

Further Differentiation

1 (1 mark)

$$\frac{d y}{d x} = 4 \cosh 4x$$

B1

Answer: $4 \cosh 4x$

1 mark

2 (2 marks)

$$\text{Use } \frac{d}{d x}(\arctan u) = \frac{1}{1+u^2} \frac{d u}{d x} \text{ with } u = 3x$$

M1

$$\frac{d y}{d x} = \frac{3}{1+9x^2}$$

A1

$$\text{Answer: } \frac{3}{1+9x^2}$$

2 marks

3 (2 marks)

$$\text{Use } \frac{d}{d x}(\coth u) = -\operatorname{cosech}^2 u \frac{d u}{d x} \text{ with } u = 2x$$

M1

$$\frac{d y}{d x} = -6 \operatorname{cosech}^2 2x$$

A1

$$\text{Answer: } -6 \operatorname{cosech}^2 2x$$

2 marks

4 (2 marks)

$$\frac{d}{d x}(\operatorname{arsinh} u) = \frac{1}{\sqrt{u^2+1}} \frac{d u}{d x}, u = 2x$$

M1

$$\frac{d y}{d x} = \frac{2}{\sqrt{4x^2+1}}$$

A1

$$\text{Answer: } \frac{2}{\sqrt{4x^2+1}}$$

2 marks

5 (2 marks)

$$f'(x) = \frac{1}{\cosh x} \times \sinh x$$

M1

$$f'(x) = \tanh x$$

A1

$$\text{Answer: } \tanh x$$

2 marks

6 (2 marks)

$$\frac{d}{d x}(\arccos u) = -\frac{1}{\sqrt{1-u^2}} \frac{d u}{d x} \text{ with } u = \frac{x}{3}$$

M1

$$\frac{d y}{d x} = -\frac{1}{3\sqrt{1-\frac{x^2}{9}}} = -\frac{1}{\sqrt{9-x^2}}$$

A1

$$\text{Answer: } -\frac{1}{\sqrt{9-x^2}}$$

2 marks

7 (3 marks)

$$y = 4 (\cosh 2x)^{-1}, \text{ so } \frac{d y}{d x} = -4 (\cosh 2x)^{-2} \times \frac{d}{d x}(\cosh 2x)$$

M1

$$\frac{d}{d x}(\cosh 2x) = 2 \sinh 2x, \text{ so } \frac{d y}{d x} = -\frac{8 \sinh 2x}{\cosh^2 2x}$$

A1

$$\frac{d y}{d x} = -8 \times \frac{1}{\cosh 2x} \times \frac{\sinh 2x}{\cosh 2x} = -8 \operatorname{sech} 2x \tanh 2x$$

A1

$$\text{Answer: } -8 \operatorname{sech} 2x \tanh 2x$$

3 marks

8 (3 marks)Product rule with $u = x$, $v = \arctan x$

M1

$$\frac{d v}{d x} = \frac{1}{1+x^2}$$

B1

$$\frac{d y}{d x} = \arctan x + \frac{x}{1+x^2}$$

A1

$$\text{Answer: } \arctan x + \frac{x}{1+x^2}$$

3 marks

9 (2 marks)Chain rule: $y = (\sinh x)^3$

M1

$$\frac{d y}{d x} = 3 \sinh^2 x \cosh x$$

A1

$$\text{Answer: } 3 \sinh^2 x \cosh x$$

2 marks

10 (3 marks)

$$\frac{d}{d x}(\operatorname{arcosh} u) = \frac{1}{\sqrt{u^2-1}} \frac{d u}{d x} \text{ with } u = 3x + 1$$

M1

$$\frac{d y}{d x} = \frac{3}{\sqrt{(3x+1)^2-1}}$$

A1

$$(3x+1)^2-1 = 9x^2+6x = 3x(3x+2), \text{ so } \frac{d y}{d x} = \frac{3}{\sqrt{3x(3x+2)}}$$

A1

$$\text{Answer: } \frac{3}{\sqrt{9x^2+6x}}$$

3 marks

11 (3 marks)

$$\text{Chain rule: } \frac{d y}{d x} = 2 \operatorname{arsinh} x \times \frac{d}{d x}(\operatorname{arsinh} x)$$

M1

$$\frac{d}{d x}(\operatorname{arsinh} x) = \frac{1}{\sqrt{x^2+1}}$$

B1

$$\frac{d y}{d x} = \frac{2 \operatorname{arsinh} x}{\sqrt{x^2+1}}$$

A1

$$\text{Answer: } \frac{2 \operatorname{arsinh} x}{\sqrt{x^2+1}}$$

3 marks

12 (4 marks)

$$\frac{d y}{d x} = \sinh 2x + 2x \cosh 2x$$

M1 A1

$$\sinh(2 \ln 2) = \frac{1}{2}\left(4 - \frac{1}{4}\right) = \frac{15}{8}, \cosh(2 \ln 2) = \frac{1}{2}\left(4 + \frac{1}{4}\right) = \frac{17}{8}$$

M1

$$\text{Gradient} = \frac{15}{8} + 2 \ln 2 \times \frac{17}{8} = \frac{15}{8} + \frac{17}{4} \ln 2$$

A1

$$\text{Answer: } \frac{15}{8} + \frac{17}{4} \ln 2$$

4 marks

13 (4 marks)

$$\text{Quotient rule: } \frac{d y}{d x} = \frac{\cosh x \times \cosh x - (1 + \sinh x) \sinh x}{\cosh^2 x}$$

M1 A1

$$\text{Numerator: } \cosh^2 x - \sinh x - \sinh^2 x$$

M1

$$\cosh^2 x - \sinh^2 x = 1, \text{ so the numerator is } 1 - \sinh x \text{ and } \frac{d y}{d x} = \frac{1 - \sinh x}{\cosh^2 x}$$

A1

$$\text{Answer: } \frac{1 - \sinh x}{\cosh^2 x}$$

4 marks

14 (4 marks)

$$x = \sinh y \text{ gives } \frac{dx}{dy} = \cosh y$$

B1

$$\frac{dy}{dx} = \frac{1}{\cosh y}$$

M1

$$\cosh^2 y = 1 + \sinh^2 y = 1 + x^2, \text{ and } \cosh y > 0$$

M1

$$\text{So } \cosh y = \sqrt{1 + x^2} \text{ and } \frac{dy}{dx} = \frac{1}{\sqrt{1 + x^2}}$$

A1

$$\text{Answer: } \frac{1}{\sqrt{1 + x^2}}$$

4 marks

15 (4 marks)

$$x = \cosh y \text{ gives } \frac{dx}{dy} = \sinh y$$

B1

$$\frac{dy}{dx} = \frac{1}{\sinh y}$$

M1

$$\sinh^2 y = \cosh^2 y - 1 = x^2 - 1, \text{ and } \sinh y > 0 \text{ for } y > 0$$

M1

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2 - 1}}$$

A1

$$\text{Answer: } \frac{1}{\sqrt{x^2 - 1}}$$

4 marks

16 (5 marks)

$$\text{Let } u = \frac{1+x}{1-x}, \frac{du}{dx} = \frac{(1-x) + (1+x)}{(1-x)^2} = \frac{2}{(1-x)^2}$$

M1 A1

$$\frac{dy}{dx} = \frac{1}{1+u^2} \times \frac{du}{dx}$$

M1

$$1 + u^2 = \frac{(1-x)^2 + (1+x)^2}{(1-x)^2} = \frac{2 + 2x^2}{(1-x)^2}$$

M1

$$\frac{dy}{dx} = \frac{(1-x)^2}{2(1+x^2)} \times \frac{2}{(1-x)^2} = \frac{1}{1+x^2}$$

A1

$$\text{Answer: } \frac{1}{1+x^2}$$

5 marks

17 (5 marks)

$$\frac{dy}{dx} = 1 - \operatorname{sech}^2 x$$

M1

$$\text{Using } \operatorname{sech}^2 x = 1 - \tanh^2 x: \frac{dy}{dx} = \tanh^2 x$$

A1

$$\tanh x = 0 \text{ only at } x = 0, \text{ so one stationary point, at } (0, 0)$$

B1

$$\frac{dy}{dx} = \tanh^2 x \geq 0 \text{ on both sides of } x = 0 \text{ (zero only at } x = 0)$$

M1

$$\text{The gradient does not change sign, so it is a (stationary) point of inflection}$$

A1

$$\text{Answer: Point of inflection at the origin}$$

5 marks

18 (5 marks)

$$\text{When } x = \frac{1}{2}, y = \arcsin \frac{1}{2} = \frac{\pi}{6}$$

B1

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

M1

$$m = \frac{1}{\sqrt{1-\frac{1}{4}}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

A1

$$y - \frac{\pi}{6} = \frac{2\sqrt{3}}{3} \left(x - \frac{1}{2}\right)$$

M1

$$y = \frac{2\sqrt{3}}{3}x + \frac{\pi}{6} - \frac{\sqrt{3}}{3}$$

A1

$$\text{Answer: } y = \frac{2\sqrt{3}}{3}x + \frac{\pi}{6} - \frac{\sqrt{3}}{3}$$

5 marks

19 (5 marks)

$$\ln y = \sinh x \ln x$$

B1

$$\frac{1}{y} \frac{dy}{dx} = \cosh x \ln x + \frac{\sinh x}{x}$$

M1 A1

$$\frac{dy}{dx} = y(\cosh x \ln x + \frac{\sinh x}{x})$$

M1

$$\frac{dy}{dx} = x^{\sinh x} (\cosh x \ln x + \frac{\sinh x}{x})$$

A1

$$\text{Answer: } x^{\sinh x} (\cosh x \ln x + \frac{\sinh x}{x})$$

5 marks

20 (6 marks)

$$(a) u_r = 3^r e^{3x}, v = x^2, v_1 = 2x, v_2 = 2 \text{ and } v_r = 0 \text{ for } r \geq 3$$

B1

$$y_n = 3^n e^{3x} x^2 + n 3^{n-1} e^{3x} (2x) + \frac{n(n-1)}{2} 3^{n-2} e^{3x} (2)$$

M1 A1

$$\text{Factor out } e^{3x} 3^{n-2}: y_n = e^{3x} 3^{n-2} (9x^2 + 6nx + n(n-1))$$

A1

$$\text{Answer: } y_n = e^{3x} 3^{n-2} (9x^2 + 6nx + n(n-1))$$

4 marks

$$(b) \text{ Put } n = 6 \text{ and } x = 0: y_6 = 3^4 \times (0 + 0 + 6 \times 5)$$

M1

$$= 81 \times 30 = 2430$$

A1

$$\text{Answer: } 2430$$

2 marks

21 (7 marks)

$$(a) \frac{dy}{dx} = 5 \sinh x - 3 \cosh x = 0 \text{ gives } \tanh x = \frac{3}{5}$$

M1 A1

$$x = \operatorname{artanh} \frac{3}{5} = \frac{1}{2} \ln(4) = \ln 2, \text{ since } (1 + \frac{3}{5}) \div (1 - \frac{3}{5}) = 4$$

M1

$$y = 5 \times \frac{1}{2} (2 + \frac{1}{2}) - 3 \times \frac{1}{2} (2 - \frac{1}{2})$$

M1

$$y = \frac{25}{4} - \frac{9}{4} = 4$$

A1

$$\text{Answer: } (\ln 2, 4)$$

5 marks

$$(b) \frac{d^2 y}{dx^2} = 5 \cosh x - 3 \sinh x = y > 0 \text{ at the stationary point}$$

M1

So it is a minimum

A1

$$\text{Answer: Minimum}$$

2 marks

22 (6 marks)

$$\frac{dy}{dx} = \frac{1}{\tanh(x/2)} \times \operatorname{sech}^2(\frac{x}{2}) \times \frac{1}{2}$$

M1 A1 A1

$$= \frac{1}{2} \times \frac{\cosh(x)}{2/\sinh(x/2)} \times \frac{1}{\cosh^2(x/2)}$$

M1

$$= \frac{1}{2 \sinh(x/2) \cosh(x/2)}$$

A1

$$\text{Using } \sinh x = 2 \sinh(\frac{x}{2}) \cosh(\frac{x}{2}): \frac{dy}{dx} = \frac{1}{\sinh x} = \operatorname{cosech} x$$

A1

$$\text{Answer: } \frac{dy}{dx} = \operatorname{cosech} x$$

6 marks

23 (6 marks)

$$(a) \frac{dy}{dx} = \frac{1}{1 + \sinh^2 x} \times \cosh x$$

M1 A1

$$1 + \sinh^2 x = \cosh^2 x, \text{ so } \frac{dy}{dx} = \frac{1}{\cosh x} = \operatorname{sech} x$$

A1

$$\text{Answer: } \frac{dy}{dx} = \operatorname{sech} x$$

3 marks

$$(b) \frac{d^2 y}{dx^2} = \frac{d}{dx} (\cosh x)^{-1} = -(\cosh x)^{-2} \sinh x$$

M1 A1

$$= -\operatorname{sech} x \tanh x. \text{ As } x \rightarrow \infty, \operatorname{sech} x \rightarrow 0 \text{ so the gradient tends to } 0$$

B1

$$\text{Answer: Shown; gradient } \rightarrow 0$$

3 marks

24 (6 marks)**(a)** The chain rule has been omitted: the derivative of the inner function x^2 was not included

E1

Answer: Chain rule omitted

1 mark

(b) $\frac{dy}{dx} = 2x \cosh(x^2)$

B1

For $y = \sinh(x^2) - 4x$: $\frac{dy}{dx} = 2x \cosh(x^2) - 4 = 0$, so $x \cosh(x^2) = 2$

M1 A1

Let $g(x) = x \cosh(x^2) - 2$: $g(1) = \cosh 1 - 2 = -0.457 < 0$

M1

$g(1.2) = 1.2 \cosh 1.44 - 2 = 0.675 > 0$, sign change so a root lies in (1, 1.2)

A1

Answer: Stationary point where $x \cosh(x^2) = 2$, with $1 < x < 1.2$

5 marks

25 (7 marks)

(a) $f'(x) = 2 + \cosh x$

M1

$\cosh x \geq 1$, so $f'(x) \geq 3 > 0$

A1

Answer: $f'(x) = 2 + \cosh x \geq 3 > 0$

2 marks

(b) $f(0) = 0$, $f'(0) = 3$

M1

$(f^{-1})'(0) = \frac{1}{f'(0)} = \frac{1}{3}$

A1

Answer: $\frac{1}{3}$

2 marks

(c) $a = f(\ln 3) = 2 \ln 3 + \sinh(\ln 3) = 2 \ln 3 + \frac{4}{3}$

B1

$f(\ln 3) = 2 + \cosh(\ln 3) = 2 + \frac{5}{3} = \frac{11}{3}$

M1

$(f^{-1})'(a) = \frac{1}{f'(\ln 3)} = \frac{3}{11}$

A1

Answer: $\frac{3}{11}$

3 marks

26 (7 marks)

$\frac{dy}{dx} = \frac{2 \arcsin x}{\sqrt{1-x^2}}$, so $\sqrt{1-x^2} \frac{dy}{dx} = 2 \arcsin x$

M1 A1

Differentiate both sides with respect to x

M1

$\sqrt{1-x^2} \frac{d^2y}{dx^2} - \frac{x}{\sqrt{1-x^2}} \frac{dy}{dx} = \frac{2}{\sqrt{1-x^2}}$

M1 A1

Multiply through by $\sqrt{1-x^2}$

M1

$(1-x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} = 2$

A1

Answer: $(1-x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} = 2$

7 marks

27 (9 marks)

(a) $\frac{dy}{dx} = \frac{1}{1+x^2}$

M1 A1

Answer: $(1+x^2) \frac{dy}{dx} = 1$

2 marks

(b) Differentiate (a) using the product rule

M1

$(1+x^2) \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} = 0$

A1

Answer: $(1+x^2) \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} = 0$

2 marks

(c) Product rule on each term

M1

$(1+x^2) y''' + 2x y'' + 2x y'' + 2y' = 0$, giving the result

A1

Answer: $(1+x^2) \frac{d^3y}{dx^3} + 4x \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} = 0$

2 marks

(d) At $x = 0$: $y' = 1$

B1

 $y'' = 0$ (from (b) with $x = 0$)

M1

$$y''' + 0 + 2 = 0, \text{ so } y''' = -2$$

Answer: -2

A1

3 marks

28 (10 marks)

(a) Chain rule: $\frac{dy}{dx} = k \cosh(k \operatorname{arsinh} x) \times \frac{d}{dx}(\operatorname{arsinh} x)$

M1 A1

$$\frac{d}{dx}(\operatorname{arsinh} x) = \frac{1}{\sqrt{1+x^2}}, \text{ so } \sqrt{1+x^2} \frac{dy}{dx} = k \cosh(k \operatorname{arsinh} x)$$

A1

Answer: $\sqrt{1+x^2} \frac{dy}{dx} = k \cosh(k \operatorname{arsinh} x)$

3 marks

(b) Differentiate the result of (a) with respect to x

M1

$$\frac{x}{\sqrt{1+x^2}} \frac{dy}{dx} + \sqrt{1+x^2} \frac{d^2y}{dx^2} = k \sinh(k \operatorname{arsinh} x) \times \frac{k}{\sqrt{1+x^2}}$$

M1 A1 A1

Multiply through by $\sqrt{1+x^2}$

M1

$$x \frac{dy}{dx} + (1+x^2) \frac{d^2y}{dx^2} = k^2 \sinh(k \operatorname{arsinh} x)$$

A1

$$= k^2 y$$

A1

Answer: $(1+x^2) \frac{d^2y}{dx^2} + x \frac{dy}{dx} = k^2 y$

7 marks

29 (10 marks)

(a) $\frac{dy}{dx} = \frac{1}{\sqrt{1-\tanh^2 x}} \times \operatorname{sech}^2 x$

M1 A1

$$1 - \tanh^2 x = \operatorname{sech}^2 x, \text{ so } \frac{dy}{dx} = \frac{\operatorname{sech}^2 x}{\operatorname{sech} x} = \operatorname{sech} x$$

A1

Answer: $\frac{dy}{dx} = \operatorname{sech} x$

3 marks

(b) $\operatorname{sech} x > 0$ for all x, so $\frac{dy}{dx} \neq 0$: no stationary points

M1 A1

$$\frac{d^2y}{dx^2} = -\operatorname{sech} x \tanh x, \text{ which is zero only when } x = 0, \text{ and changes sign there}$$

M1

Point of inflection at (0, 0)

A1

Answer: (0, 0)

4 marks

(c) $\tanh x \rightarrow 1$ so $y \rightarrow \arcsin 1 = \frac{\pi}{2}$

B1

Odd function through O, increasing, gradient 1 at O

B1

$$\text{Asymptotes } y = \frac{\pi}{2} \text{ and } y = -\frac{\pi}{2}$$

B1

Answer: $y \rightarrow \frac{\pi}{2}$; asymptotes $y = \pm \frac{\pi}{2}$

3 marks

30 (11 marks)

(a) Product rule: $\frac{d}{dx}(x \operatorname{arsinh} x) = \operatorname{arsinh} x + \frac{x}{\sqrt{1+x^2}}$

M1 A1

$$\frac{d}{dx}(\sqrt{1+x^2}) = \frac{x}{\sqrt{1+x^2}}$$

B1

The fractions cancel, giving $\frac{dy}{dx} = \operatorname{arsinh} x$

A1

Answer: $\frac{dy}{dx} = \operatorname{arsinh} x$

4 marks

(b) $\operatorname{arsinh} x = 0$ gives $x = 0$

M1

$$y = 0 - 1 = -1, \text{ so the stationary point is } (0, -1)$$

A1

$$\frac{d^2y}{dx^2} = \frac{1}{\sqrt{1+x^2}} = 1 > 0 \text{ at } x = 0$$

M1

So it is a minimum

A1

Answer: (0, -1), minimum

4 marks

(c) $\frac{dy}{dx} = \operatorname{arsinh} \frac{3}{4}$

M1

$$\begin{aligned}\operatorname{arsinh} \frac{3}{4} &= \ln\left(\frac{3}{4} + \sqrt{\frac{9}{16} + 1}\right) = \ln\left(\frac{3}{4} + \frac{5}{4}\right) \\ &= \ln 2 \\ \text{Answer: } \ln 2\end{aligned}$$

M1

A1

3 marks

Further Integration

1 (2 marks)

$$\begin{aligned}\text{Standard form } \int \frac{1}{x^2 + a^2} dx &= \frac{1}{a} \arctan\left(\frac{x}{a}\right) \text{ with } a = 3 \\ &= \frac{1}{3} \arctan\left(\frac{x}{3}\right) + c \\ \text{Answer: } \frac{1}{3} \arctan\left(\frac{x}{3}\right) + c\end{aligned}$$

M1

A1

2 marks

2 (2 marks)

$$\begin{aligned}\int \frac{1}{\sqrt{a^2 - x^2}} dx &= \arcsin\left(\frac{x}{a}\right) \text{ with } a = 2 \\ &= \arcsin\left(\frac{x}{2}\right) + c \\ \text{Answer: } \arcsin\left(\frac{x}{2}\right) + c\end{aligned}$$

M1

A1

2 marks

3 (2 marks)

$$\begin{aligned}\int \frac{1}{\sqrt{x^2 + a^2}} dx &= \operatorname{arsinh}\left(\frac{x}{a}\right) \text{ with } a = 4 \\ &= \operatorname{arsinh}\left(\frac{x}{4}\right) + c \\ \text{Answer: } \operatorname{arsinh}\left(\frac{x}{4}\right) + c\end{aligned}$$

M1

A1

2 marks

4 (2 marks)

$$\begin{aligned}[\arctan x]_0^1 \\ &= \frac{\pi}{4} - 0 = \frac{\pi}{4} \\ \text{Answer: } \frac{\pi}{4}\end{aligned}$$

M1

A1

2 marks

5 (2 marks)

$$\begin{aligned}[\arcsin(\frac{x}{3})]_0^{3/2} \\ &= \arcsin\left(\frac{1}{2}\right) - 0 = \frac{\pi}{6} \\ \text{Answer: } \frac{\pi}{6}\end{aligned}$$

M1

A1

2 marks

6 (1 mark)

$$\begin{aligned}\frac{1}{2} \sinh 2x + c \\ \text{Answer: } \frac{1}{2} \sinh 2x + c\end{aligned}$$

B1

1 mark

7 (4 marks)

$$\text{Mean value} = \frac{1}{4-0} \int_0^4 \frac{1}{\sqrt{x^2+9}} \, dx \quad \text{M1}$$

$$= \frac{1}{4} [\operatorname{arsinh}(\frac{x}{3})]_0^4 = \frac{1}{4} \operatorname{arsinh}(\frac{4}{3}) \quad \text{M1 A1}$$

$$\operatorname{arsinh}(\frac{4}{3}) = \ln(\frac{4}{3} + \sqrt{\frac{16}{9} + 1}) = \ln(\frac{4}{3} + \frac{5}{3}) = \ln 3, \text{ so the mean value is } \frac{1}{4} \ln 3 \quad \text{A1}$$

$$\text{Answer: } \frac{1}{4} \ln 3 \quad \text{4 marks}$$

8 (3 marks)

$$[\operatorname{arsinh}(\frac{x}{2})]_0^2 = \operatorname{arsinh} 1 \quad \text{M1 A1}$$

$$\operatorname{arsinh} 1 = \ln(1 + \sqrt{1+1}) = \ln(1 + \sqrt{2}) \quad \text{A1}$$

$$\text{Answer: } \ln(1 + \sqrt{2}) \quad \text{3 marks}$$

9 (4 marks)

$$[\operatorname{arcosh} x]_2^3 \quad \text{M1}$$

$$\operatorname{arcosh} 3 = \ln(3 + \sqrt{8}) = \ln(3 + 2\sqrt{2}), \operatorname{arcosh} 2 = \ln(2 + \sqrt{3}) \quad \text{M1 A1}$$

$$\text{Value} = \ln\left(\frac{3 + 2\sqrt{2}}{2 + \sqrt{3}}\right) \quad \text{A1}$$

$$\text{Answer: } \ln\left(\frac{3 + 2\sqrt{2}}{2 + \sqrt{3}}\right) \quad \text{4 marks}$$

10 (4 marks)

$$u = x^2, \, du = 2x \, dx; \text{ limits 0 to 1} \quad \text{B1}$$

$$\text{Integral} = \frac{1}{2} \int_0^1 \frac{1}{1+u^2} \, du \quad \text{M1}$$

$$= \frac{1}{2} [\arctan u]_0^1 \quad \text{A1}$$

$$= \frac{\pi}{8} \quad \text{A1}$$

$$\text{Answer: } \frac{\pi}{8} \quad \text{4 marks}$$

11 (3 marks)

$$\text{Split: } \frac{3x}{x^2+1} + \frac{2}{x^2+1} \quad \text{M1}$$

$$\frac{3}{2} \ln(x^2+1) + 2 \arctan x + c \quad \text{A1 A1}$$

$$\text{Answer: } \frac{3}{2} \ln(x^2+1) + 2 \arctan x + c \quad \text{3 marks}$$

12 (3 marks)

$$2x - x^2 = 1 - (x-1)^2 \quad \text{M1}$$

$$\int \frac{1}{\sqrt{1-(x-1)^2}} \, dx \quad \text{M1}$$

$$= \arcsin(x-1) + c \quad \text{A1}$$

$$\text{Answer: } \arcsin(x-1) + c \quad \text{3 marks}$$

13 (4 marks)

$$\cosh^2 x = \frac{1}{2}(1 + \cosh 2x) \quad \text{M1}$$

$$\int_0^{\ln 2} \cosh^2 x \, dx = \left[\frac{x}{2} + \frac{1}{4} \sinh 2x \right]_0^{\ln 2} \quad \text{A1}$$

$$\sinh(2 \ln 2) = \frac{1}{2}(4 - \frac{1}{4}) = \frac{15}{8}, \text{ so the integral is } \frac{1}{2} \ln 2 + \frac{15}{32} \quad \text{M1}$$

$$\text{Mean value} = \frac{1}{\ln 2} \left(\frac{1}{2} \ln 2 + \frac{15}{32} \right) = \frac{1}{2} + \frac{15}{32 \ln 2} \quad \text{A1}$$

$$\text{Answer: } \frac{1}{2} + \frac{15}{32 \ln 2} \quad \text{4 marks}$$

14 (4 marks)

$$\cosh 2x = 1 + 2 \sinh^2 x, \text{ so } \sinh^2 x = \frac{1}{2}(\cosh 2x - 1) \quad \text{M1}$$

$$\int_0^{\ln 2} = \frac{1}{2} \left[\frac{1}{2} \sinh 2x - x \right]_0^{\ln 2} \quad \text{M1 A1}$$

$$\sinh(2 \ln 2) = \frac{1}{2}(4 - \frac{1}{4}) = \frac{15}{8}, \text{ so value} = \frac{1}{2} \left(\frac{15}{16} - \ln 2 \right) = \frac{15}{32} - \frac{1}{2} \ln 2 \quad \text{A1}$$

$$\text{Answer: } \frac{15}{32} - \frac{1}{2} \ln 2 \quad \text{4 marks}$$

15 (5 marks)

$$u = \arctan x, \frac{dv}{dx} = 1; \frac{du}{dx} = \frac{1}{1+x^2}, v = x \quad \text{M1}$$

$$\int_0^1 \arctan x \, dx = [x \arctan x]_0^1 - \int_0^1 \frac{x}{1+x^2} \, dx \quad \text{M1 A1}$$

$$= \frac{\pi}{4} - \left[\frac{1}{2} \ln(1+x^2) \right]_0^1 \quad \text{M1}$$

$$= \frac{\pi}{4} - \frac{1}{2} \ln 2 \quad \text{A1}$$

$$\text{Answer: } \frac{\pi}{4} - \frac{1}{2} \ln 2 \quad \text{5 marks}$$

16 (5 marks)

$$\text{By parts with } u = \arcsin x, \frac{dv}{dx} = 1 \quad \text{M1}$$

$$= [x \arcsin x]_0^{1/2} - \int_0^{1/2} \frac{x}{\sqrt{1-x^2}} \, dx \quad \text{M1 A1}$$

$$= \frac{1}{2} \times \frac{\pi}{6} - [-\sqrt{1-x^2}]_0^{1/2} \quad \text{M1}$$

$$= \frac{\pi}{12} + \sqrt{\frac{3}{4}} - 1 = \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1 \quad \text{A1}$$

$$\text{Answer: } \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1 \quad \text{5 marks}$$

17 (4 marks)

$$x^2 + 4x + 13 = (x+2)^2 + 9 \quad \text{M1 A1}$$

$$\int \frac{1}{\sqrt{(x+2)^2 + 3^2}} \, dx \quad \text{M1}$$

$$= \operatorname{arsinh}\left(\frac{x+2}{3}\right) + c \quad \text{A1}$$

$$\text{Answer: } \operatorname{arsinh}\left(\frac{x+2}{3}\right) + c \quad \text{4 marks}$$

18 (5 marks)

Write $2x + 3 = (2x + 2) + 1$

M1

$$\int \frac{2x+2}{\sqrt{x^2+2x+5}} dx = 2\sqrt{x^2+2x+5}$$

M1 A1

$$\int \frac{1}{\sqrt{(x+1)^2+4}} dx = \operatorname{arsinh}\left(\frac{x+1}{2}\right)$$

M1

Total: $2\sqrt{x^2+2x+5} + \operatorname{arsinh}\left(\frac{x+1}{2}\right) + c$

A1

Answer: $2\sqrt{x^2+2x+5} + \operatorname{arsinh}\left(\frac{x+1}{2}\right) + c$

5 marks

19 (4 marks)

$du = e^x dx$; limits $u = 1$ to $u = 2$

B1

$$\int_1^2 \frac{1}{1+u^2} du$$

M1

$= \arctan 2 - \arctan 1$

A1

$= \arctan 2 - \frac{\pi}{4}$

A1

Answer: $\arctan 2 - \frac{\pi}{4}$

4 marks

20 (5 marks)

$dx = \sec \theta \tan \theta d\theta$; $x = \sqrt{2}$ gives $\theta = \frac{\pi}{4}$, $x = 2$ gives $\theta = \frac{\pi}{3}$

B1

$\sqrt{x^2-1} = \sqrt{\sec^2 \theta - 1} = \tan \theta$

M1

Integrand becomes $\frac{\sec \theta \tan \theta}{\sec \theta \tan \theta} = 1$

M1 A1

$$\int_{\pi/4}^{\pi/3} 1 d\theta = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$$

A1

Answer: $\frac{\pi}{12}$

5 marks

21 (7 marks)

Let $\frac{x+3}{(x+1)(x^2+1)} \equiv \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$

M1

$x+3 \equiv A(x^2+1) + (Bx+C)(x+1)$; $x = -1$ gives $2 = 2A$, so $A = 1$

M1 A1

Compare x^2 : $0 = A + B$ so $B = -1$; constant: $3 = A + C$ so $C = 2$

A1

$$\int_0^1 = [\ln(x+1) - \frac{1}{2} \ln(x^2+1) + 2 \arctan x]_0^1$$

M1 A1

$$= \ln 2 - \frac{1}{2} \ln 2 + \frac{\pi}{2} = \frac{1}{2} \ln 2 + \frac{\pi}{2}$$

A1

Answer: $\frac{1}{2} \ln 2 + \frac{\pi}{2}$

7 marks

22 (6 marks)

$dx = 2 \cos \theta d\theta$; limits $\theta = 0$ to $\frac{\pi}{6}$

B1

$\sqrt{4-x^2} = 2 \cos \theta$, so the integrand is $\frac{4 \sin^2 \theta}{2 \cos \theta} \times 2 \cos \theta = 4 \sin^2 \theta$

M1 A1

$= 2(1 - \cos 2\theta)$

M1

$$\int_0^{\pi/6} = [2\theta - \sin 2\theta]_0^{\pi/6}$$

A1

$$= \frac{\pi}{3} - \frac{\sqrt{3}}{2}$$

A1

Answer: $\frac{\pi}{3} - \frac{\sqrt{3}}{2}$

6 marks

23 (6 marks)

$$dx = \cosh u \, du; \text{ limits } u = 0 \text{ to } \operatorname{arsinh} 1 = \ln(1 + \sqrt{2})$$

B1

$$\sqrt{1 + \sinh^2 u} = \cosh u, \text{ integrand } \cosh^2 u$$

M1

$$\cosh^2 u = \frac{1}{2}(1 + \cosh 2u)$$

M1

$$\int = \frac{1}{2} \left[u + \frac{1}{2} \sinh 2u \right] = \frac{1}{2} [u + \sinh u \cosh u]$$

M1 A1

$$\sinh u = 1, \cosh u = \sqrt{2}, \text{ so value} = \frac{1}{2}(\ln(1 + \sqrt{2}) + \sqrt{2})$$

A1

$$\text{Answer: } \frac{1}{2}(\sqrt{2} + \ln(1 + \sqrt{2}))$$

6 marks

24 (6 marks)

$$u = \operatorname{arsinh} x, \frac{dv}{dx} = 1$$

M1

$$= [x \operatorname{arsinh} x]_0^1 - \int_0^1 \frac{x}{\sqrt{1+x^2}} dx$$

M1 A1

$$\int \frac{x}{\sqrt{1+x^2}} dx = \sqrt{1+x^2}$$

M1

$$= \operatorname{arsinh} 1 - [\sqrt{1+x^2}]_0^1 = \ln(1 + \sqrt{2}) - (\sqrt{2} - 1)$$

M1 A1

$$\text{Answer: } \ln(1 + \sqrt{2}) - \sqrt{2} + 1$$

6 marks

25 (7 marks)

(a) The standard form $\arcsin(\frac{x}{a})$ applies to $\frac{1}{\sqrt{a^2 - x^2}}$, but here there is no square root

E1

Answer: Wrong standard form

1 mark

$$(b) \frac{1}{9-x^2} \equiv \frac{1}{6} \left(\frac{1}{3-x} + \frac{1}{3+x} \right)$$

M1 A1

$$\int_0^1 = \frac{1}{6} [-\ln(3-x) + \ln(3+x)]_0^1$$

M1 A1

$$= \frac{1}{6} \left[\ln \left(\frac{3+x}{3-x} \right) \right]_0^1$$

M1

$$= \frac{1}{6} (\ln 2 - \ln 1) = \frac{1}{6} \ln 2$$

A1

$$\text{Answer: } \frac{1}{6} \ln 2$$

6 marks

26 (6 marks)

$$(a) (x+2)^2 + 9$$

M1 A1

$$\text{Answer: } (x+2)^2 + 9$$

2 marks

$$(b) \text{Area} = \int_{-2}^1 \frac{6}{(x+2)^2 + 3^2} dx$$

M1

$$= 6 \times \frac{1}{3} \left[\arctan \left(\frac{x+2}{3} \right) \right]_{-2}^1$$

M1 A1

$$= 2(\arctan 1 - \arctan 0) = \frac{\pi}{2}$$

A1

$$\text{Answer: } \frac{\pi}{2}$$

4 marks

27 (8 marks)

$$\text{By parts: } u = \arctan x, \frac{d v}{d x} = x^2; \frac{d u}{d x} = \frac{1}{1+x^2}, v = \frac{x^3}{3}$$

M1 A1

$$\int_0^1 = \left[\frac{x^3}{3} \arctan x \right]_0^1 - \frac{1}{3} \int_0^1 \frac{x^3}{1+x^2} d x$$

M1 A1

$$= \frac{\pi}{12} - \frac{1}{3} \int_0^1 \frac{x^3}{1+x^2} d x$$

A1

$$\frac{x^3}{1+x^2} = x - \frac{x}{1+x^2}$$

M1

$$\int_0^1 (x - \frac{x}{1+x^2}) d x = \left[\frac{x^2}{2} - \frac{1}{2} \ln(1+x^2) \right]_0^1 = \frac{1}{2} - \frac{1}{2} \ln 2$$

M1

$$\text{Value} = \frac{\pi}{12} - \frac{1}{3} \left(\frac{1}{2} - \frac{1}{2} \ln 2 \right) = \frac{\pi}{12} - \frac{1}{6} + \frac{1}{6} \ln 2$$

A1

$$\text{Answer: } \frac{\pi}{12} - \frac{1}{6} + \frac{1}{6} \ln 2$$

8 marks

28 (7 marks)

$$d x = 2 \sec^2 \theta d \theta; \text{ limits } \theta = 0 \text{ to } \frac{\pi}{4}$$

B1 B1

$$x^2 + 4 = 4 \sec^2 \theta, \text{ so } (x^2 + 4)^{3/2} = 8 \sec^3 \theta$$

M1 A1

$$\text{Integrand: } \frac{2 \sec^2 \theta}{8 \sec^3 \theta} = \frac{1}{4} \cos \theta$$

M1

$$\frac{1}{4} [\sin \theta]_0^{\pi/4}$$

M1

$$= \frac{\sqrt{2}}{8}$$

A1

$$\text{Answer: } \frac{\sqrt{2}}{8}$$

7 marks

29 (8 marks)

$$d x = 2 \cosh u d u; \text{ limits } u = 0 \text{ to } \operatorname{arsinh} 1$$

B1

$$\sqrt{x^2 + 4} = \sqrt{4 \sinh^2 u + 4} = 2 \cosh u$$

M1 A1

$$\text{Integrand: } \frac{4 \sinh^2 u}{2 \cosh u} \times 2 \cosh u = 4 \sinh^2 u$$

M1 A1

$$= 2(\cosh 2u - 1)$$

M1

$$[\sinh 2u - 2u]_0^{\operatorname{arsinh} 1}; \sinh 2u = 2 \sinh u \cosh u = 2 \times 1 \times \sqrt{2}$$

M1

$$= 2\sqrt{2} - 2 \ln(1 + \sqrt{2})$$

A1

$$\text{Answer: } 2\sqrt{2} - 2 \ln(1 + \sqrt{2})$$

8 marks

30 (9 marks)

$$(a) 3x^2 + 5x + 12 \equiv A(x^2 + 4) + (Bx + C)(x + 1)$$

M1

$$x = -1: 10 = 5A \text{ so } A = 2$$

M1 A1

$$x^2: 3 = A + B \text{ so } B = 1; \text{ constant: } 12 = 4A + C \text{ so } C = 4$$

A1

$$\text{Answer: } A = 2, B = 1, C = 4$$

4 marks

$$(b) \int \frac{2}{x+1} d x = 2 \ln(x+1)$$

B1

$$\int \frac{x}{x^2+4} d x = \frac{1}{2} \ln(x^2+4)$$

M1 A1

$$\int \frac{4}{x^2+4} d x = 2 \arctan\left(\frac{x}{2}\right)$$

M1

$$\text{Limits: } 2 \ln 3 + \frac{1}{2}(\ln 8 - \ln 4) + 2 \times \frac{\pi}{4} = 2 \ln 3 + \frac{1}{2} \ln 2 + \frac{\pi}{2}$$

A1

$$\text{Answer: } 2 \ln 3 + \frac{1}{2} \ln 2 + \frac{\pi}{2}$$

5 marks

Integration by Reduction Formulae

1 (2 marks)

$$I_2 = \frac{1}{2} I_0 = \frac{\pi}{4} \quad \text{M1}$$

$$I_4 = \frac{3}{4} I_2 = \frac{3\pi}{16} \quad \text{A1}$$

$$\text{Answer: } \frac{3\pi}{16} \quad \text{2 marks}$$

2 (3 marks)

$$I_0 = [-e^{-x}]_0^1 = 1 - e^{-1} \quad \text{B1}$$

$$I_1 = I_0 - e^{-1} = 1 - 2e^{-1} \quad \text{M1}$$

$$I_2 = 2I_1 - e^{-1} = 2 - 5e^{-1} \quad \text{A1}$$

$$\text{Answer: } 2 - 5e^{-1} \quad \text{3 marks}$$

3 (3 marks)

$$T_0 = \int_0^{\pi/4} 1 \, dx = \frac{\pi}{4} \quad \text{B1}$$

$$T_1 = [\ln(\sec x)]_0^{\pi/4} \quad \text{M1}$$

$$= \ln \sqrt{2} = \frac{1}{2} \ln 2 \quad \text{A1}$$

$$\text{Answer: } T_0 = \frac{\pi}{4}, T_1 = \frac{1}{2} \ln 2 \quad \text{3 marks}$$

4 (2 marks)

$$C_3 = \frac{2}{3} C_1 = \frac{2}{3} \quad \text{M1}$$

$$C_5 = \frac{4}{5} C_3 = \frac{8}{15} \quad \text{A1}$$

$$\text{Answer: } \frac{8}{15} \quad \text{2 marks}$$

5 (3 marks)

$$I_1 = \frac{1}{2} x e^{2x} - \frac{1}{2} I_0 \quad \text{M1}$$

$$I_0 = \frac{1}{2} e^{2x} \quad \text{B1}$$

$$I_1 = \frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} \quad \text{A1}$$

$$\text{Answer: } \frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} \quad \text{3 marks}$$

6 (3 marks)

$$L_0 = e - 1, L_1 = e - L_0 = 1 \quad \text{M1}$$

$$L_2 = e - 2L_1 = e - 2 \quad \text{A1}$$

$$L_3 = e - 3L_2 = 6 - 2e \quad \text{A1}$$

$$\text{Answer: } 6 - 2e \quad \text{3 marks}$$

7 (3 marks)

$$\text{By parts: } u = x^n, \frac{dv}{dx} = e^x; \frac{du}{dx} = n x^{n-1}, v = e^x \quad \text{M1}$$

$$I_n = x^n e^x - \int n x^{n-1} e^x \, dx \quad \text{A1}$$

$$= x^n e^x - n I_{n-1} \quad \text{A1}$$

$$\text{Answer: } I_n = x^n e^x - n I_{n-1} \quad \text{3 marks}$$

8 (4 marks)

$$T_n + T_{n-2} = \int_0^{\pi/4} \tan^{n-2} x (\tan^2 x + 1) dx \quad \text{M1}$$

$$= \int_0^{\pi/4} \tan^{n-2} x \sec^2 x dx \quad \text{A1}$$

$$= \left[\frac{\tan^{n-1} x}{n-1} \right]_0^{\pi/4} \quad \text{M1}$$

$$= \frac{1}{n-1} \quad \text{A1}$$

Answer: $T_n + T_{n-2} = \frac{1}{n-1}$ 4 marks

9 (4 marks)

$$u = x^n, \frac{dv}{dx} = e^{-x}; \frac{du}{dx} = n x^{n-1}, v = -e^{-x} \quad \text{M1 A1}$$

$$I_n = [-x^n e^{-x}]_0^1 + n \int_0^1 x^{n-1} e^{-x} dx \quad \text{M1}$$

$$= -e^{-1} + n I_{n-1} \quad \text{A1}$$

Answer: $I_n = n I_{n-1} - e^{-1}$ 4 marks

10 (4 marks)

$$\text{Write } (\ln x)^n = 1 \times (\ln x)^n; u = (\ln x)^n, \frac{dv}{dx} = 1 \quad \text{M1}$$

$$\frac{du}{dx} = \frac{n (\ln x)^{n-1}}{x}, v = x \quad \text{M1 A1}$$

$$L_n = [x (\ln x)^n]_1^e - n \int_1^e (\ln x)^{n-1} dx = e - n L_{n-1} \quad \text{A1}$$

Answer: $L_n = e - n L_{n-1}$ 4 marks

11 (4 marks)

$$\sin^5 x \cos^2 x = \sin^5 x - \sin^7 x \quad \text{M1}$$

$$I_1 = 1, I_5 = \frac{4}{5} \times \frac{2}{3} \times 1 = \frac{8}{15} \quad \text{M1}$$

$$I_7 = \frac{6}{7} I_5 = \frac{16}{35} \quad \text{A1}$$

$$\frac{8}{15} - \frac{16}{35} = \frac{56 - 48}{105} = \frac{8}{105} \quad \text{A1}$$

Answer: $\frac{8}{105}$ 4 marks

12 (5 marks)

$$u = x^n, \frac{dv}{dx} = e^{3x}; v = \frac{1}{3} e^{3x} \quad \text{M1}$$

$$I_n = \frac{1}{3} x^n e^{3x} - \frac{n}{3} \int x^{n-1} e^{3x} dx = \frac{1}{3} x^n e^{3x} - \frac{n}{3} I_{n-1} \quad \text{A1}$$

$$I_0 = \frac{1}{3} e^{3x}, I_1 = \frac{1}{3} x e^{3x} - \frac{1}{9} e^{3x} \quad \text{M1}$$

$$I_2 = \frac{1}{3} x^2 e^{3x} - \frac{2}{3} \left(\frac{1}{3} x e^{3x} - \frac{1}{9} e^{3x} \right) \quad \text{M1}$$

$$= e^{3x} \left(\frac{x^2}{3} - \frac{2x}{9} + \frac{2}{27} \right) + c \quad \text{A1}$$

Answer: $e^{3x} \left(\frac{x^2}{3} - \frac{2x}{9} + \frac{2}{27} \right) + c$ 5 marks

13 (3 marks)

Product rule: $\frac{d}{dx}(\sin^{n-1} x \cos x) = (n-1) \sin^{n-2} x \cos^2 x - \sin^n x$ M1 A1

Replace $\cos^2 x = 1 - \sin^2 x$: $(n-1) \sin^{n-2} x - (n-1) \sin^n x - \sin^n x = (n-1) \sin^{n-2} x - n \sin^n x$ A1

Answer: $\frac{d}{dx}(\sin^{n-1} x \cos x) = (n-1) \sin^{n-2} x - n \sin^n x$ 3 marks

14 (7 marks)

(a) $I_n = \int_0^{\pi/2} \sin^{n-1} x \sin x \, dx$; $u = \sin^{n-1} x$, $\frac{dv}{dx} = \sin x$ M1

$I_n = [-\sin^{n-1} x \cos x]_0^{\pi/2} + (n-1) \int_0^{\pi/2} \sin^{n-2} x \cos^2 x \, dx$ M1 A1

First term is 0; $\cos^2 x = 1 - \sin^2 x$ gives $I_n = (n-1)(I_{n-2} - I_n)$, so $n I_n = (n-1) I_{n-2}$ A1

Answer: $I_n = \frac{n-1}{n} I_{n-2}$ 4 marks

(b) $I_0 = \frac{\pi}{2}$; $I_2 = \frac{1}{2} I_0 = \frac{\pi}{4}$ B1

$I_4 = \frac{3}{4} I_2 = \frac{3\pi}{16}$ M1

$I_6 = \frac{5}{6} I_4 = \frac{5\pi}{32}$ A1

Answer: $\frac{5\pi}{32}$ 3 marks

15 (7 marks)

(a) $u = (1-x^2)^n$, $\frac{dv}{dx} = 1$; $\frac{du}{dx} = -2nx(1-x^2)^{n-1}$, $v = x$ M1 A1

$I_n = [x(1-x^2)^n]_0^1 + 2n \int_0^1 x^2(1-x^2)^{n-1} \, dx$; the first term is 0 M1

$x^2 = 1 - (1-x^2)$; $I_n = 2n(I_{n-1} - I_n)$ M1

$(2n+1) I_n = 2n I_{n-1}$, giving the result A1

Answer: $I_n = \frac{2n}{2n+1} I_{n-1}$ 5 marks

(b) $I_0 = 1$, $I_1 = \frac{2}{3}$, $I_2 = \frac{4}{5} \times \frac{2}{3} = \frac{8}{15}$ M1

$I_3 = \frac{6}{7} \times \frac{8}{15} = \frac{16}{35}$ A1

Answer: $\frac{16}{35}$ 2 marks

16 (7 marks)

(a) $u = \sec^{n-2} x$, $\frac{dv}{dx} = \sec^2 x$; $\frac{du}{dx} = (n-2) \sec^{n-2} x \tan x$, $v = \tan x$ M1 A1

$S_n = [\sec^{n-2} x \tan x]_0^{\pi/4} - (n-2) \int_0^{\pi/4} \sec^{n-2} x \tan^2 x \, dx$ M1

$\tan^2 x = \sec^2 x - 1$; $S_n = (\sqrt{2})^{n-2} - (n-2)(S_n - S_{n-2})$ M1

Rearranging gives $(n-1) S_n = (\sqrt{2})^{n-2} + (n-2) S_{n-2}$ A1

Answer: $(n-1) S_n = (\sqrt{2})^{n-2} + (n-2) S_{n-2}$ 5 marks

(b) $S_2 = [\tan x]_0^{\pi/4} = 1$ B1

$3 S_4 = 2 + 2 \times 1 = 4$, so $S_4 = \frac{4}{3}$ A1

Answer: $\frac{4}{3}$ 2 marks

17 (7 marks)

$$(a) u = x^n, \frac{d v}{d x} = (1 - x)^{1/2}; \frac{d u}{d x} = n x^{n-1}, v = -\frac{2}{3}(1 - x)^{3/2} \quad \text{M1 A1}$$

$$K_n = \left[-\frac{2}{3} x^n (1 - x)^{3/2}\right]_0^1 + \frac{2n}{3} \int_0^1 x^{n-1} (1 - x)^{3/2} d x; \text{ the first term is 0} \quad \text{M1}$$

$$(1 - x)^{3/2} = (1 - x)(1 - x)^{1/2}; K_n = \frac{2n}{3}(K_{n-1} - K_n) \quad \text{M1}$$

$$(3 + 2n) K_n = 2n K_{n-1}, \text{ so } K_n = \frac{2n}{2n + 3} K_{n-1} \quad \text{A1}$$

$$\text{Answer: } K_n = \frac{2n}{2n + 3} K_{n-1} \quad \text{5 marks}$$

$$(b) K_0 = \int_0^1 \sqrt{1 - x} d x = \frac{2}{3} \quad \text{B1}$$

$$K_1 = \frac{2}{5} \times \frac{2}{3} = \frac{4}{15}, K_2 = \frac{4}{7} \times \frac{4}{15} = \frac{16}{105} \quad \text{A1}$$

$$\text{Answer: } \frac{16}{105} \quad \text{2 marks}$$

18 (7 marks)

$$(a) \text{ Write } x^n e^{-x^2} = x^{n-1} \times x e^{-x^2}; u = x^{n-1}, \frac{d v}{d x} = x e^{-x^2}, v = -\frac{1}{2} e^{-x^2} \quad \text{M1 A1}$$

$$I_n = \left[-\frac{1}{2} x^{n-1} e^{-x^2}\right]_0^1 + \frac{n-1}{2} \int_0^1 x^{n-2} e^{-x^2} d x \quad \text{M1}$$

$$= -\frac{1}{2e} + \frac{n-1}{2} I_{n-2} \quad \text{A1}$$

$$\text{Answer: } I_n = \frac{n-1}{2} I_{n-2} - \frac{1}{2e} \quad \text{4 marks}$$

$$(b) I_1 = \left[-\frac{1}{2} e^{-x^2}\right]_0^1 = \frac{1}{2}(1 - e^{-1}) \quad \text{B1}$$

$$I_3 = I_1 - \frac{1}{2e} \quad \text{M1}$$

$$= \frac{1}{2} - \frac{1}{2e} - \frac{1}{2e} = \frac{1}{2} - e^{-1} \quad \text{A1}$$

$$\text{Answer: } I_3 = \frac{1}{2} - e^{-1} \quad \text{3 marks}$$

19 (6 marks)

$$(a) \cot^n x = \cot^{n-2} x (\operatorname{cosec}^2 x - 1) \quad \text{M1}$$

$$K_n = \int_{\pi/4}^{\pi/2} \cot^{n-2} x \operatorname{cosec}^2 x d x - K_{n-2} = \left[-\frac{\cot^{n-1} x}{n-1}\right]_{\pi/4}^{\pi/2} - K_{n-2} \quad \text{M1}$$

$$\text{At } \frac{\pi}{2} \cot = 0 \text{ and at } \frac{\pi}{4} \cot = 1, \text{ so } K_n = \frac{1}{n-1} - K_{n-2} \quad \text{A1}$$

$$\text{Answer: } K_n = \frac{1}{n-1} - K_{n-2} \quad \text{3 marks}$$

$$(b) K_1 = [\ln(\sin x)]_{\pi/4}^{\pi/2} = 0 - \ln\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2} \ln 2 \quad \text{B1}$$

$$K_3 = \frac{1}{2} - K_1 = \frac{1}{2} - \frac{1}{2} \ln 2 \quad \text{M1}$$

$$K_5 = \frac{1}{4} - K_3 = \frac{1}{2} \ln 2 - \frac{1}{4} \quad \text{A1}$$

$$\text{Answer: } \frac{1}{2} \ln 2 - \frac{1}{4} \quad \text{3 marks}$$

20 (7 marks)

(a) $\tanh^n x = \tanh^{n-2} x (1 - \operatorname{sech}^2 x)$, so $J_n = J_{n-2} - \int_0^{\ln 3} \tanh^{n-2} x \operatorname{sech}^2 x \, dx$ M1 A1

$$\int \tanh^{n-2} x \operatorname{sech}^2 x \, dx = \frac{\tanh^{n-1} x}{n-1}$$
 M1

$$\tanh(\ln 3) = \frac{3 - \frac{1}{3}}{3 + \frac{1}{3}} = \frac{4}{5}, \text{ so } J_n = J_{n-2} - \frac{1}{n-1} \left(\frac{4}{5}\right)^{n-1}$$
 A1

Answer: $J_n = J_{n-2} - \frac{1}{n-1} \left(\frac{4}{5}\right)^{n-1}$ 4 marks

(b) $J_0 = \ln 3$ and $J_2 = J_0 - \frac{4}{5} = \ln 3 - \frac{4}{5}$ M1 A1

$$J_4 = J_2 - \frac{1}{3} \left(\frac{4}{5}\right)^3 = \ln 3 - \frac{4}{5} - \frac{64}{375} = \ln 3 - \frac{364}{375}$$
 A1

Answer: $\ln 3 - \frac{364}{375}$ 3 marks

21 (8 marks)

(a) $u = x^n, \frac{dv}{dx} = \sin x$: $K_n = [-x^n \cos x]_0^{\pi/2} + n \int_0^{\pi/2} x^{n-1} \cos x \, dx$ M1 A1

The first term is 0 ($\cos \frac{\pi}{2} = 0$ and $x^n = 0$ at $x = 0$) B1

Again by parts: $\int_0^{\pi/2} x^{n-1} \cos x \, dx = [x^{n-1} \sin x]_0^{\pi/2} - (n-1) \int_0^{\pi/2} x^{n-2} \sin x \, dx$ M1 A1

$$= \left(\frac{\pi}{2}\right)^{n-1} - (n-1) K_{n-2}, \text{ so } K_n = n \left(\frac{\pi}{2}\right)^{n-1} - n(n-1) K_{n-2}$$
 A1

Answer: $K_n = n \left(\frac{\pi}{2}\right)^{n-1} - n(n-1) K_{n-2}$ 6 marks

(b) $K_1 = [\sin x - x \cos x]_0^{\pi/2} = 1$ B1

$$K_3 = 3 \left(\frac{\pi}{2}\right)^2 - 6 \times 1 = \frac{3\pi^2}{4} - 6$$
 A1

Answer: $\frac{3\pi^2}{4} - 6$ 2 marks

22 (7 marks)

(a) Differentiate: $\frac{d}{dx} \left(\frac{x}{(1+x^2)^n} \right) = \frac{1}{(1+x^2)^n} - \frac{2nx^2}{(1+x^2)^{n+1}}$ M1 A1

$$x^2 = (1+x^2) - 1: \text{ this equals } \frac{(1-2n)}{(1+x^2)^n} + \frac{2n}{(1+x^2)^{n+1}}$$
 M1

Integrate over $[0, 1]$: $\frac{1}{2^n} = (1-2n) I_n + 2n I_{n+1}$ M1

$$I_{n+1} = \frac{1}{n 2^{n+1}} + \frac{2n-1}{2n} I_n$$
 A1

Answer: $I_{n+1} = \frac{1}{n 2^{n+1}} + \frac{2n-1}{2n} I_n$ 5 marks

(b) $I_1 = [\arctan x]_0^1 = \frac{\pi}{4}$ B1

$$I_2 = \frac{1}{4} + \frac{1}{2} \times \frac{\pi}{4} = \frac{1}{4} + \frac{\pi}{8}$$
 A1

Answer: $\frac{1}{4} + \frac{\pi}{8}$ 2 marks

23 (8 marks)

$$(a) u = \cosh^{n-1} x, \frac{dv}{dx} = \cosh x; \frac{du}{dx} = (n-1) \cosh^{n-2} x \sinh x, v = \sinh x$$

M1 A1

$$C_n = [\cosh^{n-1} x \sinh x]_0^{\ln 2} - (n-1) \int_0^{\ln 2} \cosh^{n-2} x \sinh^2 x \, dx$$

M1

$$\sinh^2 x = \cosh^2 x - 1; C_n = [\cosh^{n-1} x \sinh x]_0^{\ln 2} - (n-1)(C_n - C_{n-2})$$

M1

$$n C_n = \cosh^{n-1}(\ln 2) \sinh(\ln 2) + (n-1) C_{n-2}$$

A1

$$\text{Answer: } n C_n = \cosh^{n-1}(\ln 2) \sinh(\ln 2) + (n-1) C_{n-2}$$

5 marks

$$(b) \cosh(\ln 2) = \frac{5}{4}, \sinh(\ln 2) = \frac{3}{4}, C_0 = \ln 2; 2C_2 = \frac{15}{16} + \ln 2, \text{ so } C_2 = \frac{15}{32} + \frac{1}{2} \ln 2$$

B1

$$4C_4 = \left(\frac{5}{4}\right)^3 \times \frac{3}{4} + 3C_2 = \frac{375}{256} + \frac{45}{32} + \frac{3}{2} \ln 2$$

M1

$$C_4 = \frac{3 \ln 2}{8} + \frac{735}{1024}$$

A1

$$\text{Answer: } \frac{3 \ln 2}{8} + \frac{735}{1024}$$

3 marks

24 (7 marks)

$$(a) u = (\ln x)^n, \frac{dv}{dx} = x^2; \frac{du}{dx} = \frac{n(\ln x)^{n-1}}{x}, v = \frac{x^3}{3}$$

M1 A1

$$M_n = \left[\frac{x^3}{3}(\ln x)^n\right]_1^e - \frac{n}{3} \int_1^e x^2(\ln x)^{n-1} \, dx$$

M1

$$= \frac{e^3}{3} - \frac{n}{3} M_{n-1}$$

A1

$$\text{Answer: } M_n = \frac{e^3}{3} - \frac{n}{3} M_{n-1}$$

4 marks

$$(b) M_0 = \left[\frac{x^3}{3}\right]_1^e = \frac{e^3 - 1}{3}; M_1 = \frac{e^3}{3} - \frac{1}{3} M_0 = \frac{2e^3 + 1}{9}$$

B1

$$M_2 = \frac{e^3}{3} - \frac{2}{3} M_1 = \frac{5e^3 - 2}{27}$$

M1

$$M_3 = \frac{e^3}{3} - M_2 = \frac{4e^3 + 2}{27} = \frac{2}{27}(2e^3 + 1)$$

A1

$$\text{Answer: } M_3 = \frac{2}{27}(2e^3 + 1)$$

3 marks

25 (7 marks)

$$(a) u = (1-x)^n, \frac{dv}{dx} = x^m; \frac{du}{dx} = -n(1-x)^{n-1}, v = \frac{x^{m+1}}{m+1}$$

M1 A1

$$B(m, n) = \left[\frac{x^{m+1}(1-x)^n}{m+1}\right]_0^1 + \frac{n}{m+1} \int_0^1 x^{m+1}(1-x)^{n-1} \, dx$$

M1

$$\text{The first term is 0, so } B(m, n) = \frac{n}{m+1} B(m+1, n-1)$$

A1

$$\text{Answer: } B(m, n) = \frac{n}{m+1} B(m+1, n-1)$$

4 marks

$$(b) B(2, 3) = \frac{3}{3} B(3, 2) = B(3, 2), B(3, 2) = \frac{2}{4} B(4, 1), B(4, 1) = \frac{1}{5} B(5, 0)$$

M1

$$B(5, 0) = \int_0^1 x^5 \, dx = \frac{1}{6}$$

B1

$$B(2, 3) = 1 \times \frac{1}{2} \times \frac{1}{5} \times \frac{1}{6} = \frac{1}{60}$$

A1

$$\text{Answer: } \frac{1}{60}$$

3 marks

26 (6 marks)

$$(a) T_n = \int_0^{\pi/4} \tan^{n-2} x \sec^2 x \, dx - T_{n-2}$$

M1

$$\text{The sign of } T_{n-2} \text{ is wrong: it should be minus, so } T_n = \frac{1}{n-1} - T_{n-2}$$

E1

$$\text{Answer: Sign error: } T_n = \frac{1}{n-1} - T_{n-2}$$

2 marks

$$(b) T_0 = \frac{\pi}{4} \quad \text{B1}$$

$$T_2 = 1 - T_0 = 1 - \frac{\pi}{4} \quad \text{M1}$$

$$T_4 = \frac{1}{3} - T_2 = \frac{\pi}{4} - \frac{2}{3}, T_6 = \frac{1}{5} - T_4 \quad \text{M1}$$

$$T_6 = \frac{13}{15} - \frac{\pi}{4} \quad \text{A1}$$

$$\text{Answer: } \frac{13}{15} - \frac{\pi}{4} \quad \text{4 marks}$$

27 (9 marks)

$$(a) \text{ Write the integrand as } (\sin^m x \cos x) \times \cos^{n-1} x; u = \cos^{n-1} x, \frac{d v}{d x} = \sin^m x \cos x, v = \frac{\sin^{m+1} x}{m+1} \quad \text{M1 A1}$$

$$W(m, n) = \left[\frac{\sin^{m+1} x \cos^{n-1} x}{m+1} \right]_0^{\pi/2} + \frac{n-1}{m+1} \int_0^{\pi/2} \sin^{m+2} x \cos^{n-2} x \, dx \quad \text{M1 A1}$$

$$\text{The first term is 0; } \sin^{m+2} x = \sin^m x (1 - \cos^2 x), \text{ so } W(m, n) = \frac{n-1}{m+1} (W(m, n-2) - W(m, n)) \quad \text{M1}$$

$$(m+1) W(m, n) + (n-1) W(m, n) = (n-1) W(m, n-2), \text{ so } W(m, n) = \frac{n-1}{m+n} W(m, n-2) \quad \text{A1}$$

$$\text{Answer: } W(m, n) = \frac{n-1}{m+n} W(m, n-2) \quad \text{6 marks}$$

$$(b) W(4, 0) = \frac{3\pi}{16} \quad \text{B1}$$

$$W(4, 2) = \frac{1}{6} W(4, 0) = \frac{\pi}{32}, W(4, 4) = \frac{3}{8} W(4, 2) = \frac{3\pi}{256} \quad \text{M1}$$

$$W(4, 6) = \frac{5}{10} W(4, 4) = \frac{3\pi}{512} \quad \text{A1}$$

$$\text{Answer: } \frac{3\pi}{512} \quad \text{3 marks}$$

28 (9 marks)

$$(a) \text{ Multiply the given result by } I_{n-1}: n I_n I_{n-1} = (n-1) I_{n-2} I_{n-1} \quad \text{M1 A1}$$

$$\text{Answer: } n I_n I_{n-1} = (n-1) I_{n-1} I_{n-2} \quad \text{2 marks}$$

$$(b) \text{ Let } u_n = n I_n I_{n-1}; (a) \text{ shows } u_n = u_{n-1} \text{ for } n \geq 2 \quad \text{M1 A1}$$

$$u_1 = 1 \times I_1 I_0 = 1 \times \frac{\pi}{2}, \text{ so } u_n = \frac{\pi}{2} \text{ for all } n \quad \text{A1}$$

$$\text{Answer: } n I_n I_{n-1} = \frac{\pi}{2} \quad \text{3 marks}$$

$$(c) 5 I_5 I_4 = \frac{\pi}{2}, \text{ so } I_5 = \frac{\pi}{2} \div (5 \times \frac{3\pi}{16}) \quad \text{M1 A1}$$

$$I_5 = \frac{8}{15} \quad \text{A1}$$

$$\text{Check: } I_5 = \frac{4}{5} I_3 = \frac{4}{5} \times \frac{2}{3} \times I_1 = \frac{8}{15} \quad \text{B1}$$

$$\text{Answer: } \frac{8}{15} \quad \text{4 marks}$$

29 (10 marks)

$$(a) dx = \cos \theta \, d\theta; x = 0 \text{ gives } \theta = 0, x = 1 \text{ gives } \theta = \frac{\pi}{2} \quad \text{B1}$$

$$(\arcsin x)^n = \theta^n, \text{ so } A_n = \int_0^{\pi/2} \theta^n \cos \theta \, d\theta \quad \text{A1}$$

$$\text{Answer: } A_n = \int_0^{\pi/2} \theta^n \cos \theta \, d\theta \quad \text{2 marks}$$

$$(b) u = \theta^n, \frac{d v}{d \theta} = \cos \theta: A_n = [\theta^n \sin \theta]_0^{\pi/2} - n \int_0^{\pi/2} \theta^{n-1} \sin \theta \, d\theta \quad \text{M1 A1}$$

$$= \left(\frac{\pi}{2}\right)^n - n \int_0^{\pi/2} \theta^{n-1} \sin \theta \, d\theta \quad \text{A1}$$

$$\int_0^{\pi/2} \theta^{n-1} \sin \theta \, d\theta = [-\theta^{n-1} \cos \theta]_0^{\pi/2} + (n-1) \int_0^{\pi/2} \theta^{n-2} \cos \theta \, d\theta \quad \text{M1 A1}$$

The bracket is 0, so the integral is $(n - 1) A_{n-2}$ and $A_n = \left(\frac{\pi}{2}\right)^n - n(n - 1) A_{n-2}$

A1

Answer: $A_n = \left(\frac{\pi}{2}\right)^n - n(n - 1) A_{n-2}$

6 marks

(c) $A_0 = \int_0^{\pi/2} \cos \theta \, d\theta = 1$

B1

$A_2 = \frac{\pi^2}{4} - 2$

A1

Answer: $\frac{\pi^2}{4} - 2$

2 marks

30 (9 marks)

(a) $m = 0$: $I_0 = \frac{\pi}{2}$; formula gives $\frac{\pi}{2}$ $\square$

B1

Assume $I_{2k} = \frac{(2k)! \pi}{2^{2k+1} (k!)^2}$

B1

$I_{2k+2} = \frac{2k+1}{2k+2} I_{2k}$

M1

$= \frac{(2k+1)}{(2k+2)} \times \frac{(2k)! \pi}{2^{2k+1} (k!)^2}$

M1

Multiply numerator and denominator by $(2k+2)$: numerator $(2k+2)! \pi$; denominator $(2k+2)^2 2^{2k+1} (k!)^2 = 2^{2k+3} ((k+1)!)^2$

M1 A1

So the formula holds for $k+1$; true for $m=0$ and k implies $k+1$, so true for all $m \geq 0$ by induction

A1

Answer: $I_{2m} = \frac{(2m)! \pi}{2^{2m+1} (m!)^2}$

7 marks

(b) $I_8 = \frac{8! \pi}{2^9 (4!)^2} = \frac{40320\pi}{512 \times 576} = \frac{35\pi}{256}$

M1

Check: $\frac{7}{8} \times \frac{5}{6} \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2} = \frac{105\pi}{768} = \frac{35\pi}{256}$

A1

Answer: $\frac{35\pi}{256}$

2 marks

Improper Integrals

1 (2 marks)

$\int_1^R \frac{1}{x^2} \, dx = \left[-\frac{1}{x}\right]_1^R = 1 - \frac{1}{R}$

M1

As $R \rightarrow \infty$, $\frac{1}{R} \rightarrow 0$, so the value is 1

A1

Answer: 1

2 marks

2 (2 marks)

$\int_0^R e^{-2x} \, dx = \left[-\frac{1}{2} e^{-2x}\right]_0^R = \frac{1}{2} - \frac{1}{2} e^{-2R}$

M1

As $R \rightarrow \infty$, $e^{-2R} \rightarrow 0$, so the value is $\frac{1}{2}$

A1

Answer: $\frac{1}{2}$

2 marks

3 (2 marks)

The integrand is undefined at $x=0$; $\int_a^1 x^{-1/2} \, dx = [2\sqrt{x}]_a^1 = 2 - 2\sqrt{a}$

M1

As $a \rightarrow 0$, $2\sqrt{a} \rightarrow 0$, so the value is 2

A1

Answer: 2

2 marks

4 (2 marks)

$$\int_1^R \frac{1}{x} dx = [\ln x]_1^R = \ln R$$

M1

As $R \rightarrow \infty$, $\ln R \rightarrow \infty$, so the integral diverges

A1

Answer: Diverges

2 marks

5 (2 marks)

$$\int_0^R \frac{1}{1+x^2} dx = \arctan R$$

M1

As $R \rightarrow \infty$, $\arctan R \rightarrow \frac{\pi}{2}$, so the value is $\frac{\pi}{2}$

A1

Answer: $\frac{\pi}{2}$

2 marks

6 (2 marks)(i) is improper: $\ln x$ is undefined at $x = 0$, the lower limit

B1

(iii) is improper because the upper limit is infinite; (ii) and (iv) are not improper as the integrands are continuous on the finite intervals

B1

Answer: (i) and (iii) only

2 marks

7 (3 marks)

$$\int_0^R x e^{-x^2} dx = \left[-\frac{1}{2} e^{-x^2}\right]_0^R$$

M1 A1

$$= \frac{1}{2} - \frac{1}{2} e^{-R^2} \rightarrow \frac{1}{2} \text{ as } R \rightarrow \infty$$

A1

Answer: $\frac{1}{2}$

3 marks

8 (3 marks)

$$\int_a^1 \ln x dx = [x \ln x - x]_a^1 = -1 - a \ln a + a$$

M1 A1

As $a \rightarrow 0$, $a \ln a \rightarrow 0$ and $a \rightarrow 0$, so the integral is -1

A1

Answer: -1

3 marks

9 (4 marks)

$$\frac{1}{x^2 - 1} \equiv \frac{1}{2} \left(\frac{1}{x-1} - \frac{1}{x+1} \right)$$

M1 A1

$$\int_2^R = \frac{1}{2} \left[\ln \left(\frac{x-1}{x+1} \right) \right]_2^R$$

M1

As $R \rightarrow \infty$, $\frac{R-1}{R+1} \rightarrow 1$, so $\ln \rightarrow 0$; value = $0 - \frac{1}{2} \ln \left(\frac{1}{3} \right) = \frac{1}{2} \ln 3$

A1

Answer: $\frac{1}{2} \ln 3$

4 marks

10 (3 marks)

$$\text{By parts: } \int_0^R x e^{-x} dx = [-x e^{-x}]_0^R + \int_0^R e^{-x} dx = -R e^{-R} + 1 - e^{-R}$$

M1 A1

As $R \rightarrow \infty$, $R e^{-R} \rightarrow 0$ and $e^{-R} \rightarrow 0$, so the value is 1

A1

Answer: 1

3 marks

11 (3 marks)

$$\text{The integrand is undefined at } x = 3: \int_0^b \frac{1}{\sqrt{9-x^2}} dx = \arcsin \left(\frac{b}{3} \right)$$

M1 A1

As $b \rightarrow 3$ from below, $\arcsin \left(\frac{b}{3} \right) \rightarrow \arcsin 1 = \frac{\pi}{2}$

A1

Answer: $\frac{\pi}{2}$

3 marks

12 (4 marks)

$$\int_1^R x^{-p} dx = \left[\frac{x^{1-p}}{1-p} \right]_1^R$$

$$= \frac{R^{1-p} - 1}{1-p}$$

M1 A1

A1

Since $p > 1$, $1 - p < 0$ so $R^{1-p} \rightarrow 0$ as $R \rightarrow \infty$, and the value is $\frac{1}{p-1}$

A1

Answer: $\frac{1}{p-1}$

4 marks

13 (4 marks)

$$\frac{1}{x(x+1)} \equiv \frac{1}{x} - \frac{1}{x+1}$$

M1 A1

$$\int_1^R = \left[\ln\left(\frac{x}{x+1}\right) \right]_1^R = \ln\left(\frac{R}{R+1}\right) + \ln 2$$

M1

As $R \rightarrow \infty$, $\ln\left(\frac{R}{R+1}\right) \rightarrow 0$, so the value is $\ln 2$

A1

Answer: $\ln 2$

4 marks

14 (5 marks)

By parts: $\int_0^R x^2 e^{-x} dx = [-x^2 e^{-x}]_0^R + 2 \int_0^R x e^{-x} dx$

M1 A1

$$\int_0^R x e^{-x} dx = -R e^{-R} + 1 - e^{-R}$$

M1

As $R \rightarrow \infty$, $R^2 e^{-R}$, $R e^{-R}$ and e^{-R} all tend to 0

B1

So the integral is $2 \times 1 = 2$

A1

Answer: 2

5 marks

15 (5 marks)

$$u = \ln x, \frac{dv}{dx} = x: \int_a^1 x \ln x dx = \left[\frac{x^2}{2} \ln x \right]_a^1 - \frac{1}{2} \int_a^1 x dx$$

M1 A1

$$= -\frac{a^2}{2} \ln a - \frac{1}{4}(1 - a^2)$$

M1 A1

As $a \rightarrow 0$, $a^2 \ln a \rightarrow 0$, so the value is $-\frac{1}{4}$

A1

Answer: $-\frac{1}{4}$

5 marks

16 (4 marks)

$$x^2 + 2x + 2 = (x+1)^2 + 1$$

B1

$$\int_0^R = [\arctan(x+1)]_0^R = \arctan(R+1) - \arctan 1$$

M1 A1

As $R \rightarrow \infty$, $\arctan(R+1) \rightarrow \frac{\pi}{2}$, so value $= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$

A1

Answer: $\frac{\pi}{4}$

4 marks

17 (4 marks)

Split at 0: $\int_{-\infty}^{\infty} = \int_{-\infty}^0 + \int_0^{\infty}$, each must converge

M1

$$\int_0^{\infty} \frac{1}{1+x^2} dx = \frac{\pi}{2}$$

B1

By symmetry (even function) $\int_{-\infty}^0 = \frac{\pi}{2}$

M1

Both converge, total π

A1

Answer: π

4 marks

18 (6 marks)

Let $I = \int_0^R e^{-x} \sin x \, dx$. By parts: $u = \sin x$, $\frac{dv}{dx} = e^{-x}$: $I = [-e^{-x} \sin x]_0^R + \int_0^R e^{-x} \cos x \, dx$ M1 A1

Again by parts: $\int_0^R e^{-x} \cos x \, dx = [-e^{-x} \cos x]_0^R - I$ M1 A1

So $2I = -e^{-R} \sin R - e^{-R} \cos R + 1$ M1

As $R \rightarrow \infty$, $e^{-R}(\sin R + \cos R) \rightarrow 0$, so $I \rightarrow \frac{1}{2}$ A1

Answer: $\frac{1}{2}$

6 marks

19 (5 marks)

$u = \ln x$, $\frac{dv}{dx} = x^{-1/2}$: $\int_a^1 = [2\sqrt{x} \ln x]_a^1 - \int_a^1 \frac{2}{\sqrt{x}} \, dx$ M1 A1

$= -2\sqrt{a} \ln a - [4\sqrt{x}]_a^1 = -2\sqrt{a} \ln a - 4 + 4\sqrt{a}$ M1 A1

As $a \rightarrow 0$, $\sqrt{a} \ln a \rightarrow 0$ and $\sqrt{a} \rightarrow 0$, so the value is -4 A1

Answer: -4

5 marks

20 (5 marks)

$u = \ln x$, $\frac{dv}{dx} = x^{-2}$: $\int_1^R = [-\frac{\ln x}{x}]_1^R + \int_1^R \frac{1}{x^2} \, dx$ M1 A1

$= -\frac{\ln R}{R} + [-\frac{1}{x}]_1^R = -\frac{\ln R}{R} + 1 - \frac{1}{R}$ M1 A1

As $R \rightarrow \infty$, $\frac{\ln R}{R} \rightarrow 0$ and $\frac{1}{R} \rightarrow 0$, so the value is 1 A1

Answer: 1

5 marks

21 (6 marks)

$dx = \sec^2 \theta \, d\theta$; $x = 0$ gives $\theta = 0$ and $x \rightarrow \infty$ gives $\theta \rightarrow \frac{\pi}{2}$ B1

$1 + x^2 = \sec^2 \theta$, so the integrand becomes $\frac{\sec^2 \theta}{\sec^4 \theta} = \cos^2 \theta$ M1 A1

$\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$ M1

$\int_0^b = \frac{1}{2}[\theta + \frac{1}{2} \sin 2\theta]_0^b$ A1

As $b \rightarrow \frac{\pi}{2}$, $\sin 2b \rightarrow 0$, so value $= \frac{1}{2} \times \frac{\pi}{2} = \frac{\pi}{4}$ A1

Answer: $\frac{\pi}{4}$

6 marks

22 (6 marks)

(a) $\int_0^R \frac{x}{1+x^2} \, dx = \frac{1}{2} \ln(1+R^2)$ M1 A1

As $R \rightarrow \infty$, $\ln(1+R^2) \rightarrow \infty$, so the integral diverges A1

Answer: Diverges

3 marks

(b) An integral over $(-\infty, \infty)$ is defined as $\int_{-\infty}^0 + \int_0^{\infty}$ with each part converging separately B1

By (a) $\int_0^{\infty}$ diverges (and similarly $\int_{-\infty}^0$ diverges to $-\infty$) B1

Taking symmetric limits $[-R, R]$ gives 0 only as a limit of a particular form; the integral itself does not exist E1

Answer: The integral does not exist

3 marks

23 (6 marks)

$$\int_0^R \left(\frac{3}{1+3x} - \frac{x}{1+x^2} \right) dx = [\ln(1+3x) - \frac{1}{2} \ln(1+x^2)]_0^R$$

M1 A1 A1

$$= \ln\left(\frac{1+3R}{\sqrt{1+R^2}}\right) - 0$$

M1

$$\frac{1+3R}{\sqrt{1+R^2}} = \frac{\frac{1}{R} + 3}{\sqrt{\frac{1}{R^2} + 1}} \rightarrow 3 \text{ as } R \rightarrow \infty$$

M1

So the value is $\ln 3$

A1

Answer: $\ln 3$

6 marks

24 (6 marks)

$$u = (\ln x)^2, \frac{dv}{dx} = 1: \int_a^1 (\ln x)^2 dx = [x(\ln x)^2]_a^1 - 2 \int_a^1 \ln x dx$$

M1 A1

$$= -a(\ln a)^2 - 2[x \ln x - x]_a^1$$

M1 A1

$$= -a(\ln a)^2 - 2(-1 - a \ln a + a)$$

A1

As $a \rightarrow 0$, $a(\ln a)^2 \rightarrow 0$, $a \ln a \rightarrow 0$, $a \rightarrow 0$, so the value is 2

A1

Answer: 2

6 marks

25 (7 marks)

(a) The integrand is positive, so any value of the integral must be positive

E1

The integrand is undefined at $x = 0$, inside the interval, so the integral is improper and the working is invalid

E1

Answer: Integrand undefined at 0

2 marks

$$(b) \int_b^1 \frac{1}{x^2} dx = \frac{1}{b} - 1 \rightarrow \infty \text{ as } b \rightarrow 0 \text{ from above}$$

M1

So the integral over $[-1, 1]$ diverges

A1

Answer: Diverges

2 marks

$$(c) \text{ By symmetry the integral is } 2 \int_0^1 x^{-2/3} dx$$

B1

$$\int_a^1 x^{-2/3} dx = 3(1 - a^{1/3}) \rightarrow 3 \text{ as } a \rightarrow 0$$

M1

Total = 6

A1

Answer: 6

3 marks

26 (6 marks)

$$(a) \text{ By parts: } \int_0^R x^n e^{-x} dx = [-x^n e^{-x}]_0^R + n \int_0^R x^{n-1} e^{-x} dx$$

M1 A1

 $R^n e^{-R} \rightarrow 0 \text{ as } R \rightarrow \infty$ (exponential growth beats any power)

B1

So $I_n = n I_{n-1}$

A1

Answer: $I_n = n I_{n-1}$

4 marks

$$(b) I_0 = \int_0^\infty e^{-x} dx = 1$$

B1

$$I_4 = 4 \times 3 \times 2 \times 1 \times I_0 = 24$$

A1

Answer: 24

2 marks

27 (7 marks)

$x = u^2$, $dx = 2u \, du$; the integral is improper at both ends

B1

$$\frac{2u \, du}{(u^2 + 1)u} = \frac{2}{u^2 + 1} \, du$$

M1 A1

Limits $u = 0$ and $u \rightarrow \infty$

B1

$$\int_0^\infty \frac{2}{1+u^2} \, du = 2[\arctan u]_0^\infty$$

M1

$\arctan R \rightarrow \frac{\pi}{2}$ as $R \rightarrow \infty$, so the value is $2 \times \frac{\pi}{2}$

M1

$= \pi$

A1

Answer: π

7 marks

28 (10 marks)

(a) Substitute $x = \frac{\pi}{2} - u$: $\sin x = \cos u$, limits swap, $dx = -du$

M1

$$I = \int_0^{\pi/2} \ln(\cos u) \, du = J$$

A1

Answer: $I = J$

2 marks

(b) $I + J = \int_0^{\pi/2} \ln(\sin x \cos x) \, dx$

M1

$$\sin x \cos x = \frac{1}{2} \sin 2x$$

M1

$$\ln\left(\frac{1}{2} \sin 2x\right) = \ln(\sin 2x) - \ln 2, \text{ and } \int_0^{\pi/2} \ln 2 \, dx = \frac{\pi}{2} \ln 2$$

A1

$$\text{Answer: } I + J = \int_0^{\pi/2} \ln(\sin 2x) \, dx - \frac{\pi}{2} \ln 2$$

3 marks

(c) Let $u = 2x$: $\int_0^{\pi/2} \ln(\sin 2x) \, dx = \frac{1}{2} \int_0^\pi \ln(\sin u) \, du$

M1

$$\sin(\pi - u) = \sin u, \text{ so } \int_0^\pi \ln(\sin u) \, du = 2 \int_0^{\pi/2} \ln(\sin u) \, du = 2I$$

M1 A1

So the integral equals I , and $2I = I - \frac{\pi}{2} \ln 2$ from (a) and (b)

M1

$$I = -\frac{\pi}{2} \ln 2$$

A1

Answer: $-\frac{\pi}{2} \ln 2$

5 marks

29 (8 marks)

(a) Let $I_R = \int_0^R e^{-px} \sin(qx) \, dx$. By parts with $u = \sin(qx)$: $I_R = \left[-\frac{1}{p} e^{-px} \sin(qx)\right]_0^R + \frac{q}{p} \int_0^R e^{-px} \cos(qx) \, dx$

M1 A1

By parts again: $\int_0^R e^{-px} \cos(qx) \, dx = \left[-\frac{1}{p} e^{-px} \cos(qx)\right]_0^R - \frac{q}{p} I_R$

M1 A1

$$I_R = -\frac{1}{p} e^{-pR} \sin(qR) + \frac{q}{p} \left(-\frac{1}{p} e^{-pR} \cos(qR) + \frac{1}{p}\right) - \frac{q^2}{p^2} I_R$$

M1

As $R \rightarrow \infty$ the terms in $e^{-pR} \rightarrow 0$, giving $I(1 + \frac{q^2}{p^2}) = \frac{q}{p^2}$

M1

$$I = \frac{q}{p^2 + q^2}$$

A1

Answer: $I = \frac{q}{p^2 + q^2}$

7 marks

(b) $p = 2$, $q = 3$: $\frac{3}{4+9} = \frac{3}{13}$

B1

Answer: $\frac{3}{13}$

1 mark

30 (9 marks)

$$u = \arctan x, \frac{d u}{d x} = x^{-2}: \int_1^R = \left[-\frac{\arctan x}{x}\right]_1^R + \int_1^R \frac{1}{x(1+x^2)} dx$$

M1 A1

$$-\frac{\arctan R}{R} + \arctan 1 = \frac{\pi}{4} - \frac{\arctan R}{R}; \frac{\arctan R}{R} \rightarrow 0 \text{ as } R \rightarrow \infty$$

M1 A1

$$\frac{1}{x(1+x^2)} \equiv \frac{1}{x} - \frac{x}{1+x^2}$$

M1 A1

$$\int_1^R = [\ln x - \frac{1}{2} \ln(1+x^2)]_1^R = \ln\left(\frac{R}{\sqrt{1+R^2}}\right) + \frac{1}{2} \ln 2$$

M1 A1

$$\ln\left(\frac{R}{\sqrt{1+R^2}}\right) \rightarrow \ln 1 = 0, \text{ so total} = \frac{\pi}{4} + \frac{1}{2} \ln 2$$

A1

$$\text{Answer: } \frac{\pi}{4} + \frac{1}{2} \ln 2$$

9 marks

Volumes of Revolution**1 (3 marks)**

$$V = \pi \int_0^2 y^2 dx = \pi \int_0^2 x^4 dx$$

M1 A1

$$= \pi \left[\frac{x^5}{5}\right]_0^2 = \frac{32\pi}{5}$$

A1

$$\text{Answer: } \frac{32\pi}{5}$$

3 marks

2 (3 marks)

$$V = \pi \int_0^4 x dx$$

M1 A1

$$= \pi \left[\frac{x^2}{2}\right]_0^4 = 8\pi$$

A1

$$\text{Answer: } 8\pi$$

3 marks

3 (3 marks)

$$V = \pi \int_0^1 e^{2x} dx$$

M1 A1

$$= \pi \left[\frac{1}{2} e^{2x}\right]_0^1 = \frac{\pi}{2}(e^2 - 1)$$

A1

$$\text{Answer: } \frac{\pi}{2}(e^2 - 1)$$

3 marks

4 (3 marks)

The solid is a cylinder of radius 3 and height 4

B1

$$V = \pi \int_0^4 3^2 dx = \pi[9x]_0^4$$

M1

$$= 36\pi$$

A1

$$\text{Answer: Cylinder, } 36\pi$$

3 marks

5 (4 marks)

$$V = \pi \int_0^\pi \sin^2 x dx = \frac{\pi}{2} \int_0^\pi (1 - \cos 2x) dx$$

M1 A1

$$= \frac{\pi}{2} \left[x - \frac{1}{2} \sin 2x\right]_0^\pi$$

M1

$$= \frac{\pi^2}{2}$$

A1

$$\text{Answer: } \frac{\pi^2}{2}$$

4 marks

6 (3 marks)

$$V = \pi \int_0^4 x^2 \, dy \text{ with } x^2 = y$$

M1

$$= \pi \int_0^4 y \, dy = \pi \left[\frac{y^2}{2} \right]_0^4$$

A1

$$= 8\pi$$

A1

Answer: 8π

3 marks

7 (3 marks)

$$V = \pi \int_1^3 \frac{1}{x^2} \, dx$$

M1 A1

$$= \pi \left[-\frac{1}{x} \right]_1^3 = \pi \left(1 - \frac{1}{3} \right) = \frac{2\pi}{3}$$

A1

Answer: Volume = $\frac{2\pi}{3}$

3 marks

8 (4 marks)At position x the side of the square is the chord $2\sqrt{9 - x^2}$

B1

Area of cross-section = $4(9 - x^2)$

M1

$$V = \int_{-3}^3 4(9 - x^2) \, dx = 4 \left[9x - \frac{x^3}{3} \right]_{-3}^3$$

M1

$$= 4 \times 36 = 144$$

A1

Answer: 144

4 marks

9 (4 marks) $y = \ln x$ gives $x = e^y$, so $x^2 = e^{2y}$

M1

$$V = \pi \int_0^2 e^{2y} \, dy$$

M1 A1

$$= \frac{\pi}{2} (e^4 - 1)$$

A1

Answer: $\frac{\pi}{2}(e^4 - 1)$

4 marks

10 (4 marks)Outer radius x , inner radius x^2 : $V = \pi \int_0^1 (x^2 - x^4) \, dx$

M1 A1

$$= \pi \left[\frac{x^3}{3} - \frac{x^5}{5} \right]_0^1$$

M1

$$= \frac{2\pi}{15}$$

A1

Answer: $\frac{2\pi}{15}$

4 marks

11 (4 marks)Curve meets the axis at $x = \pm 2$

B1

$$V = \pi \int_{-2}^2 (4 - x^2)^2 \, dx = \pi \int_{-2}^2 (16 - 8x^2 + x^4) \, dx$$

M1

$$= 2\pi \left[16x - \frac{8x^3}{3} + \frac{x^5}{5} \right]_0^2$$

A1

$$= \frac{512\pi}{15}$$

A1

Answer: $\frac{512\pi}{15}$

4 marks

12 (4 marks)

$$V = \pi \int_0^1 \cosh^2 x \, dx \quad \text{M1}$$

$$\cosh^2 x = \frac{1}{2}(\cosh 2x + 1) \quad \text{M1}$$

$$V = \frac{\pi}{2} \left[\frac{1}{2} \sinh 2x + x \right]_0^1 \quad \text{A1}$$

$$= \frac{\pi}{2} \left(\frac{1}{2} \sinh 2 + 1 \right) = \frac{\pi}{4} (2 + \sinh 2) \quad \text{A1}$$

Answer: Volume = $\frac{\pi}{4}(2 + \sinh 2)$ 4 marks

13 (4 marks)

$$V = \pi \int_0^{\pi/4} \tan^2 x \, dx \quad \text{M1}$$

$$\tan^2 x = \sec^2 x - 1 \quad \text{M1}$$

$$V = \pi [\tan x - x]_0^{\pi/4} \quad \text{A1}$$

$$= \pi \left(1 - \frac{\pi}{4} \right) \quad \text{A1}$$

Answer: $\pi(1 - \frac{\pi}{4})$ 4 marks

14 (5 marks)

$$V = \pi \int_0^1 x^2 e^{2x} \, dx \quad \text{M1}$$

By parts: $\int x^2 e^{2x} \, dx = \frac{1}{2} x^2 e^{2x} - \int x e^{2x} \, dx$ M1 A1

$$\int x e^{2x} \, dx = \frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} \quad \text{A1}$$

$$V = \pi \left[e^{2x} \left(\frac{x^2}{2} - \frac{x}{2} + \frac{1}{4} \right) \right]_0^1 = \pi \left(\frac{e^2}{4} - \frac{1}{4} \right) = \frac{\pi}{4} (e^2 - 1) \quad \text{A1}$$

Answer: Volume = $\frac{\pi}{4}(e^2 - 1)$ 5 marks

15 (5 marks)

$$V = \pi \int_0^1 \frac{1}{(1+x^2)^2} \, dx \quad \text{M1}$$

Let $x = \tan \theta$: $dx = \sec^2 \theta \, d\theta$, limits 0 to $\frac{\pi}{4}$ B1

Integrand becomes $\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$ M1

$$V = \frac{\pi}{2} \left[\theta + \frac{1}{2} \sin 2\theta \right]_0^{\pi/4} \quad \text{A1}$$

$$= \frac{\pi}{2} \left(\frac{\pi}{4} + \frac{1}{2} \right) = \frac{\pi(\pi + 2)}{8} \quad \text{A1}$$

Answer: $\frac{\pi(\pi + 2)}{8}$ 5 marks

16 (5 marks)

$$V = \pi \int_1^e \ln x \, dx \quad \text{M1}$$

$$= \pi [x \ln x - x]_1^e \quad \text{M1 A1}$$

$$= \pi ((e - e) - (0 - 1)) \quad \text{A1}$$

$$= \pi \quad \text{A1}$$

Answer: π 5 marks

17 (4 marks)

$$x^2 = 3 - y$$

B1

$$V = \pi \int_0^3 (3 - y) \, dy$$

M1

$$= \pi \left[3y - \frac{y^2}{2} \right]_0^3$$

A1

$$= \frac{9\pi}{2}$$

A1

Answer: $\frac{9\pi}{2}$

4 marks

18 (4 marks)

For $0 \leq x \leq \frac{\pi}{4}$, $\cos x \geq \sin x$, so $V = \pi \int_0^{\pi/4} (\cos^2 x - \sin^2 x) \, dx$

M1 A1

$$\cos^2 x - \sin^2 x = \cos 2x$$

M1

$$V = \pi \left[\frac{1}{2} \sin 2x \right]_0^{\pi/4} = \frac{\pi}{2}$$

A1

Answer: Volume = $\frac{\pi}{2}$

4 marks

19 (5 marks)

$y = e^{-x^2}$ gives $x^2 = -\ln y$, and the solid extends from $y = 0$ to $y = 1$

B1

$V = \pi \int_0^1 (-\ln y) \, dy$, which is improper at $y = 0$

M1

$$\int_a^1 (-\ln y) \, dy = [y - y \ln y]_a^1 = 1 - a + a \ln a$$

M1 A1

As $a \rightarrow 0$, $a \ln a \rightarrow 0$, so $V = \pi$

A1

Answer: Volume = π

5 marks

20 (4 marks)

$$y^2 = 25 - x^2$$

B1

$$V = \pi \int_2^5 (25 - x^2) \, dx$$

M1

$$= \pi \left[25x - \frac{x^3}{3} \right]_2^5$$

A1

$$= \pi \left(\frac{250}{3} - \frac{142}{3} \right) = 36\pi$$

A1

Answer: 36π

4 marks

21 (6 marks)

Outer radius e^x , inner radius e^{-x}

B1

$$V = \pi \int_0^1 (e^{2x} - e^{-2x}) \, dx$$

M1 A1

$$= \pi \left[\frac{1}{2} e^{2x} + \frac{1}{2} e^{-2x} \right]_0^1$$

M1 A1

$$= \pi \left(\frac{1}{2} e^2 + \frac{1}{2} e^{-2} - 1 \right) = \pi (\cosh 2 - 1)$$

A1

Answer: Volume = $\pi(\cosh 2 - 1)$

6 marks

22 (6 marks)

$x = y^2$; the region lies between the curve (inner) and the line $x = 4$ (outer), for $0 \leq y \leq 2$ (the curve meets $x = 4$ at $y = 2$)

B1 B1

$$V = \pi \int_0^2 (4^2 - (y^2)^2) \, dy$$

M1 A1

$$= \pi \left[16y - \frac{y^5}{5} \right]_0^2$$

A1

$$= \pi \left(32 - \frac{32}{5} \right) = \frac{128\pi}{5}$$

A1

Answer: $\frac{128\pi}{5}$

6 marks

23 (6 marks)

$$(a) V = \pi \int_{-r}^r (r^2 - x^2) dx \quad \text{M1}$$

$$= 2\pi \left[r^2 x - \frac{x^3}{3} \right]_0^r \quad \text{A1}$$

$$= 2\pi \left(r^3 - \frac{r^3}{3} \right) = \frac{4}{3} \pi r^3 \quad \text{A1}$$

Answer: Volume = $\frac{4}{3} \pi r^3$ 3 marks

$$(b) V = \pi \int_0^h \frac{r^2 x^2}{h^2} dx \quad \text{M1}$$

$$= \pi \frac{r^2}{h^2} \left[\frac{x^3}{3} \right]_0^h \quad \text{A1}$$

$$= \frac{1}{3} \pi r^2 h \quad \text{A1}$$

Answer: Volume = $\frac{1}{3} \pi r^2 h$ 3 marks

24 (6 marks)

$$y = -x^2 + 6x - 8 = 1 - (x - 3)^2, \text{ so } x = 3 \pm \sqrt{1 - y} \text{ for } 0 \leq y \leq 1 \quad \text{B1 M1}$$

$$\text{Outer radius } 3 + \sqrt{1 - y}, \text{ inner radius } 3 - \sqrt{1 - y} \quad \text{B1}$$

$$V = \pi \int_0^1 [(3 + \sqrt{1 - y})^2 - (3 - \sqrt{1 - y})^2] dy = \pi \int_0^1 12\sqrt{1 - y} dy \quad \text{M1 A1}$$

$$= 12\pi \left[-\frac{2}{3} (1 - y)^{3/2} \right]_0^1 = 8\pi \quad \text{A1}$$

Answer: 8π 6 marks

25 (5 marks)

(a) The radius y has not been squared: the integrand should be $y^2 = (x^2 + 1)^2$ E1

Answer: y not squared 1 mark

$$(b) V = \pi \int_0^1 (x^2 + 1)^2 dx = \pi \int_0^1 (x^4 + 2x^2 + 1) dx \quad \text{M1 A1}$$

$$= \pi \left[\frac{x^5}{5} + \frac{2x^3}{3} + x \right]_0^1 \quad \text{M1}$$

$$= \pi \left(\frac{1}{5} + \frac{2}{3} + 1 \right) = \frac{28\pi}{15} \quad \text{A1}$$

Answer: $\frac{28\pi}{15}$ 4 marks

26 (6 marks)

(a) $\text{sech}^2 x$ B1

Answer: $\text{sech}^2 x$ 1 mark

$$(b) V = \pi \int_0^{\ln 3} \text{sech}^2 x dx \quad \text{M1}$$

$$= \pi [\tanh x]_0^{\ln 3} \quad \text{A1}$$

$$\tanh(\ln 3) = \frac{3 - \frac{1}{3}}{3 + \frac{1}{3}} = \frac{\frac{8}{3}}{\frac{10}{3}} \quad \text{M1}$$

$$= \frac{8}{10} = \frac{4}{5} \quad \text{A1}$$

$$V = \frac{4\pi}{5} \quad \text{A1}$$

Answer: Volume = $\frac{4\pi}{5}$ 5 marks

27 (8 marks)

Upper semicircle: $y = 3 + \sqrt{1 - x^2}$; lower semicircle: $y = 3 - \sqrt{1 - x^2}$

B1 B1

$$V = \pi \int_{-1}^1 [(3 + \sqrt{1 - x^2})^2 - (3 - \sqrt{1 - x^2})^2] dx$$

M1 A1

Difference of squares: $(3 + s)^2 - (3 - s)^2 = 12s$

M1

$$V = 12\pi \int_{-1}^1 \sqrt{1 - x^2} dx$$

A1

The integral is the area of a semicircle of radius 1 (or use $x = \sin \theta$): $\frac{\pi}{2}$

M1

$$V = 12\pi \times \frac{\pi}{2} = 6\pi^2$$

A1

Answer: Volume = $6\pi^2$

8 marks

28 (7 marks)

Curve and line meet where $x^2 = 4$, so $x = \pm 2$

B1

Distances from the axis of rotation: outer radius $4 + 1 = 5$, inner radius $x^2 + 1$

M1 A1

$$V = \pi \int_{-2}^2 [25 - (x^2 + 1)^2] dx$$

M1

$$= \pi \int_{-2}^2 (24 - 2x^2 - x^4) dx$$

A1

$$= 2\pi [24x - \frac{2x^3}{3} - \frac{x^5}{5}]_0^2$$

M1

$$= 2\pi (48 - \frac{16}{3} - \frac{32}{5}) = \frac{1088\pi}{15}$$

A1

Answer: $\frac{1088\pi}{15}$

7 marks

29 (10 marks)

$$V = \pi \int_0^\pi x^2 \sin^2 x dx$$

M1

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

M1

$$V = \frac{\pi}{2} \int_0^\pi x^2 dx - \frac{\pi}{2} \int_0^\pi x^2 \cos 2x dx$$

A1

$$\int_0^\pi x^2 dx = \frac{\pi^3}{3}$$

B1

$$\text{By parts: } \int x^2 \cos 2x dx = \frac{1}{2} x^2 \sin 2x - \int x \sin 2x dx$$

M1 A1

$$\int x \sin 2x dx = -\frac{1}{2} x \cos 2x + \frac{1}{2} \int \cos 2x dx = -\frac{1}{2} x \cos 2x + \frac{1}{4} \sin 2x$$

M1 A1

$$\int_0^\pi x^2 \cos 2x dx = [\frac{1}{2} x^2 \sin 2x + \frac{1}{2} x \cos 2x - \frac{1}{4} \sin 2x]_0^\pi = \frac{\pi}{2}$$

M1

$$V = \frac{\pi}{2} (\frac{\pi^3}{3} - \frac{\pi}{2}) = \frac{\pi^2}{12} (2\pi^2 - 3)$$

A1

Answer: Volume = $\frac{\pi^2}{12} (2\pi^2 - 3)$

10 marks

30 (9 marks)

$$(a) V = \pi \int_0^h x^2 dy = \pi \int_0^h 5y dy$$

M1 A1

$$= \frac{5\pi}{2} h^2$$

A1

Capacity: $h = 10$ gives $250\pi = 785 \text{ cm}^3$ (nearest cm^3)

A1

Answer: $250\pi \approx 785 \text{ cm}^3$

4 marks

$$(b) \frac{dV}{dh} = 5\pi h$$

M1 A1

$$\frac{dh}{dt} = \frac{dV}{dt} \div \frac{dV}{dh}$$

M1

$$\text{When } h = 4: \frac{dh}{dt} = \frac{20}{20\pi}$$

A1

$$= \frac{1}{\pi} = 0.318 \text{ cm s}^{-1}$$

A1

Answer: 0.318 cm per second

5 marks

Arc Length

1 (3 marks)

$$\frac{dy}{dx} = 3, \text{ so } 1 + \left(\frac{dy}{dx}\right)^2 = 10$$

M1

$$s = \int_1^5 \sqrt{10} \, dx = 4\sqrt{10}$$

M1 A1

Answer: $4\sqrt{10}$

3 marks

2 (2 marks)

$$\frac{dy}{dx} = 2x$$

B1

$$s = \int_0^1 \sqrt{1 + 4x^2} \, dx$$

B1

Answer: $\int_0^1 \sqrt{1 + 4x^2} \, dx$

2 marks

3 (2 marks)

$$\frac{dy}{dx} = \sinh x$$

B1

$$1 + \sinh^2 x = \cosh^2 x$$

B1

Answer: $1 + \left(\frac{dy}{dx}\right)^2 = \cosh^2 x$

2 marks

4 (3 marks)

$$\frac{dx}{dt} = 3, \frac{dy}{dt} = 4, \text{ so } \sqrt{9 + 16} = 5$$

M1 A1

$$s = \int_0^2 5 \, dt = 10 \text{ (a straight line segment)}$$

A1

Answer: 10 (straight line)

3 marks

5 (2 marks)

$$\frac{dy}{dx} = \frac{\sec x \tan x}{\sec x} = \tan x$$

B1

$$1 + \tan^2 x = \sec^2 x$$

B1

Answer: $1 + \left(\frac{dy}{dx}\right)^2 = \sec^2 x$

2 marks

6 (3 marks)

$$\frac{dx}{dt} = -2 \sin t, \frac{dy}{dt} = 2 \cos t$$

B1

$$\sqrt{4 \sin^2 t + 4 \cos^2 t} = 2$$

M1

$$s = \int_0^{\pi/2} 2 \, dt = \pi$$

A1

Answer: π

3 marks

7 (4 marks)

$$\frac{dy}{dx} = \frac{3}{2} x^{1/2}, \text{ so } 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{9x}{4} \quad \text{M1}$$

$$s = \int_0^4 \sqrt{1 + \frac{9x}{4}} dx = \left[\frac{4}{9} \times \frac{2}{3} \left(1 + \frac{9x}{4}\right)^{3/2} \right]_0^4 \quad \text{M1 A1}$$

$$= \frac{8}{27} (10^{3/2} - 1) = \frac{8}{27} (10\sqrt{10} - 1) \quad \text{A1}$$

Answer: $\frac{8}{27}(10\sqrt{10} - 1)$ 4 marks

8 (3 marks)

$$1 + \sinh^2 x = \cosh^2 x, \text{ so } s = \int_0^{\ln 3} \cosh x dx \quad \text{M1 A1}$$

$$= \sinh(\ln 3) = \frac{1}{2} \left(3 - \frac{1}{3}\right) = \frac{4}{3} \quad \text{A1}$$

Answer: $\frac{4}{3}$ 3 marks

9 (4 marks)

$$\frac{dy}{dx} = \tan x, \text{ so } 1 + \left(\frac{dy}{dx}\right)^2 = \sec^2 x \quad \text{M1 A1}$$

$$s = \int_0^{\pi/3} \sec x dx = [\ln(\sec x + \tan x)]_0^{\pi/3} \quad \text{M1}$$

$$\sec \frac{\pi}{3} = 2, \tan \frac{\pi}{3} = \sqrt{3}, \text{ and } \ln(1 + 0) = 0, \text{ so } s = \ln(2 + \sqrt{3}) \quad \text{A1}$$

Answer: Length = $\ln(2 + \sqrt{3})$ 4 marks

10 (4 marks)

$$\frac{dy}{dx} = \frac{1}{\tanh(x/2)} \times \frac{1}{2} \operatorname{sech}^2\left(\frac{x}{2}\right) = \frac{1}{2 \sinh(\frac{x}{2}) \cosh(\frac{x}{2})} = \operatorname{cosech} x \quad \text{M1 A1}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \operatorname{cosech}^2 x = \coth^2 x, \text{ so } s = \int_{\ln 2}^{\ln 3} \coth x dx = [\ln(\sinh x)]_{\ln 2}^{\ln 3} \quad \text{M1}$$

$$\sinh(\ln 3) = \frac{4}{3}, \sinh(\ln 2) = \frac{3}{4}, \text{ so } s = \ln\left(\frac{4}{3} \div \frac{3}{4}\right) = \ln\left(\frac{16}{9}\right) \quad \text{A1}$$

Answer: $\ln\left(\frac{16}{9}\right)$ 4 marks

11 (3 marks)

$$\sqrt{25 \sin^2 t + 25 \cos^2 t} = 5, \text{ so } s = \int_0^{\pi/3} 5 dt = \frac{5\pi}{3} \quad \text{M1 A1}$$

$$\text{Check: } s = r\theta = 5 \times \frac{\pi}{3} \quad \text{B1}$$

Answer: $\frac{5\pi}{3}$ 3 marks

12 (4 marks)

$$\frac{dx}{dt} = 6t, \frac{dy}{dt} = 6t^2 \quad \text{B1}$$

$$\sqrt{36t^2 + 36t^4} = 6t\sqrt{1 + t^2} \quad \text{M1}$$

$$s = \int_0^1 6t\sqrt{1 + t^2} dt = [2(1 + t^2)^{3/2}]_0^1 \quad \text{A1}$$

$$= 2(2\sqrt{2} - 1) = 4\sqrt{2} - 2 \quad \text{A1}$$

Answer: Length = $4\sqrt{2} - 2$ 4 marks

13 (3 marks)

$$\frac{dy}{dx} = \cos x$$

B1

$$s = \int_0^{\pi} \sqrt{1 + \cos^2 x} \, dx$$

M1

$$= 3.820 \text{ (3 s.f.)}$$

A1

Answer: 3.820

3 marks

14 (5 marks)

$$\frac{dy}{dx} = x - \frac{1}{4x}$$

B1

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + x^2 - \frac{1}{2} + \frac{1}{16x^2} = \left(x + \frac{1}{4x}\right)^2$$

M1 A1

$$s = \int_1^2 \left(x + \frac{1}{4x}\right) dx = \left[\frac{x^2}{2} + \frac{1}{4} \ln x\right]_1^2$$

M1

$$= \left(2 + \frac{1}{4} \ln 2\right) - \frac{1}{2} = \frac{3}{2} + \frac{1}{4} \ln 2$$

A1

Answer: Length = $\frac{3}{2} + \frac{1}{4} \ln 2$

5 marks

15 (5 marks)

$$\frac{dy}{dx} = \frac{x^2}{2} - \frac{1}{2x^2}$$

B1

$$1 + \left(\frac{dy}{dx}\right)^2 = \left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2$$

M1 A1

$$s = \int_1^3 \left(\frac{x^2}{2} + \frac{1}{2x^2}\right) dx = \left[\frac{x^3}{6} - \frac{1}{2x}\right]_1^3$$

M1

$$= \left(\frac{9}{2} - \frac{1}{6}\right) - \left(\frac{1}{6} - \frac{1}{2}\right) = \frac{14}{3}$$

A1

Answer: $\frac{14}{3}$

5 marks

16 (5 marks)

$$\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = 3t^2$$

B1

$$s = \int_0^2 \sqrt{4t^2 + 9t^4} \, dt = \int_0^2 t\sqrt{4 + 9t^2} \, dt$$

M1 A1

$$= \left[\frac{1}{27}(4 + 9t^2)^{3/2}\right]_0^2$$

M1

$$= \frac{1}{27}(40^{3/2} - 8) = \frac{80\sqrt{10} - 8}{27}$$

A1

Answer: $\frac{80\sqrt{10} - 8}{27}$

5 marks

17 (5 marks)

$$\frac{dx}{dt} = e^t(\cos t - \sin t), \quad \frac{dy}{dt} = e^t(\sin t + \cos t)$$

M1 A1

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = e^{2t}(2 \cos^2 t + 2 \sin^2 t) = 2e^{2t}$$

M1 A1

$$s = \int_0^1 \sqrt{2} e^t \, dt = \sqrt{2}(e - 1)$$

A1

Answer: Length = $\sqrt{2}(e - 1)$

5 marks

18 (5 marks)

$$\frac{dy}{dx} = -\frac{2x}{1-x^2}, 1 + \left(\frac{dy}{dx}\right)^2 = \frac{(1-x^2)^2 + 4x^2}{(1-x^2)^2} = \left(\frac{1+x^2}{1-x^2}\right)^2 \quad \text{M1 A1}$$

$$s = \int_0^{1/2} \frac{1+x^2}{1-x^2} dx = \int_0^{1/2} \left(-1 + \frac{2}{1-x^2}\right) dx \quad \text{M1}$$

$$= [-x + \ln\left(\frac{1+x}{1-x}\right)]_0^{1/2} \quad \text{A1}$$

$$= -\frac{1}{2} + \ln 3 \quad \text{A1}$$

Answer: $\ln 3 - \frac{1}{2}$ 5 marks

19 (5 marks)

$$\frac{dy}{dx} = \cot x, \text{ so } 1 + \cot^2 x = \operatorname{cosec}^2 x \quad \text{M1 A1}$$

$$s = \int_{\pi/3}^{\pi/2} \operatorname{cosec} x dx = [\ln(\tan \frac{x}{2})]_{\pi/3}^{\pi/2} \quad \text{M1 A1}$$

$$\tan \frac{\pi}{4} = 1 \text{ and } \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}, \text{ so } s = 0 - \ln\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{2} \ln 3 \quad \text{A1}$$

Answer: $\frac{1}{2} \ln 3$ 5 marks

20 (5 marks)

$$\frac{dy}{dx} = x(x^2 + 2)^{1/2} \quad \text{B1}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + x^2(x^2 + 2) = x^4 + 2x^2 + 1 = (x^2 + 1)^2 \quad \text{M1 A1}$$

$$s = \int_0^3 (x^2 + 1) dx = \left[\frac{x^3}{3} + x\right]_0^3 \quad \text{M1}$$

$$= 9 + 3 = 12 \quad \text{A1}$$

Answer: Length = 12 5 marks

21 (6 marks)

$$\frac{dx}{dt} = 1 - \cos t, \frac{dy}{dt} = \sin t \quad \text{B1}$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 1 - 2\cos t + \cos^2 t + \sin^2 t = 2 - 2\cos t \quad \text{M1 A1}$$

$$2 - 2\cos t = 4\sin^2\left(\frac{t}{2}\right), \text{ so the speed is } 2\sin\left(\frac{t}{2}\right) \text{ (non-negative for } 0 \leq t \leq 2\pi) \quad \text{M1 A1}$$

$$s = \int_0^{2\pi} 2\sin\left(\frac{t}{2}\right) dt = [-4\cos\left(\frac{t}{2}\right)]_0^{2\pi} = 4 + 4 = 8 \quad \text{A1}$$

Answer: Length = 8 6 marks

22 (6 marks)

$$\frac{dx}{dt} = -3\cos^2 t \sin t, \frac{dy}{dt} = 3\sin^2 t \cos t \quad \text{B1 B1}$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 9\sin^2 t \cos^2 t (\cos^2 t + \sin^2 t) = 9\sin^2 t \cos^2 t \quad \text{M1}$$

$$\text{Speed} = 3\sin t \cos t (\geq 0 \text{ on the interval}) \quad \text{A1}$$

$$s = \int_0^{\pi/2} 3\sin t \cos t dt = \left[\frac{3}{2}\sin^2 t\right]_0^{\pi/2} = \frac{3}{2} \quad \text{A1}$$

$$\text{By symmetry the perimeter is } 4 \times \frac{3}{2} = 6 \quad \text{B1}$$

Answer: $\frac{3}{2}; 6$ 6 marks

23 (7 marks)

$$\frac{dr}{d\theta} = -2 \sin \theta$$

B1

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = 4(1 + \cos \theta)^2 + 4 \sin^2 \theta = 8 + 8 \cos \theta$$

M1 A1

$$1 + \cos \theta = 2 \cos^2\left(\frac{\theta}{2}\right), \text{ so the expression is } 16 \cos^2\left(\frac{\theta}{2}\right)$$

M1

$$\sqrt{16 \cos^2\left(\frac{\theta}{2}\right)} = 4|\cos\left(\frac{\theta}{2}\right)|, \text{ and } \cos\left(\frac{\theta}{2}\right) \geq 0 \text{ for } 0 \leq \theta \leq \pi$$

A1

$$\text{By symmetry } s = 2 \int_0^\pi 4 \cos\left(\frac{\theta}{2}\right) d\theta$$

M1

$$= 2[8 \sin\left(\frac{\theta}{2}\right)]_0^\pi = 16$$

A1

Answer: Perimeter = 16

7 marks

24 (6 marks)**(a)** The derivative must be squared: it should be $1 + (2x)^2 = 1 + 4x^2$

E1

Answer: Derivative not squared

1 mark

$$\text{(b)} s = \int_0^1 \sqrt{1 + 4x^2} dx; 2 dx = \cosh u du, \text{ limits } u = 0 \text{ to } \operatorname{arsinh} 2$$

M1 A1

$$s = \frac{1}{2} \int_0^{\operatorname{arsinh} 2} \cosh^2 u du = \frac{1}{4} \int (1 + \cosh 2u) du$$

M1

$$= \frac{1}{4} \left[u + \frac{1}{2} \sinh 2u \right] = \frac{1}{4} [u + \sinh u \cosh u]$$

A1

$$\sinh u = 2, \cosh u = \sqrt{5}, u = \ln(2 + \sqrt{5}): s = \frac{1}{4} (\ln(2 + \sqrt{5}) + 2\sqrt{5})$$

A1

$$\text{Answer: } \frac{\sqrt{5}}{2} + \frac{1}{4} \ln(2 + \sqrt{5})$$

5 marks

25 (6 marks)

$$\frac{dx}{dt} = t, \frac{dy}{dt} = 1, \text{ so } s = \int_0^{\sqrt{3}} \sqrt{t^2 + 1} dt$$

M1 A1

$$\text{Let } t = \sinh u: dt = \cosh u du, \text{ and } \sqrt{t^2 + 1} = \cosh u$$

M1

$$s = \int \cosh^2 u du = \frac{1}{2} [u + \sinh u \cosh u]$$

A1

$$\text{Upper limit: } \sinh u = \sqrt{3}, \cosh u = 2, u = \ln(\sqrt{3} + 2)$$

B1

$$s = \frac{1}{2} (\ln(2 + \sqrt{3}) + 2\sqrt{3}) = \sqrt{3} + \frac{1}{2} \ln(2 + \sqrt{3})$$

A1

$$\text{Answer: Length} = \sqrt{3} + \frac{1}{2} \ln(2 + \sqrt{3})$$

6 marks

26 (6 marks)

$$\text{(a)} y(30) = 20 \cosh 1.5 - 20$$

M1

$$= 27 \text{ m}$$

A1

Answer: 27 m

2 marks

$$\text{(b)} \frac{dy}{dx} = \sinh\left(\frac{x}{20}\right), \text{ so } 1 + \left(\frac{dy}{dx}\right)^2 = \cosh^2\left(\frac{x}{20}\right)$$

M1 A1

$$s = \int_{-30}^{30} \cosh\left(\frac{x}{20}\right) dx = [20 \sinh\left(\frac{x}{20}\right)]_{-30}^{30}$$

M1

$$= 40 \sinh 1.5 = 85.2 \text{ m}$$

A1

Answer: 85.2 m

4 marks

27 (9 marks)

$$\frac{dy}{dx} = \frac{1}{x}, \text{ so } s = \int_1^{\sqrt{3}} \sqrt{1 + \frac{1}{x^2}} dx = \int_1^{\sqrt{3}} \frac{\sqrt{x^2 + 1}}{x} dx$$

M1 A1

$$\text{Let } x = \tan \theta: dx = \sec^2 \theta d\theta, \text{ limits } \frac{\pi}{4} \text{ to } \frac{\pi}{3}$$

M1 A1

$$\frac{\sqrt{x^2 + 1}}{x} dx = \frac{\sec \theta}{\tan \theta} \sec^2 \theta d\theta = \frac{\sec^3 \theta}{\tan \theta} d\theta$$

M1

$$\frac{\sec^3 \theta}{\tan \theta} = \frac{1}{\sin \theta \cos^2 \theta} = \operatorname{cosec} \theta + \sec \theta \tan \theta$$

M1 A1

$$s = [\ln \tan(\frac{\theta}{2}) + \sec \theta]_{\pi/4}^{\pi/3}$$

M1

$$= (\ln \tan \frac{\pi}{6} + 2) - (\ln \tan \frac{\pi}{8} + \sqrt{2}) \text{ and } \tan \frac{\pi}{8} = \sqrt{2} - 1, \text{ giving } 2 - \sqrt{2} - \frac{1}{2} \ln 3 + \ln(\sqrt{2} + 1)$$

A1

$$\text{Answer: Length} = 2 - \sqrt{2} + \ln(1 + \sqrt{2}) - \frac{1}{2} \ln 3$$

9 marks

28 (8 marks)

$$\frac{dx}{dt} = 1 - \operatorname{sech}^2 t = \tanh^2 t$$

M1 A1

$$\frac{dy}{dt} = -\operatorname{sech} t \tanh t$$

B1

$$(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 = \tanh^4 t + \operatorname{sech}^2 t \tanh^2 t = \tanh^2 t (\tanh^2 t + \operatorname{sech}^2 t) = \tanh^2 t$$

M1 A1

$$\text{Speed} = \tanh t \text{ (positive for } t > 0)$$

A1

$$s = \int_0^{\ln 3} \tanh t dt = [\ln \cosh t]_0^{\ln 3}$$

M1

$$\cosh(\ln 3) = \frac{1}{2}(3 + \frac{1}{3}) = \frac{5}{3}, \text{ so } s = \ln(\frac{5}{3})$$

A1

$$\text{Answer: Length} = \ln(\frac{5}{3})$$

8 marks

29 (9 marks)

$$(a) \frac{dx}{dt} = -\sin t + \sin t + t \cos t = t \cos t$$

M1 A1

$$\frac{dy}{dt} = \cos t - \cos t + t \sin t = t \sin t$$

A1

$$\text{Answer: } \frac{dy}{dt} = t \sin t$$

3 marks

$$(b) (\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 = t^2$$

M1

$$\text{Speed} = t \text{ for } t \geq 0$$

A1

$$s = \int_0^{2\pi} t dt = [\frac{t^2}{2}]_0^{2\pi}$$

M1

$$= 2\pi^2$$

A1

$$\text{Answer: } 2\pi^2$$

4 marks

$$(c) \frac{dy}{dx} = \frac{t \sin t}{t \cos t} = \tan t$$

M1

The gradient is $\tan t$, so the tangent makes angle t with the x axis; the direction of motion turns steadily with t

E1

$$\text{Answer: } \tan t$$

2 marks

30 (9 marks)

$$(a) \frac{dy}{dx} = \sinh(\frac{x}{a}), \text{ so } 1 + (\frac{dy}{dx})^2 = \cosh^2(\frac{x}{a})$$

M1 A1

$$s = \int_0^x \cosh(\frac{u}{a}) du$$

M1

$$= a \sinh(\frac{x}{a})$$

A1

$$\text{Answer: } s = a \sinh(\frac{x}{a})$$

4 marks

$$(b) A = \int_0^x a \cosh\left(\frac{u}{a}\right) du = a^2 \sinh\left(\frac{x}{a}\right)$$

M1

$$= a \times a \sinh\left(\frac{x}{a}\right) = a s$$

A1

Answer: A = a s

2 marks

$$(c) A = a s = 5 \times 12$$

M1

$$= 60$$

A1

Check that $\sinh\left(\frac{x}{5}\right) = \frac{12}{5}$ gives a valid x : $x = 5 \operatorname{arsinh} 2.4 = 8.05$ (exists)

B1

Answer: 60

3 marks

Surface Area of Revolution

1 (3 marks)

$$\frac{dy}{dx} = 2, \text{ so } \sqrt{1 + 4} = \sqrt{5}$$

B1

$$S = 2\pi \int_0^3 2x\sqrt{5} dx = 4\pi\sqrt{5}\left[\frac{x^2}{2}\right]_0^3$$

M1

$$= 18\sqrt{5}\pi$$

A1

Answer: $18\sqrt{5}\pi$

3 marks

2 (3 marks)

$$\frac{dy}{dx} = 0, \text{ so } S = 2\pi \int_0^5 3 dx$$

M1 A1

$= 30\pi$; the curved surface of a cylinder of radius 3 and length 5

A1

Answer: 30π (cylinder)

3 marks

3 (2 marks)

$$\frac{dy}{dx} = 2x$$

B1

$$S = 2\pi \int_0^1 x^2 \sqrt{1 + 4x^2} dx$$

B1

Answer: $2\pi \int_0^1 x^2 \sqrt{1 + 4x^2} dx$

2 marks

4 (3 marks)

$$\sqrt{9 + 16} = 5$$

B1

$$S = 2\pi \int_0^1 4t \times 5 dt = 40\pi\left[\frac{t^2}{2}\right]_0^1$$

M1

$$= 20\pi$$

A1

Answer: 20π

3 marks

5 (2 marks)

$$\frac{dy}{dx} = \sinh x, \text{ so } 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \sinh^2 x = \cosh^2 x$$

B1

$$y \sqrt{\cosh^2 x} = \cosh x \times \cosh x = \cosh^2 x$$

B1

Answer: $y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \cosh^2 x$

2 marks

6 (3 marks)

$$\frac{dy}{dx} = \frac{1}{2}, \text{ so } \sqrt{1 + \frac{1}{4}} = \frac{\sqrt{5}}{2}$$

B1

$$S = 2\pi \int_0^4 x \times \frac{\sqrt{5}}{2} dx = \pi\sqrt{5}\left[\frac{x^2}{2}\right]_0^4$$

M1

$$= 8\sqrt{5}\pi$$

A1

Answer: $8\sqrt{5}\pi$

3 marks

7 (4 marks)

$$\frac{dy}{dx} = -\frac{x}{\sqrt{4-x^2}}, \text{ so } 1 + \left(\frac{dy}{dx}\right)^2 = \frac{4}{4-x^2} \quad \text{M1 A1}$$

$$y\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{4-x^2} \times \frac{2}{\sqrt{4-x^2}} = 2 \quad \text{M1}$$

$$S = 2\pi \int_{-2}^2 2 \, dx = 16\pi \quad \text{A1}$$

Answer: Surface area = 16π

4 marks

8 (4 marks)

$$\frac{dy}{dx} = 3x^2, S = 2\pi \int_1^{10} x^3 \sqrt{1+9x^4} \, dx \quad \text{M1 A1}$$

$$\text{Let } u = 1 + 9x^4; \, du = 36x^3 \, dx \quad \text{M1}$$

$$S = \frac{2\pi}{36} \int_1^{10} \sqrt{u} \, du = \frac{\pi}{18} \times \frac{2}{3} [u^{3/2}]_1^{10} = \frac{\pi}{27} (10\sqrt{10} - 1) \quad \text{A1}$$

$$\text{Answer: Surface area} = \frac{\pi}{27} (10\sqrt{10} - 1) \quad \text{4 marks}$$

9 (4 marks)

$$y\sqrt{1 + \sinh^2 x} = \cosh^2 x, \text{ so } S = 2\pi \int_0^{\ln 2} \cosh^2 x \, dx \quad \text{M1 A1}$$

$$= \pi \int_0^{\ln 2} (1 + \cosh 2x) \, dx = \pi \left[x + \frac{1}{2} \sinh 2x \right]_0^{\ln 2} \quad \text{M1}$$

$$\sinh 2x = 2 \sinh x \cosh x = 2 \times \frac{3}{4} \times \frac{5}{4} = \frac{15}{8}; S = \pi \left(\ln 2 + \frac{15}{16} \right) \quad \text{A1}$$

$$\text{Answer: } \pi \left(\ln 2 + \frac{15}{16} \right) \quad \text{4 marks}$$

10 (4 marks)

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}, \text{ so } y\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{x} \sqrt{1 + \frac{1}{4x}} = \sqrt{x + \frac{1}{4}} \quad \text{M1 A1}$$

$$S = 2\pi \int_0^3 \left(x + \frac{1}{4}\right)^{1/2} \, dx = 2\pi \left[\frac{2}{3} \left(x + \frac{1}{4}\right)^{3/2} \right]_0^3 \quad \text{M1}$$

$$= \frac{4\pi}{3} \left(\left(\frac{13}{4}\right)^{3/2} - \frac{1}{8} \right) = \frac{\pi}{6} (13\sqrt{13} - 1) \quad \text{A1}$$

$$\text{Answer: } \frac{\pi}{6} (13\sqrt{13} - 1) \quad \text{4 marks}$$

11 (4 marks)

$$S = 2\pi \int_0^1 x \sqrt{1 + 4x^2} \, dx \quad \text{M1 A1}$$

$$= 2\pi \left[\frac{1}{12} (1 + 4x^2)^{3/2} \right]_0^1 \quad \text{M1}$$

$$= \frac{\pi}{6} (5\sqrt{5} - 1) \quad \text{A1}$$

$$\text{Answer: } \frac{\pi}{6} (5\sqrt{5} - 1) \quad \text{4 marks}$$

12 (4 marks)

$$\frac{dy}{dx} = \frac{1}{\sqrt{x}}, \text{ so } y\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = 2\sqrt{x} \times \sqrt{\frac{x+1}{x}} = 2\sqrt{x+1} \quad \text{M1 A1}$$

$$S = 4\pi \int_0^3 \sqrt{x+1} \, dx = 4\pi \left[\frac{2}{3} (x+1)^{3/2} \right]_0^3 \quad \text{M1}$$

$$= \frac{8\pi}{3} (8 - 1) = \frac{56\pi}{3} \quad \text{A1}$$

$$\text{Answer: Surface area} = \frac{56\pi}{3} \quad \text{4 marks}$$

13 (3 marks)

$$\sqrt{4 \sin^2 t + 4 \cos^2 t} = 2$$

B1

$$S = 2\pi \int_0^{\pi/2} 2 \sin t \times 2 \, dt = 8\pi [-\cos t]_0^{\pi/2}$$

M1

= 8π ; a hemisphere (of radius 2)

A1

Answer: 8π (hemisphere)

3 marks

14 (5 marks)

$$\frac{dx}{dt} = 2t, \frac{dy}{dt} = 2, \text{ so } \sqrt{4t^2 + 4} = 2\sqrt{t^2 + 1}$$

B1

$$S = 2\pi \int_0^1 2t \times 2\sqrt{t^2 + 1} \, dt = 8\pi \int_0^1 t\sqrt{t^2 + 1} \, dt$$

M1 A1

$$= 8\pi \left[\frac{1}{3}(t^2 + 1)^{3/2} \right]_0^1$$

M1

$$= \frac{8\pi}{3}(2\sqrt{2} - 1)$$

A1

Answer: Surface area = $\frac{8\pi}{3}(2\sqrt{2} - 1)$

5 marks

15 (6 marks)

$$\frac{dy}{dx} = x^2 - \frac{1}{4x^2}$$

B1

$$1 + \left(\frac{dy}{dx}\right)^2 = \left(x^2 + \frac{1}{4x^2}\right)^2$$

M1 A1

$$S = 2\pi \int_1^2 \left(\frac{x^3}{3} + \frac{1}{4x}\right)\left(x^2 + \frac{1}{4x^2}\right) dx = 2\pi \int_1^2 \left(\frac{x^5}{3} + \frac{x}{3} + \frac{1}{16x^3}\right) dx$$

M1

$$= 2\pi \left[\frac{x^6}{18} + \frac{x^2}{6} - \frac{1}{32x^2} \right]_1^2$$

A1

$$= 2\pi \left(\frac{4855}{1152} - \frac{55}{288} \right) = \frac{515\pi}{64}$$

A1

Answer: $\frac{515\pi}{64}$

6 marks

16 (4 marks)

$$\frac{dy}{dx} = \frac{r}{h}, \text{ so } \sqrt{1 + \frac{r^2}{h^2}} = \frac{\sqrt{h^2 + r^2}}{h}$$

M1 A1

$$S = 2\pi \int_0^h \frac{rx}{h} \times \frac{\sqrt{h^2 + r^2}}{h} dx$$

M1

$$= \frac{2\pi r\sqrt{h^2 + r^2}}{h^2} \times \frac{h^2}{2} = \pi r\sqrt{r^2 + h^2}$$

A1

Answer: Surface area = $\pi r\sqrt{r^2 + h^2}$

4 marks

17 (4 marks)

$$y = \sqrt{25 - x^2}, \frac{dy}{dx} = -\frac{x}{\sqrt{25 - x^2}}, 1 + \left(\frac{dy}{dx}\right)^2 = \frac{25}{25 - x^2}$$

M1 A1

$$y\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = 5$$

M1

$$S = 2\pi \int_1^4 5 \, dx = 30\pi$$

A1

Answer: Surface area = 30π

4 marks

18 (5 marks)

$$\frac{dy}{dx} = \sinh x, \text{ so } \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \cosh x$$

B1

$$S = 2\pi \int_0^1 x \cosh x \, dx$$

M1

$$\text{By parts: } \int x \cosh x \, dx = x \sinh x - \cosh x$$

M1 A1

$$S = 2\pi[x \sinh x - \cosh x]_0^1 = 2\pi(\sinh 1 - \cosh 1 + 1) = 2\pi(1 - e^{-1})$$

A1

$$\text{Answer: Surface area} = 2\pi(1 - e^{-1})$$

5 marks

19 (5 marks)

$$\frac{dy}{dx} = \frac{x}{2} - \frac{1}{2x}$$

B1

$$1 + \left(\frac{dy}{dx}\right)^2 = \left(\frac{x}{2} + \frac{1}{2x}\right)^2$$

M1 A1

$$S = 2\pi \int_1^3 x \left(\frac{x}{2} + \frac{1}{2x}\right) dx = 2\pi \int_1^3 \left(\frac{x^2}{2} + \frac{1}{2}\right) dx$$

M1

$$= 2\pi \left[\frac{x^3}{6} + \frac{x}{2}\right]_1^3 = 2\pi \left(6 - \frac{2}{3}\right) = \frac{32\pi}{3}$$

A1

$$\text{Answer: } \frac{32\pi}{3}$$

5 marks

20 (4 marks)

$$(a) \frac{dy}{dx} = -\frac{x}{\sqrt{r^2 - x^2}}, \text{ so } 1 + \left(\frac{dy}{dx}\right)^2 = \frac{r^2}{r^2 - x^2}$$

M1

$$y\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{r^2 - x^2} \times \frac{r}{\sqrt{r^2 - x^2}} = r$$

A1

$$\text{Answer: } y\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = r$$

2 marks

$$(b) S = 2\pi \int_r^r r \, dx$$

M1

$$= 2\pi r \times 2r = 4\pi r^2$$

A1

$$\text{Answer: Surface area} = 4\pi r^2$$

2 marks

21 (7 marks)

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 2 - 2\cos t = 4\sin^2\left(\frac{t}{2}\right)$$

M1 A1

$$\text{Speed} = 2\sin\left(\frac{t}{2}\right) \text{ for } 0 \leq t \leq 2\pi$$

A1

$$y = 1 - \cos t = 2\sin^2\left(\frac{t}{2}\right)$$

B1

$$S = 2\pi \int_0^{2\pi} 2\sin^2\left(\frac{t}{2}\right) \times 2\sin\left(\frac{t}{2}\right) dt = 8\pi \int_0^{2\pi} \sin^3\left(\frac{t}{2}\right) dt$$

M1

$$\text{Let } u = \frac{t}{2}: = 16\pi \int_0^\pi \sin^3 u \, du = 16\pi \int_0^\pi (1 - \cos^2 u) \sin u \, du$$

M1

$$= 16\pi \left[-\cos u + \frac{\cos^3 u}{3}\right]_0^\pi = 16\pi \times \frac{4}{3} = \frac{64\pi}{3}$$

A1

$$\text{Answer: Surface area} = \frac{64\pi}{3}$$

7 marks

22 (6 marks)

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 9 \sin^2 t \cos^2 t, \text{ so the speed is } 3|\sin t \cos t|$$

M1 A1

Symmetry: $S = 2 \times (\text{area for } 0 \leq t \leq \frac{\pi}{2})$, where the speed is $3 \sin t \cos t$

B1

$$S = 2 \times 2\pi \int_0^{\pi/2} \sin^3 t \times 3 \sin t \cos t \, dt$$

M1

$$= 12\pi \int_0^{\pi/2} \sin^4 t \cos t \, dt = 12\pi \left[\frac{\sin^5 t}{5} \right]_0^{\pi/2}$$

M1

$$= \frac{12\pi}{5}$$

A1

Answer: $\frac{12\pi}{5}$

6 marks

23 (7 marks)

$$\frac{dx}{dt} = e^t(\cos t - \sin t), \frac{dy}{dt} = e^t(\sin t + \cos t)$$

M1 A1

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 2e^{2t}, \text{ speed} = \sqrt{2} e^t$$

M1 A1

$$S = 2\pi \int_0^\pi e^t \sin t \times \sqrt{2} e^t \, dt = 2\sqrt{2}\pi \int_0^\pi e^{2t} \sin t \, dt$$

M1

$$\int e^{2t} \sin t \, dt = \frac{1}{5} e^{2t}(2 \sin t - \cos t)$$

A1

$$S = 2\sqrt{2}\pi \times \frac{1}{5}(e^{2\pi} + 1) = \frac{2\sqrt{2}\pi}{5}(e^{2\pi} + 1)$$

A1

Answer: Surface area = $\frac{2\sqrt{2}\pi}{5}(e^{2\pi} + 1)$

7 marks

24 (4 marks)

(a) The gradient must be squared in the arc length element: it should be $\sqrt{1 + (2x)^2} = \sqrt{1 + 4x^2}$

E1

Answer: Gradient not squared

1 mark

(b) $S = 2\pi \int_0^1 x^2 \sqrt{1 + 4x^2} \, dx$

M1 A1

$$= 3.810$$

A1

Answer: 3.810

3 marks

25 (5 marks)

$$\frac{dy}{dx} = \frac{x}{10}, \text{ so } \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{x^2}{100}}$$

B1

$$S = 2\pi \int_0^{10} x \sqrt{1 + \frac{x^2}{100}} \, dx$$

M1

$$= 2\pi \left[\frac{100}{3} \left(1 + \frac{x^2}{100}\right)^{3/2} \right]_0^{10}$$

M1 A1

$$= \frac{200\pi}{3}(2\sqrt{2} - 1) = 383 \text{ cm}^2$$

A1

Answer: $\frac{200\pi}{3}(2\sqrt{2} - 1) = 383 \text{ cm}^2$

5 marks

26 (6 marks)

$$\frac{dy}{dx} = \sinh\left(\frac{x}{2}\right), \text{ so } \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \cosh\left(\frac{x}{2}\right)$$

B1

$$S = 2\pi \int_{-2}^2 2 \cosh^2\left(\frac{x}{2}\right) \, dx$$

M1 A1

$$\cosh^2\left(\frac{x}{2}\right) = \frac{1}{2}(1 + \cosh x)$$

M1

$$S = 2\pi [x + \sinh x]_{-2}^2$$

A1

$$= 2\pi(4 + 2 \sinh 2) = 4\pi(2 + \sinh 2)$$

A1

Answer: Surface area = $4\pi(2 + \sinh 2)$

6 marks

27 (10 marks)

$$y = \sqrt{3} \sqrt{1 - \frac{x^2}{4}}; \text{ implicit: } \frac{x}{2} + \frac{2y}{3} \frac{dy}{dx} = 0 \text{ gives } y \frac{dy}{dx} = -\frac{3x}{4}$$

M1 A1

$$y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{y^2 + \left(y \frac{dy}{dx}\right)^2} = \sqrt{3 - \frac{3x^2}{4} + \frac{9x^2}{16}}$$

M1 A1

$$= \sqrt{3 - \frac{3x^2}{16}} = \sqrt{3} \sqrt{1 - \frac{x^2}{16}}$$

A1

$$S = 2\pi\sqrt{3} \int_{-2}^2 \sqrt{1 - \frac{x^2}{16}} dx; \text{ let } x = 4 \sin \theta: dx = 4 \cos \theta d\theta, \text{ limits } \pm \frac{\pi}{6}$$

M1 A1

$$= 2\pi\sqrt{3} \times 4 \int_{-\pi/6}^{\pi/6} \cos^2 \theta d\theta = 4\pi\sqrt{3} \int_{-\pi/6}^{\pi/6} (1 + \cos 2\theta) d\theta$$

M1

$$= 4\pi\sqrt{3} \left[\theta + \frac{1}{2} \sin 2\theta \right]_{-\pi/6}^{\pi/6}$$

M1

$$= 4\pi\sqrt{3} \left(\frac{\pi}{3} + \frac{\sqrt{3}}{2} \right) = \frac{4\sqrt{3}\pi^2}{3} + 6\pi$$

A1

$$\text{Answer: Surface area} = 6\pi + \frac{4\sqrt{3}\pi^2}{3}$$

10 marks

28 (8 marks)

$$(a) \frac{dx}{dt} = -r \sin t, \frac{dy}{dt} = r \cos t, \text{ speed} = r$$

M1 A1

$$S = 2\pi \int_0^{2\pi} (R + r \sin t) r dt$$

M1

$$= 2\pi r [Rt - r \cos t]_0^{2\pi}$$

A1

$$= 2\pi r \times 2\pi R = 4\pi^2 Rr$$

A1

$$\text{Answer: Surface area} = 4\pi^2 Rr$$

5 marks

$$(b) \frac{S}{V} = \frac{4\pi^2 Rr}{2\pi^2 Rr^2} = \frac{2}{r}$$

M1 A1

$$\frac{2}{r} = 4 \text{ gives } r = \frac{1}{2} \text{ (independent of } R, \text{ provided } R > \frac{1}{2})$$

A1

$$\text{Answer: } \frac{S}{V} = \frac{2}{r}; r = \frac{1}{2}$$

3 marks

29 (8 marks)

$$\frac{dx}{dt} = 1 - \operatorname{sech}^2 t = \tanh^2 t, \frac{dy}{dt} = -\operatorname{sech} t \tanh t$$

M1 A1

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \tanh^2 t (\tanh^2 t + \operatorname{sech}^2 t) = \tanh^2 t$$

M1 A1

$$\text{Speed} = \tanh t$$

A1

$$S = 2\pi \int_0^R \operatorname{sech} t \tanh t dt = 2\pi [-\operatorname{sech} t]_0^R$$

M1 A1

$$= 2\pi(1 - \operatorname{sech} R) \rightarrow 2\pi \text{ as } R \rightarrow \infty \text{ (sech } R \rightarrow 0)$$

A1

$$\text{Answer: Surface area} = 2\pi$$

8 marks

30 (9 marks)

$$(a) V = \pi \int_1^R \frac{1}{x^2} dx = \pi \left(1 - \frac{1}{R}\right)$$

M1 A1

$$\text{As } R \rightarrow \infty, V \rightarrow \pi$$

A1

$$\text{Answer: } \pi$$

3 marks

$$(b) \frac{dy}{dx} = -\frac{1}{x^2}, S = 2\pi \int_1^R \frac{1}{x} \sqrt{1 + \frac{1}{x^4}} dx$$

M1 A1

$$\text{For } x \geq 1, \sqrt{1 + \frac{1}{x^4}} > 1, \text{ so } S > 2\pi \int_1^R \frac{1}{x} dx$$

M1 A1

$$= 2\pi \ln R$$

A1

$$\ln R \rightarrow \infty \text{ as } R \rightarrow \infty, \text{ so } S \text{ is infinite}$$

A1

$$\text{Answer: Infinite}$$

6 marks

Further Parametric Calculus

1 (2 marks)

$$\frac{dx}{dt} = 2t, \frac{dy}{dt} = 2$$

B1

$$\frac{dy}{dx} = \frac{2}{2t} = \frac{1}{t}$$

A1

Answer: $\frac{1}{t}$

2 marks

2 (3 marks)

$$\frac{dx}{dt} = -3 \sin t, \frac{dy}{dt} = 2 \cos t$$

B1

$$\%s = -\frac{2 \cos t}{3 \sin t} = -\frac{2}{3} \cot t$$

M1

$$\text{At } t = \frac{\pi}{4}: \text{gradient} = -\frac{2}{3}$$

A1

Answer: $-\frac{2}{3} \cot t; -\frac{2}{3}$

3 marks

3 (3 marks)

$$\text{When } t = 2, x = 5 \text{ and } y = 1$$

B1

$$\frac{dy}{dx} = \frac{2t}{2} = t = 2$$

M1

$$y - 1 = 2(x - 5), \text{ so } y = 2x - 9$$

A1

Answer: $y = 2x - 9$

3 marks

4 (3 marks)

$$\text{Area} = \int y \frac{dx}{dt} dt = \int_0^2 (t+1) \times 2t dt$$

M1 A1

$$= \left[\frac{2t^3}{3} + t^2 \right]_0^2 = \frac{28}{3}$$

A1

Answer: $\frac{28}{3}$

3 marks

5 (3 marks)

$$\frac{dy}{dx} = \frac{3t^2}{2t} = \frac{3t}{2}$$

B1

$$\frac{d^2y}{dx^2} = \frac{d}{dt} \left(\frac{dy}{dx} \right) \div \frac{dx}{dt}$$

M1

$$= \frac{3}{2} \div 2t = \frac{3}{4t}$$

A1

Answer: $\frac{3}{4t}$

3 marks

6 (2 marks)

$$t = \frac{x}{3}$$

M1

$$y = 1 - \frac{x^2}{9}$$

A1

Answer: $y = 1 - \frac{x^2}{9}$

2 marks

7 (5 marks)

$$\text{At } \theta = \frac{\pi}{3}: x = 2, y = \frac{3\sqrt{3}}{2} \quad \text{B1}$$

$$\frac{dy}{dx} = -\frac{3 \cos \theta}{4 \sin \theta} \quad \text{M1 A1}$$

$$\text{At } \theta = \frac{\pi}{3}: m = -\frac{3}{4\sqrt{3}} = -\frac{\sqrt{3}}{4} \quad \text{A1}$$

$$y - \frac{3\sqrt{3}}{2} = -\frac{\sqrt{3}}{4}(x - 2), \text{ so } y = -\frac{\sqrt{3}}{4}x + 2\sqrt{3} \quad \text{A1}$$

$$\text{Answer: } y = -\frac{\sqrt{3}}{4}x + 2\sqrt{3} \quad \text{5 marks}$$

8 (4 marks)

$$\frac{dy}{dx} = \frac{2t}{3t^2} = \frac{2}{3t}; \text{ at } t = 1 \text{ gradient} = \frac{2}{3} \quad \text{M1 A1}$$

$$\text{Normal gradient} = -\frac{3}{2}; \text{ point } (1, 1) \quad \text{M1}$$

$$y - 1 = -\frac{3}{2}(x - 1), \text{ so } 3x + 2y = 5 \quad \text{A1}$$

$$\text{Answer: } 3x + 2y = 5 \quad \text{4 marks}$$

9 (4 marks)

$$\frac{dy}{dt} = 3t^2 - 3 = 0 \text{ gives } t = \pm 1 \left(\frac{dx}{dt} = 2t \neq 0 \right) \quad \text{M1 A1}$$

$$t = 1: (2, -2) \quad \text{A1}$$

$$t = -1: (2, 2) \quad \text{A1}$$

$$\text{Answer: } (2, -2) \text{ and } (2, 2) \quad \text{4 marks}$$

10 (3 marks)

$$\sec t = \frac{x}{3}, \tan t = \frac{y}{2} \quad \text{B1}$$

$$\text{Use } 1 + \tan^2 t = \sec^2 t \quad \text{M1}$$

$$1 + \frac{y^2}{4} = \frac{x^2}{9}, \text{ so } \frac{x^2}{9} - \frac{y^2}{4} = 1 \quad \text{A1}$$

$$\text{Answer: } \frac{x^2}{9} - \frac{y^2}{4} = 1 \quad \text{3 marks}$$

11 (3 marks)

$$\frac{dy}{dx} = \frac{\cosh t}{\sinh t} = \coth t \quad \text{B1}$$

$$\cosh^2 t - \sinh^2 t = 1 \quad \text{M1}$$

$$x^2 - y^2 = 1 \quad \text{A1}$$

$$\text{Answer: } x^2 - y^2 = 1 \quad \text{3 marks}$$

12 (4 marks)

$$\frac{dy}{dx} = \frac{2t}{3t^2} = \frac{2}{3t} \quad \text{M1 A1}$$

$$\frac{d}{dt}\left(\frac{2}{3t}\right) = -\frac{2}{3t^2} \quad \text{M1}$$

$$\frac{d^2y}{dx^2} = -\frac{2}{3t^2} \div 3t^2 = -\frac{2}{9t^4} \quad \text{A1}$$

$$\text{Answer: } \frac{d^2y}{dx^2} = -\frac{2}{9t^4} \quad \text{4 marks}$$

13 (3 marks)

$$\frac{dy}{dt} = 2t - 4 = 0, \text{ so } t = 2$$

M1

Point: (4, -4)

A1

 $y = t^2 - 4t$ is a positive quadratic in t , so a minimum

B1

Answer: (4, -4), minimum

3 marks

14 (5 marks) $y = 0$ when $t = 0$ or $t = 2$, so $A = (4, 0)$

B1

$$\text{Area} = \int_0^2 y \frac{dx}{dt} dt = \int_0^2 t(4 - t^2) \times 2t dt$$

M1 A1

$$= 2 \int_0^2 (4t^2 - t^4) dt = 2 \left[\frac{4t^3}{3} - \frac{t^5}{5} \right]_0^2$$

M1

$$= 2 \left(\frac{32}{3} - \frac{32}{5} \right) = \frac{128}{15}$$

A1

Answer: A = (4, 0); area $\frac{128}{15}$

5 marks

15 (5 marks)By symmetry, area = $4 \times$ area of the first quadrant part (t from $\frac{\pi}{2}$ to 0 as x goes 0 to 4)

B1

$$\text{Area} = 4 \int_{\pi/2}^0 y \frac{dx}{dt} dt = 4 \int_{\pi/2}^0 3 \sin t \times (-4 \sin t) dt$$

M1 A1

$$= 48 \int_0^{\pi/2} \sin^2 t dt$$

M1

$$= 24 \int_0^{\pi/2} (1 - \cos 2t) dt = 24 \times \frac{\pi}{2} = 12\pi$$

A1

Answer: Area = 12π

5 marks

16 (5 marks) $V = \pi \int_{-2}^2 y^2 dx$; $x = -2$ when $t = \pi$ and $x = 2$ when $t = 0$, and $dx = -2 \sin t dt$

M1

$$V = \pi \int_{\pi}^0 \sin^2 t \times (-2 \sin t) dt = 2\pi \int_0^{\pi} \sin^3 t dt$$

M1 A1

$$\sin^3 t = (1 - \cos^2 t) \sin t$$

M1

$$V = 2\pi \left[-\cos t + \frac{\cos^3 t}{3} \right]_0^{\pi} = 2\pi \times \frac{4}{3} = \frac{8\pi}{3}$$

A1

Answer: $\frac{8\pi}{3}$

5 marks

17 (7 marks)

$$(a) \frac{dx}{dt} = \frac{1}{t}, \frac{dy}{dt} = 1 - \frac{1}{t^2}$$

B1

$$\frac{dy}{dx} = \left(1 - \frac{1}{t^2} \right) \times t = t - \frac{1}{t}$$

A1

Answer: $\frac{dy}{dx} = t - \frac{1}{t}$

2 marks

$$(b) \frac{d^2y}{dx^2} = \frac{d}{dt} \left(t - \frac{1}{t} \right) \div \frac{dx}{dt} = \left(1 + \frac{1}{t^2} \right) t = t + \frac{1}{t}$$

M1 A1

$$\frac{dy}{dx} = 0 \text{ gives } t^2 = 1, t = 1$$

M1

Point: (0, 2)

A1

$$\frac{d^2y}{dx^2} = 2 > 0, \text{ so a minimum}$$

A1

Answer: (0, 2), minimum

5 marks

18 (5 marks)

$$\frac{dx}{dt} = 1 - \cos t, \quad \frac{dy}{dt} = \sin t \quad \text{B1}$$

$$\%s = \frac{\sin t}{1 - \cos t} = \frac{2 \sin(\frac{t}{2}) \cos(\frac{t}{2})}{2 \sin^2(\frac{t}{2})} = \cot(\frac{t}{2}) \quad \text{M1 A1}$$

$$\text{At } t = \frac{\pi}{2}: \text{gradient} = \cot \frac{\pi}{4} = 1; \text{point } (\frac{\pi}{2} - 1, 1) \quad \text{B1}$$

$$y - 1 = x - \frac{\pi}{2} + 1, \text{ so } y = x + 2 - \frac{\pi}{2} \quad \text{A1}$$

$$\text{Answer: } y = x + 2 - \frac{\pi}{2} \quad \text{5 marks}$$

19 (4 marks)

$$2t = t^2 - 3 \text{ gives } t^2 - 2t - 3 = 0, \text{ so } (t - 3)(t + 1) = 0 \quad \text{M1 A1}$$

$$A = (9, 6), B = (1, -2) \quad \text{A1}$$

$$AB = \sqrt{8^2 + 8^2} = 8\sqrt{2} \quad \text{A1}$$

$$\text{Answer: } 8\sqrt{2} \quad \text{4 marks}$$

20 (6 marks)

$$\text{(a) } \frac{dy}{dx} = \frac{2}{2t} = \frac{1}{t}, \text{ so the gradient at P is } \frac{1}{p} \quad \text{M1 A1}$$

$$y - 2p = \frac{1}{p}(x - p^2), \text{ so } p y - 2p^2 = x - p^2, \text{ giving } p y = x + p^2 \quad \text{A1}$$

$$\text{Answer: } p y = x + p^2 \quad \text{3 marks}$$

$$\text{(b) Substitute: } 2p = -4 + p^2 \quad \text{M1}$$

$$p^2 - 2p - 4 = 0 \quad \text{A1}$$

$$p = 1 \pm \sqrt{5} \quad \text{A1}$$

$$\text{Answer: } p = 1 \pm \sqrt{5} \quad \text{3 marks}$$

21 (9 marks)

$$\text{(a) Tangent gradient } \frac{1}{p}, \text{ so the normal gradient is } -p \quad \text{M1 A1}$$

$$y - 2p = -p(x - p^2), \text{ so } y + p x = 2p + p^3 \quad \text{A1}$$

$$\text{Answer: } y + p x = 2p + p^3 \quad \text{3 marks}$$

$$\text{(b) Substitute } x = q^2, y = 2q: 2q + p q^2 = 2p + p^3 \quad \text{M1}$$

$$p q^2 + 2q - 2p - p^3 = 0, \text{ which factorises as } (q - p)(p q + p^2 + 2) = 0 \quad \text{M1 A1}$$

$$q \neq p, \text{ so } p q + p^2 + 2 = 0 \quad \text{A1}$$

$$\text{Answer: } p q + p^2 + 2 = 0 \quad \text{4 marks}$$

$$\text{(c) } q = -(1 + 2) = -3 \quad \text{M1}$$

$$Q = (9, -6) \quad \text{A1}$$

$$\text{Answer: } (9, -6) \quad \text{2 marks}$$

22 (7 marks)

$$\text{By symmetry the area is 4 times the first quadrant area, where } 0 \leq t \leq \frac{\pi}{2} \text{ and } x \text{ goes from 1 to 0} \quad \text{B1}$$

$$\frac{dx}{dt} = -3 \cos^2 t \sin t \quad \text{M1 A1}$$

$$\text{Area} = 4 \int_{\pi/2}^0 \sin^3 t \times (-3 \cos^2 t \sin t) dt \quad \text{M1}$$

$$= 12 \int_0^{\pi/2} \sin^4 t \cos^2 t dt \quad \text{M1 A1}$$

$$= 12 \times \frac{\pi}{32} = \frac{3\pi}{8} \quad \text{A1}$$

$$\text{Answer: Area} = \frac{3\pi}{8} \quad \text{7 marks}$$

23 (6 marks)

$$\text{Area} = \int_0^{2\pi} y \frac{dx}{dt} dt = \int_0^{2\pi} (1 - \cos t)(1 - \cos t) dt \quad \text{M1 A1}$$

$$(1 - \cos t)^2 = 1 - 2 \cos t + \cos^2 t = \frac{3}{2} - 2 \cos t + \frac{1}{2} \cos 2t \quad \text{M1 A1}$$

$$\int_0^{2\pi} = \left[\frac{3t}{2} - 2 \sin t + \frac{1}{4} \sin 2t \right]_0^{2\pi} \quad \text{M1}$$

$$= 3\pi \quad \text{A1}$$

Answer: Area = 3π 6 marks

24 (6 marks)

(a) The student has divided in the wrong order: $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$ E1

Answer: Division inverted 1 mark

(b) $\frac{dy}{dx} = \frac{2t}{3t^2} = \frac{2}{3t}$, at $t = 1$ the gradient is $\frac{2}{3}$ M1 A1

Point (1, 1); tangent $y - 1 = \frac{2}{3}(x - 1)$ M1

$y = 0$: $-1 = \frac{2}{3}(x - 1)$ M1

$x - 1 = -\frac{3}{2}$, so $x = -\frac{1}{2}$; the tangent meets the x axis at $(-\frac{1}{2}, 0)$ A1

Answer: $\frac{2}{3}$; $(-\frac{1}{2}, 0)$ 5 marks

25 (8 marks)

(a) $\frac{dx}{dt} = \cos t$, $\frac{dy}{dt} = 2 \cos 2t$ B1

$$\frac{dy}{dx} = \frac{2 \cos 2t}{\cos t} \quad \text{A1}$$

Answer: $\frac{dy}{dx} = \frac{2 \cos 2t}{\cos t}$, for $\cos t \neq 0$ 2 marks

(b) $\cos 2t = 0$, so $2t = \frac{\pi}{2}$ or $\frac{3\pi}{2}$ M1

$t = \frac{\pi}{4}$: $(\sqrt{2}/2, 1)$; $t = \frac{3\pi}{4}$: $(\sqrt{2}/2, -1)$ A1 A1

Answer: $(\sqrt{2}/2, 1), (\sqrt{2}/2, -1)$ 3 marks

(c) Upper half of the loop: $0 \leq t \leq \frac{\pi}{2}$, area = $\int_0^{\pi/2} y \frac{dx}{dt} dt = \int_0^{\pi/2} \sin 2t \cos t dt$ M1

$$= \int_0^{\pi/2} 2 \sin t \cos^2 t dt = \left[-\frac{2}{3} \cos^3 t \right]_0^{\pi/2} = \frac{2}{3} \quad \text{M1}$$

By symmetry in the x axis the loop has area $2 \times \frac{2}{3} = \frac{4}{3}$ A1

Answer: Area = $\frac{4}{3}$ 3 marks

26 (7 marks)

(a) $\frac{dx}{dt} = 20$, $\frac{dy}{dt} = 15 - 9.8t$ B1

$$\frac{dy}{dx} = \frac{15 - 9.8t}{20} \quad \text{M1}$$

At $t = 1$: $\frac{5.2}{20} = 0.26$ A1

Answer: 0.26 3 marks

(b) $\frac{dy}{dx} = 0$ gives $t = \frac{15}{9.8} = 1.531$ M1

$y = 15 \times 1.5306 - 4.9 \times 1.5306^2 = 11.5 \text{ m}$ A1

Answer: 11.5 m 2 marks

(c) $y = 0$: $t(15 - 4.9t) = 0$, so $t = \frac{15}{4.9} = 3.061$ M1

$x = 20 \times 3.061 = 61.2 \text{ m}$ A1

Answer: 61.2 m**2 marks****27 (10 marks)**

(a) $\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}}$

B1

$$\frac{d^2y}{dx^2} = \frac{1}{\dot{x}} \frac{d}{dt} \left(\frac{\dot{y}}{\dot{x}} \right)$$

M1

$$\frac{d}{dt} \left(\frac{\dot{y}}{\dot{x}} \right) = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^2} \text{ (quotient rule)}$$

M1 A1

Answer: $\frac{d^2y}{dx^2} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^3}$

4 marks

(b) $\dot{x} = 1 - \cos t$, $\ddot{x} = \sin t$, $\dot{y} = \sin t$, $\ddot{y} = \cos t$

B1

Numerator: $(1 - \cos t) \cos t - \sin^2 t = \cos t - \cos^2 t - \sin^2 t = \cos t - 1$

M1 A1

$$\frac{d^2y}{dx^2} = \frac{\cos t - 1}{(1 - \cos t)^3} = -\frac{1}{(1 - \cos t)^2}$$

A1

Answer: $\frac{d^2y}{dx^2} = -\frac{1}{(1 - \cos t)^2}$

4 marks

(c) $(1 - \cos t)^2 > 0$, so the expression is always negative

E1

At $t = \frac{\pi}{2}$: $-\frac{1}{1} = -1$

B1**Answer: -1****2 marks****28 (10 marks)**

(a) Same x for t and $-t$; y equal requires $t^3 - 3t = -t^3 + 3t$, so $2t(t^2 - 3) = 0$

M1 A1

$t = \pm\sqrt{3}$ ($t \neq 0$), $N = (3, 0)$

A1

Answer: $N = (3, 0)$; $t = \pm\sqrt{3}$

3 marks

(b) $\frac{dy}{dx} = \frac{3t^2 - 3}{2t}$; at $t = \sqrt{3}$ gradient = $\frac{6}{2\sqrt{3}} = \sqrt{3}$, at $t = -\sqrt{3}$ gradient = $-\sqrt{3}$

M1 A1Tangent angles 60° and 120° with the x axis, so the acute angle between them is 60° **A1****Answer: 60°** **3 marks**

(c) Area = $\left| \int_{-\sqrt{3}}^{\sqrt{3}} y \frac{dx}{dt} dt \right| = \left| \int_{-\sqrt{3}}^{\sqrt{3}} (t^3 - 3t) 2t dt \right|$

M1

$$= 2 \left| \int_{-\sqrt{3}}^{\sqrt{3}} (t^4 - 3t^2) dt \right| = 4 \left| \left[\frac{t^5}{5} - t^3 \right]_0^{\sqrt{3}} \right|$$

M1 A1

$$= 4 \left| \frac{9\sqrt{3}}{5} - 3\sqrt{3} \right| = 4 \times \frac{6\sqrt{3}}{5} = \frac{24\sqrt{3}}{5}$$

A1

Answer: Area = $\frac{24\sqrt{3}}{5}$

4 marks**29 (9 marks)**

(a) $\frac{dx}{dt} = \cos t - t \sin t$, $\frac{dy}{dt} = \sin t + t \cos t$

M1 A1

At $t = \frac{\pi}{2}$: $\frac{dy}{dx} = \frac{1}{-\pi/2} = -\frac{2}{\pi}$

A1

Answer: $-\frac{2}{\pi}$

3 marks

(b) $\frac{dy}{dt} = 0$ (with $\frac{dx}{dt} \neq 0$): $\sin t + t \cos t = 0$

M1

Divide by $\cos t$: $\tan t = -t$

A1

Answer: $\tan t = -t$

2 marks

(c) $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \cos^2 t - 2t \sin t \cos t + t^2 \sin^2 t + \sin^2 t + 2t \sin t \cos t + t^2 \cos^2 t = 1 + t^2$

M1 A1

$$s = \int_0^\pi \sqrt{1+t^2} dt = \frac{1}{2} [t\sqrt{1+t^2} + \operatorname{arsinh} t]_0^\pi$$

M1

$$= \frac{1}{2} (\pi\sqrt{1+\pi^2} + \operatorname{arsinh} \pi)$$

A1

Answer: $\frac{1}{2}(\pi\sqrt{1+\pi^2} + \operatorname{arsinh} \pi)$

4 marks

30 (10 marks)

(a) $x^3 + y^3 = \frac{27t^3 + 27t^6}{(1+t^3)^3} = \frac{27t^3}{(1+t^3)^2}$

M1 A1

$3xy = 3 \times \frac{3t}{1+t^3} \times \frac{3t^2}{1+t^3} = \frac{27t^3}{(1+t^3)^2}$, so $x^3 + y^3 = 3xy$

A1

Answer: $x^3 + y^3 = 3xy$

3 marks

(b) $\frac{dx}{dt} = \frac{3(1-2t^3)}{(1+t^3)^2}$

M1 A1

$\frac{dy}{dt} = \frac{3t(2-t^3)}{(1+t^3)^2}$

M1 A1

Answer: $\frac{dy}{dx} = \frac{t(2-t^3)}{1-2t^3}$

4 marks

(c) Horizontal: $t^3 = 2$, $t = 2^{1/3}$, giving $(2^{1/3}, 2^{2/3})$

M1 A1

Vertical: $1 - 2t^3 = 0$, $t^3 = \frac{1}{2}$, giving $(2^{2/3}, 2^{1/3})$

A1

Answer: $(2^{1/3}, 2^{2/3}); (2^{2/3}, 2^{1/3})$

3 marks

Further Trigonometric Integration

1 (3 marks)

$\cos 2x = 1 - 2\sin^2 x$, so $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

M1 A1

$\int \sin^2 x \, dx = \frac{x}{2} - \frac{\sin 2x}{4} + c$

A1

Answer: $\frac{x}{2} - \frac{\sin 2x}{4} + c$

3 marks

2 (3 marks)

$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

M1

$\left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_0^\pi$

M1

$= \frac{\pi}{2}$

A1

Answer: $\frac{\pi}{2}$

3 marks

3 (2 marks)

$\tan x = \frac{\sin x}{\cos x}$, so $\int = -\ln|\cos x| + c$

M1 A1

Answer: $-\ln|\cos x| + c$ ($= \ln|\sec x| + c$)

2 marks

4 (3 marks)

$\cos^3 x = \cos x(1 - \sin^2 x)$

M1

$\int (\cos x - \sin^2 x \cos x) \, dx$

A1

$= \sin x - \frac{\sin^3 x}{3} + c$

A1

Answer: $\sin x - \frac{\sin^3 x}{3} + c$

3 marks

5 (3 marks)

$$2 \sin A \cos B = \sin(A + B) + \sin(A - B), \text{ so } \sin 4x \cos 2x = \frac{1}{2}(\sin 6x + \sin 2x)$$

M1 A1

$$\int = -\frac{\cos 6x}{12} - \frac{\cos 2x}{4} + c$$

A1

$$\text{Answer: } -\frac{\cos 6x}{12} - \frac{\cos 2x}{4} + c$$

3 marks

6 (2 marks)

$$\int \sec^2 x \, dx = \tan x$$

M1

$$[\tan x]_0^{\pi/4} = 1$$

A1

$$\text{Answer: } 1$$

2 marks

7 (4 marks)

$$\sin^3 x = \sin x(1 - \cos^2 x)$$

M1

$$\int (\cos^2 x - \cos^4 x) \sin x \, dx$$

A1

$$\text{Let } u = \cos x: -\int (u^2 - u^4) \, du$$

M1

$$= -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + c$$

A1

$$\text{Answer: } -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + c$$

4 marks

8 (4 marks)

$$\tan^3 x = \tan x(\sec^2 x - 1)$$

M1

$$\int \tan x \sec^2 x \, dx = \frac{\tan^2 x}{2}$$

A1

$$\int \tan x \, dx = \ln|\sec x|$$

B1

$$\text{Total: } \frac{\tan^2 x}{2} - \ln|\sec x| + c = \frac{\tan^2 x}{2} + \ln|\cos x| + c$$

A1

$$\text{Answer: } \frac{\tan^2 x}{2} + \ln|\cos x| + c$$

4 marks

9 (4 marks)

$$\cos^4 x = \left(\frac{1 + \cos 2x}{2}\right)^2 = \frac{1}{4}(1 + 2 \cos 2x + \cos^2 2x)$$

M1

$$\cos^2 2x = \frac{1}{2}(1 + \cos 4x), \text{ so } \cos^4 x = \frac{3}{8} + \frac{1}{2} \cos 2x + \frac{1}{8} \cos 4x$$

M1 A1

$$\left[\frac{3x}{8} + \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x\right]_0^{\pi/2} = \frac{3\pi}{16}$$

A1

$$\text{Answer: } \frac{3\pi}{16}$$

4 marks

10 (4 marks)

$$\sec^4 x = \sec^2 x(1 + \tan^2 x)$$

M1

$$\int = \tan x + \frac{\tan^3 x}{3}$$

A1

$$\left[\tan x + \frac{\tan^3 x}{3}\right]_0^{\pi/4}$$

M1

$$= 1 + \frac{1}{3} = \frac{4}{3}$$

A1

$$\text{Answer: } \frac{4}{3}$$

4 marks

11 (4 marks)

$$2 \cos A \cos B = \cos(A + B) + \cos(A - B), \text{ so } \cos 5x \cos 2x = \frac{1}{2}(\cos 7x + \cos 3x)$$

M1 A1

$$\left[\frac{\sin 7x}{14} + \frac{\sin 3x}{6} \right]_0^{\pi/3}$$

M1

$$\sin \frac{7\pi}{3} = \frac{\sqrt{3}}{2}, \sin \pi = 0, \text{ so value} = \frac{\sqrt{3}}{28}$$

A1

$$\text{Answer: } \frac{\sqrt{3}}{28}$$

4 marks

12 (3 marks)

$$d u = \cos x \, d x; \text{ limits } u = 0 \text{ to } 1$$

B1

$$\int_0^1 \frac{1}{1+u^2} \, d u = [\arctan u]_0^1$$

M1

$$= \frac{\pi}{4}$$

A1

$$\text{Answer: } \frac{\pi}{4}$$

3 marks

13 (4 marks)

$$\sin x \cos x = \frac{1}{2} \sin 2x, \text{ so } \sin^2 x \cos^2 x = \frac{1}{4} \sin^2 2x$$

M1

$$\sin^2 2x = \frac{1}{2}(1 - \cos 4x)$$

M1 A1

$$\int = \frac{x}{8} - \frac{\sin 4x}{32} + c$$

A1

$$\text{Answer: } \frac{x}{8} - \frac{\sin 4x}{32} + c$$

4 marks

14 (6 marks)

$$(a) \sin x = 2 \sin\left(\frac{x}{2}\right) \cos\left(\frac{x}{2}\right) = \frac{2 \tan(\frac{x}{2})}{\sec^2(\frac{x}{2})} = \frac{2t}{1+t^2}$$

M1 A1

$$\frac{d t}{d x} = \frac{1}{2} \sec^2\left(\frac{x}{2}\right) = \frac{1}{2}(1+t^2), \text{ so } d x = \frac{2}{1+t^2} \, d t$$

A1

$$\text{Answer: } \sin x = \frac{2t}{1+t^2}, d x = \frac{2}{1+t^2} \, d t$$

3 marks

$$(b) \text{ Limits: } x = 0 \text{ gives } t = 0, x = \frac{\pi}{2} \text{ gives } t = 1$$

B1

$$\frac{1}{1+\sin x} \, d x = \frac{1}{1+\frac{2t}{1+t^2}} \times \frac{2}{1+t^2} \, d t = \frac{2}{(1+t)^2} \, d t$$

M1

$$\left[-\frac{2}{1+t} \right]_0^1 = -1 + 2 = 1$$

A1

$$\text{Answer: } 1$$

3 marks

15 (5 marks)

$$\text{Limits } t = 0 \text{ to } 1$$

B1

$$2 + \cos x = \frac{2(1+t^2) + 1 - t^2}{1+t^2} = \frac{3+t^2}{1+t^2}$$

M1

$$\text{Integrand: } \frac{1+t^2}{3+t^2} \times \frac{2}{1+t^2} \, d t = \frac{2}{3+t^2} \, d t$$

A1

$$\int_0^1 \frac{2}{3+t^2} \, d t = \frac{2}{\sqrt{3}} \left[\arctan\left(\frac{t}{\sqrt{3}}\right) \right]_0^1$$

M1

$$= \frac{2}{\sqrt{3}} \times \frac{\pi}{6} = \frac{\pi}{3\sqrt{3}}$$

A1

$$\text{Answer: } \frac{\pi}{3\sqrt{3}}$$

5 marks

16 (6 marks)

$$\sin x = \frac{2t}{1+t^2}, \quad dx = \frac{2}{1+t^2} dt, \quad \text{so } \operatorname{cosec} x \, dx = \frac{1+t^2}{2t} \times \frac{2}{1+t^2} dt = \frac{1}{t} dt$$

M1 A1

$$\int \frac{1}{t} dt = \ln|t| + c = \ln\left|\tan\left(\frac{x}{2}\right)\right| + c$$

A1

$$\left[\ln \tan\left(\frac{x}{2}\right)\right]_{\pi/3}^{\pi/2} = \ln 1 - \ln \tan \frac{\pi}{6}$$

M1

$$= -\ln\left(\frac{1}{\sqrt{3}}\right)$$

A1

$$= \frac{1}{2} \ln 3$$

A1

$$\text{Answer: } \frac{1}{2} \ln 3$$

6 marks

17 (5 marks)

$$\text{By parts: } u = \sec x, \quad \frac{dv}{dx} = \sec^2 x; \quad I = [\sec x \tan x]_0^{\pi/4} - \int_0^{\pi/4} \sec x \tan^2 x \, dx$$

M1 A1

$$\tan^2 x = \sec^2 x - 1; \quad I = \sqrt{2} - I + \int_0^{\pi/4} \sec x \, dx$$

M1

$$\int_0^{\pi/4} \sec x \, dx = [\ln(\sec x + \tan x)]_0^{\pi/4} = \ln(\sqrt{2} + 1)$$

A1

$$2I = \sqrt{2} + \ln(1 + \sqrt{2}), \quad \text{so } I = \frac{1}{2}(\sqrt{2} + \ln(1 + \sqrt{2}))$$

A1

$$\text{Answer: } \frac{1}{2}(\sqrt{2} + \ln(1 + \sqrt{2}))$$

5 marks

18 (4 marks)

$$\tan^4 x = \tan^2 x(\sec^2 x - 1) = \tan^2 x \sec^2 x - \tan^2 x$$

M1

$$\int \tan^2 x \sec^2 x \, dx = \frac{\tan^3 x}{3}$$

A1

$$\tan^2 x = \sec^2 x - 1, \quad \text{so } \int \tan^2 x \, dx = \tan x - x$$

M1

$$\int \tan^4 x \, dx = \frac{\tan^3 x}{3} - \tan x + x + c$$

A1

$$\text{Answer: } \frac{\tan^3 x}{3} - \tan x + x + c$$

4 marks

19 (4 marks)

$$\sec^4 x = \sec^2 x(1 + \tan^2 x); \quad \text{let } u = \tan x, \quad du = \sec^2 x \, dx$$

M1

$$\int_0^1 (1 + u^2) u^2 \, du = \int_0^1 (u^2 + u^4) \, du$$

M1 A1

$$= \frac{1}{3} + \frac{1}{5} = \frac{8}{15}$$

A1

$$\text{Answer: } \frac{8}{15}$$

4 marks

20 (5 marks)

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x), \quad \text{so } \sin^4 x = \frac{1}{4}(1 - 2 \cos 2x + \cos^2 2x)$$

M1 A1

$$\cos^2 2x = \frac{1}{2}(1 + \cos 4x)$$

M1

$$\sin^4 x = \frac{3}{8} - \frac{1}{2} \cos 2x + \frac{1}{8} \cos 4x$$

A1

$$\int = \frac{3x}{8} - \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + c$$

A1

$$\text{Answer: } \frac{3x}{8} - \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + c$$

5 marks

21 (7 marks)Limits $t = 0$ to 1

B1

$$5 + 4 \sin x = \frac{5(1+t^2) + 8t}{1+t^2}$$

M1

$$\text{Integrand: } \frac{2}{5t^2 + 8t + 5} dt$$

A1

$$5t^2 + 8t + 5 = 5\left(t + \frac{4}{5}\right)^2 + \frac{9}{5}$$

M1

$$\int_0^1 \frac{2}{5(t + 4/5)^2 + 9/5} dt = \frac{2}{5} \times \frac{5}{3} [\arctan(\frac{5t+4}{3})]_0^1$$

M1 A1

$$= \frac{2}{3} (\arctan 3 - \arctan \frac{4}{3}) = \frac{2}{3} \arctan(\frac{3-4}{3/1+4}) = \frac{2}{3} \arctan(\frac{1}{3})$$

A1

$$\text{Answer: } \frac{2}{3} \arctan(\frac{1}{3})$$

7 marks

22 (6 marks)

$$\text{Limits: } t = 0 \text{ to } \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}$$

B1

$$1 + \cos x + \sin x = \frac{(1+t^2) + (1-t^2) + 2t}{1+t^2} = \frac{2+2t}{1+t^2}$$

M1 A1

$$\text{Integrand: } \frac{1+t^2}{2+2t} \times \frac{2}{1+t^2} dt = \frac{1}{1+t} dt$$

A1

$$[\ln(1+t)]_0^{1/\sqrt{3}}$$

M1

$$= \ln(1 + \frac{1}{\sqrt{3}})$$

A1

$$\text{Answer: } \ln(1 + \frac{1}{\sqrt{3}})$$

6 marks

23 (7 marks)

$$(a) \sec x dx = \frac{1+t^2}{1-t^2} \times \frac{2}{1+t^2} dt = \frac{2}{1-t^2} dt$$

M1 A1

$$\frac{2}{1-t^2} \equiv \frac{1}{1-t} + \frac{1}{1+t}$$

M1

$$\int = -\ln|1-t| + \ln|1+t| + c = \ln|\frac{1+t}{1-t}| + c$$

A1

$$\text{Answer: } \ln|\frac{1+t}{1-t}| + c$$

4 marks

$$(b) \text{ Limits: } t = 0 \text{ to } \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}$$

B1

$$\ln(\frac{1+1}{\sqrt{3}/1 - 1/\sqrt{3}}) = \ln(\frac{\sqrt{3}+1}{\sqrt{3}-1})$$

M1

$$= \ln(\frac{(\sqrt{3}+1)^2}{2}) = \ln(\frac{4+2\sqrt{3}}{2}) = \ln(2+\sqrt{3})$$

A1

$$\text{Answer: } \ln(2+\sqrt{3})$$

3 marks

24 (6 marks)

$$(a) \frac{d}{dx}(\frac{\sin^3 x}{3}) = \sin^2 x \cos x$$

M1

This is not $\sin^2 x$, so the answer is wrong (the chain rule factor $\cos x$ is missing)

E1

Answer: Derivative is $\sin^2 x \cos x$

2 marks

$$(b) \sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

M1

$$[\frac{x}{2} - \frac{\sin 2x}{4}]_0^{\pi/6}$$

M1 A1

$$= \frac{\pi}{12} - \frac{\sqrt{3}}{8}$$

A1

$$\text{Answer: } \frac{\pi}{12} - \frac{\sqrt{3}}{8}$$

4 marks

25 (6 marks)

$d u = \cos x \, d x$; limits $u = 0$ to 1

B1

$$\frac{1}{(1+u)(2+u)} \equiv \frac{1}{1+u} - \frac{1}{2+u}$$

M1 A1

$$[\ln(1+u) - \ln(2+u)]_0^1$$

M1

$$= (\ln 2 - \ln 3) - (0 - \ln 2)$$

A1

$$= 2 \ln 2 - \ln 3 = \ln\left(\frac{4}{3}\right)$$

A1

Answer: $\ln\left(\frac{4}{3}\right)$

6 marks

26 (6 marks)

Divide numerator and denominator by $\cos^2 x$: $\frac{\sec^2 x}{\sec^2 x + 1}$

M1 A1

$\sec^2 x = 1 + u^2$, $d u = \sec^2 x \, d x$

M1

Integrand becomes $\frac{1}{u^2 + 2} \, d u$, limits 0 to 1

A1

$$\int_0^1 \frac{1}{u^2 + 2} \, d u = \frac{1}{\sqrt{2}} [\arctan(\frac{u}{\sqrt{2}})]_0^1$$

M1

$$= \frac{1}{\sqrt{2}} \arctan\left(\frac{1}{\sqrt{2}}\right)$$

A1

Answer: $\frac{1}{\sqrt{2}} \arctan\left(\frac{1}{\sqrt{2}}\right)$

6 marks

27 (9 marks)

$$(a) \, I + J = \int_0^{\pi/2} \frac{\sin x + \cos x}{\sin x + \cos x} \, d x = \int_0^{\pi/2} 1 \, d x$$

M1

$$= \frac{\pi}{2}$$

A1

Answer: $I + J = \frac{\pi}{2}$

2 marks

(b) $d u = -d x$; limits $x = 0$ gives $u = \frac{\pi}{2}$, $x = \frac{\pi}{2}$ gives $u = 0$

M1

$$\sin x = \cos u, \cos x = \sin u, \text{ so } I = \int_{\pi/2}^0 \frac{\cos u}{\cos u + \sin u} (-d u) = \int_0^{\pi/2} \frac{\cos u}{\sin u + \cos u} \, d u = J$$

M1 A1

$$2I = \frac{\pi}{2}, \text{ so } I = \frac{\pi}{4}$$

A1

Answer: $\frac{\pi}{4}$

4 marks

(c) The integral is $I - J = 0$

M1 A1

The integrand is odd about $x = \frac{\pi}{4}$ (it changes sign under $x \rightarrow \frac{\pi}{2} - x$), so the areas above and below the axis cancel

E1

Answer: 0

3 marks

28 (9 marks)

Substitute $x = \pi - u$: $I = \int_0^\pi \frac{(\pi - u) \sin u}{1 + \cos^2 u} du$ (since $\sin(\pi - u) = \sin u$ and $\cos^2(\pi - u) = \cos^2 u$) M1 A1

So $I = \pi \int_0^\pi \frac{\sin u}{1 + \cos^2 u} du - I$ M1 A1

$2I = \pi \int_0^\pi \frac{\sin x}{1 + \cos^2 x} dx$ A1

Let $w = \cos x$: $dw = -\sin x dx$, limits 1 to -1 M1

$\int_0^\pi \frac{\sin x}{1 + \cos^2 x} dx = \int_{-1}^1 \frac{1}{1 + w^2} dw = [\arctan w]_{-1}^1 = \frac{\pi}{2}$ M1 A1

$2I = \pi \times \frac{\pi}{2}$, so $I = \frac{\pi^2}{4}$ A1

Answer: $I = \frac{\pi^2}{4}$ 9 marks

29 (8 marks)

(a) $\sin A \sin B = \frac{1}{2}(\cos(A - B) - \cos(A + B))$ M1

$I(m, n) = \frac{1}{2} \int_0^{2\pi} (\cos((m - n)x) - \cos((m + n)x)) dx$ A1

$= \frac{1}{2} \left[\frac{\sin((m - n)x)}{m - n} - \frac{\sin((m + n)x)}{m + n} \right]_0^{2\pi} = 0$, as $\sin$ of a multiple of 2π is 0 A1

Answer: 0 3 marks

(b) $\sin^2(nx) = \frac{1}{2}(1 - \cos 2nx)$ M1

$I(n, n) = \frac{1}{2} \left[x - \frac{\sin 2nx}{2n} \right]_0^{2\pi} = \pi$ A1

Answer: π 2 marks

(c) Expand: $\sin^2 2x + 6 \sin 2x \sin 5x + 9 \sin^2 5x$ M1

$= I(2, 2) + 6 I(2, 5) + 9 I(5, 5)$ M1

$= \pi + 0 + 9\pi = 10\pi$ A1

Answer: 10π 3 marks

30 (10 marks)

(a) $\cos x = \frac{1 - t^2}{1 + t^2}$, $dx = \frac{2}{1 + t^2} dt$; $x = 0$ gives $t = 0$ and $x \rightarrow \pi$ gives $t \rightarrow \infty$ B1

$a + \cos x = \frac{a(1 + t^2) + 1 - t^2}{1 + t^2} = \frac{(a + 1) + (a - 1)t^2}{1 + t^2}$ M1 A1

Integrand: $\frac{2}{(a + 1) + (a - 1)t^2} dt$ A1

Answer: $I(a) = \int_0^\infty \frac{2}{(a + 1) + (a - 1)t^2} dt$ 4 marks

(b) $\frac{2}{(a + 1) + (a - 1)t^2} = \frac{2}{a - 1} \times \frac{1}{k^2 + t^2}$ with $k^2 = \frac{a + 1}{a - 1}$ M1

$\int_0^\infty = \frac{2}{a - 1} \times \frac{1}{k} [\arctan(\frac{t}{k})]_0^\infty$ M1 A1

$= \frac{2}{(a - 1)k} \times \frac{\pi}{2} = \frac{\pi}{(a - 1)k}$; $(a - 1)k = (a - 1)\sqrt{\frac{a + 1}{a - 1}} = \sqrt{a^2 - 1}$ A1

Answer: $I(a) = \frac{\pi}{\sqrt{a^2 - 1}}$ 4 marks

(c) $\sqrt{a^2 - 1} = 2$ gives $a = \sqrt{5}$ M1

$a = 3$: $I(3) = \frac{\pi}{\sqrt{8}}$, which matches the integral of $\frac{1}{3 + \cos x}$ over 0 to π A1

Answer: $a = \sqrt{5}$ 2 marks

Differentiation Under the Integral Sign

1 (2 marks)

By the fundamental theorem of calculus, the derivative of $\int_1^x f(t) \, dt$ is $f(x)$

M1

$$x^2 \ln x$$

A1

Answer: $x^2 \ln x$

2 marks

2 (2 marks)

Upper limit $g(x) = x^2$: derivative is $f(g(x)) g'(x)$

M1

$$\cos(x^2) \times 2x = 2x \cos(x^2)$$

A1

Answer: $2x \cos(x^2)$

2 marks

3 (2 marks)

Swap the limits: $\int_x^3 = -\int_3^x$

M1

$$-\sqrt{1+x^4}$$

A1

Answer: $-\sqrt{1+x^4}$

2 marks

4 (3 marks)

$f(t) = t^2$: derivative = $f(2x) \times 2 - f(x) \times 1$

M1

$$= 4x^2 \times 2 - x^2 = 7x^2$$

A1

Check: $\int_x^{2x} t^2 \, dt = \frac{8x^3}{3} - \frac{x^3}{3} = \frac{7x^3}{3}$, derivative $7x^2$

B1

Answer: $7x^2$

3 marks

5 (3 marks)

$$F(a) = \frac{1}{a+1}$$

B1

$$F'(a) = \int_0^1 \frac{\partial}{\partial a}(x^a) \, dx = \int_0^1 x^a \ln x \, dx$$

M1

$$F'(a) = -\frac{1}{(a+1)^2}$$

A1

Answer: $-\frac{1}{(a+1)^2}$

3 marks

6 (2 marks)

Differentiate inside the integral: $\int_0^1 (x^2 + \cos a) \, dx$

M1

$$= \frac{1}{3} + \cos a$$

A1

Answer: $\frac{1}{3} + \cos a$

2 marks

7 (3 marks)

Leibniz: $f(x^2) \times 2x - f(x) \times 1$

M1

$$2x \ln(x^2) - \ln x$$

A1

$$= 4x \ln x - \ln x = (4x - 1) \ln x$$

A1

Answer: $(4x - 1) \ln x$

3 marks

8 (3 marks)

$$G'(x) = \frac{1}{1+x^2} \quad \text{B1}$$

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2} \text{ so } G(x) = \arctan x + c \quad \text{M1}$$

$$G(0) = 0 \text{ gives } c = 0, \text{ so } G(x) = \arctan x \quad \text{A1}$$

$$\text{Answer: } G'(x) = \frac{1}{1+x^2}; G(x) = \arctan x \quad \text{3 marks}$$

9 (4 marks)

$$F(a) = \left[\frac{\sin(ax)}{a} \right]_0^\pi = \frac{\sin(\pi a)}{a} \quad \text{B1}$$

$$\text{Left: } F'(a) = \int_0^\pi (-x \sin(ax)) dx \quad \text{M1}$$

$$\text{Right (quotient rule): } F'(a) = \frac{\pi a \cos(\pi a) - \sin(\pi a)}{a^2} \quad \text{M1}$$

$$\text{So } \int_0^\pi x \sin(ax) dx = -F'(a) = \frac{\sin(\pi a) - \pi a \cos(\pi a)}{a^2} \quad \text{A1}$$

$$\text{Answer: } \frac{\sin(\pi a) - \pi a \cos(\pi a)}{a^2} \quad \text{4 marks}$$

10 (4 marks)

$$\text{Differentiate under the integral sign: } -\int_0^\infty x e^{-ax} dx = -\frac{1}{a^2} \quad \text{M1 A1}$$

$$\int_0^\infty x e^{-ax} dx = \frac{1}{a^2} \quad \text{A1}$$

$$\text{Differentiate again: } \int_0^\infty x^2 e^{-ax} dx = \frac{2}{a^3} \quad \text{A1}$$

$$\text{Answer: } \frac{1}{a^2} \text{ and } \frac{2}{a^3} \quad \text{4 marks}$$

11 (4 marks)

$$F'(x) = 2e^{-(2x)^2} - e^{-x^2} = 2e^{-4x^2} - e^{-x^2} \quad \text{M1 A1}$$

$$F'(x) = 0 \text{ gives } 2 = e^{3x^2} \quad \text{M1}$$

$$x = \pm \sqrt{\frac{\ln 2}{3}} \quad \text{A1}$$

$$\text{Answer: } x = \pm \sqrt{\frac{\ln 2}{3}} \quad \text{4 marks}$$

12 (3 marks)

$$\text{The limit has the form } \frac{0}{0} \quad \text{B1}$$

$$\text{L'Hôpital: differentiate numerator and denominator: } \frac{\cos(x^2)}{1} \quad \text{M1}$$

$$\text{As } x \rightarrow 0 \text{ this tends to } 1 \quad \text{A1}$$

$$\text{Answer: } 1 \quad \text{3 marks}$$

13 (3 marks)

$$F'(x) = \cos^3 x \times (-\sin x) - \sin^3 x \times \cos x \quad \text{M1 A1}$$

$$= -\sin x \cos x (\cos^2 x + \sin^2 x) = -\sin x \cos x \quad \text{A1}$$

$$\text{Answer: } F'(x) = -\sin x \cos x \quad \text{3 marks}$$

14 (5 marks)Put $x = 1$: $f(1) = 2$

B1

Differentiate: $f'(x) = x f(x)$

M1 A1

$$\frac{1}{f} \frac{df}{dx} = x \quad \ln f = \frac{x^2}{2} + c$$

M1

$$f(x) = 2e^{(x^2 - 1)/2}$$

A1

Answer: $f(x) = 2e^{(x^2 - 1)/2}$

5 marks

15 (6 marks)

$$I'(a) = \int_0^1 \frac{x^a \ln x}{\ln x} dx = \int_0^1 x^a dx$$

M1 A1

$$= \frac{1}{a+1}$$

A1

$$I(a) = \ln(a+1) + c$$

M1

$$I(0) = \int_0^1 0 dx = 0, \text{ so } c = 0$$

M1

$$I(a) = \ln(1+a)$$

A1

Answer: $I(a) = \ln(1+a)$

6 marks

16 (5 marks)

$$I'(a) = - \int_0^\infty x e^{-ax} \sin x dx$$

M1 A1

$$\frac{d}{da} \left(\frac{1}{1+a^2} \right) = - \frac{2a}{(1+a^2)^2}$$

M1

$$\text{So } \int_0^\infty x e^{-ax} \sin x dx = \frac{2a}{(1+a^2)^2}$$

A1

$$a = 2: \frac{4}{25}$$

B1

Answer: $\frac{4}{25}$

5 marks

17 (6 marks)

$$(a) F(x) = \left[-\frac{\cos(xt)}{x} \right]_0^x = \frac{1 - \cos(x^2)}{x}$$

M1 A1

$$F'(x) = \frac{2x^2 \sin(x^2) - (1 - \cos(x^2))}{x^2} = 2 \sin(x^2) - \frac{1 - \cos(x^2)}{x^2}$$

A1

Answer: $2 \sin(x^2) - \frac{1 - \cos(x^2)}{x^2}$

3 marks

$$(b) \frac{\partial}{\partial x} \sin(xt) = t \cos(xt)$$

B1

$$\int_0^x t \cos(xt) dt = \left[\frac{t \sin(xt)}{x} \right]_0^x - \int_0^x \frac{\sin(xt)}{x} dt = \sin(x^2) - \frac{1 - \cos(x^2)}{x^2}$$

M1

$$F'(x) = \sin(x^2) + \sin(x^2) - \frac{1 - \cos(x^2)}{x^2}, \text{ which agrees with (a)}$$

A1

Answer: Agrees

3 marks

18 (6 marks)

$$(a) F'(x) = 2e^{-4x^2} - e^{-x^2}$$

M1 A1

$$F'(x) = 0 \text{ gives } e^{3x^2} = 2, \text{ so } x = \sqrt{\frac{\ln 2}{3}} \quad (x > 0)$$

A1

Answer: $x = \sqrt{\frac{\ln 2}{3}}$

3 marks

$$(b) F''(x) = -16x e^{-4x^2} + 2x e^{-x^2} = 2x e^{-x^2} (1 - 8e^{-3x^2})$$

M1 A1

At the stationary point $e^{-3x^2} = \frac{1}{2}$, so $F'' = 2x e^{-x^2} (1 - 4) < 0$: a maximum

A1

Answer: Maximum

3 marks

19 (5 marks)

$$\frac{d}{dk} \int_0^{\infty} \frac{1}{x^2 + k} dx = \int_0^{\infty} \frac{\partial}{\partial k} \left(\frac{1}{x^2 + k} \right) dx = - \int_0^{\infty} \frac{1}{(x^2 + k)^2} dx$$

M1 A1

$$\frac{d}{dk} \left(\frac{\pi}{2} k^{-1/2} \right) = -\frac{\pi}{4} k^{-3/2}$$

M1

$$\text{Equating, } \int_0^{\infty} \frac{1}{(x^2 + k)^2} dx = \frac{\pi}{4k^{3/2}}$$

A1

$$\text{Put } k = 4: \frac{\pi}{4 \times 8} = \frac{\pi}{32}$$

A1

$$\text{Answer: } \frac{\pi}{32}$$

5 marks

20 (5 marks)

$$\text{Left: } \frac{d}{da} (a + \cos x)^{-1} = -(a + \cos x)^{-2}, \text{ so the derivative is } - \int_0^{\pi} \frac{1}{(a + \cos x)^2} dx$$

M1 A1

$$\text{Right: } \frac{d}{da} (\pi(a^2 - 1)^{-1/2}) = -\pi a(a^2 - 1)^{-3/2}$$

M1 A1

$$\text{Equate and cancel the minus signs; for } a = 2: \frac{2\pi}{3\sqrt{3}}$$

A1

$$\text{Answer: } \frac{2\pi}{3\sqrt{3}}$$

5 marks

21 (6 marks)

$$I'(a) = \int_0^{\infty} \frac{-x e^{-ax}}{x} dx = - \int_0^{\infty} e^{-ax} dx$$

M1 A1

$$= -\frac{1}{a}$$

A1

$$I(a) = -\ln a + c$$

M1

$$I(b) = \int_0^{\infty} 0 dx = 0, \text{ so } c = \ln b$$

M1

$$I(a) = \ln b - \ln a = \ln\left(\frac{b}{a}\right)$$

A1

$$\text{Answer: } I(a) = \ln\left(\frac{b}{a}\right)$$

6 marks

22 (8 marks)

$$(a) I'(a) = \int_0^{\infty} \frac{-x e^{-ax} \sin x}{x} dx = - \int_0^{\infty} e^{-ax} \sin x dx$$

M1

$$= -\frac{1}{1 + a^2}$$

A1

$$\text{Answer: } I'(a) = -\frac{1}{1 + a^2}$$

2 marks

$$(b) I(a) = -\arctan a + c$$

M1

$$I(a) \rightarrow 0 \text{ as } a \rightarrow \infty \text{ and } \arctan a \rightarrow \frac{\pi}{2}, \text{ so } c = \frac{\pi}{2}$$

M1

$$I(a) = \frac{\pi}{2} - \arctan a$$

A1

$$\text{Answer: } I(a) = \frac{\pi}{2} - \arctan a$$

3 marks

$$(c) a = 0: I(0) = \frac{\pi}{2}$$

B1

$$a = 1: \frac{\pi}{2} - \arctan 1$$

M1

$$= \frac{\pi}{4}$$

A1

$$\text{Answer: } \frac{\pi}{2}; \frac{\pi}{4}$$

3 marks

23 (6 marks)

$$(a) I'(a) = \int_0^1 x^a \ln x \, dx = -\frac{1}{(a+1)^2} \quad \text{M1}$$

$$I''(a) = \int_0^1 x^a (\ln x)^2 \, dx \quad \text{A1}$$

$$I''(a) = \frac{2}{(a+1)^3} \quad \text{A1}$$

$$\text{Answer: } \frac{2}{(a+1)^3} \quad \text{3 marks}$$

$$(b) I'''(a) = \int_0^1 x^a (\ln x)^3 \, dx \quad \text{B1}$$

$$I'''(a) = -\frac{6}{(a+1)^4} \quad \text{M1}$$

$$a = 2: -\frac{6}{81} = -\frac{2}{27} \quad \text{A1}$$

$$\text{Answer: } -\frac{2}{27} \quad \text{3 marks}$$

24 (6 marks)

$$\text{Leibniz: } F'(x) = \frac{e^{x \times x^2}}{x^2} \times 2x - \frac{e^{x \times x}}{x} \times 1 + \int_x^{x^2} \frac{\partial}{\partial x} \left(\frac{e^{xt}}{t} \right) dt \quad \text{M1 A1 A1}$$

$$\frac{\partial}{\partial x} \left(\frac{e^{xt}}{t} \right) = e^{xt} \quad \text{B1}$$

$$\int_x^{x^2} e^{xt} \, dt = \left[\frac{e^{xt}}{x} \right]_x^{x^2} = \frac{e^{x^3} - e^{x^2}}{x} \quad \text{M1}$$

$$F'(x) = \frac{2e^{x^3}}{x} - \frac{e^{x^2}}{x} + \frac{e^{x^3} - e^{x^2}}{x} = \frac{3e^{x^3} - 2e^{x^2}}{x} \quad \text{A1}$$

$$\text{Answer: } F'(x) = \frac{3e^{x^3} - 2e^{x^2}}{x} \quad \text{6 marks}$$

25 (6 marks)

$$(a) \text{ The integrand } t e^x \text{ depends on } x, \text{ so the term } \int_0^x \frac{\partial}{\partial x} (t e^x) \, dt \text{ must also be included in the} \quad \text{E1}$$

Leibniz rule

Answer: Missing the integral of the partial derivative 1 mark

$$(b) \text{ Leibniz: } x e^x + \int_0^x t e^x \, dt \quad \text{M1 A1}$$

$$= x e^x + e^x \frac{x^2}{2} \quad \text{M1 A1}$$

$$\text{Check: } \int_0^x t e^x \, dt = \frac{x^2}{2} e^x, \text{ whose derivative is } x e^x + \frac{x^2}{2} e^x \quad \text{B1}$$

$$\text{Answer: } e^x \left(x + \frac{x^2}{2} \right) \quad \text{5 marks}$$

26 (7 marks)

$$(a) \text{ Leibniz: } f'(x) = (x - x) f(x) + \int_0^x f(t) \, dt = \int_0^x f(t) \, dt \quad \text{M1 A1}$$

$$\text{Differentiate again: } f''(x) = f(x) \quad \text{M1 A1}$$

$$\text{Answer: } f'(x) = \int_0^x f(t) \, dt \quad \text{4 marks}$$

$$(b) f(0) = 1 + 0 = 1, f'(0) = 0 \quad \text{B1}$$

$$\text{General solution } f = A \cosh x + B \sinh x: A = 1, B = 0 \quad \text{M1}$$

$$f(x) = \cosh x \quad \text{A1}$$

$$\text{Answer: } f(x) = \cosh x \quad \text{3 marks}$$

27 (9 marks)

(a) Differentiate the integrand with respect to a : $\frac{\partial}{\partial a} \cos(ax) = -x \sin(ax)$

B1

Answer: $I'(a) = - \int_0^\infty x e^{-x^2} \sin(ax) \, dx$

1 mark

(b) $u = \sin(ax)$, $\frac{dv}{dx} = x e^{-x^2}$, $v = -\frac{1}{2} e^{-x^2}$

M1 A1

$$\int_0^\infty x e^{-x^2} \sin(ax) \, dx = \left[-\frac{1}{2} e^{-x^2} \sin(ax)\right]_0^\infty + \frac{a}{2} \int_0^\infty e^{-x^2} \cos(ax) \, dx$$

M1 A1

The bracket is 0, so the integral is $\frac{a}{2} I(a)$ and $I'(a) = -\frac{a}{2} I(a)$

A1

Answer: $I'(a) = -\frac{a}{2} I(a)$

5 marks

(c) $\frac{1}{I} \frac{dI}{da} = -\frac{a}{2}$, so $\ln I = -\frac{a^2}{4} + c$

M1

$I(0) = \frac{\sqrt{\pi}}{2}$, so $I = \frac{\sqrt{\pi}}{2} e^{-a^2/4}$

M1 A1

Answer: $I(a) = \frac{\sqrt{\pi}}{2} e^{-a^2/4}$

3 marks**28 (10 marks)**

(a) $\frac{\partial}{\partial a} \frac{\ln(1 + a \cos x)}{\cos x} = \frac{1}{\cos x} \times \frac{\cos x}{1 + a \cos x}$

M1

$= \frac{1}{1 + a \cos x}$

A1

Answer: $I'(a) = \int_0^{\pi/2} \frac{1}{1 + a \cos x} \, dx$

2 marks

(b) $\cos x = \frac{1-t^2}{1+t^2}$, $dx = \frac{2}{1+t^2} dt$, limits $t = 0$ to 1

B1

$1 + a \cos x = \frac{(1+a) + (1-a)t^2}{1+t^2}$, integrand $\frac{2}{(1+a) + (1-a)t^2} dt$

M1 A1

$= \frac{2}{\sqrt{1-a^2}} [\arctan(t \sqrt{\frac{1-a}{1+a}})]_0^1$

M1

$\arctan \sqrt{\frac{1-a}{1+a}} = \frac{1}{2} \arccos a$ (as $\tan(\frac{\theta}{2}) = \sqrt{\frac{1-\cos \theta}{1+\cos \theta}}$), giving $I'(a) = \frac{\arccos a}{\sqrt{1-a^2}}$

A1

Answer: $I'(a) = \frac{\arccos a}{\sqrt{1-a^2}}$

5 marks

(c) $I(0) = 0$ and $\frac{d}{da}(-\frac{1}{2}(\arccos a)^2) = \frac{\arccos a}{\sqrt{1-a^2}}$

M1 A1

$I(\frac{1}{2}) = \frac{\pi^2}{8} - \frac{1}{2}(\frac{\pi}{3})^2 = \frac{5\pi^2}{72}$

A1

Answer: $\frac{5\pi^2}{72}$

3 marks**29 (8 marks)**

(a) $F_n'(x) = \frac{1}{(n-1)!} [(x-x)^{n-1} f(x) + \int_0^x (n-1)(x-t)^{n-2} f(t) \, dt]$

M1 A1

The first term is 0 since $n-1 \geq 1$

B1

$= \frac{1}{(n-2)!} \int_0^x (x-t)^{n-2} f(t) \, dt = F_{n-1}(x)$

A1

Answer: $F_n'(x) = F_{n-1}(x)$ for $n \geq 2$

4 marks

(b) $F_3' = F_2$, $F_2' = F_1$, $F_1' = f$, with $F_n(0) = 0$

E1

$F_3(x) = \frac{1}{2} \int_0^x (x-t)^2 e^t \, dt$

M1

By parts twice (or by integrating e^x three times with zero constants): $F_1 = e^x - 1$, $F_2 = e^x - 1 - x$

M1

$F_3(x) = e^x - 1 - x - \frac{x^2}{2}$

A1

Answer: $e^x - 1 - x - \frac{x^2}{2}$

4 marks

30 (8 marks)

(a) $I'(a) = \int_0^\infty \frac{1}{x} \times \frac{x}{1+a^2 x^2} dx = \int_0^\infty \frac{1}{1+a^2 x^2} dx$
 $= \frac{1}{a} [\arctan(ax)]_0^\infty$
 $= \frac{\pi}{2a}$

M1 A1

M1

A1

Answer: $I'(a) = \frac{\pi}{2a}$

4 marks

(b) $I(a) = \frac{\pi}{2} \ln a + c$

M1

$I(b) = 0$ (the integrand is 0), so $c = -\frac{\pi}{2} \ln b$

M1

$I(a) = \frac{\pi}{2} \ln\left(\frac{a}{b}\right)$

A1

Answer: $I(a) = \frac{\pi}{2} \ln\left(\frac{a}{b}\right)$

3 marks

(c) $a = 3, b = 1: \frac{\pi}{2} \ln 3$

B1

Answer: $\frac{\pi}{2} \ln 3$

1 mark