4x}{9}$

6 marks

25 (6 marks)

By symmetry, Area = $4 \int_0^a y \, dx$

B1

$$x = a \cos \theta, \, dx = -a \sin \theta \, d\theta; \, x = 0 \leftrightarrow \theta = \frac{\pi}{2}, \, x = a \leftrightarrow \theta = 0$$

M1

$$\text{Area} = 4 \int_{\pi/2}^0 b \sin \theta (-a \sin \theta) \, d\theta = 4ab \int_0^{\pi/2} \sin^2 \theta \, d\theta$$

M1 A1

$$\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta), \text{ so the integral} = \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4} \right]_0^{\pi/2} = \frac{\pi}{4}$$

M1

$$\text{Area} = 4ab \times \frac{\pi}{4} = \pi ab$$

A1

Answer: Area = πab

6 marks

26 (5 marks)

Vertices $(\pm 5 \cos \theta, \pm 3 \sin \theta)$: width $10 \cos \theta$ and height $6 \sin \theta$

M1 A1

$$\text{Area} = 60 \sin \theta \cos \theta = 30 \sin 2\theta$$

A1

Greatest when $\sin 2\theta = 1$

M1

$$\text{Greatest area } 30 \text{ when } \theta = \frac{\pi}{4}$$

A1

Answer: 30 at $\theta = \frac{\pi}{4}$

5 marks

27 (7 marks)

From E, $y^2 = 4 - \frac{4x^2}{9}$. Substitute: $x^2 + 4 - \frac{4x^2}{9} = r^2$

M1 A1

$$\frac{5x^2}{9} = r^2 - 4, \text{ so } x^2 = \frac{9(r^2 - 4)}{5}$$

M1

$$\text{Need } 0 \leq x^2 \leq 9: 4 \leq r^2 \leq 9, \text{ i.e. } 2 \leq r \leq 3$$

M1 A1

$$2 < r < 3: x = \pm \frac{3\sqrt{r^2 - 4}}{\sqrt{5}}, y^2 = \frac{4(9 - r^2)}{5} > 0, \text{ giving four points}$$

A1

$$r = 2: x = 0, y = \pm 2; r = 3: x = \pm 3, y = 0 \text{ (two points each). Otherwise no points}$$

A1

Answer: 4 points if $2 < r < 3$; 2 points if $r = 2$ or 3 ; none otherwise

7 marks

28 (8 marks)

$$\text{Area} = \frac{1}{2}|x_A(y_B - y_C) + x_B(y_C - y_A) + x_C(y_A - y_B)| \quad \text{M1}$$

With $x = 4 \cos \phi$, $y = 3 \sin \phi$ each term gives $12 \cos \phi_A(\sin \phi_B - \sin \phi_C)$ etc. A1

Sum = $12[\sin(\phi_B - \phi_A) + \sin(\phi_C - \phi_B) + \sin(\phi_A - \phi_C)]$, using $\cos \phi_A \sin \phi_B - \sin \phi_A \cos \phi_B$
 = $\sin(\phi_B - \phi_A)$ after collecting terms M1 A1

The differences are $\frac{2\pi}{3}$, $\frac{2\pi}{3}$ and $-\frac{4\pi}{3}$ M1

$$\sin \frac{2\pi}{3} + \sin \frac{2\pi}{3} + \sin(-\frac{4\pi}{3}) = \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2} \quad \text{M1 A1}$$

$$\text{Area} = \frac{1}{2}(12)(\frac{3\sqrt{3}}{2}) = 9\sqrt{3}, \text{ independent of } \alpha \quad \text{A1}$$

Answer: $9\sqrt{3}$ 8 marks

29 (7 marks)

K has centre A(-3, 0) and radius 10 B1

CF = r since F is on the circle B1

Internal contact: AC = 10 - r M1

AC + CF = 10, so C moves with the sum of its distances from A and F constant A1

This is an ellipse with foci (± 3 , 0) and $2a = 10$, so $a = 5$ M1

$$b^2 = a^2 - c^2 = 25 - 9 = 16 \quad \text{A1}$$

$$\frac{x^2}{25} + \frac{y^2}{16} = 1 \quad \text{A1}$$

Answer: $\frac{x^2}{25} + \frac{y^2}{16} = 1$ 7 marks

30 (8 marks)

Closest distance $a - c = 0.6$ and greatest $a + c = 17.4$ M1

$a = 9$, $c = 8.4$ A1 A1

$$e = \frac{c}{a} = \frac{14}{15} \quad \text{M1 A1}$$

$$b^2 = a^2 - c^2 = 81 - 70.56 = 10.44 = \frac{261}{25} \quad \text{M1 A1}$$

At (0, b): distance to focus = $\sqrt{c^2 + b^2} = \sqrt{81} = 9$ AU (= a) A1

Answer: $a = 9$, $c = 8.4$, $e = \frac{14}{15}$, $b^2 = \frac{261}{25}$; 9 AU 8 marks

Hyperbolas

1 (1 mark)

$y = 0$: $x^2 = 9$, vertices (± 3 , 0) B1

Answer: (± 3 , 0) 1 mark

2 (2 marks)

$$\text{Asymptotes } \frac{x^2}{9} - \frac{y^2}{16} = 0 \quad \text{M1}$$

$$y = \pm \frac{4}{3}x \quad \text{A1}$$

Answer: $y = \pm \frac{4x}{3}$ 2 marks

3 (2 marks)

$$b^2 = a^2(e^2 - 1): 16 = 9(e^2 - 1) \quad \text{M1}$$

$$e^2 = \frac{25}{9}, e = \frac{5}{3} \quad \text{A1}$$

Answer: $\frac{5}{3}$ 2 marks

4 (2 marks)

$$\text{Foci } (\pm ae, 0) = (\pm 5, 0)$$

B1

$$\text{Directrices } x = \pm \frac{a}{e} = \pm \frac{9}{5}$$

B1

$$\text{Answer: Foci } (\pm 5, 0); x = \pm \frac{9}{5}$$

2 marks

5 (3 marks)

Asymptotes: the x-axis ($y = 0$) and the y-axis ($x = 0$)

B1

$$x^2 = 4$$

M1

(2, 2) and (-2, -2)

A1

Answer: Asymptotes $y = 0, x = 0$; (2, 2), (-2, -2)

3 marks

6 (3 marks)

$$\frac{y^2}{16} - \frac{x^2}{4} = 1 \text{ has its branches above and below the origin, with vertices } (0, \pm 4)$$

B1

$$\text{Asymptotes: } \frac{y^2}{16} - \frac{x^2}{4} = 0, \text{ so } y^2 = 4x^2$$

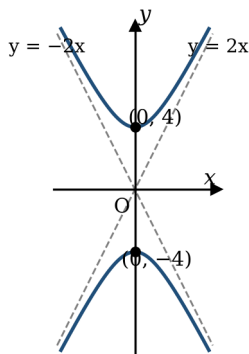
M1

$$y = \pm 2x$$

A1

Answer: Vertices (0, ± 4); asymptotes $y = \pm 2x$

3 marks

**7 (3 marks)**

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ with } a = 2$$

B1

$$\frac{b}{a} = \frac{3}{2}, \text{ so } b = 3$$

M1

$$\frac{x^2}{4} - \frac{y^2}{9} = 1$$

A1

$$\text{Answer: } \frac{x^2}{4} - \frac{y^2}{9} = 1$$

3 marks

8 (4 marks)

$$\text{Equation } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \text{ with foci } (\pm ae, 0): a^2 + b^2 = 17^2 = 289$$

M1

$$\text{Asymptotes } y = \pm \frac{b}{a}x, \text{ so } \frac{b}{a} = \frac{8}{15}$$

M1

$$a = 15k, b = 8k: 289k^2 = 289, \text{ so } k = 1 \text{ and } a = 15, b = 8$$

A1

$$\frac{x^2}{225} - \frac{y^2}{64} = 1$$

A1

$$\text{Answer: } \frac{x^2}{225} - \frac{y^2}{64} = 1$$

4 marks

9 (3 marks)

$$\frac{x^2}{4} - \frac{y^2}{9} = 1 \quad \text{M1}$$

Vertices $(\pm 2, 0)$ A1

Asymptotes $y = \pm \frac{3}{2}x$ A1

Answer: $(\pm 2, 0)$; $y = \pm \frac{3x}{2}$ 3 marks

10 (3 marks)

$$a = 3, \text{ so } |PF_1 - PF_2| = 2a = 6 \quad \text{M1}$$

$$PF_2 = 11 - 6 = 5 \text{ or } 11 + 6 = 17 \quad \text{A1}$$

(both possible: P on either branch) B1

Answer: 5 or 17 3 marks

11 (4 marks)

$$x(x + 3) = 4, \text{ so } x^2 + 3x - 4 = 0 \quad \text{M1 A1}$$

$$(x + 4)(x - 1) = 0, x = -4 \text{ or } 1 \quad \text{M1}$$

$(1, 4)$ and $(-4, -1)$ A1

Answer: $(1, 4), (-4, -1)$ 4 marks

12 (3 marks)

$$xy = 3t \times \frac{3}{t} = 9 \quad \text{M1 A1}$$

$$t = 2: (6, \frac{3}{2}) \quad \text{A1}$$

Answer: $xy = 9$; $(6, \frac{3}{2})$ 3 marks

13 (4 marks)

Centre $(1, -2)$ B1

Vertices $(1 \pm 2, -2) = (3, -2)$ and $(-1, -2)$ B1

Asymptotes $y + 2 = \pm \frac{3}{2}(x - 1)$ M1 A1

Answer: Centre $(1, -2)$; vertices $(3, -2), (-1, -2)$; $y + 2 = \pm \frac{3}{2}(x - 1)$ 4 marks

14 (4 marks)

$$\sec \theta = \frac{x}{3}, \tan \theta = \frac{y}{2} \quad \text{M1}$$

$$\sec^2 \theta - \tan^2 \theta = 1 \text{ gives } \frac{x^2}{9} - \frac{y^2}{4} = 1 \quad \text{A1}$$

$$x = 3 \sec \frac{\pi}{3} = 6 \quad \text{M1}$$

$$y = 2 \tan \frac{\pi}{3} = 2\sqrt{3}: (6, 2\sqrt{3}) \quad \text{A1}$$

Answer: $(6, 2\sqrt{3})$ 4 marks

15 (5 marks)

$$\frac{x^2}{4} - \frac{m^2x^2}{9} = 1$$

M1

$$x^2\left(\frac{1}{4} - \frac{m^2}{9}\right) = 1$$

A1

A real solution needs $\frac{1}{4} - \frac{m^2}{9} > 0$

M1

$$m^2 < \frac{9}{4}$$

A1

$$-\frac{3}{2} < m < \frac{3}{2}$$

A1

Answer: $-\frac{3}{2} < m < \frac{3}{2}$

5 marks

16 (4 marks)

$$x(6 - x) = 8, \text{ so } x^2 - 6x + 8 = 0$$

M1 A1

$$x = 2 \text{ or } 4: A(2, 4), B(4, 2)$$

A1

$$AB = \sqrt{4 + 4} = 2\sqrt{2}$$

A1

Answer: $2\sqrt{2}$

4 marks

17 (4 marks)

$$\text{Asymptotes } y = \pm \frac{4}{3}x$$

B1

Each makes an angle $\arctan \frac{4}{3} = 53.13^\circ$ with the x-axis

M1 A1

Angle between $= 2 \times 53.13^\circ = 106.3^\circ$, so the acute angle is 73.7°

A1

Answer: 73.7°

4 marks

18 (5 marks)

$$(x - 5)^2 + y^2 = \frac{25}{9}(x - \frac{9}{5})^2$$

M1 A1

$$x^2 - 10x + 25 + y^2 = \frac{25}{9}x^2 - 10x + 9$$

M1

$$\frac{16}{9}x^2 - y^2 = 16$$

A1

Divide by 16: $\frac{x^2}{9} - \frac{y^2}{16} = 1$

A1

Answer: $\frac{x^2}{9} - \frac{y^2}{16} = 1$

5 marks

19 (6 marks)

$$9(x - 2)^2 - 36 - 16(y + 1)^2 + 16 = 124$$

M1 A1

$$\frac{(x - 2)^2}{16} - \frac{(y + 1)^2}{9} = 1: \text{ a hyperbola with centre } (2, -1)$$

A1

$$9 = 16(e^2 - 1), e = \frac{5}{4}; \text{ foci } (2 \pm 5, -1) = (7, -1), (-3, -1)$$

A1 A1

$$\text{Asymptotes } y + 1 = \pm \frac{3}{4}(x - 2)$$

A1

Answer: Centre $(2, -1)$; $e = \frac{5}{4}$; foci $(7, -1), (-3, -1)$; $y + 1 = \pm \frac{3}{4}(x - 2)$

6 marks

20 (5 marks)

$$a = 30$$

B1

$$\frac{2500}{900} - \frac{1600}{b^2} = 1$$

M1

$$\frac{1600}{b^2} = \frac{16}{9}, b^2 = 900, b = 30$$

A1

$$\text{At } y = -60: \frac{x^2}{900} = 1 + \frac{3600}{900} = 5$$

M1

$$x = 30\sqrt{5} \approx 67.1 \text{ m}$$

A1

Answer: a = b = 30; 67.1 m

5 marks

21 (6 marks)

$$y^2 = b^2\left(\frac{x^2}{a^2} - 1\right) = (e^2 - 1)x^2 - a^2(e^2 - 1)$$

M1 A1

$$PF_2^2 = (x - ae)^2 + y^2 = x^2 - 2aex + a^2e^2 + e^2x^2 - x^2 - a^2e^2 + a^2$$

M1

$$= e^2x^2 - 2aex + a^2 = (ex - a)^2$$

A1

On the right branch $x \geq a$, so $ex - a > 0$ and $PF_2 = ex - a$. Replacing e by $-e$ gives $PF_1 = ex + a$

M1

$$PF_1 - PF_2 = 2a$$

A1

Answer: $PF_2 = ex - a$, $PF_1 = ex + a$, $PF_1 - PF_2 = 2a$

6 marks

22 (6 marks)

$$\text{Substitute: } \frac{x^2}{a^2} - \frac{(mx + c)^2}{b^2} = 1$$

M1

$$b^2x^2 - a^2(m^2x^2 + 2mcx + c^2) = a^2b^2$$

A1

$$(b^2 - a^2m^2)x^2 - 2a^2mcx - a^2(c^2 + b^2) = 0$$

A1

$$\text{Tangent: discriminant zero, } 4a^4m^2c^2 + 4(b^2 - a^2m^2)a^2(c^2 + b^2) = 0$$

M1

$$a^2m^2c^2 + b^2c^2 + b^4 - a^2m^2c^2 - a^2m^2b^2 = 0$$

M1

$$b^2(c^2 + b^2 - a^2m^2) = 0, \text{ so } c^2 = a^2m^2 - b^2$$

A1

Answer: $c^2 = a^2m^2 - b^2$

6 marks

23 (6 marks)

$$\sqrt{(x + 5)^2 + y^2} = 6 + \sqrt{(x - 5)^2 + y^2}$$

M1

$$\text{Square: } (x + 5)^2 + y^2 = 36 + 12\sqrt{(x - 5)^2 + y^2} + (x - 5)^2 + y^2$$

M1 A1

$$20x - 36 = 12\sqrt{(x - 5)^2 + y^2}, \text{ so } 5x - 9 = 3\sqrt{(x - 5)^2 + y^2}$$

A1

$$\text{Square: } 25x^2 - 90x + 81 = 9x^2 - 90x + 225 + 9y^2$$

M1

$$16x^2 - 9y^2 = 144, \text{ so } \frac{x^2}{9} - \frac{y^2}{16} = 1$$

A1

Answer: $\frac{x^2}{9} - \frac{y^2}{16} = 1$

6 marks

24 (6 marks)

$$\text{Let the ends of a chord be } (x_1, y_1), (x_2, y_2), \text{ with } h = \frac{x_1 + x_2}{2}, k = \frac{y_1 + y_2}{2}$$

B1

$$\frac{x_1^2}{4} - y_1^2 = 1 \text{ and } \frac{x_2^2}{4} - y_2^2 = 1; \text{ subtract}$$

M1

$$\frac{(x_1 - x_2)(x_1 + x_2)}{4} - (y_1 - y_2)(y_1 + y_2) = 0$$

A1

$$\text{Divide by } (x_1 - x_2), \text{ gradient } \frac{1}{2}: \frac{2h}{4} - 2k\left(\frac{1}{2}\right) = 0$$

M1 A1

$$h = 2k: M \text{ lies on } y = \frac{x}{2}$$

A1

Answer: $y = \frac{x}{2}$

6 marks

25 (6 marks)

Tangent at (x_1, y_1) : $\frac{xx_1}{9} - \frac{yy_1}{4} = 1$, so $\frac{x \sec \theta}{3} - \frac{y \tan \theta}{2} = 1$

B1

On $y = \frac{2x}{3}$: $\frac{x}{3}(\sec \theta - \tan \theta) = 1$, so $x = 3(\sec \theta + \tan \theta)$, $y = 2(\sec \theta + \tan \theta)$, using $\sec^2 \theta - \tan^2 \theta = 1$

M1 A1

On $y = -\frac{2x}{3}$: $\frac{x}{3}(\sec \theta + \tan \theta) = 1$, so $x = 3(\sec \theta - \tan \theta)$, $y = -2(\sec \theta - \tan \theta)$

A1

Midpoint of AB = $(3 \sec \theta, 2 \tan \theta)$, which is P

M1

Area = $\frac{1}{2}|x_A y_B - x_B y_A| = \frac{1}{2}|-6(\sec^2 \theta - \tan^2 \theta) - 6(\sec^2 \theta - \tan^2 \theta)| = 6$

A1

Answer: P is the midpoint of AB; area of triangle OAB = 6

6 marks

26 (5 marks)

$\frac{x^2}{9} - \frac{y^2}{16} = \cosh^2 u - \sinh^2 u$

M1

= 1

A1

$3 \cosh u = 5$, so $\cosh u = \frac{5}{3}$

M1

$\sinh u = \sqrt{\frac{25}{9} - 1} = \frac{4}{3}$ (matches $y = 4 \sinh u = \frac{16}{3}$)

A1

$u = \ln\left(\frac{5}{3} + \frac{4}{3}\right) = \ln 3$

A1

Answer: $u = \ln 3$

5 marks

27 (7 marks)

From H: $y^2 = \frac{16x^2}{9} - 16$

M1

$x^2 + \frac{16x^2}{9} - 16 = r^2$, so $\frac{25x^2}{9} = r^2 + 16$

M1 A1

$x^2 = \frac{9(r^2 + 16)}{25}$

A1

H requires $x^2 \geq 9$: $r^2 + 16 \geq 25$, so $r \geq 3$

M1

$r > 3$: $x = \pm\sqrt{9(r^2 + 16)}/5$ and $y^2 = \frac{16x^2}{9} - 16 > 0$, giving 4 points

A1

$r = 3$: $(\pm 3, 0)$, 2 points; $r < 3$: none

A1

Answer: 4 points if $r > 3$; 2 points if $r = 3$; none if $r < 3$

7 marks

28 (8 marks)

Asymptotes: $bx - ay = 0$ and $bx + ay = 0$

B1

Distances: $\frac{|bx_1 - ay_1|}{\sqrt{a^2 + b^2}}$ and $\frac{|bx_1 + ay_1|}{\sqrt{a^2 + b^2}}$

M1 A1

Product = $\frac{|b^2 x_1^2 - a^2 y_1^2|}{a^2 + b^2}$

M1 A1

P on H: $b^2 x_1^2 - a^2 y_1^2 = a^2 b^2$

M1 A1

Product = $\frac{a^2 b^2}{a^2 + b^2}$, independent of P

A1

Answer: Product of distances = $\frac{a^2 b^2}{a^2 + b^2}$

8 marks

29 (8 marks)

$$PF_1^2 = (2\sqrt{2}t - 4)^2 + \left(\frac{2\sqrt{2}}{t} - 4\right)^2 \quad \text{M1}$$

$$= 8t^2 - 16\sqrt{2}t + 16 + \frac{8}{t^2} - \frac{16\sqrt{2}}{t} + 16 \quad \text{A1}$$

$$= 8(u^2 - 2) - 16\sqrt{2}u + 32 = 8u^2 - 16\sqrt{2}u + 16 \quad \text{M1 A1}$$

$$= (2\sqrt{2}u - 4)^2 \quad \text{A1}$$

$$u \geq 2 \text{ so } 2\sqrt{2}u - 4 > 0 \text{ and } PF_1 = 2\sqrt{2}u - 4 \quad \text{B1}$$

$$\text{Likewise } PF_2^2 = 8u^2 + 16\sqrt{2}u + 16 = (2\sqrt{2}u + 4)^2 \quad \text{M1}$$

$$PF_2 - PF_1 = 8, \text{ a constant, so } F_1, F_2 \text{ are the foci} \quad \text{A1}$$

Answer: $F_2P - F_1P = 8$, so F_1 and F_2 are the foci 8 marks

30 (7 marks)

$$R \text{ is symmetrical about the x-axis, so Area} = 2 \int_3^6 \frac{2}{3} \sqrt{x^2 - 9} \, dx \quad \text{M1 A1}$$

$$x = 3 \cosh u, \, dx = 3 \sinh u \, du, \, \sqrt{x^2 - 9} = 3 \sinh u \quad \text{M1}$$

$$\text{Limits } u = 0 \text{ to } u = \operatorname{arcosh} 2 = \ln(2 + \sqrt{3}) \quad \text{A1}$$

$$\text{Area} = 12 \int \sinh^2 u \, du = 6 \int (\cosh 2u - 1) \, du \quad \text{M1}$$

$$= 6 \left[\frac{1}{2} \sinh 2u - u \right]_0^{\ln(2 + \sqrt{3})} = 3 \sinh 2u - 6u \quad \text{A1}$$

$$\sinh 2u = 2 \sinh u \cosh u = 2\sqrt{3}(2) = 4\sqrt{3}, \text{ so Area} = 12\sqrt{3} - 6 \ln(2 + \sqrt{3}) \quad \text{A1}$$

Answer: $12\sqrt{3} - 6 \ln(2 + \sqrt{3})$ 7 marks

Parabolas**1 (2 marks)**

$$y^2 = 4ax \text{ with } 4a = 12, \text{ so } a = 3 \quad \text{M1}$$

$$\text{Focus } (3, 0), \text{ directrix } x = -3 \quad \text{A1}$$

Answer: Focus (3, 0); $x = -3$ 2 marks

2 (2 marks)

$$64 = 4a(4) \quad \text{M1}$$

$$a = 4 \quad \text{A1}$$

Answer: $a = 4$ 2 marks

3 (1 mark)

$$x = 2(9) = 18, \, y = 4(3) = 12, \text{ so } (18, 12) \quad \text{B1}$$

Answer: (18, 12) 1 mark

4 (2 marks)

$$a = 2, \text{ and the latus rectum is the chord through the focus perpendicular to the axis, } x = 2 \quad \text{M1}$$

$$y^2 = 16, \, y = \pm 4, \text{ length } 8 (= 4a) \quad \text{A1}$$

Answer: 8 2 marks

5 (2 marks)

$$x^2 = 4ay \text{ with } a = 2 \quad \text{M1}$$

$$\text{Focus } (0, 2), \text{ directrix } y = -2 \quad \text{A1}$$

Answer: Focus (0, 2); $y = -2$ 2 marks

6 (4 marks)Vertex $(-1, 2)$

B1

 $(x + 1)^2 = 4(2)(y - 2)$, so $a = 2$: focus $(-1, 4)$ and directrix $y = 0$

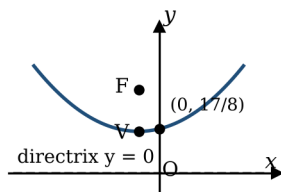
M1 A1

 $x = 0$: $1 = 8(y - 2)$, so $y = \frac{17}{8}$

A1

Answer: Vertex $(-1, 2)$; focus $(-1, 4)$; directrix $y = 0$; crosses the y-axis at $(0, \frac{17}{8})$

4 marks

**7 (3 marks)**Directrix $y = 3$

B1

$$x^2 + (y + 3)^2 = (y - 3)^2$$

M1

$$x^2 = -12y$$

A1

Answer: $x^2 = -12y$

3 marks

8 (3 marks) $a = 5$, focus $(5, 0)$, directrix $x = -5$

B1

Focal distance = distance to directrix

M1

$$= 7 + 5 = 12$$

A1

Answer: 12

3 marks

9 (4 marks)P is $t = 2$ ($x = t^2$, $y = 2t$); for a focal chord $t_1 t_2 = -1$, so $t = -\frac{1}{2}$

M1 A1

$$Q = \left(\frac{1}{4}, -1\right)$$

A1

$$PQ = x_P + x_Q + 2a = 4 + \frac{1}{4} + 2 = \frac{25}{4}$$

A1

Answer: $Q(\frac{1}{4}, -1)$; $PQ = \frac{25}{4}$

4 marks

10 (3 marks)Vertex $(1, 2)$

B1

 $4a = 8$, $a = 2$: focus $(3, 2)$

B1

Directrix $x = -1$

B1

Answer: $(1, 2)$; $(3, 2)$; $x = -1$

3 marks

11 (4 marks)Substitute: $(x + 1)^2 = 4x$

M1

$$x^2 - 2x + 1 = 0, (x - 1)^2 = 0$$

A1

A repeated root, so the line is a tangent

E1

$$x = 1, y = 2: (1, 2)$$

A1

Answer: $(1, 2)$

4 marks

12 (3 marks)

$$a = 5 - 2 = 3$$

B1

$$(y - 1)^2 = 4a(x - 2)$$

M1

$$(y - 1)^2 = 12(x - 2)$$

A1

Answer: $(y - 1)^2 = 12(x - 2)$

3 marks

13 (5 marks)

$$y^2 = 16x, \text{ so } a = 4 \text{ and } S = (4, 0); \frac{dy}{dx} = \frac{8}{y} = \frac{2}{3} \text{ at } P$$

M1 A1

$$\text{Normal gradient } -\frac{3}{2}: y - 12 = -\frac{3}{2}(x - 9)$$

M1

$$y = 0: x = 9 + 8 = 17, \text{ so } Q = (17, 0)$$

A1

$$\text{Area} = \frac{1}{2} \times SQ \times 12 = \frac{1}{2} \times 13 \times 12 = 78$$

A1

Answer: 78

5 marks

14 (5 marks)

$$(2x - 4)^2 = 8x, \text{ so } x^2 - 6x + 4 = 0$$

M1 A1

$$x = 3 \pm \sqrt{5}$$

A1

$$\text{Difference in } x = 2\sqrt{5}; \text{ gradient } 2 \text{ so difference in } y = 4\sqrt{5}$$

M1

$$AB = \sqrt{20 + 80} = 10$$

A1

Answer: 10

5 marks

15 (5 marks)

$$\text{Gradient of } PQ = \frac{2a(p - q)}{a(p^2 - q^2)} = \frac{2}{p + q}$$

M1 A1

$$\text{Gradient of } PS = \frac{2ap}{ap^2 - a} = \frac{2p}{p^2 - 1}$$

M1

$$\text{Collinear: } \frac{2}{p + q} = \frac{2p}{p^2 - 1}, \text{ so } p^2 - 1 = p^2 + pq$$

M1

$$pq = -1 \text{ (and conversely)}$$

A1

Answer: PQ is a focal chord if and only if $pq = -1$

5 marks

16 (5 marks)

$$y^2 = 36p^2 = 12x$$

M1 A1

$$M = \left(-\frac{3p^2}{2}, 3p\right)$$

M1

$$p = \frac{y}{3}, \text{ so } x = \frac{3}{2} \times \frac{y^2}{9} = \frac{y^2}{6}$$

M1

$$y^2 = 6x$$

A1

Answer: $y^2 = 12x$; locus $y^2 = 6x$

5 marks

17 (4 marks)

$$\text{Model: } y^2 = 4ax, \text{ and the rim point is } (10, 30)$$

M1

$$900 = 4a(10)$$

M1

$$a = 22.5$$

A1

The receiver is 22.5 cm from the vertex

A1

Answer: 22.5 cm

4 marks

18 (4 marks)

$$x^2 + 8x = 48, \text{ so } x^2 + 8x - 48 = 0$$

M1

$$(x + 12)(x - 4) = 0$$

A1

$$x = 4 \text{ (} x = -12 \text{ gives } y^2 < 0 \text{)}$$

A1

$$y^2 = 32: (4, 4\sqrt{2}) \text{ and } (4, -4\sqrt{2})$$

A1

Answer: $(4, \pm 4\sqrt{2})$

4 marks

19 (5 marks)

$$(y - 3)^2 - 9 - 8x + 1 = 0$$

M1

$$(y - 3)^2 = 8x + 8 = 8(x + 1)$$

A1

$$\text{Vertex } (-1, 3)$$

A1

$$a = 2: \text{ focus } (1, 3)$$

A1

$$\text{Directrix } x = -3$$

A1

Answer: Vertex $(-1, 3)$; focus $(1, 3)$; $x = -3$

5 marks

20 (5 marks)

The chord through S: $y = x - 1$

B1

$$(x - 1)^2 = 4x, \text{ so } x^2 - 6x + 1 = 0$$

M1 A1

$$x_1 + x_2 = 6$$

A1

$$\text{Focal chord length} = x_1 + x_2 + 2a = 6 + 2 = 8$$

A1

Answer: $y = x - 1$; length 8

5 marks

21 (5 marks)

$$(x - 3)^2 + (y - 2)^2 = (y + 4)^2$$

M1 A1

$$(x - 3)^2 + y^2 - 4y + 4 = y^2 + 8y + 16$$

M1

$$(x - 3)^2 = 12y + 12 = 12(y + 1)$$

A1

Vertex (3, -1)

A1

Answer: $(x - 3)^2 = 12(y + 1)$; vertex (3, -1)

5 marks

22 (5 marks)

Let the ends of a chord be $(x_1, y_1), (x_2, y_2)$; $y_1^2 = 12x_1, y_2^2 = 12x_2$

B1

$$\text{Subtract: } (y_1 - y_2)(y_1 + y_2) = 12(x_1 - x_2)$$

M1

$$\frac{y_1 - y_2}{x_1 - x_2} = 3, \text{ so } 3(y_1 + y_2) = 12$$

M1 A1

$$k = \frac{y_1 + y_2}{2} = 2: \text{ the midpoints lie on } y = 2$$

A1

Answer: $y = 2$

5 marks

23 (6 marks)

Q has parameter $q = -\frac{1}{p}$

B1

PQ = $x_P + x_Q + 2a$ (focal distances $x + a$)

M1

$$= a(p^2 + \frac{1}{p^2} + 2)$$

M1 A1

$$= a(p + \frac{1}{p})^2$$

A1

$$(p + \frac{1}{p})^2 \geq 4 \text{ (AM-GM, or } p + 1/p \geq 2 \text{ for } p > 0), \text{ so } PQ \geq 4a, \text{ with equality at } p = 1$$

A1

Answer: $PQ = a(p + \frac{1}{p})^2 \geq 4a$, so the shortest focal chord has length $4a$

6 marks

24 (6 marks)

$P(ap^2, 2ap), Q(aq^2, 2aq)$ with $pq = -1$

B1

$$\text{Centre } M = (\frac{a(p^2 + q^2)}{2}, a(p + q))$$

M1 A1

$$\text{Radius} = \frac{1}{2}PQ = \frac{1}{2}(x_P + x_Q + 2a) = \frac{a(p^2 + q^2)}{2} + a$$

M1

$$\text{Distance from } M \text{ to the directrix } x = -a \text{ is } \frac{a(p^2 + q^2)}{2} + a$$

M1

This equals the radius, so the circle touches the directrix

A1

Answer: The circle touches the directrix $x = -a$

6 marks

25 (3 marks)

The standard form is $y^2 = 4ax$, so $4a = 12$

E1

$a = 3$, the focus is (3, 0)

M1

Directrix $x = -3$

A1

Answer: Focus (3, 0); $x = -3$

3 marks

26 (6 marks)

$$y = m(x + 2): m^2(x + 2)^2 = 8x \quad \text{M1}$$

$$m^2x^2 + (4m^2 - 8)x + 4m^2 = 0 \quad \text{A1}$$

$$\text{Discriminant: } (4m^2 - 8)^2 - 16m^4 = 64 - 64m^2 \quad \text{M1 A1}$$

$$L \text{ meets } P \text{ if } 64 - 64m^2 \geq 0, \text{ so } -1 \leq m \leq 1 \quad \text{A1}$$

$$\text{Tangent when the discriminant is zero: } m = \pm 1 \quad \text{A1}$$

Answer: $-1 \leq m \leq 1$ ($m \neq 0$ for two points); tangents $m = \pm 1$

6 marks

27 (7 marks)

$$P(ap^2, 2ap), Q(aq^2, 2aq), pq = -1, \text{ so } q = -\frac{1}{p} \quad \text{B1}$$

$$SP = x_p + a = a(p^2 + 1) \quad \text{M1 A1}$$

$$SQ = a(q^2 + 1) = a\left(\frac{1}{p^2} + 1\right) = \frac{a(1 + p^2)}{p^2} \quad \text{M1 A1}$$

$$\frac{1}{SP} + \frac{1}{SQ} = \frac{1}{a(1 + p^2)} + \frac{p^2}{a(1 + p^2)} \quad \text{M1}$$

$$= \frac{1 + p^2}{a(1 + p^2)} = \frac{1}{a} \quad \text{A1}$$

Answer: $\frac{1}{SP} + \frac{1}{SQ} = \frac{1}{a}$

7 marks

28 (9 marks)

(a) Gradient of PQ = $\frac{2a(p - q)}{a(p^2 - q^2)} = \frac{2}{p + q}$ M1 A1

$$y - 2ap = \frac{2}{p + q}(x - ap^2) \quad \text{M1}$$

$$(p + q)y - 2ap(p + q) = 2x - 2ap^2, \text{ so } 2x - (p + q)y + 2apq = 0 \quad \text{A1}$$

Answer: $2x - (p + q)y + 2apq = 0$

4 marks

(b) Gradients of OP and OQ are $\frac{2}{p}$ and $\frac{2}{q}$ B1

$$\text{Perpendicular: } \frac{2}{p} \times \frac{2}{q} = -1 \quad \text{M1}$$

$$pq = -4 \quad \text{A1}$$

Answer: $pq = -4$

3 marks

(c) Substituting $pq = -4$: $2x - (p + q)y - 8a = 0$ M1

$$\text{When } y = 0, x = 4a \text{ for all such chords: the fixed point is } (4a, 0) \quad \text{A1}$$

Answer: $(4a, 0)$

2 marks

29 (7 marks)

(a) Rim point (20, 15): $225 = 4a(20)$ M1

$$a = \frac{45}{16} \quad \text{A1}$$

$$\text{The bulb is } \frac{45}{16} = 2.8125 \text{ cm from the vertex} \quad \text{A1}$$

Answer: $a = \frac{45}{16}$

3 marks

(b) $V = \pi \int_0^{20} y^2 dx = \pi \int_0^{20} 4ax dx$ M1

$$= \pi [2ax^2]_0^{20} \quad \text{M1}$$

$$= 800\pi a \quad \text{A1}$$

$$= 2250\pi \text{ cm}^3 \quad \text{A1}$$

Answer: $2250\pi \text{ cm}^3$

4 marks

30 (8 marks)

(a) Substitute $y = \frac{x^2}{4}$: $\frac{x^4}{16} = 4x, x(x^3 - 64) = 0$ M1

$$x = 4, y = 4: A(4, 4) \quad \text{A1}$$

Answer: $(4, 4)$

2 marks

$$(b) \text{ Area} = \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4}\right) dx$$

M1

$$= \left[\frac{4}{3}x^{3/2} - \frac{x^3}{12}\right]_0^4$$

M1

$$= \frac{32}{3} - \frac{16}{3} = \frac{16}{3}$$

A1

$$\text{Answer: } \frac{16}{3}$$

3 marks

$$(c) V = \pi \int_0^4 \left(4x - \frac{x^4}{16}\right) dx$$

M1

$$= \pi \left[2x^2 - \frac{x^5}{80}\right]_0^4$$

M1

$$= \pi \left(32 - \frac{64}{5}\right) = \frac{96\pi}{5}$$

A1

$$\text{Answer: } \frac{96\pi}{5}$$

3 marks

Tangents and Normals to Conics

1 (3 marks)

$$2y \frac{dy}{dx} = 4, \text{ so } \frac{dy}{dx} = \frac{2}{y} = 1$$

M1 A1

$$y - 2 = x - 1, \text{ so } y = x + 1$$

A1

$$\text{Answer: } y = x + 1$$

3 marks

2 (3 marks)

$$\frac{dy}{dx} = \frac{4}{y} = 1$$

M1

Normal gradient -1

A1

$$y - 4 = -(x - 2), \text{ so } y = 6 - x$$

A1

$$\text{Answer: } y = 6 - x$$

3 marks

3 (3 marks)

$$\text{Tangent: } \frac{4x}{25} + \frac{\left(\frac{9}{5}\right)y}{9} = 1$$

M1 A1

$$4x + 5y = 25$$

A1

$$\text{Answer: } 4x + 5y = 25$$

3 marks

4 (3 marks)

$$y = \frac{4}{x}, \frac{dy}{dx} = -\frac{4}{x^2} = -1$$

M1 A1

$$y - 2 = -(x - 2), \text{ so } x + y = 4$$

A1

$$\text{Answer: } x + y = 4$$

3 marks

5 (3 marks)

$$\text{Tangent: } \frac{5x}{9} - \frac{\left(\frac{16}{3}\right)y}{16} = 1$$

M1 A1

$$5x - 3y = 9$$

A1

$$\text{Answer: } 5x - 3y = 9$$

3 marks

6 (2 marks)

$$\frac{dy}{dt} = 2a, \frac{dx}{dt} = 2at$$

M1

$$\frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}$$

A1

$$\text{Answer: Gradient} = \frac{1}{t}$$

2 marks

7 (4 marks)

$$\frac{2x}{16} + \frac{2y}{9} \frac{dy}{dx} = 0, \text{ so } \frac{dy}{dx} = -\frac{9x}{16y}$$

M1

$$\text{At the point: } -\frac{18}{24\sqrt{3}} = -\frac{\sqrt{3}}{4}$$

A1

$$\text{Normal gradient } \frac{4}{\sqrt{3}}$$

M1

$$y - \frac{3\sqrt{3}}{2} = \frac{4}{\sqrt{3}}(x - 2), \text{ so } 8x - 2\sqrt{3}y = 7$$

A1

$$\text{Answer: } 8x - 2\sqrt{3}y = 7$$

4 marks

8 (3 marks)

$$a = 3. \text{ Tangent } y = mx + \frac{a}{m}$$

M1

$$c = \frac{3}{3} = 1$$

A1

$$y = 3x + 1$$

A1

$$\text{Answer: } y = 3x + 1$$

3 marks

9 (3 marks)

$$\text{Point } (9, 1); \frac{dy}{dt} = -\frac{3}{t^2}, \frac{dx}{dt} = 3, \text{ so } \frac{dy}{dx} = -\frac{1}{t^2} = -\frac{1}{9}$$

M1 A1

$$y - 1 = -\frac{1}{9}(x - 9), \text{ so } x + 9y = 18$$

A1

$$\text{Answer: } x + 9y = 18$$

3 marks

10 (4 marks)

$$\frac{dy}{dx} = \frac{8}{y} = 2$$

M1

$$\text{Tangent } y - 4 = 2(x - 1), \text{ so } y = 2x + 2$$

A1

$$A(-1, 0), B(0, 2)$$

A1

$$\text{Area} = \frac{1}{2}(1)(2) = 1$$

A1

$$\text{Answer: } 1$$

4 marks

11 (3 marks)

$$\text{Check: } \frac{16}{4} - \frac{27}{9} = 1$$

B1

$$\text{Tangent: } \frac{4x}{4} - \frac{3\sqrt{3}y}{9} = 1$$

M1

$$x - \frac{\sqrt{3}}{3}y = 1$$

A1

$$\text{Answer: } x - \frac{\sqrt{3}}{3}y = 1$$

3 marks

12 (4 marks)

$$\text{Tangent gradient } \frac{2}{y} = \frac{1}{2}, \text{ normal gradient } -2$$

M1

$$y - 4 = -2(x - 4)$$

A1

$$y = 0: x = 6, \text{ so } N(6, 0)$$

A1

$$SN = 5$$

A1

$$\text{Answer: } 5$$

4 marks

13 (5 marks)

Tangent $y = mx + c$: substitute into $y^2 = 8x$ to get $m^2x^2 + (2mc - 8)x + c^2 = 0$ M1

Tangent: discriminant zero, $(2mc - 8)^2 - 4m^2c^2 = 0$, so $64 - 32mc = 0$ and $c = \frac{2}{m}$ A1

Through $(-2, 3)$: $3 = -2m + \frac{2}{m}$, so $2m^2 + 3m - 2 = 0$ M1

$(2m - 1)(m + 2) = 0$, so $m = \frac{1}{2}$ or $m = -2$ A1

$y = \frac{1}{2}x + 4$ and $y = -2x - 1$ A1

Answer: $y = \frac{1}{2}x + 4$; $y = -2x - 1$ 5 marks

14 (4 marks)

$\frac{dx}{d\theta} = -a \sin \theta$, $\frac{dy}{d\theta} = b \cos \theta$, so $\frac{dy}{dx} = -\frac{b \cos \theta}{a \sin \theta}$ M1 A1

$y - b \sin \theta = -\frac{b \cos \theta}{a \sin \theta}(x - a \cos \theta)$ M1

$bx \cos \theta + ay \sin \theta = ab(\cos^2 \theta + \sin^2 \theta) = ab$; dividing by ab gives $\frac{x \cos \theta}{a} + \frac{y \sin \theta}{b} = 1$ A1

Answer: $\frac{x \cos \theta}{a} + \frac{y \sin \theta}{b} = 1$ 4 marks

15 (5 marks)

$4x^2 - 9(x + k)^2 = 36$ M1

$5x^2 + 18kx + 9k^2 + 36 = 0$ A1

Discriminant zero: $324k^2 - 20(9k^2 + 36) = 0$ M1

$144k^2 = 720$, $k^2 = 5$ A1

$k = \pm\sqrt{5}$ A1

Answer: $k = \pm\sqrt{5}$ 5 marks

16 (5 marks)

Tangent $y = mx + 3$ B1

$x^2 + 4(mx + 3)^2 = 4$, so $(1 + 4m^2)x^2 + 24mx + 32 = 0$ M1 A1

Discriminant zero: $576m^2 - 128(1 + 4m^2) = 0$, so $64m^2 = 128$ M1

$m = \pm\sqrt{2}$: $y = \pm\sqrt{2}x + 3$ A1

Answer: $y = \pm\sqrt{2}x + 3$ 5 marks

17 (5 marks)

Tangent gradient $-\frac{b \cos \theta}{a \sin \theta}$, so normal gradient $\frac{a \sin \theta}{b \cos \theta}$ M1 A1

$b \cos \theta(y - b \sin \theta) = a \sin \theta(x - a \cos \theta)$ M1

by $\cos \theta - ax \sin \theta = (b^2 - a^2)\sin \theta \cos \theta$ A1

Divide by $\sin \theta \cos \theta$ and rearrange A1

Answer: $\frac{ax}{\cos \theta} - \frac{by}{\sin \theta} = a^2 - b^2$ 5 marks

18 (5 marks)

Tangent at P: $py = x + ap^2$; at Q: $qy = x + aq^2$ B1 B1

Subtract: $(p - q)y = a(p^2 - q^2)$ M1

$y = a(p + q)$ A1

$x = apy - ap^2 = a p(p + q) - ap^2 = apq$ A1

Answer: $T = (apq, a(p + q))$ 5 marks

19 (5 marks)Tangent gradient $\frac{2}{y} = 1$, normal gradient -1

M1

Normal: $y - 2 = -(x - 1)$, so $y = 3 - x$

A1

 $(3 - x)^2 = 4x$, so $x^2 - 10x + 9 = 0$

M1

 $x = 1$ or 9

A1

Q(9, -6)

A1

Answer: (9, -6)

5 marks

20 (4 marks)A ($y = 0$): $(2ct, 0)$; B ($x = 0$): $(0, \frac{2c}{t})$

M1 A1

Area = $\frac{1}{2}|2ct| \times |\frac{2c}{t}|$

M1

 $= 2c^2$, independent of t

A1

Answer: $2c^2$

4 marks

21 (6 marks) $b^2x^2 + a^2(mx + c)^2 = a^2b^2$

M1

 $(b^2 + a^2m^2)x^2 + 2a^2mcx + a^2(c^2 - b^2) = 0$

A1

Tangent: discriminant zero

M1

 $4a^4m^2c^2 - 4a^2(b^2 + a^2m^2)(c^2 - b^2) = 0$

A1

 $a^2m^2c^2 - b^2c^2 + b^4 - a^2m^2c^2 + a^2m^2b^2 = 0$

M1

 $c^2 = b^2 + a^2m^2$

A1

Answer: $c^2 = a^2m^2 + b^2$

6 marks

22 (6 marks)Tangent $y = mx + \frac{a}{m}$; through (h, k) : $k = mh + \frac{a}{m}$

M1 A1

 $m^2h - mk + a = 0$

A1

The two tangent gradients m_1, m_2 are the roots, so $m_1m_2 = \frac{a}{h}$

M1 A1

Perpendicular: $m_1m_2 = -1$, so $h = -a$

A1

Answer: $h = -a$, so the point lies on the directrix

6 marks

23 (3 marks)

An undefined gradient means the tangent is vertical, not that it does not exist

E1

Tangent: $\frac{2x}{4} - 0 = 1$

M1

 $x = 2$

A1

Answer: $x = 2$

3 marks

24 (6 marks)Normal: $\frac{4x}{\cos \theta} - \frac{3y}{\sin \theta} = 7$

B1

A: $x = \frac{7 \cos \theta}{4}$; B: $y = -\frac{7 \sin \theta}{3}$

M1 A1

M: $x = \frac{7 \cos \theta}{8}$, $y = -\frac{7 \sin \theta}{6}$

A1

 $\cos^2 \theta + \sin^2 \theta = 1$: $(\frac{8x}{7})^2 + (\frac{6y}{7})^2 = 1$

M1

 $64x^2 + 36y^2 = 49$

A1

Answer: $64x^2 + 36y^2 = 49$

6 marks

25 (5 marks)

Tangent: $py = x + ap^2$

B1

$y = 0: x = -ap^2, \text{ so } T(-ap^2, 0)$

M1 A1

$ST = a + ap^2$

A1

$SP = x_p + a = ap^2 + a = ST$

A1

Answer: $ST = SP = a(1 + p^2)$

5 marks

26 (6 marks)

Parabola tangent: $y = mx + \frac{1}{m}$

B1

Ellipse: $c^2 = 2m^2 + 1$

B1

$\frac{1}{m^2} = 2m^2 + 1, \text{ so } 2m^4 + m^2 - 1 = 0$

M1 A1

$(2m^2 - 1)(m^2 + 1) = 0, m^2 = \frac{1}{2}$

M1

$y = \frac{x}{\sqrt{2}} + \sqrt{2} \text{ and } y = -\frac{x}{\sqrt{2}} - \sqrt{2}$

A1

Answer: $y = \pm(\frac{x}{\sqrt{2}} + \sqrt{2})$

6 marks

27 (7 marks)

Tangent at A: $\frac{xx_1}{4} + yy_1 = 1$

M1

T lies on it: $x_1 + 3y_1 = 1$

A1

Likewise $x_2 + 3y_2 = 1$

A1

So A and B both lie on $x + 3y = 1$

A1

Substitute $x = 1 - 3y$: $(1 - 3y)^2 + 4y^2 = 4$

M1

$13y^2 - 6y - 3 = 0, y = \frac{3 \pm 4\sqrt{3}}{13}$

A1

$x = 1 - 3y = \frac{4 \mp 12\sqrt{3}}{13}$

A1

Answer: $x + 3y = 1; (\frac{4 - 12\sqrt{3}}{13}, \frac{3 + 4\sqrt{3}}{13}), (\frac{4 + 12\sqrt{3}}{13}, \frac{3 - 4\sqrt{3}}{13})$

7 marks

28 (8 marks)

Substitute (h, k): $k + th = 2at + at^3$

M1

$at^3 + (2a - h)t - k = 0$

A1

$a = 1, (8, 4): t^3 - 6t - 4 = 0$

M1

$t = -2 \text{ is a root: } -8 + 12 - 4 = 0$

A1

$(t + 2)(t^2 - 2t - 2) = 0, \text{ so } t = 1 \pm \sqrt{3}$

M1 A1

Normals: $y + tx = 2t + t^3$

M1

$t = -2: y - 2x = -12; t = 1 \pm \sqrt{3} \text{ give the other two normals}$

A1

Answer: $t = -2, 1 \pm \sqrt{3}$

8 marks

29 (9 marks)

(a) Tangent gradient $-\frac{1}{t^2}$, so normal gradient t^2

M1 A1

$y - \frac{2}{t} = t^2(x - 2t)$

M1

$ty - 2 = t^3x - 2t^4, \text{ so } t^3x - ty = 2(t^4 - 1)$

A1

Answer: $t^3x - ty = 2(t^4 - 1)$

4 marks

(b) Normal: $8x - 2y = 30, \text{ so } y = 4x - 15$

M1 A1

$x(4x - 15) = 4, 4x^2 - 15x - 4 = 0$

M1

$(4x + 1)(x - 4) = 0, x = -\frac{1}{4} \text{ (} x = 4 \text{ is P)}$

A1

$$Q(-\frac{1}{4}, -16)$$

A1

Answer: $Q(-\frac{1}{4}, -16)$

5 marks

30 (8 marks)

Tangent at $(a \cos \theta, b \sin \theta)$: $bx \cos \theta + ay \sin \theta - ab = 0$

M1 A1

Denominator: $D = \sqrt{b^2 \cos^2 \theta + a^2 \sin^2 \theta}$

B1

Distances: $\frac{|abe \cos \theta - ab|}{D}$ and $\frac{|-abe \cos \theta - ab|}{D}$

M1 A1

Product = $\frac{a^2 b^2 (1 - e^2 \cos^2 \theta)}{D^2}$

M1

$D^2 = a^2 - (a^2 - b^2) \cos^2 \theta = a^2 (1 - e^2 \cos^2 \theta)$

M1

Product = b^2

A1

Answer: Product of the distances = b^2

8 marks

Parametric Forms of Conics

1 (1 mark)

$x = 3(4) = 12, y = -12$

B1

Answer: (12, -12)

1 mark

2 (2 marks)

$\cos \theta = \frac{x}{5}, \sin \theta = \frac{y}{3}$

M1

$\frac{x^2}{25} + \frac{y^2}{9} = 1$

A1

Answer: $\frac{x^2}{25} + \frac{y^2}{9} = 1$

2 marks

3 (2 marks)

$\sec \theta = \frac{x}{2}, \tan \theta = \frac{y}{3}, \text{ and } \sec^2 \theta - \tan^2 \theta = 1$

M1

$\frac{x^2}{4} - \frac{y^2}{9} = 1$

A1

Answer: $\frac{x^2}{4} - \frac{y^2}{9} = 1$

2 marks

4 (2 marks)

$xy = 4t \times \frac{4}{t}$

M1

$xy = 16$

A1

Answer: $xy = 16$

2 marks

5 (1 mark)

$x = 4 \cos \theta, y = 2 \sin \theta$

B1

Answer: $x = 4 \cos \theta, y = 2 \sin \theta$

1 mark

6 (2 marks)

$x = 2 \cos \frac{\pi}{6} = \sqrt{3}$

B1

$y = 5 \sin \frac{\pi}{6} = \frac{5}{2}$

B1

Answer: $(\sqrt{3}, \frac{5}{2})$

2 marks

7 (4 marks)

$$t = \frac{y}{10}, x = \frac{5y^2}{100} = \frac{y^2}{20}$$

M1

$$y^2 = 20x$$

A1

$$4a = 20, a = 5$$

M1

Focus (5, 0)

A1

Answer: $y^2 = 20x$; focus (5, 0)

4 marks

8 (4 marks)

$$\cos \theta = \frac{x-1}{3}, \sin \theta = \frac{y+2}{2}$$

M1

$$\frac{(x-1)^2}{9} + \frac{(y+2)^2}{4} = 1$$

A1

Centre (1, -2)

A1

Semi-axes 3 and 2

A1

Answer: Centre (1, -2), semi-axes 3 and 2

4 marks

9 (4 marks)

$$t = \frac{y}{4}, \text{ so } x = \frac{y^2}{16} + 1$$

M1

$$y^2 = 16(x-1)$$

A1

Vertex (1, 0)

A1

(the parameter gives all points since $y = 4t$ is any real number)

B1

Answer: $y^2 = 16(x-1)$; vertex (1, 0)

4 marks

10 (4 marks)

$$x = 3 \sec \frac{\pi}{3} = 6, y = 2 \tan \frac{\pi}{3} = 2\sqrt{3}$$

M1 A1

$$\frac{36}{9} - \frac{12}{4}$$

M1

$$= 4 - 3 = 1$$

A1

Answer: (6, $2\sqrt{3}$)

4 marks

11 (3 marks)

$$2at = 2a, \text{ so } t = 1$$

M1

$$x = a(1) = a$$

A1

The latus rectum is the chord $x = a$ (through the focus (a, 0))

E1

Answer: $t = 1$ gives (a, 2a), and $x = a$ is the latus rectum

3 marks

12 (4 marks)

$$\frac{3}{t} = 6t - 3$$

M1

$$6t^2 - 3t - 3 = 0, \text{ so } 2t^2 - t - 1 = 0$$

A1

$$(2t+1)(t-1) = 0$$

M1

$$t = 1 \text{ or } t = -\frac{1}{2}$$

A1

Answer: $t = 1, -\frac{1}{2}$

4 marks

13 (4 marks)

$$3 \sin \theta = 3 \cos \theta, \text{ so } \tan \theta = 1$$

M1 A1

$$\theta = \frac{\pi}{4} \text{ or } \frac{5\pi}{4}$$

A1

$$(2\sqrt{2}, \frac{3\sqrt{2}}{2}) \text{ and } (-2\sqrt{2}, -\frac{3\sqrt{2}}{2})$$

A1

Answer: $(\pm 2\sqrt{2}, \pm \frac{3\sqrt{2}}{2})$

4 marks

14 (5 marks)

$$PF^2 = (4 \cos \theta - \sqrt{7})^2 + 9 \sin^2 \theta$$

M1

$$= 16 \cos^2 \theta - 8\sqrt{7} \cos \theta + 7 + 9 - 9 \cos^2 \theta$$

M1 A1

$$= 7 \cos^2 \theta - 8\sqrt{7} \cos \theta + 16 = (4 - \sqrt{7} \cos \theta)^2$$

M1

$$4 - \sqrt{7} \cos \theta > 0, \text{ so } PF = 4 - \sqrt{7} \cos \theta$$

A1

$$\text{Answer: } PF = 4 - \sqrt{7} \cos \theta$$

5 marks

15 (5 marks)

$$x^2 + y^2 = \frac{(1 - t^2)^2 + 4t^2}{(1 + t^2)^2}$$

M1 A1

$$= \frac{(1 + t^2)^2}{(1 + t^2)^2} = 1$$

A1

$$x = -1 \text{ would need } t^2 \rightarrow \infty$$

M1

The point $(-1, 0)$ is not obtained

A1

Answer: $(-1, 0)$ is missing

5 marks

16 (5 marks)

$$\frac{dx}{d\theta} = 3 \sec \theta \tan \theta, \frac{dy}{d\theta} = 4 \sec^2 \theta$$

M1 A1

$$\frac{dy}{dx} = \frac{4 \sec \theta}{3 \tan \theta}$$

A1

$$y - 4 \tan \theta = \frac{4 \sec \theta}{3 \tan \theta} (x - 3 \sec \theta)$$

M1

Multiply by $3 \tan \theta$: $3y \tan \theta - 12 \tan^2 \theta = 4x \sec \theta - 12 \sec^2 \theta$, so $4x \sec \theta - 3y \tan \theta = 12$;
divide by 12

A1

$$\text{Answer: } \frac{x \sec \theta}{3} - \frac{y \tan \theta}{4} = 1$$

5 marks

17 (4 marks)

$$y_P - y_Q = \frac{c}{p} - \frac{c}{q} = \frac{c(q - p)}{pq}$$

M1 A1

$$x_P - x_Q = c(p - q)$$

A1

$$m = -\frac{1}{pq}$$

A1

$$\text{Answer: } m = -\frac{1}{pq}$$

4 marks

18 (4 marks)

$$M = \left(\frac{5 \cos \theta}{2}, \frac{3 \sin \theta}{2} \right)$$

M1 A1

$$\cos \theta = \frac{2x}{5}, \sin \theta = \frac{2y}{3}$$

M1

$$\frac{4x^2}{25} + \frac{4y^2}{9} = 1$$

A1

$$\text{Answer: } \frac{4x^2}{25} + \frac{4y^2}{9} = 1$$

4 marks

19 (5 marks)

$$\text{Midpoint } M = \left(\frac{a(p^2 + 1)}{2}, ap \right)$$

M1 A1

$$y^2 = a^2 p^2$$

M1

$$2a \left(x - \frac{a}{2} \right) = 2a \left(\frac{ap^2}{2} \right) = a^2 p^2$$

A1

So M lies on the parabola

A1

$$\text{Answer: } y^2 = 2a \left(x - \frac{a}{2} \right)$$

5 marks

20 (4 marks)

$$\frac{dy}{dt} = -\frac{c}{t^2}, \frac{dx}{dt} = c, \text{ so } \frac{dy}{dx} = -\frac{1}{t^2}$$

M1 A1

$$y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$$

M1

$$t^2y - ct = -x + ct, \text{ so } x + t^2y = 2ct$$

A1

$$\text{Answer: } x + t^2y = 2ct$$

4 marks

21 (6 marks)

$$\text{Tangent at } \alpha: \frac{x \cos \alpha}{a} + \frac{y \sin \alpha}{b} = 1$$

B1

$$\text{Tangent at } -\alpha: \frac{x \cos \alpha}{a} - \frac{y \sin \alpha}{b} = 1$$

B1

$$\text{Add: } \frac{2x \cos \alpha}{a} = 2, x = \frac{a}{\cos \alpha}$$

M1 A1

$$\text{Subtract: } y \sin \alpha = 0, \text{ so } y = 0$$

M1

$$T\left(\frac{a}{\cos \alpha}, 0\right) = (a \sec \alpha, 0)$$

A1

$$\text{Answer: } (a \sec \alpha, 0)$$

6 marks

22 (5 marks)

$$\text{Gradient } \frac{2}{p+q} = 1$$

M1

$$p + q = 2$$

A1

$$\text{Midpoint } y = a(p + q)$$

M1

$$= 2a$$

A1

$$(\text{independent of the chord, so all such midpoints lie on } y = 2a)$$

B1

$$\text{Answer: } p + q = 2$$

5 marks

23 (6 marks)

$$\text{Gradient AP} = \frac{2 \sin \theta}{3 \cos \theta - 3}$$

M1 A1

$$\text{Gradient BP} = \frac{2 \sin \theta}{3 \cos \theta + 3}$$

A1

$$\text{Product} = \frac{4 \sin^2 \theta}{9 \cos^2 \theta - 9}$$

M1

$$= \frac{4 \sin^2 \theta}{-9 \sin^2 \theta} = -\frac{4}{9}$$

M1 A1

$$\text{Answer: } -\frac{4}{9}$$

6 marks

24 (4 marks)

$$\frac{x^2}{9} + \frac{y^2}{4} = \cos^2 \theta + \sin^2 \theta = 1, \text{ an ellipse}$$

M1 E1

$$\text{For } \frac{x^2}{9} - \frac{y^2}{4}, \text{ use } \sec^2 \theta - \tan^2 \theta = 1$$

M1

$$x = 3 \sec \theta, y = 2 \tan \theta \left(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\right)$$

A1

$$\text{Answer: } x = 3 \sec \theta, y = 2 \tan \theta$$

4 marks

25 (6 marks)

$$\text{Tangent: } \frac{x \cos \theta}{4} + \frac{y \sin \theta}{2} = 1$$

B1

$$A(4 \sec \theta, 0), B(0, 2 \operatorname{cosec} \theta)$$

M1 A1

$$AB^2 = 16 \sec^2 \theta + 4 \operatorname{cosec}^2 \theta = 16 + 16 \tan^2 \theta + 4 + 4 \cot^2 \theta$$

M1

$$= 20 + 16 \tan^2 \theta + 4 \cot^2 \theta \geq 20 + 2\sqrt{64} = 36$$

M1

$$\text{Least } AB = 6 \left(\text{at } \tan^2 \theta = \frac{1}{2}\right)$$

A1

$$\text{Answer: } 6$$

6 marks

26 (5 marks)

$$OP^2 = c^2t^2 + \frac{c^2}{t^2}$$

B1

$$= c^2[(t - \frac{1}{t})^2 + 2]$$

M1 A1

$$OP^2 \geq 2c^2, \text{ with equality when } t = 1$$

M1

$$\text{Least } OP = c\sqrt{2} \text{ at } (c, c)$$

A1

$$\text{Answer: } c\sqrt{2} \text{ at } (c, c)$$

5 marks

27 (6 marks)

$$\text{Normal at P: } y + px = 2ap + ap^3$$

B1

$$\text{Substitute Q: } 2aq + paq^2 = 2ap + ap^3$$

M1

$$pq^2 + 2q - 2p - p^3 = 0$$

A1

$$(q - p)(pq + p^2 + 2) = 0$$

M1

$$q \neq p, \text{ so } pq = -p^2 - 2$$

A1

$$q = -p - \frac{2}{p}$$

A1

$$\text{Answer: } q = -p - \frac{2}{p}$$

6 marks

28 (8 marks)

$$\text{Chord PQ: } \frac{x}{a}\cos\frac{\alpha+\beta}{2} + \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha-\beta}{2}$$

M1 A1

$$\text{Through F: } e\cos\frac{\alpha+\beta}{2} = \cos\frac{\alpha-\beta}{2}$$

M1 A1

$$e(\cos\frac{\alpha}{2}\cos\frac{\beta}{2} - \sin\frac{\alpha}{2}\sin\frac{\beta}{2}) = \cos\frac{\alpha}{2}\cos\frac{\beta}{2} + \sin\frac{\alpha}{2}\sin\frac{\beta}{2}$$

M1

$$\text{Divide by } \cos\frac{\alpha}{2}\cos\frac{\beta}{2}: e(1 - T) = 1 + T, \text{ where } T = \tan\frac{\alpha}{2}\tan\frac{\beta}{2}$$

M1

$$T(1 + e) = e - 1$$

M1

$$T = -\frac{1 - e}{1 + e}$$

A1

$$\text{Answer: } \tan\frac{\alpha}{2}\tan\frac{\beta}{2} = -\frac{1 - e}{1 + e}$$

8 marks

29 (6 marks)

$$\text{Tangent gradient at } t \text{ is } -\frac{1}{t^2}, \text{ so the normal gradient at P is } p^2$$

M1 A1

$$\text{Chord PQ has gradient } -\frac{1}{pq}$$

M1 A1

$$\text{The chord PQ is the normal: } -\frac{1}{pq} = p^2$$

M1

$$p^3q = -1$$

A1

$$\text{Answer: } p^3q = -1$$

6 marks

30 (8 marks)

Tangent: $\frac{x \cos \theta}{2} + y \sin \theta = 1$

B1

$x = 2: y = \frac{1 - \cos \theta}{\sin \theta}; A(2, \frac{1 - \cos \theta}{\sin \theta})$

M1 A1

$x = -2: y = \frac{1 + \cos \theta}{\sin \theta}; B(-2, \frac{1 + \cos \theta}{\sin \theta})$

A1

Gradient $F_1A = \frac{\frac{1 - \cos \theta}{\sin \theta}}{2 + \sqrt{3}}$, gradient $F_1B = \frac{\frac{1 + \cos \theta}{\sin \theta}}{\sqrt{3} - 2}$

M1 A1

Product = $\frac{1 - \cos^2 \theta}{\sin^2 \theta} \times \frac{1}{(\sqrt{3} + 2)(\sqrt{3} - 2)}$

M1

$= 1 \times \frac{1}{3 - 4} = -1$, so AF_1 is perpendicular to BF_1

A1

Answer: $A(2, \tan \frac{\theta}{2}), B(-2, \cot \frac{\theta}{2}); AF_1 \perp BF_1$

8 marks**Loci and Geometrical Properties****1 (2 marks)**

$(x - 2)^2 + (y + 1)^2 = 25$

M1 A1

Answer: $(x - 2)^2 + (y + 1)^2 = 25$

2 marks**2 (2 marks)**

$x^2 + y^2 = (x - 6)^2 + y^2$

M1

$x = 3$

A1

Answer: $x = 3$

2 marks**3 (3 marks)**

$x^2 + (y - 2)^2 = (y + 2)^2$

M1 A1

$x^2 = 8y$

A1

Answer: $x^2 = 8y$

3 marks**4 (2 marks)**

Circle, centre O, radius 4

B1

$x^2 + y^2 = 16$

B1

Answer: $x^2 + y^2 = 16$

2 marks**5 (3 marks)**

$P(t, 3t)$

B1

$y = 3x$

M1 A1

Answer: $y = 3x$

3 marks**6 (1 mark)** $e = 1$ gives a parabola**B1**

Answer: **Parabola**

1 mark**7 (4 marks)**

$x^2 + y^2 = 4[(x - 6)^2 + y^2]$

M1

$3x^2 - 48x + 3y^2 + 144 = 0$

A1

$(x - 8)^2 + y^2 = 16$

M1

Centre (8, 0), radius 4

A1

Answer: **Centre (8, 0), radius 4**

4 marks

8 (3 marks)

$$(x - 2)^2 + y^2 = (x + 2)^2$$

M1 A1

$$y^2 = 8x, \text{ a parabola}$$

A1

$$\text{Answer: } y^2 = 8x$$

3 marks

9 (4 marks)

Ellipse with foci $(\pm 4, 0)$

B1

$$2a = 10, a = 5$$

M1

$$b^2 = 25 - 16 = 9$$

M1

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

A1

$$\text{Answer: } \frac{x^2}{25} + \frac{y^2}{9} = 1$$

4 marks

10 (4 marks)

Hyperbola with foci $(\pm 5, 0)$, $2a = 8$, $a = 4$

M1 A1

$$b^2 = 25 - 16 = 9$$

M1

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

A1

$$\text{Answer: } \frac{x^2}{16} - \frac{y^2}{9} = 1$$

4 marks

11 (3 marks)

$$P = A + \frac{1}{3}(B - A) = \left(\frac{2t}{3}, \frac{2t}{3}\right)$$

M1 A1

Locus $y = x$

A1

$$\text{Answer: } y = x$$

3 marks

12 (2 marks)

$$t = \frac{x}{2}, \text{ so } y = \frac{x^2}{4}$$

M1

$$x^2 = 4y, \text{ a parabola}$$

A1

$$\text{Answer: } x^2 = 4y$$

2 marks

13 (3 marks)

$$x^2 + (y - 3)^2 = y^2$$

M1 A1

$$x^2 = 6y - 9$$

A1

$$\text{Answer: } x^2 = 6y - 9$$

3 marks

14 (5 marks)

$$(x - 3)^2 + y^2 = \frac{9}{25}\left(x - \frac{25}{3}\right)^2$$

M1 A1

$$x^2 - 6x + 9 + y^2 = \frac{9}{25}x^2 - 6x + 25$$

M1

$$\frac{16}{25}x^2 + y^2 = 16$$

A1

$$\frac{x^2}{25} + \frac{y^2}{16} = 1$$

A1

$$\text{Answer: } \frac{x^2}{25} + \frac{y^2}{16} = 1$$

5 marks

15 (5 marks)

$$(x - 5)^2 + y^2 = \frac{25}{9}(x - \frac{9}{5})^2 \quad \text{M1 A1}$$

$$x^2 - 10x + 25 + y^2 = \frac{25}{9}x^2 - 10x + 9 \quad \text{M1}$$

$$\frac{16}{9}x^2 - y^2 = 16 \quad \text{A1}$$

$$\frac{x^2}{9} - \frac{y^2}{16} = 1 \quad \text{A1}$$

Answer: $\frac{x^2}{9} - \frac{y^2}{16} = 1$ 5 marks

16 (5 marks)

$$x^2 + y^2 = 4(x - 3)^2 \quad \text{M1 A1}$$

$$3x^2 - 24x + 36 - y^2 = 0 \quad \text{A1}$$

$$3(x - 4)^2 - y^2 = 12 \quad \text{M1}$$

$$\frac{(x - 4)^2}{4} - \frac{y^2}{12} = 1 \quad \text{A1}$$

Answer: $\frac{(x - 4)^2}{4} - \frac{y^2}{12} = 1$ 5 marks

17 (5 marks)

$$A(a, 0), B(0, b) \text{ with } a^2 + b^2 = 36 \quad \text{B1}$$

$$P = (\frac{2a}{3}, \frac{b}{3}) \quad \text{M1 A1}$$

$$a = \frac{3x}{2}, b = 3y \quad \text{M1}$$

$$\frac{9x^2}{4} + 9y^2 = 36, \text{ so } \frac{x^2}{16} + \frac{y^2}{4} = 1 \quad \text{A1}$$

Answer: $\frac{x^2}{16} + \frac{y^2}{4} = 1$ 5 marks

18 (4 marks)

$$(x + a)^2 + y^2 + (x - a)^2 + y^2 = 4a^2 \quad \text{M1}$$

$$2x^2 + 2y^2 + 2a^2 = 4a^2 \quad \text{A1}$$

$$x^2 + y^2 = a^2 \quad \text{A1}$$

Circle with radius a A1

Answer: $x^2 + y^2 = a^2$; radius a 4 marks

19 (4 marks)

$$P(4 \cos \theta, 4 \sin \theta), \text{ so } M = (4 + 2 \cos \theta, 2 \sin \theta) \quad \text{M1 A1}$$

$$(x - 4)^2 + y^2 = 4 \quad \text{M1}$$

A circle with centre (4, 0) and radius 2 A1

Answer: $(x - 4)^2 + y^2 = 4$ 4 marks

20 (5 marks)

$$\frac{y}{x+3} \times \frac{y}{x-3} = -\frac{4}{9} \quad \text{M1 A1}$$

$$9y^2 = -4(x^2 - 9) \quad \text{M1}$$

$$\frac{x^2}{9} + \frac{y^2}{4} = 1 \quad \text{A1}$$

A and B themselves are excluded (gradients undefined) B1

Answer: $\frac{x^2}{9} + \frac{y^2}{4} = 1$ excluding $(\pm 3, 0)$ 5 marks

21 (6 marks)

$$OC = 1 + r \text{ (external tangency)}$$

B1

$$CF = r$$

B1

$$OC - CF = 1$$

M1

Hyperbola with foci O and F, $2a = 1$, centre $(\frac{3}{2}, 0)$

M1

$$c = \frac{3}{2}, b^2 = c^2 - a^2 = \frac{9}{4} - \frac{1}{4} = 2$$

M1

$$4(x - \frac{3}{2})^2 - \frac{y^2}{2} = 1, \text{ the branch nearer F}$$

A1

$$\text{Answer: } 4(x - \frac{3}{2})^2 - \frac{y^2}{2} = 1 \text{ (branch with } x > \frac{3}{2})$$

6 marks

22 (6 marks)

$$\text{Focal chord: } pq = -1$$

B1

$$M = (\frac{a(p^2 + q^2)}{2}, a(p + q))$$

M1 A1

$$\text{Let } s = p + q: p^2 + q^2 = s^2 + 2$$

M1

$$x = \frac{a(s^2 + 2)}{2}, y = as, \text{ so } s^2 = \frac{y^2}{a^2}$$

M1

$$y^2 = 2a(x - a)$$

A1

$$\text{Answer: } y^2 = 2a(x - a)$$

6 marks

23 (6 marks)

$$\text{Tangent } y = mx + c \text{ with } c^2 = a^2m^2 + b^2$$

B1

$$\text{Through T: } (k - mh)^2 = a^2m^2 + b^2$$

M1

$$(h^2 - a^2)m^2 - 2hkm + k^2 - b^2 = 0$$

A1

$$\text{The gradients } m_1, m_2 \text{ are the roots: } m_1m_2 = \frac{k^2 - b^2}{h^2 - a^2}$$

M1

$$\text{Perpendicular: } m_1m_2 = -1$$

M1

$$k^2 - b^2 = a^2 - h^2, \text{ so } h^2 + k^2 = a^2 + b^2$$

A1

$$\text{Answer: T lies on } x^2 + y^2 = a^2 + b^2$$

6 marks

24 (4 marks)

$$AB = 6, \text{ and by the triangle inequality } PA + PB \geq AB = 6$$

M1 E1

$$\text{So } PA + PB = 4 \text{ is impossible}$$

A1

$$\text{The locus is empty } (b^2 = a^2 - c^2 = 4 - 9 < 0 \text{ confirms there is no real ellipse)}$$

A1

$$\text{Answer: No points: the locus is empty}$$

4 marks

25 (5 marks)

$$\text{Tangent } y = mx + \frac{a}{m}$$

B1

$$\text{Perpendicular from S: } y = -\frac{1}{m}(x - a)$$

M1

$$mx + \frac{a}{m} = -\frac{x - a}{m}$$

M1

$$m^2x + a = -x + a, \text{ so } x(m^2 + 1) = 0$$

A1

$$x = 0, \text{ so N lies on the y-axis}$$

A1

$$\text{Answer: The foot of the perpendicular lies on } x = 0, \text{ the y-axis}$$

5 marks

26 (5 marks)

$$(x - ae)^2 + y^2 = e^2(x - \frac{a}{e})^2 = (ex - a)^2 \quad \text{M1 A1}$$

$$x^2 - 2aex + a^2e^2 + y^2 = e^2x^2 - 2aex + a^2 \quad \text{M1}$$

$$(1 - e^2)x^2 + y^2 = a^2(1 - e^2) \quad \text{A1}$$

$$\text{Divide by } a^2(1 - e^2): \frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1 \quad \text{A1}$$

$$\text{Answer: } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{5 marks}$$

27 (7 marks)

$$(x + a)^2 + y^2 = k^2[(x - a)^2 + y^2] \quad \text{M1 A1}$$

$$(1 - k^2)(x^2 + y^2) + 2a(1 + k^2)x + a^2(1 - k^2) = 0 \quad \text{A1}$$

$$x^2 + y^2 + 2a\frac{1 + k^2}{1 - k^2}x + a^2 = 0 \quad \text{M1}$$

$$\text{Centre } x_0 = -a\frac{1 + k^2}{1 - k^2} = a\frac{k^2 + 1}{k^2 - 1}, y_0 = 0 \quad \text{A1}$$

$$r^2 = x_0^2 - a^2 = a^2\frac{(k^2 + 1)^2 - (k^2 - 1)^2}{(k^2 - 1)^2} \quad \text{M1}$$

$$= \frac{4a^2k^2}{(k^2 - 1)^2}, \text{ so } r = \frac{2ak}{|k^2 - 1|} \quad \text{A1}$$

$$\text{Answer: Circle, centre } \left(\frac{a(k^2 + 1)}{k^2 - 1}, 0\right), \text{ radius } \frac{2ak}{|k^2 - 1|} \quad \text{7 marks}$$

28 (8 marks)

$$\text{Tangent: } \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1 \quad \text{B1}$$

$$\text{At } x = \frac{a}{e}: y_T = \frac{b^2(ae - x_1)}{aey_1} \quad \text{M1 A1}$$

$$\text{FP} = (x_1 - ae, y_1), \text{ FT} = \left(\frac{a(1 - e^2)}{e}, y_T\right) \quad \text{B1}$$

$$\text{FP} \cdot \text{FT} = (x_1 - ae)\frac{a(1 - e^2)}{e} + \frac{b^2(ae - x_1)}{ae} \quad \text{M1 A1}$$

$$= \frac{(x_1 - ae)}{e}[a(1 - e^2) - \frac{b^2}{a}] \quad \text{M1}$$

$$b^2 = a^2(1 - e^2), \text{ so the bracket is zero and FP is perpendicular to FT} \quad \text{A1}$$

$$\text{Answer: } \text{FP} \cdot \text{FT} = 0, \text{ so angle PFT} = 90^\circ \quad \text{8 marks}$$

29 (9 marks)

$$\text{Tangent: } bx \cos \theta + ay \sin \theta = ab, \text{ with normal vector } n = (b \cos \theta, a \sin \theta) \quad \text{M1}$$

$$N = F + \lambda n \text{ lies on the tangent: } b \cos \theta(ae + \lambda b \cos \theta) + a \sin \theta(\lambda a \sin \theta) = ab \quad \text{M1}$$

$$\lambda D^2 = ab(1 - e \cos \theta), \text{ where } D^2 = b^2 \cos^2 \theta + a^2 \sin^2 \theta = a^2(1 - e^2 \cos^2 \theta) \quad \text{M1 A1}$$

$$\text{ON}^2 = |F|^2 + 2\lambda F \cdot n + \lambda^2 D^2 = a^2 e^2 + 2\lambda ab e \cos \theta + \lambda ab(1 - e \cos \theta) \quad \text{M1}$$

$$= a^2 e^2 + \lambda ab(1 + e \cos \theta) \quad \text{A1}$$

$$\lambda ab(1 + e \cos \theta) = \frac{a^2 b^2 (1 - e^2 \cos^2 \theta)}{a^2 (1 - e^2 \cos^2 \theta)} = b^2 \quad \text{M1 A1}$$

$$\text{ON}^2 = a^2 e^2 + b^2 = a^2 \quad \text{A1}$$

$$\text{Answer: The foot lies on } x^2 + y^2 = a^2 \quad \text{9 marks}$$

30 (7 marks)

Tangent: $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$; $y = 0$ gives $x = \frac{a^2}{x_1}$, so $OT = \frac{a^2}{x_1}$

M1 A1

Tangent gradient $-\frac{b^2x_1}{a^2y_1}$, so normal gradient $\frac{a^2y_1}{b^2x_1}$

M1 A1

Normal: $y - y_1 = \frac{a^2y_1}{b^2x_1}(x - x_1)$; $y = 0$ gives $x - x_1 = -\frac{b^2x_1}{a^2}$

M1

$ON = x_1(1 - \frac{b^2}{a^2}) = e^2x_1$

A1

$OT \times ON = \frac{a^2}{x_1} \times e^2x_1 = a^2e^2$

A1

Answer: $OT \times ON = a^2e^2$

7 marks