

Conic Sections

1 Find the coordinates of the centre and the radius of the circle with equation $x^2 + y^2 - 6x + 4y - 3 = 0$. [3]

2 Write down the equation of the circle with centre $(1, -2)$ and radius 3. [1]

3 A conic section has eccentricity e . State the type of conic when

(a) $e = \frac{1}{2}$, [1]
 (b) $e = 1$, [1]
 (c) $e = 2$. [1]

4 Name each conic.

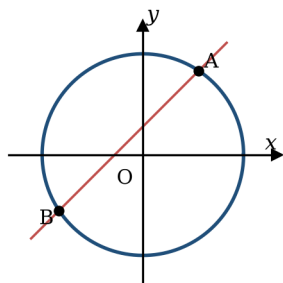
(a) $\frac{x^2}{9} + \frac{y^2}{4} = 1$ [1]
 (b) $y^2 = 8x$ [1]
 (c) $\frac{x^2}{9} - \frac{y^2}{4} = 1$ [1]

5 A parabola has focus $(3, 0)$ and directrix $x = -3$. Find its Cartesian equation. [3]

6 Show that the point $(1, \sqrt{3})$ lies on the circle $x^2 + y^2 = 4$, and write down the radius of the circle. [2]

7 A circle passes through the points $O(0, 0)$, $A(4, 0)$ and $B(0, 6)$. Find the equation of the circle. [4]

8 The line $y = x + 1$ meets the circle $x^2 + y^2 = 13$ at the points A and B. Find the coordinates of A and B.



[4]

9 Find the equation of the tangent to the circle $x^2 + y^2 = 25$ at the point $(3, 4)$. [3]

10 The points $A(-1, 2)$ and $B(5, 6)$ are the ends of a diameter of a circle. Find the equation of the circle. [4]

11 The parabola P has equation $y^2 = 16x$. Write down the coordinates of the focus and the equation of the directrix, and find the y-coordinates of the points on P with $x = 9$. [4]

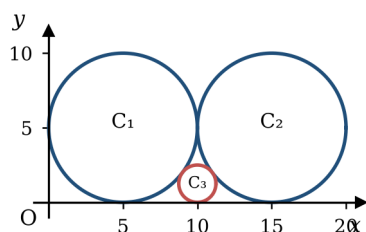
12 The line $y = 2x + k$ is a tangent to the circle $x^2 + y^2 = 5$. Find the possible values of k . [4]

13 The line $y = x + 1$ meets the ellipse $\frac{x^2}{4} + y^2 = 1$ at two points. Find their coordinates. [4]

- 14** A point $P(x, y)$ moves so that its distance from the point $F(4, 0)$ is half its distance from the line $x = 1$. Show that the locus of P is an ellipse, and find its equation in the form $\frac{(x - p)^2}{a^2} + \frac{y^2}{b^2} = 1$. [5]
- 15** A point $P(x, y)$ moves so that its distance from the point $F(4, 0)$ is twice its distance from the line $x = 1$. Show that the locus of P is a hyperbola, and find its equation in the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. [5]
- 16** The curve C has equation $4x^2 + 9y^2 - 16x + 18y - 11 = 0$. Show that C is an ellipse, and state the coordinates of its centre and the lengths of its semi-axes. [5]
- 17** The curve C has equation $y^2 - 4y - 8x + 20 = 0$. Show that C is a parabola, and find the coordinates of its vertex and its focus. [4]
- 18** A circle touches both coordinate axes and passes through the point $(1, 2)$. Find the possible equations of the circle, given that its centre is in the first quadrant. [5]
- 19** The circle C has equation $(x - 1)^2 + (y + 2)^2 = 8$. The line $y = x + c$ is a tangent to C . Find the possible values of c . [5]
- 20** The line $y = x - 3$ meets the parabola $y^2 = 4x$ at the points A and B . Find the length of AB . [5]
- 21** A point P moves so that $PA = 2PB$, where A is the point $(0, 0)$ and B is the point $(3, 0)$. Show that the locus of P is a circle, and find its centre and radius. [5]
- 22** A conic has focus $F(2, 3)$, directrix $x = -1$ and eccentricity $\frac{3}{2}$.
Show that the Cartesian equation of the conic is $5x^2 + 34x - 4y^2 + 24y - 43 = 0$, and state the type of conic. [6]
- 23** A circle passes through the points $A(1, 1)$, $B(5, 1)$ and $C(3, 5)$. Find the equation of the circle. [6]
- 24** A student claims that $x^2 + y^2 + 2x + 6y + 20 = 0$ is the equation of a circle with centre $(-1, -3)$ and radius $\sqrt{20}$.
Explain what is wrong with the student's statement, and say what the equation represents. [3]
- 25** The ellipse E has equation $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and the line L has equation $y = k$.
Find, in terms of k , the number of points of intersection of E and L , justifying your answer. [4]
- 26** For $a > 0$, the circle C_a has centre $(a, 0)$ and passes through the origin O . The parabola P has equation $y^2 = 4x$.
Find the range of values of a for which C_a and P meet at exactly one point. [6]
- 27** The points A and B have coordinates $(-3, 0)$ and $(3, 0)$. The point P moves so that $PA = 2PB$.
(a) Show that the locus of P is a circle, and find its centre and radius. [5]
(b) Find the coordinates of the points where the locus meets the x -axis, and show that they divide AB internally and externally in the ratio $2 : 1$. [3]

- 28** A conic has focus $F(2, 0)$, directrix $x = 8$ and eccentricity $\frac{1}{2}$.
- (a) Show that the Cartesian equation of the conic is $\frac{x^2}{16} + \frac{y^2}{12} = 1$. [4]
- (b) Find the coordinates of the second focus and the equation of the second directrix, and verify that the focus-directrix property holds for the point $(0, 2\sqrt{3})$ with respect to the second focus and directrix. [5]

- 29** The diagram shows three circles. The circle C_1 has centre $(5, 5)$ and touches both axes. The circle C_2 has the same radius as C_1 , touches the x -axis and touches C_1 . The circle C_3 touches the x -axis, C_1 and C_2 , and lies between them. Find an equation of C_3 .

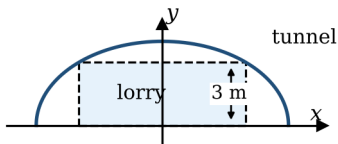


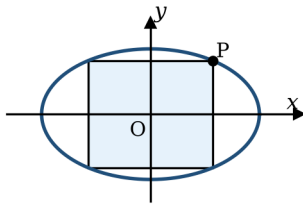
[8]

- 30** A circle with centre (h, k) touches the circle $x^2 + y^2 = 1$ externally and touches the line $x = -1$, lying to the right of this line. Show that the locus of (h, k) is a parabola, find its Cartesian equation, and state the coordinates of its focus and the equation of its directrix. [8]

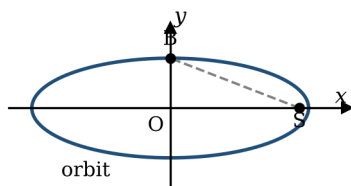
Ellipses

- 1** The ellipse E has equation $\frac{x^2}{25} + \frac{y^2}{9} = 1$. Write down the coordinates of the points where E meets the x -axis and the y -axis. [2]
- 2** The ellipse E has equation $\frac{x^2}{25} + \frac{y^2}{9} = 1$. Find the eccentricity of E . [2]
- 3** The ellipse E has equation $\frac{x^2}{25} + \frac{y^2}{9} = 1$ and eccentricity $\frac{4}{5}$. Write down the coordinates of the foci and the equations of the directrices. [2]
- 4** The ellipse E has equation $\frac{x^2}{25} + \frac{y^2}{9} = 1$ and foci $F_1(-4, 0)$ and $F_2(4, 0)$. Verify that the point $P(0, 3)$ satisfies $PF_1 + PF_2 = 10$. [2]
- 5** Write down the equation of the ellipse with centre at the origin, semi-axis 4 along the x -axis and semi-axis 2 along the y -axis. [1]
- 6** Show that the point $(3, \frac{12}{5})$ lies on the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$. [2]
- 7** The ellipse E has equation $4x^2 + 9y^2 = 36$. Find the eccentricity of E and the coordinates of its foci. [4]
- 8** An ellipse has centre O , foci at $(\pm 3, 0)$ and eccentricity $\frac{3}{5}$. Find its equation. [3]

- 9** The ellipse E has equation $\frac{x^2}{16} + \frac{y^2}{9} = 1$. Find the area enclosed by E, giving your answer in terms of π . [2]
- 10** The ellipse E has equation $\frac{x^2}{9} + \frac{y^2}{25} = 1$. State the length of the major axis, and find the coordinates of the foci and the eccentricity. [4]
- 11** An ellipse has centre $(2, -1)$, a horizontal semi-axis of length 3 and a vertical semi-axis of length 2. Write down its equation, and find the coordinates of its foci. [4]
- 12** The line $y = x$ meets the ellipse $\frac{x^2}{4} + y^2 = 1$ at two points. Find their coordinates. [4]
- 13** The ellipse E has equation $\frac{x^2}{36} + \frac{y^2}{20} = 1$. The point P on E is 4 units from the focus F_1 . Find the distance of P from the other focus F_2 , and the coordinates of the foci. [3]
- 14** An ellipse is given parametrically by $x = 3 \cos \theta$, $y = 2 \sin \theta$, $0 \leq \theta < 2\pi$. Find its Cartesian equation, and the coordinates of the point P where $\theta = \frac{\pi}{3}$. [4]
- 15** The point $P(3 \cos \theta, 2 \sin \theta)$ lies on the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$, and O is the origin. Show that $OP^2 = 4 + 5 \cos^2 \theta$, and hence find the greatest and least distances from O to a point of the ellipse. [4]
- 16** Sketch the ellipse with equation $\frac{(x-1)^2}{9} + \frac{(y+2)^2}{4} = 1$, showing the coordinates of its centre and of the points where it meets the coordinate axes. [5]
- 17** The line $y = x + k$ is a tangent to the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$. Find the possible values of k. [5]
- 18** The curve C has equation $9x^2 + 4y^2 - 36x + 8y + 4 = 0$. Show that C is an ellipse, and find its centre, its eccentricity and the coordinates of its foci. [6]
- 19** A tunnel has a cross-section in the shape of the upper half of the ellipse $\frac{x^2}{36} + \frac{y^2}{16} = 1$, where x and y are in metres and the ground is the x-axis. A lorry is 3 m high. Find the greatest width a lorry can have to pass through the tunnel, giving your answer to 3 significant figures.
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- [5]
- 20** A student says that the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ has foci at $(\pm\sqrt{5}, 0)$ because $a^2 - b^2 = 9 - 4 = 5$. Explain the error and write down the correct foci. [3]
- 21** The points $F_1(-3, 0)$ and $F_2(3, 0)$ are fixed. The point $P(x, y)$ moves so that $PF_1 + PF_2 = 10$. Show that the locus of P is the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$. [6]

- 22** The ellipse E has equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $b^2 = a^2(1 - e^2)$, and has foci $F_1(-ae, 0)$ and $F_2(ae, 0)$. The point $P(x, y)$ lies on E.
Show that $PF_2 = a - ex$, and find a similar expression for PF_1 . Hence show that $PF_1 + PF_2 = 2a$. [6]
- 23** An ellipse has focus $F(0, 3)$, directrix $y = 12$ and eccentricity $\frac{1}{2}$. Find its Cartesian equation, in the form $\frac{x^2}{p} + \frac{y^2}{q} = 1$, and state the coordinates of its centre. [6]
- 24** The chords of the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ with gradient 1 have midpoints $M(h, k)$.
Show that M lies on a straight line through the origin, and find its equation. [6]
- 25** Use the parametrisation $x = a \cos \theta$, $y = b \sin \theta$ to show that the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab . [6]
- 26** A rectangle is inscribed in the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$, with its sides parallel to the axes. One vertex is $P(5 \cos \theta, 3 \sin \theta)$, where $0 < \theta < \frac{\pi}{2}$.
Show that the area of the rectangle is $30 \sin 2\theta$, and find the greatest possible area, and the corresponding value of θ . [5]
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- 27** The circle C_r has equation $x^2 + y^2 = r^2$, where $r > 0$, and the ellipse E has equation $\frac{x^2}{9} + \frac{y^2}{4} = 1$.
Find, in terms of r , the number of points of intersection of C_r and E. [7]
- 28** Three points A, B and C lie on the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ and have coordinates $(4 \cos \alpha, 3 \sin \alpha)$, $(4 \cos(\alpha + \frac{2\pi}{3}), 3 \sin(\alpha + \frac{2\pi}{3}))$ and $(4 \cos(\alpha + \frac{4\pi}{3}), 3 \sin(\alpha + \frac{4\pi}{3}))$ respectively.
Show that the area of triangle ABC is $9\sqrt{3}$, whatever the value of α . [8]
- 29** The circle K has equation $(x + 3)^2 + y^2 = 100$. A circle with centre C and radius r passes through the point $F(3, 0)$ and touches K internally.
Show that the locus of C is an ellipse, and find its Cartesian equation. [7]

- 30** A comet moves in an elliptical orbit with the Sun at one focus. Its closest distance to the Sun is 0.6 AU and its greatest distance is 17.4 AU. The orbit has equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with the centre at the origin and the Sun at the focus $(c, 0)$, $c > 0$.
Find a , c , the eccentricity of the orbit and the exact value of b^2 , and the distance from the Sun to the comet when the comet is at the end of the minor axis.



[8]

Hyperbolas

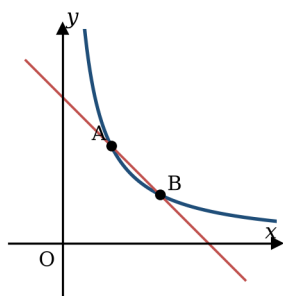
- 1** Write down the coordinates of the vertices of the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$. [1]
- 2** Write down the equations of the asymptotes of the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$. [2]
- 3** Find the eccentricity of the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$. [2]
- 4** The hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$ has eccentricity $\frac{5}{3}$. Write down the coordinates of its foci and the equations of its directrices. [2]
- 5** The rectangular hyperbola H has equation $xy = 4$. Write down the equations of its asymptotes, and find the coordinates of the points where the line $y = x$ meets H . [3]
- 6** Sketch the hyperbola with equation $\frac{y^2}{16} - \frac{x^2}{4} = 1$, showing the coordinates of its vertices and the equations of its asymptotes. [3]
- 7** A hyperbola has vertices at $(\pm 2, 0)$ and asymptotes $y = \pm \frac{3}{2}x$. Find its equation. [3]
- 8** A hyperbola has centre O , foci at $(\pm 17, 0)$ and asymptotes $y = \pm \frac{8}{15}x$. Find the Cartesian equation of the hyperbola. [4]
- 9** The hyperbola H has equation $9x^2 - 4y^2 = 36$. Find the coordinates of the vertices of H and the equations of its asymptotes. [3]
- 10** The hyperbola H has equation $\frac{x^2}{9} - \frac{y^2}{16} = 1$ and foci F_1 and F_2 . The point P lies on H , with $PF_1 = 11$. Find the possible values of PF_2 . [3]
- 11** The rectangular hyperbola H has equation $xy = 4$ and the line L has equation $y = x + 3$. Find the coordinates of the points where L meets H . [4]
- 12** A rectangular hyperbola has parametric equations $x = 3t$, $y = \frac{3}{t}$, $t \neq 0$. Find its Cartesian equation, and the coordinates of the point where $t = 2$. [3]

- 13** The hyperbola H has equation $\frac{(x-1)^2}{4} - \frac{(y+2)^2}{9} = 1$. Find the coordinates of its centre and vertices, and the equations of its asymptotes. [4]

- 14** A hyperbola is given parametrically by $x = 3 \sec \theta$, $y = 2 \tan \theta$. Show that its Cartesian equation is $\frac{x^2}{9} - \frac{y^2}{4} = 1$, and find the coordinates of the point where $\theta = \frac{\pi}{3}$. [4]

- 15** The hyperbola H has equation $\frac{x^2}{4} - \frac{y^2}{9} = 1$ and the line L has equation $y = mx$. Find the range of values of m for which L meets H . [5]

- 16** The rectangular hyperbola H has equation $xy = 8$ and the line L has equation $x + y = 6$. L meets H at the points A and B . Find the length of AB .



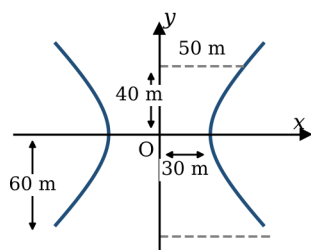
[4]

- 17** The hyperbola H has equation $\frac{x^2}{9} - \frac{y^2}{16} = 1$. Find the acute angle between its asymptotes, giving your answer in degrees to 1 decimal place. [4]

- 18** A hyperbola has focus $(5, 0)$, directrix $x = \frac{9}{5}$ and eccentricity $\frac{5}{3}$. Show that its equation is $\frac{x^2}{9} - \frac{y^2}{16} = 1$. [5]

- 19** The curve C has equation $9x^2 - 16y^2 - 36x - 32y - 124 = 0$. Show that C is a hyperbola, and find its centre, its eccentricity, the coordinates of its foci and the equations of its asymptotes. [6]

- 20** The cross-section of a cooling tower is modelled by the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where x is the radius and y the height above the narrowest part (the throat), both in metres. The throat has radius 30 m and at a height of 40 m the radius is 50 m. Find the values of a and b , and hence the radius of the tower at the base, which is 60 m below the throat, to 3 significant figures.



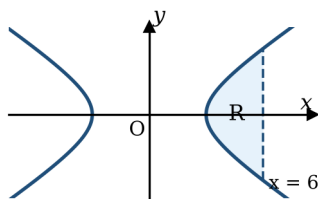
[5]

- 21** The hyperbola H has equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where $b^2 = a^2(e^2 - 1)$, and has foci $F_1(-ae, 0)$ and $F_2(ae, 0)$. The point $P(x, y)$ lies on the right-hand branch of H .
Show that $PF_2 = ex - a$ and $PF_1 = ex + a$, and hence that $PF_1 - PF_2 = 2a$. [6]
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- 22** The line $y = mx + c$ is a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.
Show that $c^2 = a^2m^2 - b^2$. [6]
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- 23** The points $F_1(-5, 0)$ and $F_2(5, 0)$ are fixed. The point $P(x, y)$ moves so that $PF_1 - PF_2 = 6$.
Show that P lies on the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$. [6]
-
- 24** The chords of the hyperbola $\frac{x^2}{4} - y^2 = 1$ with gradient $\frac{1}{2}$ have midpoints $M(h, k)$.
Show that M lies on a straight line through the origin, and find its equation. [6]
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- 25** The point $P(3 \sec \theta, 2 \tan \theta)$ lies on the hyperbola H with equation $\frac{x^2}{9} - \frac{y^2}{4} = 1$. The tangent to H at P meets the asymptotes of H at the points A and B .
Show that P is the midpoint of AB , and that the area of triangle OAB , where O is the origin, is 6. [6]
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- 26** The point $P(3 \cosh u, 4 \sinh u)$ lies on a curve, where u is a real parameter.
Show that P lies on the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$, and find the value of u , in exact form, for which P is the point $(5, \frac{16}{3})$. [5]
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- 27** The circle C_r has equation $x^2 + y^2 = r^2$, where $r > 0$, and the hyperbola H has equation $\frac{x^2}{9} - \frac{y^2}{16} = 1$.
Find, in terms of r , the number of points of intersection of C_r and H . [7]
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- 28** The hyperbola H has equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $P(x_1, y_1)$ is a point on H .
Show that the product of the perpendicular distances from P to the two asymptotes of H is $\frac{a^2b^2}{a^2 + b^2}$, which is independent of the position of P . [8]
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- 29** The rectangular hyperbola H has parametric equations $x = 2\sqrt{2}t$, $y = \frac{2\sqrt{2}}{t}$, $t > 0$, and $F_1(4, 4)$ and $F_2(-4, -4)$ are two points.
Show that the distances from the general point P of H to F_1 and F_2 are $2\sqrt{2}u - 4$ and $2\sqrt{2}u + 4$ respectively, where $u = t + \frac{1}{t}$, and hence that F_1 and F_2 are the foci of H , with $F_2P - F_1P = 8$. [8]

30

The finite region R is bounded by the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ and the line $x = 6$.

Use the substitution $x = 3 \cosh u$ to find the exact area of R .



[7]

Parabolas

- 1 The parabola P has equation $y^2 = 12x$. Write down the coordinates of its focus and the equation of its directrix. [2]

- 2 The parabola $y^2 = 4ax$ passes through the point $(4, 8)$. Find the value of a . [2]

- 3 The parabola $y^2 = 8x$ has parametric equations $x = 2t^2$, $y = 4t$. Find the coordinates of the point where $t = 3$. [1]

- 4 Find the length of the latus rectum of the parabola $y^2 = 8x$. [2]

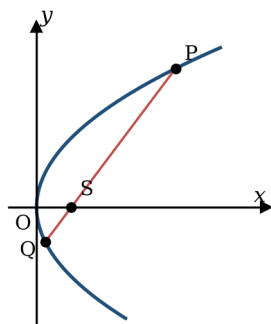
- 5 The parabola C has equation $x^2 = 8y$. Write down the coordinates of its focus and the equation of its directrix. [2]

- 6 Sketch the parabola with equation $(x + 1)^2 = 8(y - 2)$, showing the vertex, the focus, the equation of the directrix and the point where it crosses the y -axis. [4]

- 7 A parabola has its vertex at the origin, its focus at $(0, -3)$ and its axis along the y -axis. Find its equation. [3]

- 8 The point P lies on the parabola $y^2 = 20x$ and has x -coordinate 7. Find its distance from the focus. [3]

- 9 The parabola $y^2 = 4x$ has focus $S(1, 0)$. The points $P(4, 4)$ and Q lie on the parabola, with PQ a focal chord. Find the coordinates of Q , and the length of PQ .



[4]

- 10 The parabola C has equation $(y - 2)^2 = 8(x - 1)$. Write down the coordinates of its vertex and focus and the equation of its directrix. [3]

- 11 Show that the line $y = x + 1$ touches the parabola $y^2 = 4x$, and find the coordinates of the point of contact. [4]

12 A parabola has vertex (2, 1), focus (5, 1) and axis parallel to the x-axis. Find its equation. [3]

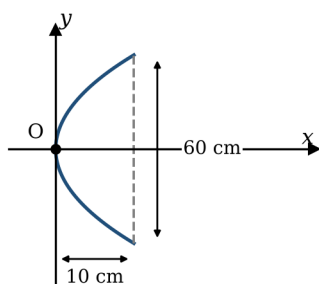
13 The point P(9, 12) lies on the parabola with equation $y^2 = 16x$. The focus of the parabola is S. The normal to the parabola at P meets the x-axis at Q. Find the area of triangle PQS. [5]

14 The line $y = 2x - 4$ meets the parabola $y^2 = 8x$ at the points A and B. Find the length of AB. [5]

15 The parabola $y^2 = 4ax$ has focus S. The points P(ap^2 , $2ap$) and Q(aq^2 , $2aq$) lie on the parabola, with $p \neq q$. Show that PQ is a focal chord (passes through S) if and only if $pq = -1$. [5]

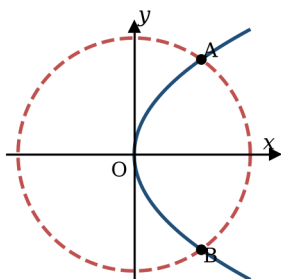
16 The point P($3p^2$, $6p$) lies on a parabola, and M is the midpoint of OP, where O is the origin. Find the Cartesian equation of the parabola, and the locus of M as p varies, giving the answer in the form $y^2 = kx$. [5]

17 A satellite dish has a cross-section in the shape of a parabola. The receiver is placed at the focus. The dish is 60 cm wide and 10 cm deep. Taking the vertex as the origin and the axis as the x-axis, find the distance of the receiver from the vertex.



[4]

18 The parabola $y^2 = 8x$ meets the circle $x^2 + y^2 = 48$ at two points. Find their coordinates.



[4]

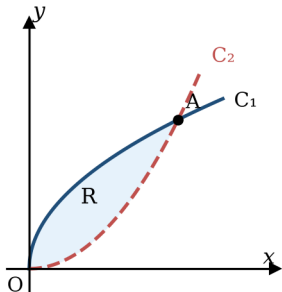
19 The curve C has equation $y^2 - 6y - 8x + 1 = 0$. Show that C is a parabola, and find the coordinates of its vertex, its focus and the equation of its directrix. [5]

20 The parabola $y^2 = 4x$ has focus S(1, 0). A focal chord has gradient 1. Find its equation and show that its length is 8. [5]

21 The point P(x, y) is equidistant from the point F(3, 2) and the line $y = -4$. Find the equation of the locus of P, and state the coordinates of its vertex. [5]

22 The chords of the parabola $y^2 = 12x$ with gradient 3 have midpoints M(h, k). Show that M lies on a straight line parallel to the x-axis, and find its equation. [5]

23 The parabola $y^2 = 4ax$ has a focal chord PQ, where P(ap^2 , $2ap$) and Q(aq^2 , $2aq$) and $pq = -1$. Show that $PQ = a(p + \frac{1}{p})^2$, and deduce that the shortest focal chord has length 4a. [6]

- 24** The parabola $y^2 = 4ax$ has a focal chord PQ. Show that the circle with PQ as diameter touches the directrix of the parabola. [6]
- 25** A student says that the focus of the parabola $y^2 = 12x$ is (12, 0), since 12 is the coefficient of x. Explain the error, find the correct focus, and find the equation of the directrix. [3]
- 26** The line L passes through the point $(-2, 0)$ and has gradient m. The parabola P has equation $y^2 = 8x$. Find the range of values of m for which L meets P, and the values of m for which L is a tangent to P. [6]
- 27** The parabola $y^2 = 4ax$ has focus S and a focal chord PQ. Show that $\frac{1}{SP} + \frac{1}{SQ} = \frac{1}{a}$. [7]
- 28** The points $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$ lie on the parabola $y^2 = 4ax$, where $p \neq q$.
 (a) Show that the chord PQ has equation $2x - (p + q)y + 2apq = 0$. [4]
 (b) Given that angle $POQ = 90^\circ$, where O is the origin, show that $pq = -4$. [3]
 (c) Hence show that every chord PQ that subtends a right angle at O passes through a fixed point, and state its coordinates. [2]
- 29** A searchlight reflector is formed by rotating an arc of a parabola about its axis. The bulb is at the focus. The reflector has an opening of diameter 30 cm and a depth of 20 cm. Model the cross-section by $y^2 = 4ax$ with the vertex at the origin.
 (a) Find the exact value of a, and the distance of the bulb from the vertex. [3]
 (b) Find the volume of the reflector bowl, in terms of π , by rotating the region bounded by the curve and the line $x = 20$ about the x-axis. [4]
- 30** The parabolas $C_1: y^2 = 4x$ and $C_2: x^2 = 4y$ meet at the origin O and at the point A.

 (a) Find the coordinates of A. [2]
 (b) Find the area of the finite region R enclosed by C_1 and C_2 . [3]
 (c) Find the exact volume generated when R is rotated through 2π about the x-axis. [3]

Tangents and Normals to Conics

- 1** Find the equation of the tangent to the parabola $y^2 = 4x$ at the point (1, 2). [3]
- 2** Find the equation of the normal to the parabola $y^2 = 8x$ at the point (2, 4). [3]
- 3** Find the equation of the tangent to the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ at the point $(4, \frac{9}{5})$. [3]
- 4** Find the equation of the tangent to the rectangular hyperbola $xy = 4$ at the point (2, 2). [3]

5 Find the equation of the tangent to the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$ at the point $(5, \frac{16}{3})$. [3]

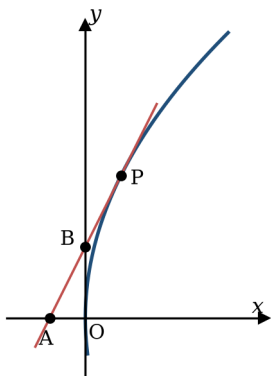
6 The point $P(at^2, 2at)$ lies on the parabola $y^2 = 4ax$. Show that the gradient of the tangent at P is $\frac{1}{t}$. [2]

7 Find the equation of the normal to the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ at the point $(2, \frac{3\sqrt{3}}{2})$, in the form $ax + by = c$. [4]

8 The line with gradient 3 is a tangent to the parabola $y^2 = 12x$. Find its equation. [3]

9 The rectangular hyperbola H has parametric equations $x = 3t, y = \frac{3}{t}$. Show that the tangent at the point where $t = 3$ is $x + 9y = 18$. [3]

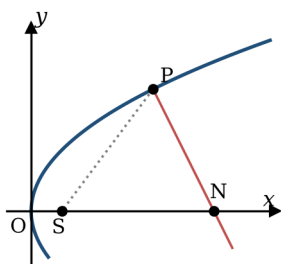
10 The tangent to the parabola $y^2 = 16x$ at the point $(1, 4)$ meets the x-axis at A and the y-axis at B. Find the area of triangle OAB, where O is the origin.



[4]

11 Find the equation of the tangent to the hyperbola $\frac{x^2}{4} - \frac{y^2}{9} = 1$ at the point $(4, 3\sqrt{3})$. [3]

12 The normal to the parabola $y^2 = 4x$ at $P(4, 4)$ meets the x-axis at N. The focus is $S(1, 0)$. Find the length SN.



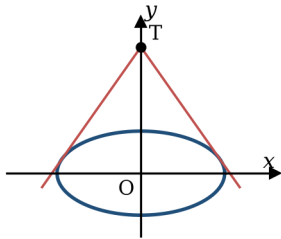
[4]

13 Find the equations of the two tangents to the parabola $y^2 = 8x$ that pass through the point $(-2, 3)$. [5]

14 The point $P(a \cos \theta, b \sin \theta)$ lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Show that the tangent at P has equation $\frac{x \cos \theta}{a} + \frac{y \sin \theta}{b} = 1$. [4]

15 The line $y = x + k$ is a tangent to the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$. Find the possible values of k. [5]

- 16** Two tangents to the ellipse $\frac{x^2}{4} + y^2 = 1$ pass through the point $(0, 3)$. Find their equations.



[5]

- 17** Show that the normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $P(a \cos \theta, b \sin \theta)$ has equation $\frac{ax}{\cos \theta} - \frac{by}{\sin \theta} = a^2 - b^2$.

[5]

- 18** The tangents to the parabola $y^2 = 4ax$ at $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$ meet at T . Show that T has coordinates $(apq, a(p + q))$.

[5]

- 19** The normal to the parabola $y^2 = 4x$ at $P(1, 2)$ meets the parabola again at Q . Find the coordinates of Q .

[5]

- 20** The tangent to the rectangular hyperbola $xy = c^2$ at $P(ct, \frac{c}{t})$ has equation $x + t^2y = 2ct$. This tangent meets the axes at A and B . Show that the area of triangle OAB is independent of t .

[4]

- 21** Show that the line $y = mx + c$ is a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if and only if $c^2 = a^2m^2 + b^2$.

[6]

- 22** Two tangents to the parabola $y^2 = 4ax$ are perpendicular and meet at the point (h, k) . Show that $h = -a$, so that the point lies on the directrix.

[6]

- 23** A student finds the tangent to the hyperbola $\frac{x^2}{4} - y^2 = 1$ at the vertex $(2, 0)$. They write $\frac{dy}{dx} = \frac{x}{4y} = \frac{2}{0}$, so "the gradient is undefined and there is no tangent". Explain the error and find the equation of the tangent at $(2, 0)$ using the formula for the tangent at (x_1, y_1) .

[3]

- 24** The normal to the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ at the point $P(4 \cos \theta, 3 \sin \theta)$ meets the x -axis at A and the y -axis at B , and M is the midpoint of AB . Find the Cartesian equation of the locus of M as θ varies.

[6]

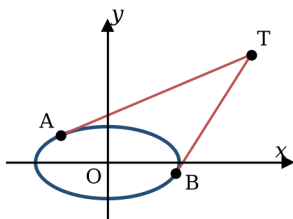
- 25** The tangent to the parabola $y^2 = 4ax$ at $P(ap^2, 2ap)$, $p > 0$, meets the x -axis at T . The focus is $S(a, 0)$. Show that $ST = SP$.

[5]

- 26** Find the equations of the two common tangents to the parabola $y^2 = 4x$ and the ellipse $\frac{x^2}{2} + y^2 = 1$.

[6]

- 27 Tangents are drawn from the point $T(4, 3)$ to the ellipse $\frac{x^2}{4} + y^2 = 1$, touching it at $A(x_1, y_1)$ and $B(x_2, y_2)$.
Show that the line AB has equation $x + 3y = 1$, and find the coordinates of the points where AB meets the ellipse.



[7]

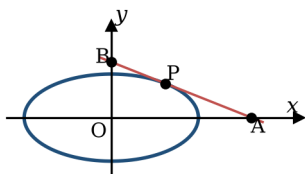
- 28 The normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$ has equation $y + tx = 2at + at^3$.
Show that the normals from the point (h, k) correspond to values of t satisfying $at^3 + (2a - h)t - k = 0$, and find the three normals to $y^2 = 4x$ that pass through $(8, 4)$. [8]
- 29 The rectangular hyperbola H has equation $xy = 4$, with parametric equations $x = 2t, y = \frac{2}{t}$.
(a) Show that the normal at the point $P(2t, \frac{2}{t})$ has equation $t^3x - ty = 2(t^4 - 1)$. [4]
(b) The normal at the point where $t = 2$ meets H again at Q . Find the coordinates of Q . [5]
- 30 The ellipse E has equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and foci $F_1(-ae, 0), F_2(ae, 0)$, where $b^2 = a^2(1 - e^2)$.
Show that the product of the perpendicular distances from F_1 and F_2 to any tangent of E is b^2 . [8]

Parametric Forms of Conics

- 1 A parabola has parametric equations $x = 3t^2, y = 6t$. Find the coordinates of the point where $t = -2$. [1]
- 2 An ellipse has parametric equations $x = 5 \cos \theta, y = 3 \sin \theta$. Find its Cartesian equation. [2]
- 3 A hyperbola has parametric equations $x = 2 \sec \theta, y = 3 \tan \theta$. Find its Cartesian equation. [2]
- 4 A curve has parametric equations $x = 4t, y = \frac{4}{t}, t \neq 0$. Find its Cartesian equation. [2]
- 5 Write down parametric equations for the ellipse $\frac{x^2}{16} + \frac{y^2}{4} = 1$. [1]
- 6 The point $P(2 \cos \theta, 5 \sin \theta)$ lies on an ellipse. Find the coordinates of P when $\theta = \frac{\pi}{6}$. [2]
- 7 A parabola has parametric equations $x = 5t^2, y = 10t$. Find its Cartesian equation, and the coordinates of its focus. [4]
- 8 A curve has parametric equations $x = 1 + 3 \cos \theta, y = -2 + 2 \sin \theta$. Show that the curve is an ellipse, and state its centre and semi-axes. [4]
- 9 A curve has parametric equations $x = t^2 + 1, y = 4t$. Find its Cartesian equation and state the coordinates of its vertex. [4]

- 10** The point $P(3 \sec \theta, 2 \tan \theta)$ lies on a hyperbola. Find the coordinates of P when $\theta = \frac{\pi}{3}$, and show that P lies on $\frac{x^2}{9} - \frac{y^2}{4} = 1$. [4]
- 11** The parabola $y^2 = 4ax$ has parametric equations $x = at^2$, $y = 2at$. Show that the point $(a, 2a)$ corresponds to $t = 1$ and that it lies on the latus rectum. [3]
- 12** The rectangular hyperbola $xy = 9$ has parametric equations $x = 3t$, $y = \frac{3}{t}$. Find the value(s) of t at which the curve meets the line $y = 2x - 3$. [4]
- 13** Find the coordinates of the points on the ellipse $x = 4 \cos \theta$, $y = 3 \sin \theta$ where the line $y = \frac{3}{4}x$ meets it, for $0 \leq \theta < 2\pi$. [4]
- 14** The point $P(4 \cos \theta, 3 \sin \theta)$ lies on an ellipse and $F(\sqrt{7}, 0)$ is a focus. Show that $PF = 4 - \sqrt{7} \cos \theta$. [5]
- 15** The curve C has parametric equations $x = \frac{1-t^2}{1+t^2}$, $y = \frac{2t}{1+t^2}$. Show that C is part of the circle $x^2 + y^2 = 1$, and state which point of the circle is not obtained. [5]
- 16** A hyperbola has parametric equations $x = 3 \sec \theta$, $y = 4 \tan \theta$. Show that the tangent at the point with parameter θ is $\frac{x \sec \theta}{3} - \frac{y \tan \theta}{4} = 1$. [5]
- 17** The rectangular hyperbola $xy = c^2$ has parametric equations $x = ct$, $y = \frac{c}{t}$. The chord joining the points with parameters p and q has gradient m . Show that $m = -\frac{1}{pq}$. [4]
- 18** A point P moves on the ellipse $x = 5 \cos \theta$, $y = 3 \sin \theta$. The point M is the midpoint of OP , where O is the origin. Find the Cartesian equation of the locus of M . [4]
- 19** The point $P(ap^2, 2ap)$ is on the parabola $y^2 = 4ax$, and $S(a, 0)$ is its focus. Show that the midpoint of SP lies on the parabola $y^2 = 2a(x - \frac{a}{2})$. [5]
- 20** The rectangular hyperbola $xy = c^2$ has parametric equations $x = ct$, $y = \frac{c}{t}$. Show that the tangent at the point with parameter t has equation $x + t^2y = 2ct$. [4]
- 21** The tangents to the ellipse $x = a \cos \theta$, $y = b \sin \theta$ at the points $P(\alpha)$ and $Q(-\alpha)$ meet at T . Find the coordinates of T . [6]
- 22** The points $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$ lie on the parabola $y^2 = 4ax$ and the chord PQ has gradient 1. Find $p + q$, and show that the midpoint of PQ lies on $y = 2a$. [5]
- 23** The point $P(3 \cos \theta, 2 \sin \theta)$ lies on an ellipse and $A(3, 0)$, $B(-3, 0)$ are the ends of the major axis. Show that the product of the gradients of AP and BP is constant, and find its value. [6]
- 24** A student writes the parametric equations of the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ as $x = 3 \cos \theta$, $y = 2 \sin \theta$. Show that these equations describe an ellipse rather than the hyperbola, and write down correct parametric equations for the right-hand branch. [4]

- 25 The ellipse $x = 4 \cos \theta$, $y = 2 \sin \theta$ has a point P in the first quadrant. The tangent at P meets the x-axis at A and the y-axis at B. Find the least possible length of AB.



[6]

- 26 The point $P(ct, \frac{c}{t})$, with $t > 0$, lies on the rectangular hyperbola $xy = c^2$, and O is the origin. Find the least possible value of OP, and the coordinates of the point where it occurs.

[5]

- 27 The point $P(ap^2, 2ap)$ lies on the parabola $y^2 = 4ax$ and the normal at P meets the parabola again at $Q(aq^2, 2aq)$. Show that $q = -p - \frac{2}{p}$.

[6]

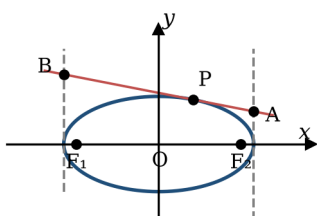
- 28 The ellipse E has parametric equations $x = a \cos \theta$, $y = b \sin \theta$. The points $P(\alpha)$ and $Q(\beta)$ lie on E, and the chord PQ passes through the focus $F(ae, 0)$. Show that $\tan \frac{\alpha}{2} \tan \frac{\beta}{2} = -\frac{1-e}{1+e}$.

[8]

- 29 The rectangular hyperbola $xy = c^2$ has parametric equations $x = ct$, $y = \frac{c}{t}$. The normal at $P(cp, \frac{c}{p})$ meets the hyperbola again at $Q(cq, \frac{c}{q})$. Show that $p^3q = -1$.

[6]

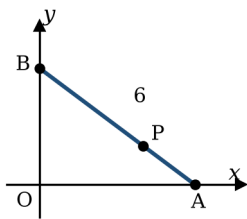
- 30 The point $P(2 \cos \theta, \sin \theta)$ lies on the ellipse $\frac{x^2}{4} + y^2 = 1$. The tangent at P meets the line $x = 2$ at A and the line $x = -2$ at B, and $F_1(-\sqrt{3}, 0)$, $F_2(\sqrt{3}, 0)$ are the foci. Find A and B, and show that AF_1 is perpendicular to BF_1 .



[8]

Loci and Geometrical Properties

- 1 A point $P(x, y)$ is always 5 units from the point $(2, -1)$. Find the equation of its locus. [2]
- 2 A point $P(x, y)$ is equidistant from $A(0, 0)$ and $B(6, 0)$. Find the equation of its locus. [2]
- 3 A point $P(x, y)$ is equidistant from the point $(0, 2)$ and the line $y = -2$. Find the equation of its locus. [3]
- 4 A point $P(x, y)$ moves so that its distance from the origin is always 4. Describe the locus and give its equation. [2]
- 5 The point P is the midpoint of $A(2t, 0)$ and $B(0, 6t)$. Find the coordinates of P and the equation of the locus of P as t varies. [3]

- 6** A point P moves so that $PF = e \times$ (perpendicular distance from P to a fixed line), with $e = 1$ and F not on the line. Name the type of curve traced by P. [1]
- 7** A point P moves so that $PA = 2PB$, where A is (0, 0) and B is (6, 0). Show that the locus of P is a circle, and find its centre and radius. [4]
- 8** The point P(x, y) is such that its distance from F(2, 0) equals its distance from the line $x = -2$. Find the equation of the locus of P and name the curve. [3]
- 9** The point P moves so that the sum of its distances from (-4, 0) and (4, 0) is 10. Name the curve, and find its equation. [4]
- 10** The point P moves so that the difference of its distances from (-5, 0) and (5, 0) is 8 in absolute value. Find the equation of the locus. [4]
- 11** The points A(t, 0) and B(0, 2t) move with t. The point P divides AB so that $AP : PB = 1 : 2$. Find the locus of P as t varies. [3]
- 12** The point $P(2t, t^2)$ moves with parameter t. Find the Cartesian equation of its locus, and name the curve. [2]
- 13** A point P(x, y) moves so that its distance from A(0, 3) equals its distance from the x-axis. Find the locus of P. [3]
- 14** A point P moves so that its distance from F(3, 0) is $\frac{3}{5}$ of its distance from the line $x = \frac{25}{3}$. Find the equation of its locus. [5]
- 15** A point P(x, y) moves so that its distance from F(5, 0) is $\frac{5}{3}$ of its distance from the line $x = \frac{9}{5}$. Find the equation of its locus. [5]
- 16** A point P(x, y) is such that its distance from the origin is twice its distance from the line $x = 3$. Show that the locus of P is a hyperbola, and find its equation in the form $\frac{(x-p)^2}{a^2} - \frac{y^2}{b^2} = 1$. [5]
- 17** A rod AB of length 6 slides with A on the x-axis and B on the y-axis. Find the locus of the point P on AB with $AP : PB = 1 : 2$. [5]
- 
- 18** The points A(-a, 0) and B(a, 0) are fixed. Show that the locus of P(x, y) with $PA^2 + PB^2 = 4a^2$ is a circle, and state its radius. [4]
- 19** The point P moves on the circle $x^2 + y^2 = 16$, and A is the fixed point (8, 0). M is the midpoint of AP. Find the locus of M. [4]
- 20** The points A(-3, 0) and B(3, 0) are fixed. P(x, y) moves so that (gradient of AP) $\times$ (gradient of BP) = $-\frac{4}{9}$. Find the locus of P, stating which points are excluded. [5]

- 21** A circle of radius r touches the circle $x^2 + y^2 = 1$ externally and passes through the point $F(3, 0)$. Find the locus of the centre C of the circle. [6]
-
- 22** The parabola $y^2 = 4ax$ has focus $S(a, 0)$. A focal chord PQ has endpoints $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$. Find the Cartesian equation of the locus of the midpoint of PQ as the chord varies. [6]
-
- 23** Tangents to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are drawn from the point $T(h, k)$ and are perpendicular. Show that T lies on the circle $x^2 + y^2 = a^2 + b^2$. [6]
-
- 24** A student claims that the locus of the point P for which $PA + PB = 4$, where $A(-3, 0)$ and $B(3, 0)$, is the ellipse $\frac{x^2}{4} + \frac{y^2}{-5} = 1$. Explain why this is wrong, and state what the locus actually is. [4]
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- 25** The tangent at a point P of the parabola $y^2 = 4ax$ is drawn, and the perpendicular from the focus $S(a, 0)$ to this tangent meets it at N . Show that the locus of N is the y -axis. [5]
-
- 26** A point $P(x, y)$ moves so that its distance from the focus $F(ae, 0)$ is e times its distance from the directrix $x = \frac{a}{e}$, where $0 < e < 1$. Show that the locus of P is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $b^2 = a^2(1 - e^2)$. [5]
-
- 27** The points $A(-a, 0)$ and $B(a, 0)$ are fixed, and $k > 0$, $k \neq 1$. A point P moves so that $PA = k PB$. Show that the locus of P is a circle with centre $(a\frac{k^2 + 1}{k^2 - 1}, 0)$ and radius $\frac{2ak}{|k^2 - 1|}$. [7]
-
- 28** The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ has focus $F(ae, 0)$ and directrix $x = \frac{a}{e}$, where $b^2 = a^2(1 - e^2)$. The tangent at $P(x_1, y_1)$ meets the directrix at T . Show that angle $PFT = 90^\circ$. [8]
-
- 29** The tangent at the point $P(a \cos \theta, b \sin \theta)$ of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{x \cos \theta}{a} + \frac{y \sin \theta}{b} = 1$. Show that the foot of the perpendicular from the focus $F(ae, 0)$ to this tangent lies on the circle $x^2 + y^2 = a^2$. [9]
-
- 30** The tangent and the normal at the point $P(x_1, y_1)$, $y_1 \neq 0$, of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meet the x -axis at T and N . Show that $OT \times ON = a^2 e^2$, where $b^2 = a^2(1 - e^2)$. [7]
-