

Further Differentiation

- 1 Differentiate $y = \sinh 4x$ with respect to x . [1]
- 2 Find the derivative of $y = \arctan 3x$. [2]
- 3 Differentiate $y = 3 \coth 2x$ with respect to x . [2]
- 4 Given that $y = \operatorname{arsinh} 2x$, find $\frac{dy}{dx}$. [2]
- 5 Differentiate $f(x) = \ln(\cosh x)$ and simplify your answer. [2]
- 6 Find the derivative of $y = \arccos\left(\frac{x}{3}\right)$ for $-3 < x < 3$. [2]
- 7 Find the derivative of $y = 4 \operatorname{sech} 2x$, giving your answer in terms of $\operatorname{sech} 2x$ and $\tanh 2x$. [3]
- 8 Differentiate $y = x \arctan x$. [3]
- 9 Find the derivative of $y = \sinh^3 x$. [2]
- 10 Find the derivative of $y = \operatorname{arcosh}(3x + 1)$ for $x > 0$. [3]
- 11 Differentiate $y = (\operatorname{arsinh} x)^2$. [3]
- 12 Find the exact gradient of the curve $y = x \sinh 2x$ at the point where $x = \ln 2$. [4]
- 13 Show that if $y = \frac{1 + \sinh x}{\cosh x}$ then $\frac{dy}{dx} = \frac{1 - \sinh x}{\cosh^2 x}$. [4]
- 14 Given that $y = \operatorname{arsinh} x$, so that $x = \sinh y$, show that $\frac{dy}{dx} = \frac{1}{\sqrt{1 + x^2}}$. [4]
- 15 Given that $y = \operatorname{arcosh} x$ for $x > 1$, use $x = \cosh y$ to show that $\frac{dy}{dx} = \frac{1}{\sqrt{x^2 - 1}}$. [4]
- 16 Show that if $y = \arctan\left(\frac{1 + x}{1 - x}\right)$ for $x \neq 1$ then $\frac{dy}{dx} = \frac{1}{1 + x^2}$. [5]
- 17 The curve C has equation $y = x - \tanh x$.
Show that C has exactly one stationary point, and determine its nature. [5]
- 18 Find the equation of the tangent to the curve $y = \arcsin x$ at the point where $x = \frac{1}{2}$.
Give your answer in the form $y = mx + c$, with exact values of m and c . [5]
- 19 Use logarithmic differentiation to find $\frac{dy}{dx}$ when $y = x^{\sinh x}$, for $x > 0$. [5]
- 20 Leibniz's theorem states that if $y = u v$ then $y_n = u_n v + n u_{n-1} v_1 + \frac{n(n-1)}{2} u_{n-2} v_2 + \dots$, where $u_r = \frac{d^r u}{dx^r}$.
The function y is given by $y = x^2 e^{3x}$.
(a) Use Leibniz's theorem, with $u = e^{3x}$ and $v = x^2$, to show that the n -th derivative of y is $e^{3x} 3^{n-2}(9x^2 + 6nx + n(n-1))$ for $n \geq 2$. [4]
(b) Hence find the value of $\frac{d^6 y}{dx^6}$ at $x = 0$. [2]

- 21** The curve C has equation $y = 5 \cosh x - 3 \sinh x$.
 (a) Show that C has a single stationary point, at $x = \ln 2$, and find its y coordinate. [5]
 (b) Determine the nature of this stationary point. [2]
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- 22** Show that if $y = \ln(\tanh \frac{x}{2})$ for $x > 0$, then $\frac{dy}{dx} = \operatorname{cosech} x$. [6]
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- 23** A curve has equation $y = \arctan(\sinh x)$.
 (a) Show that $\frac{dy}{dx} = \operatorname{sech} x$. [3]
 (b) Hence show that $\frac{d^2y}{dx^2} = -\operatorname{sech} x \tanh x$, and state the behaviour of the gradient as $x \rightarrow \infty$. [3]
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- 24** A student is asked to differentiate $y = \sinh(x^2)$ and writes:
 ' $\frac{dy}{dx} = \cosh(x^2)$ because the derivative of $\sinh$ is $\cosh$.'
 (a) Explain the error in the student's working. [1]
 (b) Find the correct derivative, and hence show that the curve $y = \sinh(x^2) - 4x$ has a stationary point where $x \cosh(x^2) = 2$. Verify that the stationary point lies between $x = 1$ and $x = 1.2$. [5]
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- 25** The function f is defined by $f(x) = 2x + \sinh x$ for $x \in \mathbb{R}$, and f is one-one so that f^{-1} exists.
 (a) Show that $f'(x) > 0$ for all x . [2]
 (b) Find $f(0)$ and hence the value of $(f^{-1})'(0)$. [2]
 (c) Find the exact value of $(f^{-1})'(a)$, where $a = f(\ln 3)$. [3]
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- 26** Given that $y = (\arcsin x)^2$ for $-1 < x < 1$, show that $(1 - x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} = 2$. [7]
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- 27** Given that $y = \arctan x$,
 (a) show that $(1 + x^2) \frac{dy}{dx} = 1$, [2]
 (b) hence show that $(1 + x^2) \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} = 0$, [2]
 (c) differentiate the result of (b) to show that $(1 + x^2) \frac{d^3y}{dx^3} + 4x \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} = 0$, [2]
 (d) find the value of $\frac{d^3y}{dx^3}$ when $x = 0$. [3]
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- 28** Given that $y = \sinh(k \operatorname{arsinh} x)$, where k is a constant and $x \in \mathbb{R}$,
 (a) show that $\sqrt{1 + x^2} \frac{dy}{dx} = k \cosh(k \operatorname{arsinh} x)$, [3]
 (b) hence show that $(1 + x^2) \frac{d^2y}{dx^2} + x \frac{dy}{dx} = k^2 y$. [7]
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- 29** The curve C has equation $y = \arcsin(\tanh x)$ for $x \in \mathbb{R}$.
 (a) Show that $\frac{dy}{dx} = \operatorname{sech} x$. [3]
 (b) Hence show that C has no stationary points, but has a point of inflection, and find its coordinates. [4]
 (c) Write down the limit of y as $x \rightarrow \infty$, and sketch C, labelling any asymptotes. [3]

- 30** A curve has equation $y = x \operatorname{arsinh} x - \sqrt{1 + x^2}$ for $x \in \mathbb{R}$.
- (a) Show that $\frac{dy}{dx} = \operatorname{arsinh} x$. [4]
- (b) Find the coordinates of the stationary point of C and show that it is a minimum. [4]
- (c) Find the exact gradient of C where $x = \frac{3}{4}$. [3]

Further Integration

- 1** Find $\int \frac{1}{x^2 + 9} dx$. [2]
- 2** Find $\int \frac{1}{\sqrt{4 - x^2}} dx$. [2]
- 3** Find $\int \frac{1}{\sqrt{x^2 + 16}} dx$, giving your answer in terms of an inverse hyperbolic function. [2]
- 4** Evaluate $\int_0^1 \frac{1}{1 + x^2} dx$, giving your answer in terms of π . [2]
- 5** Evaluate $\int_0^{3/2} \frac{1}{\sqrt{9 - x^2}} dx$, giving your answer in terms of π . [2]
- 6** Find $\int \cosh 2x dx$. [1]
- 7** Find the mean value of $f(x) = \frac{1}{\sqrt{x^2 + 9}}$ over the interval $0 \leq x \leq 4$, giving your answer in the form $k \ln 3$. [4]
- 8** Show that $\int_0^2 \frac{1}{\sqrt{x^2 + 4}} dx = \ln(1 + \sqrt{2})$. [3]
- 9** Evaluate $\int_2^3 \frac{1}{\sqrt{x^2 - 1}} dx$, giving your answer as a single logarithm. [4]
- 10** Use the substitution $u = x^2$ to evaluate $\int_0^1 \frac{x}{1 + x^4} dx$. [4]
- 11** Find $\int \frac{3x + 2}{x^2 + 1} dx$. [3]
- 12** By completing the square, find $\int \frac{1}{\sqrt{2x - x^2}} dx$ for $0 < x < 2$. [3]
- 13** Find the exact mean value of $f(x) = \cosh^2 x$ over the interval $0 \leq x \leq \ln 2$. [4]
- 14** Find the exact value of $\int_0^{\ln 2} \sinh^2 x dx$. [4]
- 15** Use integration by parts to show that $\int_0^1 \arctan x dx = \frac{\pi}{4} - \frac{1}{2} \ln 2$. [5]
- 16** Find the exact value of $\int_0^{1/2} \arcsin x dx$. [5]

- 17 Find $\int \frac{1}{\sqrt{x^2 + 4x + 13}} \, dx$. [4]
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- 18 Find $\int \frac{2x + 3}{\sqrt{x^2 + 2x + 5}} \, dx$. [5]
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- 19 Use the substitution $u = e^x$ to find the exact value of $\int_0^{\ln 2} \frac{e^x}{1 + e^{2x}} \, dx$. [4]
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- 20 Use the substitution $x = \sec \theta$ to show that $\int_{\sqrt{2}}^2 \frac{1}{x\sqrt{x^2 - 1}} \, dx = \frac{\pi}{12}$. [5]
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- 21 Express $\frac{x + 3}{(x + 1)(x^2 + 1)}$ in partial fractions, and hence find the exact value of $\int_0^1 \frac{x + 3}{(x + 1)(x^2 + 1)} \, dx$. [7]
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- 22 Use the substitution $x = 2 \sin \theta$ to find the exact value of $\int_0^1 \frac{x^2}{\sqrt{4 - x^2}} \, dx$. [6]
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- 23 Use the substitution $x = \sinh u$ to show that $\int_0^1 \sqrt{1 + x^2} \, dx = \frac{1}{2}(\sqrt{2} + \ln(1 + \sqrt{2}))$. [6]
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- 24 Use integration by parts to show that $\int_0^1 \operatorname{arsinh} x \, dx = \ln(1 + \sqrt{2}) - \sqrt{2} + 1$. [6]
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- 25 A student writes: ' $\int_0^1 \frac{1}{9 - x^2} \, dx = [\frac{1}{3} \arcsin(\frac{x}{3})]_0^1$ '.
- (a) Explain why this is incorrect. [1]
- (b) Use partial fractions to find the correct exact value of the integral. [6]
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- 26 The curve C has equation $y = \frac{6}{x^2 + 4x + 13}$.
- (a) Write $x^2 + 4x + 13$ in the form $(x + a)^2 + b$. [2]
- (b) Hence find the exact area of the region bounded by C, the x axis and the lines $x = -2$ and $x = 1$. [4]
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- 27 Show that $\int_0^1 x^2 \arctan x \, dx = \frac{\pi}{12} - \frac{1}{6} + \frac{1}{6} \ln 2$. [8]
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- 28 Use the substitution $x = 2 \tan \theta$ to find the exact value of $\int_0^2 \frac{1}{(x^2 + 4)^{3/2}} \, dx$. [7]
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- 29 Use the substitution $x = 2 \sinh u$ to show that $\int_0^2 \frac{x^2}{\sqrt{x^2 + 4}} \, dx = 2\sqrt{2} - 2 \ln(1 + \sqrt{2})$. [8]
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- 30 Given that $\frac{3x^2 + 5x + 12}{(x + 1)(x^2 + 4)} \equiv \frac{A}{x + 1} + \frac{Bx + C}{x^2 + 4}$,
- (a) find the values of the constants A, B and C, [4]
- (b) hence show that $\int_0^2 \frac{3x^2 + 5x + 12}{(x + 1)(x^2 + 4)} \, dx = 2 \ln 3 + \frac{1}{2} \ln 2 + \frac{\pi}{2}$. [5]
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Integration by Reduction Formulae

- 1 $I_n = \int_0^{\pi/2} \sin^n x \, dx$ and $I_n = \frac{n-1}{n} I_{n-2}$ for $n \geq 2$.
Given that $I_0 = \frac{\pi}{2}$, find I_4 in terms of π . [2]
- 2 $I_n = \int_0^1 x^n e^{-x} \, dx$ and $I_n = n I_{n-1} - e^{-1}$ for $n \geq 1$.
Find the exact value of I_2 . [3]
- 3 $T_n = \int_0^{\pi/4} \tan^n x \, dx$. Find the exact values of T_0 and T_1 . [3]
- 4 $C_n = \int_0^{\pi/2} \cos^n x \, dx$ and $C_n = \frac{n-1}{n} C_{n-2}$ for $n \geq 2$.
Given that $C_1 = 1$, find C_5 . [2]
- 5 $I_n = \int x^n e^{2x} \, dx$ and $I_n = \frac{1}{2} x^n e^{2x} - \frac{n}{2} I_{n-1}$ for $n \geq 1$.
Find I_1 , giving your answer without any integral signs. (Do not include a constant of integration.) [3]
- 6 $L_n = \int_1^e (\ln x)^n \, dx$ and $L_n = e - n L_{n-1}$ for $n \geq 1$.
Find the exact value of L_3 . [3]
- 7 $I_n = \int x^n e^x \, dx$, where n is a positive integer.
Show that $I_n = x^n e^x - n I_{n-1}$. [3]
- 8 $T_n = \int_0^{\pi/4} \tan^n x \, dx$ for integers $n \geq 2$.
Show that $T_n + T_{n-2} = \frac{1}{n-1}$. [4]
- 9 $I_n = \int_0^1 x^n e^{-x} \, dx$ for integers $n \geq 1$.
Show that $I_n = n I_{n-1} - e^{-1}$. [4]
- 10 $L_n = \int_1^e (\ln x)^n \, dx$ for integers $n \geq 1$.
Show that $L_n = e - n L_{n-1}$. [4]
- 11 $I_n = \int_0^{\pi/2} \sin^n x \, dx$ and $I_n = \frac{n-1}{n} I_{n-2}$ for $n \geq 2$.
Use this result to show that $\int_0^{\pi/2} \sin^5 x \cos^2 x \, dx = \frac{8}{105}$. [4]
- 12 $I_n = \int x^n e^{3x} \, dx$. Show that $I_n = \frac{1}{3} x^n e^{3x} - \frac{n}{3} I_{n-1}$, and hence find $\int x^2 e^{3x} \, dx$. [5]
- 13 Show that $\frac{d}{dx}(\sin^{n-1} x \cos x) = (n-1) \sin^{n-2} x - n \sin^n x$. [3]

- 14** $I_n = \int_0^{\pi/2} \sin^n x \, dx$, where n is an integer with $n \geq 2$.
- (a) Show that $I_n = \frac{n-1}{n} I_{n-2}$. [4]
- (b) Hence find the exact value of I_6 . [3]
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- 15** $I_n = \int_0^1 (1-x^2)^n \, dx$ for integers $n \geq 1$.
- (a) Show that $I_n = \frac{2n}{2n+1} I_{n-1}$. [5]
- (b) Hence find the exact value of I_3 . [2]
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- 16** $S_n = \int_0^{\pi/4} \sec^n x \, dx$ for integers $n \geq 2$.
- (a) Show that $(n-1) S_n = (\sqrt{2})^{n-2} + (n-2) S_{n-2}$. [5]
- (b) Hence find the exact value of S_4 . [2]
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- 17** $K_n = \int_0^1 x^n \sqrt{1-x} \, dx$ for integers $n \geq 1$.
- (a) Show that $K_n = \frac{2n}{2n+3} K_{n-1}$. [5]
- (b) Hence find the exact value of K_2 . [2]
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- 18** $I_n = \int_0^1 x^n e^{-x^2} \, dx$ for integers $n \geq 2$.
- (a) Show that $I_n = \frac{n-1}{2} I_{n-2} - \frac{1}{2e}$. [4]
- (b) Hence show that $I_3 = \frac{1}{2} - e^{-1}$. [3]
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- 19** $K_n = \int_{\pi/4}^{\pi/2} \cot^n x \, dx$ for integers $n \geq 0$.
- (a) Show that $K_n = \frac{1}{n-1} - K_{n-2}$ for $n \geq 2$. [3]
- (b) Find the exact value of K_1 and hence of K_5 . [3]
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- 20** $J_n = \int_0^{\ln 3} \tanh^n x \, dx$ for integers $n \geq 0$.
- (a) Show that $J_n = J_{n-2} - \frac{1}{n-1} \left(\frac{4}{5}\right)^{n-1}$ for $n \geq 2$. [4]
- (b) Hence find the exact value of J_4 . [3]
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- 21** $K_n = \int_0^{\pi/2} x^n \sin x \, dx$ for integers $n \geq 2$.
- (a) Show that $K_n = n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1) K_{n-2}$. [6]
- (b) Hence find the exact value of K_3 . [2]
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- 22** $I_n = \int_0^1 \frac{1}{(1+x^2)^n} \, dx$ for integers $n \geq 1$.
- (a) Show that $I_{n+1} = \frac{1}{n} \frac{1}{2^{n+1}} + \frac{2n-1}{2n} I_n$. [5]
- (b) Hence find the exact value of $\int_0^1 \frac{1}{(1+x^2)^2} \, dx$. [2]
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- 23** $C_n = \int_0^{\ln 2} \cosh^n x \, dx$ for integers $n \geq 2$.
- (a) Show that $n C_n = \cosh^{n-1}(\ln 2) \sinh(\ln 2) + (n - 1) C_{n-2}$. [5]
- (b) Hence find the exact value of C_4 . [3]
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- 24** $M_n = \int_1^e x^2 (\ln x)^n \, dx$ for integers $n \geq 1$.
- (a) Show that $M_n = \frac{e^3}{3} - \frac{n}{3} M_{n-1}$. [4]
- (b) Hence show that $M_3 = \frac{2}{27}(2e^3 + 1)$. [3]
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- 25** $B(m, n) = \int_0^1 x^m (1 - x)^n \, dx$, where m and n are non-negative integers.
- (a) Show that for $n \geq 1$, $B(m, n) = \frac{n}{m+1} B(m+1, n-1)$. [4]
- (b) Hence find the exact value of $B(2, 3)$. [3]
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- 26** $T_n = \int_0^{\pi/4} \tan^n x \, dx$. A student writes:
- $$T_n = \int_0^{\pi/4} \tan^{n-2} x (\sec^2 x - 1) \, dx = \frac{1}{n-1} + T_{n-2}$$
- (a) Identify the error in the student's working. [2]
- (b) Using the correct relation, find the exact value of T_6 . [4]
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- 27** For integers m, n with $m \geq 0$ and $n \geq 2$, let $W(m, n) = \int_0^{\pi/2} \sin^m x \cos^n x \, dx$.
- (a) Show that $W(m, n) = \frac{n-1}{m+n} W(m, n-2)$. [6]
- (b) Hence find the exact value of $\int_0^{\pi/2} \sin^4 x \cos^6 x \, dx$. [3]
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- 28** $I_n = \int_0^{\pi/2} \sin^n x \, dx$, where n is a non-negative integer. It is given that $I_n = \frac{n-1}{n} I_{n-2}$ for $n \geq 2$.
- (a) Show that $n I_n I_{n-1} = (n-1) I_{n-1} I_{n-2}$ for $n \geq 2$. [2]
- (b) Hence prove that $n I_n I_{n-1} = \frac{\pi}{2}$ for all $n \geq 1$. [3]
- (c) Given that $I_4 = \frac{3\pi}{16}$, find the exact value of I_5 and verify your result using the reduction formula. [4]
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- 29** $A_n = \int_0^1 (\arcsin x)^n \, dx$ for integers $n \geq 0$.
- (a) Use the substitution $x = \sin \theta$ to show that $A_n = \int_0^{\pi/2} \theta^n \cos \theta \, d\theta$. [2]
- (b) Show that for $n \geq 2$, $A_n = \left(\frac{\pi}{2}\right)^n - n(n-1) A_{n-2}$. [6]
- (c) Hence find the exact value of $\int_0^1 (\arcsin x)^2 \, dx$. [2]
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- 30** $I_n = \int_0^{\pi/2} \sin^n x \, dx$.
- (a) Given that $I_n = \frac{n-1}{n} I_{n-2}$ for $n \geq 2$, prove by induction that $I_{2m} = \frac{(2m)! \pi}{2^{2m+1} (m!)^2}$ for all integers $m \geq 0$. [7]
- (b) Hence evaluate I_8 , and check it using the reduction formula. [2]

Improper Integrals

- 1 Evaluate $\int_1^{\infty} \frac{1}{x^2} dx$. [2]
- 2 Evaluate $\int_0^{\infty} e^{-2x} dx$. [2]
- 3 Evaluate $\int_0^1 \frac{1}{\sqrt{x}} dx$. [2]
- 4 Show that $\int_1^{\infty} \frac{1}{x} dx$ does not converge. [2]
- 5 Evaluate $\int_0^{\infty} \frac{1}{1+x^2} dx$. [2]
- 6 State, giving a reason in each case, which of the following integrals are improper.
 - (i) $\int_0^1 \ln x dx$
 - (ii) $\int_1^2 \frac{1}{x} dx$
 - (iii) $\int_0^{\infty} e^{-x} dx$
 - (iv) $\int_0^1 x^2 dx$
 [2]
- 7 Evaluate $\int_0^{\infty} x e^{-x^2} dx$. [3]
- 8 Show that $\int_0^1 \ln x dx = -1$. [3]
- 9 Show that $\int_2^{\infty} \frac{1}{x^2 - 1} dx = \frac{1}{2} \ln 3$. [4]
- 10 Evaluate $\int_0^{\infty} x e^{-x} dx$. [3]
- 11 Evaluate $\int_0^3 \frac{1}{\sqrt{9-x^2}} dx$, giving your answer in terms of π . [3]
- 12 Given that p is a constant with $p > 1$, find $\int_1^{\infty} x^{-p} dx$ in terms of p . [4]
- 13 Find the exact value of $\int_1^{\infty} \frac{1}{x(x+1)} dx$. [4]
- 14 Show that $\int_0^{\infty} x^2 e^{-x} dx = 2$. [5]
- 15 Find the exact value of $\int_0^1 x \ln x dx$. [5]
- 16 Show that $\int_0^{\infty} \frac{1}{x^2 + 2x + 2} dx = \frac{\pi}{4}$. [4]
- 17 Show that the integral $\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$ converges, and find its value. [4]

- 18** Show that $\int_0^{\infty} e^{-x} \sin x \, dx = \frac{1}{2}$. [6]
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- 19** Find the exact value of $\int_0^1 \frac{\ln x}{\sqrt{x}} \, dx$. [5]
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- 20** Show that $\int_1^{\infty} \frac{\ln x}{x^2} \, dx = 1$. [5]
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- 21** Use the substitution $x = \tan \theta$ to show that $\int_0^{\infty} \frac{1}{(1+x^2)^2} \, dx = \frac{\pi}{4}$. [6]
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- 22** Consider the integral $\int_{-\infty}^{\infty} \frac{x}{1+x^2} \, dx$.
- (a) Show that $\int_0^{\infty} \frac{x}{1+x^2} \, dx$ diverges. [3]
- (b) A student argues that, because $x/(1+x^2)$ is an odd function, the integral over $(-\infty, \infty)$ is 0. Explain why this is not valid. [3]
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- 23** Show that $\int_0^{\infty} \left(\frac{3}{1+3x} - \frac{x}{1+x^2} \right) \, dx = \ln 3$, showing clearly the limiting process. [6]
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- 24** Find the exact value of $\int_0^1 (\ln x)^2 \, dx$. [6]
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- 25** A student calculates $\int_{-1}^1 \frac{1}{x^2} \, dx = \left[-\frac{1}{x} \right]_{-1}^1 = -1 - 1 = -2$.
- (a) Explain why this answer cannot be correct. [2]
- (b) Show that $\int_{-1}^1 \frac{1}{x^2} \, dx$ diverges. [2]
- (c) Evaluate $\int_{-1}^1 \frac{1}{x^{2/3}} \, dx$. [3]
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- 26** $I_n = \int_0^{\infty} x^n e^{-x} \, dx$ for integers $n \geq 1$.
- (a) Show that $I_n = n I_{n-1}$. [4]
- (b) Hence find I_4 . [2]
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- 27** Use the substitution $u = \sqrt{x}$ to show that $\int_0^{\infty} \frac{1}{(x+1)\sqrt{x}} \, dx = \pi$. [7]
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- 28** $I = \int_0^{\pi/2} \ln(\sin x) \, dx$. The integrand is undefined at $x = 0$, and it may be assumed that I converges.
- Let $J = \int_0^{\pi/2} \ln(\cos x) \, dx$.
- (a) Show that $I = J$. [2]
- (b) Show that $I + J = \int_0^{\pi/2} \ln(\sin 2x) \, dx - \frac{\pi}{2} \ln 2$. [3]
- (c) Show that $\int_0^{\pi/2} \ln(\sin 2x) \, dx = I$, and hence find the exact value of I . [5]
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- 29** For constants $p > 0$ and $q > 0$, let $I = \int_0^{\infty} e^{-px} \sin(qx) \, dx$.
- (a) Show that $I = \frac{q}{p^2 + q^2}$. [7]
- (b) Hence find the exact value of $\int_0^{\infty} e^{-2x} \sin 3x \, dx$. [1]
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- 30** Show that $\int_1^{\infty} \frac{\arctan x}{x^2} dx = \frac{\pi}{4} + \frac{1}{2} \ln 2$. [9]

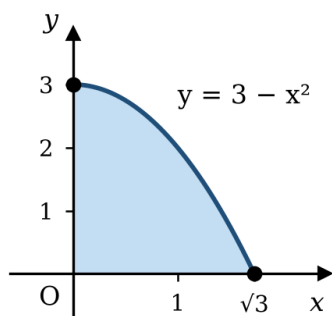
Volumes of Revolution

- 1** The region bounded by the curve $y = x^2$, the x axis and the line $x = 2$ is rotated through 2π radians about the x axis. Find the exact volume generated. [3]
- 2** The region bounded by the curve $y = \sqrt{x}$, the x axis and the line $x = 4$ is rotated through 360° about the x axis. Find the exact volume of the solid formed. [3]
- 3** The region bounded by $y = e^x$, the axes and the line $x = 1$ is rotated through 2π radians about the x axis. Find the exact volume generated. [3]
- 4** The line $y = 3$ for $0 \leq x \leq 4$ is rotated through 2π radians about the x axis. Describe the solid formed and use integration to find its exact volume. [3]
- 5** Find the exact volume generated when the region between $y = \sin x$ ($0 \leq x \leq \pi$) and the x axis is rotated through 2π radians about the x axis. [4]
- 6** The region bounded by the curve $y = x^2$ ($x \geq 0$), the y axis and the line $y = 4$ is rotated through 2π radians about the y axis. Find the exact volume generated. [3]
- 7** The region bounded by $y = \frac{1}{x}$, the x axis and the lines $x = 1$ and $x = 3$ is rotated through 2π radians about the x axis. Show that the volume generated is $\frac{2\pi}{3}$. [3]
- 8** A solid has a circular base with equation $x^2 + y^2 = 9$ in the x - y plane. Every cross-section of the solid perpendicular to the x axis is a square, with one side lying in the base. Find the volume of the solid. [4]
- 9** The region bounded by the curve $y = \ln x$, the y axis and the lines $y = 0$ and $y = 2$ is rotated through 2π radians about the y axis. Find the exact volume generated. [4]
- 10** The curves $y = x$ and $y = x^2$ intersect at the origin and at the point $(1, 1)$. The region between the curves is rotated through 2π radians about the x axis. Find the exact volume of the solid generated. [4]
- 11** The region enclosed by the curve $y = 4 - x^2$ and the x axis is rotated through 2π radians about the x axis. Find the exact volume generated. [4]
- 12** The region bounded by $y = \cosh x$, the axes and the line $x = 1$ is rotated through 2π radians about the x axis. Show that the volume generated is $\frac{\pi}{4}(2 + \sinh 2)$. [4]
- 13** The region bounded by $y = \tan x$, the x axis and the line $x = \frac{\pi}{4}$ is rotated through 2π radians about the x axis. Find the exact volume generated. [4]
- 14** The region bounded by $y = x e^x$, the x axis and the line $x = 1$ is rotated through 2π radians about the x axis. Show that the volume generated is $\frac{\pi}{4}(e^2 - 1)$. [5]

- 15** The region bounded by $y = \frac{1}{1+x^2}$, the axes and the line $x = 1$ is rotated through 2π radians about the x axis. Find the exact volume generated. [5]

- 16** The region bounded by $y = \sqrt{\ln x}$, the x axis and the line $x = e$ is rotated through 360° about the x axis. Find the exact volume generated. [5]

- 17** The diagram shows the region R bounded by the curve $y = 3 - x^2$, the x axis and the y axis, for $x \geq 0$.
The region R is rotated through 2π radians about the y axis. Find the exact volume of the solid generated.



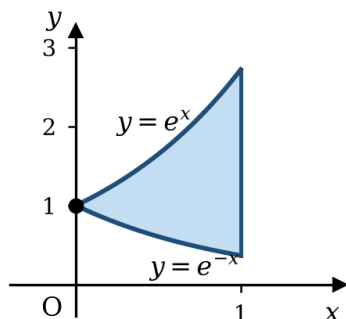
[4]

- 18** The region bounded by the curves $y = \sin x$ and $y = \cos x$ and the y axis, for $0 \leq x \leq \frac{\pi}{4}$, is rotated through 2π radians about the x axis.
Show that the volume generated is $\frac{\pi}{2}$. [4]

- 19** The region R , for $x \geq 0$, lies between the x axis, the y axis and the curve $y = e^{-x^2}$.
The region R is rotated through 2π radians about the y axis. Show that the volume of the solid generated is π . [5]

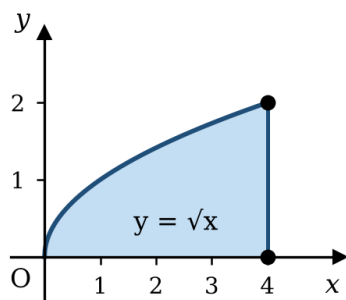
- 20** A solid is formed by rotating about the x axis the part of the circle $x^2 + y^2 = 25$ that lies above the x axis between $x = 2$ and $x = 5$.
Find the exact volume of this solid, which is a spherical cap. [4]

- 21** The diagram shows the region R bounded by the curves $y = e^x$ and $y = e^{-x}$, the y axis and the line $x = 1$.
The region R is rotated through 2π radians about the x axis. Show that the volume generated is $\pi(\cosh 2 - 1)$.



[6]

- 22** The diagram shows the region S bounded by the curve $y = \sqrt{x}$, the line $x = 4$ and the x axis. The region S is rotated through 2π radians about the y axis. Find the exact volume generated.



[6]

- 23** (In this question a, r and h are positive constants.)

- (a) The circle $x^2 + y^2 = r^2$ is rotated through 2π radians about the x axis. Use integration to show that the volume of the sphere formed is $\frac{4}{3}\pi r^3$.
- (b) The line $y = \frac{r}{h}x$ for $0 \leq x \leq h$ is rotated through 2π radians about the x axis. Show that the volume of the cone formed is $\frac{1}{3}\pi r^2 h$.

[3]

[3]

- 24** The finite region R is bounded by the curve $y = (4 - x)(x - 2)$ and the x axis. The region R is rotated through 2π radians about the y axis. Find the exact volume of the solid generated.

[6]

- 25** The region bounded by $y = x^2 + 1$, the x axis, the y axis and the line $x = 1$ is rotated through 2π radians about the x axis. A student writes:

$$V = \pi \int_0^1 (x^2 + 1) dx = \pi \left[\frac{x^3}{3} + x \right]_0^1 = \frac{4\pi}{3}$$

- (a) Identify the error in the student's working.
- (b) Find the correct volume.

[1]

[4]

- 26** The region bounded by $y = \operatorname{sech} x$, the axes and the line $x = \ln 3$ is rotated through 2π radians about the x axis.

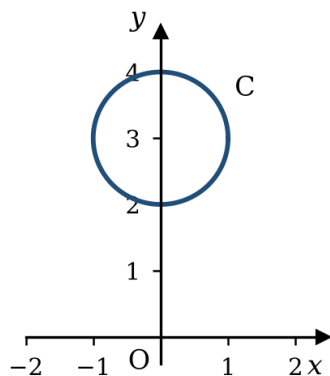
- (a) Write down $\frac{d}{dx}(\tanh x)$.

[1]

- (b) Hence show that the volume generated is $\frac{4\pi}{5}$.

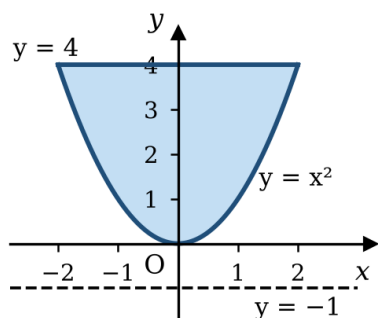
[5]

- 27** The circle C has equation $x^2 + (y - 3)^2 = 1$ and is shown in the diagram. The circle is rotated through 2π radians about the x axis to form a solid torus (ring shape). Show that the volume of the torus is $6\pi^2$.



[8]

- 28** The diagram shows the region bounded by the curve $y = x^2$ and the line $y = 4$. This region is rotated through 2π radians about the line $y = -1$. Find the exact volume of the solid generated.



[7]

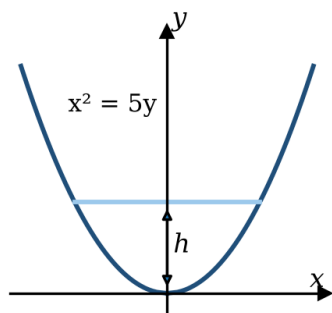
- 29** The region bounded by the curve $y = x \sin x$ for $0 \leq x \leq \pi$ and the x axis is rotated through 2π radians about the x axis.

Show that the volume generated is $\frac{\pi^2}{12}(2\pi^2 - 3)$.

[10]

- 30** A bowl is made by rotating through 2π radians about the y axis the part of the curve $x^2 = 5y$ between $y = 0$ and $y = 10$, where x and y are in centimetres. The bowl is held with its axis vertical and the vertex at the bottom, as shown.

Water is poured into the bowl so that the depth of water is h cm.



- (a) Show that the volume of water in the bowl is $\frac{5\pi}{2} h^2 \text{ cm}^3$, and find the capacity of the bowl in cm^3 , giving your answer to the nearest cm^3 .
- (b) Water flows into the bowl at a constant rate of 20 cm^3 per second. Find the rate at which the depth of the water is rising when $h = 4$. Give your answer in cm per second to 3 significant figures.

[4]

[5]

Arc Length

- 1** Use the arc length formula to find the length of the line $y = 3x - 2$ between $x = 1$ and $x = 5$. Give your answer as a surd.
- 2** The curve C has equation $y = x^2$ for $0 \leq x \leq 1$. Write down a definite integral for the length of C . (Do not evaluate it.)
- 3** Given that $y = \cosh x$, show that $1 + \left(\frac{dy}{dx}\right)^2 = \cosh^2 x$.
- 4** A curve is given parametrically by $x = 3t$, $y = 4t$, for $0 \leq t \leq 2$. Use the parametric arc length formula to find the length of the curve, and state what kind of curve it is.

[3]

[2]

[2]

[3]

- 5** Given that $y = \ln(\sec x)$ for $0 \leq x < \frac{\pi}{2}$, show that $1 + \left(\frac{dy}{dx}\right)^2 = \sec^2 x$. [2]
- 6** A curve has parametric equations $x = 2 \cos t$, $y = 2 \sin t$, for $0 \leq t \leq \frac{\pi}{2}$. Find its length. [3]
- 7** Find the exact length of the curve $y = x^{3/2}$ between $x = 0$ and $x = 4$. [4]
- 8** Find the exact length of the curve $y = \cosh x$ between $x = 0$ and $x = \ln 3$. [3]
- 9** Show that the length of the curve $y = \ln(\sec x)$ between $x = 0$ and $x = \frac{\pi}{3}$ is $\ln(2 + \sqrt{3})$. [4]
- 10** Find the exact length of the curve $y = \ln(\tanh \frac{x}{2})$ between $x = \ln 2$ and $x = \ln 3$. [4]
- 11** Find the length of the arc of the circle $x = 5 \cos t$, $y = 5 \sin t$ between $t = 0$ and $t = \frac{\pi}{3}$, and check your answer using the formula for the arc of a circle. [3]
- 12** A curve is given by $x = 3t^2$, $y = 2t^3$ for $0 \leq t \leq 1$. Show that its length is $4\sqrt{2} - 2$. [4]
- 13** Write down an integral for the length of one arch of the curve $y = \sin x$, from $x = 0$ to $x = \pi$, and use your calculator to evaluate it correct to 3 significant figures. [3]
- 14** The curve C has equation $y = \frac{x^2}{2} - \frac{1}{4} \ln x$ for $x > 0$.
Show that the length of C between $x = 1$ and $x = 2$ is $\frac{3}{2} + \frac{1}{4} \ln 2$. [5]
- 15** Find the length of the curve $y = \frac{x^3}{6} + \frac{1}{2x}$ between $x = 1$ and $x = 3$. [5]
- 16** A curve has parametric equations $x = t^2$, $y = t^3$, for $0 \leq t \leq 2$. Find the exact length of the curve. [5]
- 17** A curve is given by $x = e^t \cos t$, $y = e^t \sin t$, for $0 \leq t \leq 1$. Show that its length is $\sqrt{2}(e - 1)$. [5]
- 18** Find the exact length of the curve $y = \ln(1 - x^2)$ from $x = 0$ to $x = \frac{1}{2}$. [5]
- 19** Find the exact length of the curve $y = \ln(\sin x)$ between $x = \frac{\pi}{3}$ and $x = \frac{\pi}{2}$. [5]
- 20** Show that the length of the curve $y = \frac{1}{3}(x^2 + 2)^{3/2}$ between $x = 0$ and $x = 3$ is 12. [5]
- 21** One arch of a cycloid has parametric equations $x = t - \sin t$, $y = 1 - \cos t$, for $0 \leq t \leq 2\pi$. Show that the length of the arch is 8. [6]
- 22** The astroid has parametric equations $x = \cos^3 t$, $y = \sin^3 t$. Find the length of the part of the astroid in the first quadrant ($0 \leq t \leq \frac{\pi}{2}$) and hence write down the total perimeter of the astroid. [6]
- 23** The cardioid with polar equation $r = 2(1 + \cos \theta)$, $0 \leq \theta \leq 2\pi$, has arc length given by $s = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$.
Show that the perimeter of the cardioid is 16. [7]

- 24** A student writes: 'The length of $y = x^2$ from $x = 0$ to $x = 1$ is $\int_0^1 \sqrt{1 + 2x} \, dx$.'
- (a) Identify the error. [1]
- (b) Use the substitution $2x = \sinh u$ to find the correct length, giving the exact answer. [5]
-
- 25** A curve has parametric equations $x = \frac{t^2}{2}$, $y = t$, for $0 \leq t \leq \sqrt{3}$. Show that its length is $\sqrt{3} + \frac{1}{2} \ln(2 + \sqrt{3})$. [6]
-
- 26** A cable hangs between two supports at the same height. Taking the lowest point of the cable as the origin, with x and y in metres, its shape is $y = 20 \cosh(\frac{x}{20}) - 20$. The supports are at $x = -30$ and $x = 30$.
- (a) Find the vertical distance between the supports and the lowest point of the cable, in metres to 3 significant figures. [2]
- (b) Show that the length of the cable is $40 \sinh 1.5$ metres, and evaluate this to 3 significant figures. [4]
-
- 27** Show that the length of the curve $y = \ln x$ between $x = 1$ and $x = \sqrt{3}$ is $2 - \sqrt{2} + \ln(1 + \sqrt{2}) - \frac{1}{2} \ln 3$. [9]
-
- 28** The tractrix has parametric equations $x = t - \tanh t$, $y = \operatorname{sech} t$, for $t \geq 0$. Show that the length of the curve from $t = 0$ to $t = \ln 3$ is $\ln(\frac{5}{3})$. [8]
-
- 29** The curve with parametric equations $x = \cos t + t \sin t$, $y = \sin t - t \cos t$, for $0 \leq t \leq 2\pi$, is called an involute of the unit circle.
- (a) Show that $\frac{dx}{dt} = t \cos t$ and find $\frac{dy}{dt}$. [3]
- (b) Hence find the exact length of the curve. [4]
- (c) Show that the tangent to the curve at the point with parameter t has gradient $\tan t$, and explain what this says about the direction of travel. [2]
-
- 30** The curve $y = a \cosh(\frac{x}{a})$, where a is a positive constant, is called a catenary. For $x \geq 0$, let s be the length of the curve from $x = 0$ to a general point (x, y) and let A be the area of the region bounded by the curve, the axes and the line through (x, y) parallel to the y axis.
- (a) Show that $s = a \sinh(\frac{x}{a})$. [4]
- (b) Show that $A = a s$. [2]
- (c) Hence find the area A when $a = 5$ and the arc length is 12. [3]

Surface Area of Revolution

- 1** The line $y = 2x$ for $0 \leq x \leq 3$ is rotated through 2π radians about the x axis. Use $S = 2\pi \int \sqrt{1 + (\frac{dy}{dx})^2} \, dx$ to find the exact area of the curved surface formed. [3]
-
- 2** The line $y = 3$, for $0 \leq x \leq 5$, is rotated through 2π radians about the x axis. Use integration to find the area of the surface formed, and state which solid shape it is the curved surface of. [3]
-
- 3** The curve $y = x^2$ for $0 \leq x \leq 1$ is rotated through 2π radians about the x axis. Write down, but do not evaluate, an integral for the area of the surface generated. [2]

- 4** A line has parametric equations $x = 3t$, $y = 4t$, for $0 \leq t \leq 1$. It is rotated through 2π radians about the x axis. Use $S = 2\pi \int y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$ to find the area of the surface formed. [3]
-
- 5** Given that $y = \cosh x$, show that $y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \cosh^2 x$. [2]
-
- 6** The line $y = \frac{x}{2}$ for $0 \leq x \leq 4$ is rotated through 2π radians about the y axis. Use $S = 2\pi \int x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$ to find the exact area of the surface formed. [3]
-
- 7** The semicircle $y = \sqrt{4 - x^2}$ for $-2 \leq x \leq 2$ is rotated through 2π radians about the x axis to form a sphere. Show that the surface area of the sphere is 16π . [4]
-
- 8** The curve $y = x^3$ for $0 \leq x \leq 1$ is rotated through 2π radians about the x axis. Show that the area of the surface generated is $\frac{\pi}{27}(10\sqrt{10} - 1)$. [4]
-
- 9** The curve $y = \cosh x$ for $0 \leq x \leq \ln 2$ is rotated through 2π radians about the x axis. Find the exact area of the surface generated. [4]
-
- 10** The curve $y = \sqrt{x}$ for $0 \leq x \leq 3$ is rotated through 2π radians about the x axis. Find the exact area of the surface generated. [4]
-
- 11** The curve $y = x^2$ for $0 \leq x \leq 1$ is rotated through 2π radians about the y axis. Find the exact area of the surface generated. [4]
-
- 12** The curve $y = 2\sqrt{x}$ for $0 \leq x \leq 3$ is rotated through 2π radians about the x axis. Show that the area of the surface generated is $\frac{56\pi}{3}$. [4]
-
- 13** The curve $x = 2 \cos t$, $y = 2 \sin t$, for $0 \leq t \leq \frac{\pi}{2}$, is rotated through 2π radians about the x axis. Find the exact area of the surface formed and name the shape of the surface. [3]
-
- 14** A curve has parametric equations $x = t^2$, $y = 2t$, for $0 \leq t \leq 1$. It is rotated through 2π radians about the x axis.
Show that the area of the surface generated is $\frac{8\pi}{3}(2\sqrt{2} - 1)$. [5]
-
- 15** The curve C has equation $y = \frac{x^3}{3} + \frac{1}{4x}$ for $1 \leq x \leq 2$. The curve is rotated through 2π radians about the x axis.
Find the exact area of the surface generated. [6]
-
- 16** A cone has base radius r and perpendicular height h . The cone is formed by rotating the line $y = \frac{rx}{h}$, for $0 \leq x \leq h$, through 2π radians about the x axis.
Show that the area of the curved surface of the cone is $\pi r \sqrt{r^2 + h^2}$. [4]
-
- 17** The circle $x^2 + y^2 = 25$ is rotated about the x axis to form a sphere of radius 5. The part of the sphere between the planes $x = 1$ and $x = 4$ is called a zone.
Show that the area of the surface of this zone is 30π . [4]
-
- 18** The curve $y = \cosh x$ for $0 \leq x \leq 1$ is rotated through 2π radians about the y axis. Show that the area of the surface generated is $2\pi(1 - e^{-1})$. [5]
-

- 19** The curve $y = \frac{x^2}{4} - \frac{1}{2} \ln x$ for $1 \leq x \leq 3$ is rotated through 2π radians about the y axis. Find the exact area of the surface generated. [5]
-
- 20** The semicircle $y = \sqrt{r^2 - x^2}$ for $-r \leq x \leq r$, where $r > 0$, is rotated through 2π radians about the x axis to form a sphere.
- (a) Show that $y\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = r$. [2]
- (b) Hence prove that the surface area of the sphere is $4\pi r^2$. [2]
-
- 21** One arch of a cycloid has parametric equations $x = t - \sin t$, $y = 1 - \cos t$, for $0 \leq t \leq 2\pi$. The arch is rotated through 2π radians about the x axis. Show that the area of the surface generated is $\frac{64\pi}{3}$. [7]
-
- 22** The astroid $x = \cos^3 t$, $y = \sin^3 t$, for $0 \leq t \leq \pi$ (the part above the x axis), is rotated through 2π radians about the x axis. Find the exact area of the surface generated. [6]
-
- 23** The curve $x = e^t \cos t$, $y = e^t \sin t$, for $0 \leq t \leq \pi$, is rotated through 2π radians about the x axis. Show that the area of the surface generated is $\frac{2\sqrt{2}\pi}{5}(e^{2\pi} + 1)$. [7]
-
- 24** A student attempts to find the area of the surface formed when the curve $y = x^2$, $0 \leq x \leq 1$, is rotated about the x axis, and writes:
' $S = 2\pi \int_0^1 x^2 \sqrt{1 + 2x} \, dx$ '
- (a) Identify the error. [1]
- (b) Write down the correct integral, and use a calculator to evaluate it to 3 significant figures. [3]
-
- 25** A reflecting dish is modelled by rotating the curve $y = \frac{x^2}{20}$, for $0 \leq x \leq 10$, through 2π radians about the y axis. The units are centimetres. Find the exact area of the curved surface of the dish, and give its value to 3 significant figures. [5]
-
- 26** A soap film forms a surface of revolution obtained by rotating the curve $y = 2 \cosh\left(\frac{x}{2}\right)$, for $-2 \leq x \leq 2$, through 2π radians about the x axis. Show that the area of the surface is $4\pi(2 + \sinh 2)$. [6]
-
- 27** The ellipse $\frac{x^2}{4} + \frac{y^2}{3} = 1$ is rotated through 2π radians about the x axis to form a solid called a prolate spheroid. Show that the area of the surface of the spheroid is $6\pi + \frac{4\sqrt{3}\pi^2}{3}$. [10]
-
- 28** A torus is formed by rotating the circle $x^2 + (y - R)^2 = r^2$, where $R > r > 0$, through 2π radians about the x axis. The circle has parametric equations $x = r \cos t$, $y = R + r \sin t$, for $0 \leq t \leq 2\pi$.
- (a) Show that the surface area of the torus is $4\pi^2 Rr$. [5]
- (b) The volume of the torus is $2\pi^2 Rr^2$. Find the exact value of $\frac{S}{V}$ and hence the value of r for which $\frac{S}{V} = 4$, giving your answer in terms of R if necessary. [3]

- 29** The tractrix $x = t - \tanh t$, $y = \operatorname{sech} t$, for $t \geq 0$, is rotated through 2π radians about the x axis to form a surface called a pseudosphere.
Show that the area of the surface is 2π . [8]

- 30** Gabriel's horn is formed by rotating the curve $y = \frac{1}{x}$, for $x \geq 1$, through 2π radians about the x axis.
(a) Show that the volume of the horn is finite, and find it. [3]
(b) Show that the area of the curved surface of the horn is infinite. [6]

Further Parametric Calculus

- 1** A curve has parametric equations $x = t^2$, $y = 2t$, for $t > 0$. Find $\frac{dy}{dx}$ in terms of t . [2]
- 2** A curve has parametric equations $x = 3 \cos t$, $y = 2 \sin t$. Find $\frac{dy}{dx}$ in terms of t , and find the gradient of the curve at the point where $t = \frac{\pi}{4}$. [3]
- 3** A curve has parametric equations $x = 2t + 1$, $y = t^2 - 3$. Find an equation of the tangent to the curve at the point where $t = 2$, giving your answer in the form $y = mx + c$. [3]
- 4** The curve C has parametric equations $x = t^2$, $y = t + 1$, for $0 \leq t \leq 2$. Find the area of the region bounded by C , the x axis and the lines $x = 0$ and $x = 4$. [3]
- 5** A curve has parametric equations $x = t^2$, $y = t^3$, for $t > 0$. Show that $\frac{dy}{dx} = \frac{3t}{2}$ and find $\frac{d^2y}{dx^2}$ in terms of t . [3]
- 6** A curve has parametric equations $x = 3t$, $y = 1 - t^2$. Find a Cartesian equation of the curve. [2]
- 7** An ellipse has parametric equations $x = 4 \cos \theta$, $y = 3 \sin \theta$. Find an equation of the tangent to the ellipse at the point where $\theta = \frac{\pi}{3}$. Give your answer in the form $y = mx + c$ with exact values of m and c . [5]
- 8** A curve has parametric equations $x = t^3$, $y = t^2$, for $t > 0$. Find an equation of the normal to the curve at the point where $t = 1$, giving your answer in the form $ax + by = c$, where a , b and c are integers. [4]
- 9** A curve has parametric equations $x = t^2 + 1$, $y = t^3 - 3t$. Find the coordinates of the points on the curve where the tangent is parallel to the x axis. [4]
- 10** A curve has parametric equations $x = 3 \sec t$, $y = 2 \tan t$. Find a Cartesian equation of the curve in the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. [3]
- 11** A curve has parametric equations $x = \cosh t$, $y = \sinh t$. Show that $\frac{dy}{dx} = \coth t$, and find a Cartesian equation of the curve. [3]
- 12** A curve has parametric equations $x = t^3$, $y = t^2$, for $t > 0$. Show that $\frac{d^2y}{dx^2} = -\frac{2}{9t^4}$. [4]

- 13** A curve has parametric equations $x = 2t$, $y = t^2 - 4t$. Find the coordinates of the point on the curve where the tangent is parallel to the x axis, and state whether this point is a maximum or minimum of y . [3]
-
- 14** A curve has parametric equations $x = t^2$, $y = t(4 - t^2)$, for $0 \leq t \leq 2$. The curve meets the x axis at the origin and at the point A. Find the coordinates of A, and show that the area enclosed between the curve and the x axis is $\frac{128}{15}$. [5]
-
- 15** An ellipse has parametric equations $x = 4 \cos t$, $y = 3 \sin t$, for $0 \leq t < 2\pi$. Use integration with parametric equations to show that the area of the ellipse is 12π . [5]
-
- 16** The upper half of the ellipse with parametric equations $x = 2 \cos t$, $y = \sin t$, for $0 \leq t \leq \pi$, is rotated through 2π radians about the x axis. Find the exact volume of the solid generated. [5]
-
- 17** A curve has parametric equations $x = \ln t$, $y = t + \frac{1}{t}$, for $t > 0$.
 (a) Show that $\frac{dy}{dx} = t - \frac{1}{t}$. [2]
 (b) Find $\frac{d^2y}{dx^2}$, and hence find the coordinates of the stationary point of the curve and determine its nature. [5]
-
- 18** A cycloid has parametric equations $x = t - \sin t$, $y = 1 - \cos t$. Show that $\frac{dy}{dx} = \cot(\frac{t}{2})$, and find an equation of the tangent at the point where $t = \frac{\pi}{2}$. [5]
-
- 19** A curve has parametric equations $x = t^2$, $y = 2t$. The line $y = x - 3$ meets the curve at the points A and B. Find the exact length of the chord AB. [4]
-
- 20** A curve has parametric equations $x = t^2$, $y = 2t$, for $t \in \mathbb{R}$. The point P on the curve has parameter p , where $p \neq 0$.
 (a) Show that an equation of the tangent at P is $py = x + p^2$. [3]
 (b) The tangent at P passes through the point $(-4, 2)$. Find the possible values of p . [3]
-
- 21** The curve C has parametric equations $x = t^2$, $y = 2t$. The point P has parameter p , where $p \neq 0$.
 (a) Show that an equation of the normal to C at P is $y + px = 2p + p^3$. [3]
 (b) The normal at P meets C again at the point Q with parameter q . Show that $p^2q + p^2 + 2 = 0$. [4]
 (c) Hence find the coordinates of Q when $p = 1$. [2]
-
- 22** The astroid has parametric equations $x = \cos^3 t$, $y = \sin^3 t$, for $0 \leq t < 2\pi$. Show that the area enclosed by the astroid is $\frac{3\pi}{8}$.
 (You may use $\int_0^{\pi/2} \sin^4 t \cos^2 t \, dt = \frac{\pi}{32}$.) [7]
-
- 23** One arch of a cycloid has parametric equations $x = t - \sin t$, $y = 1 - \cos t$, for $0 \leq t \leq 2\pi$. Show that the area between the arch and the x axis is 3π . [6]
-

- 24** A student is asked to find the gradient of the curve $x = t^3$, $y = t^2$ at $t = 1$ and writes:
 $\frac{dy}{dx} = \frac{dx}{dt} \div \frac{dy}{dt} = \frac{3t^2}{2t} = \frac{3}{2}$,
 (a) Identify the error. [1]
 (b) Find the correct gradient at $t = 1$, and hence find where the tangent at $t = 1$ meets the x axis. [5]
-
- 25** The curve L has parametric equations $x = \sin t$, $y = \sin 2t$, for $0 \leq t \leq 2\pi$. It forms a figure of eight.
 (a) Show that $\frac{dy}{dx} = \frac{2 \cos 2t}{\cos t}$, for $\cos t \neq 0$. [2]
 (b) Find the exact coordinates of the points on L , for $0 < t < \pi$, where the tangent is parallel to the x axis. [3]
 (c) Show that the area of the right-hand loop of L (the loop where $x \geq 0$) is $\frac{4}{3}$. [3]
-
- 26** The path of a ball is modelled by $x = 20t$, $y = 15t - 4.9t^2$, for $t \geq 0$, where x and y are in metres and t is in seconds. Take $g = 9.8 \text{ m s}^{-2}$.
 (a) Find $\frac{dy}{dx}$ in terms of t , and the gradient of the path when $t = 1$. [3]
 (b) Find the greatest height reached, to 3 significant figures. [2]
 (c) Find the horizontal range, to 3 significant figures. [2]
-
- 27** A curve has parametric equations $x = f(t)$, $y = g(t)$. Dots denote differentiation with respect to t .
 (a) Show that $\frac{d^2 y}{dx^2} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^3}$. [4]
 (b) Hence show that for the cycloid $x = t - \sin t$, $y = 1 - \cos t$ ($0 < t < 2\pi$), $\frac{d^2 y}{dx^2} = -\frac{1}{(1 - \cos t)^2}$. [4]
 (c) Hence explain why the cycloid is concave downward (that is, $\frac{d^2 y}{dx^2} < 0$) everywhere on its arch, and find the value of $\frac{d^2 y}{dx^2}$ at $t = \frac{\pi}{2}$. [2]
-
- 28** A curve has parametric equations $x = t^2$, $y = t^3 - 3t$, for $t \in \mathbb{R}$. The curve crosses itself at the point N .
 (a) Find the coordinates of N , and the values of t at which the curve passes through N . [3]
 (b) Find the acute angle between the two tangents to the curve at N . [3]
 (c) Show that the area of the loop enclosed between $t = -\sqrt{3}$ and $t = \sqrt{3}$ is $\frac{24\sqrt{3}}{5}$. [4]
-
- 29** A spiral has parametric equations $x = t \cos t$, $y = t \sin t$, for $t \geq 0$.
 (a) Show that $\frac{dy}{dx} = \frac{\sin t + t \cos t}{\cos t - t \sin t}$, and find the gradient at $t = \frac{\pi}{2}$. [3]
 (b) Show that the tangent is parallel to the x axis when $\tan t = -t$. [2]
 (c) Show that $(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 = 1 + t^2$, and hence find the exact length of the spiral from $t = 0$ to $t = \pi$, in terms of π . [4]
-
- 30** The curve C has parametric equations $x = \frac{3t}{1+t^3}$, $y = \frac{3t^2}{1+t^3}$, for $t > 0$.
 (a) Show that $x^3 + y^3 = 3xy$. [3]
 (b) Show that $\frac{dy}{dx} = \frac{t(2-t^3)}{1-2t^3}$. [4]
 (c) Find the exact coordinates of the point of C where the tangent is parallel to the x axis, and of the point where the tangent is parallel to the y axis. [3]

Further Trigonometric Integration

- 1 Find $\int \sin^2 x \, dx$. [3]

- 2 Evaluate $\int_0^{\pi} \sin^2 x \, dx$. [3]

- 3 Find $\int \tan x \, dx$. [2]

- 4 Find $\int \cos^3 x \, dx$. [3]

- 5 Find $\int \sin 4x \cos 2x \, dx$. [3]

- 6 Evaluate $\int_0^{\pi/4} \sec^2 x \, dx$. [2]

- 7 Find $\int \sin^3 x \cos^2 x \, dx$. [4]

- 8 Find $\int \tan^3 x \, dx$. [4]

- 9 Show that $\int_0^{\pi/2} \cos^4 x \, dx = \frac{3\pi}{16}$. [4]

- 10 Evaluate $\int_0^{\pi/4} \sec^4 x \, dx$. [4]

- 11 Find the exact value of $\int_0^{\pi/3} \cos 5x \cos 2x \, dx$. [4]

- 12 Use the substitution $u = \sin x$ to show that $\int_0^{\pi/2} \frac{\cos x}{1 + \sin^2 x} \, dx = \frac{\pi}{4}$. [3]

- 13 Find $\int \sin^2 x \cos^2 x \, dx$. [4]

- 14 Let $t = \tan(\frac{x}{2})$.
 - (a) Show that $\sin x = \frac{2t}{1+t^2}$ and $dx = \frac{2}{1+t^2} dt$. [3]
 - (b) Hence show that $\int_0^{\pi/2} \frac{1}{1 + \sin x} \, dx = 1$. [3]

- 15 Use the substitution $t = \tan(\frac{x}{2})$ to show that $\int_0^{\pi/2} \frac{1}{2 + \cos x} \, dx = \frac{\pi}{3\sqrt{3}}$.
 (You may use $\cos x = \frac{1-t^2}{1+t^2}$ and $dx = \frac{2}{1+t^2} dt$.) [5]

- 16 Use the substitution $t = \tan(\frac{x}{2})$ to show that $\int \operatorname{cosec} x \, dx = \ln|\tan(\frac{x}{2})| + c$.
 Hence find the exact value of $\int_{\pi/3}^{\pi/2} \operatorname{cosec} x \, dx$. [6]

- 17 Show that $\int_0^{\pi/4} \sec^3 x \, dx = \frac{1}{2}(\sqrt{2} + \ln(1 + \sqrt{2}))$. [5]

- 18** Find $\int \tan^4 x \, dx$. [4]
-
- 19** Evaluate $\int_0^{\pi/4} \sec^4 x \tan^2 x \, dx$. [4]
-
- 20** Find $\int \sin^4 x \, dx$. [5]
-
- 21** Use the substitution $t = \tan(\frac{x}{2})$ to show that $\int_0^{\pi/2} \frac{1}{5 + 4 \sin x} \, dx = \frac{2}{3} \arctan(\frac{1}{3})$.
(You may use $\sin x = \frac{2t}{1+t^2}$ and $dx = \frac{2}{1+t^2} \, dt$.) [7]
-
- 22** Use the substitution $t = \tan(\frac{x}{2})$ to find the exact value of $\int_0^{\pi/3} \frac{1}{1 + \cos x + \sin x} \, dx$. [6]
-
- 23** Let $t = \tan(\frac{x}{2})$, so that $\cos x = \frac{1-t^2}{1+t^2}$ and $dx = \frac{2}{1+t^2} \, dt$.
(a) Show that $\int \sec x \, dx = \ln \left| \frac{1+t}{1-t} \right| + c$. [4]
(b) Hence show that $\int_0^{\pi/3} \sec x \, dx = \ln(2 + \sqrt{3})$. [3]
-
- 24** A student writes: ' $\int \sin^2 x \, dx = \frac{\sin^3 x}{3} + c$ '.
(a) Show that the student's answer is incorrect by differentiating it. [2]
(b) Find the exact value of $\int_0^{\pi/6} \sin^2 x \, dx$. [4]
-
- 25** Use the substitution $u = \sin x$, and partial fractions, to show that $\int_0^{\pi/2} \frac{\cos x}{(1 + \sin x)(2 + \sin x)} \, dx = \ln(\frac{4}{3})$. [6]
-
- 26** Show that $\int_0^{\pi/4} \frac{1}{1 + \cos^2 x} \, dx = \frac{1}{\sqrt{2}} \arctan(\frac{1}{\sqrt{2}})$, by using the substitution $u = \tan x$. [6]
-
- 27** $I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} \, dx$ and $J = \int_0^{\pi/2} \frac{\cos x}{\sin x + \cos x} \, dx$.
(a) Show that $I + J = \frac{\pi}{2}$. [2]
(b) By using the substitution $u = \frac{\pi}{2} - x$ in I , show that $I = J$, and hence find the value of I . [4]
(c) Hence find the exact value of $\int_0^{\pi/2} \frac{\sin x - \cos x}{\sin x + \cos x} \, dx$, and explain geometrically why the result is not a surprise. [3]
-
- 28** $I = \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} \, dx$. Show that $I = \frac{\pi^2}{4}$. [9]
-
- 29** For positive integers m and n , let $I(m, n) = \int_0^{2\pi} \sin(mx) \sin(nx) \, dx$.
(a) Show that $I(m, n) = 0$ when $m \neq n$. [3]
(b) Show that $I(n, n) = \pi$. [2]
(c) Hence find the exact value of $\int_0^{2\pi} (\sin 2x + 3 \sin 5x)^2 \, dx$. [3]
-

- 30** For a constant $a > 1$, let $I(a) = \int_0^{\pi} \frac{1}{a + \cos x} dx$.
- (a) Use the substitution $t = \tan(\frac{x}{2})$ to show that $I(a) = \int_0^{\infty} \frac{2}{(a+1) + (a-1)t^2} dt$. [4]
- (b) Hence show that $I(a) = \frac{\pi}{\sqrt{a^2 - 1}}$. [4]
- (c) Find the value of a for which $I(a) = \frac{\pi}{2}$, and verify your answer with the result of part (b) for $a = 3$. [2]

Differentiation Under the Integral Sign

- 1** Find $\frac{d}{dx} \int_1^x t^2 \ln t dt$. [2]
- 2** Find $\frac{d}{dx} \int_0^{x^2} \cos t dt$. [2]
- 3** Find $\frac{d}{dx} \int_x^3 \sqrt{1+t^4} dt$. [2]
- 4** Use the rule for differentiating an integral with variable limits to find $\frac{d}{dx} \int_x^{2x} t^2 dt$, and check your answer by integrating first. [3]
- 5** Let $F(a) = \int_0^1 x^a dx$ for $a > -1$. Write down $F(a)$, and by differentiating under the integral sign show that $\int_0^1 x^a \ln x dx = -\frac{1}{(a+1)^2}$. [3]
- 6** Find $\frac{d}{da} \int_0^1 (ax^2 + \sin a) dx$. [2]
- 7** Find $\frac{d}{dx} \int_x^{x^2} \ln t dt$. [3]
- 8** Given that $G(x) = \int_0^x \frac{1}{1+t^2} dt$, use the fundamental theorem of calculus to find $G'(x)$, and hence identify $G(x)$ in terms of an inverse trigonometric function. [3]
- 9** Let $F(a) = \int_0^{\pi} \cos(ax) dx$ for $a > 0$. Show that $F(a) = \frac{\sin(\pi a)}{a}$, and by differentiating both sides with respect to a , show that $\int_0^{\pi} x \sin(ax) dx = \frac{\sin(\pi a) - \pi a \cos(\pi a)}{a^2}$. [4]
- 10** Given that $\int_0^{\infty} e^{-ax} dx = \frac{1}{a}$ for $a > 0$, differentiate with respect to a to find $\int_0^{\infty} x e^{-ax} dx$ and $\int_0^{\infty} x^2 e^{-ax} dx$. [4]
- 11** $F(x) = \int_x^{2x} e^{-t^2} dt$. Find the exact values of x for which $F'(x) = 0$. [4]
- 12** Use the fundamental theorem of calculus and L'Hôpital's rule to evaluate $\lim_{x \rightarrow 0} \frac{1}{x} \int_0^x \cos(t^2) dt$. [3]

- 13** Show that if $F(x) = \int_{\sin x}^{\cos x} t^3 dt$ then $F'(x) = -\sin x \cos x$. [3]
-
- 14** The function f satisfies $f(x) = 2 + \int_1^x t f(t) dt$ for $x > 0$.
Find $f(x)$. [5]
-
- 15** For $a > -1$, let $I(a) = \int_0^1 \frac{x^a - 1}{\ln x} dx$.
By differentiating with respect to a , show that $I(a) = \ln(1 + a)$. [6]
-
- 16** Given that $I(a) = \int_0^\infty e^{-ax} \sin x dx = \frac{1}{1 + a^2}$ for $a > 0$, differentiate with respect to a to show that $\int_0^\infty x e^{-ax} \sin x dx = \frac{2a}{(1 + a^2)^2}$, and write down the value of $\int_0^\infty x e^{-2x} \sin x dx$. [5]
-
- 17** The function F is defined by $F(x) = \int_0^x \sin(xt) dt$ for $x > 0$.
(a) Show by integrating first that $F(x) = \frac{1 - \cos(x^2)}{x}$, and hence find $F'(x)$. [3]
(b) Use the Leibniz rule $F'(x) = \sin(x \times x) + \int_0^x \frac{\partial}{\partial x} \sin(xt) dt$ to obtain the same result. [3]
-
- 18** $F(x) = \int_x^{2x} e^{-t^2} dt$ for $x > 0$.
(a) Show that F has a single stationary point for $x > 0$, at $x = \sqrt{\frac{\ln 2}{3}}$. [3]
(b) Determine the nature of this stationary point. [3]
-
- 19** It is given that $I(k) = \int_0^\infty \frac{1}{x^2 + k} dx = \frac{\pi}{2\sqrt{k}}$ for $k > 0$.
By differentiating with respect to k , show that $\int_0^\infty \frac{1}{(x^2 + k)^2} dx = \frac{\pi}{4k^{3/2}}$, and hence find the exact value of $\int_0^\infty \frac{1}{(x^2 + 4)^2} dx$. [5]
-
- 20** It is given that $\int_0^\pi \frac{1}{a + \cos x} dx = \frac{\pi}{\sqrt{a^2 - 1}}$ for $a > 1$.
Differentiate both sides with respect to a to show that $\int_0^\pi \frac{1}{(a + \cos x)^2} dx = \frac{\pi a}{(a^2 - 1)^{3/2}}$, and hence find the exact value of $\int_0^\pi \frac{1}{(2 + \cos x)^2} dx$. [5]
-
- 21** For positive constants a and b , let $I(a) = \int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} dx$.
By differentiating with respect to a , show that $I(a) = \ln\left(\frac{b}{a}\right)$. [6]
-
- 22** For $a \geq 0$, let $I(a) = \int_0^\infty e^{-ax} \frac{\sin x}{x} dx$. It may be assumed that $I(a) \rightarrow 0$ as $a \rightarrow \infty$ and that differentiation under the integral sign is valid.
(a) Show that $I'(a) = -\frac{1}{1 + a^2}$, using $\int_0^\infty e^{-ax} \sin x dx = \frac{1}{1 + a^2}$. [2]
(b) Hence show that $I(a) = \frac{\pi}{2} - \arctan a$. [3]
(c) Write down the value of $\int_0^\infty \frac{\sin x}{x} dx$, and find the exact value of $\int_0^\infty e^{-x} \frac{\sin x}{x} dx$. [3]

- 23** Let $I(a) = \int_0^1 x^a dx = \frac{1}{a+1}$ for $a > -1$.
- (a) By differentiating twice with respect to a , show that $\int_0^1 x^a (\ln x)^2 dx = \frac{2}{(a+1)^3}$. [3]
- (b) Hence find the exact value of $\int_0^1 x^2 (\ln x)^3 dx$. [3]
-
- 24** Let $F(x) = \int_x^{x^2} \frac{e^{xt}}{t} dt$, for $x > 1$. Show that $F'(x) = \frac{3e^{x^3} - 2e^{x^2}}{x}$. [6]
-
- 25** A student writes: ' $\frac{d}{dx} \int_0^x t e^x dt = x e^x$ ', treating e^x as if it did not depend on x inside the integral.
- (a) Explain the error. [1]
- (b) Find the correct derivative, and check it by integrating first. [5]
-
- 26** The function f satisfies $f(x) = 1 + \int_0^x (x-t)f(t) dt$ for all x .
- (a) Show that $f'(x) = \int_0^x f(t) dt$, and hence that $f''(x) = f(x)$. [4]
- (b) Write down the values of $f(0)$ and $f'(0)$, and hence find $f(x)$. [3]
-
- 27** Let $I(a) = \int_0^\infty e^{-x^2} \cos(ax) dx$. It is given that $I(0) = \frac{\sqrt{\pi}}{2}$ and that differentiation under the integral sign is valid.
- (a) Show that $I'(a) = -\int_0^\infty x e^{-x^2} \sin(ax) dx$. [1]
- (b) Use integration by parts to show that $I'(a) = -\frac{a}{2} I(a)$. [5]
- (c) Hence show that $I(a) = \frac{\sqrt{\pi}}{2} e^{-a^2/4}$. [3]
-
- 28** For $0 \leq a < 1$, let $I(a) = \int_0^{\pi/2} \frac{\ln(1+a \cos x)}{\cos x} dx$.
- (a) Show that $I'(a) = \int_0^{\pi/2} \frac{1}{1+a \cos x} dx$. [2]
- (b) Using the substitution $t = \tan(\frac{x}{2})$, show that $I'(a) = \frac{\arccos a}{\sqrt{1-a^2}}$. [5]
- (c) Hence show that $I(a) = \frac{\pi^2}{8} - \frac{1}{2}(\arccos a)^2$, and find $I(\frac{1}{2})$ exactly. [3]
-
- 29** For $n \geq 1$ let $F_n(x) = \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} f(t) dt$, where f is a continuous function, and $F_1(x) = \int_0^x f(t) dt$.
- (a) Use the Leibniz rule to show that $F_n'(x) = F_{n-1}(x)$ for $n \geq 2$. [4]
- (b) Explain why $F_3(x)$ is the result of integrating f from 0 three times (each integral taken from 0 to x), and find $F_3(x)$ for $f(t) = e^t$. [4]
-
- 30** For positive constants a and b , let $I(a) = \int_0^\infty \frac{\arctan(ax) - \arctan(bx)}{x} dx$.
- (a) Show that $I'(a) = \frac{\pi}{2a}$. [4]
- (b) Hence show that $I(a) = \frac{\pi}{2} \ln(\frac{a}{b})$. [3]
- (c) Write down the exact value of $\int_0^\infty \frac{\arctan(3x) - \arctan x}{x} dx$. [1]